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CLASSIFICATION OF TRACES AND HYPERTRACES ON SPACES OF CLASSICAL PSEUDODIFFERENTIAL OPERATORS

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Abstract. Let M be a closed manifold and let $\mathrm{CL}^\bullet(M)$ be the algebra of classical pseudodifferential operators. The aim of this note is to classify trace functionals on the subspaces $\mathrm{CL}^a(M) \subset \mathrm{CL}^\bullet(M)$ of operators of order a. $\mathrm{CL}^a(M)$ is a $\mathrm{CL}^0(M)$–module for any real a; it is an algebra only if a is a non–positive integer. Therefore, it turns out to be useful to introduce the notions of pretrace and hypertrace. Our main result gives a complete classification of pre– and hypertraces on $\mathrm{CL}^a(M)$ for any $a \in \mathbb{R}$, as well as the traces on $\mathrm{CL}^a(M)$ for $a \in \mathbb{Z}, a \leq 0$. We also extend these results to classical pseudodifferential operators acting on sections of a vector bundle.

As a byproduct we give a new proof of the well–known uniqueness results for the Guillemin–Wodzicki residue trace and for the Kontsevich–Vishik canonical trace. The novelty of our approach lies in the calculation of the cohomology groups of homogeneous and log–polyhomogeneous differential forms on a symplectic cone. This allows to give an extremely simple proof of a generalization of a Theorem of Guillemin about the representation of homogeneous functions as sums of Poisson brackets.

This paper exposes and extends some of the results of the Ph.D. Thesis [NJ10] of the second named author. We acknowledge with gratitude the substantial help received from Sylvie Paycha.

Contents

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1. Introduction and formulation of the result

Let M be a smooth closed connected riemannian manifold of dimension $n>1$. [1] We denote by $\mathrm{CL}^a(M)$ the space of classical pseudodifferential operators of order $a\in\mathbb{R}$ on M. It is well–known that the residue trace Res, which was discovered independently by V. Guillemin [Gui85] and M. Wodzicki [Wod87b], is up to normalization the unique trace on the algebra $\mathrm{CL}^{\mathbb{Z}}(M)$ of integer order classical pseudodifferential operators ([Wod87b], [BrGe87], [FGLS96], [Les99]). Res is non–trivial only on $\mathrm{CL}^k(M)$ for integers $k\geq -n$, and it is complemented by the canonical trace, TR, of Kontsevich and Vishik [KoVi95]. The latter is defined on operators of real order $a\neq -n,-n+1,\ldots$, it extends the Hilbert space trace on smoothing operators and it vanishes on commutators (for the precise statement see Eq. (3.12) below). By [MSS08] it is the unique functional which is linear on its domain, has the trace property and coincides with the L^2–operator trace on trace–class operators.

A natural problem which arises is to characterize the traces on the spaces $\mathrm{CL}^a(M)$. First, one has to note that $\mathrm{CL}^a(M)$ is always a $\mathrm{CL}^0(M)$–module; it is an algebra if and only if $a\in\mathbb{Z}_{\leq 0}=\{0,-1,-2,\ldots\}$. Let us call a functional on $\mathrm{CL}^a(M)$ a *hypertrace* (resp. *pretrace*) if $\tau([A,B])=0$ for $A\in\mathrm{CL}^0(M), B\in\mathrm{CL}^a(M)$ (resp. $A,B\in\mathrm{CL}^{a/2}(M)$), see Definition 3.1.

The above mentioned uniqueness results for Res and TR cannot extend to $\mathrm{CL}^a(M)$ for a simple reason: let T be a distribution on the cosphere bundle S^*M and denote by $\sigma_a:\mathrm{CL}^a(M)\to \mathrm{C}^\infty(S^*M)$ the leading symbol. Due to the multiplicativity of the leading symbol (Eq. (3.3)) the map $T\circ\sigma_a$ is a pretrace and a hypertrace on $\mathrm{CL}^a(M)$, and for $a\in\mathbb{Z}_{\leq 0}$ it is a trace on $\mathrm{CL}^a(M)$. $T\circ\sigma_a$ is called a leading symbol trace, [PaRo04].

For $\mathrm{CL}^0(M)$ it was already proved by Wodzicki [Wod87a] that any trace is a linear combination of Res and a leading symbol trace, see also Lescure, Paycha [LePa07] and Ponge [Pon10].

Before stating our generalization of this result we introduce a convenient notation which combines TR and Res. Namely, fix a *linear* functional $\widetilde{\mathrm{Tr}}:\mathrm{CL}^0(M)\to\mathbb{C}$ such

[1] The case $n=1$ has some peculiarities due to the non–connectedness of the cosphere bundle of S^1. As a consequence many results need to be slightly modified in the case $n=1$. These modifications are more annoying than difficult and for the sake of a clean exposition they are left to the reader. But see Remark 2.10.

that for $a \in \mathbb{Z}_{<-n} = \{-n-1, -n-2, \dots\}$

$$\widetilde{\mathrm{Tr}}_a = \widetilde{\mathrm{Tr}} \restriction \mathrm{CL}^a(M) = \mathrm{Tr} \restriction \mathrm{CL}^a(M) = \mathrm{Tr}_a,$$

and put

$$\overline{\mathrm{TR}}_a := \begin{cases} \mathrm{TR}_a, & \text{if } a \in \mathbb{R} \setminus \mathbb{Z}_{\geq -n}, \\ \widetilde{\mathrm{Tr}}_a, & \text{if } a \in \mathbb{Z}, -n \leq a < \frac{-n+1}{2}, \\ \mathrm{Res}_a, & \text{if } a \in \mathbb{Z}, \frac{-n+1}{2} \leq a \leq 0. \end{cases} \tag{1.1}$$

In this note we will prove:

Theorem 1.1. *Let M be a closed connected riemannian manifold of dimension $n > 1$.*
1. *Let $a \in \mathbb{R}$ and let τ be a hypertrace on $\mathrm{CL}^a(M)$. Then there are uniquely determined $\lambda \in \mathbb{C}$ and a distribution $T \in \big(\mathrm{C}^\infty(S^*M)\big)^*$ such that*

$$\tau = T \circ \sigma_a + \begin{cases} \lambda\, \overline{\mathrm{TR}}_a, & \text{if } a \notin \mathbb{Z}_{>-n}, \\ \lambda\ \mathrm{Res}_a, & \text{if } a \in \mathbb{Z}_{>-n}. \end{cases} \tag{1.2}$$

2. *Let $a \in \mathbb{Z}_{\leq 0}$, and denote by*

$$\pi_a : \mathrm{CL}^a(M) \longrightarrow \mathrm{CL}^a(M)/\,\mathrm{CL}^{2a-1}(M)$$

the quotient map. Let $\tau : \mathrm{CL}^a(M) \to \mathbb{C}$ be a trace. Then there are uniquely determined $\lambda \in \mathbb{C}$ and $T \in \big(\mathrm{CL}^a(M)/\,\mathrm{CL}^{2a-1}(M)\big)^*$ such that

$$\tau = \lambda\, \overline{\mathrm{TR}}_a + T \circ \pi_a. \tag{1.3}$$

This Theorem is a summary of Theorem 4.10, Theorem 4.12 and Corollary 4.13 in the text. It extends to the vector bundle case. This requires even more notation and is therefore not reproduced here in the introduction. The interested reader is referred to Theorem 5.7 in Section 5.

It is interesting to note that Res and TR as well as the leading symbol traces have precise analogues on the symbolic level. This analogy is not only formal but is used to prove Theorem 1.1. Namely, consider the Hörmander symbols $\mathrm{CS}^a(\mathbb{R}^n)\big(= \mathrm{CS}^a(\{0\} \times \mathbb{R}^n)\big)$. This is the space of smooth functions f on $\mathbb{R}^n$ such that $f \sim \sum_{j=0}^\infty f_{a-j}$ with $f_{a-j}(\xi)$ positively homogeneous of order $a - j$ for ξ large enough. The analogue of a hypertrace is then a linear functional $\tau : \mathrm{CS}^a(\mathbb{R}^n) \to \mathbb{C}$ such that $\tau(\partial_j f) = 0$ for $j = 1, \dots, n$. Such functionals have been investigated by Paycha [Pay07] and were partially classified (up to functionals on smoothing symbols).

Functionals with the "Stokes' property", $\tau(\partial_j f) = 0$, can most naturally be classified by looking at a certain variant of de Rham cohomology. Namely, putting $T(f d\xi_1 \wedge \ldots \wedge d\xi_n) := \tau(f)$ one obtains a linear function on the top degree de Rham cohomology of forms in $\mathbb{R}^n$ whose coefficients lie in $\mathrm{CS}^a(\mathbb{R}^n)$. While the calculation of this cohomology is possible, it will be postponed to a subsequent paper. Rather it turns out that to classify the functionals with the Stokes' property it suffices to calculate the de Rham cohomology of forms with homogeneous coefficients. This, in a sense a simple extension of Euler's identity from homogeneous functions to forms, will be carried out in Section 2. There are two main consequences: first we will be able to prove a generalization of Guillemin's Theorem [Gui85] on the representation of homogeneous functions on a symplectic cone as sums of Poisson brackets (Theorem 2.9). Secondly, we will be able to completely characterize the functionals on $\mathrm{CS}^a(\mathbb{R}^n)$ with the Stokes' property or equivalently when a function in $\mathrm{CS}^{a-1}(\mathbb{R}^n)$ can be written as a sum of partial derivatives of functions in $\mathrm{CS}^a(\mathbb{R}^n)$ (Proposition 2.12). This generalizes [Pay07, Prop. 2, Thm. 2].

These results about symbols allow to prove an improved version of the known results about the representation of a pseudodifferential operator as a sum of commutators (Theorem 4.6).

The paper is organized as follows:

In Section 2 we study homogeneous differential forms on cones and calculate their de Rham cohomology. As applications we prove the aforementioned generalization of Guillemin's Theorem on homogeneous functions and a characterization of functionals with the Stokes' property.

In Section 3 we first review some basic facts about pseudodifferential operators and trace functionals. We introduce pretraces and hypertraces and we give some examples. In Section 4 we apply the results of Section 2 and provide a result about the representation of a classical pseudodifferential operator as a sum of commutators. We use this result to give the classification of hypertraces and traces on $\mathrm{CL}^a(M)$ for different values of a. For the case of integral a we give two proofs, one relying on a result due to Ponge [Pon10] and a completely self–contained one in Subsection 4.4.

Finally, in Section 5 we extend the results about tracial functionals to operators acting on sections of vector bundles over the manifold. The main result then is Theorem 5.7.

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Lemma 4.5 was communicated to the second named author by Sylvie Paycha. This Lemma is crucial for the classification of traces on the integer order algebras $\mathrm{CL}^k(M, E)$ for $k < 0$ as well as for showing that every pretrace is a hypertrace, cf. Proposition 4.9, Proposition 5.1. Furthermore, the alternative approach to Theorem 4.12 presented in Subsection 4.4 is based on joint work of the second named author and Sylvie Paycha. These results are included here with the kind permission of Sylvie Paycha whose generosity is greatly appreciated.

The results of Section 2, in particular the simple approach to the generalization of a result by Guillemin, Theorem 2.9, using homogeneous cohomology (Subsections 2.1, 2.2) is due to the first named author and a rather general version of homogeneous cohomology was announced in [Les11, Sec. 7]. The second named author acknowledges the kind permission to include these results in her thesis [NJ10]. Lemma 4.5 and Subsection 4.4 are not needed to prove the classification results about hypertraces contained in Theorem 4.10, 4.12, and Theorem 5.7. These results, due to the first named author, are also independent of [NJ10]. Section 5, written entirely by ML, supercedes the in part erroneous Section 5.2 of [NJ10].

2. Cohomology of homogeneous differential forms

In this section we calculate the de Rham cohomology of homogeneous differential forms on cones. The theory is stunningly simple. Nevertheless as corollaries we obtain generalizations of the results of V. Guillemin [Gui85] on the representation of homogeneous functions on *symplectic* cones as sums of Poisson brackets. Also our approach generalizes the theory of homogeneous functions on $\mathbb{R}^n \setminus \{0\}$ in a straightforward way. Therefore, we also obtain as a corollary the precise criterion when a homogeneous function can be written as a sum of partial derivatives of homogeneous functions, cf. [FGLS96], [Les99]. Finally, this criterion is generalized to classical symbol functions, generalizing [Pay07, Prop. 2, Thm. 2].

2.1. **Homogeneous differential forms on cones.** A *cone* over a manifold B is a principal bundle $\pi : Y \to B$ with structure group $\mathbb{R}_+^*$, the multiplicative group of positive real numbers. Denote by $\varrho_\lambda : Y \to Y$, the action of $\lambda \in \mathbb{R}_+^*$. Via $\Phi_t := \varrho_{e^t}$ we obtain a one parameter group of diffeomorphisms of Y. Let $\mathcal{X} \in \mathrm{C}^\infty(TY)$ be the infinitesimal generator of this group, which is sometimes called the *Liouville vector field.*

A differential form $\omega \in \Omega^p(Y)$ is called *homogeneous* of degree a if $\varrho_\lambda^*\omega = \lambda^a\omega$ for all $\lambda \in \mathbb{R}_+^*$. The space of differential forms of form degree p and homogeneity a is denoted by $\Omega^p\mathcal{P}^a(Y)$. $\mathcal{P}^a(Y) := \Omega^0\mathcal{P}^a(Y)$ are the smooth functions on Y which are homogeneous of degree a.

We choose a function $r \in \mathcal{P}^1(Y)$ which is everywhere positive and put $Z := \big\{y \in Y \bigm| r(y) = 1\big\}$. $\pi_{|Z}$ is a diffeomorphism from Z onto B and r induces a trivialization of Y as follows:

$$\Phi : Y \longrightarrow \mathbb{R}_+^* \times Z, \quad y \ \mapsto (r(y), \varrho_{r(y)^{-1}}y). \tag{2.1}$$

Note that

$$\Phi(\varrho_\lambda(y)) = (r(\varrho_\lambda(y)), \varrho_{r(\varrho_\lambda(y))^{-1}}\varrho_\lambda(y)) = (\lambda r(y), \varrho_{r(y)^{-1}}y). \tag{2.2}$$

Hence Φ intertwines the $\mathbb{R}_+^*$ action on Y and the natural $\mathbb{R}_+^*$ action on the product $\mathbb{R}_+^* \times Z$. For convenience we will from now on work with the trivialized bundle $\mathbb{R}_+^* \times Z$. The first coordinate will be called r, so the Liouville vector field is then given by $\mathcal{X} = r\frac{\partial}{\partial r}$.

With the projection $\pi : \mathbb{R}_+^* \times Z \to Z$, a differential form $\omega \in \Omega^p\mathcal{P}^a(\mathbb{R}_+^* \times Z)$ can be written

$$\omega = r^{a-1}dr \wedge \pi^*\tau + r^a\pi^*\eta \tag{2.3}$$

with

$$\eta = i_Z^*\omega \in \Omega^p(Z), \quad \tau = i_Z^*(\iota_{\mathcal{X}}\omega) \in \Omega^{p-1}(Z), \tag{2.4}$$

where $i_Z : Z \hookrightarrow Y$ is the inclusion map and $\iota_{\mathcal{X}}$ denotes interior multiplication by the Liouville vector field $\mathcal{X}$. We have furthermore

$$d\omega = r^{a-1}dr \wedge (a\pi^*\eta - \pi^*d_Z\tau) + r^a\pi^*d_Z\eta \in \Omega^{p+1}\mathcal{P}^a(\mathbb{R}_+^* \times Z), \tag{2.5}$$

so exterior derivation preserves the homogeneity degree. Hence we can form the *homogeneous de Rham cohomology groups*

$$H^p\mathcal{P}^a(Y) := \frac{\ker\big(d : \Omega^p\mathcal{P}^a(Y) \longrightarrow \Omega^{p+1}\mathcal{P}^a(Y)\big)}{\operatorname{im}\big(d : \Omega^{p-1}\mathcal{P}^a(Y) \longrightarrow \Omega^p\mathcal{P}^a(Y)\big)}. \tag{2.6}$$

These cohomology groups can easily be calculated:

Theorem 2.1. *Let Z be a smooth paracompact manifold, let $\pi : Y \to Z$ be a $\mathbb{R}^*_+$ principal bundle over Z.*

(1) *If $a \neq 0$ then $H^p\mathcal{P}^a(Y) = \{0\}$.*
(2) *If $a = 0$ then the map*

$$\Psi : \Omega^\bullet\mathcal{P}^0(Y) \longrightarrow \Omega^{\bullet-1}(Z) \oplus \Omega^\bullet(Z)$$
$$\omega \;\mapsto \big(i_Z^*(\iota_{\mathcal{X}}\omega), i_Z^*\omega\big)$$

is up to a sign a complex isomorphism, in particular it induces an isomorphism

$$H^p\mathcal{P}^0(Y) \cong H^{p-1}(Z) \oplus H^p(Z). \tag{2.7}$$

In terms of the everywhere positive function $r \in \mathcal{P}^1(Y)$ the inverse of Ψ is given by $(\tau, \eta) \mapsto r^{-1}dr \wedge \pi^*\tau + \pi^*\eta$.

