Stability Analysis of Geometric Evolution Equations with Triple Lines and Boundary Contact

DISSERTATION ZUR ERLANGUNG DES DOKTORGRADES
DER NATURWISSENSCHAFTEN (Dr. rer. nat.)
AN DER NWF I - MATHEMATIK
DER UNIVERSITÄT REGENSBURG

vorgelegt von
Daniel Depner
Regensburg, April 2010

Promotionsgesuch eingereicht am 13. April 2010.

Die Arbeit wurde angeleitet von Prof. Dr. H. Garcke.

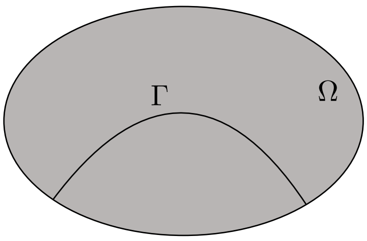
Prüfungsausschuss:	Vorsitzender:	Prof. Dr. B. Amann
	1. Gutachter:	Prof. Dr. H. Garcke
	2. Gutachter:	Prof. Dr. K. Deckelnick (Universität Magdeburg)
	weiterer Prüfer:	Prof. Dr. G. Dolzmann
	Ersatzprüfer:	Prof. Dr. H. Abels

Contents

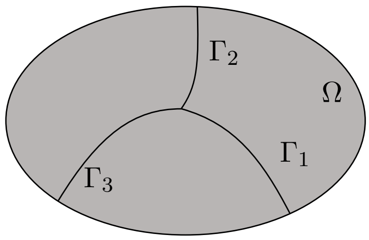
Chapter 1

Introduction

The subject of the present work is the study of geometric evolution laws for evolving hypersurfaces with boundary contact and triple lines. The considered hypersurfaces lie inside a fixed bounded region and are in contact with its boundary through a $90°$ angle. In case of triple lines they also meet each other with some prescribed angle conditions, see Figure 1.1 for a sketch of the arising situations for curves in the plane.



(a) one hypersurface



(b) three hypersurfaces

Figure 1.1: A sketch of the arising situations.

The geometric evolution laws that we want to consider are the mean curvature flow

$$V = H\,, \tag{1.1}$$

the surface diffusion flow

$$V = -\Delta H \tag{1.2}$$

and the volume preserving mean curvature flow

$$V = H - \overline{H}\,. \tag{1.3}$$

Here V is the normal velocity of the evolving hypersurface, H is the mean curvature, Δ is the Laplace-Beltrami operator and $\overline{H}$ is the average mean curvature. Our sign convention is that H is negative for spheres provided with outer unit normal. For a review concerning geometric evolution equations, in particular for the mean curvature flow, we want to refer the reader to the work of Deckelnick, Dziuk and Elliott [DDE05].

Mean curvature flow (1.1) was first studied by Brakke [Bra78] from a point of view of geometric measure theory. Gage and Hamilton [GH86] showed that convex curves in the plane under this flow shrink to round points and Grayson [Gray87] generalized this result to embedded plane curves. Huisken [Hui84] generalized the result of [GH86] to show that convex, compact hypersurfaces retain their convexity and become asymptotically round. Finally we mention that this flow is the L^2-gradient flow of the area functional, it is area decreasing and for curves in the plane it is therefore also called curve shortening flow.

Surface diffusion flow (1.2) was first proposed by Mullins [Mu57] to model motion of interfaces where this motion is governed purely by mass diffusion within the interfaces. Davi and Gurtin [DG90] derived the above law within rational thermodynamics and Cahn, Elliott and Novick-Cohen [CEN96] identified it as the sharp interface limit of a Cahn-Hilliard equation with degenerate mobility. An existence result for curves in the plane and stability of circles has been shown by Elliott and Garcke [EG97] and this result was generalized to the higher dimensional case by Escher, Mayer and Simonett [EMS98]. Cahn and Taylor [CT94] showed that (1.2) is the H^{-1}-gradient flow of the area functional and we finally mention that for closed embedded hypersurfaces the enclosed volume is preserved and the surface area decreases in time as can be seen for example in [EG97] or [EMS98].

The volume preserving mean curvature flow (1.3) was considered for example in the work of Huisken [Hui87] and in Escher and Simonett [ES98]. The idea behind this flow is to overcome the lack of volume conservation in the mean curvature flow by enforcing it with the help of a nonlocal term.

We will examine the above evolution laws with boundary conditions by considering evolving hypersurfaces Γ that meet the boundary of a fixed bounded region Ω or even intersect each other at triple lines inside of this region. In the case of the surface diffusion flow these boundary conditions were derived by Garcke and Novick-Cohen [GN00] as the asymptotic limit of a Cahn-Hilliard system with a degenerate mobility matrix. At the outer boundary this yields natural boundary conditions given by a 90° angle condition and a no-flux condition, i.e. we require at $\Gamma \cap \partial\Omega$

$$\Gamma \perp \partial\Omega\,, \tag{1.4}$$

$$n_{\partial\Gamma} \cdot \nabla H = 0\,. \tag{1.5}$$

Here ∇ is the surface gradient and $n_{\partial\Gamma}$ is the outer unit conormal of Γ at boundary points. The conditions (1.4) and (1.5) are the natural boundary conditions when viewing surface diffusion (1.2) with outer boundary contact as the H^{-1}-gradient flow of the area functional.

For the evolution law (1.2) for one evolving curve in the plane with boundary conditions (1.4) and (1.5) Garcke, Ito and Kohsaka gave in [GIK05] a linearized stability criterion for spherical arcs resp. lines, which are the stationary states in this case. In [GIK08] the same authors showed nonlinear stability results for the above situation.

For the mean curvature flow (1.1), one can also consider situations where an evolving hypersurface is attached to an outer fixed boundary. In this case, instead of the two conditions (1.4) and (1.5), only an angle condition has to be fulfilled. This is due to the fact that surface diffusion is a fourth order and mean curvature flow is a second order geometric evolution law. For the stability analysis for mean curvature flow (1.1) with boundary condition (1.4) we refer to [EY93, ESY96], where the results heavily depend on maximum principles.

When we now draw our attention to the appearance of triple lines, we want to change the considered evolution laws slightly by including some constants that allow different contact angles between the hypersurfaces. We assume that three evolving hypersurfaces Γ_i either fulfill the weighted mean curvature flow

$$V_i \quad = \quad \gamma_i \, H_i \,, \tag{1.6}$$

or the weighted surface diffusion flow

$$V_i \quad = \quad -m_i \, \gamma_i \, \Delta H_i \,, \tag{1.7}$$

each for $i = 1, 2, 3$. Here the constants $\gamma_i, m_i > 0$ are the surface energy density and the mobility of the evolving hypersurface Γ_i. If the three evolving hypersurfaces meet at a triple line $L(t)$, we require that there the following conditions hold.

$$\angle(\Gamma_1(t), \Gamma_2(t)) = \theta_3 \,, \ \angle(\Gamma_2(t), \Gamma_3(t)) = \theta_1 \,, \ \angle(\Gamma_3(t), \Gamma_1(t)) = \theta_2 \,, \tag{1.8}$$

$$\gamma_1 \, H_1 + \gamma_2 \, H_2 + \gamma_3 \, H_3 = 0 \,, \tag{1.9}$$

$$m_1 \, \gamma_1 \, \nabla H_1 \cdot n_{\partial\Gamma_1} = m_2 \, \gamma_2 \, \nabla H_2 \cdot n_{\partial\Gamma_2} = m_3 \, \gamma_3 \, \nabla H_3 \cdot n_{\partial\Gamma_3} \,, \tag{1.10}$$

where the quantity $\angle(\Gamma_i(t), \Gamma_j(t))$ denotes the angle between $\Gamma_i(t)$ and $\Gamma_j(t)$ and the angles $\theta_1, \theta_2, \theta_3$ with $0 < \theta_i < \pi$ are related through the identity $\theta_1 + \theta_2 + \theta_3 = 2\pi$ and Young's law, which is

$$\frac{\sin\theta_1}{\gamma_1} = \frac{\sin\theta_2}{\gamma_2} = \frac{\sin\theta_3}{\gamma_3} \,. \tag{1.11}$$

We can show that Young's law (1.11) is equivalent to

$$\gamma_1 \, n_{\partial\Gamma_1} + \gamma_2 \, n_{\partial\Gamma_2} + \gamma_3 \, n_{\partial\Gamma_3} = 0 \,, \tag{1.12}$$

which is the force balance at the triple line.

For the derivation of the conditions (1.8)-(1.10) at the triple line, we refer to Garcke and Novick-Cohen [GN00]. The angle condition (1.8) follows from the balance of forces (1.12) at the triple line, the second condition (1.9) follows from the continuity of chemical potentials and the conditions (1.10) are the flux balance at the triple line $L(t)$.

We remark that for three hypersurfaces evolving due to the weighted mean curvature flow (1.6), only the angle condition (1.8) has to be fulfilled. In this case together with outer boundary contact for the three evolving hypersurfaces, linearized stability was considered in Ikota and Yanagida [IY03]. Nonlinear stability for the weighted curvature flow for curves in the plane with triple junction and boundary contact was shown by Garcke, Kohsaka and Ševčovič [GKS09].

In the following situations there are some results on stability for surface diffusion. Let three plane curves lie in the fixed region Ω, where $\partial\Omega$ is a rectangle, and evolve due to the weighted surface diffusion flow (1.7) such that the outer boundary conditions (1.4) and (1.5) are fulfilled for each curve. The three plane curves shall also have a triple junction where the conditions (1.8)-(1.10) are fulfilled. In this case Ito and Kohsaka [IK01a] and also Escher, Garcke and Ito [EGI03] showed global existence results when the initial curve is a small perturbation of a certain stationary curve. The same is true if $\partial\Omega$ is a triangle and was shown in [IK01b] from Ito and Kohsaka. In these cases also nonlinear stability of the stationary curve can be shown. The above described curve situation was also considered without the special geometry of Ω in the work of Garcke, Ito and Kohsaka [GIK10], where the authors formulate a linearized stability criterion for stationary curves.

For numerical results we want to refer to the work of Deckelnick and Elliott [DE98], where the authors considered the curve shortening flow with outer boundary contact and to Bronsard and Wetten [BW95], where curvature flow for a network of curves is the subject. We also want to refer to a series of papers by Barrett, Garcke and Nürnberg. For example they considered in [BGN07] surface diffusion with triple lines and outer boundary contact for curves in the plane and extended this work to the case of hypersurfaces in [BGN09]. In all cases the authors derive numerical schemes and give also a lot of examples which indicate the stability behaviour.

The main goal in this work is the extension of the linearized stability analysis in [GIK05] and [GIK10] from curves to hypersurfaces. In detail this means that we will consider the surface diffusion (1.2) for one evolving hypersurface Γ lying in a bounded region Ω such that Γ fulfills the boundary conditions (1.4) and (1.5). The second important part will consist in regarding three evolving hypersurfaces Γ_i lying in a bounded region Ω, such that each of the Γ_i fulfills (1.4) and (1.5) and such that the Γ_i meet at a triple line inside of Ω, where the conditions (1.8)-(1.10) hold. In both cases we generalize the necessary steps of [GIK05] and [GIK10] to the higher dimensional setting.

The first main difference to the curve case considered in these papers is the parametrization of the hypersurfaces, which is needed to derive partial differential equations for unknown functions from the geometric evolution laws. In contrast to the very explicit given parametrization in the curve case, we set up for the situation of one evolving hypersurface as described above an abstract curvilinear coordinate system from Vogel [Vog00], that takes into account a possibly curved outer boundary $\partial\Omega$. In short, we fix a stationary solution Γ^* and consider a mapping $\Psi : \Gamma^* \times (-d, d) \to \Omega$ with the properties $\Psi(q, 0) = q$ and $\Psi(q, w) \in \partial\Omega$ for $q \in \partial\Gamma^*$. In the case of three evolving hypersurfaces as described above we also fix a stationary solution $\Gamma^* = \bigcup_{i=1}^{3} \Gamma_i^*$ and use an explicit parametrization with two parameters w and s near the triple line $L^* = \partial\Gamma_1^* = \partial\Gamma_2^* = \partial\Gamma_3^*$ given by $q \mapsto q + w\, n_i^*(q) + s\, t_i^*(q)$, where n_i^* is a unit normal of Γ_i^* and t_i^* is a tangent vector field on Γ_i^* with support in a neighbourhood of L^*, that equals the outer unit conormal of Γ_i^* at $\partial\Gamma_i^*$. By introducing functions on Γ^*, whose values take the place of the parameters w and s, we will denote the considered evolving hypersurfaces as graphs over Γ^*, although in the literature, for example in [DDE05], also the term parametric approach is used.

Another difference compared to the curve case is the linearization of the arising partial differential equations. Instead of the explicit calculations in [GIK05] we use the concept of normal time derivative to get the linearization of mean curvature in Lemma 3.5. The treatment of the angle conditions in Lemmata 3.7 and 4.11 is considerably harder than in the curve case. Here we write the arising normals with the help of the cross product and use a local parametrization for the hypersurfaces with well chosen properties at a fixed point.

It is very important that we can describe the linearized problem as in the curve case as an H^{-1}-gradient flow, because this is the main reason that the linearized operator is self-adjoint. Also in the situation with triple lines we find an energy such that the system of partial differential equations on different hypersurfaces can be viewed as an H^{-1}-gradient flow with respect to this energy. Then we are in a good position to apply results from spectral theory. We can relate the asymptotic stability of the zero solution of the linearized problem to the fact that the eigenvalues of the linearized operator are negative. Since we can describe the largest eigenvalue with the help of a bilinear form arising due to the gradient flow structure, we can finally give a criterion for linearized stability of the original geometric problems. The main results from this work appear in the Theorems 3.17, 3.42, 4.21 and 4.43 and will be summarized further down in the description of each chapter in bordered frames.

Since the above method works very fine without use of any maximum principle, we also apply it to the case of mean curvature flow with and without triple lines and, as a corollary, to volume preserving mean curvature flow.

The remaining part of this introduction will be a summary of the contents from the following chapters. The second Chapter contains an overview of the used concepts from differential geometry for hypersurfaces such as curvature terms, differential operators and the theorem of Gauß on hypersurfaces with boundary. We also introduce with great care the notion of an evolving hypersurface. Thereby we explain the term normal velocity, give a representation of the tangent space and consider the normal time derivative for functions resp. vector fields defined on an evolving hypersurface. We also describe evolving hypersurfaces that arise as a graph over a fixed reference hypersurface. Then we continue this part with the presentation of the transport equation that gives a formula for the time derivative of a spatial integral $\int_{\Gamma(t)} f$ in geometric terms. Finally we use the transport equation to calculate the evolution of area and volume in an abstract setting that is adapted to the geometry of the evolution equations that are considered in later parts of this work. We will apply these formulas in Chapter 3 and extend them for the evolution equations for three evolving hypersurfaces in Chapter 4.

In the third Chapter we consider the situation in which one evolving hypersurface Γ stays inside a fixed bounded region Ω, fulfills the boundary conditions (1.4) and (1.5) at the outer boundary and evolves due to different area decreasing evolution laws. We give the used parametrization that will lead to partial differential equations for functions defined on a fixed stationary reference hypersurface Γ^*. Then we consider the mean curvature flow with boundary condition (1.4) and linearize the resulting equations, which in particular involves the linearization of mean curvature and the 90° angle condition at the outer boundary. This will lead to the following equations

$$\left\{\begin{array}{rcll} \partial_t\rho & = & \Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho & \text{in } \Gamma^*\,, \\ 0 & = & \partial_\mu\rho - S(n^*,n^*)\rho & \text{on } \partial\Gamma^*\cap\partial\Omega\,. \end{array}\right. \tag{1.13}$$

Here Δ_{Γ^*} is the Laplace-Beltrami operator on Γ^*, σ^* is the second fundamental form on Γ^* with respect to a chosen normal n^*, $|\sigma^*|^2$ is the sum of the squared principal curvatures of Γ^*, μ is the outer unit normal of Ω, $\partial_\mu\rho$ is the directional derivative of ρ in direction of μ and S is the second fundamental form on $\partial\Omega$ with respect to $(-\mu)$. We remark that the right side of these equations is also derived and examined with respect to stability in a time independent formulation in the papers of Barbosa and doCarmo [BdoC84], Ros and Souam [RS97] and Vogel [Vog00] by considering the second variation of the area functional. The reason that we regard these equations is the desire to adapt the notion of the later Section 3.4, which is a generalization of the work of Garcke, Ito and Kohsaka [GIK05], also to this case of mean curvature flow and to have therefore a common description and derivation for linearized stability of a larger class of evolution equations. The approach to get an asymptotic stability criterion for the linearized equation (1.13) was summarized above. We also consider results for the volume preserving mean curvature flow, which we obtain by similar methods. The arising linear equations for surface diffusion flow with boundary conditions (1.4) and (1.5) are given by

$$\left\{\begin{array}{rcll} \partial_t\rho &=& -\Delta_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho+|\sigma^*|^2\rho\right) & \text{in } \Gamma^*\,, \\ 0 &=& \partial_\mu\rho - S(n^*,n^*)\rho & \text{on } \partial\Gamma^*\cap\partial\Omega\,, \\ 0 &=& \nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho+|\sigma^*|^2\rho\right)\cdot\mu & \text{on } \partial\Gamma^*\cap\partial\Omega\,. \end{array}\right. \tag{1.14}$$

By using the approach as described above we get the following stability result.

$$\boxed{\begin{array}{l} \text{The zero solution of (1.14) is asymptotically stable} \\ \Longleftrightarrow \left\{\begin{array}{l} I(\rho,\rho) := \int_{\Gamma^*}\left(|\nabla_{\Gamma^*}\rho|^2-|\sigma^*|^2\rho^2\right) - \int_{\partial\Gamma^*} S(n^*,n^*)\rho^2 \\ \text{is positive for all } \rho\in H^{1,2}(\Gamma^*)\backslash\{0\} \text{ with } \int_{\Gamma^*}\rho = 0\,. \end{array}\right. \end{array}}$$

The last two parts of this chapter consist of some remarks concerning the nonlinear stability of the considered surface diffusion problem and examples for explicit situations where we examine the linearized stability.

In the fourth Chapter we consider the situation in which three evolving hypersurfaces Γ_i stay inside a fixed bounded region Ω, meet each other at a triple line inside of Ω and fulfill the boundary conditions (1.4) and (1.5) at the outer boundary and (1.8)-(1.10) at the triple line. In Section 4.1 we consider the mean curvature flow with outer boundary contact. In detail we regard three evolving hypersurfaces that meet each other at a triple line, evolve due to the weighted mean curvature flow (1.6) and fulfill the angle condition (1.8) at the triple line and the right angle condition (1.4) at the three outer boundary parts. Here we use a parametrization that is more explicit near the triple line than in the previous chapter. More precisely, near the triple line we use a mapping depending on two parameters where one is responsible for a normal direction and the other one for a tangential movement. This gives us eventually the possibility to rewrite the geometric evolution law as a system of partial differential equations for functions ρ_i and μ_i defined on fixed stationary reference hypersurfaces Γ_i^*, that meet each other at a triple line L^* and touch the outer boundary at a right angle at S_i^*. The linearization of these equations leads to the following linear problem.

$$\left\{\begin{array}{ll} \partial_t\rho_i = \gamma_i\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right) & \text{in } \Gamma_i^*\,, \\ 0 = \left(\partial_\mu - S(n_i^*,n_i^*)\right)\rho_i & \text{on } S_i^*\,, \\ 0 = \gamma_1\rho_1+\gamma_2\rho_2+\gamma_3\rho_3 & \text{on } L^*\,, \\ \left(\nabla_{\Gamma_i^*}\rho_i\cdot n_{\partial\Gamma_i^*}\right)+a_i\rho_i = \left(\nabla_{\Gamma_j^*}\rho_j\cdot n_{\partial\Gamma_j^*}\right)+a_j\rho_j & \text{on } L^*\,, \end{array}\right. \tag{1.15}$$

where $i=1,2,3$ in the first and second line, $(i,j)=(1,2),(2,3)$ in the third line and where the a_i are defined in (4.35)-(4.37). Stability analysis with the help of spectral theory gives here the condition

The zero solution of (1.15) is asymptotically stable

$$\iff \left\{ \begin{array}{l} I(\rho,\rho) := \sum_{i=1}^3 \gamma_i \int_{\Gamma_i^*} \left(|\nabla_{\Gamma_i^*}\rho_i|^2 - |\sigma_i^*|^2 \rho_i^2 \right) - \sum_{i=1}^3 \gamma_i \int_{S_i^*} S(n_i^*, n_i^*)\, \rho_i^2 \\ \qquad\qquad + \sum_{i=1}^3 \gamma_i \int_{L^*} a_i\, \rho_i^2 \\ \text{ is positive for all } 0 \neq \rho = (\rho_1,\rho_2,\rho_3) \text{ with } \rho_i \in H^1(\Gamma_i^*) \\ \text{ and } \gamma_1\rho_1 + \gamma_2\rho_2 + \gamma_3\rho_3 = 0 \text{ at } L^* . \end{array} \right.$$

In Section 4.2 we consider finally the weighted surface diffusion flow (1.7) with outer boundary contact. We use the same parametrization as in Section 4.1 and get thereby equations for functions ρ_i and μ_i whose linearization lead to the following linear problem in Γ_i^*

$$\partial_t \rho_i = -m_i\gamma_i \Delta_{\Gamma_i^*} \left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2 \rho_i \right) \tag{1.16}$$

for $i = 1,2,3$ with the following boundary conditions at the outer boundary $\Gamma_i^* \cap \partial\Omega$

$$\left\{ \begin{array}{l} 0 = \partial_\mu \rho_i - S(n_i^*, n_i^*)\rho_i\,, \\ 0 = \nabla_{\Gamma_i^*} \left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2 \rho_i \right) \cdot \mu\,, \end{array} \right. \tag{1.17}$$

for $i = 1,2,3$ and the following boundary conditions at the triple line L^*

$$\left\{ \begin{array}{l} 0 = \gamma_1\rho_1 + \gamma_2\rho_2 + \gamma_3\rho_3\,, \\ \left(\nabla_{\Gamma_i^*}\rho_i \cdot n_{\partial\Gamma_i^*} \right) + a_i\rho_i = \left(\nabla_{\Gamma_j^*}\rho_j \cdot n_{\partial\Gamma_j^*} \right) + a_j\rho_j\,, \\ 0 = \sum_{i=1}^3 \gamma_i \left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i \right)\,, \\ m_i\gamma_i \left(\nabla_{\Gamma_i^*} \left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i \right) \cdot n_{\partial\Gamma_i^*} \right) = m_j\gamma_j \left(\nabla_{\Gamma_j^*} \left(\Delta_{\Gamma_j^*}\rho_j + |\sigma_j^*|^2\rho_j \right) \cdot n_{\partial\Gamma_j^*} \right)\,, \end{array} \right. \tag{1.18}$$

where $(i,j) = (1,2)$ and $(2,3)$ in the second and fourth line. We proceed with stability analysis as prescribed above and get the following result, which is a direct generalization of [GIK10] to the higher dimensional case, as expected.

The zero solution of (1.16)-(1.18) is asymptotically stable

$$\iff \left\{ \begin{array}{l} I(\rho,\rho) \;:=\; \sum_{i=1}^3 \gamma_i \int_{\Gamma_i^*} \left(|\nabla_{\Gamma_i^*}\rho_i|^2 - |\sigma_i^*|^2 \rho_i^2 \right) - \sum_{i=1}^3 \gamma_i \int_{S_i^*} S(n_i^*, n_i^*)\, \rho_i^2 \\ \qquad\qquad\qquad + \sum_{i=1}^3 \gamma_i \int_{L^*} a_i\, \rho_i^2 \\ \text{is positive for all } 0 \neq \rho = (\rho_1,\rho_2,\rho_3) \text{ with } \rho_i \in H^1(\Gamma_i^*) \text{ such that} \\ \int_{\Gamma_i^*} \rho_i = \int_{\Gamma_j^*} \rho_j \text{ for } (i,j) = (1,2),(2,3) \text{ and } \gamma_1\rho_1 + \gamma_2\rho_2 + \gamma_3\rho_3 = 0 \text{ on } L^* . \end{array} \right.$$

We remark that the corresponding bilinear form without the integrals over the outer boundary parts S_i^* also arises in the proof of the double bubble conjecture from Hutchings, Morgan, Ritoré and Ros [HMRR02].

At last we give in the appendix detailed proofs for the normal time derivative of mean curvature and the unit normal and mention some facts about the vector product in $\mathbb{R}^{n+1}$, which is used in the text to describe the arising unit normals for the linearization.

Acknowledgement

I want to thank everyone who had an influence somehow on finishing this work. First of all, I want to thank my supervisor Harald Garcke for introducing me to the interesting field of geometric evolution equations and for providing me with a clear formulated and simultaneously challenging topic for my dissertation. Then I want to thank Yoshihito Kohsaka from Muroran Institute of Technology, with whom I discussed several topics related to this work, including a local existence result for mean curvature flow with triple lines. I also want to mention Martin Butz and thank him for the careful reading of a preliminary version of this dissertation and for his suggestions to improve it.

Finally I want to thank my family including my wife and my two sons, who showed me that there is also some life different from the mathematical world but at the same time gave me enough space to finish this work.

Chapter 2

Facts about Hypersurfaces

In this chapter we will introduce our notation for hypersurfaces Γ, the geometric curvature quantities on Γ and the relevant differential operators up to the Gauß' theorem on hypersurfaces. This means we gather together facts from textbooks as for example Amann and Escher [AE09], Bröcker [Broe92], Jänich [Jae01] and Spivak [Spi65]. For the differential geometry we refer to Kühnel [Kue06] and Eschenburg and Jost [EJ07], although there are lots of other excellent written books on this subject.

We will also explain carefully the concept of evolving hypersurfaces, which are, roughly speaking, hypersurfaces that move in time. In this case, special attention has to be given to the time derivative, which we introduce as normal time derivative in the sense of Gurtin [Gur93]. We also consider evolving hypersurfaces as a graph over some fixed reference hypersurface Γ^*. Later on, these will be the solutions of the geometric evolution equations that we consider, and Γ^* will be a stationary solution.

Finally we formulate the Transport theorem which is a formula for the time derivative of some integrated function $\frac{d}{dt}\int_{\Gamma(t)} f(t,p)$. With the help of this formula we give equations for the evolution of area and volume for evolving hypersurfaces that lie inside a fixed bounded region $\Omega \subset \mathbb{R}^{n+1}$ and meet the boundary $\partial\Omega$ with a right angle. These evolutions will lead to a better understanding of some geometric properties of the considered evolution equations.

2.1 Differential operators and curvature terms

For the convenience of the reader we want to introduce in this first section basic terms concerning hypersurfaces in $\mathbb{R}^{n+1}$. These include our definition of hypersurfaces with boundary, the differential operators surface gradient, surface divergence and Laplace-Beltrami operator and the first and second fundamental form. We also introduce important curvature terms as normal curvature and mean curvature for later use. Finally we give a version of Gauß' theorem on hypersurfaces with nonempty boundary involving a curvature term.

Definition 2.1 (Regular submanifold)**.** *Let M be a subset of $\mathbb{R}^n$. M is called a* **regular submanifold** *of dimension m with $1 \leq m \leq n$, iff for every $p \in M$ there is an open neighbourhood*

V of p in $\mathbb{R}^n$, an open subset $U \subset \mathbb{R}^m$ and a smooth mapping

$$\gamma : U \longrightarrow \mathbb{R}^n$$

such that

(*i*) *$\gamma(U) = M \cap V$ and $\gamma : U \to \gamma(U)$ is a homeomorphism and*

(*ii*) *the Jacobian matrix $D\gamma(u) : \mathbb{R}^m \to \mathbb{R}^n$ has rank m (i.e. full rank) for all $u \in U$.*

(U, γ, V) is called a **local representation** *of M around p.*

We give a remark about other possibilities to describe a submanifold locally.

Remark 2.2. *Additionally to the local parametrization there are equivalent formulations for the local description of a submanifold as a graph, as a zero-level set or with the help of diffeomorphisms from subsets in $\mathbb{R}^{n+1}$ to subsets in $\mathbb{R}^m \times \{0\}$. In most of the following explanations the characterization with a local parametrization will suffice, but we use also the other possibilities whenever necessary. In particular, it will be convenient to use diffeomorphisms to show the correctness of the Definitions 2.3 and 2.7 of differentiability and the differential, the zero-level set description in the Definition 2.32 of normal velocity and the graph representation in the calculation 5.1 of normal time derivative of mean curvature in the appendix.*

Differentiability properties of mappings having their domain of definition respectively their range on hypersurfaces will always be defined on the euclidian space with the help of a local parametrization.

Definition 2.3 (Differentiability)**.**

(*i*) *A mapping*

$$f : M_1 \longrightarrow M_2\,,$$

where $M_1 \subset \mathbb{R}^{n_1}, M_2 \subset \mathbb{R}^{n_2}$ are regular submanifolds, is called ***smooth****, iff for every local parametrization (U, γ, V) of M_1 the composition*

$$f \circ \gamma : U \longrightarrow \mathbb{R}^{n_2}$$

is smooth.

(*ii*) *The same definition as in (i) applies to mappings with range in an euclidian space*

$$f : M \longrightarrow \mathbb{R}^l\,,$$

where $M \subset \mathbb{R}^n$ is a regular submanifold.

In the next definition we formulate exactly our use of the term hypersurface, which in particular includes the possibility of a nonempty boundary.

Definition 2.4 (Regular hypersurface)**.** *In this work, $\Gamma \subset \mathbb{R}^{n+1}$ with $n \geq 2$ is called a* **regular hypersurface***, if Γ is a regular submanifold of dimension n, connected and closed as a subset of $\mathbb{R}^{n+1}$, orientable and the boundary $\partial\Gamma$ of Γ is either empty, $\partial\Gamma = \emptyset$, or is a regular submanifold of dimension $n-1$, such that Γ lies on one side of the boundary.*
Analytically, this means that around every point $p \in \partial\Gamma$ there exists an open neighbourhood $U \subset \mathbb{R}^{n+1}$, an open set $V \subset \mathbb{R}^{n+1}$ and a diffeomorphism $\varphi : U \to V$ such that

$$\varphi(U \cap \Gamma) = V \cap \left(\mathbb{R}^n_+ \times \{0\}\right) \quad \textit{with} \quad \left(\varphi(p)\right)_n = 0\,,$$

where $\left(\varphi(p)\right)_n$ is the n-th coordinate of $\varphi(p)$. In particular we want to remark the fact that here the boundary of Γ belongs to Γ, i.e. $\partial\Gamma \subset \Gamma$.
From now on, we will call such a Γ simply **hypersurface***.*

As an important concept we define the linearization of a hypersurface, called the tangent space.

Definition 2.5 (Tangent space)**.** *Let Γ be a hypersurface and fix $p \in \Gamma$. The* **tangent space** *$T_p\Gamma$ of Γ at p is then defined as*

$$\begin{aligned} T_p\Gamma \;=\; \{v \in \mathbb{R}^{n+1} \mid & \textit{ There exists a smooth curve } c : I \to \Gamma \textit{ with } c(0) = p, c'(0) = v, \\ & \textit{where } I = (-\varepsilon, \varepsilon),\ I = [0, \varepsilon) \textit{ or } I = (-\varepsilon, 0]\}\ . \end{aligned}$$

The halfopen intervals I in the above definition make sure that even for points on the boundary $p \in \partial\Gamma$ the tangent space is a subspace, which is summarized in the next remark. Although we skip the details here, we want to note that for $v \in T_p\Gamma$ with $p \in \partial\Gamma$ the following construction also yields $-v \in T_p\Gamma$. Indeed, let $c : [0, \varepsilon) \to \Gamma$ be a curve with $c(0) = p$ and $c'(0) = v$. Then the curve $\alpha : (-\varepsilon, 0] \to \Gamma$, $\alpha(\tau) := c(-\tau)$ fulfills $\alpha(0) = c(0) = p$ and $\alpha'(0) = -c'(0) = -v$ and therefore $-v \in T_p\Gamma$. That's the reason why both intervals $[0, \varepsilon)$ and $(-\varepsilon, 0]$ appear in the above definition of the tangent space.

Remark 2.6. *If (U, γ, V) is a local parametrization of Γ around p with $u = \gamma^{-1}(p)$, then one can show that*

$$T_p\Gamma \;=\; D\gamma(u)\left(\mathbb{R}^n\right)\ ,$$

or in another notation with $(e_1, \ldots, e_n)$ the standard basis of $\mathbb{R}^n$

$$\begin{aligned} T_p\Gamma \;&=\; \mathit{span}\left(D\gamma(u)(e_1), \ldots, D\gamma(u)(e_n)\right) \\ &=\; \mathit{span}\left(\frac{\partial\gamma}{\partial u_1}(u), \ldots, \frac{\partial\gamma}{\partial u_n}(u)\right) \\ &=\; \mathit{span}\left(\partial_1\gamma(u), \ldots, \partial_n\gamma(u)\right)\ . \end{aligned}$$

This means in particular that $T_p\Gamma$ is an n-dimensional subspace of $\mathbb{R}^{n+1}$. Here appears a slight abuse of notation, since for points $p \in \partial\Gamma$ the parametrization γ is not yet defined. In this case we consider the inverse of a diffeomorphism φ from Definition 2.4 and restrict it to the first n variables to get a natural parametrization at the boundary through $\gamma = \varphi^{-1}\big|_{\mathbb{R}^n}$.

Now we can introduce the differential of a mapping between hypersurfaces.

Definition 2.7 (Differential)**.** *Let $f : \Gamma_1 \to \Gamma_2$ be a smooth mapping between hypersurfaces. The* **differential** *of f at $p \in \Gamma_1$ is defined as the mapping*

$$d_p f : T_p\Gamma_1 \longrightarrow T_{f(p)}\Gamma_2$$

through the following rule:
To $v \in T_p\Gamma_1$ choose a smooth curve $c : (-\varepsilon, \varepsilon) \to \Gamma_1$ with $c(0) = p$, $c'(0) = v$ and define

$$d_p f(v) \quad := \quad \frac{d}{d\tau}(f \circ c)(\tau)\bigg|_{\tau=0} \in T_{f(p)}\Gamma_2 \ .$$

Analogously we define the **differential** *of f at p in Γ for a mapping $f : \Gamma \to \mathbb{R}^l$. Then we have*

$$d_p f : T_p\Gamma \longrightarrow \mathbb{R}^l \ .$$

We summarize some important properties of the differential in the next remark.

Remark 2.8. *One can show that the differential is independent of the curve, is a linear mapping between the tangent spaces and that there is a chain rule. More precisely, for mappings between hypersurfaces $f : \Gamma_1 \to \Gamma_2$ and $g : \Gamma_2 \to \Gamma_3$ it holds that*

$$d_p(g \circ f) \quad = \quad d_{f(p)}g \circ d_p f \ .$$

A proof of these basic statements is best done with the help of a local description of the hypersurfaces with diffeomorphisms and will be skipped here. □

In the next definition we introduce the directional derivative for arbitrary mappings and not just for tangent vector fields as is done in lots of textbooks.

