



*p*-adic Beilinson conjecture for ordinary  
Hecke motives associated to imaginary  
quadratic fields

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# **$p$ -ADIC BEILINSON CONJECTURE FOR ORDINARY HECKE MOTIVES ASSOCIATED TO IMAGINARY QUADRATIC FIELDS**

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## 1. INTRODUCTION

The purpose of this article is to give an overview of the series of papers [BK1] [BK2] concerning the  $p$ -adic Beilinson conjecture of motives associated to Hecke characters of an imaginary quadratic field  $K$ , for a prime  $p$  which splits in  $K$ . The  $p$ -adic  $L$ -function for such  $p$  interpolating *critical values* of  $L$ -functions of Hecke characters associated to imaginary quadratic fields was first constructed by Vishik and Manin [VM], and a different construction using  $p$ -adic Eisenstein series was given by Katz [Katz]. The  $p$ -adic Beilinson conjecture, as formulated by Perrin-Riou in [PR], gives a precise conjecture concerning the *non-critical values* of  $p$ -adic  $L$ -functions associated to general motives. The purpose of our research is to investigate the interpolation property at non-critical points of the  $p$ -adic  $L$ -function constructed by Vishik-Manin and Katz.

For simplicity, we assume in this article that the imaginary quadratic field  $K$  has class number *one* and that the Hecke character  $\psi$  we consider corresponds to an elliptic curve with complex multiplication defined over  $\mathbb{Q}$ . Let  $a$  be an integer  $> 0$ . The main theorem of this article (Theorem 6.9) is a proof of the  $p$ -adic Beilinson conjecture for  $\psi^a$  (see Conjecture 2.3), when the prime  $p \geq 5$  is an *ordinary* prime. The authors would like to thank the organizers Takashi Ichikawa, Masanari Kida and Takao Yamazaki for the opportunity to present our research at the RIMS “Algebraic Number Theory and Related Topics 2009” conference.

## 2. THE $p$ -ADIC BEILINSON CONJECTURE

Assume that  $K$  is an imaginary quadratic field of *class number one*. Let  $E$  be an elliptic curve defined over  $\mathbb{Q}$ . We assume in addition that  $E$  has complex multiplication by the ring of integers  $\mathcal{O}_K$  of  $K$ . We let  $\psi := \psi_{E/K}$  be the Grossencharacter of  $K$  associated to  $E_K := E \otimes_{\mathbb{Q}} K$  by the theory of complex multiplication, and we denote by  $\mathfrak{f}$  the conductor of  $\psi$ .

We let  $M(\psi)$  be the motive over  $K$  with coefficients in  $K$  associated to the Grossencharacter  $\psi$ . Then we have  $M(\psi) = H^1(E_K)$ , where  $H^1(E_K)$

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is the motive associated to  $E_K$ . The Hasse-Weil  $L$ -function of  $M(\psi)$  is a function with values in  $K \otimes_{\mathbb{Q}} \mathbb{C}$  given by

$$L(M(\psi), s) = (L(\psi_\tau, s))_{\tau:K \hookrightarrow \mathbb{C}},$$

where  $\tau : K \hookrightarrow \mathbb{C}$  are the embeddings of the coefficient  $K$  of  $M(\psi)$  into  $\mathbb{C}$  and  $L(\psi_\tau, s)$  is the Hecke  $L$ -function

$$L(\psi_\tau, s) = \prod_{(\mathfrak{q}, \mathfrak{f})=1} \left( 1 - \frac{\psi_\tau(\mathfrak{q})}{N\mathfrak{q}^s} \right)^{-1}$$

associated to the character  $\psi_\tau : \mathbb{A}_K^\times \xrightarrow{\psi} K \xrightarrow{\tau} \mathbb{C}$ . Here, the product is over the prime ideals  $\mathfrak{q}$  of  $K$  which are prime to  $\mathfrak{f}$ .

For integers  $a > 0$  and  $n$ , we let  $M^a = M(\psi^a) := M(\psi)^{\otimes_{K^a}}$ , which is a motive over  $K$  with coefficients in  $K$ . Then the Hasse-Weil  $L$ -function  $L(M^a, s)$  is given by the Hecke  $L$ -function

$$L(M^a, s) = (L(\psi_\tau^a, s))_{\tau:K \hookrightarrow \mathbb{C}}$$

with values in  $K \otimes_{\mathbb{Q}} \mathbb{C}$ . We let  $M_B^a$  be the Betti realization of  $M^a$ , which is a  $K$ -vector space of dimension *one*. We fix a  $K$ -basis  $\omega_B^a$  of  $M_B^a$ . The de Rham realization  $M_{\text{dR}}^a(n)$  of  $M^a(n)$  is the rank one  $K \otimes_{\mathbb{Q}} K$ -module

$$M_{\text{dR}}^a(n) = K\omega^{n-a,n} \bigoplus K\omega^{n,n-a},$$

with Hodge filtration given by

$$F^m M_{\text{dR}}^a(n) = \begin{cases} M_{\text{dR}}^a(n) & m \leq -n \\ K\omega^{n,n-a} & -n < m \leq a-n \\ 0 & \text{otherwise.} \end{cases}$$

In what follows, we consider the case when  $n > a$ , which implies in particular that our motive is *non-critical*. We have in this case  $F^0 M_{\text{dR}}^a(n) = 0$ . The tangent space of our motive is given by

$$t_n^a := M_{\text{dR}}^a(n) / F^0 M_{\text{dR}}^a(n) \cong M_{\text{dR}}^a(n),$$

which is again a  $K \otimes_{\mathbb{Q}} K$ -module of rank one. Note that  $\omega_{\text{ig},n}^a := \omega^{n-a,n} + \omega^{n,n-a}$  gives a basis of  $t_n^a$  as a  $K \otimes_{\mathbb{Q}} K$ -module.

We denote by  $V_\infty^a(n)$  the  $\mathbb{R}$ -Hodge realization of  $M^a(n)$ . The Beilinson-Deligne cohomology  $H_{\mathcal{D}}^1(K \otimes_{\mathbb{Q}} \mathbb{R}, V_\infty^a(n))$  is given as the cokernel of the natural inclusion

$$M_B^a(n) \otimes_{\mathbb{Q}} \mathbb{R} \rightarrow t_n^a \otimes_{\mathbb{Q}} \mathbb{R}.$$

The Beilinson regulator map gives a homomorphism

$$(1) \quad r_\infty : H_{\text{mot}}^1(K, M^a(n)) \rightarrow H_{\mathcal{D}}^1(K \otimes_{\mathbb{Q}} \mathbb{R}, V_\infty^a(n)),$$

from the motivic cohomology  $H_{\text{mot}}^1(K, M^a(n))$  of  $K$  with coefficients in  $M^a(n)$  to  $H_{\mathcal{D}}^1(K \otimes_{\mathbb{Q}} \mathbb{R}, V_\infty^a(n))$ . Then  $r_\infty \otimes_{\mathbb{Q}} \mathbb{R}$  is known to be surjective and is conjectured to be an isomorphism. We let  $c_n^a$  be an element of  $H_{\text{mot}}^1(K, M^a(n))$  such that  $r_\infty(c_n^a)$  generates  $H_{\mathcal{D}}^1(K_{\mathfrak{p}}, V_\infty^a(n))$  as a  $K_\infty :=$

$K \otimes_{\mathbb{Q}} \mathbb{R}$ -module. We define the complex period  $\Omega_{\infty}(n)$  of  $M^a(n)$  to be the determinant of the exact sequence

$$(2) \quad 0 \rightarrow M_B^a(n) \otimes_{\mathbb{Q}} \mathbb{R} \rightarrow t_n^a \otimes_{\mathbb{Q}} \mathbb{R} \rightarrow H_{\mathcal{D}}^1(K \otimes_{\mathbb{Q}} \mathbb{R}, V_{\infty}^a(n)) \rightarrow 0$$

for the basis  $r_{\infty}(c_n^a)$ ,  $\omega_{\text{tg},n}^a$ , and  $\omega_B^a$ . The complex period is an element in  $K_{\infty}$  and is independent of the choice of the basis up to multiplication by an element in  $K^{\times}$ . The value  $L(M^a, n)$  is in  $K \otimes_{\mathbb{Q}} \mathbb{R}$ , and the weak Beilinson conjecture for  $M^a(n)$  as proved by Deninger [De1] gives the following (see Theorem 6.2 and Corollary 6.4 for the precise statement.)

**Theorem 2.1.** *The value*

$$\frac{L(M^a, n)}{\Omega_{\infty}(n)}$$

*is an element in  $K^{\times}$ .*

For any prime  $p$ , the étale realization  $V_p^a(n)$  of our motive is a  $K \otimes_{\mathbb{Q}} \mathbb{Q}_p$ -vector space with continuous action of  $\text{Gal}(\bar{K}/K)$ . We fix a prime  $p \geq 5$  relatively prime to  $\mathfrak{f}$  such that  $E$  has good *ordinary* reduction at  $p$ . In this case, the ideal generated by  $p$  splits as  $(p) = \mathfrak{p}\mathfrak{p}^*$  in  $K$ . We fix a prime ideal  $\mathfrak{p}$  of  $K$  above  $p$ . Then the Bloch-Kato exponential map gives an isomorphism

$$(3) \quad \exp_p : t_n^a \otimes_K K_{\mathfrak{p}} \xrightarrow{\cong} H_{\mathfrak{f}}^1(K_{\mathfrak{p}}, V_p^a(n)),$$

and the inverse of this isomorphism is denoted by  $\log_p$ . The  $p$ -adic étale regulator map gives a homomorphism

$$(4) \quad r_p : H_{\text{mot}}^1(K, M^a(n)) \rightarrow H_{\mathfrak{f}}^1(K_{\mathfrak{p}}, V_p^a(n)),$$

and the map  $r_p \otimes \mathbb{Q}_p$  is conjectured to be an isomorphism. Assuming that the  $p$ -adic regulator map  $r_p$  is *injective*, we define the  $p$ -adic period  $\Omega_p(n)$  of  $M^a(n)$  to be the determinant of the map  $\log_p$  for the basis  $r_p(c_n^a)$  and  $\omega_{\text{tg},n}^a$ . In other words,  $\Omega_p(n)$  is an element in  $K_p := K \otimes_{\mathbb{Q}} K_{\mathfrak{p}} \cong K_{\mathfrak{p}} \oplus K_{\mathfrak{p}^*}$  satisfying

$$(5) \quad \log_p \circ r_p(c_n^a) = \Omega_p(n) \omega_{\text{tg},n}^a.$$

The  $p$ -adic period  $\Omega_p(n)$  is independent of the choice of basis up to multiplication by an element in  $K^{\times}$ .

