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on motivic  $K$ -theory spectra

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# Existence and uniqueness of $E_\infty$ -structures on motivic $K$ -theory spectra

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## Abstract

The algebraic  $K$ -theory spectrum  $KGL$ , the motivic Adams summand  $ML$  and their connective covers have unique  $E_\infty$ -structures refining their naive multiplicative structures in the motivic stable homotopy category. These results are deduced from  $\Gamma$ -homology computations in motivic obstruction theory.

## 1 Introduction

Motivic homotopy theory intertwines classical algebraic geometry and stable algebraic topology. In this paper we study obstruction theory for  $E_\infty$ -structures in the motivic setup. An  $E_\infty$ -structure on a spectrum refers as usual to a ring structure which is not just given up to homotopy, but where the homotopies encode a coherent homotopy commutative multiplication. Many of the examples of motivic ring spectra begin life as commutative monoids in the motivic stable homotopy category. We are interested in the following questions: When can the multiplicative structure of a given commutative monoid in the motivic stable homotopy category be refined to an  $E_\infty$ -ring spectrum? And if such a refinement exists, is it unique? The questions of existence and uniqueness of  $E_\infty$ -structures and their many ramifications have been studied extensively in topology. The first motivic examples worked out in this paper are of  $K$ -theoretic interest.

The complex cobordism spectrum  $MU$  and its motivic analogue  $MGL$  have natural  $E_\infty$ -structures. In the topological setup, Baker and Richter [1] have shown that the complex  $K$ -theory spectrum  $KU$ , the Adams summand  $L$  and the real  $K$ -theory spectrum  $KO$  admit unique  $E_\infty$ -structures. The results in [1] are approached via the obstruction theory developed by Robinson in [11], where it is shown that existence and uniqueness of  $E_\infty$ -structures are guaranteed provided certain  $\Gamma$ -cohomology groups vanish.

In our approach we rely on analogous results in the motivic setup, see [12] for a further generalization. We show that the relevant motivic  $\Gamma$ -cohomology groups vanish in the case of the algebraic  $K$ -theory spectrum  $\mathrm{KGL}$  (Theorem 2.6) and the motivic Adams summand  $\mathrm{ML}$  (see §4). The main ingredients in the proofs are new computations of the  $\Gamma$ -homology complexes of  $\mathrm{KU}$  and  $\mathrm{L}$ , see Theorem 2.3 and Lemma 4.3, and the Landweber base change formula for the motivic cooperations of  $\mathrm{KGL}$  and  $\mathrm{ML}$ . Our main result for  $\mathrm{KGL}$  can be formulated as follows:

**Theorem:** *The algebraic  $K$ -theory spectrum  $\mathrm{KGL}$  has a unique  $E_\infty$ -structure refining its multiplication in the motivic stable homotopy category.*

The existence of the  $E_\infty$ -structure on  $\mathrm{KGL}$  was already known using the Bott inverted model for algebraic  $K$ -theory, see [13], [15], [2], but the analogous result for  $\mathrm{ML}$  is new. The uniqueness part of the Theorem is new, and it rules out the existence of any exotic  $E_\infty$ -structures on  $\mathrm{KGL}$ . We note that related motivic  $E_\infty$ -structures have proven useful in the recent constructions of Atiyah-Hirzebruch types of spectral sequences for motivic twisted  $K$ -theory [14].

One may ask if the uniqueness of  $E_\infty$ -structures on  $\mathrm{KGL}$  has any consequences for the individual algebraic  $K$ -theory spectra of smooth schemes over a fixed base scheme. If the base scheme is regular, consider the following presheaves of  $E_\infty$ -ring spectra. The first one arises from evaluating the  $E_\infty$ -spectrum  $\mathrm{KGL}$  on individual smooth schemes, and the second one from a functorial construction of algebraic  $K$ -theory spectra, cf. [7]. It is natural to ask if these two presheaves are equivalent in some sense. If the second presheaf is obtained from a motivic  $E_\infty$ -spectrum, then our uniqueness result would answer this question in the affirmative. The  $K$ -theory presheaf has this property when viewed as an  $A_\infty$ -object, see [8], but as an  $E_\infty$ -object this is still an open problem.