Definition 2.9 (Directional derivative)**.** *For a mapping $f : \Gamma \to \mathbb{R}^l$ we define the* **directional derivative** *of f in direction of $v \in T_p\Gamma$ through*

$$\partial_v f \quad := \quad d_p f(v) \in T_{f(p)}\mathbb{R}^l = \mathbb{R}^l \ .$$

For a tangent vector field $v(p)$, that is a mapping $v : \Gamma \to \mathbb{R}^{n+1}$ with $v(p) \in T_p\Gamma$, we call the mapping

$$\partial_v f : \Gamma \longrightarrow \mathbb{R}^l \ , \quad \partial_v f(p) := \partial_{v(p)} f$$

also the **directional derivative** *of f in direction of v.*

To do some geometry on a hypersurface Γ, for example measuring the length of curves or the angle between curves respectively tangent vectors, the concept of a Riemannian metric is crucial. Since we have to consider pullback metrics in Lemma 3.26 and in Lemma 3.27, we introduce this notion arbitrarily and not just as a restriction of the euclidian inner product on $\mathbb{R}^{n+1}$.

Definition 2.10 (Riemannian metric, first fundamental form)**.**

(*i*) *Let $p \mapsto g_p$ be a mapping, where*

$$g_p \quad : \quad T_p\Gamma \times T_p\Gamma \longrightarrow \mathbb{R} \tag{2.1}$$

is an inner product on $T_p\Gamma$, i.e. g_p is a symmetric, positive definite bilinear form on $T_p\Gamma$. If additionally g_p depends smoothly on its basis point p, we call g a ***Riemannian metric****. Here the smooth dependence of g_p on p means that the representation g_{ij} given below is smooth with respect to every parametrization. For a local parametrization (U, γ, V) of Γ around $p \in \Gamma$ with $\gamma(u) = p$ we say that*

$$g_{ij}(u) \quad := \quad g_p(\frac{\partial \gamma}{\partial u_i}(u), \frac{\partial \gamma}{\partial u_j}(u))$$

for $i, j = 1, \ldots, n$ is the ***representation*** *or* ***matrix representation*** *of g_p.*

(*ii*) *If the Riemannian metric g is simply the restriction of the euclidian inner product $(.,.)$ on $\mathbb{R}^{n+1}$ to $T_p\Gamma$, i.e.*

$$g_p \quad := \quad (.,.)|_{T_p\Gamma \times T_p\Gamma} \ , \tag{2.2}$$

we call g also the ***first fundamental form*** *of Γ. In this case the representation with respect to a local parametrization as above is given through*

$$g_{ij}(u) \quad = \quad \left(\frac{\partial \gamma}{\partial u_i}(u), \frac{\partial \gamma}{\partial u_j}(u)\right) = \frac{\partial \gamma}{\partial u_i}(u) \cdot \frac{\partial \gamma}{\partial u_j}(u) \, ,$$

where we often replace the brackets by a dot between the vectors.

Remark 2.11. *Actually, the mapping g_p from point (ii) of the above Definition 2.10 can be defined more generally for arbitrary smooth manifolds M without the surrounding space $\mathbb{R}^{n+1}$. g is then also called* ***Riemannian metric*** *and the pair (M, g) is a* ***Riemannian manifold****. But we will always use hypersurfaces lying in $\mathbb{R}^{n+1}$ and therefore our Definition 2.10 is justified. The case of an inner product different than the euclidian one is important for the concept of compatibility of mappings with the metric, which will be defined below. These mappings have nice properties concerning the transformation of differential operators, which will become important later.*

Definition 2.12 ((local) isometry)**.** *A smooth mapping $f : \Gamma \to \tilde{\Gamma}$ between hypersurfaces (Γ, g) and $(\tilde{\Gamma}, \tilde{g})$ is called a* ***local isometry****, if for all $p \in \Gamma$, $v, w \in T_p\Gamma$*

$$\tilde{g}_{f(p)}\left(d_p f(v), d_p f(w)\right) \quad = \quad g_p(v, w) \ .$$

If f is additionally a diffeomorphism, that is f is bijective and the inverse f^{-1} is smooth, we call it an ***isometry****.*

With the help of the above new scalar product on the tangent spaces we define some geometric quantities as promised.

Definition 2.13 (Length, angle, orthonormal moving frame). *For a tangent vector $v \in T_p\Gamma$ we set its length through*

$$\|v\|^2 \;:=\; g_p(v,v)$$

and the angle between two tangent vectors $v, w \in T_p\Gamma$ through

$$\vartheta \;:=\; \arccos\left(\frac{g_p(v,w)}{\|v\|\|w\|}\right) \in [0,\pi]\,.$$

We will often use an **orthonormal basis** *$v_1,\dots,v_n$ of the tangent space $T_p\Gamma$. This means that $v_1,\dots,v_n$ is a basis of the n-dimensional subspace $T_p\Gamma \subset \mathbb{R}^{n+1}$, and that $g_p(v_i,v_j) = \delta_{ij}$ for $i,j = 1,\dots,n$, where δ_{ij} denotes the Kronecker-symbol.*
Tangent vector fields $v_i : \Gamma \to \mathbb{R}^{n+1}$, $v_i(p) \in T_p\Gamma$ for all $p \in \Gamma$, such that for every $p \in \Gamma$ the vectors $v_1(p),\dots,v_n(p)$ form an orthonormal basis of $T_p\Gamma$, will be called an **orthonormal moving frame** *of Γ.*

Guided by experience from curves, one expects that curvature of hypersurfaces results from a change of the tangent space, respectively its one-dimensional orthogonal complement. This leads to the following definitions of the Gauß mapping and its differential, the so-called shape operator.

Definition 2.14 (Gauß mapping). *Since we assumed that our hypersurfaces Γ are oriented, there exists a smooth normal n on Γ, called the* ***Gauß mapping***

$$n : \Gamma \longrightarrow S^n\,,$$

where S^n is the n-dimensional sphere in $\mathbb{R}^{n+1}$, such that $g_p(n(p),v) = 0$ for all $v \in T_p\Gamma$ and $\|n(p)\| = 1$ for all $p \in \Gamma$. We call $N_p\Gamma = (T_p\Gamma)^\perp = \{w \in \mathbb{R}^{n+1} \,|\, g_p(w,v) = 0\}$ the **normal space** *of Γ.*

Note that $T_{n(p)}S^n = n(p)^\perp = T_p\Gamma$, therefore we can regard the differential of the Gauß mapping n as an endomorphism.

Definition 2.15 (Shape operator). *The* **shape operator** *W_p, also called the* **Weingarten map***, is defined with the help of the differential of the Gauß mapping through*

$$W_p : T_p\Gamma \longrightarrow T_p\Gamma\,, \quad W_p(v) := -d_p n(v)\,.$$

With the help of a local parametrization one can see that W_p is a self-adjoint endomorphism with respect to the first fundamental form, which means

$$g_p(v, W_p(w)) \;=\; g_p(W_p(v), w) \;\; \textit{for all } v,w \in T_p\Gamma\,.$$

Before we proceed with our differential geometric notations, we want to mention an important distinction between **inner geometry** and **outer geometry** on the hypersurface.

The first one means terms that can be derived just with knowledge of the hypersurface as for example the Riemannian metric, the differential and the tangent space. The second one describes expressions, for which the knowledge of the surrounding space, $\mathbb{R}^{n+1}$ in our case, is essential. In this class we have the Gauß mapping and the so-called second fundamental form, which describes the change of the tangent space $T_p\Gamma$ in dependence of p and therefore contains some curvature information.

Definition 2.16 (Second fundamental form)**.** *The* **second fundamental form** σ_p *of the hypersurface* Γ *at* $p \in \Gamma$ *is defined as the related bilinear form with respect to* g_p *of the shape operator* W_p*, that is*

$$\sigma_p : T_p\Gamma \times T_p\Gamma \longrightarrow \mathbb{R}\,, \quad \sigma_p(v,w) := g_p(W_p(v),w) = -g_p(d_pn(v),w)\;.$$

The ***representation*** *or* ***matrix representation*** *of the second fundamental form* σ_p *with respect to the basis* $\frac{\partial\gamma}{\partial u_1}(u),\ldots,\frac{\partial\gamma}{\partial u_n}(u)$*, where* (U,γ,V) *is a local representation of* Γ *around* $p \in \Gamma$ *with* $\gamma(u) = p$*, is given by*

$$\begin{aligned} h_{ij}(u) &:= \sigma_p\left(\frac{\partial\gamma}{\partial u_i}(u), \frac{\partial\gamma}{\partial u_j}(u)\right) \\ &= g_p\left(W_p(\frac{\partial\gamma}{\partial u_i}(u)), \frac{\partial\gamma}{\partial u_j}(u)\right)\;. \end{aligned}$$

In case of g_p *being the restriction of the euclidian scalar product this representation equals*

$$\begin{aligned} h_{ij}(u) &= W_p(\frac{\partial\gamma}{\partial u_i}(u)) \cdot \frac{\partial\gamma}{\partial u_j}(u) = -d_pn(\frac{\partial\gamma}{\partial u_i}(u)) \cdot \frac{\partial\gamma}{\partial u_j}(u) \\ &= -\frac{\partial}{\partial u_i}(n\circ\gamma)(u) \cdot \frac{\partial\gamma}{\partial u_j}(u) = \frac{\partial}{\partial u_i}\underbrace{\left((n\circ\gamma)(u)\cdot\frac{\partial\gamma}{\partial u_j}(u)\right)}_{=0} + n(\gamma(u)) \cdot \frac{\partial^2\gamma}{\partial u_j \partial u_i}(u) \\ &= n(\gamma(u)) \cdot \frac{\partial^2\gamma}{\partial u_j \partial u_i}(u)\,. \end{aligned}$$

The next step is to introduce the basic curvature terms that will be needed.

Definition 2.17 (Normal curvature)**.** *For a tangent vector* $v \in T_p\Gamma$ *with length* $\|v\|^2 = g_p(v,v) = 1$ *we define the* **normal curvature** κ_v *of* Γ *in direction* v *at* p *through*

$$\kappa_v(p) \;=\; \sigma_p(v,v)\;.$$

Remark 2.18. *If* g_p *is the restriction of the euclidian scalar product, then one can show that the normal curvature of* Γ *in direction of a unit tangent vector* $v \in T_p\Gamma$ *at* p *is the curvature of the arclength-parametrized curve* c*, which arises from the intersection of* Γ *and the plane spanned by* v *and* $n(p)$*, so that the name is justified.*

Proof. Let c be this arclength-parametrized curve, which lies in the plane E spanned by v and $n(p)$ with $c(0) = p$ and $c'(0) = v$. Then of course $c''(0)$ also lies in the plane E and due to the arclength-parametrization $c''(0) \cdot c'(0) = \frac{d}{dt}\frac{1}{2}\left(c'(t) \cdot c'(t)\right)\big|_{t=0} = 0$. This means that $c''(0)$ has no tangential part and can be given as $c''(0) = (c''(0))^{\perp} = (c''(0) \cdot n(p))\, n(p)$. A further calculation shows then

$$c''(t) \cdot n(c(t)) = -c'(t) \cdot \frac{d}{dt} n(c(t)) = -c'(t) \cdot d_{c(t)} n(c'(t)) = c'(t) \cdot W_{c(t)}(c'(t)) = \sigma_{c(t)}(c'(t), c'(t))$$

and therefore for $t = 0$

$$\big(c''(0) \cdot n(p)\big) = \sigma_p(v, v)\,.$$

Since $c''(0) \cdot n(p)$ is the curvature of the plain curve, we get the claim. $\square$

Because we know that the shape operator W_p is self-adjoint, there exists an orthonormal basis of eigenvectors, so that we can give the following definition.

Definition 2.19 (Principal curvatures, Gauß curvature, mean curvature)**.** *Let $v_1, \ldots, v_n$ be an orthonormal basis of $T_p\Gamma$ consisting of eigenvectors of W_p. The normal curvatures of Γ in direction of v_i at p are called the* **principal curvatures** *κ_i of Γ at p, that is*

$$\kappa_i(p) \quad = \quad \sigma_p(v_i, v_i)\,.$$

So the principal curvatures are defined as the eigenvalues of the shape operator W_p.
The **Gauß curvature** *K of Γ at p is then introduced as the determinant of W_p,*

$$K(p) \quad = \quad \det(W_p) \quad = \quad \kappa_1 \cdot \ldots \cdot \kappa_n\,.$$

Another important quantity is the **mean curvature** *H of Γ at p as the trace of W_p,*

$$H(p) \quad = \quad \mathrm{trace}(W_p) \quad = \quad \kappa_1 + \ldots + \kappa_n\,.$$

We will also need the **mean curvature vector** *$\vec{H}$ defined as*

$$\vec{H}(p) \quad = \quad H(p)\, n(p)\,,$$

so that $\vec{H}$ is a normal field.

Example 2.20. *To illustrate our sign convention for the mean curvature, which is different from book to book, we calculate H for the sphere $S^n = \{p \in \mathbb{R}^{n+1} \,|\, \|p\| = 1\}$ with unit outer normal $n(p) = p$. Since this is the restriction of the identity, we can derive $d_p n = Id$ for the differential to get*

$$W_p(v) = -d_p n(v) = -v\,.$$

This means that W_p equals $-Id$ and has n eigenvalues -1. So we get $H(p) \equiv -n$ for the sphere with unit normal pointing outside the unit ball.

Next we want to introduce some differential operators on hypersurfaces. In generalization to the usual gradient of a function we define the surface gradient.

Definition 2.21 (Surface gradient). *For a smooth function $f : \Gamma \to \mathbb{R}$ the* **surface gradient** *$\nabla_\Gamma f$ at a point $p \in \Gamma$ is defined through*

$$\nabla_\Gamma f(p) \quad := \quad \sum_{i=1}^{n} (\partial_{v_i} f)\, v_i \ \in T_p\Gamma \ ,$$

where $v_1, \ldots, v_n$ is an orthonormal basis of $T_p\Gamma$. In particular this means $g_p(v_i, v_j) = \delta_{ij}$ and the dependence of the surface gradient on the metric becomes apparent.

In the next remark we give some useful descriptions of the surface gradient.

Remark 2.22. *Equivalent to the above definition one could also define the surface gradient $\nabla_\Gamma f(p)$ as the unique vector $v(p) \in T_p\Gamma$, such that*

$$d_p f(w) \quad = \quad g_p(v(p), w) \quad \textit{for all } w \in T_p\Gamma \ .$$

With Definition 2.9 of the directional derivative we also have

$$g_p(\nabla_\Gamma f(p), w) \quad = \quad \partial_w f \quad \textit{for all } w \in T_p\Gamma \ .$$

In a local parametrization (U, γ, V) with $\gamma(u) = p$ of Γ around p there is the following representation

$$\nabla_\Gamma f(p) \quad = \quad \sum_{i,j=1}^{n} g^{ij}(u)\, \partial_i (f \circ \gamma)(u)\, \partial_j \gamma(u) \ ,$$

where $\left(g^{ij}(u)\right)_{ij}$ is the inverse of the matrix $(g_{ij}(u))_{ij}$.

If we can extend $f : \Gamma \to \mathbb{R}$ to an open neighbourhood of Γ and g_p is the restriction of the euclidian scalar product, then the following formula involving the usual gradient ∇ on $\mathbb{R}^{n+1}$

$$\nabla_\Gamma f(p) \quad = \quad (\nabla f(p))^T = \nabla f(p) - (\nabla f(p) \cdot n(p))\; n(p)\, ,$$

is true. Here, $()^T$ is the orthogonal projection onto $T_p\Gamma$. One could also use the above formula with an arbitrary extension of f as definition for the surface gradient and observe that it depends only on values of f on the hypersurface.

Next we define the surface divergence. For hypersurfaces (Γ, g) equipped with an arbitrary Riemannian metric we need therefor the notion of covariant derivative $\nabla_w v$ of a tangent vector field in direction of $w \in T_p\Gamma$. If g_p is the restriction of the euclidian scalar product, the covariant derivative reduces to orthogonal projection of the directional derivative onto the tangent space, that is $\nabla_w v = (\partial_w v)^T \in T_p\Gamma$.

Definition 2.23 (Surface divergence)**.** *For a smooth tangent vector field $f : \Gamma \to \mathbb{R}^{n+1}$ (which means $f(p) \in T_p\Gamma$) on an arbitrary Riemannian hypersurface (Γ_g) we define the* **surface divergence** *of f on Γ through*

$$\mathrm{div}_\Gamma f(p) \quad := \quad \nabla_\Gamma \cdot f(p) \quad := \quad \sum_{i=1}^{n} g_p\left(\nabla_{v_i} f(p), v_i\right) \ ,$$

where $v_1, \ldots, v_n$ is an orthonormal basis of $T_p\Gamma$. If g_p is the restriction of the euclidian scalar product, this definition reduces to

$$\mathrm{div}_\Gamma f(p) = \sum_{i=1}^{n} (\partial_{v_i} f)^T \cdot v_i = \sum_{i=1}^{n} \partial_{v_i} f \cdot v_i \ .$$

The last line makes sense also for nontangent vector fields, i.e. arbitrary smooth mappings $f : \Gamma \to \mathbb{R}^{n+1}$. This notion will be used in Theorem 2.29, the so-called Gauß' theorem on hypersurfaces. We remark that even if we consider tangent vector fields, the tangential part $(\partial_{v_i} f)^T$ from the definition does not equal the directional derivative $\partial_{v_i} f$, in general.

As we did for the surface gradient we give some useful descriptions.

Remark 2.24. *If g_p is the restriction of the euclidian scalar product and (U, γ, V) is a local parametrization of Γ around p with $\gamma(u) = p$, it holds*

$$\nabla_\Gamma \cdot f(p) \quad = \quad \sum_{i,j=1}^{n} g^{ij}(u)\ \left(\partial_i (f \circ \gamma)(u) \cdot \partial_j \gamma(u)\right) \ .$$

If we can additionally extend $f : \Gamma \to \mathbb{R}^{n+1}$ to an open neighbourhood of Γ, we have the formula as above for the surface gradient of the components $f = (f_1, \ldots, f_{n+1})$ given by

$$\begin{aligned} \nabla_\Gamma f_i(p) \quad &= \quad \nabla f_i(p) - (\nabla f_i(p), n(p))\ n(p) \\ &=: \quad \Big(\underline{D_1} f_i(p), \ldots, \underline{D_{n+1}} f_i(p)\Big) \ . \end{aligned}$$

With this notation we can write

$$\nabla_\Gamma \cdot f(p) \quad = \quad \sum_{i=1}^{n+1} \underline{D_i} f_i(p) \ ,$$

so there is a similar appearance as for the usual divergence $\nabla \cdot f = \sum_i \partial_i f_i$ in euclidian space. As in the case of the surface gradient, one could also use the above formula with an arbitrary extension of f as definition for the surface divergence and observe that it depends only on values of f on the hypersurface.

Now we want to define the Laplace-Beltrami operator, which is an extension of the usual Laplace operator $\sum_i \partial_{ii}$ to hypersurfaces and will be needed for surface diffusion in later sections.

Definition 2.25 (Laplace-Beltrami operator)**.** *For a smooth function $f : \Gamma \to \mathbb{R}$ we define the* **Laplace-Beltrami operator** *on Γ through*

$$\Delta_\Gamma f(p) \quad := \quad \nabla_\Gamma \cdot \nabla_\Gamma f(p) \, .$$

Also for this differential operator we give some descriptions which are useful for calculations and for a better understanding.

Remark 2.26. *In a local parametrization (U, γ, V) of Γ around p with $\gamma(u) = p$ we have*

$$\begin{aligned} \Delta_\Gamma f(p) &= \frac{1}{\sqrt{g(u)}} \sum_{i,j=1}^{n} \partial_i \left(\sqrt{g(u)}\, g^{ij}(u)\, \partial_j (f \circ \gamma)(u) \right) \\ &= \sum_{i,j=1}^{n} g^{ij}(u) \left(\partial_{ij}(f \circ \gamma)(u) - \sum_{k=1}^{n} \Gamma_{ij}^k(u) \partial_k (f \circ \gamma)(u) \right) \, , \end{aligned}$$

where $g(u) := \det \left((g_{ij}(u))_{ij} \right)$ and Γ_{ij}^k are the ***Christoffel symbols*** *given by*

$$\Gamma_{ij}^k(u) \quad := \quad \sum_{l=1}^{n} \frac{1}{2} g^{kl}(u) \left(\partial_i g_{jl}(u) + \partial_j g_{il}(u) - \partial_l g_{ij}(u) \right) \, .$$

If g_p is the restriction of the euclidian scalar product, we get with the help of an orthonormal moving frame $v_1, \dots, v_n$ of Γ the following representation

$$\begin{aligned} \Delta_\Gamma f(p) &= \operatorname{div}_\Gamma \left(\sum_{i=1}^{n} \partial_{v_i} f \, v_i \right) = \sum_{j=1}^{n} \partial_{v_j} \left(\sum_{i=1}^{n} \partial_{v_i} f \, v_i \right) \cdot v_j \\ &= \sum_{i,j=1}^{n} \left(\partial_{v_j} \partial_{v_i} f \; (v_i \cdot v_j) + \partial_{v_i} f \; (\partial_{v_j} v_i \cdot v_j) \right) \\ &= \sum_{i=1}^{n} \partial_{v_i} \partial_{v_i} f + \sum_{i=1}^{n} \partial_{v_i} f \sum_{j=1}^{n} \partial_{v_j} v_i \cdot v_j \, . \end{aligned}$$

If additionally f admits an extension to an open neighbourhood of Γ, we see with the above notations:

$$\Delta_\Gamma f(p) \quad = \quad \sum_{i=1}^{n} \underline{D_i} \left(\underline{D_i} f(p) \right) \, .$$

In the next lemma we want to describe the mean curvature with the help of the introduced differential operators and give a local representation.

Lemma 2.27. *If g_p is the restriction of the euclidian scalar product, the following formulas for the mean curvature and mean curvature vector hold true.*

(i) $H(p) = -\nabla_\Gamma \cdot n(p)$,

(ii) $\vec{H}(p) = \Delta_\Gamma id(p)$, *in particular* $H(p) = \Delta_\Gamma id(p) \cdot n(p)$, *where* $id : \Gamma \to \Gamma$ *is the identity map on* Γ *and* $\Delta_\Gamma id(p)$ *is defined component wise through* $\Delta_\Gamma f(p) := (\Delta_\Gamma f_i(p))_{i=1,\ldots,n}$ *for mappings* $f : \Gamma \to \mathbb{R}^{n+1}$.

(iii) *With a local parametrization* (U, γ, V) *of* Γ *around* $p \in \Gamma$ *with* $\gamma(u) = p$ *it holds that*

$$\vec{H}(p) \quad = \quad \frac{1}{\sqrt{g(u)}} \sum_{i,j=1}^{n} \partial_i \left(\sqrt{g(u)} g^{ij}(u) \partial_j \gamma(u) \right) \, .$$

Proof. ad (i): With the help of an orthonormal basis $v_1, \ldots, v_n$ of the tangent space $T_p\Gamma$ we see from our definition of the mean curvature H:

$$H(p) \quad = \quad \mathrm{trace}(W_p) = \sum_{i=1}^{n} (W_p(v_i) \cdot v_i) = \sum_{i=1}^{n} - (d_p n(v_i) \cdot v_i) = - \sum_{i=1}^{n} (\partial_{v_i} n \cdot v_i) = -\nabla_\Gamma \cdot n(p) \, .$$

ad (ii): We use the first two of the following product rules (the third one is given for completeness). Let $f, h : \Gamma \to \mathbb{R}$, $v : \Gamma \to \mathbb{R}^{n+1}$ (not necessary tangential) be smooth mappings. Then it holds

$$\begin{array}{lrcl} (a) & \nabla_\Gamma(f\,h) & = & f\,\nabla_\Gamma h + h\,\nabla_\Gamma f \, , \\ (b) & \mathrm{div}_\Gamma(f\,v) & = & \nabla_\Gamma f \cdot v + f \; \mathrm{div}_\Gamma\, v \, , \\ (c) & \Delta_\Gamma(f\,h) & = & f\,\Delta_\Gamma h + 2\,\nabla_\Gamma f \cdot \nabla_\Gamma h + h\,\Delta_\Gamma f \, . \end{array}$$

ad (a): For fixed $p \in \Gamma$ let $w \in T_p\Gamma$ and a curve c on Γ with $c(0) = p$ and $c'(0) = w$ as in the definition of the differential be given. Then it holds that

$$\begin{aligned} (\nabla_\Gamma(f\,h)(p), w) \quad &= \quad d_p(f\,h)(w) = \frac{d}{d\tau}(f\,h)(c(\tau))\bigg|_{\tau=0} \\ &= \quad \frac{d}{d\tau} f(c(\tau))\bigg|_{\tau=0} h(c(0)) + f(c(0)) \; \frac{d}{d\tau} h(c(\tau))\bigg|_{\tau=0} \\ &= \quad d_p f(w)\, h(p) + f(p)\, d_p h(w) \\ &= \quad (h(p)\,\nabla_\Gamma f(p) + f(p)\,\nabla_\Gamma h(p), w) \; , \end{aligned}$$

and since w was arbitrary the claim holds.
ad (b): For fixed $p \in \Gamma$ let $v_1, \ldots, v_n$ be an orthonormal basis of $T_p\Gamma$ and c_i curves on Γ with

$c_i(0) = p$ and $c_i'(0) = v_i$. Then it holds that

$$\begin{aligned}
\operatorname{div}_\Gamma(f\,v) &= \sum_{i=1}^{n} \partial_{v_i}(f\,v)\cdot v_i = \sum_{i=1}^{n} d_p(f\,v)(v_i)\cdot v_i = \sum_{i=1}^{n} \frac{d}{d\tau}(f\,v)(c_i(\tau))\Big|_{\tau=0}\cdot v_i \\
&= \sum_{i=1}^{n} \frac{d}{d\tau} f(c_i(\tau))\Big|_{\tau=0} (v(p)\cdot v_i) + \sum_{i=1}^{n} f(p)\left(\frac{d}{d\tau}v(c_i(\tau))\Big|_{\tau=0}\cdot v_i\right) \\
&= \sum_{i=1}^{n} \partial_{v_i} f\ (v(p)\cdot v_i) + \sum_{i=1}^{n} f(p)\ (\partial_{v_i} v\cdot v_i) \\
&= v(p)\cdot \sum_{i=1}^{n} \partial_{v_i} f\, v_i + f(p)\sum_{i=1}^{n} \partial_{v_i} v\cdot v_i \\
&= v(p)\cdot \nabla_\Gamma f(p) + f(p)\ \operatorname{div}_\Gamma v(p)\ .
\end{aligned}$$

ad (c): With the help of (a) and (b) we have

$$\begin{aligned}
\Delta_\Gamma(f\,h) &= \operatorname{div}_\Gamma(\nabla_\Gamma(f\,h)) \\
&\overset{(a)}{=} \operatorname{div}_\Gamma(h\,\nabla_\Gamma f) + \operatorname{div}_\Gamma(f\cdot\nabla_\Gamma h) \\
&\overset{(b)}{=} \nabla_\Gamma h\cdot\nabla_\Gamma f + h\ \operatorname{div}_\Gamma(\nabla_\Gamma f) + \nabla_\Gamma f\cdot\nabla_\Gamma h + f\ \operatorname{div}_\Gamma(\nabla_\Gamma h) \\
&= h\,\Delta_\Gamma f + 2\,\nabla_\Gamma f\cdot\nabla_\Gamma h + f\,\Delta_\Gamma h\ .
\end{aligned}$$

To show finally (ii), we use the obvious extension of $id : \Gamma \to \mathbb{R}^{n+1}$ to all of $\mathbb{R}^{n+1}$, set $f_i(p) = (id(p))_i = p_i$ for $i = 1,\ldots,n+1$ and proceed with the euclidian gradient $\nabla f_i(p) \equiv e_i$ as follows.

$$\begin{aligned}
\Delta_\Gamma f_i(p) &= \operatorname{div}_\Gamma(\nabla_\Gamma f_i(p)) \\
&= \operatorname{div}_\Gamma\left(\nabla f_i(p) - (\nabla f_i(p), n(p))\ n(p)\right) \\
&= \operatorname{div}_\Gamma\left(e_i - (e_i, n(p))\ n(p)\right) \\
&= -\operatorname{div}_\Gamma\left((e_i, n(p))\ n(p)\right) \\
&= -\nabla_\Gamma\left(e_i, n(p)\right)\cdot n(p) - (e_i, n(p))\ \underbrace{\operatorname{div}_\Gamma n(p)}_{=-H(p)} \\
&= H(p)\cdot (n(p))_i\ ,
\end{aligned}$$

where we used (i) and the fact that the term $-\nabla_\Gamma\left(e_i, n(p)\right)\cdot n(p)$ vanishes, since $\nabla_\Gamma\left(e_i, n(p)\right)$ lies in $T_p\Gamma$ and $n(p) \in N_p\Gamma$. So we get $\Delta_\Gamma id(p) = H(p)\cdot n(p) = \vec{H}(p)$ and by taking the scalar product with $n(p)$ we arrive at $H(p) = \Delta_\Gamma id(p)\cdot n(p)$.
ad (iii): With the help of (ii) and the local representation of $\Delta_\Gamma id(p)$ from Remark 2.26 we see the last point of the lemma. □

From now on g_p is always the restriction of the euclidian scalar product, unless otherwise noted.

Definition 2.28 (Outer unit conormal)**.** *With our notation of a hypersurface* Γ *it holds that for* $p \in \partial\Gamma$ *the tangent space* $T_p\partial\Gamma$ *is* $(n-1)$*-dimensional and* $T_p\Gamma$ *is* n*-dimensional. Since also* $T_p\partial\Gamma \subset T_p\Gamma$*, there exists an one-dimensional subspace* L *such that*

$$T_p\Gamma \;=\; T_p\partial\Gamma \cup L\,.$$

We can therefore choose the unique vector $n_{\partial\Gamma}(p)$ *in* L *with the following three properties.*

(i) $|n_{\partial\Gamma}(p)| = 1$,

(ii) $n_{\partial\Gamma}(p) \cdot v = 0$ *for all* $v \in T_p\partial\Gamma$ *and*

(iii) there exists a curve $c : (-\varepsilon, 0] \to \Gamma$ *with* $c(0) = p$ *and* $c'(0) = n_{\partial\Gamma}(p)$.

$n_{\partial\Gamma}(p)$ *is then called the* ***outer unit conormal*** *of* Γ *at* $p \in \partial\Gamma$.

With the above notation we can state the Gauß' theorem for hypersurfaces Γ with possibly nonempty boundary $\partial\Gamma$. This is an extension of the Gauß' theorem for regions in euclidian space to the setting of manifolds and it contains an additional term involving the mean curvature vector. This theorem will be used in the calculation of the evolution of volume in Lemmata 2.46 and 4.22.

Theorem 2.29 (Gauß' theorem on hypersurfaces)**.** *Let* Γ *be a bounded hypersurface and* $f : \Gamma \to \mathbb{R}^{n+1}$ *a smooth mapping. Then we have*

$$\int\limits_{\Gamma} \left(\operatorname{div}_\Gamma f + f \cdot \vec{H}\right) d\mathcal{H}^n \;=\; \int\limits_{\partial\Gamma} f \cdot n_{\partial\Gamma}\, d\mathcal{H}^{n-1}\,.$$

Remark 2.30. *The assumption that* Γ *is bounded can be skipped, if one assures the existence of the arising integrals in another way.*

We also want to give the following useful expressions that are derived directly from the above Theorem 2.29. For a tangent vector field, that is $f : \Gamma \to \mathbb{R}^{n+1}$ *with* $f(p) \in T_p\Gamma$ *and functions* $h_1, h_2 : \Gamma \to \mathbb{R}$ *it holds*

$$\begin{aligned}
&(i) & \int\limits_{\Gamma} \operatorname{div}_\Gamma f\, d\mathcal{H}^n &= \int\limits_{\partial\Gamma} f \cdot n_{\partial\Gamma}\, d\mathcal{H}^{n-1}\,,\\
&(ii) & \int\limits_{\Gamma} \Delta_\Gamma h_1\, d\mathcal{H}^n &= \int\limits_{\partial\Gamma} \nabla_\Gamma h_1 \cdot n_{\partial\Gamma}\, d\mathcal{H}^{n-1}\,,\\
&(iii) & \int\limits_{\Gamma} (\nabla_\Gamma h_1 \cdot f + \operatorname{div}_\Gamma f\, h_1)\, d\mathcal{H}^n &= \int\limits_{\partial\Gamma} h_1\, (f \cdot n_{\partial\Gamma})\, d\mathcal{H}^{n-1} \text{ and}\\
&(iv) & \int\limits_{\Gamma} (\nabla_\Gamma h_1 \cdot \nabla_\Gamma h_2 + h_1\, \Delta_\Gamma h_2)\, d\mathcal{H}^n &= \int\limits_{\partial\Gamma} h_1\, (\nabla_\Gamma h_2 \cdot n_{\partial\Gamma})\, d\mathcal{H}^{n-1}\,,
\end{aligned}$$

where the last equation is Greens formula and will be used frequently in this work for integration by parts.

2.2 Evolving hypersurfaces

In this section we will explain the concept of evolving hypersurfaces, i.e. hypersurfaces that move in time. We also introduce the normal velocity and show a representation of the tangent space. Furthermore we give our use of time derivative in this setting, the so-called normal time derivative. The idea here is to follow the evolution of a fixed point in a specified direction and differentiate along the arising curve, see Definition 2.36 for the details. Then we compare the normal time derivative with a related expression and give formulas for mean curvature and normal, which are proven in the appendix.

Finally we consider evolving hypersurfaces that arise as a graph over some fixed reference hypersurface Γ^*, which is the situation that will be considered in later chapters of this work. In this situation we give a formula for the normal velocity and write down transformation rules from $\Gamma(t)$ to Γ^* for some differential operators.

For more information we refer the reader to Gurtin [Gur93], but we also used lecture notes from Alt [Alt04].

Definition 2.31 (Evolving hypersurface)**.** Γ *is called a (smooth)* **evolving hypersurface** *of* $\mathbb{R}^{n+1}$*, if there exists a* $T > 0$ *or* $T = \infty$*, such that*

(*i*) Γ *is a hypersurface of* $\mathbb{R} \times \mathbb{R}^{n+1}$,

(*ii*) *there exist hypersurfaces* $\Gamma(t) = \Gamma_t$ *of* $\mathbb{R}^{n+1}$*, such that*

$$\Gamma = \{(t,p) \mid t \in [0,T)\,,\ p \in \Gamma(t)\}\,, \qquad \textit{and}$$

(*iii*) *the tangent spaces* $T_{(t,p)}\Gamma$ *of* Γ *are nowhere spacelike, that is*

$$T_{(t,p)}\Gamma \neq \{0\} \times \mathbb{R}^{n+1} \qquad \textit{for all } (t,p) \in \Gamma\,.$$

Note that from now on Γ is an evolving hypersurface and $\Gamma(t) = \Gamma_t$ are hypersurfaces in $\mathbb{R}^{n+1}$ in contrast to the previous Section 2.1, where Γ was a hypersurface in $\mathbb{R}^{n+1}$.
To define the normal velocity, we choose smooth unit normal fields $n(t,\cdot) : \Gamma(t) \to \mathbb{R}^{n+1}$ such that $n(t,p)$ is a unit normal to $\Gamma(t)$ at $p \in \Gamma(t)$ in a way that they fit together to give a smooth vector field $n : \Gamma \to \mathbb{R}^{n+1}$.