**Remark 2.2.** We need to assume the injectivity of the  $p$ -adic regulator  $r_p$  to insure that the  $p$ -adic period  $\Omega_p(n)$  is non-zero. Kato has proved in [Kato] 15.15 the weak Leopoldt conjecture for any Hecke character of  $K$ . Hence by a result of Jannsen ([Jan], Lemma 8), we may then conclude that

$$H^2(\mathcal{O}_K[1/p\mathfrak{f}_a], V_p^a(n)) = 0$$

for almost all  $n$ . This implies that  $r_p$  is injective for such  $n$ .

The  $p$ -adic Beilinson conjecture as formulated by Perrin-Riou (see [Col] Conjecture 2.7) specialized to our setting is given as follows.

**Conjecture 2.3.** *Let  $a$  be an integer  $> 0$ . Then there exists a  $p$ -adic pseudo-measure  $\mu^a$  on  $\mathbb{Z}_p$  with values in  $K_p$  such that the value*

$$L_p(\psi^a \otimes \chi_{\text{cyc}}^n) := \int_{\mathbb{Z}_p^\times} w^n \mu^a(w)$$

*in  $K_p$  for any integer  $n > a$  satisfies*

$$\frac{L_p(\psi^a \otimes \chi_{\text{cyc}}^n)}{\Omega_p(n)} = \left(1 - \frac{\psi(\mathfrak{p})^a}{p^n}\right) \left(1 - \frac{\overline{\psi}(\mathfrak{p}^*)^a}{p^{a+1-n}}\right) \frac{\Gamma(n)L(\psi^a, n)}{\Omega_\infty(n)},$$

*where  $\mathfrak{p}$  is a fixed prime in  $K$  above  $p$ .*

If  $\mathfrak{f}_a \neq (1)$  for the conductor  $\mathfrak{f}_a$  of  $\psi^a$ , then  $\mu^a$  should in fact be a  $p$ -adic measure. Note that the dependence of the pseudo-measure on the choices of the basis  $\omega_{\text{tg},n}^a$  and  $c_n^a$  cancel, where as the pseudo-measure depends on the choice of the basis  $\omega_B^a$ . The main goal of our research is to prove that the  $p$ -adic measure constructed by Vishik-Manin and Katz gives the pseudo-measure of the above conjecture when the prime  $p$  is split in  $K$ .

The main theorem of this article (Theorem 6.9) is the proof of the above conjecture for integers  $n$  such that the corresponding  $p$ -adic regulator map  $r_p$  is injective.

### 3. CONSTRUCTION OF THE EISENSTEIN CLASS

The main difficulty in the proof of the Beilinson and  $p$ -adic Beilinson conjectures is to construct the element  $c_n^a \in H_{\text{mot}}^1(K, M^a(n))$  for  $M^a := M(\psi^a)$  and to calculate the images  $r_\infty(c_n^a)$  and  $r_p(c_n^a)$  with respect to the Beilinson-Deligne and  $p$ -adic regulator maps. We will use the Eisenstein symbol as constructed by Beilinson.

We fix an integer  $N \geq 3$ , and let  $M(N)$  be the modular curve defined over  $\mathbb{Z}[1/N]$  parameterizing for any scheme  $S$  over  $\mathbb{Z}[1/N]$  the pair  $(E, \nu)$ , where  $E$  is an elliptic curve over  $S$  and

$$\nu : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\cong} E[N]$$

is a full level  $N$ -structure on  $E$ , where  $E[N]$  is the group of  $N$ -torsion points of  $E$ . We let  $\text{pr} : \tilde{E} \rightarrow M$  be the universal elliptic curve over  $M$  with universal level  $N$ -structure  $\tilde{\nu} : (\mathbb{Z}/N\mathbb{Z})^2 \cong \tilde{E}[N]$ , and consider the motivic sheaf  $\mathbb{Q}(1)$  on  $\tilde{E}$ . We let

$$(6) \quad \mathcal{H} := R^1 \text{pr}_* \mathbb{Q}(1),$$

and we denote by  $\text{Sym}^k \mathcal{H}$  the  $k$ -th symmetric product of  $\mathcal{H}$ . Let  $\varphi = \sum_{\rho \in \tilde{E}[N] \setminus \{0\}} a_\rho[\rho]$  be a  $\mathbb{Q}$ -linear sum of non-zero elements in  $\tilde{E}[N]$ . For any integer  $k > 0$ , the Eisenstein class  $\text{Eis}_{\text{mot}}^{k+2}(\varphi)$  is an element

$$(7) \quad \text{Eis}_{\text{mot}}^{k+2}(\varphi) \in H_{\text{mot}}^1(M, \text{Sym}^k \mathcal{H}(1)).$$

Although the formalism of mixed motivic sheaves or motivic cohomology with coefficients have not yet been fully developed, one can give meaning to the above sheaves and cohomology (see [BL], [BK1] for details).

Then the class  $c_n^a$  may be constructed from the Eisenstein class as follows. Let  $K$  be an imaginary quadratic field of class number one, and let  $E$  be an elliptic curve defined over  $\mathbb{Q}$  with complex multiplication by the ring of integers  $\mathcal{O}_K$  of  $K$ . We denote again by  $\psi$  the Hecke character of  $K$  corresponding to  $E_K$  with conductor  $\mathfrak{f}$ . We take  $N \geq 3$  such that  $N$  is divisible by  $\mathfrak{f}$ . For the extension  $F := K(E[N])$  of  $K$  generated by the coordinates of the points in  $E[N]$ , we let  $G_{F/K} := \text{Gal}(F/K)$  the Galois group of  $F$  over  $K$ . We fix a level  $N$ -structure  $\nu : (\mathbb{Z}/N\mathbb{Z})^2 \cong E[N]$  of  $E$  over  $F$ , and we denote by  $\nu^\sigma$  the composition of  $\nu$  with the action of  $\sigma \in G_{F/K}$ . Then for any  $\sigma \in G_{F/K}$ , we denote by  $\iota^{\sigma*}$  the pull-back with respect to the  $F$ -valued point  $\iota^\sigma : \text{Spec } F \rightarrow M$  of  $M$  corresponding to  $(E, \nu^\sigma)$ . Then the image of the sum  $\iota^* := \sum_{\sigma \in G_{F/K}} \iota^{\sigma*}$  is invariant by the action of the Galois group, hence gives a pull-back morphism

$$H_{\text{mot}}^1(M, \text{Sym}^k \mathcal{H}(1)) \xrightarrow{\iota^*} H_{\text{mot}}^1(K, \text{Sym}^k \iota^* \mathcal{H}(1)).$$

Note that on  $\text{Spec } K$ , the motivic sheaf  $\iota^* \mathcal{H}$  is given by the motive  $H^1(E)(1)$ , which by definition corresponds to the motive  $M(\psi)(1)$ . The structure of  $K$ -coefficients on  $\iota^* \mathcal{H}$  gives the following decomposition.

**Lemma 3.1.** *For integers  $j$  satisfying  $0 \leq j \leq k/2$ , we have the decomposition of motives*

$$\text{Sym}^k \iota^* \mathcal{H} = \bigoplus_{0 \leq j \leq \frac{k}{2}} M(\psi^{k-2j})(k-j),$$

where we take the convention that for  $k = 2j$ , we let  $M(\psi^0)(k/2)$  be the Tate motive  $\mathbb{Q}(k/2)$  with coefficients in  $\mathbb{Q}$ .

Let  $a > 0$  be an integer and we let  $\mathfrak{f}_a$  be the conductor of  $\psi^a$ . We let  $F_a := K(E[\mathfrak{f}_a])$  be the extension of  $K$  generated by the coordinates of the points in  $E[\mathfrak{f}_a]$ , and we let  $w_{F/F_a}$  be the order of the Galois group  $\text{Gal}(F/F_a)$ . The Eisenstein classes  $\text{Eis}_{\text{mot}}^{k+2}(\rho)$  are defined for points  $\rho \in \tilde{E}[N] \setminus \{0\}$  but is *not* defined for  $\rho = 0$ . Hence in defining  $c_n^a$ , we differentiate between the case when  $\mathfrak{f}_a \neq (1)$  and  $\mathfrak{f}_a = (1)$ .

**Definition 3.2.** We define  $\varphi_a$  as follows.

- (1) If  $\mathfrak{f}_a \neq (1)$ , then we fix a primitive  $\mathfrak{f}_a$ -torsion point  $\rho_a$  of  $E$  and let

$$\varphi_a := \frac{1}{w_a w_{F/F_a}} [\rho_a],$$

where we denote again by  $\rho_a$  the  $N$ -torsion point of  $\tilde{E}$  corresponding to  $\rho_a$  through  $\nu$  and  $\tilde{\nu}$ , and  $w_a$  is the number of units in  $\mathcal{O}_K$  which are congruent to *one* modulo  $\mathfrak{f}_a$ .

(2) If  $\mathfrak{f}_a = (1)$ , then we let

$$\varphi_a := \frac{1}{w_a w_{F/F_a}} \sum_{\rho \in \tilde{E}[N] \setminus \{0\}} [\rho].$$

We define the class  $c_n^a$  as follows.