In topology, the Goerss-Hopkins-Miller obstruction theory [3] allows to gain control over moduli spaces of  $E_\infty$ -structures. In favorable cases, such as for Lubin-Tate spectra, the moduli spaces are  $K(\pi, 1)$ 's giving rise to actions of certain automorphism groups as  $E_\infty$ -maps. A motivic analogue of this obstruction theory has not been worked out. One reason for doing so is that having a homotopy ring structure on a spectrum is often not sufficient in order to form homotopy fixed points under a group action. In Subsection 2.3 we note an interesting consequence concerning  $E_\infty$ -structures on hermitian  $K$ -theory.

In Section 3 we show that the connective cover  $\mathrm{kgf}$  of the algebraic  $K$ -theory spectrum has a unique  $E_\infty$ -structure, and ditto in Section 4 for the connective cover of the Adams summand.

## 2 Algebraic $K$ -theory KGL

In this section we shall present the  $\Gamma$ -cohomology computation showing there is a unique  $E_\infty$ -structure on the algebraic  $K$ -theory spectrum KGL. Throughout we work over some noetherian base scheme of finite Krull dimension, which we omit from the notation.

There are two main ingredients which make this computation possible: First, the  $\Gamma$ -homology computation of  $\mathrm{KU}_0\mathrm{KU}$  over  $\mathrm{KU}_0 = \mathbf{Z}$ , where  $\mathrm{KU}$  is the complex  $K$ -theory spectrum. Second, we employ base change for the motivic cooperations of algebraic  $K$ -theory, as shown in our previous work [9].

### 2.1 The $\Gamma$ -homology of $\mathrm{KU}_0\mathrm{KU}$ over $\mathrm{KU}_0$

For a map  $A \rightarrow B$  between commutative algebras we denote Robinson's  $\Gamma$ -homology complex by  $\tilde{\mathcal{K}}(B|A)$  [11, Definition 4.1]. Recall that  $\tilde{\mathcal{K}}(B|A)$  is a homological double complex of  $B$ -modules concentrated in the first quadrant. The same construction can be performed for maps between graded and bigraded algebras. In all cases we let  $\mathcal{K}(B|A)$  denote the total complex associated with the double complex  $\tilde{\mathcal{K}}(B|A)$ .

The  $\Gamma$ -cohomology

$$\mathrm{H}\Gamma^*(\mathrm{KU}_0\mathrm{KU}|\mathrm{KU}_0, -) = \mathbf{H}^*\mathbf{R}\mathrm{Hom}_{\mathrm{KU}_0\mathrm{KU}}(\mathcal{K}(\mathrm{KU}_0\mathrm{KU}|\mathrm{KU}_0), -)$$

has been computed for various coefficients in [1]. In what follows we require precise information about the complex  $\mathcal{K}(\mathrm{KU}_0\mathrm{KU}|\mathrm{KU}_0)$ , since it satisfies a motivic base change property, cf. Lemma 2.4.

**Lemma 2.1:** *Let  $X \in \mathrm{Ch}_{\geq 0}(\mathrm{Ab})$  be a non-negative chain complex of abelian groups. The following are equivalent:*

- i) The canonical map  $X \rightarrow X \otimes_{\mathbf{Z}} \mathbf{Q} = X \otimes_{\mathbf{Z}} \mathbf{Q}$  is a quasi isomorphism.*
- ii) For every prime  $p$ , there is a quasi isomorphism  $X \otimes_{\mathbf{Z}} \mathbf{F}_p \simeq 0$ .*

*Proof.* It is well known that  $X$  is formal [4, pg. 164], i.e. there is a quasi isomorphism

$$X \simeq \bigoplus_{n \geq 0} H_n(X)[n].$$

(For an abelian group  $A$  and integer  $n$ , we let  $A[n]$  denote the chain complex that consists of  $A$  concentrated in degree  $n$ .) Hence for every prime  $p$ ,

$$X \otimes_{\mathbf{Z}} \mathbf{F}_p \simeq \bigoplus_{n \geq 0} (H_n(X)[n] \otimes_{\mathbf{Z}} \mathbf{F}_p).$$