Proof. As before we work with the trivialized bundle $\mathbb{R}^*_+ \times Z$. If ω is closed then (2.5) implies that

$$d_Z\tau = a\eta, \quad d_Z\eta = 0, \tag{2.8}$$

and hence we obtain a form analogue of *Euler's identity* (see Eq. (2.19) below)

$$d\big(i_{\mathcal{X}}\omega\big) = d(r^a\pi^*\tau) = ar^{a-1}dr \wedge \pi^*\tau + r^a\pi^*d_Z\tau = a\omega. \tag{2.9}$$

Thus ω is exact if $a \neq 0$, explicitly

$$\omega = \frac{1}{a}d\big(i_{\mathcal{X}}\omega\big). \tag{2.10}$$

Now let $a = 0$ and consider $\omega \in \Omega^p\mathcal{P}^0(\mathbb{R}^*_+ \times Z)$. Since $\varrho^*_{e^t}\omega = \omega$,

$$\mathcal{L}_{\mathcal{X}}\omega = \frac{d}{dt}\Big|_{t=0}\varrho^*_{e^t}\omega = 0, \tag{2.11}$$

and Cartan's magic formula $d\iota_{\mathcal{X}} + \iota_{\mathcal{X}}d = \mathcal{L}_{\mathcal{X}}$ implies that $d\iota_{\mathcal{X}}\omega = -\iota_{\mathcal{X}}d\omega$. Thus

$$d\big(i_Z^*(\iota_{\mathcal{X}}\omega), i_Z^*\omega\big) = \big(-i_Z^*(\iota_{\mathcal{X}}d\omega), i_Z^*d\omega\big), \tag{2.12}$$

and hence the exterior derivative on $\Omega^{\bullet-1}(Z) \oplus \Omega^\bullet(Z)$ can be modified by a sign such that Ψ becomes a complex homomorphism. Furthermore,

$$\omega = r^{-1}dr \wedge \pi^*\tau + \pi^*\eta \tag{2.13}$$

and, since $r^{-1}dr$ is closed,

$$d\omega = \pi^*d_Z\eta - r^{-1}dr \wedge \pi^*d_Z\tau, \tag{2.14}$$

from which (2.7) is now obvious. □

Remark 2.2. We comment on a special case of Theorem 2.1 which combines the constructions of the residue of a homogeneous function on $\mathbb{R}^n\setminus\{0\}$ (see the next Subsection) and of Guillemin's symplectic residue (Subsection 2.3).

Let $\dim Y = n$ and suppose that $\omega \in \Omega^n\mathcal{P}^a(Y)$ is a homogeneous volume form. Then $i_Z^*(\iota_{\mathcal{X}}\omega)$ is a volume form on Z. In particular Z is orientable and we choose the orientation such that $i_Z^*(\iota_{\mathcal{X}}\omega)$ is positively oriented. If additionally Z is compact, then integration yields an isomorphism $H^{n-1}(Z) \cong \mathbb{C}$.

For $f \in \mathcal{P}^{-a}(Y)$ the closed form $f\omega \in \Omega^n\mathcal{P}^0(Y)$ defines a class $[f\omega] \in H^n\mathcal{P}^0(Y)$ which under the isomorphism Ψ corresponds to the class $\big[i_Z^*(f\iota_{\mathcal{X}}\omega)\big] \in H^{n-1}(Z)$.

Definition 2.3. For $f \in \mathcal{P}^{-a}(Y)$ we define the *residue* with respect to the fixed volume form $\omega \in \Omega^n\mathcal{P}^a(Y)$ to be the complex number corresponding to the class $[f\omega] \in H^n\mathcal{P}^0(Y)$ under the composition of the isomorphisms $H^n\mathcal{P}^0(Y) \cong H^{n-1}(Z) \cong \mathbb{C}$:

$$\operatorname{res}_\omega(f) := \int_Z i_Z^*\big(f\iota_{\mathcal{X}}\omega\big). \tag{2.15}$$

For $f \in \mathcal{P}^b(Y), b \neq -a$, we put $\operatorname{res}_\omega(f) = 0$.

Example: $Y = \mathbb{R}^n \setminus \{0\} \cong \mathbb{R}_+^* \times S^{n-1}, B = Z = S^{n-1}$. We elaborate on this interesting special case. Denote by $(\xi_1, \dots, \xi_n)$ the coordinates on $\mathbb{R}^n \setminus \{0\}$ and put $\omega := d\xi_1 \wedge \dots \wedge d\xi_n \in \Omega^n\mathcal{P}^n(\mathbb{R}^n \setminus \{0\})$. Then

$$\mathcal{X} = \sum_{i=1}^n \xi_i \frac{\partial}{\partial \xi_i}, \qquad \iota_{\mathcal{X}}\omega = \sum_{i=1}^n (-1)^{i-1}\xi_i\, d\xi_1 \wedge \dots \wedge \widehat{d\xi_i} \wedge \dots \wedge d\xi_n. \tag{2.16}$$

The form $\iota_{\mathcal{X}}\omega$ is in $\Omega^{n-1}\mathcal{P}^n(\mathbb{R}^n \setminus \{0\})$, and $i_{S^{n-1}}^*(\iota_{\mathcal{X}}\omega)$ is the standard volume form on S^{n-1}. Moreover, by (2.9) we have for $f \in \mathcal{P}^a(\mathbb{R}^n \setminus \{0\})$

$$d(f\iota_{\mathcal{X}}\omega) = (a+n)f\,\omega, \tag{2.17}$$

on the other hand by (2.16)

$$d\big(f\iota_{\mathcal{X}}\omega\big) = \sum_{i=1}^n \partial_{\xi_i}(f\xi_i)\, d\xi_1 \wedge \dots \wedge d\xi_n = \Big(\sum_{i=1}^n (\partial_{\xi_i} f)\xi_i + n\, f\Big)\omega, \tag{2.18}$$

and thus we arrive at Euler's identity for homogeneous functions:

$$\sum_{i=1}^n (\partial_{\xi_i} f)\xi_i = a\, f. \tag{2.19}$$

Corollary 2.4. *Let* $\operatorname{res}_\omega$ *be the residue associated to* $\omega = d\xi_1 \wedge \dots \wedge d\xi_n \in \Omega^n\mathcal{P}^n(\mathbb{R}^n \setminus \{0\})$ *according to Definition* 2.3. *Then for a homogeneous function* $f \in \mathcal{P}^a(\mathbb{R}^n \setminus \{0\})$ *the following holds:*

(1) $\operatorname{res}_\omega(\partial_{\xi_j} f) = 0$.

(2) *There exist* $\sigma_j \in \mathcal{P}^{a+1}(\mathbb{R}^n \setminus \{0\})$ *such that* $f = \sum_{j=1}^n \partial_{\xi_j}\sigma_j$ *if and only if* $\operatorname{res}_\omega(f) = 0$.

Note that $\operatorname{res}_\omega(f) \neq 0$ *at most if* $a = -n$.

Proof. It follows from the remarks before Definition 2.3 that for a function $g \in \mathcal{P}^a(\mathbb{R}^n \setminus \{0\})$ the residue vanishes if and only if the class $[g\,\omega] \in H^n\mathcal{P}^{a+n}(\mathbb{R}^n \setminus \{0\})$ vanishes.

To prove (1) we note that $(\partial_{\xi_j} f) d\xi_1 \wedge \dots \wedge \xi_n = d\eta$ with the form

$$\eta = (-1)^{j-1} f d\xi_1 \wedge \dots \wedge \widehat{d\xi_j} \wedge \dots \wedge d\xi_n \in \Omega^{n-1}\mathcal{P}^{a+n-1}(\mathbb{R}^n \setminus \{0\})$$

and hence $\operatorname{res}_\omega(\partial_{\xi_j} f) = 0$.

(1) shows that for the σ_j in (2) to exist it is necessary that $\operatorname{res}_\omega(f) = 0$. To prove sufficiency consider $f \in \mathcal{P}^a(\mathbb{R}^n \setminus \{0\})$ with $\operatorname{res}_\omega(f) = 0$. Then there is $\eta \in \Omega^{n-1}\mathcal{P}^{a+n}(\mathbb{R}^n \setminus \{0\})$ with $d\eta = f\omega$. We write

$$\eta = \sum_{j=1}^{n} (-1)^{j-1} \sigma_j d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n \tag{2.20}$$

with $\sigma_j \in \mathcal{P}^{a+1}(\mathbb{R}^n \setminus \{0\})$. Then $f = \sum_{j=1}^{n} \partial_{\xi_j} \sigma_j$. □

2.2. **Extension to** log**–polyhomogeneous forms.** We generalize our previous considerations to log–polyhomogeneous forms.

A p–form $\omega \in \Omega^p(\mathbb{R}_+^* \times Z)$ is called log*–polyhomogeneous* of degree (a, k) if

$$\omega = \sum_{j=0}^{k} \omega_j \ \log^j r, \tag{2.21}$$

with $\omega_j \in \Omega^p \mathcal{P}^a(\mathbb{R}_+^* \times Z)$, cf. [Les99]. The set of all such forms is denoted by $\Omega^p\mathcal{P}^{a,k}(\mathbb{R}_+^* \times Z)$.

The exterior derivative preserves the (a, k)–degree. More explicitly,

$$\begin{aligned} d\Big(&\big(r^{a-1} dr \wedge \pi^*\tau + r^a \pi^*\eta\big) \ \log^j r\Big) \\ &= \big(r^{a-1} dr \wedge (a\pi^*\eta - \pi^* d_Z \tau) + r^a \pi^* d_Z \eta\big) \ \log^j r + j r^{a-1} dr \wedge \pi^*\eta \ \log^{j-1} r. \end{aligned} \tag{2.22}$$

Hence analogously to Eq. (2.6) we define the log*–homogeneous de Rham cohomology groups*

$$H^p\mathcal{P}^{a,k}(Y) := \frac{\ker\big(d : \Omega^p\mathcal{P}^{a,k}(Y) \longrightarrow \Omega^{p+1}\mathcal{P}^{a,k}(Y)\big)}{\operatorname{im}\big(d : \Omega^{p-1}\mathcal{P}^{a,k}(Y) \longrightarrow \Omega^p\mathcal{P}^{a,k}(Y)\big)}, \tag{2.23}$$

for which we can prove the following analogue of Theorem 2.1:

Theorem 2.5. *Let Z be a smooth paracompact manifold, let $\pi : Y \to Z$ be a $\mathbb{R}_+^*$ principal bundle over Z. Let $r \in \mathcal{P}^1(Y)$ be everywhere positive.*

(1) *If $a \neq 0$ then $H^p\mathcal{P}^{a,k}(Y) = \{0\}$.*
(2) *If $a = 0$ then the map*

$$\begin{aligned} \Phi^k : \Omega^{\bullet-1}(Z) \oplus \Omega^\bullet(Z) &\longrightarrow \Omega^\bullet\mathcal{P}^{0,k}(Y) \\ (\tau, \eta) &\mapsto r^{-1} dr \wedge (\pi^*\tau) \log^k r \ + \pi^*\eta \end{aligned}$$

induces an isomorphism

$$H(\Phi^k) : H^{p-1}(Z) \oplus H^p(Z) \cong H^p\mathcal{P}^{0,k}(Y). \tag{2.24}$$

If $\omega = \sum_{j=0}^{k} \omega_j \log^j r \in \Omega^p\mathcal{P}^{0,k}(Y), \omega_j \in \Omega^p\mathcal{P}^0(Y)$, is a closed form then the inverse of $H(\Phi^k)$ applied to $[\omega]$ is given by $\big([i_Z^*(\iota_{\mathcal{X}}\omega_k)], [i_Z^*\omega_0]\big)$.

Proof. We consider a *closed* form $\omega \in \Omega^p \mathcal{P}^{a,k}(\mathbb{R}_+^* \times Z)$ and write

$$\omega = \omega_k \; \log^k r + \chi \tag{2.25}$$

with $\chi \in \Omega^p \mathcal{P}^{a,k-1}(\mathbb{R}_+^* \times Z)$. Then

$$0 = d\omega = (d\omega_k) \; \log^k r + \text{ lower log degree,} \tag{2.26}$$

thus ω_k is closed and Euler's identity (2.9) gives

$$\begin{aligned} d\big(\iota_{\mathcal{X}}\omega\big) &= d\big(\iota_{\mathcal{X}}\omega_k \; \log^k r\big) + \text{ lower log degree} \\ &= a\,\omega_k \; \log^k r + \text{ lower log degree} \\ &= a\,\omega + \text{ lower log degree.} \end{aligned} \tag{2.27}$$

If $a \neq 0$ then ω is cohomologous to $\omega - \frac{1}{a} d\big(i_{\mathcal{X}}\omega\big) \in \Omega^p \mathcal{P}^{a,k-1}(\mathbb{R}_+^* \times Z)$. By induction and Theorem 2.1 one then shows that ω is exact.

Next let $a = 0$ and consider a closed form $\omega \in \Omega^p \mathcal{P}^{0,k}(\mathbb{R}_+^* \times Z)$:

$$\omega = \sum_{j=0}^{k} \big(r^{-1} dr \wedge \pi^* \tau_j + \pi^* \eta_j\big) \; \log^j r, \tag{2.28}$$

$$\begin{aligned} 0 &= d\omega \\ &= \sum_{j=0}^{k} \big(-r^{-1} dr \wedge \pi^* d_Z \tau_j + \pi^* d_Z \eta_j\big) \; \log^j r + j r^{-1} dr \wedge \pi^* \eta_j \; \log^{j-1} r \\ &= \big(-r^{-1} dr \wedge \pi^* d_Z \tau_k + \pi^* d_Z \eta_k\big) \; \log^k r \\ &\quad + \sum_{j=0}^{k-1} \big(r^{-1} dr \wedge ((j+1)\pi^* \eta_{j+1} - \pi^* d_Z \tau_j) + \pi^* d_Z \eta_j\big) \; \log^j r, \end{aligned} \tag{2.29}$$

hence

$$\begin{aligned} &d_Z \tau_k = 0, \quad d_Z \eta_k = 0, \\ &d_Z \eta_j = 0, \quad d_Z \tau_j = (j+1) \eta_{j+1}, \quad j = 0, ..., k-1. \end{aligned} \tag{2.30}$$

This shows that $H(\Phi^k)$ is well–defined. Furthermore,

$$\omega - d\left(\sum_{j=0}^{k-1} \frac{1}{j+1} \pi^* \tau_j \; \log^{j+1} r\right) = r^{-1} dr \wedge \pi^* \tau_k \; \log^k r + \pi^* \eta_0, \tag{2.31}$$

showing that $H(\Phi^k)$ is surjective.

Finally, consider a closed form

$$\omega = r^{-1} dr \wedge \pi^* \tau_k \; \log^k r + \pi^* \eta_0 \tag{2.32}$$

and suppose that $d\chi = \omega$ with

$$\chi = \sum_{j=0}^{k} \big(r^{-1} dr \wedge \pi^* \alpha_j + \pi^* \beta_j\big) \; \log^j r. \tag{2.33}$$

Then $d_Z \beta_0 = \eta_0$ and $-d_Z \alpha_k = \tau_k$. This proves the remaining claims. □

Example: $Y = \mathbb{R}^n \setminus \{0\}, B = Z = S^{n-1}$. As in the homogeneous case we put:

Definition 2.6. Let $f \in \mathcal{P}^{-n,k}(\mathbb{R}^n \setminus \{0\})$. We define the *residue* of f to be the integral

$$\operatorname{res}_{\omega,k}(f) := \operatorname{res}_\omega(f_k) = \int_{S^{n-1}} i^*_{S^{n-1}}\big(f_k\, \iota_{\mathcal{X}}\omega\big), \quad \omega = d\xi_1 \wedge \cdots \wedge d\xi_n. \tag{2.34}$$

Note that by Theorem 2.5, $H^n\mathcal{P}^{0,k}(\mathbb{R}^n \setminus \{0\}) \cong H^{n-1}(S^{n-1}) \cong \mathbb{C}$, and that $\operatorname{res}_{\omega,k}(f)$ is the image in $\mathbb{C}$ of the class $[f\omega]$ under this isomorphism. Therefore exactly as Corollary 2.4 one now proves:

Corollary 2.7. *For a* log*–polyhomogeneous function* $f \in \mathcal{P}^{a,k}(\mathbb{R}^n \setminus \{0\})$ *the following holds:*

(1) $\operatorname{res}_{\omega,k}(\partial_{\xi_j} f) = 0$.

(2) *There exist* $\sigma_j \in \mathcal{P}^{a+1,k}(\mathbb{R}^n \setminus \{0\})$ *such that* $f = \sum_{j=1}^n \partial_{\xi_j}\sigma_j$ *if and only if* $\operatorname{res}_{\omega,k}(f) = 0$. *Note that* $\operatorname{res}_{\omega,k}(f) \neq 0$ *at most if* $a = -n$.

2.3. **Homogeneous functions on symplectic cones.** In this section we give an explicit expression of a homogeneous function in terms of Poisson brackets. This generalizes work of V. Guillemin [Gui85, Thm. 6.2].