**Definition 3.3.** For any integer  $a, n$  such that  $n > a > 0$ , we let  $k = 2n - a - 2$ . Then the motive  $M^a(n) := M(\psi^a)(n)$  is a direct summand of  $\mathrm{Sym}^k t^* \mathcal{H}(1)$ . We define the motivic class  $c_n^a$  to be the image of  $\mathrm{Eis}_{\mathrm{mot}}^{k+2}(\varphi_a)$  with respect to the projection

$$H_{\mathrm{mot}}^1(K, \mathrm{Sym}^k t^* \mathcal{H}(1)) \rightarrow H_{\mathrm{mot}}^1(K, M^a(n)),$$

where  $\varphi_a$  is as in Definition 3.2.

Let  $p$  be a rational prime which does not divide  $\mathfrak{f}$ , and we take  $N \geq 3$  to be an integer divisible by  $\mathfrak{f}$  and prime to  $p$ . In order to prove the  $p$ -adic Beilinson conjecture, it is necessary to calculate the images of  $c_n^a$  with respect to the Beilinson-Deligne and  $p$ -adic regulator maps. The image  $r_\infty(c_n^a)$  by the Beilinson-Deligne regulator map was calculated by Deninger [De1]. We will calculate the image  $r_p(c_n^a)$  by the  $p$ -adic regulator map using rigid syntomic cohomology.

Denote by  $M_{\mathrm{cris}}^a(n)$  the crystalline realization of  $M^a(n)$ , which is a filtered module with a  $\sigma$ -linear action of Frobenius, and let  $H_{\mathrm{syn}}^1(K_{\mathfrak{p}}, M_{\mathrm{cris}}^a(n))$  be the syntomic cohomology of  $K_{\mathfrak{p}}$  with coefficients in  $M_{\mathrm{cris}}^a(n)$ . Then noting that  $t_n^a \otimes_{\mathbb{Q}} \mathbb{Q}_p = M_{\mathrm{cris}}^a(n)$  in this case, there exists a canonical isomorphism

$$(8) \quad t_n^a \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\cong} H_{\mathrm{syn}}^1(K_{\mathfrak{p}}, M_{\mathrm{cris}}^a(n)).$$

If we let  $V_p^a(n)$  be the  $p$ -adic étale realization of  $M^a(n)$ , then we have a canonical isomorphism

$$(9) \quad H_{\mathrm{syn}}^1(K_{\mathfrak{p}}, M_{\mathrm{cris}}^a(n)) \xrightarrow{\cong} H_f^1(K_{\mathfrak{p}}, V_p^a(n)),$$

which combined with (8) gives the exponential map (3). The syntomic regulator map

$$r_{\mathrm{syn}} : H_{\mathrm{mot}}^1(K, M^a(n)) \rightarrow H_{\mathrm{syn}}^1(K_{\mathfrak{p}}, M_{\mathrm{cris}}^a(n))$$

defined by Besser ([Bes] §7) is compatible with the  $p$ -adic regulator  $r_p$  through the isomorphism (9) ([Bes] Proposition 9.9). Therefore, in order to calculate  $\log_p \circ r_p(c_n^a)$ , it is sufficient to calculate the image of  $r_{\mathrm{syn}}(c_n^a)$  with respect to (8). We will calculate this image using the explicit determination of the syntomic Eisenstein class given in [BK1].

#### 4. EISENSTEIN CLASS AND $p$ -ADIC EISENSTEIN SERIES

In this section, we review the explicit description of the syntomic Eisenstein class in terms of  $p$ -adic Eisenstein series given in [BK1]. Let  $M := M(N)$  be the modular curve over  $\mathbb{Z}[1/N]$  given in the previous section. We will first describe a certain real analytic Eisenstein series  $E_{k+2, l, \varphi}^\infty$ .

Let  $\Gamma \subset \mathbb{C}$  be a lattice, and we denote by  $A$  the area of the fundamental domain of  $\Gamma$  divided by  $\pi := 3.14159 \dots$ . For any integer  $a$  and complex number  $s$  satisfying  $\operatorname{Re}(s) > a/2 + 1$ , the Eisenstein-Kronecker-Lerch series  $K_a^*(z, w, s; \Gamma)$  to be the series

$$K_a^*(z, w, s; \Gamma) := \sum_{\gamma \in \Gamma}^* \frac{(\bar{z} + \bar{\gamma})^a}{|z + \gamma|^{2s}} \langle \gamma, w \rangle$$

where  $\sum^*$  denotes the sum over  $\gamma \in \Gamma$  satisfying  $\gamma \neq -z$  and  $\langle z, w \rangle := \exp((\bar{w}z - w\bar{z})/A)$ . By [Wei] VIII §12 (see [BKT] Proposition 2.4 for the case  $a < 0$ ), this series for  $s$  continues meromorphically to a function on the whole  $s$ -plane, holomorphic except for a simple pole at  $s = 1$  when  $a = 0$  and  $w \in \Gamma$ . This function satisfies the functional equation

$$(10) \quad \Gamma(s) K_a^*(z, w, s; \Gamma) = A^{a+1-2s} \Gamma(a+1-s) K_a^*(w, z, a+1-s) \langle w, z \rangle.$$

We fix a level  $N$ -structure  $\nu : (\mathbb{Z}/N\mathbb{Z})^2 \cong \frac{1}{N}\Gamma/\Gamma$ , and let  $\rho \in \frac{1}{N}\Gamma/\Gamma$ . For integers  $k$  and  $l$ , we define the real analytic Eisenstein series  $E_{k+2,l,\rho}^\infty$  to be the modular form on  $M_{\mathbb{C}} := M(N) \otimes_{\mathbb{Q}} \mathbb{C}$  whose value at the test object  $(\mathbb{C}/\Gamma, dz, \nu)$  is given by

$$(11) \quad E_{k+2,l,\rho}^\infty(\mathbb{C}/\Gamma, dz, \nu) := A^{-l} \Gamma(s) K_{k+l+2}^*(0, \rho, s; \Gamma) \Big|_{s=k+2}.$$

We let  $E_{k+2,l,\varphi}^\infty := \sum_{\rho} a_{\rho} E_{k+2,l,\rho}^\infty$  for the  $\mathbb{Q}$ -linear sum  $\varphi = \sum_{\rho} a_{\rho} [\rho]$ .

When  $l = 0$ , then  $E_{k+2,0,\varphi}^\infty$  is a holomorphic Eisenstein series of weight  $k+2$  on  $M_{\mathbb{C}}$ . From the  $q$ -expansion, we see in this case that this Eisenstein series is defined over  $\mathbb{Q}$ , and hence defines a section  $E_{k+2,0,\varphi}$  in  $\Gamma(M_{\mathbb{Q}}, \omega^{\otimes k} \otimes \Omega_{M_{\mathbb{Q}}}^1)$  for  $\omega := \operatorname{pr}_* \Omega_{\tilde{E}/M}^1$ . Denote by  $\mathcal{H}_{\text{dR}}$  the de Rham realization of  $\mathcal{H}$ , which is the coherent  $\mathcal{O}_{M_{\mathbb{Q}}}$ -module  $R^1 \operatorname{pr}_* \Omega_{\tilde{E}}^\bullet$  with Gauss-Manin connection

$$\nabla : \mathcal{H}_{\text{dR}} \rightarrow \mathcal{H}_{\text{dR}} \otimes \Omega_{M_{\mathbb{Q}}}^1,$$

and let  $\operatorname{Sym}^k \mathcal{H}_{\text{dR}}$  be the  $k$ -th symmetric product of  $\mathcal{H}_{\text{dR}}$  with the induced connection. From the natural inclusion  $\omega^{\otimes k} \hookrightarrow \operatorname{Sym}^k \mathcal{H}_{\text{dR}}$ , we see that  $E_{k+2,0,\varphi}$  defines a section in  $\Gamma(M_{\mathbb{Q}}, \operatorname{Sym}^k \mathcal{H}_{\text{dR}} \otimes \Omega_{M_{\mathbb{Q}}}^1)$ .

Let  $p$  be a prime number not dividing  $N$ . We denote by  $\mathcal{H}_{\text{rig}}$  the filtered overconvergent  $F$ -isocrystal associated to  $\mathcal{H}$  on  $M_{\mathbb{Z}_p}$ , which is given by  $\mathcal{H}_{\text{dR}}$  with an additional structure of Hodge filtration and Frobenius. Let  $H_{\text{syn}}^1(M_{\mathbb{Z}_p}, \operatorname{Sym}^k \mathcal{H}_{\text{rig}}(1))$  be the rigid syntomic cohomology of  $M_{\mathbb{Z}_p}$  with coefficients in  $\operatorname{Sym}^k \mathcal{H}_{\text{rig}}(1)$ . The rigid syntomic regulator is a map

$$r_{\text{syn}} : H_{\text{mot}}^1(M, \operatorname{Sym}^k \mathcal{H}(1)) \rightarrow H_{\text{syn}}^1(M_{\mathbb{Z}_p}, \operatorname{Sym}^k \mathcal{H}_{\text{rig}}(1)),$$

and we define the syntomic Eisenstein class  $\operatorname{Eis}_{\text{syn}}^{k+2}(\varphi)$  to be the image by the syntomic regulator of the motivic Eisenstein class. We let  $M_{\mathbb{Z}_p}^{\text{ord}}$  be the ordinary locus in  $M_{\mathbb{Z}_p}$ , and  $M_{\mathbb{Q}_p}^{\text{ord}} := M_{\mathbb{Z}_p}^{\text{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ . By [BK1] Proposition



A.16, a class in  $H_{\text{syn}}^1(M_{\mathbb{Z}_p}^{\text{ord}}, \text{Sym}^k \mathcal{H}_{\text{rig}}(1))$  is given by a pair  $(\alpha, \xi)$  of sections

$$(12) \quad \begin{aligned} \alpha &\in \Gamma(M_{\mathbb{Q}_p}^{\text{ord}}, j^\dagger \text{Sym}^k \mathcal{H}_{\text{rig}}(1)) \\ \xi &\in \Gamma(M_{\mathbb{Q}_p}^{\text{ord}}, \text{Sym}^k \mathcal{H}_{\text{dR}} \otimes_{\mathbb{Q}_p} \Omega_{M^{\text{ord}}}^1) \end{aligned}$$

satisfying  $\nabla(\alpha) = (1 - \phi^*)\xi$ . The  $\alpha$  for the class  $(\alpha, \xi)$  corresponding to the restriction to the ordinary locus of the syntomic Eisenstein class  $\text{Eis}_{\text{syn}}^{k+2}(\varphi)$  is given as follows.