By resolving  $\mathbf{F}_p = (\mathbf{Z} \xrightarrow{\cdot p} \mathbf{Z})$  one finds an isomorphism

$$H_*(A[n] \otimes_{\mathbf{Z}}^{\mathbf{L}} \mathbf{F}_p) \cong (A/pA)[n] \oplus A\{p\}[n+1]$$

for every abelian group  $A$  and integer  $n$ . Here  $A\{p\}$  is shorthand for  $\{x \in A \mid px = 0\}$ . In summary, ii) holds if and only if the multiplication by  $p$  map

$$\cdot p : H_*(X) \longrightarrow H_*(X)$$

is an isomorphism for every prime  $p$ . The latter is equivalent to i). □

We shall use the previous lemma in order to study cotangent complexes introduced by Illusie in [6]. Let  $R$  be a ring and set  $R_{\mathbf{Q}} := R \otimes_{\mathbf{Z}} \mathbf{Q}$ . Then there is a canonical map

$$\tau_R : \mathbb{L}_{R/\mathbf{Z}} \longrightarrow \mathbb{L}_{R/\mathbf{Z}} \otimes_{\mathbf{Z}}^{\mathbf{L}} \mathbf{Q} \simeq \mathbb{L}_{R/\mathbf{Z}} \otimes_R^{\mathbf{L}} R_{\mathbf{Q}} \xrightarrow{\simeq} \mathbb{L}_{R_{\mathbf{Q}}/\mathbf{Q}}$$

of cotangent complexes in  $\mathrm{Ho}(\mathrm{Ch}_{\geq 0}(\mathbf{Z}))$ . The first quasi isomorphism is obvious, while the second one is an instance of flat base change for cotangent complexes.

**Lemma 2.2:** *The following are equivalent:*

- i)  $\tau_R$  is a quasi isomorphism.
- ii) For every prime  $p$ , there is a quasi isomorphism  $\mathbb{L}_{R/\mathbf{Z}} \otimes_{\mathbf{Z}}^{\mathbf{L}} \mathbf{F}_p \simeq 0$ .

If the abelian group underlying  $R$  is torsion free, then i) and ii) are equivalent to

- iii) For every prime  $p$ ,  $\mathbb{L}_{(R/pR)/\mathbf{F}_p} \simeq 0$ .

*Proof.* The equivalence of i) and ii) follows by applying Lemma 2.1 to  $X = \mathbb{L}_{R/\mathbf{Z}}$ . If  $R$  is torsion free, then it is flat as a  $\mathbf{Z}$ -algebra. Hence, by flat base change, there exists a quasi isomorphism

$$\mathbb{L}_{R/\mathbf{Z}} \otimes_{\mathbf{Z}}^{\mathbf{L}} \mathbf{F}_p \simeq \mathbb{L}_{(R/pR)/\mathbf{F}_p}.$$

□

The following is our analogue for Robinson's  $\Gamma$ -homology complex of the Baker-Richter result [1, Theorem 5.1].

**Theorem 2.3:** *i) Let  $R$  be a torsion free ring such that  $\mathbb{L}_{(R/pR)/\mathbf{F}_p} \simeq 0$  for every prime  $p$ , e.g. assume that  $\mathbf{F}_p \rightarrow R/pR$  is ind-étale for all  $p$ . Then there is a quasi isomorphism*

$$\mathcal{K}(R|\mathbf{Z}) \simeq \mathcal{K}(R_{\mathbf{Q}}|\mathbf{Q})$$

*in the derived category of  $R$ -modules.*

ii) *There is a quasi isomorphism*

$$\mathcal{K}(\mathrm{KU}_0\mathrm{KU}|\mathrm{KU}_0) \simeq (\mathrm{KU}_0\mathrm{KU})_{\mathbf{Q}}[0]$$

*in the derived category of  $\mathrm{KU}_0\mathrm{KU}$ -modules.*

*Proof.* i) The Atiyah-Hirzebruch spectral sequence noted in [10, Remark 2.3] takes the form

$$E_{p,q}^2 = H^p(\mathbb{L}_{R/\mathbf{Z}} \otimes_{\mathbf{Z}}^{\mathbf{L}} \Gamma^q(\mathbf{Z}[x]|\mathbf{Z})) \Rightarrow H^{p+q}(\mathcal{K}(R|\mathbf{Z})).$$

Our assumptions on  $R$  and Lemma 2.2 imply that the  $E^2$ -page is comprised of  $\mathbf{Q}$ -vector spaces. Hence so is the abutment, and there exists a quasi isomorphism between complexes of  $R$ -modules

$$\mathcal{K}(R|\mathbf{Z}) \xrightarrow{\simeq} \mathcal{K}(R|\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