Definition 2.32 (Normal velocity)**.** *For a fixed point* $(t,p) \in \Gamma$*, we choose a curve*

$$c : (t-\varepsilon, t+\varepsilon) \longrightarrow \mathbb{R}^{n+1}$$

with $c(\tau) \in \Gamma(\tau)$ *and* $c(t) = p$*. Then we set*

$$V(t,p) \quad := \quad n(t,p) \cdot \frac{d}{d\tau} c(\tau)\bigg|_{\tau=t}$$

and call V *the* **normal velocity** *of the evolving hypersurface* Γ *at* (t,p).

Of course we have to show independence of the curve in the previous definition.

Lemma 2.33. *The normal velocity V from the above definition is independent of the chosen curve.*

Proof. Because of point (ii) of the definition of an evolving hypersurface we can describe Γ locally around $(t,p) \in \Gamma$ as the zero level set of a function $h : I \times D \to \mathbb{R}$, $(s,x) \mapsto h(s,x)$ where $I \times D \subset \mathbb{R} \times \mathbb{R}^{n+1}$ is an open set. That is, for some open neighbourhood $W \subset \mathbb{R} \times \mathbb{R}^{n+1}$ of (t,p) it holds that

$$\Gamma \cap W \quad = \quad \{(s,x) \in I \times D \mid h(s,x) = 0\}$$

and the function $h_t : D \to \mathbb{R}$ defined through $h_t(x) = h(t,x)$ is the zero level function of $\Gamma(t)$, that is there exists an open neighbourhood $\widetilde{W} \subset \mathbb{R}^{n+1}$ of $p \in \Gamma(t)$, such that

$$\Gamma(t) \cap \widetilde{W} \quad = \quad \{x \in D \mid h_t(x) = 0\} \, .$$

Also the condition $\nabla h_t(x) \neq 0$ for $x \in D$ with $h_t(x) = 0$ is fulfilled. With a sign convention on h the normal $n(t,p)$ to $\Gamma(t)$ at p can be written as

$$n(t,p) \quad = \quad \frac{\nabla_x h(t,p)}{|\nabla_x h(t,p)|} \, .$$

This can be seen with the help of a local parametrization (U, γ, V) of $\Gamma(t)$ for fixed t with $\gamma(u) = p$. We observe that $h(t, \gamma(u)) = 0$ and therefore by differentiating with respect to u_i we get

$$\nabla_x h(t,p) \cdot \frac{\partial}{\partial u_i} \gamma(u) \quad = \quad 0 \, .$$

Since $\frac{\partial}{\partial u_i} \gamma(u)$ for $i = 1, \ldots, n$ is a basis for $T_p\Gamma(t)$ we see that $\nabla_x h(t,p) \in N_p\Gamma(t)$. Normalizing gives the above formula.

Now we choose a curve c as in the Definition 2.32 of normal velocity and observe that $h(\tau, c(\tau)) = 0$. Differentiating with respect to τ yields

$$0 \quad = \quad \frac{d}{d\tau} h(\tau, c(\tau))\bigg|_{\tau=t} = \partial_t h(t,p) + \nabla_x h(t,p) \cdot \frac{d}{d\tau} c(\tau)\bigg|_{\tau=t} \, .$$

Multiplying this equation with $|\nabla_x h(t,p)|^{-1}$ gives

$$\frac{\nabla_x h(t,p)}{|\nabla_x h(t,p)|} \cdot \frac{d}{d\tau} c(\tau)\bigg|_{\tau=t} \quad = \quad -\frac{\partial_t h(t,p)}{|\nabla_x h(t,p)|} \, ,$$

where the left side equals the definition of the normal velocity $V(t,p)$. To summarize, we achieved with the above construction a different representation of V independent of the curve and therefore showed the lemma. $\square$

For a better understanding of evolving hypersurfaces we want to describe the tangent space $T_{(t,p)}\Gamma$ with the help of tangent vectors from $T_p\Gamma(t)$ and one other quantity.

Lemma 2.34. *For a point $(t,p) \in \Gamma$, there holds a splitting of the tangent space $T_{(t,p)}\Gamma$ of Γ. More explicitly, there exists a function $v_\Gamma : \Gamma \to \mathbb{R}^{n+1}$, such that*

$$T_{(t,p)}\Gamma \quad = \quad \{r\,(1, v_\Gamma(t,p)) + (0,\tau) \mid r \in \mathbb{R}\,,\ \tau \in T_p\Gamma(t)\}\ .$$

Here $v_\Gamma(t,p)$ is the uniquely determined mapping with the properties:

(i) $(1, v_\Gamma(t,p)) \in T_{(t,p)}\Gamma$ *and*

(ii) $v_\Gamma(t,p) \in (T_p\Gamma(t))^\perp = N_p\Gamma(t)$.

In particular, we see a representation of the normal space of Γ as

$$N_{(t,p)}\Gamma \quad = \quad \{s\,(-v_\Gamma \cdot n, n)(t,p) \mid s \in \mathbb{R}\}\ .$$

Proof. Fix $(t,p) \in \Gamma$. Because of point (iii) of the Definition 2.31 of an evolving hypersurface, there exists a tangent vector $(r,v) \in T_{(t,p)}\Gamma$ with $r \in \mathbb{R}$, $v \in \mathbb{R}^{n+1}$ and $r \neq 0$. Without loss of generality we assume $r = 1$.

Now we can extend $(1,v)$ to a basis of $T_{(t,p)}\Gamma$, that is there exist linearly independent vectors

$$(1,v), (\varepsilon_1, w_1), \ldots (\varepsilon_n, w_n)$$

in $T_{(t,p)}\Gamma$. Then it holds that the definition

$$(\varepsilon_i, w_i) - \varepsilon_i\,(1,v) \quad = \quad (0, w_i - \varepsilon_i v) \quad =: \quad (0,\tau_i)$$

yields vectors $(0,\tau_i)$ with $\tau_i \neq 0$, because $(1,v)$ and (ε_i, w_i) are linearly independent. With this small trick, we have a new basis of $T_{(t,p)}\Gamma$ given through

$$(1,v), (0,\tau_1), \ldots, (0,\tau_n)\ .$$

Since $(0,\tau_1), \ldots, (0,\tau_n)$ are linearly independent, the subspace $W := \operatorname{span}\{\tau_1, \ldots, \tau_n\}$ is an n-dimensional subspace of $\mathbb{R}^{n+1}$.

Until now, we have the representation

$$T_{(t,p)}\Gamma \quad = \quad \{r \cdot (1,v) + (0,\tau) \mid r \in \mathbb{R}\,,\ \tau \in W\}\ .$$

With the help of the orthogonal projection P of $\mathbb{R}^{n+1}$ to W we define

$$v_\Gamma(t,p) \quad := v - Pv \in W^\perp$$

and we see that $(1, v_\Gamma) = (1,v) - (0, Pv) \in T_{(t,p)}\Gamma$ because of the above representation. So we can write

$$T_{(t,p)}\Gamma \quad = \quad \{r \cdot (1, v_\Gamma(t,p)) + (0,\tau) \mid r \in \mathbb{R}\,,\ \tau \in W\}\ .$$

If we now show that

$$W \quad = \quad T_p\Gamma(t)$$

then v_Γ is well defined and unique and fulfils point (ii).
The uniqueness can be seen as follows. If we take $\tilde{v}$ instead of v with $(1,v)$ and $(1,\tilde{v})$ in $T_{(t,p)}\Gamma$ at the beginning of the proof, we see from

$$(0, v-\tilde{v}) = (1,v) - (1,\tilde{v}) \in T_{(t,p)}\Gamma$$

that $v - \tilde{v} \in W$ and therefore $v - Pv = \tilde{v} - P\tilde{v}$. This uniqueness is the reason why we work with $v_\Gamma = v - Pv$ instead of the arbitrary v.
By using the decomposition

$$\mathbb{R}\times\mathbb{R}^{n+1} \quad = \quad \{(s, v+w) \mid s\in\mathbb{R}\,,\, v\in W\,,\, w\in W^\perp\}$$

we write Γ locally around (t,p) with a parametrization over $\mathbb{R}\times W$ such that

$$\Gamma\cap U \quad = \quad \big\{(s, v+w)\in\mathbb{R}\times\mathbb{R}^{n+1} \mid w = h(v)\big\}$$

for a mapping $h : W \to W^\perp$ and an open neighbourhood $U\subset\mathbb{R}\times\mathbb{R}^{n+1}$ of (t,p).
This representation gives us the possibility to denote a basis for $T_{(s,v+h(s,v))}\Gamma$ with the help of an orthonormal basis $v_1,\dots,v_n$ of W through

$$\partial_s(s, v+h(s,v))\,,\ \partial_{v_1}(s, v+h(s,v))\,,\ \dots\,,\ \partial_{v_n}(s, v+h(s,v))\,.$$

Calculating the derivatives gives

$$(1, \partial_s h(s,v))\,,\ (0, v_1 + \partial_{v_1} h(s,v))\,,\ \dots\,,\ (0, v_n + \partial_{v_n} h(s,v))\,.$$

Due to the above representation of the tangent space at the point $(t,p) = (t, w+h(t,w)) \in T_{(t,p)}\Gamma$ for some $w\in W$, and due to the fact that $\partial_{v_i} h(s,w)\in W^\perp$, we see that

$$\partial_{v_i} h(t,w) = 0\,.$$

The above parametrization of Γ leads to a parametrization of $\Gamma(t)$ through

$$\Gamma(t)\cap\widetilde{U} \quad = \quad \big\{v+w\in\mathbb{R}^{n+1} \mid w = h(t,v)\big\}\ ,$$

where $\widetilde{U}\subset\mathbb{R}^{n+1}$ is an open neighbourhood of $p\in\Gamma(t)$. A basis of tangent vectors of $T_p\Gamma(t)$ is then given by

$$(v_1 + \partial_{v_1} h(s,v))\,,\ \dots\,,\ (v_n + \partial_{v_n} h(s,v))\,,$$

where we used the above notation and get therefore that this basis equals $v_1,\dots,v_n$.
This shows that an orthonormal basis of W is also an orthonormal basis of $T_p\Gamma(t)$ and since they are both n-dimensional subspaces of $\mathbb{R}\times\mathbb{R}^{n+1}$, we have

$$T_p\Gamma(t) = W\ .$$

Since this was the only missing part, we proved the lemma. $\square$

The vector v_Γ from the above Lemma 2.34 can be given more explicitly and gets a name, the **velocity vector**.

Lemma 2.35. *The vector* v_Γ *from the above Lemma 2.34 is related to the normal velocity of Definition 2.32 by*

$$v_\Gamma(t,p) \quad = \quad V(t,p)\, n(t,p) \quad \text{for } (t,p) \in \Gamma\,.$$

That's the reason why v_Γ *is also called* **velocity vector**.

Proof. The term $V(t,p)\,n(t,p)$ can be described with the help of a curve $c:(t-\varepsilon,t+\varepsilon)\to\mathbb{R}^{n+1}$ with $c(\tau)\in\Gamma(\tau)$ and $c(t)=p$ as:

$$\begin{aligned} V(t,p)\,n(t,p) &= \left(\frac{d}{d\tau}c(\tau)\Big|_{\tau=t}\cdot n(t,p)\right) n(t,p) \\ &= P^\perp\left(\frac{d}{d\tau}c(\tau)\Big|_{\tau=t}\right), \end{aligned}$$

where $P^\perp:\mathbb{R}^{n+1}\to N_p\Gamma(t)$ is the orthonormal projection onto the normal space.
On the other hand, we have $v_\Gamma(t,p)=P^\perp(v)$, where $v\in\mathbb{R}^{n+1}$ such that $(1,v)\in T_{(t,p)}\Gamma$ and where we have seen in the previous proof that v_Γ is independent of the choice of v.
So we have to show that $\frac{d}{d\tau}c(\tau)\big|_{\tau=t}=c'(t)$ is such a v from above, which means $(1,c'(t))\in T_{(t,p)}\Gamma$. To this end, we just have to give a curve on Γ whose derivative equals $(1,c'(t))$. One possible choice is

$$\tilde{c}:(t-\varepsilon,t+\varepsilon)\longrightarrow\mathbb{R}^{n+1}\,,\quad \tilde{c}(\tau)=(\tau,c(\tau))\,.$$

For this curve $\tilde{c}$, it holds that $\tilde{c}(\tau)\in\Gamma$, $\tilde{c}(t)=(t,p)$ and $\tilde{c}'(t)=(1,c'(t))$. □

When we consider a function on an evolving hypersurface Γ through

$$f:\Gamma\longrightarrow\mathbb{R}\,,\quad (t,p)\mapsto f(t,p)\,,$$

we describe values of f just at points (t,p) with $p\in\Gamma(t)$, which means in particular that the second variable depends on the first one. Since Γ itself is a hypersurface in $\mathbb{R}\times\mathbb{R}^{n+1}$, we have the possibilities of derivatives from the previous section 2.1 as for example directional derivatives with respect to a tangent vector in $T_{(t,p)}\Gamma$. But if we want to differentiate separately in time and space, we have to think about the right derivatives of such functions.
Of course, we can differentiate in space for fixed t, because then the function

$$f_t := f(t,\cdot):\Gamma(t)\longrightarrow\mathbb{R}$$

is defined on a fixed hypersurface $\Gamma(t)$ and we have all concepts from 2.1 as the differential d_pf_t, the directional derivative $\partial_v f_t$ for a tangent vector $v\in T_p\Gamma(t)$, the surface gradient $\nabla_{\Gamma(t)}f_t$, the divergence $\mathrm{div}_{\Gamma(t)}\,f_t$ and the Laplace-Beltrami operator $\Delta_{\Gamma(t)}f_t$ each on $\Gamma(t)$.
But what is the right kind of derivative in time at a point $(t,p)\in\Gamma$? Of course, one can not fix p and look at small variations of t, because the function f is just defined for $p\in\Gamma(t)$. So if we vary t, we also have to vary p. The essential concept is the normal time derivative in the sense of Gurtin [Gur93]. Therefore we consider for fixed $(t,p)\in\Gamma$ a curve

$$\eta:(t-\varepsilon,t+\varepsilon)\longrightarrow\mathbb{R}^{n+1}$$

with $\eta(t) = p$ such that $(\tau, \eta(\tau)) \in \Gamma$, i.e. $\eta(\tau) \in \Gamma(\tau)$. Furthermore, η shall point in normal direction scaled with normal velocity at every point $\eta(\tau) \in \Gamma(\tau)$, which means

$$\eta'(\tau) \quad = \quad V(\tau, \eta(\tau))\, n(\tau, \eta(\tau)) \ .$$

Then we can define the above mentioned time derivative.

Definition 2.36 (Normal time derivative)**.** *We define the* **normal time derivative** $\partial^\circ f(t,p)$ *at a point* $(t,p) \in \Gamma$ *with the help of a curve* η *as above through*

$$\partial^\circ f(t,p) \quad := \quad \frac{d}{d\tau} f(\tau, \eta(\tau))\bigg|_{\tau=t} \ .$$

With the same curve and notation this definition can be extended to vector fields.

To see the existence of such a curve η as above, the independence of Definition 2.36 from this curve and to obtain another useful representation of the normal time derivative we observe the following lemma.

Lemma 2.37. *The normal time derivative can be written with the help of a directional derivative as follows.*

$$\partial^\circ f \quad = \quad \partial_{(1,V\,n)} f \ ,$$

where we use the fact $(1, V(t,p)\, n(t,p)) \in T_{(t,p)}\Gamma$ *from Lemmata 2.34 and 2.35. In particular this means that the definition of* $\partial^\circ f$ *is independent of the chosen curve.*

Proof. Consider the directional derivative of f in direction $(1, V(t,p)\, n(t,p)) \in T_{(t,p)}\Gamma$ at a fixed point $(t,p) \in \Gamma$, which is given by

$$\partial_{(1,V\,n)} f \quad = \quad \frac{d}{d\varepsilon} f(\zeta(\varepsilon))\bigg|_{\varepsilon=0} \ ,$$

where $\zeta : (-\varepsilon_0, \varepsilon_0) \to \Gamma$, $\zeta(0) = (t,p)$ and $\zeta'(0) = (1, V(t,p)\, n(t,p))$. Because we know from Lemma 2.33 that this definition is independent of the curve, we specify ζ more detailed as solution to the following ordinary differential equation.
Find $\zeta : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R} \times \mathbb{R}^{n+1}$ such that with an arbitrary extension of $V\,n$

$$\left\{ \begin{array}{rcl} \zeta'(\varepsilon) & = & (1, (V\,n)(\zeta(\varepsilon))) \ , \\ \zeta(0) & = & (t,p) \ . \end{array} \right.$$

Because we know that $(1, (V\,n)(t,p))$ lies in the tangent space $T_{(t,p)}\Gamma$, we can deduce that a solution fulfills $\zeta(\varepsilon) \in \Gamma$ (compare for example Hildebrandt [Hil03]) and in fact we don't need the above mentioned arbitrary extension of $V\,n$. For small ε_0 such a ζ is uniquely determined and we use this to calculate the above directional derivative.

With point (*ii*) of the Definition of an evolving hypersurface 2.31 we can divide $\zeta(\varepsilon) \in \Gamma$ into

$$\zeta(\varepsilon) \quad = \quad (\alpha(\varepsilon), \eta(\varepsilon)) \ ,$$

where $\alpha : (-\varepsilon_0, \varepsilon_0) \to (0, T)$ and $\eta : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^{n+1}$ with $\eta(\varepsilon) \in \Gamma(\alpha(\varepsilon))$.
Because of the starting point $\zeta(0) = (t, p)$ we can conclude

$$\alpha(0) = t \quad \text{and} \quad \eta(0) = p$$

and with the expression $\zeta'(\varepsilon) = (1, (V\,n)(\zeta(\varepsilon)))$ we arrive at

$$\alpha'(\varepsilon) = 1 \quad \text{and} \quad \eta'(\varepsilon) = (V\,n)(\alpha(\varepsilon), \eta(\varepsilon)) \ .$$

Then it has to hold that $\alpha(\varepsilon) = \varepsilon + t$ and therefore $\eta'(\varepsilon) = (V\,n)(\varepsilon + t, \eta(\varepsilon))$. With the help of the reparametrized curve $\tilde{\eta} : (t - \varepsilon_0, t + \varepsilon_0) \to \mathbb{R}^{n+1}$, $\tilde{\eta}(\tau) := \eta(\tau - t)$, we see that

$$\begin{aligned} \tilde{\eta}(\tau) &= \eta(\tau - t) \in \Gamma(\alpha(\tau - t)) = \Gamma(\tau) \ , \\ \tilde{\eta}(t) &= \eta(0) = p \ \text{ and} \\ \tilde{\eta}'(\tau) &= \eta'(\varepsilon)\big|_{\varepsilon = \tau - t} = (1, (V\,n)(\varepsilon + t, \eta(\varepsilon)))\big|_{\varepsilon = \tau - t} = (1, (V\,n)(\tau, \tilde{\eta}(\tau))) \ . \end{aligned}$$

So we constructed a curve $\tilde{\eta}$ as in Definition 2.36 of the normal time derivative $\partial^\circ f$ and since $(\tau, \tilde{\eta}(\tau))$ is just a shift in time for $\zeta(\varepsilon)$ we observe at the fixed point $(t, p) \in \Gamma$ finally

$$\partial^\circ f(t, p) \quad = \quad \tfrac{d}{d\tau} f(\tau, \tilde{\eta}(\tau))\big|_{\tau = t} = \tfrac{d}{d\varepsilon} f(\zeta(\varepsilon))\big|_{\varepsilon = 0} \quad = \quad \partial_{(1,(V\,n))} f(t, p) \,.$$

Hence the proof is complete. $\square$

We want to calculate the normal time derivative in an instructive special case.

Lemma 2.38. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be smooth and consider the restriction $\tilde{f}$ of f to an evolving hypersurface Γ through*

$$\tilde{f} : \Gamma \to \mathbb{R} \,, \ (t, p) \mapsto f(p) \ .$$

Then the following formula for the normal time derivative of $\tilde{f}$ holds

$$\partial^\circ \tilde{f}(t, p) = V(t, p) \left(\nabla f(p) \cdot n(t, p) \right) \ .$$

For smooth vector fields $g : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and an analogue definition for $\tilde{g}$ as for $\tilde{f}$ we observe the identity

$$\partial^\circ \tilde{g}(t, p) = V(t, p) \left(Dg(p) \cdot n(t, p) \right) \ ,$$

where $Dg(p)$ is the Jacobian matrix of g. In particular for the identity map $g = id$ it holds $\partial^\circ id = V\,n$.

Proof. This is seen immediately with the formula from Lemma 2.37 through

$$\partial^\circ \tilde{f}(t,p) = \partial_{(1,(Vn)(t,p))}\tilde{f} = \partial_{((Vn)(t,p)}f = \Big(\nabla f(p)\cdot (Vn)(t,p)\Big) = V(t,p)\left(\nabla f(p)\cdot n(t,p)\right) ,$$

and the proof is done. □

We will need the following lemma for the linearization of mean curvature in later chapters. In this lemma, we compare the normal time derivative $\partial^\circ f(t,p)$ with a term $\frac{d}{d\tau}f(\tau,\vartheta(\tau))\big|_{\tau=t}$, where ϑ has the same properties as η from Definition 2.36 of the normal time derivative, but points not necessarily in normal direction.

Lemma 2.39. *For a fixed point $(t,p)\in\Gamma$ let $\vartheta:(t-\varepsilon,t+\varepsilon)\to\mathbb{R}^{n+1}$ with $\vartheta(\tau)\in\Gamma(\tau)$ and $\vartheta(t)=p$. In this case it holds*

$$\frac{d}{d\tau}f(\tau,\vartheta(\tau))\bigg|_{\tau=t} \quad=\quad \partial^\circ f(t,p)+\nabla_{\Gamma(t)}f(t,p)\cdot\left(\vartheta'(t)\right)^T ,$$

where the tangential projection $()^T$ on $T_p\Gamma(t)$ can also be skipped due to the fact that $\nabla_{\Gamma(t)}f(t,p)$ lies in tangential direction. Therefore we get

$$\frac{d}{d\tau}f(\tau,\vartheta(\tau))\bigg|_{\tau=t} \quad=\quad \partial^\circ f(t,p)+\nabla_{\Gamma(t)}f(t,p)\cdot\vartheta'(t) .$$

Proof. With the help of the tangential projection on $T_p\Gamma(t)$, we write the derivative $\vartheta'(t)$ of the curve ϑ as

$$\begin{aligned}\vartheta'(t) &= \left(\vartheta'(t)\cdot n(t,p)\right)\, n(t,p)+\left(\vartheta'(t)\right)^T\\ &= V(t,p)\,n(t,p)+\left(\vartheta'(t)\right)^T ,\end{aligned}$$

where the normal velocity appears by its definition.
With similar calculations as in Lemma 2.37 we write the term $\frac{d}{d\tau}f(\tau,\vartheta(\tau))\big|_{\tau=t}$ as a directional derivative and use linearity of the directional derivative to get

$$\frac{d}{d\tau}f(\tau,\vartheta(\tau))\bigg|_{\tau=t} = \partial_{(1,\vartheta'(t))}f = \partial_{(0,(\vartheta'(t))^T)}f+\partial_{(1,Vn)(t,p)}f = \partial_{(0,(\vartheta'(t))^T)}f+\partial^\circ f(t,p) .$$

It remains to show the identity

$$\partial_{(0,(\vartheta'(t))^T)}f \quad=\quad \nabla_{\Gamma(t)}f(t,p)\cdot\left(\vartheta'(t)\right)^T .$$

To this end, we consider a curve $\alpha:(t-\varepsilon,t+\varepsilon)\to\Gamma$ given through $\alpha(\tau)=(t,\beta(\tau))$, where $\beta:(t-\varepsilon,t+\varepsilon)\to\Gamma(t)$ lies completely in $\Gamma(t)$ and fulfills $\beta(t)=p$ and $\beta'(t)=(\vartheta'(t))^T$. Such a curve β can be found because $(\vartheta'(t))^T)\in T_p\Gamma(t)$.

Since the definition of the directional derivative is independent of the chosen curve, we can finally conclude

$$\begin{aligned}\partial_{(0,(\vartheta'(t))^T)} f &= \frac{d}{d\tau} f(\alpha(\tau))\Big|_{\tau=t} = \frac{d}{d\tau} f(t,\beta(\tau))\Big|_{\tau=t} = \nabla_{\Gamma(t)} f(t,p)\cdot\beta'(\tau)\\ &= \nabla_{\Gamma(t)} f(t,p)\cdot\left(\vartheta'(t)\right)^T ,\end{aligned}$$

which proves the lemma. □

We proceed by giving the normal time derivatives of mean curvature and the normal. We denote by H the function on an evolving hypersurface

$$H:\Gamma\longrightarrow\mathbb{R}\,,$$

where $H(t,p)$ is the mean curvature of the hypersurface $\Gamma(t)$ at the point $p\in\Gamma(t)$. We will need the following identity

$$\partial^\circ H(t,p) \;=\; \Delta_{\Gamma(t)} V(t,p) + |\sigma|^2(t,p)\, V(t,p)\,, \tag{2.3}$$

where $|\sigma|^2(t,p)$ is the square of the second fundamental form of $\Gamma(t)$ at $p\in\Gamma(t)$. In terms of principal curvatures κ_i of $\Gamma(t)$ this is given by

$$|\sigma|^2(t,p) \;=\; \sum_{i=1}^{n}\kappa_i^2(t,p)\,,$$

where $\kappa_i(t,p)$ is the i-th principal curvature of $\Gamma(t)$ at $p\in\Gamma(t)$. A proof of equation (2.3) is given in the appendix.

For vector fields

$$g:\Gamma\longrightarrow\mathbb{R}^l$$

we also defined the normal time derivative with the help of the same curve as for functions. Therefore we can ask for the normal time derivative of

$$n:\Gamma\longrightarrow\mathbb{R}^{n+1}\,,$$

where $n(t,p)$ is the normal of $\Gamma(t)$ at $p\in\Gamma(t)$. There is the following result

$$\partial^\circ n(t,p) \;=\; -\nabla_{\Gamma(t)} V(t,p)\,, \tag{2.4}$$

which is proven in the appendix.

In later chapters we will often describe evolving hypersurfaces $\Gamma=\cup_{t\in[0,T)}\{t\}\times\Gamma(t)$ as graphs over some fixed reference hypersurface Γ^* with additional properties. Here we want to introduce this description, calculate the normal velocity and give some formulas for the transformation of differential operators to Γ^*.

Consider a smooth mapping

$$\Phi : [0,T) \times \Gamma^* \longrightarrow \mathbb{R}^{n+1}\,,\ (t,q) \mapsto \Phi(t,q)\,, \tag{2.5}$$

where for fixed t the mapping

$$\Phi_t : \Gamma^* \longrightarrow \mathbb{R}^{n+1}\,,\ (t,q) \mapsto \Phi_t(q) := \Phi(t,q) \tag{2.6}$$

is a diffeomorphism onto its image. Then we define

$$\Gamma(t) \ := \ \{\Phi_t(q) \,|\, q \in \Gamma^*\} \tag{2.7}$$

for $t \in [0,T)$ and in this way we get an evolving hypersurface

$$\Gamma := \cup_{t\in[0,T)}\{t\} \times \Gamma(t) = \cup_{t\in[0,T)}\{t\} \times \Phi_t(\Gamma^*)\,. \tag{2.8}$$

In this case the normal velocity can be calculated with the help of the time derivative of Φ in the following way.

Lemma 2.40. *For an evolving hypersurface given by (2.8) the normal velocity at a point $(t,p) \in \Gamma$ with $p = \Phi_t(q)$ for some $q \in \Gamma^*$ reads as*

$$V(t,p) \ = \ n(t,p) \cdot \partial_t\Phi(t,q)\,.$$

Proof. The normal velocity for such a Γ at a point (t,p) with $\Phi(t,q) = (t,p)$ can be calculated in the following way. With the help of a curve on the fixed hypersurface Γ^*

$$\tilde{c} : (t-\varepsilon, t+\varepsilon) \longrightarrow \Gamma^*$$

with $\tilde{c}(t) = q$ we define a curve on $\Gamma(\tau)$ through

$$c : (t-\varepsilon, t+\varepsilon) \longrightarrow \mathbb{R}^{n+1}\,,\ \tau \mapsto \Phi(\tau, \tilde{c}(\tau))\,,$$

so that we have $c(\tau) \in \Gamma(\tau)$, $c(t) = p$ and

$$c'(\tau) \ = \ \partial_t\Phi(\tau,\tilde{c}(\tau)) + (d_{\tilde{c}(\tau)}\Phi_t)(\tilde{c}'(\tau))\,.$$

At time t, this gives

$$c'(t) \ = \ \partial_t\Phi(t,q) + (d_q\Phi_t)(\tilde{c}'(t))\,,$$

where the second term in the above sum lies in $T_p\Gamma(t)$. To compute the normal velocity $V(t,p)$, we use the normal $n(t,p)$ of $\Gamma(t)$ at $p \in \Gamma(t)$ and derive

$$\begin{aligned} V(t,p) \ &= \ n(t,p) \cdot \frac{d}{d\tau}c(\tau)\bigg|_{\tau=t} \\ &= \ n(t,p) \cdot \partial_t\Phi(t,q) + n(t,p) \cdot (d_q\Phi_t)(\tilde{c}'(t)) \\ &= \ n(t,p) \cdot \partial_t\Phi(t,q)\,, \end{aligned}$$

because the second term vanishes. $\square$

With the help of the above description of an evolving hypersurface Γ as a graph over the reference hypersurface Γ^* we can formulate quantities on Γ also on the fixed hypersurface Γ^* using an induced diffeomorphism. This will be helpful in later chapters about evolution equations on Γ, which we rewrite as equations on a fixed hypersurface Γ^* with additional properties. To be precise with our notation, we consider the following diffeomorphism

$$\hat{\Phi} : [0,T) \times \Gamma^* \longrightarrow \Gamma\,, \qquad (t,q) \mapsto \hat{\Phi}(t,q) := (t, \Phi(t,q))\ . \tag{2.9}$$

The fact that $\hat{\Phi}$ is a diffeomorphism follows from (2.5) to (2.8).

For a function

$$f : \Gamma \longrightarrow \mathbb{R}\,, \qquad (t,p) \mapsto f(t,p) \tag{2.10}$$

we define the according expression on $[0,T) \times \Gamma^*$ as

$$\tilde{f} : [0,T) \times \Gamma^* \longrightarrow \mathbb{R}\,, \qquad (t,q) \mapsto \tilde{f}(t,q) := \left(f \circ \hat{\Phi}\right)(t,q)\,, \tag{2.11}$$

which gives $\tilde{f}(t,q) = f(\hat{\Phi}(t,q)) = f(t, \Phi_t(q))$.

In the literature, the hat on $\hat{\Phi}$ is often omitted and the expressions

$$\left(f \circ \hat{\Phi}\right)(t,q) \quad \text{and} \quad (f \circ \Phi)\,(t,q) \tag{2.12}$$

are identified by a slight abuse of notation. Analog transformations also hold for vector fields $g : \Gamma \to \mathbb{R}^l$.

In the last remark of this section we want to give transformation rules for some differential operators, which will be needed in the Lemmata 3.26 and 3.27 about the linearization of surface diffusion and the natural boundary condition. Here we equip Γ^* for fixed t with the pull-back metric $g = (\Phi_t)^*\,\eta$, where η is just a symbol for the euclidian scalar product in $\mathbb{R}^{n+1}$, restricted to $\Gamma(t)$. This means for tangent vectors $v, w \in T_q\Gamma^*$

$$g(v,w) := \eta\left(d_q\Phi_t(v), d_q\Phi_t(w)\right) = d_q\Phi_t(v) \cdot d_q\Phi_t(w)\,. \tag{2.13}$$

In this way the diffeomorphism Φ_t from (2.6) gets an isometry

$$\Phi_t : (\Gamma^*, g) \longrightarrow (\Gamma(t), \eta) \tag{2.14}$$

as in Definition 2.12.

Remark 2.41. *With the above notation the following transformation rules for the surface gradient and for the Laplace-Beltrami operator hold.*

(i) $\nabla_{\Gamma(t)} f(t,p) = d_q\Phi_t\left(\nabla_{\Gamma^*}\tilde{f}(t,q)\right)$ *and*

(ii) $\Delta_{\Gamma(t)} f(t,p) = \Delta_{\Gamma^*}\tilde{f}(t,q)$,

where $(t,p) = \hat{\Phi}(t,q) \in \Gamma$ *for some* $q \in \Gamma^*$.

To make sure that we do not confuse the reader, we remark again that in these formulas Γ^* *has a different metric than the euclidian one.*

2.3 Transport equation

In the last two sections of this chapter we focus our attention again on arbitrary evolving hypersurfaces Γ not necessarily given as a graph over some fixed reference hypersurface. In this section we formulate the Transport theorem, which specifies the rate of change of the spatial integral $\int_{\Gamma(t)} f$ of a function defined on Γ.

Therefore we need an additional expression, called the normal boundary velocity $v_{\partial\Gamma}$, which is a term that describes the local decrease or increase of the surface area of $\Gamma(t)$ due to the tangential velocity of the boundary $\partial\Gamma(t)$. The definition is formally the same as Definition 2.32 for the normal velocity but instead of the unit normal of $\Gamma(t)$ we need the outer unit conormal of $\Gamma(t)$ at the boundary $\partial\Gamma(t)$.

Definition 2.42 (Normal boundary velocity)**.** *For a fixed point $(t,p) \in \partial\Gamma$, i.e. $p \in \partial\Gamma(t)$, let $c : (t-\varepsilon, t+\varepsilon) \to \mathbb{R}^{n+1}$ be a curve with $c(t) = p$ and $c(\tau) \in \partial\Gamma(\tau)$. Then we define the* **normal boundary velocity** *$v_{\partial\Gamma}$ as*

$$v_{\partial\Gamma}(t,p) \quad := \quad n_{\partial\Gamma}(t,p) \cdot \frac{d}{d\tau} c(\tau)\Big|_{\tau=t} ,$$

where $n_{\partial\Gamma}(t,p)$ is the outer unit conormal of $\Gamma(t)$ at $p \in \partial\Gamma(t)$.

Remark 2.43. *We must not confuse the normal boundary velocity $v_{\partial\Gamma}$ with the velocity vector v_Γ from Lemma 2.35. We think of $v_{\partial\Gamma}$ as an extension of the normal velocity V to the boundary, as the definition indicates. One could also call this term $V_{\partial\Gamma}$, but we stick to the literature. As in Lemma 2.33 one can show independence of the curve in Definition 2.42.*

Finally we are able to present the promised Transport theorem.

Theorem 2.44 (Transport theorem)**.** *For a smooth function $f : \Gamma \to \mathbb{R}$ there holds the following formula for the time-derivative of the spatial integral of f:*

$$\begin{aligned} \frac{d}{dt} \int\limits_{\Gamma(t)} f(t,p)\, d\mathcal{H}^n(p) \quad &= \quad \int\limits_{\Gamma(t)} \Big(\partial^\circ f(t,p) - f(t,p)\, V(t,p)\, H(t,p) \Big)\, d\mathcal{H}^n(p) \\ &\quad + \int\limits_{\partial\Gamma(t)} f(t,p)\, v_{\partial\Gamma}(t,p)\, d\mathcal{H}^{n-1}(p) \, . \end{aligned}$$

For a proof we refer to the paper [GW06] of Garcke and Wieland, where the above formula is shown for surfaces in $\mathbb{R}^3$ and can be extended directly to arbitrary dimensions.

2.4 Evolution of area and volume

In this last section of the chapter we consider evolving hypersurfaces $\Gamma = \cup_{t\in[0,T)} \{t\} \times \Gamma(t)$, which lie inside a fixed bounded region $\Omega \subset \mathbb{R}^{n+1}$ and meet the boundary $\partial\Omega$ with a right angle.