We let  $p \geq 5$  be a prime not dividing  $N$ , and we let  $\mathcal{M}$  be the  $p$ -adic modular curve defined over  $\mathbb{Z}_p$  parameterizing the triples  $(E_B, \eta, \nu)$  consisting of an elliptic curve  $E_B$  over a  $p$ -adic ring  $B$ , an isomorphism

$$(13) \quad \eta : \widehat{\mathbb{G}}_m \cong \widehat{E}_B$$

of formal groups over  $B$ , and a level  $N$ -structure  $\nu$ . The ring of  $p$ -adic modular forms  $V_p(\mathbb{Q}_p, \Gamma(N))$  is defined as the global section

$$V_p(\mathbb{Q}_p, \Gamma(N)) := \Gamma(\mathcal{M}, \mathcal{O}_{\mathcal{M}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

The  $q$ -expansion gives an injection

$$V_p(\mathbb{Q}_p, \Gamma(N)) \hookrightarrow \mathbb{Q}_p(\zeta_N)[[q]].$$

There exists a Frobenius action  $\phi^*$  on  $V_p(\mathbb{Q}_p, \Gamma(N))$  given on the  $q$ -expansion as  $\phi^* = \text{Frob} \otimes \sigma$ , where  $\text{Frob}(q) = q^p$  and  $\sigma$  is the absolute Frobenius acting on  $\mathbb{Q}_p(\zeta_N)$ . The Eisenstein series  $E_{k+2,0,\varphi}$  naturally defines an element in  $V_p(\mathbb{Q}_p, \Gamma(N))$ , and using the fact that the differential  $\partial_{\log q} := q \frac{d}{dq}$  preserves the space of  $p$ -adic modular forms, we let for any integer  $l \geq 0$

$$E_{k+l+2,l,\varphi} := \partial_{\log q}^l E_{k+2,0,\varphi}.$$

We let  $E_{k+2,0,\varphi}^{(p)} := (1 - \phi^*)E_{k+2,0,\varphi}$  and  $E_{k+l+2,l,\varphi}^{(p)} := \partial_{\log q}^l E_{k+2,0,\varphi}^{(p)}$  for any integer  $l \geq 0$ . Then the calculation of the  $q$ -expansion shows that we have

$$(14) \quad E_{k+l+2,l,\varphi}^{(p)} = (1 - p^l \phi^*)E_{k+l+2,l,\varphi}.$$

Following the method of Katz [Katz], we may construct a  $p$ -adic measure on  $\mathbb{Z}_p \times \mathbb{Z}_p^\times$  with values in  $V_p(\mathbb{Q}_p, \Gamma(N))$  satisfying the following interpolation property.

**Theorem 4.1.** *There exists a  $p$ -adic measure  $\mu_\varphi$  on  $\mathbb{Z}_p \times \mathbb{Z}_p^\times$  with values in  $V_p(\mathbb{Q}_p, \Gamma(N))$  such that*

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} x^{k+1} y^l \mu_\varphi(x, y) = E_{k+2,l,\varphi}^{(p)}$$

for integers  $k > 0, l \geq 0$ .

Using this measure, we define  $E_{k+2,l,\varphi}^{(p)}$  for  $l < 0$  as follows.

**Definition 4.2** ( $p$ -adic Eisenstein series). Let  $k$  be an integer  $\geq -1$ . We let

$$E_{k+2,l,\varphi}^{(p)} := \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} x^{k+1} y^l \mu_\varphi(x, y) \in V_p(\mathbb{Q}_p, \Gamma(N)),$$

where  $l$  is any integer in  $\mathbb{Z}$ .

The  $p$ -adic Eisenstein series satisfies the differential equation

$$\partial_{\log q} E_{k+2,l,\varphi}^{(p)} = E_{k+3,l+1,\varphi}^{(p)},$$

and the weight of  $E_{k+2,l,\varphi}^{(p)}$  is  $k+l+2$ . The syntomic Eisenstein class may be described using these  $p$ -adic Eisenstein series. The moduli problem for  $\mathcal{M}$  implies that there exists a universal trivialization

$$\eta : \widehat{\mathbb{G}}_m \cong \widehat{E}$$

of the universal elliptic curve on  $\mathcal{M}$ , which gives rise to a canonical section  $\tilde{\omega}$  of  $\omega := \text{pr}_* \Omega_{\widehat{E}/\mathcal{M}}^1$  corresponding to the invariant differential  $d \log(1+T)$  on  $\widehat{\mathbb{G}}_m$ . Since  $\mathcal{M}$  is affine, there exists sections  $x$  and  $y$  of  $\widehat{E}$  such that the elliptic curve  $\tilde{E}_{\mathbb{Q}_p} := \tilde{E} \otimes \mathbb{Q}_p$  is given by the Weierstrass equation

$$\tilde{E}_{\mathbb{Q}_p} : y^2 = 4x^3 - g_2x - g_3, \quad g_2, g_3 \in V_p(\mathbb{Q}_p, \Gamma(N))$$

satisfying  $\tilde{\omega} = dx/y$ . Then the pull back of the  $F$ -isocrystal  $\mathcal{H}_{\text{rig}}$  to  $\mathcal{M}_{\mathbb{Q}_p}$  is given as

$$\mathcal{H}_{\text{rig}} = \mathcal{O}_{\mathcal{M}_{\mathbb{Q}_p}} \tilde{\omega}^\vee \oplus \mathcal{O}_{\mathcal{M}_{\mathbb{Q}_p}} \tilde{u}^\vee,$$

with connection  $\nabla(\tilde{u}^\vee) = \tilde{\omega}^\vee \otimes d \log q$ ,  $\nabla(\tilde{\omega}^\vee) = 0$ , Frobenius  $\phi^*(\tilde{\omega}^\vee) = p^{-1}\tilde{\omega}^\vee$ ,  $\phi^*(\tilde{u}^\vee) = \tilde{u}^\vee$  and Hodge filtration  $\text{Fil}^{-1} \mathcal{H}_{\text{rig}} = \mathcal{H}_{\text{rig}}$ ,  $\text{Fil}^0 \mathcal{H}_{\text{rig}} = \mathcal{O}_{\mathcal{M}_{\mathbb{Q}_p}} \tilde{u}^\vee$ ,  $\text{Fil}^1 \mathcal{H}_{\text{rig}} = 0$  (See [BK1] §4.3). If let  $\tilde{\omega}^{m,n} := \tilde{\omega}^{\vee m} \tilde{u}^{\vee n}$ , then the filtered  $F$ -isocrystal  $\text{Sym}^k \mathcal{H}_{\text{rig}}(1)$  on  $\mathcal{M}_{\mathbb{Q}_p}$  is given by the coherent module

$$\text{Sym}^k \mathcal{H}_{\text{rig}}(1) = \bigoplus_{j=0}^k \mathcal{O}_{\mathcal{M}_{\mathbb{Q}_p}} \tilde{\omega}^{k-j,j}(1)$$

with connection  $\nabla(\tilde{\omega}^{k-j,j}(1)) = j \tilde{\omega}^{k-j+1,j-1}(1) \otimes d \log q$ , Frobenius

$$\phi^*(\tilde{\omega}^{k-j,j}(1)) = p^{j-k-1} \tilde{\omega}^{k-j,j}(1),$$

and Hodge filtration

$$\text{Fil}^m(\text{Sym}^k \mathcal{H}_{\text{rig}}(1)) = \bigoplus_{j=m+k+1}^k \mathcal{O}_{\mathcal{M}_{\mathbb{Q}_p}} \tilde{\omega}^{k-j,j}(1).$$

If we let  $\tilde{\alpha}_{\text{Eis}}^{k+2}$  be the section

$$\tilde{\alpha}_{\text{Eis}}^{k+2}(\varphi) := \sum_{j=0}^k \frac{(-1)^{k-j}}{j!} E_{j+1,j-k-1,\varphi}^{(p)} \tilde{\omega}^{k-j,j}(1),$$

then we have

$$\nabla(\tilde{\alpha}_{\text{Eis}}^{k+2}(\varphi)) = \frac{(1 - \phi^*)E_{k+2,0,\varphi}}{k!} \tilde{\omega}^{0,k}(1) \otimes d \log q.$$

The main result of [BK1] is the following.

**Theorem 4.3** ([BK1] Theorem 5.11). *For any integer  $k > 0$ , the syntomic Eisenstein class*

$$\text{Eis}_{\text{syn}}^{k+2}(\varphi) \in H_{\text{syn}}^1(M_{\mathbb{Z}_p}, \text{Sym}^k \mathcal{H}_{\text{rig}}(1))$$

*restricted to the ordinary locus  $H_{\text{syn}}^1(M_{\mathbb{Z}_p}^{\text{ord}}, \text{Sym}^k \mathcal{H}_{\text{rig}}(1))$  is represented by the pair  $(\alpha, \xi)$  as in (12), where  $\xi = E_{k+2,0,\varphi} \tilde{\omega}^{0,k}(1)/k! \otimes d \log q$  and  $\alpha$  is a section which maps to  $\tilde{\alpha}_{\text{Eis}}^{k+2}(\varphi)$  in  $\Gamma(\mathcal{M}_{\mathbb{Q}_p}, \text{Sym}^k \mathcal{H}_{\text{rig}}(1))$ .*

The main ingredient in the proof of the above theorem is the characterization of  $\xi$  by the residue, which by [BL] 2.2.3 (see also [HK] C.1.1) and the compatibility of the Beilinson-Deligne regulator map with the residue morphism shows that  $\xi$  represents the de Rham Eisenstein class in de Rham cohomology. See [BK1] Proposition 3.6 and Proposition 4.1 for details concerning this point.

