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Mathematik

Hebey-Vaugon conjecture II

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Preprint Nr. 14/2011

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ABSTRACT. In this paper we consider the remaining cases of Hebey–Vaugon conjecture. We give a positive answer to the conjecture.

1. INTRODUCTION

Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. Denote by $I(M,g)$, $C(M,g)$ and R_g the isometry group, the conformal transformations group and the scalar curvature, respectively. Let G be a subgroup of the isometry group $I(M,g)$. The equivariant Yamabe problem can be formulated as follows: *in the conformal class of g, there exists a $G-$invariant metric with constant scalar curvature.* Assuming the positive mass theorem and the Weyl vanishing conjecture (for more details on the subject, see [7], [10] and the references therein), E. Hebey and M. Vaugon [4] proved that this problem has solutions. Moreover, they proved that the infimum of Yamabe functional

$$I_g(\varphi) = \frac{\int_M |\nabla\varphi|^2 + \frac{n-2}{4(n-1)} R_g \varphi^2 \mathrm{d}v}{\|\varphi\|^2_{\frac{2n}{n-2}}} \tag{1}$$

over $G-$invariant nonnegative functions is achieved by a smooth positive $G-$invariant function. This function is a solution of the Yamabe equation, which is the Euler–Lagrange equation of I_g:

$$\Delta_g \varphi + \frac{n-2}{4(n-1)} R_g \varphi = \mu \varphi^{\frac{n+2}{n-2}}$$

One of the consequences of these results is that the following conjecture due to Lichnerowicz [6] is true.

LICHNEROWICZ CONJECTURE. *For every compact Riemannian manifold (M,g) which is not conformal to the unit sphere S^n endowed with its standard metric g_s, there exists a metric $\tilde{g}$ conformal to g for which $I(M,\tilde{g}) = C(M,g)$, and the scalar curvature $R_{\tilde{g}}$ is constant.*

The classical Yamabe problem, which consists of finding a conformal metric with constant scalar curvature on a compact Riemannian manifold, is a particular case of the equivariant Yamabe problem (it corresponds to $G = \{\mathrm{id}\}$). This problem was completely solved by H. Yamabe [13], N. Trudinger [12], T. Aubin [1] and R. Schoen [11]. The main idea to prove the existence of positive minimizers for I_g is to show that if (M,g) is not conformal to the sphere endowed with its standard metric, then

$$\mu(g) := \inf_{C^\infty(M)} I_g(\varphi) < \frac{1}{4} n(n-2)\omega_n^{2/n} \tag{2}$$

where ω_n is the volume of the unit sphere S^n.
T. Aubin [1] proved (2) in some cases by constructing a test function u_ε satisfying:

$$I_g(u_\varepsilon) < \frac{1}{4}n(n-2)\omega_n^{2/n}$$

He conjectured that (2) always holds except for the sphere. R. Schoen constructed another test function which involves the Green function of the conformal Laplacian $\Delta_g + \frac{n-2}{4(n-1)}R_g$. Using the positive mass theorem, R. Schoen proved (2) for all compact manifolds which are not conformal to (S^n, g_s). The solution of the Yamabe problem follows.

Later, E. Hebey and M. Vaugon [4] showed that we can generalize (2) for the equivariant case as follows:
Denote by $O_G(P)$ the orbit of $P \in M$ under G and by $\operatorname{card} O_G(P)$ its cardinal. Let $C_G^\infty(M)$ be the set of smooth G-invariant functions and

$$\mu_G(g) := \inf_{C_G^\infty(M)} I_g(\varphi)$$

Following E. Hebey and M. Vaugon [3, 4], we define the integer $\omega(P)$ at a point P as

$$\omega(P) = \inf\{i \in \mathbb{N} / \|\nabla^i W_g(P)\| \neq 0\} \ (\omega(P) = +\infty \text{ if } \forall i \in \mathbb{N},\ \|\nabla^i W_g(P)\| = 0)$$

Hebey–Vaugon conjecture. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ and G be a subgroup of $I(M, g)$. If (M, g) is not conformal to (S^n, g_s) or if the action of G has no fixed point, then the following inequality holds*

$$\mu_G(g) < \frac{1}{4}n(n-2)\omega_n^{2/n}(\inf_{Q \in M} \operatorname{card} O_G(Q))^{2/n} \tag{3}$$

E. Hebey and M. Vaugon showed that if this conjecture holds, then it implies that the equivariant Yamabe problem has minimizing solutions and the Lichnerowicz conjecture is also true. Notice that if $G = \{\mathrm{id}\}$, then this conjecture corresponds to (2).
Let us recall the results already known about this conjecture. Assuming the positive mass theorem, E. Hebey and M. Vaugon [4] proved the following:

Theorem 1.1 (E. Hebey and M. Vaugon)**.** *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ and G be a subgroup of $I(M, g)$. We always have :*

$$\mu_G(g) \leq \frac{1}{4}n(n-2)\omega_n^{2/n}(\inf_{Q \in M} \operatorname{card} O_G(Q))^{2/n}$$

and inequality (3) holds if at least one of the following conditions is satisfied.