In formulas, this reads as

$$\Gamma(t) \subset \Omega\,,\ \partial\Gamma(t) \subset \partial\Omega \ \text{ and } \ n(t,p)\cdot\mu(p) = 0$$

for all $t \in [0,T]$ and $p \in \partial\Gamma(t)$, where $n(t,p)$ is a unit normal of $\Gamma(t)$ at $p \in \partial\Gamma(t)$ and $\mu(p)$ is the outer unit normal of Ω at $p \in \partial\Omega$.

Evolving hypersurfaces of this kind will appear in the next two chapters as solutions to geometric evolution equations. For a better understanding we want to know the evolution of area and volume of these solutions and in this section we give the corresponding abstract formulas involving the normal velocity V and the mean curvature H. In later parts the normal velocity is prescribed by some evolution law and we will get more explicit formulas.

We let $R(t) \subset \mathbb{R}^{n+1}$ with outer unit normal $\nu(t)$ be the region surrounded by $\Gamma(t)$ and $\partial\Omega$, so that

$$\partial R(t) = \Gamma(t) \cup \Lambda(t)\ ,$$

where $\Lambda(t)$ is the corresponding part of $\partial\Omega$. Note that we always consider embedded hypersurfaces, therefore no intersections of $\Gamma(t)$ with itself are allowed. We illustrate our notation in Figure 2.1.

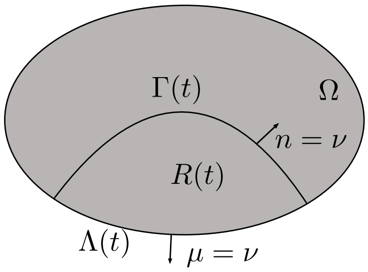


Figure 2.1: Choice of $R(t)$.

At points $p \in \Gamma(t) \subset \partial R(t)$ the outer unit normal of $R(t)$ shall equal the unit normal of $\Gamma(t)$, which means in the above notations $\nu(t,p) = n(t,p)$.

Then we can show evolution equations for the surface area $A(t)$ of $\Gamma(t)$ given through

$$A(t) \ := \ \int\limits_{\Gamma(t)} 1\, d\mathcal{H}^n \tag{2.15}$$

and for the volume of $R(t)$, also called the volume of $\Gamma(t)$, given by

$$Vol(t) \ := \ \int\limits_{R(t)} 1\, dx\ . \tag{2.16}$$

Remark 2.45. *In the literature, for example in Grosse-Brauckmann [Gro96] one often finds a different definition of the volume of* $\Gamma(t)$ *through*

$$Vol(t) = \frac{1}{n+1}\int_{\partial\Gamma(t)} p\cdot n(t,p)\, d\mathcal{H}^{n-1}\,.$$

We just want to remark that we have to take care of the fixed region Ω, where the evolving hypersurface lies in. Therefore the definition of this remark would not make sense in our case, since this would neglect terms appearing due to the right angle at $\Gamma(t)\cap\partial\Omega$, which can be observed in the proof of the following lemma.

For the convenience of the reader we present the formulas for the evolution of area and volume with proof in our notation.

Lemma 2.46. *With the above notations the following formulas for the time derivative of area and volume hold true.*

(i) $\frac{d}{dt}A(t) = -\int_{\Gamma(t)} V\,H\,d\mathcal{H}^n$ *and*

(ii) $\frac{d}{dt}Vol(t) = \int_{\Gamma(t)} V\,d\mathcal{H}^n$.

Proof. ad (i): Due to the Transport theorem 2.44 with $f \equiv 1$ we can conclude

$$\begin{aligned}\frac{d}{dt}A(t) &= \frac{d}{dt}\int\limits_{\Gamma(t)} 1\,d\mathcal{H}^n = -\int_{\Gamma(t)} V\,H\,d\mathcal{H}^n + \int_{\partial\Gamma(t)} \underbrace{v_{\partial\Gamma}}_{=0}\,d\mathcal{H}^{n-1}\\ &= -\int_{\Gamma(t)} V\,H\,d\mathcal{H}^n\,,\end{aligned}$$

where the normal boundary velocity vanishes due to the right angle condition. In fact it holds for $(t,p)\in\Gamma$ with $p\in\partial\Gamma(t)$

$$v_{\partial\Gamma}(t,p) = n_{\partial\Gamma}(t,p)\cdot\frac{d}{d\tau}c(\tau)\Big|_{\tau=t}$$

for a curve $c:(t-\varepsilon,t+\varepsilon)\to\mathbb{R}^{n+1}$ with $c(t)=p$ and $c(\tau)\in\partial\Gamma(\tau)$. Since $\partial\Gamma(\tau)\subset\partial\Omega$ it holds that $c'(t)\in T_p\partial\Omega$ and the facts $n_{\partial\Gamma}(t,p)\in T_p\Gamma(t)$ and $T_p\Gamma(t)\perp T_p\partial\Omega$ then imply $v_{\partial\Gamma}(t,p)=0$.

ad (ii): For the identity on $\mathbb{R}^{n+1}$, $id:\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$, we get $\operatorname{div} id = n+1$ and therefore with the help of Gauß' theorem for regions with Lipschitz boundary

$$\begin{aligned}(n+1)\,Vol(t) &= (n+1)\int\limits_{R(t)} 1\,dx = \int_{R(t)} \operatorname{div} id(x)\,dx\\ &= \int_{\partial R(t)} p\cdot\nu(t,p)\,d\mathcal{H}^n(p)\\ &= \underbrace{\int_{\Lambda(t)} p\cdot\mu(p)\,d\mathcal{H}^n(p)}_{(1)} + \underbrace{\int_{\Gamma(t)} p\cdot n(t,p)\,d\mathcal{H}^n(p)}_{(2)}\ .\end{aligned}$$

We consider the above terms separately and get for the first one due to the Transport theorem 2.44 with $f(p) = (p \cdot \mu(p))$ the following identity

$$\begin{aligned} \frac{d}{dt}\int_{\Lambda(t)} p\cdot\mu(p)\, d\mathcal{H}^n(p) &= \int_{\Lambda(t)} \left(\partial^\circ_\Lambda (p\cdot\mu(p)) - (p\cdot\mu(p))V_\Lambda\, H_\Lambda\right)\, d\mathcal{H}^n(p) \\ &\quad + \int_{\partial\Lambda(t)} (p\cdot\mu(p))v_{\partial\Lambda}\, d\mathcal{H}^{n-1}\;, \end{aligned}$$

where V_Λ, H_λ and $v_{\partial\Lambda}$ are the normal velocity, the mean curvature and the normal boundary velocity of the evolving hypersurface $\Lambda = \cup_{t\in[0,T]}\{t\}\times\Lambda(t)$. Here we have extended $f(p) = (p\cdot\mu(p))$ as in Lemma 2.38 to Λ through $f(t,p) = f(p)$.

Since the hypersurfaces $\Lambda(t)$ do not move in normal direction, we get

$$V_\Lambda \equiv 0\;.$$

In fact, for $p \in \Lambda(t)$, we have

$$V_\Lambda(t,p) = n_\Lambda(t,p)\cdot \frac{d}{d\tau}c(\tau)\bigg|_{\tau=t}\;,$$

where n_Λ is a normal of $\Lambda(t)$ chosen such that $n_\Lambda(t,p) = \mu(p) \in N_p\partial\Omega$ and $c : (t-\varepsilon, t+\varepsilon) \to \mathbb{R}^{n+1}$ is a curve with $c(t) = p$ and $c(\tau) \in \Lambda(\tau)$. Since $\Lambda(\tau) \subset \partial\Omega$, we conclude $c'(t) \in T_p\partial\Omega$ and therefore the above scalar product $n_\Lambda(t,p)\cdot c'(t)$ vanishes.

The normal time derivative $\partial^\circ_\Lambda f$ for $f(t,p) = p\cdot\mu(p)$ is derived with the help of Lemma 2.38. So we get

$$\partial^\circ_\Lambda\,(p\cdot\mu(p)) = V_\Lambda(t,p)\, n_\Lambda(t,p) = 0\,.$$

Altogether this means for the first term (1) in the above formula for the volume

$$\frac{d}{dt}\int_{\Lambda(t)} p\cdot\mu(p)\, d\mathcal{H}^n(p) = \int_{\partial\Lambda(t)} (p\cdot\mu(p))v_{\partial\Lambda}\, d\mathcal{H}^{n-1}\;.$$

For the second term (2), we observe again with the help of the Transport theorem 2.44 the following identity

$$\begin{aligned} \frac{d}{dt}\int_{\Gamma(t)} (p\cdot n(t,p))\, d\mathcal{H}^n(p) &= \int_{\Gamma(t)} \left(\partial^\circ(p\cdot n(t,p)) - (p\cdot n(t,p))\, V\, H\right)\, d\mathcal{H}^n(p) \\ &\quad + \int_{\partial\Gamma(t)} (p\cdot n(t,p))\, v_{\partial\Gamma}\, d\mathcal{H}^{n-1}\;. \end{aligned}$$

As in (i) we get $v_{\partial\Gamma} = 0$. For the normal time derivative of $(p\cdot n(t,p)$, we observe with the help of Lemma 2.38 and the formula (2.4) for the normal time derivative of the normal

$$\begin{aligned} \partial^\circ\Big(p\cdot n(t,p)\Big) &= \partial^\circ id(p)\cdot n(t,p) + p\cdot\partial^\circ n(t,p) \\ &= V(t,p)\, n(t,p)\cdot n(t,p) - p\cdot\nabla_{\Gamma(t)}V(t,p) \\ &= V(t,p) - p\cdot\nabla_{\Gamma(t)}V(t,p)\,. \end{aligned}$$

Therefore we conclude for the second term (2) in the formula for the volume

$$\frac{d}{dt}\int_{\Gamma(t)} (p\cdot n(t,p))\, d\mathcal{H}^n(p) \;=\; \int_{\Gamma(t)} \big(V - p\cdot\nabla_{\Gamma(t)}V - (p\cdot n(t,p))\,V\,H\big)\; d\mathcal{H}^n(p)\,.$$

The term in the middle can be integrated by parts with the help of Gauß' theorem on hypersurfaces 2.29 with $f(p) = p\,V(t,p)$ for fixed t to get

$$\begin{aligned} -\int_{\Gamma(t)} p\cdot\nabla_{\Gamma(t)}V\, d\mathcal{H}^n \;&=\; \int_{\Gamma(t)} \left(\underbrace{\operatorname{div}_{\Gamma(t)} id}_{=\,\mathrm{n}}\; V + (p\cdot n(t,p))\,V\,H\right) d\mathcal{H}^n \\ &\quad - \int_{\partial\Gamma(t)} V\,p\cdot n_{\partial\Gamma}(t,p)\mathrm{d}\mathcal{H}^{n-1}\,. \end{aligned}$$

The identity $\operatorname{div}_{\Gamma(t)} id = n$ can be seen with a local parametrization (U,γ,V) of $\Gamma(t)$ with $\gamma(u) = p$ through

$$\begin{aligned} \operatorname{div}_{\Gamma(t)} id(p) \;&=\; \sum_{i,j=1}^{n} g^{ij}(u)\left(\underbrace{\partial_i(id\circ\gamma)(u)}_{=\,\partial_i\gamma(u)}\cdot\partial_j\gamma(u)\right) \\ &=\; \sum_{i,j=1}^{n} g^{ij}(u)g_{ij}(u) = \operatorname{trace}(G^{-1}\cdot G) = n\,. \end{aligned}$$

For the second term (2) in the formula for the volume we get therefore

$$\frac{d}{dt}\int_{\Gamma(t)} (p\cdot n(t,p))\, d\mathcal{H}^n(p) \;=\; (n+1)\int_{\Gamma(t)} V\, d\mathcal{H}^n - \int_{\partial\Gamma(t)} (p\cdot n_{\partial\Gamma}(t,p))\,V\, d\mathcal{H}^{n-1}\,.$$

Altogether for the derivative of volume we observe

$$\begin{aligned} \frac{d}{dt}(n+1)Vol(t) \;&=\; (n+1)\int_{\Gamma(t)} V\, d\mathcal{H}^n \\ &\quad + \int_{\partial\Lambda(t)} (p\cdot\mu(p))v_{\partial\Lambda}\, d\mathcal{H}^{n-1} - \int_{\partial\Gamma(t)} (p\cdot n_{\partial\Gamma}(t,p))\,V\, d\mathcal{H}^{n-1}\,. \end{aligned}$$

To proceed with the proof we remark that $\partial\Lambda(t) = \partial\Gamma(t)$ and that $\mu(p) = n_{\partial\Gamma}(t,p)$ for $p\in\Gamma(t)$ due to the right angle condition. Furthermore we have for $p\in\partial\Gamma(t)$

$$\begin{aligned} v_{\partial\Lambda}(t,p) \;&=\; \mu_{\partial\Lambda}(t,p)\cdot c'(t) \quad\text{and} \\ V(t,p) \;&=\; n(t,p)\cdot a'(t)\,, \end{aligned}$$

where $\mu_{\partial\Lambda}$ is the unit outer conormal of Λ at $\partial\Lambda$, c is a curve with $c(\tau)\in\partial\Lambda(\tau) = \partial\Gamma(\tau)$ and a is a curve with $a(\tau)\in\Gamma(\tau)$. Since the normal velocity is independent of the chosen curve from Lemma 2.33, we can also use c instead of a. Again due to the right angle condition we observe that $\mu_{\partial\Lambda}(t,p) = n(t,p)$ and so we arrive at $v_{\partial\Lambda}(t,p) = V(t,p)$ for $p \in \partial\Gamma(t)$. This leads us to the desired derivative

$$\frac{d}{dt}Vol(t) \quad = \quad \int_{\Gamma(t)} V \, d\mathcal{H}^n$$

and we finished the proof. $\square$

Chapter 3

Evolution Equations with Boundary Contact

In this chapter we consider evolution laws for evolving hypersurfaces which lie inside a fixed region and meet the boundary at a right angle. These laws are the mean curvature flow, the volume preserving mean curvature flow and the surface diffusion flow. The main goal is to extend the linearized stability analysis of Garcke, Ito and Kohsaka [GIK05] for surface diffusion with boundary contact for curves in $\mathbb{R}^2$ to the case of hypersurfaces in $\mathbb{R}^{n+1}$.

To this end, we first have to introduce a setting that allows us to formulate the geometric evolution laws as partial differential equations for functions defined on a fixed reference hypersurface, which will be a stationary solution. As the parametrization for the curve situation in [GIK05] does not extend to the higher dimensional case, we use a curvilinear coordinate system from the work of Vogel [Vog00] that takes into account a possible curved boundary.

Then we linearize the resulting partial differential equations and with the help of abstract spectral theory we give a criterion for the stability of these linear equations using the positivity of some explicitly given bilinear form. An important ingredient will be to recognize the linearizations as gradient flows, since only then we can show self-adjointness of the linearized operator.

In the first Section 3.1 we give the representation of the evolving hypersurfaces in detail without demanding an explicit evolution law. Therefore we can refer to this representation in later sections.

The mean curvature flow is considered in Section 3.2, but we have to mention that a lot of work is already done in this case. For example we refer to Ei, Sato and Yanagida [ESY96] and to Stahl [Sta95, Sta96], whose work is based on results of Huisken [Hui84, Hui86] and others. Our aim in this section is to do linearized stability analysis in the spirit of [GIK05], which does not use a maximum principle and therefore can be extended to the later sections.

In the third Section 3.3 of this chapter we consider the volume preserving mean curvature flow, for which the linearized stability analysis is a straightforward extension of the previous one.

Then in Section 3.4 we treat the surface diffusion flow for which some additional considerations in the stability analysis have to be done. In particular we introduce the H^{-1}-inner product and identify the linearization as an H^{-1}-gradient flow. We also give some remarks concerning

nonlinear stability, although we don't formulate precise results in this case.
In the final Section 3.5 we consider specific situations where the linearized stability can be determined.

3.1 Parametrization

In this first section of the chapter we want to introduce the setting for the considered hypersurfaces. With the notation of Chapter 2 we want to describe evolving hypersurfaces

$$\Gamma = \bigcup_{t\in[0,T)} \{t\} \times \Gamma(t) \quad \text{with} \quad \Gamma(t) \subset \mathbb{R}^{n+1}$$

as in Definition 2.31, which evolve due to some evolution law. As in the remark after Definition 2.31 we choose a smooth unit field $n : \Gamma \to S^n \subset \mathbb{R}^{n+1}$, such that $n(t,.) : \Gamma(t) \to S^n$ are unit normal fields of $\Gamma(t)$. We remark that due to our basic assumptions for hypersurfaces from Definition 2.4 these normals can be extended smoothly up to the boundary.
The hypersurfaces $\Gamma(t)$ shall lie inside a fixed bounded region $\Omega \subset \mathbb{R}^{n+1}$, i.e. $\Gamma(t) \subset \Omega$, and the boundary $\partial\Gamma(t)$ of each of the hypersurfaces shall intersect the boundary $\partial\Omega$ of the fixed region at a right angle, i.e. $\partial\Gamma(t) \subset \partial\Omega$ and $\Gamma(t) \perp \partial\Omega$. This angle condition will be described with the help of the unit normal $n(t)$ of $\Gamma(t)$ and the unit outer normal μ of $\partial\Omega$ through

$$n(t) \cdot \mu = 0 \quad \text{on} \;\; \partial\Gamma(t) \cap \partial\Omega$$

for all $t \in [0, T)$. Imposing also a smooth starting configuration Γ_0, which lies in Ω and fulfills the angle condition, we will consider motions of the following type

$$\begin{cases} \text{some evolution law} & \text{in } \Gamma(t) & \text{for all } \; t > 0 \,, \\ \text{corresponding boundary conditions} & \text{on } \partial\Gamma(t) & \text{for all } \; t > 0 \,, \\ \Gamma(t) \subset \Omega & & \text{for all } \; t > 0 \,, \\ \partial\Gamma(t) \subset \partial\Omega & & \text{for all } \; t > 0 \,, \\ n(t) \cdot \mu = 0 & \text{on } \partial\Gamma(t) & \text{for all } \; t > 0 \,, \\ \Gamma(0) = \Gamma_0 \,. & & \end{cases} \tag{3.1}$$

A basic assumption to formulate (3.1) as a partial differential equation for some unknown function ρ concerns the appearance of the hypersurfaces $\Gamma(t)$, that we want to consider. We suppose that these can be written as a graph over a fixed reference hypersurface Γ^* with the help of a function

$$\rho : [0, T) \times \Gamma^* \longrightarrow \mathbb{R} \,.$$

The reference hypersurface Γ^* has to fulfill the same geometric properties as $\Gamma(t)$, i.e. $\Gamma^* \subset \Omega$, $\partial\Gamma^* \subset \partial\Omega$ and $\Gamma^* \perp \partial\Omega$. In later sections, Γ^* will be a stationary solution of (3.1), i.e. it will also fulfill the stationary evolution law and corresponding boundary conditions.
As a first step to describe the regarded hypersurfaces $\Gamma(t)$, we set up a specific curvilinear coordinate system as in the work of Vogel [Vog00], that takes into account a possible curved

boundary $\partial\Omega$ and the fact, that the considered hypersurfaces have to stay inside Ω and their boundary has to lie on $\partial\Omega$. Therefore, we postulate for small $d > 0$ the existence of a smooth mapping

$$\Psi : \Gamma^* \times (-d, d) \longrightarrow \Omega\,, \qquad (q, w) \longmapsto \Psi(q, w)\,, \tag{3.2}$$

such that

$$\Psi(q, 0) = q \quad \text{for all } \; q \in \Gamma^* \tag{3.3}$$

and

$$\Psi(q, w) \in \partial\Omega \quad \text{for all } \; q \in \partial\Gamma^*\,. \tag{3.4}$$

We also assume that for every (local) parametrization $q : D \to \Gamma^*$ with $D \subset \mathbb{R}^n$ open, the mapping

$$(y, w) \mapsto \Psi(q(y), w)$$

is a locally invertible map from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$. At last, we choose a normal n^* of Γ^* and impose the condition that

$$\partial_w \Psi(q, 0) \cdot n^*(q) \neq 0 \;\text{ for }\; q \in \Gamma^*\,,$$

which means that there is some movement in normal direction. With a rescaling in the w-coordinate we can then even assume that

$$\partial_w \Psi(q, 0) \cdot n^*(q) = 1 \;\text{ for }\; q \in \Gamma^*\,. \tag{3.5}$$

In [Vog00] there are some examples for situations when such a curvilinear coordinate system exists. Due to the angle condition at the boundary of Γ^*, we can conclude even more than (3.5) at the boundary $\partial\Gamma^*$.

Lemma 3.1. *For $q \in \partial\Gamma^*$, it holds that $\partial_w \Psi(q, 0) = n^*(q)$.*

Proof. We see that for fixed $q \in \partial\Gamma^*$ the curve $c(w) := \Psi(q, w)$ lies on the boundary $\partial\Omega$, and with $c(0) = \Psi(q, 0) = q$ it therefore holds $\partial_w \Psi(q, 0) \in T_q(\partial\Omega)$. With the help of the angle condition we get $T_q\Gamma^* \perp T_q(\partial\Omega)$ and so we observe that $\partial_w \Psi(q, 0) \cdot v = 0$ for all $v \in T_q\Gamma^*$. So $\partial_w \Psi(q, 0)$ has just a normal part, that is $\partial_w \Psi(q, 0) = (\partial_w \Psi(q, 0) \cdot n^*(q))\, n^*(q)$. With the rescaling condition of the normal (3.5) the claim follows. $\Box$

With the help of the mapping Ψ from (3.2) we are in a position to define the hypersurfaces, that we want to consider. For a given smooth function

$$\rho : [0, T) \times \Gamma^* \longrightarrow (-d, d) \tag{3.6}$$

we introduce the mapping

$$\Phi^\rho : [0, T) \times \Gamma^* \longrightarrow \Omega\,, \qquad \Phi^\rho(t, q) := \Psi(q, \rho(t, q))\,. \tag{3.7}$$

Then we observe that for fixed t due to the assumptions on Ψ, the function

$$\Phi_t^\rho : \Gamma^* \longrightarrow \Omega\,, \qquad \Phi_t^\rho(q) := \Phi^\rho(t,q) \tag{3.8}$$

is a diffeomorphism onto its image. We denote this image by $\Gamma_\rho(t)$, that is

$$\Gamma_\rho(t) \;:=\; \{\Phi_t^\rho(q) \mid q \in \Gamma^*\}\,. \tag{3.9}$$

In such a way we get an evolving hypersurface

$$\Gamma \;=\; \bigcup_{t\in[0,T)} \{t\} \times \Gamma_\rho(t)$$

and we made sure that the hypersurfaces $\Gamma_\rho(t)$ always fulfill the conditions

$$\Gamma_\rho(t) \subset \Omega \quad \text{and} \quad \partial\Gamma_\rho(t) \subset \partial\Omega\,.$$

We also observe that for $\rho \equiv 0$ it holds

$$\Gamma_{\rho\equiv 0}(t) = \Gamma^* \quad \text{for all} \ \in [0,T)\,. \tag{3.10}$$

Remark 3.2. *If the hypersurfaces would be closed without boundary, it would make sense to prescribe graphs over Γ^* with the help of a variation in normal direction. This is done for example in the work of Escher, Mayer and Simonett [EMS98]. With the above notations they use a mapping Ψ of the special form*

$$\Psi : \Gamma^* \times (-d,d) \longrightarrow \mathbb{R}^{n+1}\,, \qquad \Psi(q,w) := q + w\, n^*(q)\;,$$

where n^* is one chosen normal of Γ^*. For small $d > 0$, this mapping is a diffeomorphism onto its image and one could set hypersurfaces $\Gamma(t)$ defined through

$$\Gamma(t) \;:=\; im(q \mapsto \Psi(q, \rho(q,t))\,,$$

where ρ is a function as in (3.6). With the help of this setting the authors of [EMS98] use center manifold theory to get a stability criterion. Due to the highly nonlinear structure of the corresponding boundary conditions for the evolution laws, that are considered in this work, it seems to be very difficult to generalize their approach to the setting with boundary.

The last assumption concerns the starting hypersurface Γ_0. We impose that it is given with the help of a smooth function $\rho_0 : \Gamma^* \to \mathbb{R}$ through

$$\Gamma_0 \;=\; \{\Psi(q, \rho_0(q)) \,|\, q \in \Gamma^*\}\,. \tag{3.11}$$

With the help of the diffeomorphisms Φ_t^ρ, we can finally formulate (3.1) over the fixed stationary hypersurface Γ^* as follows. Find ρ as in (3.6) as a solution to the problem of the type

$$\begin{cases} \text{some evolution law} & \text{in } \Gamma^* \quad \text{for all } t > 0\,, \\ \text{corresponding boundary conditions} & \text{on } \partial\Gamma^* \quad \text{for all } t > 0\,, \\ (n \cdot \mu)\,(\Phi^\rho(t,q)) = 0 & \text{on } \partial\Gamma^* \quad \text{for all } t > 0\,, \\ \rho(0,q) = \rho_0(q) & \text{in } \Gamma^*\,. \end{cases} \tag{3.12}$$

Here we used the common abbreviation $n(\Phi^\rho(t,q)) = n(\hat{\Phi}^\rho(t,q)) = n(t, \Phi^\rho(t,q))$ with $\hat{\Phi}^\rho(t,q) = (t, \Phi^\rho(t,q)) \in \Gamma$ a point on the evolving hypersurface, as explained in (2.12). Note that with the definition $\Gamma_\rho(t) = \Phi_t^\rho(\Gamma^*)$ for a solution ρ of problem (3.12) the geometric properties of (3.1) are automatically fulfilled.

3.2 Mean curvature flow

In this section we consider the mean curvature flow $V = H$ with the boundary conditions described in the previous Section 3.1. We postulate the special representation of the evolving hypersurface as a graph and formulate the geometric equation as a partial differential equation for an unknown function ρ. Then we linearize this equation around a stationary state given by $\rho \equiv 0$ and describe the stability with the help of a bilinear form.

First we want to specify the problem in detail. In words, we want to find an evolving hypersurface

$$\Gamma = \bigcup_{t\in[0,T)} \{t\} \times \Gamma(t) \quad \text{with} \quad \Gamma(t) \subset \mathbb{R}^{n+1}\,, \tag{3.13}$$

as in Definition 2.31, evolving due to the mean curvature flow, such that $\Gamma(t)$ lies in a fixed bounded region $\Omega \subset \mathbb{R}^{n+1}$ and the boundary $\partial\Gamma(t)$ of each of the hypersurfaces intersects the boundary $\partial\Omega$ of the fixed region at a right angle.

In formulas, the problem reads as follows. Find Γ as in (3.13), such that

$$\left\{\begin{array}{rcllr} V &=& H & \text{in } \Gamma(t) & \text{for all } t>0\,, \\ \Gamma(t) &\subset& \Omega & & \text{for all } t>0\,, \\ \partial\Gamma(t) &\subset& \partial\Omega & & \text{for all } t>0\,, \\ n(t)\cdot\mu &=& 0 & \text{on } \partial\Gamma(t) & \text{for all } t>0\,, \\ \Gamma(0) &=& \Gamma_0\,. & & \end{array}\right. \tag{3.14}$$

Here V and H are the normal velocity and the mean curvature of the evolving hypersurface Γ, which means that $H(t)$ is the mean curvature of $\Gamma(t)$. We denote by n a vector field on Γ such that $n(t,p)$ is a unit normal vector of $\Gamma(t)$ at $p \in \Gamma(t)$ and μ is the outer unit normal of Ω. Γ_0 is a starting hypersurface which fulfills the geometric properties $\Gamma_0 \subset \Omega$, $\partial\Gamma_0 \subset \partial\Omega$ and $\Gamma_0 \perp \partial\Omega$.

Note that the line "corresponding boundary conditions on $\partial\Gamma(t)$" from (3.1) reduces to the angle condition, since no additional boundary conditions are needed.

We want to determine criteria for linearized stability around a stationary solution Γ^* of (3.14). This means that Γ^* fulfills a time-independent version of (3.14), i.e. for the mean curvature H^* of Γ^* it holds $H^* \equiv 0$ and furthermore Γ^* has the geometric properties $\Gamma^* \subset \Omega$, $\partial\Gamma^* \subset \partial\Omega$ and fulfills the right angle condition, i.e. $n^* \cdot \mu = 0$ on $\partial\Gamma^*$, where n^* is a unit normal of Γ^*.

The condition $H^* \equiv 0$ is the characterization of a minimal hypersurface. Although there is a large amount of literature concerning the theory and examples of minimal hypersurfaces, mostly for the case of surfaces in $\mathbb{R}^3$, we will not use anything aside from the vanishing of mean curvature.

3.2.1 Resulting partial differential equation

To rewrite the geometric evolution law (3.14) as a partial differential equation for an unknown function, we consider special solutions Γ of (3.14). Therefore we fix a stationary solution Γ^* as above and consider hypersurfaces $\Gamma_\rho(t)$ given as in Section 3.1 with the help of a function

$$\rho : [0,T) \times \Gamma^* \longrightarrow (-d,d)$$

through a diffeomorphism onto its image

$$\Phi_t^\rho : \Gamma^* \longrightarrow \Omega$$

by

$$\Gamma_\rho(t) = \Phi^\rho(t, \Gamma^*)\,.$$

The details of this construction were given in Section 3.1.

The corresponding equation for ρ on the fixed stationary hypersurface Γ^* as in (3.12) reads here

$$\left\{ \begin{array}{rcll} V(\Phi^\rho(t,q)) & = & H(\Phi^\rho(t,q)) & \text{in } \Gamma^* \quad \text{for all } t > 0\,, \\ (n\cdot\mu)(\Phi^\rho(t,q)) & = & 0 & \text{on } \partial\Gamma^* \quad \text{for all } t > 0\,, \\ \rho(0,q) & = & \rho_0(q) & \text{in } \Gamma^*\,. \end{array} \right. \tag{3.15}$$

As explained in (2.12), we use the common abbreviation $V(\Phi(t,q)) = V(t, \Phi(t,q))$ and analogously for H and n.

We also give equation (3.15) in terms of the mapping Ψ

$$\left\{ \begin{array}{rcll} V(\Psi(q,\rho(t,q))) & = & H(\Psi(q,\rho(t,q))) & \text{in } \Gamma^* \quad \text{for all } t > 0\,, \\ (n\cdot\mu)(\Psi(q,\rho(t,q))) & = & 0 & \text{on } \partial\Gamma^* \quad \text{for all } t > 0\,, \\ \rho(0,q) & = & \rho_0(q) & \text{in } \Gamma^*\,, \end{array} \right. \tag{3.16}$$

where the dependence on ρ can be seen directly.

3.2.2 Linearization around a stationary state

The idea of linearized stability is the following. For a starting hypersurface Γ_0 close to the stationary solution Γ^* we consider a solution $\Gamma(t)$ of a linearized version of (3.14) and try to find criteria for the convergence of $\Gamma(t)$ to Γ^* in some sense for $t \to \infty$. By a linearized version around Γ^* of some geometric evolution equation, here (3.14), we always mean the linearization of the corresponding equation for the unknown function ρ, here (3.16). Since Γ^* corresponds to $\rho \equiv 0$ this means more precisely the linearization of (3.16) around $\rho \equiv 0$. The criterion that we will give in Theorem 3.17 yields asymptotic stability of the linearized equation by the positivity of some bilinear form.

To get the linearization of (3.16) around $\rho \equiv 0$, we write $\varepsilon\rho$ instead of ρ in (3.16), differentiate with respect to ε and set $\varepsilon = 0$. This gives a linear partial differential equation for ρ, which will be examined further.

Remark 3.3. *Although the above explanation is exactly the usual approach to build a linearization, we give a formally correct description with the help of the first variation. Therefore we consider each term in the first line of (3.16) as operator*

$$F : C^\infty(\Gamma^*) \to C^\infty(\Gamma^*)\,, \qquad \rho \mapsto F(\rho)\,,$$

(omit the t-variable) and define the first variation of F at $\rho \equiv 0$ in direction ρ as

$$\delta F(\rho) := \frac{\partial F}{\partial \rho}(0)(\rho) = \frac{d}{d\varepsilon} F(\varepsilon\rho)\bigg|_{\varepsilon=0}\,.$$

An analogous description is done for the boundary equation in the second line of (3.16). This is the formulation that we always have in mind when building the linearization of (3.16) around $\rho \equiv 0$.

The next steps consist in building the linearization of each term in (3.16). For the normal velocity, we have the following formula.

Lemma 3.4. *The linearization of normal velocity in (3.16) is given through the following expression for* $q \in \Gamma^*$ *and* $t > 0$

$$\frac{d}{d\varepsilon} V(t, \Psi(q, \varepsilon\rho(t,q)))\Big|_{\varepsilon=0} = \partial_t \rho(t,q)\,. \tag{3.17}$$

Proof. From Lemma 2.40 of Chapter 2 we have the following representation of normal velocity

$$\begin{aligned} V(t, \Psi(q, \rho(t,q))) &= n(t, \Psi(q, \rho(t,q))) \cdot \frac{d}{dt} \Psi(q, \rho(t,q)) \\ &= \Big(n(t, \Psi(q, \rho(t,q))) \cdot \partial_w \Psi(q, \rho(t,q)) \Big)\, \partial_t \rho(t,q)\,. \end{aligned}$$

Therefore we can calculate

$$\begin{aligned} \frac{d}{d\varepsilon} V(t, \Psi(q, \varepsilon\rho(t,q)))\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon} \Big(n(t, \Psi(q, \varepsilon\rho(t,q))) \cdot \partial_w \Psi(q, \varepsilon\rho(t,q)) \Big)\Big|_{\varepsilon=0} \underbrace{(\partial_t \varepsilon\rho(t,q))|_{\varepsilon=0}}_{=0} \\ &\quad + \Big(n(t, \Psi(q, \varepsilon\rho(t,q))) \cdot \partial_w \Psi(q, \varepsilon\rho(t,q)) \Big)\Big|_{\varepsilon=0} \partial_t \rho(t,q) \\ &= \big(n(t, \Psi(q,0)) \cdot \partial_w \Psi(q,0) \big)\, \partial_t \rho(t,q) \\ &\overset{(3.3)}{=} \big(n^*(q) \cdot \partial_w \Psi(q,0) \big)\, \partial_t \rho(t,q) \\ &= \partial_t \rho(t,q)\,, \end{aligned}$$

where we used (3.5) in the last line. To see $n(t, \Psi(q,0)) = n^*(q)$ in the line before, we observe the fact that $n(t, \Psi(q, \varepsilon\rho(t,q)))$ is the normal of $\Gamma_{\varepsilon\rho}(t)$ at $\Psi(q, \varepsilon\rho(t,q)) \in \Gamma_{\varepsilon\rho}(t)$, so that for $\varepsilon = 0$ the term $n(t, \Psi(q,0))$ is the normal of $\Gamma_{\rho\equiv 0}(t)$ at $\Psi(q,0) \in \Gamma_{\rho\equiv 0}$. With (3.3) and $\Gamma_{\rho\equiv 0}(t) = \Gamma^*$ for all t we find that $n(t, \Psi(q,0)) = n(t,q) = n^*(q)$ is the normal of Γ^* at $q \in \Gamma^*$. □

In the next note, we show a formula for the linearization of mean curvature, which is well-known in the literature, but we give a basic proof with the help of the abstract results from Chapter 2.