## 5. SPECIAL VALUES OF HECKE $L$ -FUNCTIONS

In this section, we give in Propositions 5.2 and 5.4 the precise relation between the special values of the Hecke  $L$ -function  $L(\psi^a, s)$  and Eisenstein-Kronecker-Lerch series. Assume that  $K$  is an imaginary quadratic field of class number one, and let  $E$  be an elliptic curve over  $\mathbb{Q}$  with good ordinary reduction at a prime  $p$  with complex multiplication by the ring of integers  $\mathcal{O}_K$  of  $K$ . We let  $\psi$  be the Grossencharacter of  $K$  associated to  $E_K := E \otimes_{\mathbb{Q}} K$ , and we denote by  $\mathfrak{f}$  the conductor of  $\psi$ . We fix an invariant differential  $\omega$  of  $E$  defined over  $K$ . We fix once and for all a complex embedding  $\tau : K \hookrightarrow \mathbb{C}$  of the base field  $K$  into  $\mathbb{C}$ , and we let  $\Gamma$  be the period lattice of  $E := E \otimes_{K,\tau} \mathbb{C}$  with respect to  $\omega$ . Then we have a complex uniformization

$$(15) \quad \mathbb{C}/\Gamma \xrightarrow{\cong} E(\mathbb{C})$$

such that the pull-back of the invariant differential  $\omega$  coincides with  $dz$ . Note that since  $E$  has complex multiplication, we have  $\Gamma = \Omega \mathcal{O}_K$  for some complex period  $\Omega \in \mathbb{C}^\times$ .

By abuse of notation, we will denote by  $\psi$  and  $\bar{\psi}$  the complex Hecke characters  $\psi_\tau$  and  $\bar{\psi}_\tau$  associated to  $\psi$ , where  $\tau$  is the fixed embedding given above. Let  $-d_K$  denote the discriminant of  $K$ , so that  $K = \mathbb{Q}(\sqrt{-d_K})$ . The Hecke character  $\psi$  is of the form  $\psi((u)) = \varepsilon(u)u$  for any  $u \in \mathcal{O}_K$  prime to  $\mathfrak{f}$ , where  $\varepsilon : (\mathcal{O}_K/\mathfrak{f})^\times \rightarrow K^\times$  is a primitive character on  $(\mathcal{O}_K/\mathfrak{f})^\times$ .

Let  $\chi : (\mathcal{O}_K/\mathfrak{f}_\chi)^\times \rightarrow K^\times$  be a primitive character of conductor  $\mathfrak{f}_\chi$ , and let  $f_\chi$  be a generator of  $\mathfrak{f}_\chi$ . Then for any  $u$  in  $\mathcal{O}_K$ , we define the *Gauss sum*

$G(\chi, u)$  by

$$G(\chi, u) := \sum_{v \in \mathcal{O}_K/\mathfrak{f}_\chi} \bar{\chi}(v) \exp\left(2\pi i \operatorname{Tr}_{K/\mathbb{Q}}\left(uv/f_\chi \sqrt{-d_K}\right)\right)$$

(see [Lan] Chapter 22 §1), where we extend  $\chi$  to a function on  $\mathcal{O}_K/\mathfrak{f}_\chi$  by taking  $\chi(u) := 0$  for any  $u \in \mathcal{O}_K$  not prime to  $\mathfrak{f}_\chi$ . We let  $G(\chi) := G(\chi, 1)$ . Then the standard fact concerning Gauss sums are as follows (see for example [Lan] Chapter 22 §1.)

**Lemma 5.1.** *Let the notations be as above.*

- (1) *We have  $|G(\chi)|^2 = N(\mathfrak{f}_\chi)$ .*
- (2) *For any  $u \in \mathcal{O}_K$ , we have  $G(\chi, u) = \chi(u)G(\chi)$ .*

As in §3, we let  $a > 0$  be an integer and  $\mathfrak{f}_a$  be the conductor of  $\psi^a$ . Then the finite part  $\varepsilon^a$  of  $\psi^a$  is a primitive character  $\varepsilon^a : (\mathcal{O}_K/\mathfrak{f}_a)^\times \rightarrow \mathbb{C}^\times$  of conductor  $\mathfrak{f}_a$ . We fix a generator  $f_a$  of  $\mathfrak{f}_a$  and we denote by  $G(\varepsilon^a, u)$  the corresponding Gauss sum for any  $u$  in  $\mathcal{O}_K$ .

We let the notations be as in §3. In particular, we let  $w_a$  be the number of units in  $\mathcal{O}_K$  which are congruent to *one* mod  $\mathfrak{f}_a$ , and we let  $w_{F/F_a}$  be the order of the Galois group  $\operatorname{Gal}(F/F_a)$ . We again let  $N \geq 3$  be a rational integer divisible by  $\mathfrak{f}$  and prime to  $p$ , and  $F := K(E[N])$ . We fix an isomorphism  $\nu : (\mathbb{Z}/N\mathbb{Z})^2 \cong \frac{1}{N}\Gamma/\Gamma$ .

We first consider the case when  $\mathfrak{f}_a \neq (1)$ . We let  $\rho_a := \Omega/f_a$  be a primitive  $\mathfrak{f}_a$ -torsion point, which corresponds through the uniformization (15) to a point  $\rho_a \neq 0 \in E(\bar{K})$ . We then have the following.

**Proposition 5.2.** *Suppose  $\mathfrak{f}_a \neq (1)$ . Then we have*

$$\frac{(-1)^a}{w_a w_{F/F_a}} \sum_{\sigma \in \operatorname{Gal}(F/K)} E_{n, a-n, \rho_a^\sigma}^\infty(\mathbb{C}/\Gamma, dz, \nu) = \frac{G(\varepsilon^a) \bar{\Omega}^a}{A^{a-n} |\Omega|^{2n}} \Gamma(s) L(\psi^a, s)|_{s=n}.$$

*Proof.* Let  $w_0$  be the number of units in  $\mathcal{O}_K$ . By definition, we have

$$L(\psi^a, s) = \frac{1}{w_0} \sum_{u \in \mathcal{O}_K} \frac{\psi^a(u)}{N(u)^s} = \frac{1}{w_0} \sum_{u \in \mathcal{O}_K} \frac{\varepsilon^a(u) u^a}{N(u)^s}$$

Then Lemma 5.1 (2) gives the equality  $\varepsilon^a(u) = G(\varepsilon^a, u)/G(\varepsilon^a)$ . If we expand the definition of the Gauss sum, we see that

$$L(\psi^a, s) = \frac{1}{w_0} \sum_{\substack{u \in \mathcal{O}_K \\ v \in \mathcal{O}_K/\mathfrak{f}_a}} \frac{\bar{\varepsilon}^a(v) u^a}{N(u)^s} \exp\left(2\pi i \operatorname{Tr}_{K/\mathbb{Q}}\left(\frac{uv}{f_a \sqrt{-d_K}}\right)\right).$$

Noting that  $\mathcal{O}_K$  is preserved by complex conjugation, we see that the above is equal to

$$\frac{(-1)^a}{w_0} \sum_{v \in \mathcal{O}_K/\mathfrak{f}_a} \sum_{u \in \mathcal{O}_K} \frac{\bar{\varepsilon}^a(v) \bar{u}^a}{|u|^{2s}} \exp\left(\frac{2\pi}{\sqrt{d_K}} \left(\frac{u\bar{v}}{f_a} - \frac{\bar{u}v}{f_a}\right)\right).$$

For any  $\sigma \in \text{Gal}(F/K)$ , we have  $\rho_a^\sigma = \rho_a^{\sigma'}$ , where  $\sigma'$  is the class of  $\sigma$  in  $\text{Gal}(F_a/K)$ . If  $\sigma'_v := (v, F_a/F)$  is the element in  $\text{Gal}(F_a/K)$  corresponding to  $v \in (\mathcal{O}_K/\mathfrak{f}_a)^\times$  through the *inverse* of the Artin map, then by the theory of complex multiplication, we have  $\rho_a^{\sigma'_v} = \psi(v)\rho_a$ . Hence

$$\begin{aligned} \sum_{\sigma \in \text{Gal}(F/K)} K_a^*(0, \rho_a^\sigma, s; \Gamma) &= w_{F/F_a} \sum_{\sigma' \in \text{Gal}(F_a/K)} K_a^*(0, \rho_a^{\sigma'}, s; \Gamma) \\ &= w_{F/F_a} \frac{w_a}{w_0} \sum_{v \in (\mathcal{O}_K/\mathfrak{f}_a)^\times} \sum_{\gamma \in \Gamma} \frac{\bar{\gamma}^a}{|\gamma|^{2s}} \langle \gamma, \psi(v)\rho_a \rangle \\ &= w_{F/F_a} \frac{w_a}{w_0} \sum_{v \in (\mathcal{O}_K/\mathfrak{f}_a)^\times} \sum_{\gamma \in \Gamma} \frac{\bar{\varepsilon}^a(v)\bar{\gamma}^a}{|\gamma|^{2s}} \langle \gamma, v\rho_a \rangle. \end{aligned}$$

Our assertion follows from the fact that  $\Gamma = \Omega\mathcal{O}_K$ ,  $A = |\Omega|^2\sqrt{d_K}/2\pi$  and the definition (11) of the Eisenstein-Kronecker-Lerch series.  $\square$

The right hand side of Proposition 5.2 may be used to express the Hecke  $L$ -function on the other side of the functional equation as follows.