To fix some notation and to fix some (sign) conventions let us briefly collect some basic facts from symplectic geometry:

Let Y be a symplectic manifold with symplectic form ω. The Hamiltonian vector field X_f associated to $f \in \mathrm{C}^\infty(Y)$ is characterized by $\iota_{X_f}\omega = -df$. The Poisson bracket of two functions $f, g \in \mathrm{C}^\infty(Y)$ is defined by

$$\{f, g\} := \omega(X_f, X_g).$$

If X_1 and X_2 are Hamiltonian vector fields, then $[X_1, X_2]$ is also a Hamiltonian vector field with Hamiltonian function $\omega(X_1, X_2)$ (see Def. 18.5 in [CdS01]):

$$\iota_{[X_1,X_2]}\omega = \iota_{X_{\omega(X_1,X_2)}}\omega,$$

hence

$$X_{\{f,g\}} = X_{\omega(X_f,X_g)} = [X_f, X_g], \tag{2.35}$$

and $(\mathrm{C}^\infty(Y), \{,\})$ is a Poisson algebra.

Proposition 2.8 (1.2 in [Wod87b])**.** *The Poisson bracket of two functions* $f, g \in \mathrm{C}^\infty(Y)$ *satisfies:*

$$\{f, g\}\, \omega^n = n\, df \wedge dg \wedge \omega^{n-1} = d(g\, \iota_{X_f}\omega^n). \tag{2.36}$$

Let Y be a *symplectic cone*, i.e. a cone $\pi : Y \to Z$ with a symplectic form $\omega \in \Omega^2\mathcal{P}^1(Y)$. We assume furthermore that Z is compact and connected; of course, Y is then connected, too. The main example we have in mind is the cotangent bundle with the zero section removed, $T^*M \setminus M$, of a compact connected manifold M of dimension $\dim M > 1$, with its standard symplectic structure. The base manifold Z is then the cosphere bundle S^*M. In the case $M = S^1$ (the only compact connected one–dimensional manifold!), each of the two connected components of $T^*S^1 \setminus S^1$ is a symplectic cone over S^1.

2.3.1. *The symplectic residue.* Let $\dim Y =: 2n$, so $\omega^n \in \Omega^{2n}\mathcal{P}^n(Y)$ is a volume form on Y. The form $\alpha := \iota_{\mathcal{X}}\omega$ is in $\Omega^1\mathcal{P}^1(Y)$ and by Euler's identity for forms Eq. (2.9), it satisfies $\omega = d\alpha$. Hence we can apply Definition 2.3 and define the *symplectic residue* of a function $f \in \mathcal{P}^a(Y)$ to be the residue with respect to the volume form ω^n. That is

$$\operatorname{res}_Y(f) := \operatorname{res}_{\omega^n}(f) = \begin{cases} \int_Z i_Z^*(f\,\iota_{\mathcal{X}}\omega^n), & \text{if } a = -n, \\ 0, & \text{if } a \neq -n. \end{cases} \tag{2.37}$$

Recall that by construction $\operatorname{res}_Y(f) = 0$ if and only if there is a form $\beta \in \Omega^{2n-1}\mathcal{P}^{a+n}(Y)$ such that $d\beta = f\omega^n$.

Our definition of the symplectic residue differs from the original one by Guillemin [Gui85] by a factor.

2.3.2. *Homogeneous functions in terms of Poisson brackets.* Now we prove the following generalization of [Gui85, Thm. 6.2]. The proof we present is based on the homogeneous cohomology developed in Subsection 2.1. To the best of our knowledge this approach is completely new.

In the following we will for brevity write $\mathcal{P}^a$ instead of $\mathcal{P}^a(Y)$.

Theorem 2.9. *Let Y be a connected symplectic cone of dimension $2n > 2$ with compact base. Then for any real numbers l, m the following holds*

$$\begin{aligned} \{\mathcal{P}^l, \mathcal{P}^m\} &= \ker(\operatorname{res}_Y) \cap \mathcal{P}^{l+m-1} \\ &= \begin{cases} \mathcal{P}^{l+m-1}, & \text{if } l+m \neq -n+1, \\ \ker(\operatorname{res}_Y) \cap \mathcal{P}^{l+m-1}, & \text{if } l+m = -n+1. \end{cases} \end{aligned} \tag{2.38}$$

Proof. We first note that Proposition 2.8 implies that $\{\mathcal{P}^l, \mathcal{P}^m\} \subset \mathcal{P}^{l+m-1}$. Furthermore, by loc. cit. we have $\{f, g\}\,\omega^n = d(g\,\iota_{X_f}\omega^n)$, and if $f \in \mathcal{P}^l, g \in \mathcal{P}^m$ then $g\,\iota_{X_f}\omega^n \in \Omega^{2n-1}\mathcal{P}^{l+m+n-1}$. Thus the homogeneous cohomology class of $\{f, g\}\omega^n$ vanishes and hence $\operatorname{res}_Y(\{f, g\}) = 0$. So $\{\mathcal{P}^l, \mathcal{P}^m\} \subset \ker(\operatorname{res}_Y)$.

Conversely, let $f \in \mathcal{P}^{l+m-1}$ be given with $\operatorname{res}_Y(f) = 0$. That is, the homogeneous cohomology class of $f\omega^n \in \Omega^{2n}\mathcal{P}^{n+l+m-1}$ vanishes and hence there is a $\beta \in \Omega^{2n-1}\mathcal{P}^{n+l+m-1}$ such that

$$f\omega^n = d\beta. \tag{2.39}$$

1. $l \neq 0$ or $m \neq 0$. Since the claim is symmetric in l and m we may, without loss of generality, assume that $l \neq 0$.

Choose functions $g_1, \ldots, g_N \in \mathcal{P}^l$ such that at every point y of Y their differentials $dg_1|_y, \ldots, dg_N|_y$ span the cotangent space T_y^*Y. Let $X_1, \ldots, X_N$ be the Hamiltonian vector fields of $g_1, \ldots, g_N$. Since ω^n is a volume form also $\iota_{X_1}\omega^n|_y, \ldots \iota_{X_N}\omega^n|_y$ span $\Lambda^{2n-1}T_y^*Y$.

Consequently, there are functions $f_1, \ldots, f_N \in \mathrm{C}^\infty(Y)$ such that

$$\beta = \sum_{j=1}^N f_j\,\iota_{X_j}\omega^n. \tag{2.40}$$

Since β, X_j, ω^n are homogeneous it is clear that also f_j can be chosen to be homogeneous. Counting degrees then shows $f_j \in \mathcal{P}^m$. Thus by Proposition 2.8

$$\begin{aligned} f\,\omega^n = d\beta &= \sum_{j=1}^N d(f_j\,\iota_{X_j}\omega^n) \\ &= n\sum_{j=1}^N dg_j \wedge df_j \wedge \omega^{n-1} = \sum_{j=1}^N \{g_j, f_j\}\,\omega^n, \end{aligned} \tag{2.41}$$

and hence $f = \sum_{j=1}^N \{g_j, f_j\} \in \{\mathcal{P}^l, \mathcal{P}^m\}$.

2. $l = m = 0$. In this case $f \in \mathcal{P}^{-1}$. By assumption, $n > 1$ and thus by Eq. (2.9)

$$f\,\omega^n = \frac{1}{n-1} d(f\iota_{\mathcal{X}}\omega^n) = \frac{n}{n-1} d(f\alpha \wedge \omega^{n-1}), \quad \alpha = \iota_{\mathcal{X}}\omega. \tag{2.42}$$

The 1–form $f\alpha$ is homogeneous of degree 0 and since $\alpha = \iota_{\mathcal{X}}\omega$, it is the pullback of a 1–form on Z.

We now choose $g_1, \dots, g_N \in \mathcal{P}^0$ such that at every point z of Z, their differentials span the cotangent space $T^*_z Z$. Of course it is impossible to find homogeneous functions of degree 0 such that their differentials span $T^*_y Y$ at every $y \in Y$ [2].

Therefore there are functions $f_1, \dots, f_N \in C^\infty(Y)$ such that

$$f\alpha = \sum_{i=1}^N f_i\, dg_i.$$

As before, we see that f_i can be chosen such that $f_i \in \mathcal{P}^0$. Moreover, continuing Eq. (2.42) and again using Proposition 2.8

$$\begin{aligned} f\,\omega^n &= \frac{n}{n-1} d(f\alpha) \wedge \omega^{n-1} = \frac{n}{n-1} d\left(\sum_{i=1}^N f_i\, dg_i\right) \wedge \omega^{n-1} \\ &= \frac{1}{n-1} \sum_{i=1}^N \{f_i, g_i\}\,\omega^n, \end{aligned}$$

and we reach the conclusion $f = \frac{1}{n-1}\sum_{i=1}^N \{f_i, g_i\} \in \{\mathcal{P}^0, \mathcal{P}^0\}$. □

Remark 2.10. If $n = 1$, then $\{\mathcal{P}^0, \mathcal{P}^0\} = 0$. Indeed, by Eq. (2.36) with $n = 1$, $\{f, g\}\,\omega = df \wedge dg$, so if $f, g \in \mathcal{P}^0$ we have $\{f, g\} = 0$. In this one–dimensional case, there are two different symplectic residues (res^+, res^-), corresponding to each connected component of $T^*S^1 \setminus S^1$; then, when $l \neq 0$ or $m \neq 0$ we can argue as in the corresponding part of

[2]The second named author would like to thank Prof. Jean-Marie Lescure for pointing this out to her.

the proof of Theorem 2.9, to conclude that

$$\{\mathcal{P}^l, \mathcal{P}^m\} = \begin{cases} \mathcal{P}^{l+m-1}, & \text{if } l+m \neq 0, \\ \ker(\operatorname{res}^+) \cap \ker(\operatorname{res}^-) \cap \mathcal{P}^{l+m-1}, & \text{if } l+m = 0. \end{cases} \tag{2.43}$$

2.4. **The residue of a classical symbol function.** As an application of homogeneous cohomology we give a precise criterion when a classical symbol function is a sum of partial derivatives. A more thorough discussion of de Rham cohomology of forms whose coefficients are symbol functions will be given in a subsequent publication.

2.4.1. *Classes of symbols.* Let $U \subset \mathbb{R}^n$ be an open subset. We denote by $\mathrm{S}^m(U \times \mathbb{R}^N)$, $m \in \mathbb{R}$, the space of symbols of Hörmander type $(1,0)$ (Hörmander [Hör71], Grigis–Sjøstrand [GrSj94]). More precisely, $\mathrm{S}^m(U \times \mathbb{R}^N)$ consists of those $a \in \mathrm{C}^\infty(U \times \mathbb{R}^N)$ such that for multi–indices $\alpha \in \mathbb{Z}_+^n, \gamma \in \mathbb{Z}_+^N$ and compact subsets $K \subset U$ we have an estimate

$$\left|\partial_x^\alpha \partial_\xi^\gamma a(x,\xi)\right| \leq C_{\alpha,\gamma,K}(1+|\xi|)^{m-|\gamma|}, \quad x \in K,\ \xi \in \mathbb{R}^N. \tag{2.44}$$

The best constants in (2.44) provide a set of semi–norms which endow $\mathrm{S}^\infty(U \times \mathbb{R}^N) := \bigcup_{m \in \mathbb{R}} \mathrm{S}^m(U \times \mathbb{R}^N)$ with the structure of a Fréchet algebra. A symbol $a \in \mathrm{S}^m(U \times \mathbb{R}^N)$ is called *classical* if there are $a_{m-j} \in \mathrm{C}^\infty(U \times \mathbb{R}^N)$ with

$$a_{m-j}(x, r\xi) = r^{m-j} a_{m-j}(x,\xi), \quad r \geq 1, |\xi| \geq 1, \tag{2.45}$$

such that for $N \in \mathbb{Z}_+$

$$a - \sum_{j=0}^{N-1} a_{m-j} \in \mathrm{S}^{m-N}(U \times \mathbb{R}^N). \tag{2.46}$$

The latter property is usually abbreviated $a \sim \sum\limits_{j=0}^{\infty} a_{m-j}$.

Homogeneity and smoothness at 0 contradict each other except for monomials. Our convention is that symbols should always be smooth functions, thus the a_{m-j} are smooth everywhere but homogeneous only in the restricted sense of Eq. (2.45). The homogeneous extension of a_{m-j} to $U \times \mathbb{R}^n \setminus \{0\}$ will also be needed: we put

$$a_{m-j}^h(x,\xi) := a_{m-j}(x, \xi/|\xi|)\, |\xi|^{m-j}, \quad (x,\xi) \in U \times \mathbb{R}^n \setminus \{0\}. \tag{2.47}$$

Furthermore, we denote by $\mathrm{S}^{-\infty}(U \times \mathbb{R}^n) := \bigcap_{a \in \mathbb{R}} \mathrm{S}^a(U \times \mathbb{R}^n)$ the space of *smoothing symbols.*

For brevity we write $\mathrm{CS}^a(\mathbb{R}^n)$ ($\mathrm{S}^a(\mathbb{R}^n)$) instead of $\mathrm{CS}^a(\{\mathrm{pt}\} \times \mathbb{R}^n)$ ($\mathrm{S}^a(\{\mathrm{pt}\} \times \mathbb{R}^n)$). Note that $\mathrm{S}^{-\infty}(\mathbb{R}^n) = \mathscr{S}(\mathbb{R}^n)$ is nothing but the Schwartz space of rapidly decaying functions.

We will now discuss the analogue of Corollary 2.4 for the space $\mathrm{CS}^a(\mathbb{R}^n)$. We start with smoothing symbols:

Lemma 2.11. *Let $f \in \mathscr{S}(\mathbb{R}^n)$ be a Schwartz function. Then there are functions $\sigma_j \in \mathrm{CS}^{-n+1}(\mathbb{R}^n)$ such that $f = \sum\limits_{j=1}^{n} \partial_{\xi_j} \sigma_j$.*

One can choose the σ_j to be Schwartz functions if and only if $\int_{\mathbb{R}^n} f = 0$.

Proof. We start with the first claim and note that if $n = 1$ then the function $\sigma(\xi) = \int_{-\infty}^{\xi} f(t)\,dt$ is in $\mathrm{CS}^0(\mathbb{R})$ and $\partial_\xi \sigma = f$.

For general n we infer from the standard proof of the Poincaré Lemma in $\mathbb{R}^n$ applied to the closed form $f\,d\xi_1 \wedge \cdots \wedge d\xi_n$, that we can put

$$\sigma_j(\xi) = \int_0^1 f(t\xi)\,\xi_j\,t^{n-1}\,dt.$$

Indeed,

$$\partial_{\xi_j}\sigma_j(\xi) = \int_0^1 f(t\xi)\,t^{n-1}\,dt + \int_0^1 \partial_{\xi_j}(f)(t\xi)\,\xi_j\,t^n\,dt,$$

thus

$$\begin{aligned}\sum_{j=1}^n \partial_{\xi_j}\sigma_j(\xi) &= \int_0^1 f(t\xi)\,n\,t^{n-1}\,dt + \int_0^1 \sum_{j=1}^n \partial_{\xi_j}(f)(t\xi)\,\xi_j\,t^n\,dt \\ &= \int_0^1 \partial_t\Big(f(t\xi)\,t^n\Big)\,dt = f(\xi).\end{aligned}$$

It remains to show that $\sigma_j \in \mathrm{CS}^{-n+1}(\mathbb{R}^n)$. The function σ_j is certainly smooth. For $\xi \neq 0$, $|\xi| \geq 1$ we have by change of variables $r = t|\xi|$:

$$\begin{aligned}\sigma_j(\xi) &= \int_0^{|\xi|} f\Big(r\frac{\xi}{|\xi|}\Big) r^{n-1}\,dr\;|\xi|^{-n}\,\xi_j \\ &= \int_0^{\infty} f\Big(r\frac{\xi}{|\xi|}\Big) r^{n-1}\,dr\,|\xi|^{-n}\,\xi_j - \int_{|\xi|}^{\infty} f\Big(r\frac{\xi}{|\xi|}\Big) r^{n-1}\,dr\,|\xi|^{-n}\,\xi_j.\end{aligned}$$

The first summand is homogeneous of degree $-n+1$ while the second summand satisfies the estimates of a Schwartz function at ∞ (it is not a Schwartz function since it is not smooth at 0). Thus $\sigma_j \in \mathrm{CS}^{-n+1}(\mathbb{R}^n)$ and its homogeneous expansion consists only of one term of homogeneity $-n+1$:

$$\sigma_j(\xi) \sim \int_0^\infty f\Big(r\frac{\xi}{|\xi|}\Big) r^{n-1}\,dr\,|\xi|^{-n}\,\xi_j,$$

proving the first claim.