Lemma 3.5. *The linearization of mean curvature in (3.16) for* $q \in \Gamma^*$ *and* $t > 0$ *is given by*

$$\frac{d}{d\varepsilon} H(t, \Psi(q, \varepsilon\rho(t,q)))\Big|_{\varepsilon=0} = \Delta_{\Gamma^*} \rho(t,q) + |\sigma^*|^2(q)\, \rho(t,q)\,, \tag{3.18}$$

where Δ_{Γ^*} *is the Laplace-Beltrami operator on* Γ^* *and* $|\sigma^*|^2$ *is the square of the norm of the second fundamental form of* Γ^*, *given through* $|\sigma^*|^2 = \sum_{i=1}^n (\kappa_i^*)^2$ *with the principal curvatures* κ_i^* *of* Γ^*.

Proof. We prove this formula by using the concept of the normal time derivative from Definition 2.36 and in particular formula (2.3) for the normal time derivative of mean curvature. We will also need the relationship between different types of derivatives from Lemma 2.39. To use these facts from Chapter 2 we have to change our notation slightly, since we consider derivatives with respect to ε instead of t as in Chapter 2. Therefore we will introduce an evolving hypersurface

$$\widetilde{\Gamma} \quad = \quad \bigcup_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)} \{\varepsilon\} \times \widetilde{\Gamma}(\varepsilon)$$

parametrized by ε instead of t in the following way. We fix t and consider for small $\varepsilon_0 > 0$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ the mapping

$$\Phi_t^{\varepsilon\rho} : \Gamma^* \longrightarrow \Omega\,, \qquad \Phi_t^{\varepsilon\rho}(q) := \Psi(q, \varepsilon\rho(t,q))\,,$$

which is a diffeomorphism onto its image and set

$$\widetilde{\Gamma}(\varepsilon) := \operatorname{im}(\Phi_t^{\varepsilon\rho})\,.$$

There is a one-to-one relation between $\widetilde{\Gamma}$ and Γ given by

$$\widetilde{\Gamma}(\varepsilon) \quad = \quad \Gamma_{\varepsilon\rho}(t)\,,$$

so that we see that the hypersurfaces $\widetilde{\Gamma}(\varepsilon)$ are just a renaming of the previous ones. In particular it holds $\widetilde{\Gamma}(0) = \Gamma_{\rho\equiv 0}(t) = \Gamma^*$.

To calculate the normal-velocity of $\widetilde{\Gamma}$ at (ε, p), we let $p = \Phi_t^{\varepsilon\rho}(q)$ for some $q \in \Gamma^*$ and proceed

$$\begin{aligned}
\widetilde{V}(\varepsilon, p) = \widetilde{V}(\varepsilon, \Phi_t^{\varepsilon\rho}(q)) \quad &= \quad \widetilde{n}(\varepsilon, \Phi_t^{\varepsilon\rho}(q)) \cdot \frac{d}{d\varepsilon}\Phi_t^{\varepsilon\rho}(q) \\
&= \quad \Big(\widetilde{n}(\varepsilon, \Phi_t^{\varepsilon\rho}(q)) \cdot \partial_w \Psi(q, \varepsilon\rho(t,q))\Big)\; \rho(t,q)\;,
\end{aligned}$$

where $\widetilde{n}(\varepsilon, p)$ is the normal of $\widetilde{\Gamma}(\varepsilon)$ at $p \in \widetilde{\Gamma}(\varepsilon)$. With the fact $\widetilde{n}(0,q) = n^*(q)$, which is justified with the same lines as $n(t,q) = n^*(q)$ in the proof of the linearization of normal velocity in Lemma 3.4, we get for $\varepsilon = 0$

$$\widetilde{V}(0,q) \quad = \quad \big(\widetilde{n}(0,q) \cdot \partial_w \Psi(q,0)\big)\; \rho(t,q) \quad = \quad \rho(t,q)\;,$$

where we used (3.5).

The next point is to observe that the mean curvature of $\widetilde{\Gamma}(\varepsilon)$ at $p \in \widetilde{\Gamma}(\varepsilon)$, denoted by $\widetilde{H}(\varepsilon, p)$, is due to the fact $\widetilde{\Gamma}(\varepsilon) = \Gamma_{\varepsilon\rho}(t)$ also the mean curvature of $\Gamma_{\varepsilon\rho}(t)$ at $p \in \Gamma_{\varepsilon\rho}(t)$, denoted by $H(t,p)$. With $p = \Phi_t^{\varepsilon\rho}(q)$ for some $q \in \Gamma^*$ (note that t is always fixed at the moment) this gives

$$\widetilde{H}(\varepsilon, \Phi_t^{\varepsilon\rho}(q)) \quad = \quad H(t, \Phi_t^{\varepsilon\rho}(q))\,,$$

where $\widetilde{H}$ is defined on $\widetilde{\Gamma}$ and H on Γ.

So we get for the linearization of mean curvature, which we wanted to calculate,

$$\frac{d}{d\varepsilon} H(t, \Phi_{\varepsilon\rho}(t,q)) \quad = \quad \frac{d}{d\varepsilon} \widetilde{H}(\varepsilon, \Phi_{\varepsilon\rho}(t,q))\,.$$

The right side can be expressed with the help of $\widetilde{\partial}^\circ$, the normal time derivative on $\widetilde{\Gamma}$ and another term as follows from Lemma 2.39 with $\widetilde{\Gamma}$ and ε instead of Γ and t through

$$\begin{aligned}
\frac{d}{d\varepsilon}\widetilde{H}(\varepsilon,\Phi_t^{\varepsilon\rho}(q)) &= \widetilde{\partial}^\circ\,\widetilde{H}(\varepsilon,\Phi_t^{\varepsilon\rho}(q)) + \nabla_{\widetilde{\Gamma}(\varepsilon)}\widetilde{H}(\varepsilon,\Phi_t^{\varepsilon\rho}(q)\cdot\left(\frac{d}{d\varepsilon}\Psi(q,\varepsilon\rho(t,q))\right)^T \\
&= \Delta_{\widetilde{\Gamma}(\varepsilon)}\widetilde{V}(\varepsilon,\Phi_t^{\varepsilon\rho}(q)) + |\widetilde{\sigma}|^2(\varepsilon,\Phi_t^{\varepsilon\rho}(q))\,\widetilde{V}(\varepsilon,\Phi_t^{\varepsilon\rho}(q)) \\
&\quad +\nabla_{\widetilde{\Gamma}(\varepsilon)}\widetilde{H}(\varepsilon,\Phi_t^{\varepsilon\rho}(q)\cdot\Big(\partial_w\Psi(q,\varepsilon\rho(t,q))\Big)^T\ \rho(t,q)\ .
\end{aligned}$$

In the last equation we used formula (2.3) for the normal time derivative of mean curvature from Chapter 2. For $\varepsilon=0$, this gives

$$\begin{aligned}
\frac{d}{d\varepsilon}\widetilde{H}(\varepsilon,\Phi_t^{\varepsilon\rho}(q))\bigg|_{\varepsilon=0} &= \Delta_{\Gamma^*}\widetilde{V}(0,q) + |\widetilde{\sigma}|^2(0,q)\widetilde{V}(0,q) \\
&\quad +\nabla_{\Gamma^*}\widetilde{H}(0,q)\cdot\Big(\partial_w\Psi(q,0)\Big)^T\ \rho(t,q) \\
&= \Delta_{\Gamma^*}\rho(t,q) + |\sigma^*|^2(q)\rho(t,q) + \nabla_{\Gamma^*}H^*(q)\cdot\Big(\partial_w\Psi(q,0)\Big)^T\ \rho(t,q) \\
&= \Delta_{\Gamma^*}\rho(t,q) + |\sigma^*|^2(q)\rho(t,q)\ ,
\end{aligned}$$

where we used $\widetilde{V}(0,q)=\rho(t,q)$, the same relation of $\widetilde{\sigma}$ and σ as for $\widetilde{H}$ and H and the fact that $H^*\equiv 0$ on Γ^*. For later use in Section 3.4 of surface diffusion we remark that we just need $H^*\equiv const$. This yields formula (3.18). $\square$

We proceed with the linearization of the boundary condition

$$n(t,\Psi(q,\rho(t,q)))\cdot\mu(\Psi(q,\rho(t,q)))=0 \qquad \text{on } \partial\Gamma^* \tag{3.19}$$

for $t>0$ around $\rho\equiv 0$, that is around the stationary state Γ^*.

To calculate this linearization at $q_0\in\partial\Gamma^*$ and $t_0>0$, we choose a local parametrization of Γ^* around q_0 with nice properties. More precisely, let $U\subset\mathbb{R}^{n+1}$ be an open neighbourhood of q_0, $V\subset\mathbb{R}^{n+1}$ open and $\varphi:U\to V$ a diffeomorphism from Definition 2.4, such that

$$\varphi(U\cap\Gamma^*)=V\cap\left(\mathbb{R}^n_+\times\{0\}\right) \quad \text{with} \quad (\varphi(q_0))_n=0\,.$$

We set $D\times\{0\}:=V\cap\left(\mathbb{R}^n_+\times\{0\}\right)$ and let $F=\left(\varphi^{-1}\right)\big|_D$, i.e.

$$F:D\longrightarrow\Gamma^*\subset\mathbb{R}^{n+1}\,, \qquad x\mapsto F(x)\,. \tag{3.20}$$

This is a local parametrization extended up to the boundary around q_0 with $F(x_0)=q_0$ for some $x_0\in\partial D$. At the fixed point x_0, we can demand the following properties.

(A) $\partial_1F(x_0),\ldots,\partial_nF(x_0)$ is an orthonormal basis of $T_{q_0}\Gamma^*$,

(B) $\partial_1F(x_0)=n_{\partial\Gamma^*}(q_0)$, where $n_{\partial\Gamma^*}$ is the outer unit conormal of Γ^* at $\partial\Gamma^*$ and

(C) $\left(\partial_1F\times\ldots\times\partial_nF\right)(x_0)=n^*(F(x_0))$, where we just fix the sign.

The third assumption (C) uses the cross product for n vectors in $\mathbb{R}^{n+1}$, which in this case due to the orthonormality of $\partial_1 F(x_0), \dots, \partial_n F(x_0)$ lies by definition in normal direction and we just want to fix the sign. Note that with our Definition 2.5 of the tangent space even for points $q_0 \in \partial\Gamma^*$ on the boundary the tangent space $T_{q_0}\Gamma^*$ is an n-dimensional subspace in contrary to an halfspace, as considered in some literature.

With the parametrization F of Γ^* we also get a parametrization of $\Gamma_\rho(t)$ using the diffeomorphism $\Phi_t^\rho : \Gamma^* \to \Gamma_\rho(t)$ with $\Phi_{t_0}^\rho(q_0) = p_0$ for $p_0 \in \Gamma_\rho(t)$, which we denote by

$$G_t : D \longrightarrow \Gamma_\rho(t)\,, \qquad G_t(x) := \Phi_t^\rho(F(x)) = \Psi(F(x), \rho(t, F(x)))\,.$$

Locally around (t_0, p_0), the normal

$$n(t,p) \;=\; n(t, \Phi_t^\rho(q)) \;=\; n(t, \Phi_t^\rho(F(x)))$$

of $\Gamma_\rho(t)$ is given with the help of the cross product of n vectors in $\mathbb{R}^{n+1}$ through

$$n(t, \Phi_t^\rho(F(x))) \;=\; \frac{\partial_1 G_t \times \ldots \times \partial_n G_t}{|\partial_1 G_t \times \ldots \times \partial_n G_t|}(x) = \frac{\partial_1 \Phi_t^\rho \times \ldots \times \partial_n \Phi_t^\rho}{|\partial_1 \Phi_t^\rho \times \ldots \times \partial_n \Phi_t^\rho|}(F(x))\,,$$

where ∂_i is the partial derivative with respect to x_i. For the convenience of the reader, we summarize the used properties of the cross product in the appendix.

To calculate the linearization of the boundary condition (3.19), we need the following properties of Ψ at $w = 0$.

Lemma 3.6. *With the help of the parametrization F it holds for $F(x) = q \in \Gamma^*$*

(i) $\Psi(F(x), 0) = F(x)$, $\partial_i \Psi(F(x), 0) = \partial_i F(x)$,

and for $F(x) = q \in \partial\Gamma^*$ we have

(ii) $\partial_w \Psi(F(x), 0) = n^*(F(x))$, $\partial_i \partial_w \Psi(F(x), 0) \cdot n^*(F(x)) = 0$.

Additionally, for the fixed $F(x_0) = q_0 \in \partial\Gamma^*$ it holds

(iii) $(\partial_1 \Psi \times \ldots \times \partial_n \Psi)(F(x_0), 0) = n^*(F(x_0))$,

(iv) $\left(\partial_1 \Psi \times \ldots \times \overbrace{\partial_w \Psi}^{i\text{-th pos.}} \times \ldots \times \partial_n \Psi\right)(F(x_0), 0) = (-1)\partial_i F(x_0)$ *and*

(v) $\left(\partial_1 \Psi \times \ldots \times \overbrace{\partial_i \partial_w \Psi}^{i\text{-th pos.}} \times \ldots \times \partial_n \Psi\right)(F(x_0), 0) = \Big(\partial_i \partial_w \Psi(F(x_0), 0) \cdot \partial_i F(x_0)\Big)\, n^*(F(x_0))$,

where $i = 1, \dots, n$ in each case.

Proof. (i) follows directly from (3.3).

The first part in *(ii)* is just Lemma 3.1 and the second part can be derived by the first part and by differentiating (3.5). In fact, it holds

$$\begin{aligned}
0 &= \partial_i \Big(\partial_w \Psi(F(x),0) \cdot n^*(F(x)) \Big) \\
&= \partial_i \partial_w \Psi(F(x),0) \cdot n^*(F(x)) + \partial_w \Psi(F(x),0) \cdot \partial_i n^*(F(x)) \\
&= \partial_i \partial_w \Psi(F(x),0) \cdot n^*(F(x)) + \underbrace{n^*(F(x)) \cdot \partial_i n^*(F(x))}_{=0} \\
&= \partial_i \partial_w \Psi(F(x),0) \cdot n^*(F(x)) \,,
\end{aligned}$$

where we used $2\,(n^* \cdot \partial_i n^*) = \partial_i \left(|n^*|^2 \right) = 0$.
(iii) is achieved due to

$$(\partial_1 \Psi \times \ldots \times \partial_n \Psi)\,(F(x_0),0) \;\overset{(i)}{=}\; (\partial_1 F \times \ldots \times \partial_n F)\,(x_0)$$

and the above sign convention (C) for the parametrization F at x_0.
(iv) follows from

$$(\partial_1 \Psi \times \ldots \times \partial_w \Psi \times \ldots \times \partial_n \Psi)\,(F(x_0),0) \;\overset{(ii)}{=}\; (\partial_1 F \times \ldots \times (n^* \circ F) \times \ldots \times \partial_n F)\,(x_0)$$

and Lemma 5.6 in the appendix.
(v) can be shown in the following way. Due to the second part of *(ii)* at $x = x_0$, we can write

$$\partial_i \partial_w \Psi(F(x_0),0) \;=\; \sum_{l=1}^{n} \Big(\partial_i \partial_w \Psi(F(x_0),0) \cdot \partial_l F(x_0) \Big)\, \partial_l F(x_0)$$

because $\Big(\partial_1 F(x_0), \ldots, \partial_n F(x_0), n^*(F(x_0)) \Big)$ is an orthonormal basis of $\mathbb{R}^{n+1}$. This gives

$$\begin{aligned}
&\Bigg(\partial_1 \Psi \times \ldots \times \overset{i\text{-th pos.}}{\widehat{\partial_i \partial_w \Psi}} \times \ldots \times \partial_n \Psi \Bigg)\,(F(x_0),0) \\
&= \Bigg(\partial_1 \Psi(F(x_0),0) \times \ldots \times \overset{i\text{-th pos.}}{\widehat{\sum_{l=1}^{n} \Big(\partial_i \partial_w \Psi(F(x_0),0) \cdot \partial_l F(x_0) \Big)\, \partial_l F(x_0)}} \times \ldots \times \partial_n \Psi(F(x_0),0) \Bigg) \\
&= \sum_{l=1}^{n} \underbrace{\Bigg(\partial_1 F \times \ldots \times \overset{i\text{-th pos.}}{\widehat{\partial_l F(x_0)}} \times \ldots \times \partial_n F \Bigg)\,(x_0)}_{=\delta_{il}\, n^*(F(x_0))} \Big(\partial_i \partial_w \Psi(F(x_0),0) \cdot \partial_l F(x_0) \Big) \\
&= \Big(\partial_i \partial_w \Psi(F(x_0),0) \cdot \partial_i F(x_0) \Big)\; n^*(F(x_0)) \,.
\end{aligned}$$

□

With the help of the above notation for the normal in terms of a parametrization, we can write the boundary condition (3.19) locally around (t_0, x_0) through

$$\left(\left(\partial_1\Phi_t^\rho \times \ldots \times \partial_n\Phi_t^\rho\right) \cdot \left(\mu \circ \Phi_t^\rho\right)\right)(F(x)) \quad = \quad 0\,. \tag{3.21}$$

For the linearization of (3.19) we can therefore consider (3.21) and linearize this equation. To proceed with a precise result, we introduce some notation that will be convenient to shorten the calculations. We write

$$\begin{aligned}
\partial_i\Psi(q,\rho(t,q)) \quad &:= \quad \partial_i\Psi(F(x),w)|_{w=\rho(t,F(x))}\,, \\
&\qquad \text{that is, the derivative acts only on the first variable of } \Psi\,, \\
\partial_w\Psi(q,\rho(t,q)) \quad &:= \quad \partial_w\Psi\left(F(x),\rho(t,F(x))\right) \text{ and} \\
\partial_i\rho(t,q) \quad &:= \quad \partial_i\rho(t,F(x))\,,
\end{aligned}$$

or even briefer

$$\partial_i\Psi := \partial_i\Psi(q,\rho(t,q))\,, \quad \partial_w\Psi := \partial_w\Psi(q,\rho(t,q)) \text{ and } \partial_i\rho := \partial_i\rho(t,q)\,.$$

Now we can show the following linearization of (3.19).

Lemma 3.7. *The linearization of the angle condition (3.19) for $t > 0$ and $q \in \partial\Gamma^*$ is given by*

$$\begin{aligned}
&\frac{d}{d\varepsilon}\left(n(t,\Psi(q,\varepsilon\rho(t,q)))\cdot\mu(\Psi(q,\varepsilon\rho(t,q)))\right)\Big|_{\varepsilon=0} \\
= \quad &-\nabla_{\Gamma^*}\rho(t,q)\cdot\mu(q) + S_q(n^*(q),n^*(q))\rho(t,q)\,,
\end{aligned} \tag{3.22}$$

where S is the second fundamental form of $\partial\Omega$ with respect to $-\mu$. Note that $n^*(q) \in T_q\partial\Omega$ because due to the angle condition for the stationary state Γ^* the relation $n^*(q)\cdot\mu(q) = 0$ for $q \in \partial\Gamma^*$ holds true.

Proof. We calculate the linearization at a fixed point $q_0 \in \partial\Gamma^*$ and $t_0 > 0$. Using the above notation for the parametrization F we are led to the linearization of (3.21), i.e. we have to calculate

$$\frac{d}{d\varepsilon}\left[\left(\partial_1\Phi_t^{\varepsilon\rho} \times \ldots \times \partial_n\Phi_t^{\varepsilon\rho}\right) \cdot \left(\mu \circ \Phi_t^{\varepsilon\rho}\right)(F(x))\right]\Big|_{\varepsilon=0} \tag{3.23}$$

at the fixed point (t_0, x_0).

For the vector product in the above formula we do firstly some calculations without ε to get

$$\partial_i\Phi_t^\rho(F(x)) = \partial_i\left(\Psi(F(x),\rho(t,F(x)))\right) = \partial_i\Psi + \partial_w\Psi\,\partial_i\rho\,, \tag{3.24}$$

where we used the above short notation. Furthermore we observe

$$
\begin{aligned}
& \Big(\partial_1 \Phi_t^\rho \times \ldots \times \partial_n \Phi_t^\rho\Big) \\
= \; & \Big((\partial_1 \Psi + \partial_i \rho\, \partial_w \Psi) \times \ldots \times (\partial_n \Psi + \partial_i \rho\, \partial_w \Psi)\Big) \\
= \; & \Big(\partial_1 \Psi \times \ldots \times \partial_n \Psi\Big) \\
& + \sum_{i=1}^{n} \partial_i \rho \left(\partial_1 \Psi \times \ldots \times \overbrace{\partial_w \Psi}^{\text{i-th pos.}} \times \ldots \times \partial_n \Psi \right) \\
& + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \partial_i \rho\, \partial_j \rho \underbrace{\left(\partial_1 \Psi \times \ldots \times \overbrace{\partial_w \Psi}^{\text{i-th pos.}} \times \ldots \times \overbrace{\partial_w \Psi}^{\text{j-th pos.}} \times \ldots \times \partial_n \Psi \right)}_{=0} \\
& + \text{terms with more than two } \partial_w \Psi \text{ in the cross product, which also vanish} \\
= \; & \Big(\partial_1 \Psi \times \ldots \times \partial_n \Psi\Big) + \sum_{i=1}^{n} \partial_i \rho \left(\partial_1 \Psi \times \ldots \times \overbrace{\partial_w \Psi}^{\text{i-th pos.}} \times \ldots \times \partial_n \Psi \right) .
\end{aligned}
$$

Inserting the last identity into (3.23) for the fixed (t_0, x_0) with $F(x_0) = q_0$, we can do the following calculation

$$
\begin{aligned}
& \frac{d}{d\varepsilon} \left[\Big(\partial_1 \Phi_{t_0}^{\varepsilon\rho} \times \ldots \times \partial_n \Phi_{t_0}^{\varepsilon\rho}\Big) \cdot \Big(\mu \circ \Phi_{t_0}^{\varepsilon\rho}\Big) (F(x_0)) \right] \Bigg|_{\varepsilon=0} \\
= \; & \frac{d}{d\varepsilon} \Bigg\{ \Bigg[\Big(\partial_1 \Psi \times \ldots \times \partial_n \Psi\Big) (q_0, \varepsilon\rho(t_0, q_0)) \\
& + \sum_{i=1}^{n} \partial_i \varepsilon\rho(t_0, q_0) \left(\partial_1 \Psi \times \ldots \times \overbrace{\partial_w \Psi}^{\text{i-th pos.}} \times \ldots \times \partial_n \Psi \right) (q_0, \varepsilon\rho(t_0, q_0)) \Bigg] \cdot \mu(\Psi(q_0, \varepsilon\rho(t_0, q_0))) \Bigg\} \Bigg|_{\varepsilon=0} \\
= \; & \frac{d}{d\varepsilon} \Bigg[\underbrace{\Big(\partial_1 \Psi \times \ldots \times \partial_n \Psi\Big) (q_0, \varepsilon\rho(t_0, q_0))}_{(1)} \\
& + \underbrace{\sum_{i=1}^{n} \partial_i \varepsilon\rho(t_0, q_0) \left(\partial_1 \Psi \times \ldots \times \overbrace{\partial_w \Psi}^{\text{i-th pos.}} \times \ldots \times \partial_n \Psi \right) (q_0, \varepsilon\rho(t_0, q_0))}_{(2)} \Bigg] \Bigg|_{\varepsilon=0} \cdot \mu(\Psi(q_0, 0)) \\
& + \Bigg[\underbrace{\Big(\partial_1 \Psi \times \ldots \times \partial_n \Psi\Big) (q_0, 0)}_{(3)} + 0 \Bigg] \cdot \frac{d}{d\varepsilon} \underbrace{\mu(\Psi(q_0, \varepsilon\rho(t_0, q_0)))}_{(4)} \Bigg|_{\varepsilon=0} .
\end{aligned}
$$

We will consider the above numbered terms separately. For the first one, we calculate

$$\begin{aligned}\frac{d}{d\varepsilon}(1)\Big|_{\varepsilon=0} &= \sum_{k=1}^{n}\Big(\partial_1\Psi\times\ldots\times\overbrace{\partial_w\partial_k\Psi}^{\text{k-th pos.}}\times\ldots\times\partial_n\Psi\Big)(q_0,0)\,\rho(t_0,q_0)\\ &\overset{3.6,(v)}{=} \sum_{k=1}^{n} n^*(q_0)\Big(\partial_k\partial_w\Psi(F(x_0),0)\cdot\partial_kF(x_0)\Big)\,\rho(t_0,q_0)\,.\end{aligned}$$

Therefore we get

$$\begin{aligned}\frac{d}{d\varepsilon}(1)\Big|_{\varepsilon=0}\cdot\mu(q_0) &= \sum_{k=1}^{n}\Big(\partial_k\partial_w\Psi(F(x_0),0)\cdot\partial_kF(x_0)\Big)\,\rho(t_0,q_0)\,\underbrace{(n^*(q_0)\cdot\mu(q_0))}_{=0}\\ &= 0\,,\end{aligned}$$

where we used $\mu(\Psi(q_0,0)) = \mu(q_0)$ due to (3.3) and the angle condition for Γ^* to conclude $n^*\cdot\mu = 0$.
For the second term, we observe

$$\begin{aligned}\frac{d}{d\varepsilon}(2)\Big|_{\varepsilon=0} &= \sum_{i=1}^{n}\partial_i\rho(t_0,q_0)\Big(\partial_1\Psi\times\cdots\times\overbrace{\partial_w\Psi}^{\text{i-th pos.}}\times\cdots\times\partial_n\Psi\Big)(F(x_0),0)\\ &\overset{3.6,(iv)}{=} -\sum_{i=1}^{n}\partial_i\rho(t_0,q_0)\partial_iF(x_0)\\ &= -\nabla_{\Gamma^*}\rho(t_0,q_0)\,,\end{aligned}$$

where the last identity can be seen with the representation of the surface gradient in local coordinates from Remark 2.22 due to assumption (A) for F at the fixed x_0. Taking the scalar product with the normal yields

$$\frac{d}{d\varepsilon}(2)\Big|_{\varepsilon=0}\cdot\mu(q_0) = -\nabla_{\Gamma^*}\rho(t_0,q_0)\cdot\mu(q_0)\,,$$

which is the directional derivative $-\partial_\mu\rho(t_0,q_0)$ of ρ in direction of the outer unit conormal μ of Γ^* at $\partial\Gamma^*$. Here we used the fact $\mu(q) = n_{\partial\Gamma^*}(q)$ on $\partial\Gamma^*$, that is the outer unit normal of Ω equals the outer unit conormal of Γ^* at $\partial\Gamma^*$ due to the angle condition.
For the remaining terms we observe

$$\begin{aligned}(3)\cdot\frac{d}{d\varepsilon}(4)\Big|_{\varepsilon=0} &= (\partial_1\Psi\times\cdots\partial_n\Psi)(F(x_0),0)\cdot\frac{d}{d\varepsilon}\mu(\Psi(q_0,\varepsilon\rho(t_0,q_0)))\Big|_{\varepsilon=0}\\ &\overset{3.6,(iii)}{=} n^*(q_0)\cdot\partial_{\left(n^*(q_0)\,\rho(t_0,q_0)\right)}\mu\,,\end{aligned}$$

where the directional derivative appears by definition with the help of the curve $c(\varepsilon) = \Psi(q_0,\varepsilon\rho(t_0,q_0))$, which fulfills

$$\begin{aligned}c(\varepsilon) &\in \partial\Omega\,,\\ c(0) &= \Psi(q_0,0) = q_0\,,\\ c'(0) &= \partial_w\Psi(q_0,0)\,\rho(t_0,q_0) \overset{3.6,(ii)}{=} n^*(q_0)\rho(t_0,q_0)\,.\end{aligned}$$

Due to linearity of the directional derivative, we finally get

$$\begin{aligned}(3) \cdot \frac{d}{d\varepsilon}(4)\bigg|_{\varepsilon=0} &= \left(n^*(q_0) \cdot \partial_{n^*(q_0)}\mu\right)\ \rho(t_0, q_0) \\ &= S_{q_0}(n^*(q_0), n^*(q_0))\, \rho(t_0, q_0)\,,\end{aligned}$$

where S is the second fundamental form of $\partial\Omega$ equipped with normal $-\mu$, see Definition 2.16. Note that $n^*(q_0) \in T_{q_0}\partial\Omega$ due to the angle condition for the stationary state Γ^*.
Altogether, the linearization of the boundary condition

$$n(t, \Psi(q, \rho(t, q))) \cdot \mu(\Psi(q, \rho(t, q)) \;=\; 0$$

at the fixed point (t_0, q_0) yields

$$\begin{aligned}0 &= \frac{d}{d\varepsilon}(1)\bigg|_{\varepsilon=0} \cdot \mu(q_0) + \frac{d}{d\varepsilon}(2)\bigg|_{\varepsilon=0} \cdot \mu(q_0) + (3) \cdot \frac{d}{d\varepsilon}(4)\bigg|_{\varepsilon=0} \\ &= 0 - \nabla_{\Gamma^*}\rho(t_0, q_0) \cdot \mu(q_0) + S_{q_0}(n^*(q_0), n^*(q_0))\, \rho(t_0, q_0)\ ,\end{aligned}$$

Since the fixed point (t_0, q_0) was arbitrary, we can conclude the above linearization for every $q \in \partial\Gamma^*$ and $t > 0$, which completes the proof of Lemma 3.7. □

From the above Lemma 3.7 together with Lemmata 3.5 and 3.4 about mean curvature and normal velocity, we get the following linearization of (3.16).

$$\left\{\begin{array}{rcll} \partial_t\rho(t, q) &=& \Delta_{\Gamma^*}\rho(t, q) + |\sigma^*|^2(q)\, \rho(t, q) & \text{in } \Gamma^* \quad \text{for all } t > 0\,, \\ \nabla_{\Gamma^*}\rho(t, q) \cdot \mu(q) &=& S(n^*, n^*)(q)\, \rho(t, q) & \text{on } \partial\Gamma^* \quad \text{for all } t > 0\,, \\ \rho(0, q) &=& 0 & \text{in } \Gamma^*\ , \end{array}\right. \tag{3.25}$$

or in abbreviated form

$$\left\{\begin{array}{rcll} \partial_t\rho &=& \left(\Delta_{\Gamma^*} + |\sigma^*|^2\right)\rho & \text{in } \Gamma^* \quad \text{for all } t > 0\,, \\ 0 &=& (\partial_\mu - S(n^*, n^*))\, \rho & \text{on } \partial\Gamma^* \quad \text{for all } t > 0\,, \\ \rho(0) &=& 0 & \text{in } \Gamma^*\,. \end{array}\right. \tag{3.26}$$

Remark 3.8. *For the above linearization of the right angle condition we chose the second fundamental form S of $\partial\Omega$ with respect to $-\mu$ to have the same notation for the bilinear form from Definition 3.9 as in the work of Ros and Souam [RS97] and Vogel [Vog00].*

3.2.3 Conditions for linearized stability

In this important subsection we give conditions for the asymptotic stability of (3.26), which was the linearization of the geometric problem mean curvature flow with outer boundary contact. To this end we generalize the work of Garcke, Ito and Kohsaka [GIK05], where they considered surface diffusion flow with outer boundary contact for curves in the plane. This method, based on spectral theory for a specific linear operator, is independent of a maximum principle and can therefore be generalized to later sections about volume-preserving mean curvature flow, surface diffusion flow or even to cases with triple junctions, when a coupled system of partial differential equations does appear.

We want to give the necessary steps for this part firstly in words and formulate the result already. At the beginning, we are going to show that the linearized problem (3.26) is the gradient flow of a functional $E(\rho)$ which is given with the help of a certain bilinear form I through $E(\rho) = I(\rho,\rho)/2$. Then we can show that the linearized operator $\mathcal{A}$, which describes solutions of (3.26), is self-adjoint and we will study its spectrum. This spectrum will consist of countable many eigenvalues, that can be related to the bilinear form I with the help of Courant's maximum-minimum principle. Finally, we can describe asymptotic stability of the zero solution of the linearized problem (3.26) through the condition that I is positive and we achieved our goal:

$$\boxed{\begin{array}{ll} & \Gamma^* \text{ is linearly asymptotically stable} \\ \Longleftrightarrow & \left\{ \begin{array}{l} I(\rho,\rho) := \int_{\Gamma^*} \left(|\nabla_{\Gamma^*}\rho|^2 - |\sigma^*|^2\rho^2 \right) d\mathcal{H}^n - \int_{\partial\Gamma^*} S(n^*,n^*)\rho^2 \, d\mathcal{H}^{n-1} \\ \quad \text{is positive for all } \rho \in H^1(\Gamma^*)\backslash\{0\}\,. \end{array} \right. \end{array}}$$

We recall shortly the term asymptotic stability as in the book of Lunardi [Lun95]. The zero solution of (3.26) is called **stable**, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for all solutions ρ of (3.26) with starting condition $\rho(0) = \rho_0$ in Γ^* and $\|\rho_0\| < \delta$ the inequality $\|\rho(t)\| < \varepsilon$ holds for all $t > 0$. It is called **asymptotically stable**, if it is stable and in addition $\lim_{t\to\infty} \|\rho(t)\| = 0$ uniformly for ρ_0 in a neighbourhood of 0. The norm $\|.\|$ is the norm of the subspace $\mathcal{D}(\mathcal{A})$ in our upcoming notation and will be different from section to section.

Problem (3.14), that is the mean curvature flow with outer boundary contact as a right angle condition, can be interpreted as the L^2-gradient flow of the area functional $A(t)$, which follows from

$$\frac{d}{dt}A(t) \quad = \quad -\int_{\Gamma(t)} V\,H$$

from Lemma 2.46. Here we demonstrate that the linearization (3.26) can also be interpreted as a gradient flow, which will be an important observation for our stability analysis.

Therefore we introduce the following symmetric bilinear form on $H^1(\Gamma^*)$ and the associated energy.

Definition 3.9. *For ρ_1, $\rho_2 \in H^1(\Gamma^*)$ we define*

$$I(\rho_1,\rho_2) \quad := \quad \int_{\Gamma^*} \left(\nabla_{\Gamma^*}\rho_1 \cdot \nabla_{\Gamma^*}\rho_2 - |\sigma^*|^2 \rho_1\,\rho_2 \right) d\mathcal{H}^n - \int_{\partial\Gamma^*} S(n^*,n^*)\rho_1\,\rho_2\, d\mathcal{H}^{n-1} \tag{3.27}$$

and the associated energy for $\rho \in H^1(\Gamma^*)$

$$E(\rho) \quad := \quad \frac{1}{2} I(\rho,\rho)\,. \tag{3.28}$$

Now we can show that the linearized problem (3.26) is the L^2-gradient flow of E. Here we say that a time dependent function ρ with values in $H^1(\Gamma^*)$ is a solution to the gradient flow equation to E and $(.\,,.)_{L^2}$ if and only if

$$(\partial_t \rho(t), \xi)_{L^2} \quad = \quad -\partial E(\rho(t))(\xi) \tag{3.29}$$

for all $\xi \in H^1(\Gamma^*)$ and all t. The above derivative of E in a direction ξ is given by $I(\rho(t), \xi)$. Then formula (3.29) is just a weak version of the linearized problem (3.26). In fact, if we consider for fixed t the equation

$$\partial_t \rho(t) \quad = \quad \Delta_{\Gamma^*} \rho(t) + |\sigma^*|^2 \rho(t)$$

for a solution $\rho \in L^2(0, T; H^2(\Gamma^*)) \cap H^1(0, T; L^2(\Gamma^*))$ of (3.26), multiply this identity with some $\xi \in H^1(\Gamma^*)$, integrate over Γ^* and use the boundary condition, we get

$$\begin{aligned}
\int_{\Gamma^*} \partial_t \rho(t) \cdot \xi \quad &= \quad \int_{\Gamma^*} \left(\Delta_{\Gamma^*} \rho(t) \cdot \xi + |\sigma^*|^2 \rho(t)\, \xi \right) \\
&= \quad \int_{\Gamma^*} \left(-\nabla_{\Gamma^*} \rho(t) \cdot \nabla_{\Gamma^*} \xi + |\sigma^*|^2 \rho(t)\, \xi \right) + \int_{\partial\Gamma^*} \underbrace{\nabla_{\Gamma^*} \rho(t) \cdot n_{\partial\Gamma^*}}_{=\partial_\mu \rho(t)} \xi \\
&= \quad -\int_{\Gamma^*} \left(\nabla_{\Gamma^*} \rho(t) \cdot \nabla_{\Gamma^*} \xi - |\sigma^*|^2 \rho(t)\, \xi \right) + \int_{\partial\Gamma^*} S(n^*, n^*) \rho(t)\, \xi \\
&= \quad -I(\rho(t), \xi)\,.
\end{aligned}$$

Here we used integration by parts on hypersurfaces from Remark 2.30. This remark was formulated for smooth functions, but by a usual approximation of Sobolev functions it also holds in this case.