1. *The action of G on M is free.*
2. *$3 \leq \dim M \leq 11$.*
3. *There exists a point $P \in M$ with finite minimal orbit under G such that $\omega(P) > (n-6)/2$ or $\omega(P) \in \{0, 1, 2\}$.*

We have also the following result obtained by the author in [9]:

Theorem 1.2. *Hebey–Vaugon conjecture holds for every smooth compact Riemannian manifold (M, g) of dimension $n \leq 37$.*

The main result of this paper is the following:

Theorem 1.3. *If there exists a point $P \in M$ such that $\omega(P) \leq (n-6)/2$, then*

$$\mu_G(g) < \frac{1}{4} n(n-2) \omega_n^{2/n} (\operatorname{card} O_G(P))^{2/n} \tag{4}$$

Note that if we assume the positive mass theorem, then Theorem 1.3 and Theorem 1.1 implies that Hebey–Vaugon conjecture holds.
The proof of Theorem 1.3 doesn't require the positive mass theorem. If $\operatorname{card} O_G(Q) = +\infty$ for all $Q \in M$, then (3) holds. So we have to consider only the case when there exists a point in M with finite orbit. From now until the end of this paper, we suppose that $P \in M$ is contained in a finite orbit and $\omega(P) \leq \frac{n-6}{2}$. The assumption $\omega(P) \leq \frac{n-6}{2}$ deletes the case (M, g) is conformal to (S^n, g_s).

2. G-invariant test function

In order to prove Theorem 1.3 and 1.2, we construct from the function $\varphi_{\varepsilon,P}$ defined below a G-invariant test function ϕ_ε such that

$$I_g(\phi_\varepsilon) < \frac{1}{4} n(n-2) \omega_n^{2/n} (\operatorname{card} O_G(P))^{2/n} \tag{5}$$

Let us recall the construction in [9] of $\varphi_{\varepsilon,P}$. Let $\{x^j\}$ be the geodesic normal coordinates in the neighborhood of P and define $r = |x|$ and $\xi^j = x^j/r$. Without loss of generality, we suppose that $\det g = 1 + O(r^N)$, with $N > 0$ sufficiently large (for the existence of such coordinates for a G−invariant conformal class, see [4], [5]).

$$\varphi_{\varepsilon,P}(Q) = (1 - r^{\omega(P)+2} f(\xi)) u_{\varepsilon,P}(Q)$$

$$u_{\varepsilon,P}(Q) = \begin{cases} \left(\dfrac{\varepsilon}{r^2+\varepsilon^2}\right)^{\frac{n-2}{2}} - \left(\dfrac{\varepsilon}{\delta^2+\varepsilon^2}\right)^{\frac{n-2}{2}} & \text{if } Q \in B_P(\delta) \\ 0 & \text{if } Q \in M - B_P(\delta) \end{cases}$$

for all $Q \in M$, where $r = d(Q,P)$ is the distance between P and Q, and $B_P(\delta)$ is the geodesic ball of center P and radius δ fixed sufficiently small. f is a function depending only on ξ (defined on S^{n-1}), chosen such that $\int_{S^{n-1}} f d\sigma = 0$.

Let $\bar{R}$ be the leading part in the Taylor expansion of the scalar curvature R_g in a neighborhood of P and $\mu(P)$ is its degree. Hence,

$$R_g(Q) = \bar{R} + O(r^{\mu(P)+1})$$

$$\bar{R} = r^{\mu(P)} \sum_{|\beta| = \mu(P)} \nabla_\beta R_g(P) \xi^\beta$$

We summarize some properties of $\bar{R}$ in the following proposition.

Proposition 2.1. *1. $\bar{R}$ is a homogeneous polynomial of degree $\mu(P)$ and is invariant under the action of the stabilizer group of P.*
2. We always have $\mu(P) \geq \omega(P)$

3. *if* $\mu(P) \geq \omega(P)+1$, *then* $\int_{S^{n-1}(r)} R\mathrm{d}\sigma < 0$ *for* $r>0$ *sufficiently small.*
4. *If* $\mu(P)=\omega(P)$, *then there exist eigenfunctions* φ_k *of the Laplacian on* S^{n-1} *such that the restriction of* $\bar{R}$ *to the sphere is given by*
$$\bar{R}|_{S^{n-1}} = \sum_{k=1}^{q} \nu_k \varphi_k$$
where $q \leq [\omega(P)/2]$, $\Delta_s \varphi_k = \nu_k \varphi_k$ *and* $\nu_k = (\omega - 2k + 2)(n+\omega-2k)$ *are the eigenvalues of* Δ_s *with respect to the standard metric* g_s *of* S^{n-1}.