For the second claim the necessity of $\int_{\mathbb{R}^n} f = 0$ is clear. In fact the proof of the Poincaré Lemma with compact supports [BoTu82, Sec. I.4] works verbatim for the forms $\Omega^\bullet\mathscr{S}(\mathbb{R}^n)$ with coefficients in $\mathscr{S}(\mathbb{R}^n)$. Thus the closed n–form $f d\xi_1 \wedge \cdots \wedge d\xi_n$ is exact in $\Omega^\bullet\mathscr{S}(\mathbb{R}^n)$ if and only if $\int_{\mathbb{R}^n} f = 0$. If this is the case then $f d\xi_1 \wedge \cdots \wedge d\xi_n = d\eta$ with an $(n-1)$–form $\eta \in \Omega^{n-1}\mathscr{S}(\mathbb{R}^n)$. Expanding η as in (2.20) we see that $f = \sum_{j=1}^n \partial_{\xi_j}\sigma_j$ with Schwartz functions σ_j. □

2.4.2. *The residue and the regularized (cut–off) integral.* We now extend the residue (Def. 2.3) from homogeneous functions to $\mathrm{CS}^a(\mathbb{R}^n)$:

Let $\sigma \in \mathrm{CS}^a(\mathbb{R}^n)$ be with asymptotic expansion $\sigma \sim \sum_{j=0}^{\infty} \sigma_{a-j}$, cf. Eq. (2.45) and (2.47). Then $\sigma^h_{a-j} \in \mathcal{P}^{a-j}(\mathbb{R}^n \setminus \{0\})$. Put

$$\begin{aligned} \operatorname{res}(\sigma) := \operatorname{res}_\omega(\sigma^h_{-n}) &= \int_{S^{n-1}} i^*_{S^{n-1}}(\sigma^h_{-n})\, d\operatorname{vol}_{S^{n-1}} \\ &= \int_{S^{n-1}} i^*_{S^{n-1}}(\sigma^h_{-n} \iota_{\mathcal{X}} \omega), \quad \omega = d\xi_1 \wedge \ldots \wedge d\xi_n. \end{aligned} \tag{2.48}$$

In other words the residue of σ equals the residue of its homogeneous component of homogeneity degree $-n$. Thus $\operatorname{res}(\sigma) \neq 0$ at most if a is an integer $\geq -n$. The functional res was studied by S. Paycha in [Pay07].

We also recall the *regularized integral* or cut–off integral $⨍ : \mathrm{CS}^a(\mathbb{R}^n) \longrightarrow \mathbb{C}$ (cf. e.g. [Les11, Sec. 4.2]): If $f \in \mathrm{CS}^a(\mathbb{R}^n)$ then the asymptotic expansion $f \sim \sum_{j=0}^{\infty} f_{a-j}$ implies that as $R \to \infty$ one has an asymptotic expansion

$$\int_{|\xi| \leq R} f(\xi)\, d\xi \underset{R\to\infty}{\sim} \sum_{\substack{j=0 \\ a-j+n\neq 0}}^{\infty} c_{a-j} R^{a-j+n} + \widetilde{c}\, R^0 + \operatorname{res}(f) \log R. \tag{2.49}$$

The regularized integral $⨍_{\mathbb{R}^n} f(\xi)\, d\xi$ is, by definition, the constant term in this asymptotic expansion, i.e. $\widetilde{c}$. It has the property that $⨍_{\mathbb{R}^n} \partial_{\xi_j} f \neq 0$ at most if a is an integer $\geq -n+1$.

The following result generalizes [Pay07, Prop. 2, Thm. 2] where it was proved modulo smoothing symbols.

Proposition 2.12. 1. *Let $a \in \mathbb{Z}$. For a symbol $f \in \mathrm{CS}^a(\mathbb{R}^n)$ there exist symbols $\sigma_j \in \mathrm{CS}^{r(a)}(\mathbb{R}^n), r(a) := \max(a, -n) + 1$, such that $f = \sum_{j=1}^{n} \partial_{\xi_j} \sigma_j$ if and only if* $\operatorname{res}(f) = 0$.

2. *Let $a \in \mathbb{R} \setminus \mathbb{Z}$. For a symbol $f \in \mathrm{CS}^a(\mathbb{R}^n)$ there exist symbols $\sigma_j \in \mathrm{CS}^{a+1}(\mathbb{R}^n)$ such that $f = \sum_{j=1}^{n} \partial_{\xi_j} \sigma_j$ if and only if $⨍_{\mathbb{R}^n} f = 0$.*

Proof. 1. We will repeatedly use that by construction the asymptotic relation Eq. (2.46) may be differentiated, i.e. if $g \in \mathrm{CS}^a(\mathbb{R}^n)$ with $g \sim \sum_{l=0}^{\infty} g_{a-l}$ then

$$\partial_{\xi_j} g \sim \sum_{l=0}^{\infty} \partial_{\xi_j} g_{a-l}.$$

Now let $a \in \mathbb{Z}$ and $f \in \mathrm{CS}^a(\mathbb{R}^n)$ with $f \sim \sum_{l=0}^{\infty} f_{a-l}$. If $f = \sum_{j=1}^{n} \partial_{\xi_j} \tau_j$ with $\tau_j \in \mathrm{CS}^{r(a)}(\mathbb{R}^n)$ then certainly $f^h_{-n} = \sum_{j=1}^{n} \partial_{\xi_j} \tau^h_{j,-n+1}$ and hence $\operatorname{res}(f) = \operatorname{res}(f^h_{-n}) = 0$ by Corollary 2.4.

Conversely, if $\operatorname{res}(f)=0$ then again by Corollary 2.4 there are $\tau^h_{j,a-l+1}\in\mathcal{P}^{a-l+1}(\mathbb{R}^n\setminus\{0\})$ such that $f^h_{a-l}=\sum_{j=1}^n\partial_{\xi_j}\tau^h_{j,a-l+1}$.

We fix a cut–off function $\chi\in C^\infty(\mathbb{R}^n)$ such that

$$\chi(\xi)=\begin{cases}1, & \text{if } |\xi|\geq 1/2,\\ 0, & \text{if } |\xi|\leq 1/4.\end{cases} \tag{2.50}$$

Now asymptotic summation [GrSj94, Prop. 1.8] guarantees the existence of $\tau_j\in \mathrm{CS}^{a+1}(\mathbb{R}^n)$ such that $\tau_j\sim\sum_{l=0}^\infty\chi\tau^h_{j,a-l+1}$ and hence

$$\sum_{j=1}^n\partial_{\xi_j}\tau_j\sim\sum_{l=0}^\infty\sum_{j=1}^n\chi\partial_{\xi_j}\tau^h_{j,a-l+1}\sim\sum_{l=0}^\infty f_{a-l}\sim f, \tag{2.51}$$

thus

$$f-\sum_{j=1}^n\partial_{\xi_j}\tau_j=:g\in \mathrm{S}^{-\infty}(\mathbb{R}^n)=\mathscr{S}(\mathbb{R}^n). \tag{2.52}$$

Applying Lemma 2.11 to g the case $a\in\mathbb{Z}$ is settled.

2. Let $a\notin\mathbb{Z}$. It was remarked before Proposition 2.12 that the condition $⨍_{\mathbb{R}^n}f=0$ is necessary. To prove sufficiency consider $f\in \mathrm{CS}^a(\mathbb{R}^n)$ with $⨍_{\mathbb{R}^n}f=0$. Since $a\notin\mathbb{Z}$, $\operatorname{res}(f)=0$ trivially. Therefore, as before we arrive at (2.52) (this is the content of [Pay07, Prop. 2]). Still we have $\int_{\mathbb{R}^n}g=⨍_{\mathbb{R}^n}f-\sum_{j=1}^n⨍_{\mathbb{R}^n}\partial_{\xi_j}\tau_j=0$. Now apply the second part of Lemma 2.11 to g and the proof is complete. □

3. Pseudodifferential operators and tracial functionals

Standing assumptions. Unless otherwise said in the rest of the paper M will denote a smooth closed connected riemannian manifold of dimension n. The riemannian metric is chosen for convenience only to have an L^2–structure at our disposal. One could avoid choosing a metric by working with densities.

Given $b\in\mathbb{R}$, we use the notation $\mathbb{Z}_{\leq b}:=\mathbb{Z}\cap(-\infty,b]$, $\mathbb{Z}_{>b}:=\mathbb{Z}\cap(b,+\infty)$.

3.1. **Classical pseudodifferential operators.** We denote by $\mathrm{L}^\bullet(M)$ the algebra of pseudodifferential operators with complete symbols of Hörmander type $(1,0)$ (Hörmander [Hör71], Shubin [Shu01]), see Subsection 2.4.1. The subalgebra of classical pseudodifferential operators is denoted by $\mathrm{CL}^\bullet(M)$.

Let $U\subset\mathbb{R}^n$ be an open subset. Recall that for a symbol $\sigma\in \mathrm{S}^m(U\times\mathbb{R}^n)$, the canonical pseudodifferential operator associated to σ is defined by

$$\begin{aligned}\operatorname{Op}(\sigma)\,u(x)&:=\int_{\mathbb{R}^n}e^{i\langle x,\xi\rangle}\,\sigma(x,\xi)\,\hat u(\xi)\,đ\xi\\ &=\int_{\mathbb{R}^n}\int_U e^{i\langle x-y,\xi\rangle}\,\sigma(x,\xi)\,u(y)\,dy\,đ\xi,\end{aligned}\qquad đ\xi:=(2\pi)^{-n}d\xi. \tag{3.1}$$

For a manifold M, elements of $\mathrm{L}^\bullet(M)$ (resp. $\mathrm{CL}^\bullet(M)$) can locally be written as $\mathrm{Op}(\sigma)$ with $\sigma \in \mathrm{S}^\bullet(U \times \mathbb{R}^n)$ (resp. $\mathrm{CS}^\bullet(U \times \mathbb{R}^n)$).

Recall that there is an exact sequence

$$0 \longrightarrow \mathrm{CL}^{m-1}(M) \hookrightarrow \mathrm{CL}^m(M) \xrightarrow{\sigma_m} \mathcal{P}^m(T^*M \setminus M) \longrightarrow 0, \tag{3.2}$$

where $\sigma_m(A)$ is the homogeneous leading symbol of $A \in \mathrm{CL}^m(M)$. σ_m has a (non–canonical) global right inverse Op which is obtained by patching together the locally defined maps in Eq. (3.1). $\sigma_m(A)$ is a homogeneous function on the symplectic cone $T^*M \setminus M$ (cf. Subsection 2.3). We will tacitly identify $\mathcal{P}^m(T^*M \setminus M)$ by restriction with $\mathrm{C}^\infty(S^*M)$. Here, S^*M is the cosphere bundle, i.e. the unit sphere bundle $\subset T^*M$.

Recall that the leading symbol map is multiplicative in the sense that

$$\sigma_{a+b}(A \circ B) = \sigma_a(A)\, \sigma_b(B) \tag{3.3}$$

for $A \in \mathrm{CL}^a(M), B \in \mathrm{CL}^b(M)$. Furthermore, we record the important formula

$$\sigma_{a+b-1}\big([A,B]\big) = \frac{1}{i}\{\sigma_a(A), \sigma_b(B)\}, \tag{3.4}$$

which is a consequence of the asymptotic formula for the *complete* symbol of a product, cf. e.g. [Shu01, Thm. 3.4].

3.2. **Tracial functionals on subspaces of** $\mathrm{CL}^\bullet(M)$**.** Let $a \in \mathbb{R}$. $\mathrm{CL}^a(M)$ is an algebra if and only if $a \in \mathbb{Z}_{\le 0}$. In this case a linear functional $\tau : \mathrm{CL}^a(M) \longrightarrow \mathbb{C}$ is a trace if and only if

$$\tau\big([A,B]\big) = 0, \qquad \text{for all } A, B \in \mathrm{CL}^a(M). \tag{3.5}$$

Therefore, in order to characterize traces on $\mathrm{CL}^a(M)$, one has to understand the space of commutators $[\mathrm{CL}^a(M), \mathrm{CL}^a(M)]$. Note that the commutator $[A,B] \in \mathrm{CL}^{2a}(M)$. Here, in the situation of operators with scalar coefficients, one even has $[A,B] \in \mathrm{CL}^{2a-1}(M)$. However, AB and BA are only in $\mathrm{CL}^{2a}(M)$ and that $[A,B] \in \mathrm{CL}^{2a-1}(M)$ is only due to the fact that the leading symbols of A and B commute. If A, B are pseudodifferential operators acting on sections of a vector bundle (see Section 5) then one can only conclude that $[A,B]$ is of order $2a$.

Conversely, if $\tau : \mathrm{CL}^{2a}(M) \longrightarrow \mathbb{C}$ is a linear functional satisfying Eq. (3.5) then any linear extension $\widetilde{\tau}$ of τ to $\mathrm{CL}^a(M)$ is a trace on $\mathrm{CL}^a(M)$.

$\mathrm{CL}^{2a}(M)$ is a subspace of $\mathrm{CL}^a(M)$ if and only if $a \in \mathbb{Z}_{\le 0}$. However, for any $a \in \mathbb{R}$ it makes sense to consider linear functionals on $\mathrm{CL}^{2a}(M)$ satisfying (3.5):

Definition 3.1. Let $b \in \mathbb{R}$ and let $\tau : \mathrm{CL}^b(M) \longrightarrow \mathbb{C}$ be a linear functional.

1. τ is called a *pretrace* if $\tau\big([A,B]\big) = 0$ for all $A, B \in \mathrm{CL}^{b/2}(M)$.
2. τ is called a *hypertrace* if $\tau\big([A,B]\big) = 0$ for all $A \in \mathrm{CL}^0(M), B \in \mathrm{CL}^b(M)$.

If $\mathrm{CL}^a(M) \subset \mathrm{CL}^b(M)$ we sometimes use the abbreviation $\tau_a := \tau \restriction \mathrm{CL}^a(M)$.

Remark 3.2. If $b \in \mathbb{Z}_{\le 0}$ then any hypertrace on $\mathrm{CL}^b(M)$ is a trace on $\mathrm{CL}^b(M)$ since $\mathrm{CL}^b(M) \subset \mathrm{CL}^0(M)$. The restriction of a trace on $\mathrm{CL}^b(M)$ to $\mathrm{CL}^{2b}(M)$ is obviously a pretrace.

Next we discuss the canonical (pre, hyper)traces which exist on $\mathrm{CL}^a(M)$ for various a.

3.2.1. *The L^2–trace.* A pseudodifferential operator A of order $\operatorname{ord}(A) < -n = -\dim M$ is a trace–class operator. The standard Hilbert space trace on operators acting on $L^2(M)$ is denoted by Tr. Note that

$$\operatorname{Tr}(A) = \int_M K_A(x,x)\, d\operatorname{vol}(x), \tag{3.6}$$

where K_A is the Schwartz kernel of the operator A. If K_A is supported in a coordinate chart U where A is given as $\operatorname{Op}(\sigma)$ with $\sigma \in \operatorname{CS}^a(U \times \mathbb{R}^n)$ then by Eq. (3.1)

$$\operatorname{Tr}(A) = \int_U \int_{\mathbb{R}^n} \sigma(x,\xi)\, đ\xi\, dx. \tag{3.7}$$

Since for any trace–class operator K in the Hilbert space $L^2(M)$ and any bounded operator T in $L^2(M)$ one has $\operatorname{Tr}(KT) = \operatorname{Tr}(TK)$ it follows that Tr is a hypertrace on $\operatorname{CL}^a(M)$ for any real $a < -n$. Furthermore, if $p, q \geq 1$ are real numbers such that $1/p + 1/q = 1$ and if $A \in \mathscr{L}^p(L^2(M))$, the p–th Schatten ideal of operators in $L^2(M)$, and $B \in \mathscr{L}^q(L^2(M))$ then also $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$. From $\operatorname{CL}^a(M) \subset \mathscr{L}^p(L^2(M))$ for $a < -n/p$ it then follows that

$$\operatorname{Tr}\big([A,B]\big) = 0, \text{ for } A \in \operatorname{CL}^a(M), B \in \operatorname{CL}^b(M) \text{ if } a + b < -n, \tag{3.8}$$

in particular $\operatorname{Tr}_a = \operatorname{Tr} \restriction \operatorname{CL}^a(M)$ is a pretrace for any $a < -n$. In fact, Eq. (3.8) can be improved slightly:

Lemma 3.3. *Let $A \in \operatorname{CL}^a(M)$, $B \in \operatorname{CL}^b(M)$ with $a + b < -n + 1$. Then $[A,B]$ is of trace–class and $\operatorname{Tr}\big([A,B]\big) = 0$.*

Proof. We follow Sect. 4 of [Les99]. Let $P \in \operatorname{CL}^1(M)$ be an elliptic pseudodifferential operator whose leading symbol is positive and let $A \in \operatorname{CL}^a(M), B \in \operatorname{CL}^b(M)$. We put

$$\nabla_P^0(B) := B, \quad \nabla_P^{j+1} B := [P, \nabla_P^j B],$$

and by induction, for all $j \in \mathbb{N}$ we have

$$\nabla_P^j B \in \operatorname{CL}^b(M).$$

Then, for N large enough one has

$$e^{-tP} B = \sum_{j=0}^{N-1} \frac{(-t)^j}{j!} (\nabla_P^j B) e^{-tP} + R_N(t),$$

where $R_N(t)$ is a smoothing operator such that $\operatorname{Tr}\big(A R_N(t)\big) = \operatorname{Tr}\big(R_N(t) A\big) = O(t)$ as $t \to 0+$; therefore

$$\operatorname{Tr}\big([A,B] e^{-tP}\big) = -\sum_{j=1}^{N-1} \frac{(-t)^j}{j!} \operatorname{Tr}\big(A (\nabla_P^j B) e^{-tP}\big) + O(t), \quad t \to 0+. \tag{3.9}$$