If on the other hand the equation

$$(\partial_t \rho(t), \xi)_{L^2} \quad = \quad -I(\rho(t), \xi)$$

holds for all $\xi \in H^1(\Gamma^*)$, we get with the help of regularity theory that $\rho(t) \in H^2(\Gamma^*)$ is a solution of (3.26).

Now we define the corresponding linearized operator of (3.26) through

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \longrightarrow H$$

with

$$\left\{ \begin{array}{rcl} \mathcal{D}(\mathcal{A}) & = & \{\rho \in H^2(\Gamma^*) \mid (\partial_\mu - S(n^*, n^*))\, \rho = 0 \text{ on } \partial\Gamma^*\}\,, \\ H & = & L^2(\Gamma^*)\,, \end{array} \right. \tag{3.30}$$

by

$$\mathcal{A}\rho \quad := \quad \Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho \tag{3.31}$$

for all $\rho \in \mathcal{D}(\mathcal{A})$. With the help of the above gradient flow structure, we see that for $\rho \in \mathcal{D}(\mathcal{A})$ and $\xi \in H^1(\Gamma^*)$ the identity

$$(\mathcal{A}\rho, \xi)_{L^2} \quad = \quad -I(\rho, \xi) \tag{3.32}$$

holds true. This implies the symmetry of the operator $\mathcal{A}$.

Lemma 3.10. *The operator $\mathcal{A}$ is symmetric with respect to the inner product $(.\,,.)_{L^2}$.*

Proof. For $\rho, \xi \in \mathcal{D}(\mathcal{A})$ we have

$$(\mathcal{A}\rho, \xi)_{L^2} = -I(\rho, \xi) = -I(\xi, \rho) = (\mathcal{A}\xi, \rho)_{L^2} = (\rho, \mathcal{A}\xi)_{L^2} \ ,$$

so that $\mathcal{A}$ is symmetric. □

As in Garcke, Ito and Kohsaka [GIK05], we need to analyze the spectrum of $\mathcal{A}$ in order to decide on the stability behaviour of the linearized problem (3.26). This spectrum can be described with the help of the functional I from above. In fact, if $\rho \in \mathcal{D}(\mathcal{A})$ is an eigenfunction of $\mathcal{A}$ to the eigenvalue λ, it holds

$$\lambda\,(\rho, \xi)_{L^2} = (\mathcal{A}\rho, \xi)_{L^2} = -I(\rho, \xi)$$

for all $\xi \in H^1(\Gamma^*)$.

The next important step is to show boundedness of eigenvalues of $\mathcal{A}$ from above. Therefore, the following lemma is needed.

Lemma 3.11. *There exist positive constants C_1 and C_2 such that*

$$\|\rho\|^2_{H^1(\Gamma^*)} \quad \leq \quad C_1\,(\rho, \rho)_{L^2(\Gamma^*)} + C_2 I(\rho, \rho)$$

for all $\rho \in H^{1,2}(\Gamma^*)$.

Proof. At first, we want to use the following inequality. For all $\delta > 0$ there exists a $C_\delta > 0$, such that

$$\|\rho\|^2_{L^2(\partial\Gamma^*)} \quad \leq \quad \delta\|\nabla_{\Gamma^*}\rho\|^2_{L^2(\Gamma^*)} + C_\delta\|\rho\|^2_{L^2(\Gamma^*)} \tag{3.33}$$

for all $\rho \in H^1(\Gamma^*)$.

To see this inequality, assume by contradiction that there exists a $\delta > 0$, such that we can find a sequence $(\tilde{\rho}_n)_{n\in\mathbb{N}} \subset H^1(\Gamma^*)$ with

$$\|\tilde{\rho}_n\|^2_{L^2(\partial\Gamma^*)} \quad > \quad \delta\|\nabla_{\Gamma^*}\tilde{\rho}_n\|^2_{L^2(\Gamma^*)} + n\|\tilde{\rho}_n\|^2_{L^2(\Gamma^*)} \ .$$

In particular this means $\|\tilde{\rho}_n\|_{L^2(\partial\Gamma^*)} > 0$ and we can build $\rho_n := \left(\|\tilde{\rho}_n\|_{L^2(\partial\Gamma^*)}\right)^{-1}\tilde{\rho}_n$ to get

$$1 \quad > \quad \delta\|\nabla_{\Gamma^*}\rho_n\|^2_{L^2(\Gamma^*)} + n\|\rho_n\|^2_{L^2(\Gamma^*)} \ .$$

This implies

$$\|\rho_n\|^2_{L^2(\Gamma^*)} < \frac{1}{n} \longrightarrow 0 \qquad \text{as } n \to \infty$$

and

$$\|\nabla_{\Gamma^*}\rho_n\|^2_{L^2(\Gamma^*)} < \frac{1}{\delta} .$$

Therefore ρ_n is bounded uniformly in $H^1(\Gamma^*)$, so a subsequence converges weakly

$$\rho_n \rightharpoonup \overline{\rho} \qquad \text{in } H^1(\Gamma^*)$$

to some $\overline{\rho} \in H^1(\Gamma^*)$. Due to $\rho_n \to 0$ in $L^2(\Gamma^*)$, we conclude $\overline{\rho} = 0$. From the compact embedding $H^1(\Gamma^*) \hookrightarrow L^2(\partial\Gamma^*)$ we get then the strong convergence

$$\rho_n \to 0 \qquad \text{in } L^2(\partial\Gamma^*) .$$

This is a contradiction to the fact $\|\rho_n\|_{L^2(\partial\Gamma^*)} = 1$ for all $n \in \mathbb{N}$ and therefore we proved inequality (3.33).

Now we proceed with the estimate

$$\begin{aligned}
I(\rho,\rho) \quad &= \quad \int_{\Gamma^*} |\nabla_{\Gamma^*}\rho|^2 - \int_{\Gamma^*} |\sigma^*|^2\rho^2 - \int_{\partial\Gamma^*} S(n^*,n^*)\rho^2 \\
&\geq \quad \int_{\Gamma^*} |\nabla_{\Gamma^*}\rho|^2 - \||\sigma^*|^2\|_{L^\infty(\Gamma^*)} \|\rho\|^2_{L^2(\Gamma^*)} - \|S(n^*,n^*)\|_{L^\infty(\partial\Gamma^*)} \|\rho\|^2_{L^2(\partial\Gamma^*)} \\
&\overset{(3.33)}{\geq} \quad \int_{\Gamma^*} |\nabla_{\Gamma^*}\rho|^2 - \||\sigma^*|^2\|_{L^\infty(\Gamma^*)} \|\rho\|^2_{L^2(\Gamma^*)} \\
&\quad -\delta\|S(n^*,n^*)\|_{L^\infty(\partial\Gamma^*)} \|\nabla_{\Gamma^*}\rho\|^2_{L^2(\Gamma^*)} - C_\delta\|S(n^*,n^*)\|_{L^\infty(\partial\Gamma^*)} \|\rho\|^2_{L^2(\Gamma^*)} \\
&= \quad \Big(1 - \delta\|S(n^*,n^*)\|_{L^\infty(\partial\Gamma^*)}\Big) \|\nabla_{\Gamma^*}\rho\|^2_{L^2(\Gamma^*)} \\
&\quad - \Big(\||\sigma^*|^2\|_{L^\infty(\Gamma^*)} + C_\delta\|S(n^*,n^*)\|_{L^\infty(\partial\Gamma^*)}\Big) \|\rho\|^2_{L^2(\Gamma^*)} .
\end{aligned}$$

By choosing $\delta > 0$ small enough, so that $\big(1 - \delta\|S(n^*,n^*)\|_{L^\infty(\partial\Gamma^*)}\big) > 0$ we get the inequality

$$I(\rho,\rho) + C\|\rho\|^2_{L^2(\Gamma^*)} \quad \geq \quad \|\nabla_{\Gamma^*}\rho\|_{L^2(\Gamma^*)} .$$

Adding $\|\rho\|^2_{L^2(\Gamma^*)}$ gives the assertion. □

Due to the previous lemma we can show boundedness from above for the largest eigenvalue of $\mathcal{A}$.

Lemma 3.12. *Let λ be an eigenvalue of $\mathcal{A}$. Then the following inequality holds*

$$\lambda \quad \leq \quad \frac{C_1}{C_2} , \tag{3.34}$$

where C_1 and C_2 are the positive constants of the above Lemma 3.11.

Proof. Let $\rho \in \mathcal{D}(\mathcal{A})$ be an eigenvector corresponding to the eigenvalue λ, which in particular means $\rho \neq 0$. This implies

$$\lambda\,(\rho,\rho)_{L^2} = (\mathcal{A}\rho,\rho)_{L^2} = -I(\rho,\rho)\,.$$

If we now assume that $\lambda > \frac{C_1}{C_2}$, we would have

$$\begin{aligned} 0 &= I(\rho,\rho) + \lambda\,(\rho,\rho)_{L^2} > I(\rho,\rho) + \frac{C_1}{C_2}\,(\rho,\rho)_{L^2} \overset{3.11}{\geq} \frac{1}{C_2}\|\rho\|^2_{H^1(\Gamma^*)} \\ &> 0\,, \end{aligned}$$

which is a contradiction. □

The next step is to show that $\mathcal{A}$ is self-adjoint with respect to the L^2-inner product $(.\,,.)_{L^2}$. This will be done without explicit work with the adjoint $\mathcal{A}^*$, but with a property, that implies the equivalence of symmetry and self-adjointness. For this abstract theorem we refer to the book of Weidmann [Weid76]. Due to Lemma 3.10, which shows symmetry of $\mathcal{A}$, we then proved self-adjointness.

Lemma 3.13. *The operator $\mathcal{A}$ is self-adjoint with respect to the L^2-inner product.*

Proof. We use the following theorem of operator theory. If there exists a $\lambda \in \mathbb{R}$, such that

$$\operatorname{im}(\lambda Id - \mathcal{A}) \;=\; L^2(\Gamma^*)\,,$$

the properties symmetry and self-adjointness of $\mathcal{A}$ are equivalent, see [Weid76].

So we have to show that there exists a $\lambda \in \mathbb{R}$, such that for a given $f \in L^2(\Gamma^*)$ there exists a $\rho \in \mathcal{D}(\mathcal{A})$ with

$$\lambda\rho - \mathcal{A}\rho \;=\; f \quad \text{on } \Gamma^*\,,$$

that is

$$(*)\begin{cases} -\Delta_{\Gamma^*}\rho - |\sigma^*|^2\rho + \lambda\rho &= f \quad \text{on } \Gamma^*\,, \\ (\partial_\mu - S(n^*,n^*))\,\rho &= 0 \quad \text{on } \partial\Gamma^*\,. \end{cases}$$

The weak formulation of $(*)$ is given through the following problem. For given $f \in L^2(\Gamma^*)$ find a $\rho \in H^1(\Gamma^*)$ such that

$$\int_{\Gamma^*} \nabla_{\Gamma^*}\rho \cdot \nabla_{\Gamma^*}\psi - \int_{\Gamma^*} |\sigma^*|^2\rho\,\psi + \lambda\int_{\Gamma^*}\rho\,\psi - \int_{\partial\Gamma^*} S(n^*,n^*)\rho\,\psi \;=\; \int_{\Gamma^*} f\,\psi$$

for all $\psi \in H^1(\Gamma^*)$. When we define the left side of the above equation as a bilinear form $a(\rho,\psi)$

with $a : H^1(\Gamma^*) \times H^1(\Gamma^*) \to \mathbb{R}$ we conclude the following inequalities. For $\rho \in H^1(\Gamma^*)$ it holds

$$\begin{aligned} a(\rho,\rho) &= I(\rho,\rho) + \int_{\Gamma^*} \lambda\,\rho^2 \\ &\overset{3.11}{\geq} \frac{1}{C_2}\|\rho\|_{H^1}^2 - \frac{C_1}{C_2}\int_{\Gamma^*}\rho^2 + \int_{\Gamma^*}\lambda\,\rho^2 \\ &= \frac{1}{C_2}\|\rho\|_{H^1}^2 + \left(\lambda - \frac{C_1}{C_2}\right)\int_{\Gamma^*}\rho^2 \\ &\geq \frac{1}{C_2}\|\rho\|_{H^1}^2\,, \end{aligned}$$

where the last inequality can be achieved by choosing λ large enough, such that $\lambda - \frac{C_1}{C_2}$ is positive.
The above inequality shows coercivity of a for large λ and as in the theory of elliptic operators in $\mathbb{R}^n$ one can show that the above problem $(*)$ has a unique solution $\rho \in H^1(\Gamma^*)$, see Aubin [Aub82]. Regularity theory for elliptic partial differential equations on manifolds, which is due to the fact that differentiability is a local property roughly is the same as on open sets in $\mathbb{R}^n$, shows $\rho \in H^2(\Gamma^*)$. The boundary condition $\partial_\mu\rho - S(n^*, n^*)\rho = 0$ on $\partial\Gamma^*$ is then fulfilled in a strong sense.
Altogether we found a solution $\rho \in \mathcal{D}(\mathcal{A})$ of $\lambda\rho - \mathcal{A}\rho = f$ on Γ^*. With the above explanation, we proved self-adjointness of $\mathcal{A}$. □

As next point we want to give a first criterion for stability of (3.25) around the zero solution.

Theorem 3.14.

(*i*) *The spectrum of $\mathcal{A}$ consists of countable many real eigenvalues.*

(*ii*) *The initial value problem (3.25) is solvable for initial data in $H = L^2(\Gamma^*)$.*

(*iii*) *The zero solution of (3.25) is asymptotically stable if and only if the largest eigenvalue of $\mathcal{A}$ is negative, in short notation $\sigma(\mathcal{A}) < 0$.*

Proof. ad (*i*): For $\lambda > \frac{C_1}{C_2}$ we have shown surjectivity of

$$(\lambda Id - \mathcal{A}) : \mathcal{D}(\mathcal{A}) \to H$$

in the proof of the last Lemma 3.13.
With the identity

$$\sigma(\lambda Id - \mathcal{A}) = \lambda - \sigma(\mathcal{A})$$

for the spectrum, together with the fact that $\mu \leq \frac{C_1}{C_2}$ for every $\mu \in \sigma(\mathcal{A})$ from Lemma 3.12, we see that there exists no eigenvalue zero of $\lambda Id - \mathcal{A}$. For a linear operator this means in particular that it is injective.
Continuity of the resolvent

$$(\lambda Id - \mathcal{A})^{-1} : H \to \mathcal{D}(\mathcal{A})$$

can be seen by observing that

$$(\lambda Id - \mathcal{A})^{-1}(f) = \rho \quad \Leftrightarrow \quad (\lambda Id - \mathcal{A})(\rho) = f\,,$$

which means that ρ solves the elliptic partial differential equation $(*)$ from the proof of Lemma 3.13. So a standard inequality for solutions of elliptic partial differential equations

$$\|\rho\|_{H^2} \quad \le \quad \|f\|_{L^2}$$

gives the desired continuity.

Since the embedding $\mathcal{D}(\mathcal{A}) \hookrightarrow L^2$ is compact, we get a compact operator

$$(\lambda Id - \mathcal{A})^{-1} : H \to H\,.$$

Together with the self-adjointness of $\mathcal{A}$ from Lemma 3.13, we get the claim (i) with the help of an abstract theorem of operator theory, for example we refer to the book of Kato [Kat95].

ad (ii) and (iii): Existence and stability of the problem

$$\text{Find } \rho(t) \in \mathcal{D}(\mathcal{A})\,, \text{ such that } \partial_t \rho(t) = \mathcal{A}(\rho(t))$$

can be treated with the theory of analytic semigroups as is done for example, in the book of Lunardi [Lun95]. We just show that $\mathcal{A}$ generates an analytic semigroup.

Firstly, we know that for $\omega \in \mathbb{R}$ the operator $\tilde{\mathcal{A}} := \mathcal{A} - \omega Id$ is self-adjoint, since from Lemma 3.13 the operator $\mathcal{A}$ has this property. Second, we can show that $\tilde{\mathcal{A}}$ is dissipative, which means that

$$(\tilde{\mathcal{A}}\rho, \rho)_{L^2} \le 0 \qquad \text{for all } \rho \in \mathcal{D}(\mathcal{A})\,.$$

In fact, this can be seen with the help of Lemma 3.11 through

$$\begin{aligned}
(\tilde{\mathcal{A}}\rho, \rho)_{L^2} &= (\mathcal{A}\rho, \rho)_{L^2} - \omega(\rho, \rho)_{L^2} \\
&= -I(\rho, \rho) - \omega(\rho, \rho)_{L^2} \\
&\le -\frac{1}{C_2}\|\rho\|_{H^1}^2 + \left(\frac{C_1}{C_2} - \omega\right)\|\rho\|_{L^2}^2 \\
&\le 0\,,
\end{aligned}$$

where the last inequality can be achieved by choosing ω large enough. Now we use an abstract theorem from [Weid76], which states that a linear, densely defined, self-adjoint and dissipative operator is in particular sectorial and therefore generates an analytic semigroup $T(t)$. For completeness we mention finally that $S(t) := e^{\omega t}T(t)$ is the analytic semigroup with generator $\mathcal{A}$. □

As a characterization of the eigenvalues of $\mathcal{A}$, we can directly generalize a result of [GIK05], where the authors used the classical Courant's maximum-minimum principle from [CH68].

Lemma 3.15. *Let*

$$\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \dots$$

be the eigenvalues of $\mathcal{A}$ (taken multiplicity into account).

(*i*) *For all $n \in \mathbb{N}$, the following description of the eigenvalues holds*

$$\begin{aligned} \lambda_n &= \inf_{W \in \Sigma_{n-1}} \sup_{\rho \in W \setminus \{0\}} -\frac{I(\rho,\rho)}{(\rho,\rho)_{L^2}}\,, \\ -\lambda_n &= \sup_{W \in \Sigma_{n-1}} \inf_{\rho \in W^\perp \setminus \{0\}} \frac{I(\rho,\rho)}{(\rho,\rho)_{L^2}}\,, \end{aligned}$$

where Σ_n is the collection of n-dimensional subspaces of $H^{1,2}(\Gamma^*)$ and $W^\perp$ is the orthogonal complement with respect to the $(.,.)_{L^2}$- inner product.

(*ii*) *The eigenvalues λ_n depend continuously on $S(n^*,n^*)$ and $|\sigma^*|$ in the L^∞-norm.*

Proof. The first part follows with the help Courant's maximum-minimum principle from [CH68] and the second part follows due to the structure of I, which is

$$I(\rho,\rho) = \int_{\Gamma^*} \left(|\nabla_{\Gamma^*}\rho|^2 - |\sigma^*|^2\rho^2\right) d\mathcal{H}^n - \int_{\partial\Gamma^*} S(n^*,n^*)\rho^2\, d\mathcal{H}^{n-1}\,.$$

Since the generation of the infimum and supremum in (*i*) preserves continuity, we see the continuous dependence of λ_n to $\|\sigma^*\|_{L^\infty(\Gamma^*)}$ and $\|S(n^*,n^*)\|_{L^\infty(\partial\Gamma^*)}$. □

Remark 3.16. *For the largest eigenvalue λ_1 of $\mathcal{A}$ we have the description*

$$-\lambda_1 = \min_{\rho \in H^1(\Gamma^*)\setminus\{0\}} \frac{I(\rho,\rho)}{(\rho,\rho)_{L^2}}\,, \tag{3.35}$$

which can be seen directly from the second description of λ_1 in Lemma 3.15 through $-\lambda_1 = \sup_{W\in\Sigma_0}\inf_{\rho\in W^\perp\setminus\{0\}} \frac{I(\rho,\rho)}{(\rho,\rho)_{L^2}}$ and $\Sigma_0 = \{\emptyset\}$ and therefore $W^\perp = H^1(\Gamma^*)$. The fact that the minimum is attained also follows from the classical work Courant and Hilbert [CH68].

From Theorem 3.14 we have asymptotic stability of the linearized problem (3.25) if and only if $\lambda_1 < 0$. This leads to the following main conclusion.

Theorem 3.17. *The linearized problem (3.25) is asymptotically stable if and only if*

$$I(\rho,\rho) > 0$$

for all $\rho \in H^1(\Gamma^*)\setminus\{0\}$, where

$$I(\rho,\rho) = \int_{\Gamma^*} \left(|\nabla_{\Gamma^*}\rho|^2 - |\sigma^*|^2\rho^2\right) d\mathcal{H}^n - \int_{\partial\Gamma^*} S(n^*,n^*)\rho^2\, d\mathcal{H}^{n-1}\,.$$

3.3 Volume preserving mean curvature flow

Let us consider here the so called volume preserving mean curvature flow with outer boundary contact, which is a direct generalization of the previous Section 3.2. With the same notations as before, we assume the special representation of the evolving hypersurface as a graph from Section 3.1 and linearize the resulting equation. For the stability analysis we can then refer to the last section, where we used methods that are also applicable in this case.

With the same notations as in the section of the mean curvature flow we consider here the problem of finding an evolving hypersurface

$$\Gamma = \bigcup_{t\in[0,T)} \{t\} \times \Gamma(t) \quad \text{with} \quad \Gamma(t) \subset \mathbb{R}^{n+1}\,, \tag{3.36}$$

as in Definition 2.31, evolving due to the volume preserving mean curvature flow, such that $\Gamma(t)$ lies in a fixed bounded region $\Omega \subset \mathbb{R}^{n+1}$ and the boundary $\partial\Gamma(t)$ of each of the hypersurfaces intersects the boundary $\partial\Omega$ of the fixed region at a right angle.

In formulas, the problem reads as follows. Find Γ as in (3.36), such that

$$\begin{cases} V &= & H - \overline{H} & \text{in } \Gamma(t) & \text{for all } \ t > 0\,, \\ \Gamma(t) &\subset & \Omega & & \text{for all } \ t > 0\,, \\ \partial\Gamma(t) &\subset & \partial\Omega & & \text{for all } \ t > 0\,, \\ n(t)\cdot\mu &= & 0 & \text{on } \partial\Gamma(t) & \text{for all } \ t > 0\,, \\ \Gamma(0) &= & \Gamma_0\,. & & \end{cases} \tag{3.37}$$

Here V, H, n and μ are the normal velocity, the mean curvature, a unit normal of the evolving hypersurface Γ and the outer unit normal to $\partial\Omega$ as explained in Sections 3.1 and 3.2.

$\overline{H}$ is the mean value of mean curvature, that is

$$\overline{H}(t) \quad = \quad \fint_{\Gamma(t)} H \, d\mathcal{H}^n\,. \tag{3.38}$$

Γ_0 is a given starting surface, which lies in Ω and intersects the boundary $\partial\Omega$ at a right angle. We observe that stationary surfaces of this flow satisfy $H \equiv \overline{H}$, so they are hypersurfaces with constant mean curvature, so-called *H-hypersurfaces.*

With the notations of Section 2.4 for area $A(t)$ and volume $Vol(t)$ of $\Gamma(t)$, we can justify the name of the flow in (3.37).

Lemma 3.18. *For a solution Γ of the flow (3.37) the following estimates hold true*

(*i*) $\frac{d}{dt}A(t) \le 0$ *and*

(*ii*) $\frac{d}{dt}Vol(t) = 0.$

Therefore the flow is area minimizing and also volume preserving, as the name of the flow already indicated.

Proof. ad (i): With the result of Lemma 2.46 we get

$$\begin{aligned}\frac{d}{dt}A(t) &= -\int_{\Gamma(t)} V\,H\,d\mathcal{H}^n = -\int_{\Gamma(t)} (H-\overline{H})\,H\,d\mathcal{H}^n \\ &= -\int_{\Gamma(t)} H^2\,d\mathcal{H}^n + \frac{1}{|\Gamma(t)|}\left(\int_{\Gamma(t)} H\,d\mathcal{H}^n\right)^2 \\ &\le 0\,,\end{aligned}$$

where the last inequality follows from the Cauchy-Schwarz inequality. In fact, we have

$$\left(\int_{\Gamma(t)} H\,1\,d\mathcal{H}^n\right)^2 \le \left(\int_{\Gamma(t)} H^2\,d\mathcal{H}^n\right)\left(\int_{\Gamma(t)} 1^2\,d\mathcal{H}^n\right) = |\Gamma(t)|\int_{\Gamma(t)} H^2\,d\mathcal{H}^n\,.$$

ad (ii): Again with the result of Lemma 2.46, we get

$$\frac{d}{dt}Vol(t) = \int_{\Gamma(t)} V\,d\mathcal{H}^n = \int_{\Gamma(t)} \left(H-\overline{H}\right)\,d\mathcal{H}^n = 0$$

and the proof is finished. □

As in the previous Section 3.2 we consider special solutions Γ of (3.37). Recalling the notation, we fix a stationary solution Γ^* of (3.37) and consider hypersurfaces $\Gamma_\rho(t)$ given as in Section 3.1 with the help of a function

$$\rho : [0,T)\times\Gamma^* \longrightarrow (-d,d)$$

through a diffeomorphism onto its image

$$\Phi_t^\rho : \Gamma^* \longrightarrow \Omega$$

by

$$\Gamma_\rho(t) = \Phi^\rho(t,\Gamma^*)\,.$$

For the details we refer again to Section 3.1.

The corresponding equation to (3.12) for ρ on the fixed stationary hypersurface Γ^* is given through

$$\left\{\begin{array}{rcll} V(\Psi(q,\rho(t,q))) &=& H(\Psi(q,\rho(t,q))) - \overline{H}(\rho,t) & \text{in } \Gamma^* \quad \text{for all } t>0\,, \\ (n\cdot\mu)\,(\Psi(q,\rho(t,q))) &=& 0 & \text{on } \partial\Gamma^* \quad \text{for all } t>0\,, \\ \rho(0,q) &=& \rho_0(q) & \text{in } \Gamma^*\,. \end{array}\right. \tag{3.39}$$

Here the dependence of the mean value of mean curvature $\overline{H}$ on ρ reads as follows

$$\overline{H}(\rho,t) = ⨍_{\Gamma_\rho(t)} H\,\mathrm{d}\mathcal{H}^n\,,$$

such that this is an additional nonlocal term compared to mean curvature flow.

For the linearization of (3.39) around $\rho \equiv 0$, which is our notation for linearization of (3.37) around the given stationary state Γ^*, we can use the results and notation of the previous section. In particular, we use the linearization of normal velocity, mean curvature and the angle condition. For the mean value of mean curvature, we have the following result.

Lemma 3.19. *The linearization of the mean value of mean curvature is given through*

$$\frac{d}{d\varepsilon}\overline{H}(\varepsilon\rho, t)\bigg|_{\varepsilon=0} \quad = \quad \fint_{\Gamma^*} \big(\Delta_{\Gamma^*}\rho(t,q) + |\sigma^*|^2(q)\rho(t,q)\big)\, \mathrm{d}\mathcal{H}^n\,.$$

Proof. For fixed t, we use the mapping from the proof of the linearization of mean curvature in Lemma 3.5 in the previous section

$$\Phi_t^{\varepsilon\rho} : \Gamma^* \to \Omega \quad , \quad q \mapsto \Phi_t^{\varepsilon\rho}(q) = \Psi(q, \varepsilon\rho(t,q))\,,$$

where ε is small. This mapping is a diffeomorphism onto its image and we get evolving hypersurfaces in ε through $\widetilde{\Gamma}(\varepsilon) = \mathrm{im}(\Phi_{\varepsilon\rho}) = \Gamma_{\varepsilon\rho}(t)$ and in particular $\widetilde{\Gamma}(0) = \Gamma_{\rho\equiv 0}(t) = \Gamma^*$. With this notation, we can write the mean value of mean curvature as

$$\begin{aligned}
\overline{H}(\varepsilon\rho, t) \quad &= \quad \fint_{\Gamma_{\varepsilon\rho}(t)} H(t,p)\,\mathrm{d}\mathcal{H}^n = \left(\int_{\Gamma_{\varepsilon\rho}(t)} 1\,\mathrm{d}\mathcal{H}^n\right)^{-1} \left(\int_{\Gamma_{\varepsilon\rho}(t)} H(t,p)\,\mathrm{d}\mathcal{H}^n\right) \\
&= \quad \left(\int_{\widetilde{\Gamma}(\varepsilon)} 1\,\mathrm{d}\mathcal{H}^n\right)^{-1} \left(\int_{\widetilde{\Gamma}(\varepsilon)} \widetilde{H}(\varepsilon,p)\,\mathrm{d}\mathcal{H}^n\right)\,,
\end{aligned}$$

where $\widetilde{H}(\varepsilon,p)$ denotes the mean curvature of $\widetilde{\Gamma}(\varepsilon)$ at $p \in \widetilde{\Gamma}(\varepsilon)$, which is due to $\widetilde{\Gamma}(\varepsilon) = \Gamma_{\varepsilon\rho}(t)$ for fixed t really the same as the mean curvature $H(t,p)$ of $\Gamma_{\varepsilon\rho}(t)$ at $p \in \Gamma_{\varepsilon\rho}(t)$. We just use this notation to describe an evolving hypersurface in ε.

For the derivative with respect to ε at $\varepsilon \equiv 0$ we calculate with Lemma 2.46

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{\widetilde{\Gamma}(\varepsilon)} 1\,\mathrm{d}\mathcal{H}^n\bigg|_{\varepsilon=0} \quad &= \quad -\int_{\Gamma^*} \widetilde{V}(0,p)\,\widetilde{H}(0,q)\,\mathrm{d}\mathcal{H}^n = -\int_{\Gamma^*} \rho(t,p) H^*(p)\,\mathrm{d}\mathcal{H}^n \\
&= \quad -H^* \int_{\Gamma^*} \rho(t,p)\,\mathrm{d}\mathcal{H}^n\,,
\end{aligned}$$

where we used the result $\widetilde{V}(0,p) = \rho(t,p)$ from the proof of Lemma 3.5 from the previous section and the fact, that the mean curvature H^* of the stationary surface Γ^* is a constant.

Furthermore we get with the help of the Transport theorem 2.44 and the formula (2.3) for the normal time derivative for mean curvature

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{\widetilde{\Gamma}(\varepsilon)} \widetilde{H}(\varepsilon,p)\,\mathrm{d}\mathcal{H}^n &= \int_{\widetilde{\Gamma}(\varepsilon)} \left(\widetilde{\partial}^\circ\,\widetilde{H}(\varepsilon,p) - \widetilde{H}(\varepsilon,p)\widetilde{V}(\varepsilon,p)\widetilde{H}(\varepsilon,p)\right)\,\mathrm{d}\mathcal{H}^n \\
&\quad + \int_{\partial\widetilde{\Gamma}(\varepsilon)} \widetilde{H}(\varepsilon,p)v_{\partial\widetilde{\Gamma}}(\varepsilon,p)\,\mathrm{d}\mathcal{H}^{n-1} \\
&= \int_{\widetilde{\Gamma}(\varepsilon)} \left(\Delta_{\widetilde{\Gamma}(\varepsilon)}\widetilde{V}(\varepsilon,p) + |\widetilde{\sigma}|^2(\varepsilon,p)\widetilde{V}(\varepsilon,p)\right) \\
&\quad - \int_{\widetilde{\Gamma}(\varepsilon)} \widetilde{H}(\varepsilon,p)\widetilde{V}(\varepsilon,p)\widetilde{H}(\varepsilon,p)\,\mathrm{d}\mathcal{H}^n + \int_{\partial\widetilde{\Gamma}(\varepsilon)} \widetilde{H}(\varepsilon,p)v_{\partial\widetilde{\Gamma}}(\varepsilon,p)\,\mathrm{d}\mathcal{H}^{n-1}\,.
\end{aligned}$$

For $\varepsilon = 0$ we get as in the proof of Lemma 2.46 the identity $v_{\partial\Gamma^*}(0,p) = 0$ due to the 90° angle condition, which gives

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{\widetilde{\Gamma}(\varepsilon)} \widetilde{H}(\varepsilon,p)\,\mathrm{d}\mathcal{H}^n\bigg|_{\varepsilon=0} = \int_{\Gamma^*} \Delta_{\Gamma^*}\rho(t,p) + |\sigma^*|^2(t,p)\rho(t,p)\,\mathrm{d}\mathcal{H}^n - (H^*)^2\int_{\Gamma^*}\rho(t,p)\,\mathrm{d}\mathcal{H}^n\,.$$

Altogether, we get for the linearization of the mean value of mean curvature

$$\begin{aligned}
\frac{d}{d\varepsilon}\overline{H}(\varepsilon\rho,t)\bigg|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\left[\left(\int_{\widetilde{\Gamma}(\varepsilon)} 1\,\mathrm{d}\mathcal{H}^n\right)^{-1}\left(\int_{\widetilde{\Gamma}(\varepsilon)}\widetilde{H}(\varepsilon,p)\,\mathrm{d}\mathcal{H}^n\right)\right]\Bigg|_{\varepsilon=0} \\
&= \frac{1}{\left(\int_{\Gamma^*}1\right)^2}\left(\int_{\Gamma^*}1\,\left[\int_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right) - (H^*)^2\int_{\Gamma^*}\rho\right]\right. \\
&\qquad \left. - \int_{\Gamma^*}(H^*)\int_{\Gamma^*}(-\rho H^*)\right) \\
&= ⨍_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right) - (H^*)^2 ⨍_{\Gamma^*}\rho + (H^*)^2 ⨍_{\Gamma^*}\rho \\
&= ⨍_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right)\,.
\end{aligned}$$

Here we omitted the volume form $\mathrm{d}\mathcal{H}^n$ for reasons of shortness. □

So together with the results of the previous section we get for the linearization of (3.39) around the stationary hypersurface Γ^* represented through $\rho \equiv 0$ the following equations.

$$\left\{\begin{array}{rcll}
\partial_t\rho &=& \Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho - ⨍_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right) & \text{in } \Gamma^* \quad \text{for all } t > 0\,, \\
0 &=& \left(\partial_\mu - S(n^*,n^*)\right)\rho & \text{on } \partial\Gamma^* \quad \text{for all } t > 0\,, \\
\rho(0,q) &=& 0 & \text{in } \Gamma^*\,.
\end{array}\right. \tag{3.40}$$

We give a remark concerning a term in the linearization of the mean value of mean curvature.