Since the scalar curvature is invariant under the action of the isometry group $I(M,g)$, $\bar{R}$ is invariant under the action of the stabilizer of P. The second statement of Proposition 2.1 is proven by E. Hebey and M. Vaugon ([4], Section 8) and the third one by T. Aubin ([2], Section 3). So, in the case $\mu(P) \geq \omega(P)+1$, the conjecture holds immediately, by choosing $f=0$, $\varphi_{\varepsilon,P} = u_{\varepsilon,P}$ (see [8, 9] for more details).
From now we suppose that $\mu(P)=\omega(P)$. Using the fact that $\bar{R}$ is homogeneous polynomial of degree $\omega(P)$ and the fact that for all $j \leq \omega(P)-1$

$$|\nabla^j R_g(P)| = 0, \quad \Delta_g^{j+1} R_g(P) = 0 \text{ and } |\nabla \Delta_g^{j+1} R_g(P)| = 0 \tag{6}$$

we deduce that $\Delta_{\mathcal{E}}^{[\omega(P)/2]} \bar{R} = 0$. Hence, if we restrict $\bar{R}$ to the sphere, we get the decomposition of item 4. in Proposition 2.1. The proof of (6) is given in [4], Section 8.
Using the split of $\bar{R}$ given in Proposition 2.1, we proved in [9] that if the cardinal of $O_G(P)$ is minimal and $\omega(P) \leq 15$, then there exists $c \in \mathbb{R}$ such that for $f = c\bar{R}|_{S^{n-1}}$, the function
$$\phi_\varepsilon = \sum_{P_i \in O_G(P)} \varphi_{\varepsilon,P_i}$$
is G-invariant and satisfies (5), which proves Theorem 1.2. Moreover, we proved the following theorem:

Theorem 2.1. *If* $\omega(P) \leq (n-6)/2$, *then there exist* $c_k \in \mathbb{R}$, *such that for* $f = \sum_{k=1}^q c_k \varphi_k$, *the function* $\varphi_{\varepsilon,P}$ *satisfies*

$$I_g(\varphi_{\varepsilon,P}) < \frac{1}{4} n(n-2)\omega_n^{2/n} \tag{7}$$

The proof of Theorem 2.1 is technical and uses Proposition 2.1. It is given in [9] (see also [8] for a detailed proof).
Below, we show that using Theorem 2.1, we can construct a $G-$invariant function ϕ_ε which satisfies (5) for $\omega(P) \leq \frac{n-6}{2}$ (the cardinal of $O_G(P)$ is not necessarily minimal). It implies Theorem 1.3.

Proof of Theorem 1.3. Let $H \subset G$ be the stabilizer of P. We consider the function $f = \sum_{k=1}^q c_k \varphi_k$ of Theorem 2.1. Using the exponential map on P as a local chart, we can view f and φ_k as functions defined over the unit sphere of $T_P M$, the tangent space of M on P. Let h be an isometry in H.
$$h_*(P) : (T_P M, g_P) \to (T_P M, g_P)$$

is the linear tangent map of h on P. It is a linear isometry with respect to the inner product g_P which is Euclidean. $h_*(P)$ conserves the unit sphere $S^{n-1} \subset T_P M$ and the Laplacian. We already know that the function $\bar{R} = r^{\omega(P)} \sum_{k=1}^{q} \nu_k \varphi_k$ is H-invariant. Notice that φ_k and φ_j belong to two different eigenspaces if $k \neq j$. Since, isometries conserve the Laplacian and φ_k are eigenfunctions of the Laplacian on the sphere endowed with its standard metric, it yields that φ_k and f are H-invariant. On the other hand, we have the following bijective map:

$$G/H \longrightarrow O_G(P)$$
$$\sigma H \longmapsto \sigma(P)$$

Since f is H-invariant, $\varphi_{\varepsilon,P}$ is $H-$invariant and the function

$$\phi_\varepsilon = \sum_{\sigma \in G/H} \varphi_{\varepsilon,P} \circ \sigma^{-1}$$

is $G-$invariant and satisfies (5). □

Acknowledgments

The author would like to thank Bernd Ammann for his remarks and the helpful discussions. He also thanks Emmanuel Hebey for his suggestions.

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