Moreover, by Lemma 2.4, there is a quasi isomorphism

$$\mathcal{K}(R|\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Q} \simeq \mathcal{K}(R_{\mathbf{Q}}|\mathbf{Q}).$$

ii) According to [1, Theorem 3.1, Corollary 3.4, (a)] and the Hopf algebra isomorphism  $A^{st} \simeq \mathrm{KU}_0\mathrm{KU}$  [1, Proposition 6.1], the ring  $R := \mathrm{KU}_0\mathrm{KU}$  satisfies the assumptions of part i)<sup>1</sup>. Now since  $\mathrm{KU}_0 \cong \mathbf{Z}$ ,

$$\mathcal{K}(\mathrm{KU}_0\mathrm{KU}|\mathrm{KU}_0) \simeq \mathcal{K}((\mathrm{KU}_0\mathrm{KU})_{\mathbf{Q}}|\mathbf{Q}).$$

We have that  $(\mathrm{KU}_0\mathrm{KU})_{\mathbf{Q}} \simeq \mathbf{Q}[w^{\pm 1}]$  [1, Theorem 3.2, (c)] is a smooth  $\mathbf{Q}$ -algebra. Hence, since  $\Gamma$ -cohomology agrees with André-Quillen cohomology over  $\mathbf{Q}$ , there are quasi isomorphisms

$$\mathcal{K}(\mathrm{KU}_0\mathrm{KU}|\mathrm{KU}_0) \simeq \Omega_{\mathbf{Q}[w^{\pm 1}]|\mathbf{Q}}^1[0] \simeq (\mathrm{KU}_0\mathrm{KU})_{\mathbf{Q}}[0].$$

□

## 2.2 The $\Gamma$ -homology of $\mathrm{KGL}_{**}\mathrm{KGL}$ over $\mathrm{KGL}_{**}$

The strategy in what follows is to combine the computations for  $\mathrm{KU}$  in §2.1 with motivic Landweber exactness [9]. To this end we require the following general base change result, which was also used in the proof of Theorem 2.3.

<sup>1</sup>This follows also easily from Landweber exactness of  $\mathrm{KU}$ .

**Lemma 2.4:** For a pushout of ordinary, graded or bigraded commutative algebras

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

there are isomorphisms between complexes of  $D$ -modules

$$\mathcal{K}(D|C) \cong \mathcal{K}(B|A) \otimes_B D \cong \mathcal{K}(B|A) \otimes_A C.$$

If  $B$  is flat over  $A$ , then  $\tilde{\mathcal{K}}(B|A)$  is a first quadrant homological double complex of flat  $B$ -modules; thus, in the derived category of  $D$ -modules there are quasi isomorphisms

$$\mathcal{K}(D|C) \simeq \mathcal{K}(B|A) \otimes_B^{\mathbf{L}} D \simeq \mathcal{K}(B|A) \otimes_A^{\mathbf{L}} C.$$

*Proof.* Following the notation in [11, §4], let  $(B|A)^\otimes$  denote the tensor algebra of  $B$  over  $A$ . Then  $(B|A)^\otimes \otimes_A B$  has a natural  $\Gamma$ -module structure over  $B$ , cf. [11, §4]. Here  $\Gamma$  denotes the category of finite based sets and basepoint preserving maps. It follows that  $((B|A)^\otimes \otimes_A B) \otimes_B D$  is a  $\Gamma$ -module over  $D$ . Moreover, by base change for tensor algebras, there exists an isomorphism of  $\Gamma$ -modules in  $D$ -modules

$$((B|A)^\otimes \otimes_A B) \otimes_B D \cong (D|C)^\otimes \otimes_C D.$$

Here we use that the  $\Gamma$ -module structure on  $(B|A)^\otimes \otimes_A M$ , for  $M$  a  $B$ -module, is given as follows: For a map  $\varphi: [m] \rightarrow [n]$  between finite pointed sets,

$$(B \otimes_A B \otimes_A \cdots \otimes_A B) \otimes_A M \rightarrow (B \otimes_A B \otimes_A \cdots \otimes_A B) \otimes_A M$$

sends  $b_1 \otimes \cdots \otimes b_m \otimes m$  to

$$\left( \prod_{i \in \varphi^{-1}(1)} b_i \right) \otimes \cdots \otimes \left( \prod_{i \in \varphi^{-1}(n)} b_i \right) \otimes \left( \left( \prod_{i \in \varphi^{-1}(0)} b_i \right) \cdot m \right).$$