Remark 3.20. *It is also possible to write the term* $\fint_{\Gamma^*}\Delta_{\Gamma^*}\rho$ *as*

$$\fint_{\Gamma^*}\Delta_{\Gamma^*}\rho = \frac{1}{|\Gamma^*|}\int_{\Gamma^*}\nabla_{\partial\Gamma^*}\rho\cdot\mu = \frac{1}{|\Gamma^*|}\int_{\partial\Gamma^*}S(n^*,n^*)\rho\,.$$

A solvability condition for solutions of the linearized problem (3.40) gives here $\int_{\Gamma^*}\rho \equiv 0$.

Lemma 3.21. *Solutions of the linearized problem (3.40) fulfill*

$$\int_{\Gamma^*}\rho\,\mathrm{d}\mathcal{H}^n \quad\equiv\quad 0\,. \tag{3.41}$$

Proof. Integrating the first line in (3.40) gives

$$\int_0^t\int_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho+|\sigma^*|^2\rho-\fint_{\Gamma^*}(\Delta_{\Gamma^*}\rho+|\sigma^*|^2\rho)\right)\mathrm{d}\mathcal{H}^n \quad=\quad \int_0^t\int_{\Gamma^*}\partial_t\rho\,\mathrm{d}\mathcal{H}^n\,,$$

where the left side equals 0 and the right side gives

$$\int_0^t\int_{\Gamma^*}\partial_t\rho\,\mathrm{d}\mathcal{H}^n \quad=\quad \int_0^t\partial_t\int_{\Gamma^*}\rho\,\mathrm{d}\mathcal{H}^n = \int_{\Gamma^*}\rho(t,q)\,\mathrm{d}\mathcal{H}^n - \underbrace{\int_{\Gamma^*}\rho(0,q)\,\mathrm{d}\mathcal{H}^n}_{=0}\,.$$

Together we get

$$\int_{\Gamma^*}\rho(t,q)\,\mathrm{d}\mathcal{H}^n \quad=\quad 0 \quad \text{for all } t\,,$$

which shows the claim. □

We introduce the same bilinear form on $H^{1,2}(\Gamma^*)$ as in the previous chapter

$$I(\rho_1,\rho_2)=\int_{\Gamma^*}\left(\nabla_{\Gamma^*}\rho_1\cdot\nabla_{\Gamma^*}\rho_2-|\sigma^*|^2\rho_1\rho_2\right)\mathrm{d}\mathcal{H}^n-\int_{\partial\Gamma^*}S(n^*,n^*)\rho_1\rho_2\,\mathrm{d}\mathcal{H}^{n-1} \tag{3.42}$$

for $\rho_1,\rho_2\in H^1(\Gamma^*)$.

Due to the solvability condition, we introduce the space

$$V \quad:=\quad H^1(\Gamma^*)\cap\{\rho\mid \textstyle\int_{\Gamma^*}\rho=0\}\,. \tag{3.43}$$

and supply it with the L^2-inner product.

In analogy to the previous section, we want to show that the linearized problem (3.40) is the L^2-gradient flow of the functional $E(\rho):=\frac{1}{2}I(\rho,\rho)$ defined on V, which means exactly the identity stated in the next lemma.

Lemma 3.22. *The time dependent function ρ with values in V is a solution of the linearized equation (3.40) if and only if*

$$(\partial_t\rho(t),\xi)_{L^2} \;=\; -\partial E(\rho(t))(\xi)$$

holds for all $\xi \in V$ and all t and if $\rho(0,q) \equiv 0$ is fulfilled on Γ^*.

Proof. At first we remark the identity $\partial E(\rho(t))(\xi) = I(\rho(t),\xi)$.

Now let ρ be a solution of (3.40). From Lemma 3.21 we get $\rho(t) \in V$. We multiply the first line in (3.40) with $\xi \in V$ and integrate over Γ^* to get

$$\begin{aligned}\int_{\Gamma^*}\partial_t\rho\,\xi \;&=\; \int_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho\,\xi + |\sigma^*|^2\rho\,\xi\right) - \underbrace{\int_{\Gamma^*}⨍_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\right)\,\xi}_{=0}\\ &=\; -I(\rho(t),\xi)\,,\end{aligned}$$

where the last equality follows with the same calculation as in the case of mean curvature flow.

On the other hand, let $\rho(t) \in V$ fulfill the identity

$$(\rho(t),\xi)_{L^2} \;=\; -\partial E(\rho(t))(\xi)$$

for all $\xi \in V$ and all t. In detail, this gives

$$\int_{\Gamma^*}\partial_t\rho(t)\,\xi \;=\; \int_{\Gamma^*}\left(-\nabla_{\Gamma^*}\rho\cdot\nabla_{\Gamma^*}\xi + |\sigma^*|^2\rho\,\xi\right) + \int_{\partial\Gamma^*}S(n^*,n^*)\rho\,\xi \tag{3.44}$$

for all $\xi \in H^1(\Gamma^*)$ with $\int_{\Gamma^*}\xi = 0$. Regularity theory for weak solutions of (3.44) leads to $\rho \in H^2(\Gamma^*)$. After integration by parts, we observe

$$\int_{\Gamma^*}\partial_t\rho(t)\,\xi \;=\; \int_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho\,\xi + |\sigma^*|^2\rho\,\xi\right) - \int_{\partial\Gamma^*}\left(\partial_\mu\rho\,\xi - S(n^*,n^*)\rho\,\xi\right)\,. \tag{3.45}$$

First we consider $\xi \in H^1_0(\Gamma^*)$ with $\int_{\Gamma^*}\xi = 0$ and get with the help of the fundamental lemma

$$\partial_t\rho(t) \;=\; \Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho + \lambda(t) \quad \text{on } \Gamma^*\,,$$

where $\lambda(t) = -⨍_{\Gamma^*}\!\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right) + ⨍_{\Gamma^*}\partial_t\rho(t)$. Here, the integral over $\partial_t\rho$ vanishes due to

$$\frac{1}{|\Gamma^*|}\int_{\Gamma^*}\partial_t\rho(t) = \frac{1}{|\Gamma^*|}\partial_t\underbrace{\int_{\Gamma^*}\rho(t)}_{=0} = 0\,,$$

because $\rho(t) \in V$. So the terms in (3.45) simplify to

$$0 \;=\; \lambda(t)\underbrace{\int_{\Gamma^*}\xi}_{=0} - \int_{\partial\Gamma^*}\Big(\partial_\mu\rho - S(n^*,n^*)\rho\Big)\,\xi \tag{3.46}$$

for all $\xi \in H^1(\Gamma^*)$ with $\int_{\Gamma^*} \xi = 0$. Since the values of ξ on $\partial\Gamma^*$ are arbitrary, we observe again with the help of the fundamental lemma

$$\partial_\mu \rho - S(n^*, n^*)\rho \quad \text{on } \partial\Gamma^* .$$

Altogether, this means that ρ is a solution of (3.40) and we proved the lemma. □

Similarly as in the section about mean curvature flow we introduce the linearized operator of (3.40) through

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \longrightarrow H$$

with

$$\begin{cases} \mathcal{D}(\mathcal{A}) &= \{\rho \in H^{2,2}(\Gamma^*) \mid (\partial_\mu - S(n^*, n^*))\,\rho = 0 \text{ on } \partial\Gamma^* \text{ and } \int_{\Gamma^*} \rho = 0\}\,, \\ H &= L^2(\Gamma^*)\,, \end{cases}$$

by

$$\mathcal{A}\rho \;:=\; \Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho - \fint_{\Gamma^*} \left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right) \mathrm{d}\mathcal{H}^n .$$

For this operator, we can show as in the previous Section 3.2, that $\mathcal{A}$ is symmetric, self-adjoint with respect to the L^2-inner product, the spectrum consists of a countable system of real eigenvalues, the initial value problem (3.40) is solvable for initial data in H and the zero solution is an asymptotically stable solution of (3.40) if and only if the largest eigenvalue of $\mathcal{A}$ is negative.

Finally we get the following description of linearized stability.

Corollary 3.23. *The linearized problem (3.40) is asymptotically stable if and only if*

$$I(\rho, \rho) \;>\; 0$$

for all $\rho \in H^1(\Gamma^*)\backslash\{0\}$ with $\int_{\Gamma^*} \rho = 0$, where

$$I(\rho,\rho) \;=\; \int_{\Gamma^*} \left(|\nabla_{\Gamma^*}\rho|^2 - |\sigma^*|^2\rho^2\right) d\mathcal{H}^n - \int_{\partial\Gamma^*} S(n^*, n^*)\rho^2 \, d\mathcal{H}^{n-1} .$$

3.4 Surface diffusion flow

In this section we consider a fourth order geometric evolution equation, the well-known surface diffusion, in our case with boundary conditions as described in Section 3.1. As in the previous two sections we introduce a specific setting for the considered evolving hypersurface, linearize the resulting partial differential equation around a stationary state and analyze the stability of the linearized problem with the help of spectral theory. This section is a direct generalization of the paper of Garcke, Ito and Kohsaka [GIK05] about surface diffusion with boundary contact for curves in the plane to hypersurfaces, although we need another parametrization as described in Section 3.1.

We also want to give some remarks concerning the nonlinear stability as is described in the work of Garcke, Ito and Kohsaka [GIK08] for the curve case, which we do in Subsection 3.4.2.
To specify the problem in detail, we use the same notation as in the last two sections. So we want to find an evolving hypersurface

$$\Gamma = \bigcup_{t\in[0,T)} \{t\} \times \Gamma(t) \quad \text{with} \quad \Gamma(t) \subset \mathbb{R}^{n+1}\,, \tag{3.47}$$

as in Definition 2.31, evolving due to the surface diffusion flow, such that $\Gamma(t)$ lies in a fixed bounded region $\Omega \subset \mathbb{R}^{n+1}$ and the boundary $\partial\Gamma(t)$ of each of the hypersurfaces intersects the boundary $\partial\Omega$ of the fixed region at a right angle.
In formulas, the problem reads as follows. Find Γ as in (3.47), such that

$$\begin{cases} V &= -\Delta_{\Gamma(t)} H & \text{in } \Gamma(t) & \text{for all } t > 0\,, \\ \nabla_{\Gamma(t)} H \cdot n_{\partial\Gamma}(t) &= 0 & \text{on } \partial\Gamma(t) & \text{for all } t > 0\,, \\ \Gamma(t) &\subset \Omega & & \text{for all } t > 0\,, \\ \partial\Gamma(t) &\subset \partial\Omega & & \text{for all } t > 0\,, \\ n(t)\cdot\mu &= 0 & \text{on } \partial\Gamma(t) & \text{for all } t > 0\,, \\ \Gamma(0) &= \Gamma_0\,. & & \end{cases} \tag{3.48}$$

Here V, H, n, $n_{\partial\Gamma}(t)$ and μ are the normal velocity, the mean curvature, a unit normal of the evolving hypersurface Γ, the outer unit conormal of $\Gamma(t)$ at $\partial\Gamma(t)$ and the outer unit normal to $\partial\Omega$. $\nabla_{\Gamma(t)}$ is the surface gradient and $\Delta_{\Gamma(t)}$ the Laplace-Beltrami operator on $\Gamma(t)$. Γ_0 is a given starting surface, which lies in Ω and intersects the boundary $\partial\Omega$ at a right angle.
As in the Section 3.3 of volume-preserving mean curvature flow the area $A(t)$ and volume $Vol(t)$ as described in Section 2.4 of a solution $\Gamma(t)$ of (3.48) are decreasing and preserved.

Lemma 3.24. *For a solution Γ of the flow (3.48) the following estimates hold true*

(i) $\frac{d}{dt}A(t) \le 0$ *and*

(ii) $\frac{d}{dt}Vol(t) = 0$.

Therefore the flow is area minimizing and also volume preserving.

Proof. ad (i): With the result of Lemma 2.46 and the formula for integration by parts from Remark 2.30 we get

$$\begin{aligned} \frac{d}{dt}A(t) &= -\int_{\Gamma(t)} V\,H\,d\mathcal{H}^n = \int_{\Gamma(t)} \Delta_{\Gamma(t)} H\,H\,d\mathcal{H}^n \\ &= -\int_{\Gamma(t)} |\nabla_{\Gamma(t)} H|^2\,d\mathcal{H}^n + \int_{\partial\Gamma(t)} \underbrace{\nabla_{\Gamma(t)} H \cdot n_{\partial\Gamma(t)}}_{=0}\,H\,d\mathcal{H}^{n-1} \le 0\,. \end{aligned}$$

ad (ii): Again with the result of Lemma 2.46 and the formula for integration by parts, we get

$$\frac{d}{dt}Vol(t) = \int_{\Gamma(t)} V\,d\mathcal{H}^n = -\int_{\Gamma(t)} \Delta_{\Gamma(t)} H\,d\mathcal{H}^n = \int_{\partial\Gamma(t)} \nabla_{\Gamma(t)} H \cdot n_{\partial\Gamma}(t)\,d\mathcal{H}^{n-1} = 0\,.$$

This shows the assertions. □

We observe in the next lemma that stationary surfaces of the flow (3.48) satisfy $H \equiv const$, so they are hypersurfaces with constant mean curvature, so-called *H-hypersurfaces.*

Lemma 3.25. *Stationary surfaces of the flow (3.48) are H-hypersurfaces, that is they have constant mean curvature H.*

Proof. Let Γ^* be a stationary surface of (3.48), that is Γ^* lies in Ω, intersects $\partial\Omega$ at a right angle and fulfills the surface diffusion equation (3.48) with $V = 0$, so that we have

$$\Delta_{\Gamma^*} H^* \quad = \quad 0 \quad \text{in } \Gamma^*$$

and

$$\nabla_{\Gamma^*} H^* \cdot \mu \quad = \quad 0 \quad \text{on } \partial\Gamma^* \,.$$

Here we used the fact $n_{\partial\Gamma^*} = \mu$ due to the right angle condition. From a calculation due to the integration by parts formula from Remark 2.30

$$\begin{aligned} 0 \quad &= \quad \int_{\Gamma^*} \Delta_{\Gamma^*} H^* \, H^* \, d\mathcal{H}^n = -\int_{\Gamma^*} \nabla_{\Gamma^*} H^* \cdot \nabla_{\Gamma^*} H^* \, d\mathcal{H}^n + \int_{\partial\Gamma^*} \nabla_{\Gamma^*} H^* \cdot \mu \, d\mathcal{H}^{n-1} \\ &= \quad -\int_{\Gamma^*} |\nabla_{\Gamma^*} H^*|^2 \; d\mathcal{H}^n \end{aligned}$$

we get the equality $\nabla_{\Gamma^*} H^* = 0$ on Γ^*. With the representation

$$\nabla_{\Gamma^*} H^*(p) = \sum_{i=1}^{n} \partial_{\tau_i} H^*(p) \, \tau_i$$

for an orthonormal basis $\tau_1, \ldots, \tau_n$ of $T_p\Gamma^*$ we get $\partial_{\tau_i} H^* = 0$ for any tangent vector $\tau_i \in T_p\Gamma^*$. This is true because every unit tangent vector can be extended to an orthonormal basis.

For a given $p_0 \in \Gamma^*$ we set $A = \{p \in \Gamma^* \mid H^*(p) = H^*(p_0)\}$ and with standard analysis arguments we show that A is nonempty, open and closed in the relative topology of Γ^*. Since our general assumption says that Γ^* is connected, we conclude $A = \Gamma^*$ and therefore H^* is a constant. □

3.4.1 Linearized stability analysis

From now on let Γ^* be a stationary hypersurface of (3.48), i.e. Γ^* lies in Ω, intersects $\partial\Omega$ at a right angle, fulfills the natural boundary condition $\nabla_{\Gamma^*} H^* \cdot n_{\partial\Gamma^*} = \nabla_{\Gamma^*} H^* \cdot \mu = 0$ on $\partial\Gamma^*$ and the surface diffusion equation with $V = 0$, which is

$$\Delta_{\Gamma^*} H^* \quad = \quad 0 \quad \text{in } \Gamma^* \,.$$

Here H^* is the mean curvature of Γ^*.

As in the introductory section of this chapter we introduce a specific curvilinear coordinate system, such that the surfaces $\Gamma(t) = \Gamma_\rho(t)$ can be described with the help of a function

$$\rho : [0,T) \times \Gamma^* \to (-d,d)$$

as graphs over the fixed stationary surface Γ^*. We recall the notation

$$\Psi : \Gamma^* \times (-d,d) \longrightarrow \Omega\ ,\ (q,w) \mapsto \Psi(q,w)\ ,$$

such that $\Psi(q,0) = q$ for all $q \in \Gamma^*$, $\Psi(q,w) \in \partial\Omega$ for all $q \in \partial\Gamma^*$ and $\partial_w \Psi(q,0) \cdot n^*(q) = 1$ for all $q \in \Gamma^*$, where n^* is a unit normal to Γ^*. Then we built the mapping

$$\Phi^\rho : [0,T) \times \Gamma^* \longrightarrow \Omega\ ,\ \Phi^\rho(t,q) := \Psi(q,\rho(t,q))\ ,$$

which is a diffeomorphism onto its image for fixed t and we defined hypersurfaces

$$\Gamma_\rho(t) \;\; := \;\; \{\Phi^\rho(t,q) \mid q \in \Gamma^*\}\ .$$

The corresponding equation to surface diffusion (3.48) written for ρ on the fixed hypersurface Γ^* is given here through

$$\left\{ \begin{array}{rcllr} V(\Psi(q,\rho(t,q))) & = & -\Delta_{\Gamma_\rho(t)} H(\Psi(q,\rho(t,q))) & \text{in } \Gamma^* & \text{for all } t > 0\,, \\ 0 & = & \left(\nabla_{\Gamma_\rho(t)} H \cdot n_{\partial\Gamma_\rho(t)}\right)(\Psi(q,\rho(t,q))) & \text{on } \partial\Gamma^* & \text{for all } t > 0\,, \\ 0 & = & (n(t)\cdot\mu)(\Psi(q,\rho(t,q))) & \text{on } \partial\Gamma^* & \text{for all } t > 0\,, \\ \rho(0,q) & = & \rho_0(q) & \text{in } \Gamma^*\,. & \end{array} \right. \tag{3.49}$$

As in Section 3.2 for mean curvature flow and as explained in (2.12) we use the common abbreviation $V(\Psi(t,\rho(t,q))) = V(t,\Psi(t,\rho(t,q)))$ and analogously for H and n. We also assume as in the previous sections that the starting hypersurface Γ_0 is given through

$$\Gamma_0 = \{\Psi(q,\rho_0(q)) \mid q \in \Gamma^*\}\,.$$

For the linearization of (3.49) it will be useful to transform the surface gradient $\nabla_{\Gamma_\rho(t)}$ and the Laplace-Beltrami operator $\Delta_{\Gamma_\rho(t)}$ on $\Gamma_\rho(t)$ to the fixed stationary hypersurface Γ^*. To this end, we equip Γ^* with the pull-back metric $g := (\Phi_t^\rho)^*\,\eta$, where η is a symbol for the euclidian scalar product in $\mathbb{R}^{n+1}$. This means for $v, w \in T_q\Gamma^*$ that

$$g(v,w) = \eta\left(d_q\Phi_t^\rho(v), d_q\Phi_t^\rho(w)\right) = \left(d_q\Phi_t^\rho(v)\cdot d_q\Phi_t^\rho(w)\right).$$

From Remark 2.41 we obtain then with $p = \Phi_t^\rho(q) = \Psi(t,\rho(t,q)) \in \Gamma_\rho(t)$ for some $q \in \Gamma^*$ the following formulas.

$$\Delta_{\Gamma_\rho(t)} H(\Psi(t,\rho(t,q))) \;\; = \;\; \Delta^\rho_{\Gamma^*}\widetilde{H}_\rho(t,q) \;\text{ and} \tag{3.50}$$

$$\nabla_{\Gamma_\rho(t)} H(\Psi(t,\rho(t,q))) \;\; = \;\; d_q\Phi_t^\rho\left(\nabla^\rho_{\Gamma^*}\widetilde{H}_\rho(t,q)\right), \tag{3.51}$$

where we use $\widetilde{H}_\rho(t,q) = H(\Psi(t,\rho(t,q)))$ and we indicated with an index ρ on $\nabla^\rho_{\Gamma^*}$ and $\Delta^\rho_{\Gamma^*}$ that these differential operators depend on the function ρ. Therefore we will also have to differentiate these operators when building the linearization of (3.49).

For the linearization of (3.49) around $\rho \equiv 0$, which means around the given stationary state Γ^*, we can use the results and notation of Section 3.2. In particular, we use the linearization of normal velocity and mean curvature from Lemma 3.4 and 3.5.

Lemma 3.26. *The linearization of the surface diffusion equation from (3.49)*

$$V(\Psi(t,\rho(t,q))) \quad = \quad -\Delta_{\Gamma_\rho(t)} H(\Psi(t,\rho(t,q)))$$

around the stationary state represented through $\rho \equiv 0$ is given by

$$\partial_t \rho(t,q) \quad = \quad -\Delta_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho(t,q) + |\sigma^*(q)|^2\rho(t,q)\right),$$

where $q \in \Gamma^*$ and $t > 0$.

Proof. The linerization of normal velocity

$$\frac{d}{d\varepsilon} V(\Psi(t,\varepsilon\rho(t,q)))\bigg|_{\varepsilon=0} \quad = \quad \partial_t\rho(t,q)$$

follows as in Lemma 3.4.

We write the Laplace-Beltrami operator of mean curvature with the help of formula (3.50) as

$$-\Delta_{\Gamma_\rho(t)} H(\Psi(t,\rho(t,q))) \quad = \quad -\Delta^\rho_{\Gamma^*}\left(\widetilde{H}_\rho(t,q)\right).$$

Then we observe that for $\rho \equiv 0$ due to $\Phi^0_t = id|_{\Gamma^*}$ the identity

$$\Delta^0_{\Gamma^*} \quad = \quad \Delta_{\Gamma^*}$$

holds, where Δ_{Γ^*} is the Laplace-Beltrami operator of Γ^* with respect to the restriction of the euclidian scalar product. We also have

$$\widetilde{H}_0 \quad = \quad H^*,$$

where H^* is the constant mean curvature of Γ^* due to Lemma 3.25. Therefore we get with a similar calculation as in the work of Escher, Mayer and Simonett [EMS98]

$$\frac{d}{d\varepsilon}\Delta^{\varepsilon\rho}_{\Gamma^*}\bigg|_{\varepsilon=0}\widetilde{H}_0 = \frac{d}{d\varepsilon}\Delta^{\varepsilon\rho}_{\Gamma^*}\bigg|_{\varepsilon=0} H^* = \frac{d}{d\varepsilon}\underbrace{\left(\Delta^{\varepsilon\rho}_{\Gamma^*}H^*\right)}_{=0}\Bigg|_{\varepsilon=0} = 0\,.$$

Finally, this gives for the right side of the surface diffusion equation

$$\begin{aligned}\frac{d}{d\varepsilon}\left(-\Delta^{\varepsilon\rho}_{\Gamma^*}\widetilde{H}_{\varepsilon\rho}(t,q)\right)\bigg|_{\varepsilon=0} \quad &= \quad -\frac{d}{d\varepsilon}\Delta^{\varepsilon\rho}_{\Gamma^*}\bigg|_{\varepsilon=0} H^* - \Delta_{\Gamma^*}\left(\frac{d}{d\varepsilon}\widetilde{H}_{\varepsilon\rho}(t,q)\bigg|_{\varepsilon=0}\right)\\ &= \quad -\Delta_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho(t,q) + |\sigma^*(q)|^2\rho(t,q)\right),\end{aligned}$$

where we used the linearization of mean curvature from Lemma 3.5. □

The next point is to linearize the natural boundary condition.

Lemma 3.27. *The linearization of the natural boundary condition from (3.49)*

$$0 = \left(\nabla_{\Gamma_\rho(t)} H \cdot n_{\partial\Gamma_\rho(t)}\right)(\Psi(q,\rho(t,q)))$$

around the stationary state represented through $\rho \equiv 0$ *is given by*

$$\begin{aligned} 0 &= \nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho(t,q) + |\sigma^*|^2\rho(t,q)\right)\cdot\mu(q) \\ &= \partial_\mu\left(\Delta_{\Gamma^*}\rho(t,q) + |\sigma^*(q)|^2\rho(t,q)\right), \end{aligned}$$

where $q \in \partial\Gamma^*$ *and* $t > 0$.

Proof. With the help of formula (3.51) we can correlate the surface gradient on $\Gamma_\rho(t)$ and on Γ^* equipped with the pull-back metric $(\Phi_t^\rho)^*\eta$ via

$$\nabla_{\Gamma_\rho(t)} H(\Psi(q,\rho(t,q))) = d_q\Phi_t^\rho\left(\nabla^\rho_{\Gamma^*}\widetilde{H}_\rho(t,q)\right),$$

where $p = \Phi_t^\rho(q) = \Psi(q,\rho(t,q)) \in \Gamma_\rho(t)$. We observe for $\rho \equiv 0$

$$n_{\partial\Gamma_0(t)}(\Psi(q,0)) = n_{\partial\Gamma^*}(q) = \mu(q)$$

and

$$\nabla_{\Gamma_0(t)} H(\Psi(q,0)) = \nabla_{\Gamma^*} H^* = 0\,,$$

where we used the angle condition in the first equation and the fact that Γ^* is an H-surface from Lemma 3.25.

Then we can conclude for the linearization

$$\frac{d}{d\varepsilon}\left(\nabla_{\Gamma_{\varepsilon\rho(t)}} H \cdot n_{\partial\Gamma_{\varepsilon\rho(t)}}\right)(\Psi(q,\varepsilon\rho(t,q)))\bigg|_{\varepsilon=0} = 0 + \underbrace{\frac{d}{d\varepsilon}\left(\nabla_{\Gamma_{\varepsilon\rho(t)}} H(\Psi(q,\varepsilon\rho(t,q)))\right)\bigg|_{\varepsilon=0}}_{=(*)} \cdot\mu(q)$$

and the term $(*)$ can be calculated with an analogue argumentation as for the Laplace-Beltrami operator in the proof of Lemma 3.26 as follows

$$\begin{aligned} (*) &= \frac{d}{d\varepsilon}\left[(d_q\Phi_t^{\varepsilon\rho})\left(\nabla^{\varepsilon\rho}_{\Gamma^*}\widetilde{H}_{\varepsilon\rho}(t,q)\right)\right]\bigg|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon}(d_q\Phi_t^{\varepsilon\rho})\bigg|_{\varepsilon=0}\underbrace{(\nabla_{\Gamma^*}H^*)}_{=0} + \underbrace{d_q\Phi_t^0}_{=Id}\left(\frac{d}{d\varepsilon}\left(\nabla^{\varepsilon\rho}_{\Gamma^*}\widetilde{H}_{\varepsilon\rho}(t,q)\right)\right)\bigg|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon}\nabla^{\varepsilon\rho}_{\Gamma^*}\bigg|_{\varepsilon=0}\widetilde{H_0}(t,q) + \nabla_{\Gamma^*}\frac{d}{d\varepsilon}\widetilde{H}_{\varepsilon\rho}(t,q)\bigg|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon}\underbrace{(\nabla^{\varepsilon\rho}_{\Gamma^*}H^*)}_{=0}\Bigg|_{\varepsilon=0} + \nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho(t,q) + |\sigma^*(q)|^2\rho(t,q)\right) \\ &= \nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho(t,q) + |\sigma^*(q)|^2\rho(t,q)\right), \end{aligned}$$

where we used $\Phi_t^0 = id|_{\Gamma^*}$, i.e. $d_q\Phi_t^0 = Id|_{T_q\Gamma^*}$, and the linearization of mean curvature from Lemma 3.5. $\square$

So together with the results of the previous section about the linearization of the angle condition in Lemma 3.7 we get for the linearization of (3.49) around $\rho \equiv 0$ the following equations.

$$\left\{\begin{array}{rcllll} \partial_t\rho &=& -\Delta_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right) & \text{in } \Gamma^* & \text{for all } t>0\,, \\ 0 &=& \left(\partial_\mu - S(n^*,n^*)\right)\rho & \text{on } \partial\Gamma^* & \text{for all } t>0\,, \\ 0 &=& \partial_\mu\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right) & \text{on } \partial\Gamma^* & \text{for all } t>0\,, \\ \rho(0,q) &=& 0 & \text{in } \Gamma^*\,. & \end{array}\right. \tag{3.52}$$

Note that as in the previous sections S is the second fundamental form of $\partial\Omega$ with respect to the inwards pointing unit normal $(-\mu)$ of Ω and due to the angle condition for the stationary hypersurface Γ the unit normal n^* of Γ^* fulfills $n(p) \in T_p\partial\Omega$ on $\partial\Omega \cap \Gamma^*$, so that the term $S(n^*,n^*)$ does make sense.
A solvability condition for solutions of the linearized problem (3.52) gives here as in Section 3.3 for the volume preserving mean curvature flow $\int_{\Gamma^*}\rho \equiv 0$.

Lemma 3.28. *Solutions of the linearized problem (3.52) fulfill*

$$\int_{\Gamma^*} \rho\, d\mathcal{H}^n \quad \equiv \quad 0\,.$$

Proof. Integrating the first equation in (3.52) gives with the help of partial integration by Remark 2.30

$$\int_0^t\int_{\Gamma^*}\partial_t\rho = \int_0^t\int_{\Gamma^*} -\Delta_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right) = \int_0^t\int_{\partial\Gamma^*} -\nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right)\cdot\mu = 0\,.$$

For the time derivative we also have

$$\int_0^t\int_{\Gamma^*}\partial_t\rho = \int_0^t\partial_t\int_{\Gamma^*}\rho = \int_{\Gamma^*}\rho(t,q) - \int_{\Gamma^*}\underbrace{\rho(0,q)}_{=0}\,,$$

so that we showed the assertion. $\square$

To derive conditions for stability of the zero solution of the linearized problem (3.52) we proceed in an analogue way as in the previous sections. First, we show that (3.52) can be interpreted as a gradient flow with respect to an energy E given by a symmetric bilinear form I. Then we relate the eigenvalues of the linearized operator with respect to (3.52) to the positivity of the bilinear form to achieve the following result.

$$\boxed{\begin{array}{ll} & \Gamma^* \text{ is linearly asymptotically stable} \\ \Longleftrightarrow & \left\{\begin{array}{l} I(\rho,\rho) := \int_{\Gamma^*}\left(|\nabla_{\Gamma^*}\rho|^2 - |\sigma^*|^2\rho^2\right) - \int_{\partial\Gamma^*} S(n^*,n^*)\rho^2 \\ \text{ is positive for all } \rho \in H^{1,2}(\Gamma^*)\backslash\{0\} \text{ with } \int_{\Gamma^*}\rho = 0\,. \end{array}\right. \end{array}}$$

Here we generalize directly the work of Garcke, Ito and Kohsaka [GIK05] from curves to higher dimensions. Since the problem (3.52) will be a gradient flow with respect to the H^{-1}-inner product, we give its definition. We denote by $\langle .\,,.\rangle$ the duality pairing between the dual space $\big(H^1(\Gamma^*)\big)'$ and $H^1(\Gamma^*)$ and we define the space $H^{-1}(\Gamma^*)$ by

$$H^{-1}(\Gamma^*) \quad := \quad \left\{\rho \in \big(H^1(\Gamma^*)\big)' \mid \langle \rho, 1\rangle = 0\right\} . \tag{3.53}$$

Definition 3.29. *We say that $u_v \in H^1(\Gamma^*)$ with $\int_{\Gamma^*} u_v = 0$ for a given $v \in H^{-1}(\Gamma^*)$ is a weak solution of*

$$\left\{ \begin{array}{rcll} -\Delta_{\Gamma^*} u_v & = & v & \text{in } \Gamma^* , \\ \nabla_{\Gamma^*} u_v \cdot n_{\partial\Gamma^*} & = & 0 & \text{on } \partial\Gamma^* , \end{array} \right. \tag{3.54}$$

if and only if u_v satisfies

$$\langle v, \xi\rangle \quad = \quad \int_{\Gamma^*} \nabla_{\Gamma^*} u_v \cdot \nabla_{\Gamma^*} \xi$$

for all $\xi \in H^1(\Gamma^*)$.

For $\rho_i \in H^{-1}(\Gamma^*)$, $i = 1, 2$, we introduce the inner product

$$(\rho_1, \rho_2)_{-1} \quad := \quad \int_{\Gamma^*} \nabla_{\Gamma^*} u_{\rho_1} \cdot \nabla_{\Gamma^*} u_{\rho_2} , \tag{3.55}$$

called the H^{-1}-inner product, where u_{ρ_i} is defined as the weak solution of (3.54) with respect to ρ_i. This makes $H^{-1}(\Gamma^*)$ to a Hilbertspace and we also introduce the notation for the corresponding norm

$$\|\rho\|_{-1} := \sqrt{(\rho, \rho)_{-1}} \qquad \text{for } \rho \in H^{-1}(\Gamma^*) .$$

By definition, we have the identity

$$(\rho_1, \rho_2)_{-1} \quad = \quad \langle \rho_1, u_{\rho_2}\rangle \tag{3.56}$$

for $\rho_i \in H^{-1}(\Gamma^*)$.
For further use we also introduce the notation

$$V \quad := \quad \left\{\rho \in H^1(\Gamma^*) \mid \int_{\Gamma^*} \rho = 0\right\} ,$$

so that V is a subspace of $H^1(\Gamma^*)$.

Remark 3.30. *We remark that in the literature the space $H^{-1}_{lit}(\Gamma^*)$ is usually defined as the dual space $H^{-1}_{lit}(\Gamma^*) := V'$. For $v \in H^{-1}_{lit}(\Gamma^*)$ the duality pairing $\langle v\,, \xi\rangle$ would then be defined just for functions $\xi \in H^1(\Gamma^*)$ with the constraint $\int_{\Gamma^*} \xi = 0$. But this functional $v \in H^{-1}_{lit}(\Gamma^*)$ can be extended naturally to all of $H^1(\Gamma^*)$ by $\langle v, 1\rangle = 0$. Together with this extension, the dual space $H^{-1}_{lit}(\Gamma^*)$ then equals our definition of $H^{-1}(\Gamma^*)$.*

We also define as in the previous section a symmetric bilinear form on $H^1(\Gamma^*)$ and the corresponding energy. The definition equals the one from Definition 3.9, but we state it again for easy readability.

Definition 3.31. *For $\rho_1, \rho_2 \in H^1(\Gamma^*)$ we define*

$$I(\rho_1, \rho_2) \quad := \quad \int_{\Gamma^*} \left(\nabla_{\Gamma^*}\rho_1 \cdot \nabla_{\Gamma^*}\rho_2 - |\sigma^*|^2 \rho_1\, \rho_2\right) - \int_{\partial\Gamma^*} S(n^*, n^*)\rho_1\, \rho_2 \tag{3.57}$$

and the associated energy for $\rho \in H^1(\Gamma^*)$ by

$$E(\rho) \quad := \quad \frac{1}{2} I(\rho, \rho)\,. \tag{3.58}$$

The next point is to show that the linearized problem (3.52) is the gradient flow of E with respect to the H^{-1}-inner product $(\,.\,,.)_{-1}$. This means that a solution ρ of (3.52) fulfils

$$(\partial_t \rho, \xi)_{-1} \quad = \quad -\partial E(\rho(t))(\xi)$$

for all $\xi \in H^1(\Gamma^*)$ with $\int_{\Gamma^*} \xi = 0$. Here, $\partial E(\rho(t))(\xi)$ denotes the derivative of E at $\rho(t)$ in direction of ξ. Because of the definition of E via the bilinear form I, this derivative is given by

$$\partial E(\rho(t))(\xi) \quad = \quad I(\rho(t), \xi)\,.$$

To simplify notation, we introduce the following time independent problem.