Invoking the short time heat kernel asymptotics, cf. e.g. [GrSe95],

$$\operatorname{Tr}\big(A(\nabla_P^j B) e^{-tP}\big) \sim_{t \to 0+} \sum_{k=0}^{\infty} (c_k + d_k \log t) t^{k-a-b-n} + \sum_{k=0}^{\infty} e_k t^k \tag{3.10}$$

we see that for $j \geq 1$, thanks to $j - a - b - n > 0$,

$$\lim_{t\to 0+} \operatorname{Tr}\big(A(\nabla_P^j B)e^{-tP}\big) = 0. \tag{3.11}$$

Since $[A, B] \in \mathrm{CL}^{a+b-1}(M)$ and $a + b - 1 < -n$ the operator $[A, B]$ is of trace–class and from (3.9), (3.10), and (3.11) we thus infer

$$\operatorname{Tr}\big([A, B]\big) = \lim_{t\to 0+} \operatorname{Tr}\big([A, B]e^{-tP}\big) = 0. \qquad \square$$

3.2.2. *The Kontsevich–Vishik canonical trace.* For non–integer a there is a regularization procedure which allows to extend the L^2–trace in a canonical way to $\mathrm{CL}^a(M)$ (see [KoVi95], [Les99], [Les11, Sec. 4.3]). In brief for $a \in \mathbb{R} \setminus \mathbb{Z}_{\geq -n}$ there is a canonical linear functional, *the Kontsevich–Vishik canonical trace*, $\mathrm{TR} : \mathrm{CL}^a(M) \to \mathbb{C}$ such that

$$\begin{aligned} &\mathrm{TR}_a = \mathrm{TR} \restriction \mathrm{CL}^a(M) = \operatorname{Tr} \restriction \mathrm{CL}^a(M) = \operatorname{Tr}_a, \text{ if } a < -n, \\ &\mathrm{TR}\big([A, B]\big) = 0, \text{ if } A \in \mathrm{CL}^a(M), B \in \mathrm{CL}^b(M), a + b \notin \mathbb{Z}_{\geq -n+1}. \end{aligned} \tag{3.12}$$

Usually, the second property is stated only for $a+b \notin \mathbb{Z}$. However, if $a+b < -n$ then AB is of trace–class and $\mathrm{TR}(AB) = \operatorname{Tr}(AB) = \operatorname{Tr}(BA) = \mathrm{TR}(BA)$ follows from the theory of the trace in Schatten ideals (see an analogous discussion in the previous Subsection). If only $a+b-1 < -n$ then $[A, B]$ is still of trace–class and $\mathrm{TR}\big([A, B]\big) = \operatorname{Tr}\big([A, B]\big) = 0$ follows from Lemma 3.3.

The properties (3.12) immediately imply that the canonical trace TR is a hypertrace and a pretrace on $\mathrm{CL}^a(M)$ for $a \in \mathbb{R} \setminus \mathbb{Z}_{\geq -n}$.

3.2.3. *The residue trace.* The *residue trace*, called by some authors the noncommutative residue, somehow complements the canonical trace. In terms of the complete symbol, the residue trace of an operator $A \in \mathrm{CL}^{\bullet}(M)$ is given by (see [Wod87b]):

$$\operatorname{Res}(A) = \frac{1}{(2\pi)^n} \operatorname{res}(\sigma(A)) = \frac{1}{(2\pi)^n} \int_M \int_{S_x^* M} \sigma_{-n}(A)(x, \xi)\, \nu(\xi) \wedge dx,$$

where $\nu(\xi)$ is a volume form on $S_x^* M$. This is the unique trace on the whole algebra $\mathrm{CL}^{\bullet}(M)$ whenever $n > 1$ ([Wod87b], [BrGe87], [FGLS96], [Les99]). By definition, this trace vanishes on trace–class pseudodifferential operators and non–integer order pseudodifferential operators.

The residue trace Res is a pretrace and a hypertrace on $\mathrm{CL}^a(M)$ for all $a \in \mathbb{R}$. It is non–trivial, however, only if $a \in \mathbb{Z}_{\geq -n}$.

4. Operators as sums of commutators

In order to classify traces and (pre, hyper)traces on $\mathrm{CL}^a(M)$ we first study the representation of an operator as a sum of commutators.

4.1. **Smoothing operators.** The closure of the algebra $\mathrm{CL}^{-\infty}(M)$ of smoothing operators in $\mathcal{B}(L^2(M))$ is the algebra of compact operators. The latter is known to be simple. Indeed one has the following, which is in a sense an analogue of the second part of Lemma 2.11:

Theorem 4.1 ([Gui93, Thm. A.1])**.** *Let M be a closed manifold. Then for any $J \in \mathrm{CL}^{-\infty}(M)$ with $\mathrm{Tr}(J) = 1$ the following holds: for $R \in \mathrm{CL}^{-\infty}(M)$ there exist smoothing operators $S_1, \dots, S_N, T_1, \dots, T_N \in \mathrm{CL}^{-\infty}(M)$, such that*

$$R = \mathrm{Tr}(R)\; J + \sum_{j=1}^{N} [S_j, T_j].$$

Briefly, we have an exact sequence

$$0 \longrightarrow [\mathrm{CL}^{-\infty}(M), \mathrm{CL}^{-\infty}(M)] \longrightarrow \mathrm{CL}^{-\infty}(M) \xrightarrow{\mathrm{Tr}} \mathbb{C} \longrightarrow 0. \tag{4.1}$$

Can we write J as a sum of commutators of general pseudodifferential operators? Since Res is up to constants the only trace on $\mathrm{CL}^{\bullet}(M)$ (for M compact and connected of dimension > 1) the answer is yes. A more precise answer is the following:

Proposition 4.2 (See Prop. 4.2 in [Pon10])**.** *Let M be a compact riemannian manifold of dimension $n > 1$. Then $\mathrm{CL}^{-\infty}(M) \subset [\mathrm{CL}^{0}(M), \mathrm{CL}^{-n+1}(M)]$.*

We present here a brief variant of the proof of Ponge; our proof is based on

Lemma 4.3. *Let $n \geq 2$.*

1. *The operator Q_j of convolution by the function*

$$f_j(y) := \frac{y_j}{|y|^2} = \partial_{y_j}(\log|y|)$$

is a classical pseudodifferential operator of order $-n+1$ on $\mathbb{R}^n$.

2. *For any smoothing operator $R \in \mathrm{CL}^{-\infty}(\mathbb{R}^n)$ there exist $B_j \in \mathrm{CL}^{-n+1}(\mathbb{R}^n)$, $j = 1, \dots, n$, such that $R = \sum_{j=1}^{n} [\mathrm{Op}(x_j), B_j]$.*

Remark 4.4. $\mathrm{Op}(x_j)$ is the pseudodifferential operator associated to the symbol function $(x, \xi) \mapsto x_j$. Of course, this is nothing but the operator of multiplication by the coordinate x_j. Therefore, $\mathrm{Op}(x_j)$ commutes with multiplication operators, a fact which will often be used below.

Proof. 1. We have $f_j \restriction \mathbb{R}^n \setminus \{0\} \in \mathcal{P}^{-1}(\mathbb{R}^n \setminus \{0\})$. Since f_j is locally integrable in $\mathbb{R}^n$, it defines a distribution in $\mathscr{D}'(\mathbb{R}^n)$ which is homogeneous of degree -1. Then by [Hör03, Thm. 7.1.18 and 7.1.16], $f_j \in \mathscr{S}'(\mathbb{R}^n)$ and its Fourier transform $\hat{f}_j$ is a homogeneous distribution of degree $-n+1$ in $\mathbb{R}^n$ which is smooth in $\mathbb{R}^n \setminus \{0\}$. With the cut–off function χ of Eq. (2.50) we therefore have $\chi \hat{f}_j \in \mathrm{CS}^{-n+1}(\mathbb{R}^n)$. Furthermore, $(1-\chi)$ is compactly supported and thus $(1-\chi) = \hat{\psi}$ with $\psi \in \mathscr{S}(\mathbb{R}^n)$. For $u \in \mathrm{C}_c^{\infty}(\mathbb{R}^n)$ we now have

$$Q_j u = f_j * u = \mathrm{Op}(\chi \hat{f}_j) u + (\psi * f_j) * u.$$

Convolution by the Schwartz function $\psi * f_j$ is smoothing and thus $Q_j \in \mathrm{CL}^{-n+1}(\mathbb{R}^n)$.

2. A smoothing operator R has a smooth kernel $K_R(x, y)$, and therefore, $(x, y) \mapsto K_R(x, y) - K_R(x, x)$ is smooth and vanishes on the diagonal. It follows that there are smooth functions $K_1, \dots, K_n$ such that

$$K_R(x, y) = K_R(x, x) + \sum_{j=1}^{n} (x_j - y_j) K_j(x, y).$$

Let Q be the operator defined by the kernel $K_Q(x,y) = K_R(x,x)$, and let R_j be the smoothing operators defined by the kernels $K_j(x,y)$, then

$$R = Q + \sum_{j=1}^{n} [\mathrm{Op}(x_j), R_j].$$

Let H_j be the operator with kernel $(x,y) \mapsto f_j(x-y)K_R(x,x)$. H_j is Q_j followed by multiplication by the smooth function $x \mapsto K_R(x,x)$ and is therefore, by the proved part 1., a classical pseudodifferential operator of order $-n+1$. Since

$$\begin{aligned}\sum_{j=1}^{n}(x_j - y_j) f_j(x-y) K_R(x,x) &= \sum_{j=1}^{n} \frac{(x_j-y_j)^2}{|x-y|^2} K_R(x,x) \\ &= K_R(x,x) = K_Q(x,y),\end{aligned}$$

it follows that $Q = \sum\limits_{j=1}^{n} [\mathrm{Op}(x_j), H_j]$. The result of the lemma follows with $B_j := R_j + H_j \in \mathrm{CL}^{-n+1}(\mathbb{R}^n)$. □

Proof of Proposition 4.2. Let $U \subseteq \mathbb{R}^n$ be an open set and let $R \in \mathrm{CL}^{-\infty}_{\mathrm{comp}}(U)$ be a smoothing operator with compactly supported Schwartz kernel $K_R \in \mathrm{C}^\infty_c(U \times U)$. Let $\psi \in \mathrm{C}^\infty_c(U)$ be such that $\psi(x)\psi(y) = 1$ in a neighborhood of the support of the kernel of R, then $\psi R \psi = R$.

By Lemma 4.3 there exist $P_i \in \mathrm{CL}^{-n+1}(U)$ such that $R = \sum\limits_{i=1}^{n} [\mathrm{Op}(x_i), P_i]$. Let $\chi \in \mathrm{C}^\infty_c(U)$ be such that $\chi = 1$ in a neighborhood of $\mathrm{supp}(\psi)$. Then we have

$$\psi[\mathrm{Op}(x_i), P_i]\psi = \mathrm{Op}(x_i)\chi\psi P_i \psi - \psi P_i \psi \, \mathrm{Op}(x_i)\chi = [\mathrm{Op}(x_i)\chi, \psi P_i \psi],$$

thus

$$R = \sum_{i=1}^{n} [\mathrm{Op}(x_i\chi), \psi P_i \psi]. \tag{4.2}$$

Note that $x_i\chi \in \mathrm{C}^\infty_c(U)$ and $\psi P_i \psi \in \mathrm{CL}^{-n+1}_{\mathrm{comp}}(U)$.

Now let $(\varphi_j) \subset \mathrm{C}^\infty(M)$ be a partition of unity subordinate to a finite open covering (U_j) of M by coordinate charts. Furthermore, choose $\psi_j \in \mathrm{C}^\infty_c(U_j)$ such that $\psi_j = 1$ in a neighborhood of $\mathrm{supp}(\varphi_j)$. Then for any $R \in \mathrm{CL}^{-\infty}(M)$ we have

$$R = \sum_{j=1}^{N} \varphi_j R \psi_j + \sum_{j=1}^{N} \varphi_j R(1-\psi_j). \tag{4.3}$$

For each index j the operator $\varphi_j R \psi_j$ belongs to $\mathrm{CL}^{-\infty}_{\mathrm{comp}}(U_j)$, so by the previous argument it can be written as a sum of commutators of the form (4.2). Moreover, the operator $S := \sum\limits_{j=1}^{N} \varphi_j R(1-\psi_j)$ is smoothing and its Schwartz kernel vanishes on the diagonal, so its trace vanishes and by Theorem 4.1 it can be written as a sum of commutators in $[\mathrm{CL}^{-\infty}(M), \mathrm{CL}^{-\infty}(M)]$. Hence R belongs to the space $[\mathrm{CL}^0(M), \mathrm{CL}^{-n+1}(M)]$ as claimed. □

The degrees 0 and $-n+1$ in the commutator $[\mathrm{CL}^0(M), \mathrm{CL}^{-n+1}(M)]$ in Proposition 4.2 can be traded against each other as the following simple but very useful Lemma, which is based on joint work of the second named author with Sylvie Paycha, shows:

Lemma 4.5. *For any $\alpha, \beta \in \mathbb{R}$*

$$[\mathrm{CL}^0(M), \mathrm{CL}^{\alpha+\beta}(M)] \subset [\mathrm{CL}^\alpha(M), \mathrm{CL}^\beta(M)],$$

meaning that any commutator in $[\mathrm{CL}^0(M), \mathrm{CL}^{\alpha+\beta}(M)]$ can be written as a sum of commutators in $[\mathrm{CL}^\alpha(M), \mathrm{CL}^\beta(M)]$.

Proof. Let $A \in \mathrm{CL}^0(M)$, $B \in \mathrm{CL}^{\alpha+\beta}(M)$. Fix a first order positive definite elliptic operator $\Lambda \in \mathrm{CL}^1(M)$. Then $A\Lambda^\alpha, \Lambda^\alpha A, \Lambda^\alpha \in \mathrm{CL}^\alpha(M)$, $B\Lambda^{-\alpha}, \Lambda^{-\alpha}B$, $AB\Lambda^{-\alpha}, \Lambda^{-\alpha}BA \in \mathrm{CL}^\beta(M)$. Moreover,

$$[A\Lambda^\alpha, \Lambda^{-\alpha}B] = AB - \Lambda^{-\alpha}BA\Lambda^\alpha, \tag{4.4}$$

$$[\Lambda^\alpha A, B\Lambda^{-\alpha}] = \Lambda^\alpha AB\Lambda^{-\alpha} - BA, \tag{4.5}$$

$$[AB\Lambda^{-\alpha}, \Lambda^\alpha] = AB - \Lambda^\alpha AB\Lambda^{-\alpha}, \tag{4.6}$$

$$[\Lambda^{-\alpha}BA, \Lambda^\alpha] = \Lambda^{-\alpha}BA\Lambda^\alpha - BA. \tag{4.7}$$

Adding up (4.4)–(4.7) yields twice the commutator $[A, B]$, whence $[A, B] \in [\mathrm{CL}^\alpha(M), \mathrm{CL}^\beta(M)]$. □

4.2. **General classical pseudodifferential operators.** We now combine the main result of Subsection 2.3, Theorem 2.9, and the results of the previous Subsection to obtain statements about general pseudodifferential operators as sums of commutators. This improves, for classical pseudodifferential operators, [Les99, Prop. 4.7 and Prop. 4.9]; for such operators loc. cit. in fact goes back to [Wod84]. In [Les99] the more general class of pseudodifferential operators with log–polyhomogeneous symbol expansions was considered.

Theorem 4.6. *Let M be a compact connected riemannian manifold of dimension $n > 1$. Fix $Q \in \mathrm{CL}^{-n}(M)$ with $\mathrm{Res}(Q) = 1$. Then for any real numbers m, a there exist $P_1, \ldots, P_N \in \mathrm{CL}^m(M)$, such that for any $A \in \mathrm{CL}^a(M)$ there exist $Q_1, \ldots, Q_N \in \mathrm{CL}^{a-m+1}(M)$ and $R \in \mathrm{CL}^{-\infty}(M)$ such that*

$$A = \sum_{j=1}^{N} [P_j, Q_j] + \mathrm{Res}(A)\, Q + R. \tag{4.8}$$

Proof. We follow the proof of [Les99, Prop. 4.7], where the case $m = 1$ is discussed, with a few modifications and improvements.

First, replacing A by $A - \mathrm{Res}(A)\, Q$ if necessary, we may, without loss of generality, assume that $\mathrm{Res}(A) = 0$.