By convention, if  $\varphi^{-1}(j) = \emptyset$  then  $\prod_{i \in \varphi^{-1}(j)} b_i = 1$ . Robinson's  $\Xi$ -construction yields an isomorphism between double complexes of  $D$ -modules

$$\tilde{\mathcal{K}}(D|C) = \Xi((D|C)^\otimes \otimes_C D) \cong \Xi(((B|A)^\otimes \otimes_A B) \otimes_B D).$$

Inspection of the  $\Xi$ -construction reveals there is an isomorphism

$$\Xi(((B|A)^\otimes \otimes_A B) \otimes_B D) \cong \Xi((B|A)^\otimes \otimes_A B) \otimes_B D.$$

By definition, this double complex of  $D$ -modules is  $\tilde{\mathcal{K}}(B|A) \otimes_B D \cong \tilde{\mathcal{K}}(B|A) \otimes_A C$ . This proves the first assertion by comparing the corresponding total complexes. The remaining claims follow easily.  $\square$

Next we recall the structure of the motivic cooperations of the algebraic  $K$ -theory spectrum  $\mathbf{KGL}$ . The algebras we shall consider are bigraded as follows:  $\mathbf{KU}_0 \cong \mathbf{Z}$  in bidegree  $(0, 0)$  and  $\mathbf{KU}_* \cong \mathbf{Z}[\beta^{\pm 1}]$  with the Bott-element  $\beta$  in bidegree  $(2, 1)$ . With these conventions, there is a canonical bigraded map

$$\mathbf{KU}_* \rightarrow \mathbf{KGL}_{**}.$$

**Lemma 2.5:** *There are pushouts of bigraded algebras*

$$\begin{array}{ccc} \mathbf{KU}_* & \xrightarrow{\eta_L} & \mathbf{KU}_* \mathbf{KU} & & \mathbf{KU}_0 & \xrightarrow{(\eta_L)_0} & \mathbf{KU}_0 \mathbf{KU} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbf{KGL}_{**} & \xrightarrow{\eta_L} & \mathbf{KGL}_{**} \mathbf{KGL} & & \mathbf{KU}_* & \xrightarrow{\eta_L} & \mathbf{KU}_* \mathbf{KU} \end{array}$$

and a quasi isomorphism in the derived category of  $\mathbf{KGL}_{**} \mathbf{KGL}$ -modules

$$\mathcal{K}(\mathbf{KGL}_{**} \mathbf{KGL} | \mathbf{KGL}_{**}) \simeq \mathcal{K}(\mathbf{KU}_0 \mathbf{KU} | \mathbf{KU}_0) \otimes_{\mathbf{KU}_0 \mathbf{KU}}^{\mathbf{L}} \mathbf{KGL}_{**} \mathbf{KGL}.$$

*Proof.* Here,  $\eta_L$  is a generic notation for the left unit of some flat Hopf-algebroid. The first pushout is shown in [9, Proposition 9.1, (c)]. The second pushout is in [1]. Applying Lemma 2.4 twice gives the claimed quasi isomorphism.  $\square$

Next we compute the  $\Gamma$ -cohomology of the motivic cooperations of  $\mathbf{KGL}$ .

**Theorem 2.6:** *i) There is an isomorphism*

$$\mathbf{H}^{\Gamma^*,*,*}(\mathbf{KGL}_{**} \mathbf{KGL} | \mathbf{KGL}_{**}; \mathbf{KGL}_{**}) \cong \mathbf{H}^* \mathbf{RHom}_{\mathbf{Z}}(\mathbf{Q}[0], \mathbf{KGL}_{**}).$$

*ii) For all  $s \geq 2$ ,*

$$\mathbf{H}^{\Gamma^{s,*},*}(\mathbf{KGL}_{**} \mathbf{KGL} | \mathbf{KGL}_{**}; \mathbf{KGL}_{**}) = 0.$$