**Lemma 5.3.** *We have*

$$(16) \quad \frac{1}{w_a w_{F/F_a}} \sum_{\sigma \in \text{Gal}(F/K)} E_{n, a-n, \rho_a^\sigma}^\infty(\mathbb{C}/\Gamma, dz, \nu) = \frac{A^{1-n} N(\mathfrak{f}_a)^{a+1-n} \bar{\Omega}^a}{\bar{f}_a^a |\Omega|^{2(a+1-n)}} \Gamma(s) L(\bar{\psi}^a, s) \Big|_{s=a+1-n}.$$

*Proof.* We have by definition

$$\begin{aligned} \sum_{v \in \mathcal{O}_K/\mathfrak{f}_a} K_a^*(\psi(v)\rho_a, 0, a+1-s; \Gamma) &= \sum_{v \in \mathcal{O}_K/\mathfrak{f}_a} \sum_{\gamma \in \Gamma}^* \frac{(\overline{\psi(v)\rho_a} + \bar{\gamma})^a}{|\psi(v)\rho_a + \gamma|^{2(a+1-s)}} \\ &= \frac{N(\mathfrak{f}_a)^{a+1-s} \bar{\Omega}^a}{\bar{f}_a^a |\Omega|^{2(a+1-s)}} L(\bar{\psi}^a, a+1-s) \end{aligned}$$

for  $\text{Re}(s) < a/2$ , hence for any  $s \in \mathbb{C}$  by analytic continuation. Our assertion follows from the functional equation

$$\Gamma(s) K_a^*(0, \psi(v)\rho_a, s; \Gamma) = A^{a+1-2s} \Gamma(a+1-s) K_a^*(\psi(v)\rho_a, 0, a+1-s; \Gamma)$$

and the definition (11) of the Eisenstein-Kronecker-Lerch series.  $\square$

The case when  $\mathfrak{f}_a = (1)$  is given as follows.

**Proposition 5.4.** *Suppose  $\mathfrak{f}_a = (1)$ . Then we have*

$$\begin{aligned} \frac{1}{w_a} \sum_{\rho \in E[N] \setminus \{0\}} E_{n, a-n, \rho}^\infty(\mathbb{C}/\Gamma, dz, \nu) &= \left( \frac{N^{a+2}}{N^{2n}} - 1 \right) \frac{\bar{\Omega}^a}{A^{a-n} |\Omega|^{2n}} \Gamma(s) L(\psi^a, s) \Big|_{s=n}. \end{aligned}$$

*Proof.* By definition, we have

$$\sum_{\rho \in \frac{1}{N}\Gamma/\Gamma} K_a^*(0, \rho, s; \Gamma) = \sum_{\gamma \in \Gamma} \sum_{\rho \in \frac{1}{N}\Gamma/\Gamma} \frac{\bar{\gamma}^a}{|\gamma|^{2s}} \langle \gamma, \rho \rangle = \frac{N^{a+2}}{N^{2s}} \sum_{\gamma \in \Gamma} \frac{\gamma^a}{|\gamma|^{2s}},$$

where the last equality follows from the equality

$$\sum_{\rho \in \frac{1}{N}\Gamma/\Gamma} \langle \gamma, \rho \rangle = \begin{cases} N^2 & \gamma \in N\Gamma \\ 0 & \text{otherwise} \end{cases}$$

and the fact that complex conjugation acts bijectively on  $\Gamma$ . Our assertion follows from the definition (11) of the Eisenstein-Kronecker-Lerch series.  $\square$

Similarly to Lemma 5.3, we have the following.

**Lemma 5.5.** *We have*

$$(17) \quad \frac{1}{w_a} \sum_{\rho \in E[N] \setminus \{0\}} E_{n, a-n, \rho}^\infty(\mathbb{C}/\Gamma, dz, \nu) = \left( \frac{N^{a+2}}{N^{2n}} - 1 \right) \frac{A^{1-n} \bar{\Omega}^a}{|\Omega|^{2(a+1-n)}} \Gamma(s) L(\bar{\psi}^a, s) \Big|_{s=a+1-n}.$$

## 6. THE MAIN RESULT

In this section, we give an outline of the proof of our main theorem. We will mainly deal with the case when  $\mathfrak{f}_a \neq (1)$ , as the case for  $\mathfrak{f}_a = (1)$  is essentially the same except for the factor  $(N^{a+2}/N^{2n} - 1)$ . We first calculate the  $p$ -adic and complex periods  $\Omega_p(n)$  and  $\Omega(n)$ . From the definition of  $c_n^a$  and from the compatibility of the syntomic regulator with respect to pull-back morphisms, the restriction of the syntomic Eisenstein class through the decomposition of Lemma 3.1 gives the image by the syntomic regulator of the element  $c_n^a$  in  $H_{\text{mot}}^1(K, M^a(n))$ .

Let the notations be as in the previous section. We denote by  $\omega^*$  the class in  $H_{\text{dR}}^1(E/\mathbb{C})$  corresponding to  $d\bar{z}/A$ , which is in fact a class in  $H_{\text{dR}}^1(E/K)$ . Let  $k = 2n - a - 2$ . Then  $\omega^{k-j+1, j+1} := \omega^{\vee k-j} \omega^{*\vee j}(1)$  for  $0 \leq j \leq k$  form a basis of  $\text{Sym}^k \iota^* \mathcal{H}_{\text{dR}}(1)$ . The relation between the basis  $\tilde{\omega}^{m, n}$  and  $\omega^{m, n}$  is given by  $\tilde{\omega}^{m, n} = \Omega_{\mathfrak{p}}^{n-m} \omega^{m, n}$ . In what follows, let  $\varphi_a$  be as in Definition 3.2. By Theorem 4.3, the pull-back of the syntomic Eisenstein class  $\text{Eis}_{\text{syn}}^{k+2}(\varphi_a)$  to  $H_{\text{syn}}^1(K_{\mathfrak{p}}, \text{Sym}^k \iota^* \mathcal{H}(1))$  is expressed by the element

$$\iota^* \tilde{\alpha}_{\text{Eis}}^{k+2}(\varphi_a) = \sum_{j=0}^k \frac{(-1)^{k-j}}{j!} \Omega_{\mathfrak{p}}^{2j-k} E_{j+1, j-k-1, \varphi_a}^{(p)}(E, \omega, \nu) \omega^{k-j+1, j+1}.$$

Hence the element  $r_{\text{syn}}(c_n^a)$  in  $H_{\text{syn}}^1(K_{\mathfrak{p}}, M_{\text{cris}}^a(n))$  corresponding by definition to the direct factor  $j = n - 1$  and  $j = n - a - 1$  is represented by

$$\begin{aligned} & \frac{(-1)^{n-a+1}}{\Gamma(n)} \Omega_{\mathfrak{p}}^a E_{n,a-n,\varphi_a}^{(p)}(E, \omega, \nu) \omega^{n-a,n} \\ & \quad + \frac{(-1)^{n-1}}{\Gamma(n-a)} \Omega_{\mathfrak{p}}^{-a} E_{n-a,-n,\varphi_a}^{(p)}(E, \omega, \nu) \omega^{n,n-a}. \end{aligned}$$

By definition of the exponential map, the element in  $t_n^a \otimes \mathbb{Q}_p$  corresponding to  $r_{\text{syn}}(c_n^a)$  through the isomorphism (8) is

$$\begin{aligned} & \frac{(-1)^{n-a+1}}{\Gamma(n)} \Omega_{\mathfrak{p}}^a E_{n,a-n,\varphi_a}(E, \omega, \nu) \omega^{n-a,n} \\ & \quad + \frac{(-1)^{n-1}}{\Gamma(n-a)} \Omega_{\mathfrak{p}}^{-a} E_{n-a,-n,\varphi_a}(E, \omega, \nu) \omega^{n,n-a}, \end{aligned}$$

where  $E_{n,a-n,\varphi_a}(E, \omega, \nu)$  is the element in  $\widehat{K}_{\mathfrak{p}}^{\text{ur}}$  satisfying

$$(18) \quad (1 - p^{a-n} \sigma^*) E_{n,a-n,\varphi_a}(E, \omega, \nu) = E_{n,a-n,\varphi_a}^{(p)}(E, \omega, \nu).$$

From the definition of  $c_n^a$  and the discussion at the end of §3, this shows that we have

$$\begin{aligned} \log_p \circ r_p(c_n^a) &= \frac{(-1)^{n-a+1}}{\Gamma(n)} \Omega_{\mathfrak{p}}^a E_{n,a-n,\varphi_a}(E, \omega, \nu) \omega^{n-a,n} \\ & \quad + \frac{(-1)^{n-1}}{\Gamma(n-a)} \Omega_{\mathfrak{p}}^{-a} E_{n-a,-n,\varphi_a}(E, \omega, \nu) \omega^{n,n-a}. \end{aligned}$$

This gives the following.