We choose $p_1, \ldots, p_N \in \mathcal{P}^m(T^*M \setminus M)$ such that their differentials span the cotangent bundle of $T^*M \setminus M$ at every point if $m \neq 0$; if $m = 0$ we choose the p_j such that their differentials restricted to S^*M span the cotangent bundle of S^*M (cf. the proof of Theorem 2.9). Choose $P_j \in \mathrm{CL}^m(M)$ with leading symbols p_j. Consider the leading symbol $\sigma_a(A) \in \mathcal{P}^a(T^*M \setminus M)$ of A. Its symplectic residue is 0 if $a \neq -n$, and if $a = -n$ it is up to a normalization equal to $\mathrm{Res}(A)$, hence it is also 0 in that case.

Then by Theorem 2.9 and its proof there are $q_j^{(1)} \in \mathcal{P}^{a-m+1}(T^*M \setminus M)$ such that $\sigma_a(A) = \frac{1}{i}\sum\limits_{j=1}^{N}\{p_j, q_j^{(1)}\}$. Thus choosing $Q_j^{(1)} \in \mathrm{CL}^{a-m+1}(M)$ with leading symbol $q_j^{(1)}$ we find, see Eq. (3.4),

$$A^{(1)} = A - \sum_{j=1}^{N}[P_j, Q_j^{(1)}] \in \mathrm{CL}^{a-1}(M).$$

We iterate the procedure: inductively, assume that we have operators $Q_j^{(l)} \in \mathrm{CL}^{a-m+1}(M)$, $1 \le l \le l_0$, such that

$$A^{(l)} = A - \sum_{j=1}^{N}[P_j, Q_j^{(l)}] \in \mathrm{CL}^{a-l}(M)$$

and

$$Q_j^{(l)} - Q_j^{(l+1)} \in \mathrm{CL}^{a-m+1-l}(M), \quad 1 \le l \le l_0 - 1. \tag{4.9}$$

As for A we then choose $B_j \in \mathrm{CL}^{a-m-l_0+1}(M)$ such that

$$A^{(l_0+1)} = A^{(l_0)} - \sum_{j=1}^{N}[P_j, B_j] \in \mathrm{CL}^{a-l_0-1}(M).$$

Now put $Q_j^{(l_0+1)} = Q_j^{(l_0)} + B_j$. Then (4.9) holds for all l and we can invoke the asymptotic summation principle [GrSj94, Prop. 1.8] and choose $Q_j \in \mathrm{CL}^{a-m+1}(M)$ such that for all $l \in \mathbb{N}$, $Q_j - Q_j^{(l)} \in \mathrm{CL}^{s-m+1-l}(M)$. Then

$$A - \sum_{j=1}^{N}[P_j, Q_j] \in \mathrm{CL}^{-\infty}(M). \qquad \square$$

Combining Theorem 4.6 and Lemma 4.5 we find

Theorem 4.7. *Under the assumptions of Theorem 4.6 let $a \in \mathbb{Z}, -n \le a < 0$. Then*

$$\mathrm{CL}^a(M) = [\mathrm{CL}^{(a+1)/2}(M), \mathrm{CL}^{(a+1)/2}(M)] \oplus \mathbb{C} \cdot Q, \tag{4.10}$$

$$= [\mathrm{CL}^0(M), \mathrm{CL}^{a+1}(M)] \oplus \mathbb{C} \cdot Q. \tag{4.11}$$

In other words for $A \in \mathrm{CL}^a(M)$ there exist operators $P_1, \ldots, P_N, Q_1, \ldots, Q_N \in \mathrm{CL}^{(a+1)/2}(M)$ resp. $P_1, \ldots, P_N \in \mathrm{CL}^0(M)$, $Q_1, \ldots, Q_N \in \mathrm{CL}^{a+1}(M)$ such that

$$A = \sum_{j=1}^{N}[P_j, Q_j] + \mathrm{Res}(A)\, Q. \tag{4.12}$$

Proof. Apply Theorem 4.6 with $m = (a+1)/2$ (resp. $m = 0$). This yields $P_1, \ldots, P_{N'}$ in $\mathrm{CL}^{(a+1)/2}(M)$ (resp. $\mathrm{CL}^0(M)$), $Q_1, \ldots, Q_{N'} \in \mathrm{CL}^{(a+1)/2}(M)$ (resp. $\mathrm{CL}^{a+1}(M)$) and $R \in \mathrm{CL}^{-\infty}(M)$ such that

$$A = \sum_{j=1}^{N'}[P_j, Q_j] + \mathrm{Res}(A)\, Q + R.$$

By Proposition 4.2 we have

$$\mathrm{CL}^{-\infty}(M) \subset [\mathrm{CL}^0(M), \mathrm{CL}^{-n+1}(M)] \subset [\mathrm{CL}^0(M), \mathrm{CL}^{a+1}(M)]$$

and hence there are $P_{N'+1}, \ldots, P_N \in \mathrm{CL}^0(M)$ and $Q_{N'+1}, \ldots, Q_N \in \mathrm{CL}^{a+1}(M)$ such that $R = \sum_{j=N'+1}^{N} [P_j, Q_j]$ proving Eq. (4.11).

To prove Eq. (4.10) we apply Lemma 4.5 with $\alpha = (a+1)/2, \beta = -n+1-\alpha$. Then $\alpha - \beta = a + n \in \mathbb{Z}_{\geq 0}$, hence $\mathrm{CL}^\beta(M) \subset \mathrm{CL}^\alpha(M)$ and we find

$$\begin{aligned} R \in \mathrm{CL}^{-\infty}(M) &\subset [\mathrm{CL}^0(M), \mathrm{CL}^{-n+1}(M)] \\ &\subset [\mathrm{CL}^\alpha(M), \mathrm{CL}^\beta(M)] \subset [\mathrm{CL}^{(a+1)/2}(M), \mathrm{CL}^{(a+1)/2}(M)]. \end{aligned}$$ □

4.3. **Classification of traces on $\mathrm{CL}^a(M)$.** We are now going to classify the pretraces and the hypertraces on $\mathrm{CL}^a(M)$ for all $a \in \mathbb{R}$, as well as the traces on $\mathrm{CL}^a(M)$ for $a \in \mathbb{Z}_{\leq 0}$. The following definition will be convenient:

Definition 4.8. We fix once and for all, a *linear* functional $\widetilde{\mathrm{Tr}} : \mathrm{CL}^0(M) \to \mathbb{C}$ such that for $a \in \mathbb{Z}_{<-n}$

$$\widetilde{\mathrm{Tr}}_a = \widetilde{\mathrm{Tr}} \restriction \mathrm{CL}^a(M) = \mathrm{Tr} \restriction \mathrm{CL}^a(M) = \mathrm{Tr}_a,$$

cf. Definition 3.1. Furthermore put

$$\overline{\mathrm{TR}}_a := \begin{cases} \mathrm{TR}_a, & \text{if } a \in \mathbb{R} \setminus \mathbb{Z}_{\geq -n}, \\ \widetilde{\mathrm{Tr}}_a, & \text{if } a \in \mathbb{Z}, -n \leq a < \frac{-n+1}{2}, \\ \mathrm{Res}_a, & \text{if } a \in \mathbb{Z}, \frac{-n+1}{2} \leq a \leq 0. \end{cases} \tag{4.13}$$

$\overline{\mathrm{TR}}_a$ conveniently combines the Kontsevich-Vishik trace and the residue trace. The notation is slightly abusive since for $a, b \in \mathbb{Z}, a < (-n+1)/2 \leq b$ one has $\overline{\mathrm{TR}}_b \restriction \mathrm{CL}^{2a-1}(M) = \mathrm{Res} \restriction \mathrm{CL}^{2a-1}(M) = 0 \neq \mathrm{Tr} \restriction \mathrm{CL}^{2a-1}(M) = \overline{\mathrm{TR}}_{2a-1}$. The disadvantages of this notational conflict are outweighed by the convenience of having a common notation for the Kontsevich-Vishik trace and the residue trace. This will free us from repetitively having to make a distinction between the cases $a \in \mathbb{R} \setminus \mathbb{Z}_{>-n}$ and $a \in \mathbb{Z}_{>-n}$.

We also emphasize that the choice of $\widetilde{\mathrm{Tr}}$ is not canonical but certainly possible.

Proposition 4.9. *Let $a \in \mathbb{R}$.*

1. *Any pretrace on $\mathrm{CL}^a(M)$ is a hypertrace on $\mathrm{CL}^a(M)$.*
2. *If τ is a hypertrace on $\mathrm{CL}^a(M)$ then there is a unique constant $\lambda \in \mathbb{C}$ such that $\tau \restriction \mathrm{CL}^{-\infty}(M) = \lambda\, \mathrm{Tr}$.*
3. *If $a \in \mathbb{Z}_{\leq 0}$ and τ is a trace on $\mathrm{CL}^a(M)$ then $\tau \restriction \mathrm{CL}^{2a}(M)$ is a pretrace (and hence a hypertrace). Conversely, given a pretrace on $\mathrm{CL}^{2a}(M)$, any linear extension $\widetilde{\tau}$ of τ to $\mathrm{CL}^a(M)$ is a trace.*
4. *For $a \in \mathbb{Z}_{\leq 0}$, $\overline{\mathrm{TR}}_a$ is a trace on $\mathrm{CL}^a(M)$. For $a \in \mathbb{R} \setminus \big(\mathbb{Z} \cap [-n+1, -n/2]\big)$ it is a pretrace (and hence a hypertrace).*

Proof. 1. follows from Lemma 4.5.

2. follows from Theorem 4.1.

3. is obvious.

4. For $\frac{-n+1}{2} \leq a \leq 0$ the claim follows from the properties of the residue trace.

Except for $a = -n$ the fact that $\overline{\mathrm{TR}}_a$ is a pretrace follows since Res_a and TR_a are pretraces.

Next consider $a \in \mathbb{R}$, $a < \frac{-n+1}{2}$. Then for $A, B \in \mathrm{CL}^a(M)$ it follows from Lemma 3.3 that $[A, B] \in \mathrm{CL}^{2a-1}(M)$ is of trace–class and that

$$\overline{\mathrm{TR}}_{2a-1}\big([A,B]\big) = \mathrm{Tr}\big([A,B]\big) = 0.$$

This proves the remaining claims under 4. □

Thus to classify traces on $\mathrm{CL}^a(M)$ (for $a \in \mathbb{Z}_{\leq 0}$) it suffices to classify pretraces on $\mathrm{CL}^{2a}(M)$. And to classify pretraces on $\mathrm{CL}^b(M)$ (for any $b \in \mathbb{R}$!) it suffices to classify hypertraces on $\mathrm{CL}^b(M)$.

The following considerably improves a uniqueness result by Maniccia, Schrohe, Seiler [MSS08].

Theorem 4.10. *Let M be a closed connected riemannian manifold of dimension $n > 1$, $a \in \mathbb{R} \setminus \mathbb{Z}_{>-n}$ and let τ be a hypertrace on $\mathrm{CL}^a(M)$. Then there are uniquely determined $\lambda \in \mathbb{C}$ and a distribution $T \in \big(\mathrm{C}^\infty(S^*M)\big)^*$ such that $\tau = \lambda\,\overline{\mathrm{TR}}_a + T \circ \sigma_a$.*

Consequently, a linear functional on $\mathrm{CL}^a(M)$ is a hypertrace if and only if it is a pretrace.

Remark 4.11. Recall from Eq. (3.2) that σ_a denotes the leading symbol map. Since the leading symbol is multiplicative (see Eq. (3.3)) it follows that for any $T \in \big(\mathrm{C}^\infty(S^*M)\big)^*$ the functional $T \circ \sigma_a$ is a pretrace and a hypertrace on $\mathrm{CL}^a(M)$. Some authors (see [PaRo04]) call such traces *leading symbol traces*.

Proof. We note that if τ is a hypertrace on $\mathrm{CL}^a(M)$ then by Proposition 4.9 (2.), there is a unique $\lambda \in \mathbb{C}$ such that $\tau \restriction \mathrm{CL}^{-\infty}(M) = \lambda\,\mathrm{Tr}$.

We apply Theorem 4.6 with $m = 0$. Then for $A \in \mathrm{CL}^{a-1}(M)$ we find

$$A = \sum_{j=1}^{N} [P_j, Q_j] + R, \tag{4.14}$$

with $P_j \in \mathrm{CL}^0$, $Q_j \in \mathrm{CL}^a(M)$. Note that $\mathrm{Res}(A) = 0$ since $a - 1 \in \mathbb{R} \setminus \mathbb{Z}_{\geq -n}$. From Eq. (4.14) we infer $\tau(A) = \tau(R) = \lambda\,\mathrm{Tr}(R) = \lambda\,\mathrm{TR}(R) = \lambda\,\mathrm{TR}(A)$.

Thus we have $\tau \restriction \mathrm{CL}^{a-1}(M) = \lambda\ \mathrm{TR} \restriction \mathrm{CL}^{a-1}(M) = \lambda\ \mathrm{TR}_{a-1} = \lambda\,\overline{\mathrm{TR}}_{a-1}$. Put $\widetilde{\tau} := \tau - \lambda\,\overline{\mathrm{TR}}_a$. Then $\widetilde{\tau}$ vanishes on $\mathrm{CL}^{a-1}(M)$ and thus in view of the exact sequence Eq. (3.2) there is indeed a unique linear functional $T \in \big(\mathrm{C}^\infty(S^*M)\big)^*$ such that $\widetilde{\tau} = T \circ \sigma_a$.

The last statement follows from Proposition 4.9 and the fact that Res_a and $T \circ \sigma_a$ are pretraces on $\mathrm{CL}^a(M)$. □

The remaining cases of integral values are dealt with in the following:

Theorem 4.12. *Let M be a closed connected riemannian manifold of dimension $n > 1$, $a \in \mathbb{Z}_{>-n}$ and let τ be a hypertrace on $\mathrm{CL}^a(M)$. Then there are uniquely determined $\lambda \in \mathbb{C}$ and a distribution $T \in \big(\mathrm{C}^\infty(S^*M)\big)^*$ such that $\tau = \lambda \operatorname{Res}_a + T \circ \sigma_a$.*

Consequently, a linear functional on $\mathrm{CL}^a(M)$ is a hypertrace if and only if it is a pretrace.

Proof. We apply Theorem 4.7 and find for $A \in \mathrm{CL}^{a-1}(M)$

$$A = \sum_{j=1}^N [P_j, Q_j] + \operatorname{Res}(A)\, Q, \tag{4.15}$$

with $P_j \in \mathrm{CL}^0(M), Q_j \in \mathrm{CL}^a(M)$. Thus $\tau(A) = \tau(Q) \operatorname{Res}(A)$. As in the proof of Theorem 4.10 one now concludes $\tau = \tau(Q) \operatorname{Res}_a + T \circ \sigma_a$.

The last statement follows from Proposition 4.9 and the fact that Res_a and $T \circ \sigma_a$ are pretraces on $\mathrm{CL}^a(M)$. □

Combining Theorem 4.10, Theorem 4.12 and Proposition 4.9 we now obtain a complete classification of traces on the algebras $\mathrm{CL}^a(M)$, $a \in \mathbb{Z}_{\leq 0}$.

Corollary 4.13. *Let $a \in \mathbb{Z}_{\leq 0}$, and denote by*

$$\pi_a : \mathrm{CL}^a(M) \longrightarrow \mathrm{CL}^a(M)/\,\mathrm{CL}^{2a-1}(M)$$

the quotient map. Let $\tau : \mathrm{CL}^a(M) \to \mathbb{C}$ be a trace. Then there are uniquely determined $\lambda \in \mathbb{C}$ and $T \in \big(\mathrm{CL}^a(M)/\,\mathrm{CL}^{2a-1}(M)\big)^*$ such that

$$\tau = \lambda\, \overline{\mathrm{TR}}_a + T \circ \pi_a. \tag{4.16}$$

Remark 4.14. Note that for $a = 1$, the space $\mathrm{CL}^1(M)$ is not an algebra but it is a Lie algebra and it makes sense to talk about traces; in this case, the quotient map π_1 is trivial and the proof below shows that Res is up to normalization the unique trace on $\mathrm{CL}^1(M)$.

In the case $a = 0$ this result was known, see [LePa07] (and also [Wod87a]).

If $2a \leq -n \leq a$, Res_a is a non–trivial trace on $\mathrm{CL}^a(M)$, however since $\operatorname{Res} \restriction \mathrm{CL}^{2a-1}(M) = 0$ (since $2a-1 < -n$) there is $\lambda \in \big(\mathrm{CL}^a(M)/\,\mathrm{CL}^{2a-1}(M)\big)^*$, such that $\operatorname{Res}_a = \lambda \circ \pi_a$.

By choosing right inverses $\theta_a : \mathrm{C}^\infty(S^*M) \to \mathrm{CL}^a(M)$ to the symbol map one iteratively obtains an isomorphism

$$\begin{aligned}\mathrm{CL}^a(M)/\,\mathrm{CL}^{2a-1}(M) &\cong \bigoplus_{k=0}^{|a|} \mathrm{CL}^{a-k}(M)/\,\mathrm{CL}^{a-k-1}(M)\\ &\cong \bigoplus_{k=0}^{|a|} \mathrm{C}^\infty(S^*M).\end{aligned} \tag{4.17}$$

Under this (non–canonical) isomorphism $T \in \big(\mathrm{CL}^a(M)/\,\mathrm{CL}^{2a-1}(M)\big)^*$ corresponds to a $(|a|+1)$–tuple $(T_j)_{j=0}^{|a|}$ of distributions $T_j \in \big(\mathrm{C}^\infty(S^*M)\big)^*$.