*Proof.* i) By the definition of  $\Gamma$ -cohomology and the results in this Subsection there are isomorphisms

$$\begin{aligned} & \mathbf{H}^{\Gamma^*,*,*}(\mathbf{KGL}_{**} \mathbf{KGL} | \mathbf{KGL}_{**}; \mathbf{KGL}_{**}) \\ &= \mathbf{H}^* \mathbf{RHom}_{\mathbf{KGL}_{**} \mathbf{KGL}}(\mathcal{K}(\mathbf{KGL}_{**} \mathbf{KGL} | \mathbf{KGL}_{**}), \mathbf{KGL}_{**}) \\ &\cong \mathbf{H}^* \mathbf{RHom}_{\mathbf{KGL}_{**} \mathbf{KGL}}(\mathcal{K}(\mathbf{KU}_0 \mathbf{KU} | \mathbf{Z}) \otimes_{\mathbf{KU}_0 \mathbf{KU}}^{\mathbf{L}} \mathbf{KGL}_{**} \mathbf{KGL}, \mathbf{KGL}_{**}) \\ &\cong \mathbf{H}^* \mathbf{RHom}_{\mathbf{KU}_0 \mathbf{KU}}(\mathcal{K}(\mathbf{KU}_0 \mathbf{KU} | \mathbf{Z}), \mathbf{KGL}_{**}) \\ &\cong \mathbf{H}^* \mathbf{RHom}_{\mathbf{KU}_0 \mathbf{KU}}((\mathbf{KU}_0 \mathbf{KU})_{\mathbf{Q}}[0], \mathbf{KGL}_{**}) \\ &\cong \mathbf{H}^* \mathbf{RHom}_{\mathbf{Z}}(\mathbf{Q}[0], \mathbf{KGL}_{**}). \end{aligned}$$



ii) This follows from i) since  $\mathbf{Z}$  has global dimension 1. □

**Remark 2.7:** *It is an exercise to compute  $\mathbf{RHom}_{\mathbf{Z}}(\mathbf{Q}, -)$  for finitely generated abelian groups. This explicates our Gamma-cohomology computation in degrees 0 and 1 for base schemes with finitely generated algebraic  $K$ -groups, e.g. finite fields and number rings. The computation  $\mathbf{RHom}_{\mathbf{Z}}(\mathbf{Q}, \mathbf{Z}) \simeq \hat{\mathbf{Z}}/\mathbf{Z}[1]$  shows our results imply [1, Corollary 5.2].*

The vanishing result Theorem 2.6, ii) together with the motivic analogues of the results in [11, Theorem 5.6], as detailed in [12], conclude the proof of the Theorem for KGL formulated in the Introduction.

### 2.3 A remark on hermitian $K$ -theory $\mathbf{KQ}$

In this short Subsection we discuss one instance in which the motivic obstruction theory used here falls short of a putative motivic analogue of the obstruction theory of Goerss, Hopkins and Miller [3]. By [9, Theorem 9.7, (ii), Remark 9.8, (iii)] we may realize the stable Adams operation  $\Psi^{-1}$  on algebraic  $K$ -theory by a motivic ring spectrum map

$$\Psi^{-1} : \mathbf{KGL} \longrightarrow \mathbf{KGL}. \tag{1}$$

In many cases of interest one expects that  $\mathbf{fib}(\psi^{-1} - 1)$  represents Hermitian  $K$ -theory  $\mathbf{KQ}$ . A motivic version of the Goerss-Hopkins-Miller obstruction theory in [3] implies, in combination with Theorem 2.6, that (1) can be modelled as an  $E_{\infty}$ -map. With this result in hand, it would follow that  $\mathbf{KQ}$  admits an  $E_{\infty}$ -structure.

It seems the obstruction theory we use is intrinsically unable to provide such results by “computing”  $E_{\infty}$ -mapping spaces. However, there might be a more direct way of showing that  $\mathbf{KQ}$  has a unique  $E_{\infty}$ -structure, using the obstruction theory in this paper. A first step would be to compute the motivic cooperations of  $\mathbf{KQ}$ .

## 3 Connective algebraic $K$ -theory $\mathbf{kgl}$

We define the connective algebraic  $K$ -theory spectrum  $\mathbf{kgl}$  as the effective part  $\mathbf{f}_0\mathbf{KGL}$  of  $\mathbf{KGL}$ . Recall that the functor  $\mathbf{f}_i$  defined in [16] projects from the motivic stable homotopy category to its  $i$ th effective part. Note that  $\mathbf{f}_0\mathbf{KGL}$  is a commutative monoid in the motivic stable homotopy category since projection to the effective part is a lax symmetric monoidal functor (because it is right adjoint to a monoidal functor). For  $i \in \mathbf{Z}$  there exists a natural map  $\mathbf{f}_{i+1}\mathbf{KGL} \rightarrow \mathbf{f}_i\mathbf{KGL}$  in the motivic stable homotopy category with