**Theorem 6.1.** *Let  $n$  be an integer  $> a$  and assume that the  $p$ -adic regulator  $r_p$  is injective. Then the  $p$ -adic period  $\Omega_p(n) \in K_{\mathfrak{p}} \oplus K_{\mathfrak{p}^*}$  of the motive  $M^a(n)$  is given by*

$$\begin{aligned} \Omega_p(n) &= \frac{(-1)^{n-a+1}}{\Gamma(n)} \Omega_{\mathfrak{p}}^a E_{n,a-n,\varphi_a}(E, \omega, \nu) \\ & \quad \oplus \frac{(-1)^{n-1}}{\Gamma(n-a)} \Omega_{\mathfrak{p}}^{-a} E_{n-a,-n,\varphi_a}(E, \omega, \nu). \end{aligned}$$

*Proof.* The theorem follows from the definition given in (5) of the  $p$ -adic period, noting that we have an isomorphism

$$(K \otimes_{\mathbb{Q}} \mathbb{Q}_p) \omega_{\text{tg},n}^a \cong \mathbb{Q}_p \omega^{n,n-a} \oplus \mathbb{Q}_p \omega^{n-a,n}$$

induced from the canonical splitting  $K \otimes \mathbb{Q}_p = K_{\mathfrak{p}} \oplus K_{\mathfrak{p}^*} \cong \mathbb{Q}_p \oplus \mathbb{Q}_p$ .  $\square$

The calculation of the complex period, originally due to Deninger [De1] may be done in a similar fashion. If we let  $\Gamma = \Omega \mathcal{O}_K$  as in the previous section, then the Betti homology of  $E$  is given by  $H_1^B(E(\mathbb{C}), \mathbb{Z}) = \Gamma$ . We let  $\gamma_1 := \Omega \in \Gamma$ , which is a generator of  $\Gamma$  as a  $\mathcal{O}_K$ -module. If we fix a  $v \in \mathcal{O}_K$

such that  $\mathcal{O}_K := \mathbb{Z} \oplus \mathbb{Z}v$  and if we let  $\gamma_2 := v(\gamma_1)$  where  $v$  acts through the complex multiplication of  $E$ , then we have  $\Gamma := \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$  as a  $\mathbb{Z}$ -module. The period relation gives the equality

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \Omega & \bar{\Omega}/A \\ \tau\Omega & \bar{\tau}\bar{\Omega}/A \end{pmatrix} \begin{pmatrix} \omega^\vee \\ \omega^{*\vee} \end{pmatrix}.$$

The  $K$ -basis  $\gamma_1$  induces a  $K$ -basis of  $M_B^a(n) \subset \text{Sym}^k H_B^1(E(\mathbb{C}), \mathbb{Q}(1))$ , which we denote by  $\omega_B$ . Then the inclusion

$$M_B^a(n) \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow t_n^a \otimes_{\mathbb{Q}} \mathbb{R}$$

maps  $\omega_B$  to

$$\Omega^{n-a} \frac{\bar{\Omega}^n}{A^n} \omega^{n-a,n} + \Omega^n \frac{\bar{\Omega}^{n-a}}{A^{n-a}} \omega^{n,n-a}.$$

Furthermore, one may prove the following.

**Theorem 6.2** (Deninger [De1]). *The image  $r_\infty(c_n^a)$  in*

$$H_{\mathcal{D}}^1(K \otimes_{\mathbb{Q}} \mathbb{R}, V_\infty^a(n)) \cong (t_n^a \otimes_{\mathbb{Q}} \mathbb{R}) / (M_B^a(n) \otimes_{\mathbb{Q}} \mathbb{R})$$

*of  $c_n^a$  by the Beilinson regulator (1) is represented by the element*

$$\begin{aligned} & \frac{(-1)^{n-1}}{\Gamma(n)} A^{n-a} E_{n,a-n,\varphi_a}^\infty(\mathbb{C}/\Gamma, dz, \nu) \omega^{n-a,n} \\ & + \frac{(-1)^{n-a+1}}{\Gamma(n-a)} A^n E_{n-a,-n,\varphi_a}^\infty(\mathbb{C}/\Gamma, dz, \nu) \omega^{n,n-a} \end{aligned}$$

*in  $t_n^a \otimes_{\mathbb{Q}} \mathbb{C}$ .*

*Proof.* The Eisenstein class in this paper defined using the elliptic polylogarithm is related to the Eisenstein class defined by Beilinson and Deninger. The theorem is then a special case of the weak Beilinson conjecture for Hecke character associated to imaginary quadratic fields proved by Deninger [De1] (see also [DW] for the case of an elliptic curve defined over  $\mathbb{Q}$  with complex multiplication.) The theorem may also be proved by explicitly calculating the Hodge realization of the elliptic polylogarithm [BL] (see also [BKT] Theorem A 29.)  $\square$

By taking the determinant of the complex (2) with respect to the basis  $r_\infty(c_n^a)$ ,  $\omega_B$ ,  $\omega^{n-a,n}$  and  $\omega^{n,n-a}$ , the above calculation and the definition of the complex period give the following.

**Proposition 6.3.** *The complex period  $\Omega_\infty(n)$  of  $M^a(n)$  in  $K \otimes_{\mathbb{Q}} \mathbb{R}$  is given by*

$$\Omega_\infty(n) = (-1)^{a+n-1} G(\varepsilon^a) L(\psi^a, n) \bigoplus (-1)^{n+1} G(\bar{\varepsilon}^a) L(\bar{\psi}^a, n)$$

*in  $K \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C} \bigoplus \mathbb{C}$  if  $\mathfrak{f}_a \neq (1)$ . A similar formula holds for the case when  $\mathfrak{f}_a = (1)$ , but with a factor  $(N^{a+2}/N^{2n} - 1)$  multiplied to the  $L$ -value.*



*Proof.* The assertion follows from Theorem 6.2 by explicit calculation, using the definition of  $\varphi_a$  (Definition 3.2), the calculation of the complex period above, and the relation between Eisenstein-Kronecker-Lerch series and special values of  $L$ -functions (Proposition 5.2 if  $\mathfrak{f}_a \neq (1)$ , or Proposition 5.4 if  $\mathfrak{f}_a = (1)$ ).  $\square$

This gives the following corollary, which we stated in Theorem 2.1.

**Corollary 6.4.** *If  $\mathfrak{f}_a \neq (1)$ , then we have*

$$\frac{L(\psi^a, n)}{\Omega_\infty(n)} = \frac{(-1)^{a+n-1}}{G(\varepsilon^a)} \in K^\times \subset K \otimes_{\mathbb{Q}} \mathbb{C}.$$

*A similar formula holds for the case when  $\mathfrak{f}_a = (1)$ , but with multiplication by  $(N^{a+2}/N^{2n} - 1)^{-1}$  on the right hand side.*

*Proof.* The equality follow from the calculation of the complex period in Proposition 6.3. Since  $\varepsilon^a$  is a primitive Hecke character with values in  $K$ , we see that this value is in  $K$ .  $\square$

We next construct the  $p$ -adic measure  $\mu^a$  which appears in the formulation of Conjecture 2.3. Since  $E$  has good ordinary reduction at  $p$ , the prime  $p$  splits as  $p = \mathfrak{p}\mathfrak{p}^*$  in  $K$ . In what follows, we fix once and for all complex and  $p$ -adic embeddings of our coefficient  $K$  as follows. We let  $\tau : K \hookrightarrow \mathbb{C}$  as in §5 and an embedding  $K \hookrightarrow \mathbb{C}_p$  mapping  $\mathfrak{p}$  to a prime in  $\mathbb{C}_p$ . With this convention, we may regard the complex and  $p$ -adic periods as elements respectively in  $\mathbb{C}$  and  $\mathbb{C}_p$ , by taking the first components in Proposition 6.3 and Theorem 6.1.

Let  $\widehat{E}$  be the formal group of  $E$  over  $\mathcal{O}_K$ , and let  $\widehat{K}_{\mathfrak{p}}^{\text{ur}}$  be the  $p$ -adic completion of the maximal unramified extension  $K_{\mathfrak{p}}^{\text{ur}}$  of  $K_{\mathfrak{p}}$ , which we regard as a subfield of  $\mathbb{C}_p$  through our fixed embedding. Since  $p$  is an ordinary prime, there exists an isomorphism of formal groups  $\eta$  over  $\mathcal{O}_{\widehat{K}_{\mathfrak{p}}^{\text{ur}}}$

$$\eta : \widehat{E} \xrightarrow{\cong} \widehat{\mathbb{G}}_m$$

given by a power series  $\eta(t) = \exp(\lambda(t)/\Omega_{\mathfrak{p}}) - 1$ , where  $\Omega_{\mathfrak{p}}$  is a  $p$ -adic period of  $E$  which is an element in  $\mathcal{O}_{\widehat{K}_{\mathfrak{p}}^{\text{ur}}}$  satisfying

$$(19) \quad \Omega_{\mathfrak{p}}^{\sigma-1} = \psi(\mathfrak{p})p^{-1}.$$

The above isomorphism gives the equality

$$(20) \quad \eta^*(d \log(1 + T)) = \omega/\Omega_{\mathfrak{p}}.$$

Again let  $N \geq 3$  be an integer as in §3 divisible by  $\mathfrak{f}$  and prime to  $p$ . By [Katz] 5.10.1, the value of the  $p$ -adic Eisenstein series  $E_{k+2,l,\varphi_a}$  at the test object  $(E, \omega, \nu)$  is defined by

$$(21) \quad E_{k+2,l,\varphi_a}(E, \omega, \nu) := \Omega_{\mathfrak{p}}^{k+l+2} E_{k+2,l,\varphi_a}(E, \eta, \nu).$$

In addition, the comparison theorem [Katz] 8.0.9 states that this value for integers  $k > 0, l \geq 0$  is an element in  $F := K(\mathfrak{f})$  satisfying the equality

$$E_{k+2,l,\varphi_a}(E, \omega, \nu) = E_{k+2,l,\varphi_a}^\infty(E, \omega, \nu).$$

Then the calculation above and Theorem 4.1 gives the following.

**Proposition 6.5.** *We let  $\varphi_a$  be as in Definition 3.2, and we denote again by  $\mu_{\varphi_a}$  the  $p$ -adic measure on  $\mathbb{Z}_p \times \mathbb{Z}_p^\times$  obtained as the value of  $\mu_{\varphi_a}$  of Theorem 4.1 at  $(E, \omega, \nu)$ . If  $\mathfrak{f}_a \neq (1)$ , then we have*

$$\frac{(-1)^a}{\Omega_p^a} \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} x^{n-1} y^{a-n} \mu_{\varphi_a}(x, y) = G(\varepsilon^a) \left( 1 - \frac{\psi(\mathfrak{p})^a}{p^n} \right) \frac{\overline{\Omega}^a \Gamma(n) L(\psi^a, n)}{A^{a-n} |\Omega|^{2n}}$$

for integers  $a \geq n > 0$ . A similar formula holds for the case when  $\mathfrak{f}_a = (1)$ , but with multiplication by  $(N^{a+2}/N^{2n} - 1)$  on the right hand side.