Proof. By Proposition 4.9, $\tau_{2a} = \tau \restriction \mathrm{CL}^{2a}(M)$ is a hypertrace on $\mathrm{CL}^{2a}(M)$. By Theorem 4.10 (if $2a < -n+1$) resp. Theorem 4.12 (if $-n+1 \le 2a \le 0$) there is a unique $\lambda \in \mathbb{C}$ such that

$$\tau_{2a-1} = \begin{cases} \lambda \, \mathrm{Tr}_{2a-1}, & \text{if } 2a < -n+1, \\ \lambda \, \mathrm{Res}_{2a-1}, & \text{if } -n+1 \le 2a \le 0. \end{cases} \tag{4.18}$$

Putting

$$\widetilde{\tau} = \tau - \lambda \, \overline{\mathrm{TR}}_a \tag{4.19}$$

it follows that $\widetilde{\tau}$ vanishes on $\mathrm{CL}^{2a-1}(M)$ and hence is of the form $T \circ \pi_a$ for a unique $T \in \big(\mathrm{CL}^a(M)/\,\mathrm{CL}^{2a-1}(M)\big)^*$. □

4.4. **Alternative approach to Theorem 4.12 (Joint work of the second named author with Sylvie Paycha).** The proof of the uniqueness of the canonical trace TR (Theorem 4.10) relied solely on the results of Section 2 and Theorem 4.1. The proof of the uniqueness of the residue trace (Theorem 4.12), however, relied additionally on Theorem 4.7 and thus on Proposition 4.2 due to Ponge. We will give here an alternative completely self–contained proof of Theorem 4.12 which does not make use of Proposition 4.2.

Given a hypertrace τ on $\mathrm{CL}^a(M)$, $a \in \mathbb{Z}, -n < a \le 0$, apply Theorem 4.6 with $m = 0$. Then for $A \in \mathrm{CL}^{a-1}(M)$

$$A = \sum_{j=1}^{N} [P_j, Q_j] + \mathrm{Res}(A)\, Q + R \tag{4.20}$$

with $P_j \in \mathrm{CL}^0(M), Q_j \in \mathrm{CL}^a(M)$ and $R \in \mathrm{CL}^{-\infty}(M)$. If one can conclude that $\tau(R) = 0$ then one can proceed as after (4.15). So we have to prove directly

Proposition 4.15. *Let M be a closed riemannian manifold and for $a \in \mathbb{Z}$, $-n+1 \le a \le 0$, let τ be a hypertrace on $\mathrm{CL}^a(M)$. Then $\tau \restriction \mathrm{CL}^{-\infty}(M) = 0$.*

Proof. Let $(U, x_1, \dots, x_n)$ be a local coordinate chart of M. Recall that by $\mathrm{CS}^a_{\mathrm{comp}}(U \times \mathbb{R}^n)$ we denote the set of classical symbols of order a on U with U–compact support, and $\mathrm{CL}^a_{\mathrm{comp}}(U)$ denotes the space of classical pseudodifferential operators of order a on U whose Schwartz kernel has compact support in $U \times U$. Any operator in $\mathrm{CL}^a_{\mathrm{comp}}(U)$ can be extended by zero to an operator in $\mathrm{CL}^a(M)$, and we have the natural inclusion $\mathrm{CL}^a_{\mathrm{comp}}(U) \subset \mathrm{CL}^a(M)$.

Note, however, that although for $\sigma \in \mathrm{CS}^a_{\mathrm{comp}}(U \times \mathbb{R}^n)$ the operator $\mathrm{Op}(\sigma)$ maps $\mathrm{C}^\infty_c(U) \to \mathrm{C}^\infty_c(U)$, it does not necessarily lie in $\mathrm{CL}^a_{\mathrm{comp}}(U)$. Below we will take care of this fact by multiplying by some cut–off function from the right.

Let $\tau \in \mathscr{S}(\mathbb{R}^n)$ be a Schwartz function with $\int_{\mathbb{R}^n} \tau(\xi) d\xi = 1$. By Lemma 2.11 there exist $\tau_1, \dots, \tau_n \in \mathrm{CS}^a(\mathbb{R}^n)$ such that

$$\tau = \sum_{k=1}^{n} \partial_{\xi_k} \tau_k. \tag{4.21}$$

We note in passing that since the function τ has non–vanishing integral, at least one of the functions τ_k does not lie in $\mathscr{S}(\mathbb{R}^n)$.

Next we choose $f \in \mathrm{C}_c^\infty(U)$ with $\int_U f(x)dx = 1$. Then $\sigma := f \otimes \tau$, defined by $\sigma(x,\xi) := f(x)\tau(\xi)$, is a smoothing symbol with U–compact support. Furthermore,

$$\sigma = f \otimes \tau = f \otimes \sum_{k=1}^n \partial_{\xi_k}\tau_k = \sum_{k=1}^n \partial_{\xi_k}(f \otimes \tau_k), \tag{4.22}$$

$$\int_{U\times\mathbb{R}^n} \sigma(x,\xi)\, d\xi\, dx = 1. \tag{4.23}$$

Integration by parts shows that (cf. [Hör03, Thm. 18.1.6], (3.4))

$$\mathrm{Op}(\sigma) = \sum_{k=1}^n \mathrm{Op}(\partial_{\xi_k}(f \otimes \tau_k)) = -i \sum_{k=1}^n [\mathrm{Op}(x_k), \mathrm{Op}(f \otimes \tau_k)]. \tag{4.24}$$

Let $\psi \in \mathrm{C}_c^\infty(U)$ be a function with $\psi = 1$ in a neighborhood of $\operatorname{supp}(f)$; then $\psi f = f$. Moreover, for all $k = 1, \ldots, n$,

$$\begin{aligned}[]
[\mathrm{Op}(x_k), \mathrm{Op}(f \otimes \tau_k)]\,\mathrm{Op}(\psi) &= [\mathrm{Op}(x_k), \mathrm{Op}(f \otimes \tau_k)\,\mathrm{Op}(\psi)] \\
&= [\mathrm{Op}(\psi x_k), \mathrm{Op}(f \otimes \tau_k)\,\mathrm{Op}(\psi)] + A_k,
\end{aligned} \tag{4.25}$$

with

$$\begin{aligned}
A_k &:= \mathrm{Op}(f \otimes \tau_k)\,\mathrm{Op}(\psi)\,\mathrm{Op}(x_k)\,\mathrm{Op}(\psi) - \mathrm{Op}(\psi)\,\mathrm{Op}(f \otimes \tau_k)\,\mathrm{Op}(\psi)\,\mathrm{Op}(x_k) \\
&= \mathrm{Op}(f \otimes \tau_k)\,\mathrm{Op}(\psi)\,\mathrm{Op}(x_k)(\mathrm{Op}(\psi) - 1).
\end{aligned} \tag{4.26}$$

Here, we used that the operator $\mathrm{Op}(x_k)$ commutes with the operator of multiplication by ψ, $\mathrm{Op}(\psi)$, cf. Remark 4.4, and that $\psi f = f$.

Since $f \otimes \tau_k \in \mathrm{CS}^a_{\mathrm{comp}}(U \times \mathbb{R}^n)$, the operator $\mathrm{Op}(f \otimes \tau_k)\,\mathrm{Op}(\psi)$ lies in $\mathrm{CL}^a_{\mathrm{comp}}(U)$; similarly, $\psi x_k \in \mathrm{CS}^0_{\mathrm{comp}}(U \times \mathbb{R}^n)$ and the operator of multiplication by ψx_k, $\mathrm{Op}(\psi x_k)$, lies in $\mathrm{CL}^0_{\mathrm{comp}}(U)$.

Let τ be a hypertrace on $\mathrm{CL}^a(M)$. Then τ vanishes on $[\mathrm{CL}^0_{\mathrm{comp}}(U), \mathrm{CL}^a_{\mathrm{comp}}(U)]$. In particular, for all $k = 1, \ldots, n$,

$$\tau\big([\mathrm{Op}(\psi x_k), \mathrm{Op}(f \otimes \tau_k)\,\mathrm{Op}(\psi)]\big) = 0.$$

By Proposition 4.9 (2.), we have $\tau \restriction \mathrm{CL}^{-\infty}(M) = \lambda\,\mathrm{Tr}$ for some $\lambda \in \mathbb{C}$. Now, since $\psi = 1$ near the support of f, by (4.26) the operator A_k is smoothing and its Schwartz kernel vanishes on the diagonal. Hence, its L^2–trace vanishes and thus also $\tau(A_k) = \lambda\,\mathrm{Tr}(A_k) = 0$.

Thus, for $\mathrm{Op}(\sigma)\,\mathrm{Op}(\psi) \in \mathrm{CL}^{-\infty}_{\mathrm{comp}}(U)$, from (4.24) and (4.25) we conclude

$$\begin{aligned}
\tau(\mathrm{Op}(\sigma)\,\mathrm{Op}(\psi)) &= -i \sum_{k=1}^n \tau\big([\mathrm{Op}(x_k), \mathrm{Op}(f \otimes \tau_k)]\,\mathrm{Op}(\psi)\big) \\
&= -i \sum_{k=1}^n \Big(\tau\big([\mathrm{Op}(\psi x_k), \mathrm{Op}(f \otimes \tau_k)\,\mathrm{Op}(\psi)]\big) + \tau(A_k)\Big) = 0.
\end{aligned} \tag{4.27}$$

On the other hand, by (3.7) and Proposition 4.9 (2.),

$$\tau(\mathrm{Op}(\sigma)\,\mathrm{Op}(\psi)) = \lambda\ \mathrm{Tr}(\mathrm{Op}(\sigma)\,\mathrm{Op}(\psi)) = \lambda \int_{U\times\mathbb{R}^n} \sigma(x,\xi)\, d\xi\, dx = \lambda. \tag{4.28}$$

Therefore, by (4.27) we obtain $\lambda = 0$. □

5. Extension to vector bundles

In this final section we extend the classification of traces and hypertraces to the spaces $\mathrm{CL}^a(M, E)$ of pseudodifferential operators acting on sections of the vector bundle E over M.

5.1. **Preliminaries.** Unless otherwise said, during the whole section M denotes a smooth closed connected riemannian manifold of dimension n. Let $E \to M$ be a smooth hermitian vector bundle over M. We denote by $\mathrm{CL}^a(M, E)$ the space of classical pseudodifferential operators of order a acting on the sections of E. $\mathrm{CL}^a(M, E)$ acts naturally as (unbounded) operators on the Hilbert space $L^2(M, E)$ of square integrable sections of E. The elementary discussion of traces, pretraces and hypertraces in Subsection 3.2 extends verbatim to $\mathrm{CL}^a(M, E)$. However, as noted there, we now only have $[\mathrm{CL}^a(M, E), \mathrm{CL}^b(M, E)] \subset \mathrm{CL}^{a+b}(M, E)$ as opposed to $[\mathrm{CL}^a(M), \mathrm{CL}^b(M)] \subset \mathrm{CL}^{a+b-1}(M)$ in the scalar case $E = M \times \mathbb{C}$. Lemma 4.5 holds with the same proof for $\mathrm{CL}^\bullet(M, E)$ instead of $\mathrm{CL}^\bullet(M)$. Finally, Theorem 4.1 holds for $\mathrm{CL}^{-\infty}(M, E)$ too; this follows directly from [Gui93, Thm. A.1], which is stated in a Hilbert space context and therefore flexible enough.

In sum, also Proposition 4.9 (1.–3.) holds accordingly:

Proposition 5.1. *Let $a \in \mathbb{R}$.*

1. *Any pretrace on $\mathrm{CL}^a(M, E)$ is a hypertrace on $\mathrm{CL}^a(M, E)$.*
2. *If τ is a hypertrace on $\mathrm{CL}^a(M, E)$ then there is a unique constant $\lambda \in \mathbb{C}$ such that $\tau \restriction \mathrm{CL}^{-\infty}(M, E) = \lambda\, \mathrm{Tr}$.*
3. *If $a \in \mathbb{Z}_{\leq 0}$ and τ is a trace on $\mathrm{CL}^a(M, E)$ then $\tau \restriction \mathrm{CL}^{2a}(M, E)$ is a pretrace (and hence a hypertrace). Conversely, given a pretrace on $\mathrm{CL}^{2a}(M, E)$ then any linear extension $\widetilde{\tau}$ of τ to $\mathrm{CL}^a(M, E)$ is a trace.*

For the analogue of Proposition 4.9 (4.) see Proposition 5.5.

The main task now is to classify the hypertraces on $\mathrm{CL}^a(M, E)$.

5.2. **Trivial vector bundles.** Let $M_N(\mathbb{C})$ be the space of $N \times N$ matrices with coefficients in $\mathbb{C}$. For all $i, j = 1, \ldots, N$, we denote by E_{ij} the elementary matrix in $M_N(\mathbb{C})$ with 1 in the (i, j)–position and 0 everywhere else. The matrices E_{ij} form a basis of $M_N(\mathbb{C})$ and we have

$$E_{ij}\, E_{kl} \;=\; \delta_{jk}\, E_{il}. \tag{5.1}$$

Let us denote by tr_N the unique trace on the algebra $M_N(\mathbb{C})$ such that for all $i = 1, \ldots, N$, $\mathrm{tr}_N(E_{ii}) = 1$.

For a complex vector space V we will tacitly identify $M_N(V)$ with $V \otimes M_N(\mathbb{C})$ via

$$x := (x_{ij})_{i,j} \longmapsto \sum_{i,j=1}^{N} x_{ij} \otimes E_{ij}. \tag{5.2}$$

Obviously, we have $\mathrm{CL}^a(M, \mathbb{C}^N) = M_N\big(\mathrm{CL}^a(M)\big) \cong \mathrm{CL}^a(M) \otimes M_N(\mathbb{C})$.

Definition 5.2. Let $a \in \mathbb{R}$ and let τ be a linear functional on $\mathrm{CL}^a(M)$. Then we put

$$\begin{aligned} &\tau \otimes \mathrm{tr}_N : \mathrm{CL}^a(M, \mathbb{C}^N) \longrightarrow \mathbb{C}, \\ &\qquad A := (A_{ij})_{i,j} \mapsto \sum_{i,j=1}^{N} (\tau \otimes \mathrm{tr}_N)(A_{ij} \otimes E_{ij}) = \sum_{i=1}^{N} \tau(A_{ii}). \end{aligned} \tag{5.3}$$

It is straightforward to check that if τ is a hypertrace (pretrace, trace) on $\mathrm{CL}^a(M)$ then $\tau \otimes \mathrm{tr}_N$ is a hypertrace (pretrace, trace) on $\mathrm{CL}^a(M, \mathbb{C}^N)$.

Proposition 5.3. *Let $a \in \mathbb{R}$. Then every hypertrace on $\mathrm{CL}^a(M, \mathbb{C}^N)$ is of the form $\tau \otimes \mathrm{tr}_N$ with a unique hypertrace τ on $\mathrm{CL}^a(M)$.*

Proof. Let T be a hypertrace on $\mathrm{CL}^a(M, \mathbb{C}^N)$. For $i, j = 1, \ldots, N$ we put $T_{ij} : \mathrm{CL}^a(M) \to \mathbb{C}$, $T_{ij}(A) := T(A \otimes E_{ij})$.

Since $\mathrm{Id} \in \mathrm{CL}^0(M, \mathbb{C}^N)$ we infer from the hypertrace property

$$\begin{aligned} T_{ij}(A) &= T(A \otimes E_{ij}) = T\big((A \otimes E_{i1})\,(\mathrm{id} \otimes E_{1j})\big) \\ &= T\big((\mathrm{id} \otimes E_{1j})\,(A \otimes E_{i1})\big) = \delta_{ij}\, T_{11}(A), \end{aligned} \tag{5.4}$$

thus $T_{ij} = 0$ for $i \neq j$ and $T_{11} = T_{22} = \ldots = T_{NN} =: \tau$.

τ is a hypertrace on $\mathrm{CL}^a(M)$. Namely, for $A \in \mathrm{CL}^a(M), B \in \mathrm{CL}^0(M)$ we have

$$\begin{aligned} \tau(AB) &= T((AB) \otimes E_{11}) = T\big((A \otimes E_{11})\,(B \otimes E_{11})\big) \\ &= T\big((B \otimes E_{11})\,(A \otimes E_{11})\big) = \tau(BA). \end{aligned} \tag{5.5}$$

Certainly, we have $T = \tau \otimes \mathrm{tr}_N$.

For the uniqueness we only have to note that if $T = \tau \otimes \mathrm{tr}_N$ then $\tau(A) = T(A \otimes E_{11})$. □

5.3. **General vector bundles.** Let E be a vector bundle over M. By Swan's Theorem there is a positive integer N, such that E is a direct summand of $M \times \mathbb{C}^N$; let $e \in M_N(\mathrm{C}^\infty(M)) = \mathrm{C}^\infty(M, M_N(\mathbb{C}))$ be a smooth projection onto E. Then the $\mathrm{C}^\infty(M)$–module of smooth sections of E is given by

$$\Gamma^\infty(M, E) \cong e(\mathrm{C}^\infty(M)^N). \tag{5.6}$$

Note that since we assumed M to be connected (cf. Subsection 5.1), the idempotent valued function e has constant rank.