cofiber the  $i$ th slice of  $\mathbf{KGL}$ . With these definitions,  $\mathbf{KGL} \cong \text{hocolim } f_i \mathbf{KGL}$  (this is true for any motivic spectrum, cf. [16, Lemma 4.2]). Bott periodicity for algebraic  $K$ -theory implies that  $f_{i+1} \mathbf{KGL} \cong \Sigma^{2,1} f_i \mathbf{KGL}$ . This allows to recast the colimit as  $\text{hocolim } \Sigma^{2i,i} \mathbf{kgl}$  with multiplication by the Bott element  $\beta$  in  $\mathbf{kgl}^{-2,-1} \cong \mathbf{KGL}^{-2,-1}$  as the transition map at each stage. We summarize these observations in a lemma.

**Lemma 3.1:** *The algebraic  $K$ -theory spectrum  $\mathbf{KGL}$  is isomorphic in the motivic stable homotopy category to the Bott inverted connective algebraic  $K$ -theory spectrum  $\mathbf{kgl}[\beta^{-1}]$ .*

**Theorem 3.2:** *The connective algebraic  $K$ -theory spectrum  $\mathbf{kgl}$  has a unique  $E_\infty$ -structure refining its multiplication in the motivic stable homotopy category.*

*Proof.* The connective cover functor  $f_0$  preserves  $E_\infty$ -structures [5]. Thus the existence of an  $E_\infty$ -structure on  $\mathbf{kgl}$  is ensured. We note that inverting the Bott element can be refined to the level of motivic  $E_\infty$ -ring spectra by the methods employed in [13]. Thus, by Lemma 3.1, starting out with any two  $E_\infty$ -structures on  $\mathbf{kgl}$  produces two  $E_\infty$ -structures on  $\mathbf{KGL}$ , which coincide by the uniqueness result for  $E_\infty$ -structures on  $\mathbf{KGL}$ . Applying  $f_0$  recovers the two given  $E_\infty$ -structures on  $\mathbf{kgl}$ : If  $X$  is  $E_\infty$  with  $\varphi: X \simeq \mathbf{kgl}$  as ring spectra, then there is a canonical  $E_\infty$ -map  $X \rightarrow X[\beta'^{-1}]$ , where  $\beta'$  is the image of the Bott element under  $\varphi$ . Since  $X$  is an effective motivic spectrum, this map factors as an  $E_\infty$ -map  $X \rightarrow f_0(X[\beta'^{-1}])$ . By construction of  $\mathbf{kgl}$  the latter map is an equivalence. This shows the two given  $E_\infty$ -structures on  $\mathbf{kgl}$  coincide.  $\square$

## 4 The motivic Adams summands $\mathbf{ML}$ and $\mathbf{ml}$

Let  $\mathbf{BP}$  denote the Brown-Peterson spectrum for a fixed prime number  $p$ . Then the coefficient ring  $\mathbf{KU}_{(p)*}$  of the  $p$ -localized complex  $K$ -theory spectrum is a  $\mathbf{BP}_*$ -module via the ring map  $\mathbf{BP}_* \rightarrow \mathbf{MU}_{(p)*}$  which classifies the  $p$ -typicalization of the formal group law over  $\mathbf{MU}_{(p)*}$ . The  $\mathbf{MU}_{(p)*}$ -algebra structure on  $\mathbf{KU}_{(p)*}$  is induced from the natural orientation  $\mathbf{MU} \rightarrow \mathbf{KU}$ . With this  $\mathbf{BP}_*$ -module structure,  $\mathbf{KU}_{(p)*}$  splits into a direct sum of the  $\Sigma^{2i} \mathbf{L}_*$  for  $0 \leq i \leq p-2$ , where  $\mathbf{L}$  is the Adams summand of  $\mathbf{KU}_{(p)}$ . Thus motivic Landweber exactness [9] over the motivic Brown-Peterson spectrum  $\mathbf{MBP}$  produces a splitting of motivic spectra

$$\mathbf{KGL}_{(p)} = \bigvee_{i=0}^{p-2} \Sigma^{2i,i} \mathbf{ML}.$$

We refer to  $\mathbf{ML}$  as the motivic Adams summand of algebraic  $K$ -theory.