*Proof.* The relation between the action of the Frobenius on  $V_p(\mathbb{Q}_p, \Gamma(N))$  and its specialization is given by

$$\phi^*(E_{k+2,l,\varphi})(E, \eta, \nu) = E_{k+2,l,\varphi}(E, \eta, \nu^{\sigma_p}),$$

since  $(E, \omega)$  is define over  $K$  and hence  $E^{\sigma_p} = E$  and  $\omega^{\sigma_p} = \omega$ . Then from the definition of the specialization of  $p$ -adic modular forms (21) and the action of the Frobenius on the  $p$ -adic period (19), we have

$$\begin{aligned} p^l \phi^*(E_{k+2,l,\varphi})(E, \omega, \nu) &= p^l (\Omega_p^{\sigma_p})^{k+l+2} E_{k+2,l,\varphi}(E, \eta, \nu^{\sigma_p}) \\ &= \psi(\mathfrak{p})^{k+l+2} p^{-k-2} \Omega_p^{k+l+2} E_{k+2,l,\varphi^{\sigma_p}}(E, \eta, \nu). \end{aligned}$$

Applying the above calculation to the case  $a = k + l + 2$  and  $n = k + 2$ , our assertion now follows from Theorem 4.1, noting the definition of the  $p$ -adic Eisenstein series (14), the definition of  $\varphi_a$ , and the fact that the sum over all  $\sigma \in G_{F/K}$  of  $\varphi_a^\sigma$  is invariant by the action of  $\sigma_p$ .  $\square$

**Proposition 6.6.** *Let  $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \rightarrow \mathbb{Z}_p^\times$  be the surjection defined by  $\rho : (x, y) \mapsto x/y$ . We define the measure  $\mu'^a$  on  $\mathbb{Z}_p^\times$  by*

$$\int_{\mathbb{Z}_p^\times} f(w) \mu'^a(w) = \frac{(-1)^a}{G(\varepsilon^a)} \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} x^{-1} y^a \rho^*(f)(x, y) \mu_{\varphi_a}(x, y).$$

If  $\mathfrak{f}_a \neq (1)$ , then this measure satisfies the interpolation property

$$\frac{1}{\Omega_p^a} \int_{\mathbb{Z}_p^\times} w^n \mu'^a(w) = \left( 1 - \frac{\psi(\mathfrak{p})^a}{p^n} \right) \left( 1 - \frac{\overline{\psi}(\mathfrak{p}^*)^a}{p^{a+1-n}} \right) \frac{\overline{\Omega}^a \Gamma(n) L(\psi^a, n)}{A^{a-n} |\Omega|^{2n}}.$$

for any integers  $a$  and  $n$  such that  $a \geq n > 0$ . A similar formula holds for the case when  $\mathfrak{f}_a = (1)$ , but with multiplication by  $(N^{a+2}/N^{2n} - 1)$  on the right hand side.

*Proof.* The equality is obtained from the definition of  $\mu'^a$  and in calculating the restriction of the measure  $\mu_{\varphi_a}$  to  $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ . The calculation directly follows from Katz [Katz] 8.7.6, using the functional equation (see Remark 6.7 below.) One may also do the calculation using an alternative construction of

Katz  $p$ -adic measure ([BKo] Proposition 3.5 and Theorem 3.7), again after using the functional equation.  $\square$

**Remark 6.7.** Combining Proposition 5.2 and (16) (or if  $\mathfrak{f}_a = (1)$ , then Proposition 5.4 and (17)), we obtain the functional equation

$$\frac{\Gamma(n)L(\psi^a, n)}{A^{a-n}|\Omega|^{2n}} = \frac{N(\mathfrak{f}_a)^{a+1-n}\Gamma(a+1-n)L(\bar{\psi}^a, a+1-n)}{(-1)^a G(\varepsilon^a) \bar{f}_a^a A^{n-1} |\Omega|^{2(a+1-n)}}.$$

We regard  $f_a$  and  $\bar{f}_a$  in  $\mathcal{O}_K$  as elements in  $\mathbb{Z}_p^\times$  through the canonical isomorphism  $\mathcal{O}_{K_p} \cong \mathbb{Z}_p$ . Denote by  $\tilde{\mu}_{\varphi_a}$  the measure on  $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$  obtained as the pull-back of  $f_a^{-1}\mu_{\varphi_a}$  through the isomorphism

$$\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \xrightarrow{\cong} \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$$

given by  $(x, y) \mapsto (\bar{f}_a x, f_a^{-1} y)$ . Since  $A = |\Omega|^2 \sqrt{d_K} / 2\pi$ , if we let  $k_1 = a+1-n$  and  $k_2 = 1-n$ , then the interpolation property of  $\tilde{\mu}_{\varphi_a}$  at  $(E, \omega, \nu)$  becomes

$$\begin{aligned} & \frac{1}{\Omega_p^{k_1-k_2}} \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} x^{-k_2} y^{k_1-1} \tilde{\mu}_{\varphi_a}(x, y) \\ &= \left(1 - \frac{\psi(\mathfrak{p})^{k_1-k_2}}{p^{1-k_2}}\right) \left(1 - \frac{\bar{\psi}(\mathfrak{p}^*)^{k_1-k_2}}{p^{k_1}}\right) \left(\frac{\sqrt{d_K}}{2\pi}\right)^{k_2} \frac{\Gamma(k_1)L(\bar{\psi}^{k_1-k_2}, k_1)}{\Omega^{k_1-k_2}} \end{aligned}$$

for  $k_1 > -k_2 \geq 0$ . This coincides with the interpolation property of the two-variable  $p$ -adic measure constructed by Katz and Yager (see [Yag] §1.)

If  $\mathfrak{f}_a \neq (1)$ , then the measure  $\mu'^a$  defined in Proposition 6.6 satisfies the condition of Conjecture 2.3. If  $\mathfrak{f}_a = (1)$ , then we need to cancel the factor  $(N^{a+2}/N^{2n} - 1)$  which appears in the interpolation formula.

**Definition 6.8.** We define the pseudo-measure  $\mu_a$  on  $\mathbb{Z}_p^\times$  as follows.

- (1) If  $\mathfrak{f}_a \neq (1)$ , then we let  $\mu^a := \mu'^a$ .
- (2) If  $\mathfrak{f}_a = (1)$ , then we let  $\mu_N^a$  be the measure on  $\mathbb{Z}_p^\times$  defined by

$$\int_{\mathbb{Z}_p^\times} x^n \mu_N^a = \left( \frac{N^{a+2}}{N^{2n}} - 1 \right)$$

for any integer  $n$ . We define  $\mu^a$  to be the pseudo-measure on  $\mathbb{Z}_p^\times$  obtained as the quotient of  $\mu'^a$  by  $\mu_N^a$  (see for example [Col] §1.2 for the definition of a pseudo-measure).

When  $\mathfrak{f}_a \neq (1)$ , then  $\mu_a$  is by definition a  $p$ -adic measure on  $\mathbb{Z}_p^\times$ .

We now have the following.

**Theorem 6.9.** *Let  $a > 0$  be an integer and let  $\mu^a$  be the pseudo-measure on  $\mathbb{Z}_p^\times$  defined in Definition 6.8. If we let*

$$L_p(\psi^a \otimes \chi_{\text{cyc}}^n) := \int_{\mathbb{Z}_p^\times} w^n \mu^a(w),$$

then we have

$$\frac{L_p(\psi^a \otimes \chi_{\text{cyc}}^n)}{\Omega_p(n)} = \left(1 - \frac{\psi(\mathfrak{p})^a}{p^n}\right) \left(1 - \frac{\bar{\psi}(\mathfrak{p}^*)^a}{p^{a+1-n}}\right) \frac{\Gamma(n)L(\psi^a, n)}{\Omega_\infty(n)}$$

for any integer  $n > a$  such that the  $p$ -adic regulator map  $r_p$  is injective.

*Proof.* We first consider the case when  $\mathfrak{f}_a \neq (1)$ . By definition of the  $p$ -adic Eisenstein series in Definition 4.2, the moments of the measure  $\mu_{\varphi_a}$  constructed in Theorem 4.1 is given by  $p$ -adic Eisenstein series. As in (18), let  $E_{n,a-n,\varphi_a}(E, \omega, \nu)$  be the element in  $\widehat{K}_p^{\text{ur}}$  satisfying

$$(1 - p^{a-n}\sigma^*) E_{n,a-n,\varphi_a}(E, \omega, \nu) = E_{n,a-n,\varphi_a}^{(p)}(E, \omega, \nu).$$

Then the compatibility between the Frobenius on the modular curve and a point, as well as the restriction of the measure on  $\mathbb{Z}_p \times \mathbb{Z}_p^\times$  to  $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$  shows that we have the relation

$$\begin{aligned} \frac{1}{\Omega_p^a} \int_{\mathbb{Z}_p^\times} w^n \mu^a(w) \\ = \frac{1}{G(\varepsilon^a)} \left(1 - \frac{\psi(\mathfrak{p})^a}{p^n}\right) \left(1 - \frac{\bar{\psi}(\mathfrak{p}^*)^a}{p^{a+1-n}}\right) E_{n,a-n,\varphi_a}(E, \omega, \nu). \end{aligned}$$

The calculation of the  $p$ -adic period in Theorem 6.1 shows that

$$\Omega_p(n) = \frac{(-1)^{a-n+1}}{\Gamma(n)} \Omega_p^a E_{n,a-n,\varphi_a}(E, \omega, \nu)$$

for our fixed embedding  $K \hookrightarrow \mathbb{C}_p$ . This proves in particular that

$$\frac{(-1)^{a-n+1}}{\Omega_p(n)} \int_{\mathbb{Z}_p^\times} w^n \mu^a(w) = \left(1 - \frac{\psi(\mathfrak{p})^a}{p^n}\right) \left(1 - \frac{\bar{\psi}(\mathfrak{p}^*)^a}{p^{a+1-n}}\right) \frac{\Gamma(n)}{G(\varepsilon^a)}.$$

Our assertion now follows from Corollary 6.4. The case for  $\mathfrak{f}_a = (1)$  follows in a similar fashion, noting the interpolation property of  $\mu'^a$  and  $\mu_N^a$ .  $\square$

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