The following lemma is well–known. Since we could not find a place where it is stated as needed we provide, for convenience, a quick proof:

Lemma 5.4. *Let $\mathcal{A} := \mathrm{C}^\infty(M, M_N(\mathbb{C}))$. Then $\mathcal{A}\, e\, \mathcal{A} = \mathcal{A}$. In other words there exist $p_j, q_j \in \mathrm{C}^\infty(M, M_N(\mathbb{C}))$, $j = 1, \ldots, r$ such that*

$$\sum_{j=1}^{r} p_j\, e\, q_j = 1_M \otimes I_N, \tag{5.7}$$

where 1_M denotes the function which is constant 1 on M and I_N is the $N \times N$ identity matrix.

Proof. It obviously suffices to prove Eq. (5.7). Choose a finite partition of unity $\psi_j, j = 1, \ldots, s$, smooth functions $\chi_j \in \mathrm{C}^\infty(M)$ such that $\chi_j = 1$ in a neighborhood of $\operatorname{supp}(\psi_j)$ and such that in a neighborhood U_j of $\operatorname{supp}(\chi_j)$ there is a smooth map $v : U_j \to M_N(\mathbb{C})$ such that

$$v\, e\, v^{-1} = e_k := \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}.$$

Choose $N \times N$ matrices $a_l, b_l, l = 1, \ldots, t$, with

$$\sum_{l=1}^{t} a_l\, e_k\, b_l = I_N.$$

We tacitly view a_l, b_l also as constant matrix valued functions on M. Slightly abusing notation we now find the decomposition

$$\begin{aligned} 1_M \otimes I_N &= \sum_{j=1}^{s} \psi_j \chi_j \otimes I_N = \sum_{j=1}^{s} \sum_{l=1}^{t} \psi_j\, v^{-1}\, a_l\, e_k\, b_l\, v\, \chi_j \\ &= \sum_{j=1}^{s} \sum_{l=1}^{t} \big(\psi_j\, v^{-1}\, a_l\, v\big)\, e\, \big(v^{-1}\, b_l\, v\, \chi_j\big). \end{aligned}$$

□

For a linear functional τ on $\mathrm{CL}^a(M, \mathbb{C}^N)$ we now put

$$\tau_E(A) := \tau(eAe). \tag{5.8}$$

This definition depends on the choice of the idempotent e and is therefore not canonical. As in the scalar case if $\mathrm{CL}^a(M, E) \subset \mathrm{CL}^b(M, E)$ we write $\tau_{E,a} := \tau_E \restriction \mathrm{CL}^a(M, E)$.

The canonical trace TR and the residue trace Res are naturally defined on $\mathrm{CL}^\bullet(M, E)$ for any vector bundle E (cf. [Les99]). To distinguish them let us for the moment denote by $\mathrm{TR}^{(N)}, \mathrm{Res}^{(N)}$ the corresponding functionals on $\mathrm{CL}^\bullet(M, \mathbb{C}^N)$ and by $\mathrm{TR}^{(E)}, \mathrm{Res}^{(E)}$ the corresponding functionals on $\mathrm{CL}^\bullet(M, E)$.

Then one immediately checks that

$$\mathrm{TR}^{(N)} = \mathrm{TR} \otimes \mathrm{tr}_N, \qquad \mathrm{TR}^{(E)} = \big(\mathrm{TR} \otimes \mathrm{tr}_N\big)_E, \tag{5.9}$$

$$\mathrm{Res}^{(N)} = \mathrm{Res} \otimes \mathrm{tr}_N, \qquad \mathrm{Res}^{(E)} = \big(\mathrm{Res} \otimes \mathrm{tr}_N\big)_E, \tag{5.10}$$

hence TR and Res are compatible with the operations $\tau \mapsto \tau \otimes \mathrm{tr}_N$ and $\tau \mapsto \tau_E$ in the most natural way.

From now on we will write TR_E for $\mathrm{TR}^{(E)}$, and Res_E for $\mathrm{Res}^{(E)}$. A confusion with the notation introduced in Definition 3.1 should not arise.

We also extend the linear functional $\widetilde{\mathrm{Tr}}$ of Definition 4.8 to $\mathrm{CL}^0(M, E)$ by defining

$$\widetilde{\mathrm{Tr}}_E := \big(\widetilde{\mathrm{Tr}} \otimes \mathrm{tr}_N\big)_E.$$

Since $\widetilde{\mathrm{Tr}}$ is not a trace, this definition may depend on the choice of the idempotent e, hence is not canonical; but $\widetilde{\mathrm{Tr}}$ already depended on a choice.

Finally we put $\overline{\mathrm{TR}}_{E,a} := \big(\overline{\mathrm{TR}}_a \otimes \mathrm{tr}_N\big)_E$ on $\mathrm{CL}^a(M,E)$. From Subsection 4.3 we see

$$\overline{\mathrm{TR}}_{E,a} := \begin{cases} \mathrm{TR}_{E,a}, & \text{if } a \in \mathbb{R}\setminus \mathbb{Z}_{\geq -n},\\ \widetilde{\mathrm{Tr}}_{E,a}, & \text{if } a \in \mathbb{Z}, -n \leq a < \frac{-n+1}{2},\\ \mathrm{Res}_{E,a}, & \text{if } a\in\mathbb{Z}, \frac{-n+1}{2} \leq a \leq 0.\end{cases} \tag{5.11}$$

Proposition 5.5. 1. *Let $a \in \mathbb{R}$ and let τ be a hypertrace (resp. pretrace, trace) on $\mathrm{CL}^a(M,\mathbb{C}^N)$. Then $\tau_E : \mathrm{CL}^a(M,E) \longrightarrow \mathbb{C}, \quad A \mapsto \tau(eAe)$ is a hypertrace (resp. pretrace, trace) on $\mathrm{CL}^a(M,E)$.*

2. *Any hypertrace on $\mathrm{CL}^a(M,E)$ is of the form $\big(\tau\otimes\mathrm{tr}_N)_E$ for a unique hypertrace τ on $\mathrm{CL}^a(M)$.*

3. *For $a \in \mathbb{Z}_{\leq 0}$, $\overline{\mathrm{TR}}_{E,a}$ is a trace on $\mathrm{CL}^a(M,E)$. For $a\in\mathbb{R}\setminus\big(\mathbb{Z}\cap[-n+1,-n/2]\big)$ it is a pretrace (and hence a hypertrace).*

Proof. 1. To prove that the linear functional τ_E is a hypertrace consider $A \in \mathrm{CL}^a(M,E), B\in \mathrm{CL}^0(M,E)$. Then

$$\begin{aligned} \tau_E(AB) &= \tau(eABe) = \tau\big((eAe)(eBe)\big)\\ &= \tau\big((eBe)(eAe)\big) = \tau(eBAe) = \tau_E(BA). \end{aligned} \tag{5.12}$$

Note that $eAe \in \mathrm{CL}^a(M,\mathbb{C}^N), eBe \in \mathrm{CL}^0(M,\mathbb{C}^N)$. Repeating the argument with $A,B\in\mathrm{CL}^{a/2}(M,E)$ shows that if τ is a pretrace then so is τ_E. Similarly if $a \in \mathbb{Z}_{\leq 0}$ and τ is a trace then τ_E is a trace.

2. Conversely, let T be a hypertrace on $\mathrm{CL}^a(M,E)$. We choose $p_j,q_j, j=1,\ldots,r$ according to Lemma 5.4. We will repeatedly use that multiplication by p_j,q_j is in $\mathrm{CL}^0(M,\mathbb{C}^N)$, resp. $ep_je, eq_je\in\mathrm{CL}^0(M,E)$.

Suppose we had a hypertrace $\widetilde{T}$ on $\mathrm{CL}^a(M,\mathbb{C}^N)$ such that $\widetilde{T}_E = T$. Then for $A \in \mathrm{CL}^a(M,\mathbb{C}^N)$

$$\widetilde{T}(A) = \widetilde{T}\big((1_M\otimes I_N)A\big) = \sum_{j=1}^r \widetilde{T}\big(p_jeq_jA\big) = \sum_{j=1}^r T\big(eq_jAp_je\big). \tag{5.13}$$

Thus there is at most such a $\widetilde{T}$. We now *define* $\widetilde{T}$ by the right hand side of Eq. (5.13). We have $\widetilde{T}_E = T$. Indeed, for $A\in\mathrm{CL}^a(M,E)$

$$\begin{aligned} \widetilde{T}_E(A) = \widetilde{T}(eAe) &= \sum_{j=1}^r T\big((e\,q_j\,e\,A\,e)\,(e\,p_j\,e)\big)\\ &= \sum_{j=1}^r T\big(e\,p_j\,e\,q_j\,e\,A\,e\big) = T(eAe) = T(A). \end{aligned} \tag{5.14}$$

In the last line we used Eq. (5.7).

Next we show that $\widetilde{T}$ is a hypertrace on $\mathrm{CL}^a(M,\mathbb{C}^N)$. Indeed, for $A \in \mathrm{CL}^a(M,\mathbb{C}^N), B \in \mathrm{CL}^0(M,\mathbb{C}^N)$ we find using Eq. (5.7),

$$\begin{aligned}\widetilde{T}(AB) &= \sum_{j=1}^{r} T\big(e\, q_j A\, (1_M \otimes I_N)\, B\, p_j\, e\big) = \sum_{j,k=1}^{r} T\big(e\, q_j\, A\, p_k\, e\, q_k\, B\, p_j\, e\big)\\ &= \sum_{j,k=1}^{r} T\big(e\, q_k\, B\, p_j\, e\, q_j\, A\, p_k\, e\big) = \sum_{k=1}^{r} T\big(e\, q_k\, B\, A\, p_k\, e\big) = \widetilde{T}(BA).\end{aligned} \tag{5.15}$$

By Proposition 5.3 there is now a unique hypertrace τ on $\mathrm{CL}^a(M)$ such that $\widetilde{T} = \tau \otimes \mathrm{tr}_N$. Then we conclude $T = \widetilde{T}_E = \big(\tau \otimes \mathrm{tr}_N\big)_E$. Recall that $\widetilde{T}$ is uniquely determined by T and τ is uniquely determined by $\widetilde{T}$, whence τ is uniquely determined by T.

3. follows from the proved part 1., Eq. (5.11) and Proposition 4.9. □

Before stating the final result, we have to clarify how leading symbol traces on $\mathrm{CL}^a(M,E)$ look like. For the moment consider a closed manifold X with a vector bundle $E \to X$. We can construct traces on the noncommutative algebra $\Gamma^\infty(X, \mathrm{End}\, E)$ as follows: first the fiberwise trace induces a linear map

$$\begin{aligned}\mathrm{tr}_E : \Gamma^\infty(X, \mathrm{End}\, E) &\to \mathrm{C}^\infty(X)\\ \mathrm{tr}_E(s)(x) &= \mathrm{tr}_{E_x}\big(s(x)\big).\end{aligned} \tag{5.16}$$

tr_E vanishes on commutators. Thus for any $T \in \big(\mathrm{C}^\infty(X)\big)^*$ the composition $T \circ \mathrm{tr}_E$ is a trace on $\Gamma^\infty(X, \mathrm{End}\, E)$.

It is straightforward to see that indeed all traces on $\Gamma^\infty(X, \mathrm{End}\, E)$ are of this form. Since we will not use this fact we leave the details of proof to the reader:

Proposition 5.6. *Let X be a closed manifold and let E be a vector bundle over X. Then for any trace τ on $\Gamma^\infty(X, \mathrm{End}\, E)$ there is a unique distribution $T \in \big(\mathrm{C}^\infty(X)\big)^*$ such that $\tau = T \circ \mathrm{tr}_E$.*

The final result is now a consequence of Theorems 4.10, 4.12, Corollary 4.13, and Propositions 5.1, 5.5.

Theorem 5.7. *Let M be a closed connected riemannian manifold of dimension $n > 1$ and let E be a complex vector bundle over M. Denote by $\Pi : E \to M$ the projection map, by $\sigma_a : \mathrm{CL}^a(M,E) \to \Gamma^\infty\big(S^*M, \Pi^*\mathrm{End}\, E\big)$ the leading symbol map, and by tr_E the fiberwise trace $\Gamma^\infty\big(S^*M, \Pi^*\mathrm{End}\, E\big) \to \mathrm{C}^\infty(S^*M)$. Fix N and an idempotent e as in* Eq. (5.6) *and let $\overline{\mathrm{TR}}_{E,a}$ be as defined in* Eq. (5.11).

1. *Let $a \in \mathbb{R}$ and let τ be a hypertrace on $\mathrm{CL}^a(M,E)$. Then there are uniquely determined $\lambda \in \mathbb{C}$ and a distribution $T \in \big(\mathrm{C}^\infty(S^*M)\big)^*$ such that*

$$\tau = T \circ \mathrm{tr}_E \circ \sigma_a + \begin{cases} \lambda\, \overline{\mathrm{TR}}_{E,a}, & \text{if } a \notin \mathbb{Z}_{>-n},\\ \lambda\, \mathrm{Res}_{E,a}, & \text{if } a \in \mathbb{Z}_{>-n}.\end{cases} \tag{5.17}$$

2. *Let $a \in \mathbb{Z}_{\leq 0}$ and denote by*

$$\pi_a : \mathrm{CL}^a(M,E) \to \mathrm{CL}^a(M,E)/\,\mathrm{CL}^{2a}(M,E)$$

the quotient map. Furthermore, let

$$\theta_a : \mathrm{CL}^a(M,E)/\mathrm{CL}^{2a}(M,E) \to \mathrm{CL}^a(M,E)$$

be a right inverse to π_a.

Let $\tau : \mathrm{CL}^a(M,E) \to \mathbb{C}$ *be a trace. Then there are uniquely determined* $\lambda \in \mathbb{C}$, $T \in \big(\mathrm{C}^\infty(S^*M)\big)^*$ *and* $\Phi \in \big(\mathrm{CL}^a(M,E)/\mathrm{CL}^{2a}(M,E)\big)^*$ *such that*

$$\tau = \lambda\,\overline{\mathrm{TR}}_{E,a} + T \circ \mathrm{tr}_E \circ \sigma_{2a}(\mathrm{id} - \theta_a \circ \pi_a) + \Phi \circ \pi_a. \tag{5.18}$$

In the first line of Eq. (5.17) the case $a = -n$ is included, thus we write $\overline{\mathrm{TR}}_{E,a}$ instead of $\mathrm{TR}_{E,a}$ there.

Proof. The right inverse θ_a can be constructed successively from the map Op, cf. Remark 4.14.

1. By Proposition 5.5 there is a unique hypertrace $\widetilde{\tau}$ on $\mathrm{CL}^a(M)$ such that $\tau = (\widetilde{\tau} \otimes \mathrm{tr}_N)_E$. The claim now follows from Theorem 4.10 and Theorem 4.12 applied to $\widetilde{\tau}$. Note that $T \circ \mathrm{tr}_E \circ \sigma_a = \big((T \circ \sigma_a) \otimes \mathrm{tr}_N\big)_E$, cf. Eq. (5.8) and Definition 5.2.

2. Let $a \in \mathbb{Z}_{\le 0}$. By Proposition 5.1, $\tau \restriction \mathrm{CL}^{2a}(M,E)$ is a hypertrace. Thus, by the proved part 1. we have

$$\tau \restriction \mathrm{CL}^{2a}(M,E) = T \circ \mathrm{tr}_E \circ \sigma_{2a} + \begin{cases} \lambda\,\overline{\mathrm{TR}}_{E,2a}, & \text{if } 2a \le -n, \\ \lambda\,\mathrm{Res}_{E,2a}, & \text{if } 2a > -n. \end{cases} \tag{5.19}$$

We emphasize that by Eq. (5.11)

$$\overline{\mathrm{TR}}_{E,a} \restriction \mathrm{CL}^{2a}(M,E) = \begin{cases} \lambda\,\overline{\mathrm{TR}}_{E,2a}, & \text{if } 2a \le -n, \\ \lambda\,\mathrm{Res}_{E,2a}, & \text{if } 2a > -n. \end{cases} \tag{5.20}$$

Consider for $A \in \mathrm{CL}^a(M,E)$

$$\widetilde{\tau}(A) := \tau(A) - \lambda\,\overline{\mathrm{TR}}_{E,a}(A) - T \circ \mathrm{tr}_E \circ \sigma_{2a}(A - \theta_a \circ \pi_a(A)).$$

Then due to Eq. (5.20) and Eq. (5.19) the functional $\widetilde{\tau}$ vanishes on $\mathrm{CL}^{2a}(M,E)$ and thus is of the form $\Phi \circ \pi_a$ with $\Phi \in \big(\mathrm{CL}^a(M,E)/\mathrm{CL}^{2a}(M,E)\big)^*$. Then $\tau = \Phi \circ \pi_a + \widetilde{\tau}$ and the theorem is proved. □

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