Since  $L_*$  is an  $\mathbf{BP}_*$ -algebra and there are no nontrivial phantom maps from any smash power of  $\mathbf{ML}$  to  $\mathbf{ML}$ , which follows from [9, Remark 9.8, (ii)] since  $\mathbf{ML}$  is a retract of  $\mathbf{KGL}_{(p)}$ , we deduce that the corresponding ring homology theory induces a commutative monoid structure on  $\mathbf{ML}$  in the motivic stable homotopy category.

We define the connective motivic Adams summand  $\mathbf{ml}$  to be  $f_0\mathbf{ML}$ . It is also a commutative monoid in the motivic homotopy category.

**Theorem 4.1:** *The motivic Adams summand  $\mathbf{ML}$  has a unique  $E_\infty$ -structure refining its multiplication in the motivic stable homotopy category. The same result holds for the connective motivic Adams summand  $\mathbf{ml}$ .*

The construction of  $\mathbf{ML}$  as a motivic Landweber exact spectrum makes the following result evident on account of the proof of Lemma 2.5.

**Lemma 4.2:** *There exist pushout squares of bigraded algebras*

$$\begin{array}{ccc} L_* & \xrightarrow{\eta_L} & L_*L \\ \downarrow & & \downarrow \\ \mathbf{ML}_{**} & \xrightarrow{\eta_L} & \mathbf{ML}_{**}\mathbf{ML} \end{array} \quad \begin{array}{ccc} L_0 & \xrightarrow{(\eta_L)_0} & L_0L \\ \downarrow & & \downarrow \\ L_* & \xrightarrow{\eta_L} & L_*L \end{array}$$

and a quasi isomorphism in the derived category of  $\mathbf{ML}_{**}\mathbf{ML}$ -modules

$$\mathcal{K}(\mathbf{ML}_{**}\mathbf{ML}|\mathbf{ML}_{**}) \simeq \mathcal{K}(L_0L|L_0) \otimes_{L_0L}^L \mathbf{ML}_{**}\mathbf{ML}.$$

Next we show the analog of Theorem 2.3, ii) for the motivic Adams summand.

**Lemma 4.3:** *In the derived category of  $L_0L$ -modules, there is a quasi isomorphism*

$$\mathcal{K}(L_0L|L_0) \simeq (L_0L)_{\mathbf{Q}}[0].$$

*Proof.* In the notation of [1, Proposition 6.1] there is an isomorphism between Hopf algebras  $L_0L \cong {}^\zeta A_{(p)}^{st}$ . Recall that  ${}^\zeta A_{(p)}^{st}$  is a free  $\mathbf{Z}_{(p)}$ -module on a countable basis and  ${}^\zeta A_{(p)}^{st}/p{}^\zeta A_{(p)}^{st}$  is a formally étale  $\mathbf{F}_p$ -algebra [1, Theorem 3.3(c), Corollary 4.2]. Applying Theorem 2.3, i) to  $R = L_0L$  and using that  $(L_0L)_{\mathbf{Q}} \simeq \mathbf{Q}[v^{\pm 1}]$  by Landweber exactness, where  $v = w^{p-1}$  and  $(\mathbf{KU}_0\mathbf{KU})_{\mathbf{Q}} \cong \mathbf{Q}[w^{\pm 1}]$ , we find

$$\mathcal{K}(L_0L|L_0) \simeq \Omega_{\mathbf{Q}[v^{\pm 1}]|\mathbf{Q}}^1[0] \simeq (L_0L)_{\mathbf{Q}}[0].$$

□

Lemmas 4.2 and 4.3 imply there is a quasi isomorphism

$$\mathbf{H}\Gamma^{*,*,*}(\mathbf{ML}_{**}\mathbf{ML}|\mathbf{ML}_{**}; \mathbf{ML}_{**}) \simeq \mathbf{H}^*\mathbf{R}\mathbf{Hom}_{\mathbf{Z}}(\mathbf{Q}[0], \mathbf{ML}_{**}).$$

Thus the part of Theorem 4.1 dealing with  $\mathbf{ML}$  follows, since for all  $s \geq 2$ ,

$$\mathbf{H}\Gamma^{s,*,*}(\mathbf{ML}_{**}\mathbf{ML}|\mathbf{ML}_{**}; \mathbf{ML}_{**}) = 0. \quad (2)$$

The assertion about  $\mathbf{ml}$  follows by the exact same type of argument as for  $\mathbf{kgl}$ . The periodicity operator in this case is  $v_1 \in \mathbf{ml}^{2(1-p), 1-p} = \mathbf{ML}^{2(1-p), 1-p}$ .

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