Definition 3.32. *For a given $v \in H^{-1}(\Gamma^*)$ we say that $\rho \in H^3(\Gamma^*)$ with $\int_{\Gamma^*} \rho = 0$ is a weak solution of the boundary value problem*

$$\left\{ \begin{array}{lcll} v & = & -\Delta_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right) & \text{in } \Gamma^*\,, \\ 0 & = & \partial_\mu \rho - S(n^*, n^*)\rho & \text{on } \partial\Gamma^*\,, \\ 0 & = & \nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right) \cdot n_{\partial\Gamma^*} & \text{on } \partial\Gamma^*\,, \end{array} \right. \tag{3.59}$$

if and only if ρ satisfies

$$\langle v, \xi \rangle \quad = \quad \int_{\Gamma^*} \nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right) \cdot \nabla_{\Gamma^*}\xi$$

for all $\xi \in H^1(\Gamma^*)$ and

$$0 \quad = \quad \partial_\mu \rho - S(n^*, n^*)\rho \quad \text{on } \partial\Gamma^*\,.$$

In the case that $v \in L^2(\Gamma^*)$ with $\int_{\Gamma^*} v = 0$, we obtain from elliptic regularity theory on manifolds that $v = -\Delta_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right)$ is fulfilled almost everywhere on Γ^* and $\nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right) \cdot n_{\partial\Gamma^*} = 0$ is fulfilled almost everywhere on $\partial\Gamma^*$.

The fact that the linearized problem is the gradient flow of E with respect to the H^{-1}-inner product follows from the next lemma.

Lemma 3.33. *Let $v \in H^{-1}(\Gamma^*)$ and $\rho \in H^1(\Gamma^*)$ with $\int_{\Gamma^*} \rho = 0$ be given. Then ρ is a weak solution of (3.59) if and only if*

$$(v, \xi)_{-1} \quad = \quad -I(\rho, \xi)$$

holds for all $\xi \in H^1(\Gamma^*)$ with $\int_{\Gamma^*} \xi = 0$.

Proof. The proof of [GIK05] directly generalizes to the higher dimensional situation. For the convenience of the reader, we give the details.

Let $\rho \in H^3(\Gamma^*)$ with $\int_{\Gamma^*} \rho = 0$ be a weak solution of (3.59). By (3.56) and Definition 3.32, we deduce for $\xi \in H^1(\Gamma^*)$ with $\int_{\Gamma^*} \xi = 0$ the identities

$$\begin{aligned} (v, \xi)_{-1} &= \langle v, u_\xi \rangle \\ &= \int_{\Gamma^*} \nabla_{\Gamma^*} \left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho \right) \cdot \nabla_{\Gamma^*} u_\xi \,. \end{aligned}$$

Here, $u_\xi \in H^1(\Gamma^*)$ is the weak solution of (3.54) for the given $\xi \in H^1(\Gamma^*)$. Then, by virtue of $\left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho\right) \in H^1(\Gamma^*)$ we see from the definition of the weak solution u_ξ with $\left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho\right)$ as testfunction

$$\int_{\Gamma^*} \nabla_{\Gamma^*} \left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho \right) \cdot \nabla_{\Gamma^*} u_\xi \quad = \quad \int_{\Gamma^*} \left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho \right) \xi \,.$$

This is true since the right side in (3.54) lies in this case in the function space $H^1(\Gamma^*)$ instead of $\left(H^1(\Gamma^*)\right)'$ and therefore we can give the duality pairing as integral.

Now we conclude with integration by parts.

$$\begin{aligned} (v, \xi)_{-1} &= \int_{\Gamma^*} \left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho \right) \xi \\ &= -\int_{\Gamma^*} \left(\nabla_{\Gamma^*} \rho \cdot \nabla_{\Gamma^*} \xi - |\sigma^*|^2 \rho \xi \right) + \int_{\partial \Gamma^*} \nabla_{\Gamma^*} \rho \cdot n_{\partial \Gamma^*} \xi \\ &= -\int_{\Gamma^*} \left(\nabla_{\Gamma^*} \rho \cdot \nabla_{\Gamma^*} \xi - |\sigma^*|^2 \rho \xi \right) + \int_{\partial \Gamma^*} S(n^*, n^*) \, \rho \xi \\ &= -I(\rho, \xi) \,, \end{aligned}$$

where we used the boundary condition $\nabla_{\Gamma^*} \rho \cdot n_{\partial \Gamma^*} = \partial_\mu \rho = S(n^*, n^*) \rho$ on $\partial \Gamma^*$ for ρ.

Conversely, assume that $\rho \in H^1(\Gamma^*)$ with $\int_{\Gamma^*} \rho = 0$ satisfies

$$(v, \xi)_{-1} \quad = \quad -I(\rho, \xi)$$

for all $\xi \in H^1(\Gamma^*)$ with $\int_{\Gamma^*} \xi = 0$. Now we choose $\xi = -\Delta_{\Gamma^*} \eta$ for a given function $\eta \in H^3(\Gamma^*)$ with $\nabla_{\Gamma^*} \eta \cdot n_{\partial \Gamma^*} = 0$ on $\partial \Gamma^*$. From Definition 3.29 we can write $\eta = u_\xi$ and with (3.56) it holds

$$\begin{aligned} \langle v, \eta \rangle &= (v, \xi)_{-1} = -I(\rho, \xi) \\ &= -\int_{\Gamma^*} \left(\nabla_{\Gamma^*} \rho \cdot \nabla_{\Gamma^*} \xi - |\sigma^*|^2 \rho \xi \right) + \int_{\partial \Gamma^*} S(n^*, n^*) \rho \xi \\ &= \int_{\Gamma^*} \left(\nabla_{\Gamma^*} \rho \cdot \nabla_{\Gamma^*} (\Delta_{\Gamma^*} \eta) - |\sigma^*|^2 \rho \, (\Delta_{\Gamma^*} \eta) \right) + \int_{\partial \Gamma^*} S(n^*, n^*) \rho \, (\Delta_{\Gamma^*} \eta) \,. \end{aligned}$$

Since $v \in \left(H^1(\Gamma^*)\right)'$ we deduce from the above identity and elliptic regularity theory that $\rho \in H^3(\Gamma^*)$. Integration by parts gives then

$$\begin{aligned}
\langle v, \eta\rangle &= -\int_{\Gamma^*} \left(\Delta_{\Gamma^*}\rho\, \Delta_{\Gamma^*}\eta - \nabla_{\Gamma^*}(|\sigma^*|^2\rho)\cdot\nabla_{\Gamma^*}\eta\right) \\
&\quad + \int_{\partial\Gamma^*}\left(\nabla_{\Gamma^*}\rho\cdot n_{\partial\Gamma^*}\,\Delta_{\Gamma^*}\eta - |\sigma^*|^2\rho\,\underbrace{\nabla_{\Gamma^*}\eta\cdot n_{\partial\Gamma^*}}_{=0} - S(n^*,n^*)\rho\,\Delta_{\Gamma^*}\eta\right) \\
&= \int_{\Gamma^*}\nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right)\cdot\nabla_{\Gamma^*}\eta - \int_{\partial\Gamma^*}\Delta_{\Gamma^*}\rho\,\underbrace{\nabla_{\Gamma^*}\eta\cdot n_{\partial\Gamma^*}}_{=0} \\
&\quad + \int_{\partial\Gamma^*}\left(\partial_\mu\rho - S(n^*,n^*)\rho\right)\Delta_{\Gamma^*}\eta \\
&= \int_{\Gamma^*}\nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right)\cdot\nabla_{\Gamma^*}\eta + \int_{\partial\Gamma^*}\left(\partial_\mu\rho - S(n^*,n^*)\rho\right)\Delta_{\Gamma^*}\eta\,.
\end{aligned}$$

To show that ρ is a weak solution of (3.59), we choose for a given test function $\varphi \in H^1(\Gamma^*)$ a sequence $\eta_n \in H^3(\Gamma^*)$ with $\nabla_{\Gamma^*}\eta_n\cdot n_{\partial\Gamma^*} = 0$ and $\Delta_{\Gamma^*}\eta_n = 0$ each on $\partial\Gamma^*$ such that

$$\eta_n \to \varphi \qquad \text{in } H^1(\Gamma^*)\,.$$

For such η_n we get from the last equation

$$\langle v, \eta_n\rangle = \int_{\Gamma^*}\nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right)\cdot\nabla_{\Gamma^*}\eta_n + 0\,,$$

where the left side converges to $\langle v, \varphi\rangle$ and the right side to $\int_{\Gamma^*}\nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right)\cdot\nabla_{\Gamma^*}\varphi$ due to the convergence $\eta_n \to \varphi$ in $H^1(\Gamma^*)$. So we conclude

$$\langle v, \varphi\rangle = \int_{\Gamma^*}\nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right)\cdot\nabla_{\Gamma^*}\varphi$$

for arbitrary $\varphi \in H^1(\Gamma^*)$. Inserting this into the last equation for η finally gives

$$0 = \int_{\partial\Gamma^*}\left(\partial_\mu\rho - S(n^*,n^*)\rho\right)\Delta_{\Gamma^*}\eta$$

for all $\eta \in H^3(\Gamma^*)$ with $\nabla_{\Gamma^*}\eta\cdot n_{\partial\Gamma^*} = 0$ on $\partial\Gamma^*$. Since $\Delta_{\Gamma^*}\eta$ is arbitrary on $\partial\Gamma^*$, we conclude with the fundamental lemma

$$\partial_\mu\rho - S(n^*,n^*)\rho = 0 \qquad \text{on } \partial\Gamma^*\,.$$

This shows that ρ is a weak solution of (3.59) and concludes the proof. □

The next steps consist in showing that the linearized operator is self-adjoint and to study its spectrum. This linearized operator corresponding to (3.52) is given by

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \longrightarrow H\,,$$

with

$$\left\{\begin{array}{rcl} \mathcal{D}(\mathcal{A}) & = & \{\rho \in H^3(\Gamma^*) \,|\, (\partial_\mu - S(n^*,n^*))\,\rho = 0 \text{ on } \partial\Gamma^* \text{ and } \int_{\Gamma^*} \rho = 0\}\,, \\ H & = & \{\rho \in \left(H^1(\Gamma^*)\right)' \,|\, \langle \rho, 1\rangle = 0\} \end{array}\right. \tag{3.60}$$

by

$$\langle \mathcal{A}\rho, \xi\rangle := \int_{\Gamma^*} \nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right)\cdot\nabla_{\Gamma^*}\xi\,. \tag{3.61}$$

Then we can relate the boundary value problem (3.59) to the problem of finding a $\rho \in \mathcal{D}(\mathcal{A})$ with

$$\mathcal{A}\rho = v\,.$$

By Lemma 3.33 we also have for all $\xi \in H^1(\Gamma^*)$ with $\int_{\Gamma^*}\xi = 0$

$$(\mathcal{A}\rho,\xi)_{-1} = -I(\rho,\xi)\,.$$

Lemma 3.34. *The operator $\mathcal{A}$ is symmetric with respect to the inner product $(.\,,.)_{-1}$.*

Proof. For $\rho, \xi \in \mathcal{D}(\mathcal{A})$ we have

$$(\mathcal{A}\rho,\xi)_{-1} = -I(\rho,\xi) = -I(\xi,\rho) = (\mathcal{A}\xi,\rho)_{-1} = (\rho,\mathcal{A}\xi)_{-1}\ ,$$

so that $\mathcal{A}$ is symmetric. □

As in Section 3.2, we want to analyze the spectrum of $\mathcal{A}$ to decide on the stability behaviour of the linearized problem (3.52). This spectrum is related to the functional I with the help of the inner product $(.\,,.)_{-1}$. In fact, for an eigenfunction $\rho \in \mathcal{D}(\mathcal{A})$ to the eigenvalue λ of $\mathcal{A}$, it holds

$$\lambda\,(\rho,\xi)_{-1} = (\mathcal{A}\rho,\xi)_{-1} = -I(\rho,\xi)$$

for all $\xi \in H^1(\Gamma^*)$ with $\int_{\Gamma^*}\xi = 0$.

The next point is to show boundedness of eigenvalues of $\mathcal{A}$ from above. Therefore we need the following two lemmata.

Lemma 3.35. *For all $\delta > 0$ there exists a $C_\delta > 0$, such that for all functions $\rho \in V$ the inequality*

$$\|\rho\|^2_{L^2(\partial\Gamma^*)} \ \le\ \delta\,\|\nabla_{\Gamma^*}\rho\|^2_{L^2(\Gamma^*)} + C_\delta\,\|\rho\|^2_{-1}$$

holds.

Proof. Assume by contradiction that there exists $\delta > 0$ such that we can find a sequence $(\widetilde{\rho}_n)_{n\in\mathbb{N}} \subset V$ such that

$$\|\widetilde{\rho}_n\|^2_{L^2(\partial\Gamma^*)} \quad > \quad \delta \,\|\nabla_{\Gamma^*}\widetilde{\rho}_n\|^2_{L^2(\Gamma^*)} + n\,\|\widetilde{\rho}_n\|^2_{-1}\,.$$

In particular we observe $\|\widetilde{\rho_n}\|_{L^2(\partial\Gamma^*)} > 0$ for all $n \in \mathbb{N}$. Therefore, we get for the scaled functions $\rho_n = \widetilde{\rho_n}\,\left(\|\widetilde{\rho_n}\|_{L^2(\partial\Gamma^*)}\right)^{-1}$ by multiplying with $\left(\|\widetilde{\rho_n}\|_{L^2(\partial\Gamma^*)}\right)^{-2}$ the inequality

$$1 \quad > \quad \delta\,\|\nabla_{\Gamma^*}\rho_n\|^2_{L^2(\Gamma^*)} + n\,\|\rho_n\|^2_{-1}\,.$$

This implies

$$\|\rho_n\|^2_{-1} < \frac{1}{n} \longrightarrow 0 \qquad \text{as } n \to \infty$$

and

$$\|\nabla_{\Gamma^*}\rho_n\|^2_{L^2(\Gamma^*)} < \frac{1}{\delta}\,.$$

Since $\int_{\Gamma^*}\rho_n = 0$, we conclude from Poincaré's inequality that ρ_n is bounded uniformly in $H^1(\Gamma^*)$. Therefore it converges weakly for a subsequence

$$\rho_n \rightharpoonup \overline{\rho} \qquad \text{in } H^1(\Gamma^*)$$

to some $\overline{\rho} \in H^1(\Gamma^*)$. Due to

$$0 = (\rho_n, 1)_{L^2} \to (\overline{\rho}, 1)_{L^2} = \int_{\Gamma^*}\overline{\rho}$$

we observe $\int_{\Gamma^*}\overline{\rho} = 0$. Furthermore from the compact embedding

$$\left\{\rho \in H^1(\Gamma^*) \mid \int_{\Gamma^*}\rho = 0\right\} \hookrightarrow H^{-1}(\Gamma^*)$$

we see the strong convergence $\rho_n \to \overline{\rho}$ in $H^{-1}(\Gamma^*)$. By uniqueness of the limit and $\|\rho_n\|_{H^{-1}} \to 0$ we get finally $\overline{\rho} = 0$. So we have

$$\rho_n \rightharpoonup 0 \qquad \text{in } H^1(\Gamma^*)\,.$$

By another compact embedding $H^1(\Gamma^*) \hookrightarrow L^2(\partial\Gamma^*)$ we see $\rho_n \to 0$ in $L^2(\partial\Gamma^*)$, which at last contradicts the fact $\|\rho_n\|_{L^2(\partial\Gamma^*)} = 1$ for all $n \in \mathbb{N}$. ☐

Lemma 3.36. *There exist positive constants C_1 and C_2, such that*

$$\|\rho\|^2_{H^1(\Gamma^*)} \quad \leq \quad C_1\,\|\rho\|^2_{-1} + C_2\,I(\rho,\rho)$$

for all $\rho \in V$.

Proof. With an analogue argumentation as in the previous lemma we get the following inequality. For all $\delta > 0$ there exists a $C_\delta > 0$, such that

$$\|\rho\|^2_{L^2(\Gamma^*)} \leq \delta \, \|\nabla_{\Gamma^*}\rho\|^2_{L^2(\Gamma^*)} + C_\delta \, \|\rho\|^2_{-1}$$

holds for all $\rho \in V$. For this inequality we just need the compact embedding $H^1(\Gamma^*) \hookrightarrow L^2(\Gamma^*)$ instead of $H^1(\Gamma^*) \hookrightarrow L^2(\partial\Gamma^*)$. Now we obtain with the help of the above inequality and Lemma 3.35

$$\begin{aligned}
I(\rho,\rho) &= \int_{\Gamma^*} |\nabla_{\Gamma^*}\rho|^2 - \int_{\Gamma^*} |\sigma^*|^2 \, \rho^2 - \int_{\partial\Gamma^*} S(n^*, n^*) \, \rho^2 \\
&\geq \|\nabla_{\Gamma^*}\rho\|^2_{L^2(\Gamma^*)} - \||\sigma^*|^2\|_{L^\infty(\Gamma^*)} \cdot \|\rho\|^2_{L^2(\Gamma^*)} - \|S(n^*, n^*)\|_{L^\infty(\partial\Gamma^*)} \cdot \|\rho\|^2_{L^2(\partial\Gamma^*)} \\
&\geq \Big(1 - \delta_1 \, \|S(n^*, n^*)\|_{L^\infty(\partial\Gamma^*)}\Big) \cdot \|\nabla_{\Gamma^*}\rho\|^2_{L^2(\Gamma^*)} - \||\sigma^*|^2\|_{L^\infty(\Gamma^*)} \cdot \|\rho\|^2_{L^2(\Gamma^*)} \\
&\quad -\|S(n^*, n^*)\|_{L^\infty(\partial\Gamma^*)} \cdot C_{\delta_1} \, \|\rho\|^2_{-1} \\
&\geq \Big(1 - \delta_1\|S(n^*, n^*)\|_{L^\infty(\partial\Gamma^*)} - \delta_2 \, \||\sigma^*|^2\|_{L^\infty(\Gamma^*)}\Big) \cdot \|\nabla_{\Gamma^*}\rho\|^2_{L^2(\Gamma^*)} \\
&\quad - \Big(\||\sigma^*|^2\|_{L^\infty(\Gamma^*)} \, C_{\delta_2} + \|S(n^*, n^*)\|_{L^\infty(\partial\Gamma^*)} \, C_{\delta_1}\Big) \cdot \|\rho\|^2_{-1} \, .
\end{aligned}$$

With the help of the Poincaré inequality on V and by choosing δ_1 and δ_2 small enough, we get the assertion. □

With the previous two lemmata we can show boundedness from above for the largest eigenvalue of $\mathcal{A}$.

Lemma 3.37. *Let λ be an eigenvalue of $\mathcal{A}$. Then the following inequality holds*

$$\lambda \leq \frac{C_1}{C_2} \, ,$$

where C_1 and C_2 are the positive constants of the above Lemma 3.36.

Proof. Let $\rho \in \mathcal{D}(\mathcal{A})$ be an eigenvector to the eigenvalue λ, which in particular means $\rho \neq 0$. It holds

$$\lambda \, (\rho, \rho)_{-1} = (\mathcal{A}\rho, \rho)_{-1} = -I(\rho, \rho) \, .$$

Assuming that $\lambda > \frac{C_1}{C_2}$, we would have

$$\begin{aligned}
0 &= I(\rho,\rho) + \lambda \, (\rho,\rho)_{-1} > I(\rho,\rho) + \frac{C_1}{C_2} \, (\rho,\rho)_{-1} \geq \frac{1}{C_2} \|\rho\|^2_{H^1(\Gamma^*)} \\
&> 0 \, ,
\end{aligned}$$

which is a contradiction. □

Now we are able to show that $\mathcal{A}$ is self-adjoint with respect to the $(.\,,.)_{-1}$ inner product. As in Section 3.2 of mean curvature flow, we proceed with a property that implies the equivalence of symmetry and self-adjointness from [Weid76]. Since we know from Lemma 3.34 that $\mathcal{A}$ is symmetric, this will provide us even with self-adjointness.

Lemma 3.38. *The operator $\mathcal{A}$ is self-adjoint with respect to the $(.\,,.)_{-1}$ inner product.*

Proof. We use the following theorem of operator theory. If there exists an $\omega \in \mathbb{R}$, such that

$$\operatorname{im}(\omega\, Id - \mathcal{A}) \;=\; H^{-1}(\Gamma)\,,$$

the properties symmetry and self-adjointness of $\mathcal{A}$ are equivalent, see for example [Weid76].

So we have to show that there exists an $\omega \in \mathbb{R}$, such that for given $f \in H^{-1}(\Gamma^*)$ there exists a $\rho \in \mathcal{D}(\mathcal{A})$ with

$$\omega\rho - \mathcal{A}\rho \;=\; f\,.$$

This means that $\rho \in H^3(\Gamma^*)$ is a weak solution of the boundary value problem

$$\left\{\begin{aligned} \Delta_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right) + \omega\rho &= f \quad \text{in } \Gamma^*\,,\\ \partial_\mu\rho - S(n^*,n^*)\rho &= 0 \quad \text{on } \partial\Gamma^*\,,\\ \nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right)\cdot n_{\partial\Gamma^*} &= 0 \quad \text{on } \partial\Gamma^*\,. \end{aligned}\right. \tag{3.62}$$

The weak formulation consists in finding a $\rho \in H^3(\Gamma^*)$ with $\partial_\mu\rho - S(n^*,n^*)\rho = 0$ on $\partial\Gamma^*$ and

$$\int_{\Gamma^*} -\nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right)\cdot\nabla_{\Gamma^*}\xi + \omega\int_{\Gamma^*}\rho\,\xi \;=\; \langle f,\xi\rangle$$

for all $\xi \in H^1(\Gamma^*)$. Due to $\langle f,1\rangle = 0$, inserting $\xi \equiv 1$ in this equation yields $\int_{\Gamma^*}\rho = 0$, so that a solution ρ really belongs to $\mathcal{D}(\mathcal{A})$.

To obtain such a solution ρ, we use the minimization problem

$$F(\rho) := \frac{1}{2}\int_{\Gamma^*}\left(|\nabla_{\Gamma^*}\rho|^2 - |\sigma^*|^2\rho^2\right) - \int_{\partial\Gamma^*} S(n^*,n^*)\,\rho^2 + \frac{\omega}{2}\,\|\rho\|_{-1}^2 - \int_{\Gamma^*} u_f\,\rho \;\;\rightarrow\;\; \min$$

under all $\rho \in H^1(\Gamma^*)$ with $\int_{\Gamma^*}\rho = 0$. Here, $u_f \in H^1(\Gamma^*)$ is the weak solution of (3.54) with respect to $f \in H^{-1}(\Gamma^*)$.

F is coercive if and only if

$$\liminf_{\substack{\|\rho\|_{H^1(\Gamma^*)}\to\infty\\ \rho\in V}} \frac{F(\rho)}{\|\rho\|_{H^1(\Gamma^*)}^2} \;>\; 0\,.$$

Since for the linear term in F it holds that

$$\left|\left(\|\rho\|_{H^1(\Gamma^*)}\right)^{-2}\int_{\Gamma^*} u_f\,\rho\right| \le \|u_f\|_{L^2(\Gamma^*)}\cdot\frac{\|\rho\|_{L^2(\Gamma^*)}}{\|\rho\|_{H^1(\Gamma^*)}^2} \longrightarrow 0 \qquad \text{as } \|\rho\|_{H^1(\Gamma^*)}\to\infty\,,$$

the coercivity condition is equivalent to

$$\frac{1}{2}\int_{\Gamma^*}\left(|\nabla_{\Gamma^*}\rho|^2-|\sigma^*|^2\rho^2\right)-\int_{\partial\Gamma^*}S(n^*,n^*)\,\rho^2+\frac{\omega}{2}\,\|\rho\|_{-1}^2 \quad\geq\quad C\,\|\rho\|_{H^1(\Gamma^*)}^2$$

for all $\rho\in V$ for some $C>0$.

With the definition of the bilinear form I this reads as

$$\frac{1}{2}I(\rho,\rho)+\frac{\omega}{2}\,\|\rho\|_{-1}^2 \quad\geq\quad C\,\|\rho\|_{H^1(\Gamma^*)}^2\,.$$

To show this inequality for arbitrary $\rho\in V$, we proceed with the help of Lemma 3.36 for $\omega>\frac{C_1}{C_2}$, where C_1 and C_2 are the positive constants from Lemma 3.36.

$$\begin{aligned}\frac{1}{2}I(\rho,\rho)+\frac{\omega}{2}\,\|\rho\|_{-1}^2 &\geq \frac{1}{2C_2}\,\|\rho\|_{H^1(\Gamma^*)}^2-\frac{C_1}{2C_2}\,\|\rho\|_{-1}^2+\frac{\omega}{2}\,\|\rho\|_{-1}^2\\ &= \frac{1}{2C_2}\,\|\rho\|_{H^1(\Gamma^*)}^2+\frac{1}{2}\left(\omega-\frac{C_1}{C_2}\right)\,\|\rho\|_{-1}^2\\ &\geq \frac{1}{2C_2}\,\|\rho\|_{H^1(\Gamma^*)}^2\,,\end{aligned}$$

where we used $\omega-\frac{C_1}{C_2}>0$ in the last inequality. This shows coercivity. To apply an abstract existence theorem for the minimization problem from, for example the book of Jost [Jo98], it is now enough to show that the corresponding bilinear form

$$B(\rho_1,\rho_2):=\frac{1}{2}\int_{\Gamma^*}\left(\nabla_{\Gamma^*}\rho_1\cdot\nabla_{\Gamma^*}\rho_2-|\sigma^*|^2\rho_1\rho_2\right)-\frac{1}{2}\int_{\partial\Gamma^*}S(n^*,n^*)\rho_1\rho_2+\frac{\omega}{2}\,(\rho_1,\rho_2)_{-1}$$

is bounded on bounded sets in $V\times V$. To this end, we use the boundedness of $|\sigma^*|^2$ and $S(n^*,n^*)$ since we assumed that Γ^* is smooth enough and we remark that for the last term with the help of the Cauchy-Schwarz inequality and the continuous embedding $V\hookrightarrow H^{-1}(\Gamma^*)$ for $\rho_1,\rho_2\in V$ it holds

$$\left|(\rho_1,\rho_2)_{-1}\right|\leq\|\rho_1\|_{H^{-1}(\Gamma^*)}\cdot\|\rho_1\|_{H^{-1}(\Gamma^*)}\leq C\,\|\rho_1\|_{H^1(\Gamma^*)}\cdot\|\rho_1\|_{H^1(\Gamma^*)}\,.$$

Therefore there exists a unique minimizer $\overline{\rho}\in V$ of F on V. Since V is a subspace, this minimizer is characterized by the first variation of F through

$$\begin{aligned}0 &= \left.\frac{d}{d\varepsilon}F(\overline{\rho}+\varepsilon v)\right|_{\varepsilon=0}\\ &= \int_{\Gamma^*}\left(\nabla_{\Gamma^*}\overline{\rho}\cdot\nabla_{\Gamma^*}v-|\sigma^*|^2\overline{\rho}\,v\right)-\int_{\partial\Gamma^*}S(n^*,n^*)\overline{\rho}\,v+\omega\,(\overline{\rho},v)_{-1}-\int_{\Gamma^*}u_f\,v\,,\end{aligned}$$

where $v\in V$ is arbitrary. By the Definition of $u_{\overline{\rho}}$ in (3.54) and the identity (3.56), we observe that $\omega\,(\overline{\rho},v)_{-1}=\omega\,\langle v,u_{\overline{\rho}}\rangle=\omega\int_{\Gamma^*}u_{\overline{\rho}}\,v$. Since in the above equation the testfunctions v have to fulfill the constraint $\int_{\Gamma^*}v=0$, the identity is the weak version of the boundary value problem

$$\begin{cases}-\left(\Delta_{\Gamma^*}\overline{\rho}+|\sigma^*|^2\,\overline{\rho}\right)+\omega u_{\overline{\rho}}+\lambda &= u_f \quad\text{in }\Gamma^*\,,\\ \partial_\mu\overline{\rho}-S(n^*,n^*)\overline{\rho} &= 0 \quad\text{on }\partial\Gamma^*\,.\end{cases}\tag{3.63}$$

Here the Lagrange-multiplier λ is given through

$$\lambda \;=\; \frac{1}{|\Gamma^*|}\left(\int_{\Gamma^*}\left(|\sigma^*|^2\,\overline{\rho} - \omega\, u_{\overline{\rho}} + u_f\right) + \int_{\partial\Gamma^*} S(n^*, n^*)\overline{\rho}\right).$$

Since $u_{\overline{\rho}}$ and u_f are in $H^1(\Gamma^*)$, we obtain from elliptic regularity theory that $\overline{\rho} \in H^3(\Gamma^*)$. Therefore we can differentiate the first line in (3.63) and take the L^2-inner product with $\nabla_{\Gamma^*}\xi$ for some arbitrary $\xi \in H^1(\Gamma^*)$ to obtain

$$-\int_{\Gamma^*}\nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\overline{\rho} + |\sigma^*|^2\,\overline{\rho}\right)\cdot\nabla_{\Gamma^*}\xi + \omega\int_{\Gamma^*}\nabla_{\Gamma^*}u_{\overline{\rho}}\cdot\nabla_{\Gamma^*}\xi = \int_{\Gamma^*}\nabla_{\Gamma^*}u_f\cdot\nabla_{\Gamma^*}\xi\,.$$

With the Definition of the weak solutions $u_{\overline{\rho}}$ and u_f from (3.54) we finally get

$$-\int_{\Gamma^*}\nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\overline{\rho} + |\sigma^*|^2\,\overline{\rho}\right)\cdot\nabla_{\Gamma^*}\xi + \omega\int_{\Gamma^*}\overline{\rho}\,\xi = \int_{\Gamma^*}\langle f, \xi\rangle$$

for all $\xi \in H^1(\Gamma^*)$. So together with the boundary condition from (3.63), we found a $\rho \in \mathcal{D}(\mathcal{A})$ with

$$\omega\rho - \mathcal{A}\rho \;=\; f\,,$$

provided $\omega > \frac{C_1}{C_2}$, where C_1 and C_2 are the positive constants from Lemma 3.36. □

For the following theorem, the characterization of the eigenvalues of $\mathcal{A}$ and the asymptotic stability of the linearized problem (3.52) in terms of the positivity of the bilinear form I, we could in principle refer to Section 3.2 about mean curvature flow and advise the reader to do the necessary modifications. But we want to keep this section as complete as possible and therefore we give the remaining proofs for linearized stability analysis in detail.

The next point is to give a stability criterion for the zero solution of the linearized operator $\mathcal{A}$.

Theorem 3.39.

(i) The spectrum of $\mathcal{A}$ consists of countable many real eigenvalues.

(ii) The initial value problem (3.52) is solvable for initial data in $H^{-1}(\Gamma^*)$.

(iii) The zero solution of (3.52) is asymptotically stable if and only if the largest eigenvalue of $\mathcal{A}$ is negative, in short notation $\sigma(\mathcal{A}) < 0$.

Proof. ad (i). We want to show that for some $\lambda \in \mathbb{R}$, the operator $(\lambda\, I - \mathcal{A})^{-1} : H \to H$ exists and is compact.

For $\lambda > \frac{C_1}{C_2}$, where C_1 and C_2 the positive constants from Lemma 3.36 we showed surjectivity of

$$\lambda\, I - \mathcal{A} : \mathcal{D}(\mathcal{A}) \longrightarrow H$$

in the last Lemma 3.38. Since every eigenvalue $\mu \in \sigma(\mathcal{A})$ fulfills $\mu \leq \frac{C_1}{C_2}$ from Lemma 3.37, we see from the identity

$$\sigma(\lambda I - \mathcal{A}) = \lambda - \sigma(\mathcal{A})$$

for the spectrum that there exists no eigenvalue zero of $\lambda I - \mathcal{A}$ provided $\lambda > \frac{C_1}{C_2}$. For a linear operator this means in particular that it is injective.

Continuity of the resolvent

$$(\lambda I - \mathcal{A})^{-1} : H \longrightarrow \mathcal{D}(\mathcal{A})$$

for $\lambda > \frac{C_1}{C_2}$ can be seen by observing that

$$(\lambda I - \mathcal{A})^{-1}(f) = \rho \quad \Leftrightarrow \quad (\lambda I - \mathcal{A})(\rho) = f\,,$$

which means that $\rho \in \mathcal{D}(\mathcal{A})$ is a weak solution for the boundary value problem (3.62) with $\omega = \lambda$. Solutions of this problem fulfill an inequality

$$\|\rho\|_{H^3(\Gamma^*)} \quad \leq \quad C\,\|f\|_{H^{-1}(\Gamma^*)}\,,$$

which gives continuity of the resolvent. Since the embedding $\mathcal{D}(\mathcal{A}) \hookrightarrow H^{-1}(\Gamma^*)$ is compact, we get by composition a compact operator

$$(\lambda I - \mathcal{A})^{-1} : H \longrightarrow H\,,$$

provided $\lambda > \frac{C_1}{C_2}$. Together with the self-adjointness of $\mathcal{A}$ from Lemma 3.38, we get the claim (i) with the help of an abstract operator theorem from the book of Kato [Kat95].

ad (ii) and (iii). Existence and stability of the problem

$$\text{Find } \rho(t) \in \mathcal{D}(\mathcal{A})\,, \text{ such that } \partial_t \rho(t) = \mathcal{A}(t)$$

can be treated with the theory of analytic semigroups as, for example, in the book of Lunardi [Lun95]. We just show that $\mathcal{A}$ generates an analytic semigroup.

On the one hand, we know that for $\omega \in \mathbb{R}$ the operator $\widetilde{\mathcal{A}} := \mathcal{A} - \omega I$ is self-adjoint, since from Lemma 3.38 the operator $\mathcal{A}$ has this property. On the other hand, we can show that $\widetilde{\mathcal{A}}$ is dissipative, which means that

$$(\widetilde{\mathcal{A}}\rho, \rho)_{-1} \leq 0 \qquad \text{for all } \rho \in \mathcal{D}(\mathcal{A})\,.$$

In fact, this can be seen with the help of Lemmata 3.33 and 3.36.

$$\begin{aligned}
(\widetilde{\mathcal{A}}\rho, \rho)_{-1} &= (\mathcal{A}\rho, \rho)_{-1} - \omega\,(\rho, \rho)_{-1} \\
&= -I(\rho, \rho) - \omega\,(\rho, \rho)_{-1} \\
&\leq -\frac{1}{C_2}\|\rho\|^2_{H^1(\Gamma^*)} + \left(\frac{C_1}{C_2} - \omega\right)\|\rho\|_{L^2(\Gamma^*)} \\
&\leq 0\,,
\end{aligned}$$

where the last inequality can be achieved by choosing ω large enough. Now we use an abstract theorem of [Weid76] which states that a linear, densely defined, self-adjoint and dissipative operator is in particular sectorial and therefore generates an analytic semigroup $T(t)$. For completeness we mention finally that $S(t) := e^{\omega t}T(t)$ is the analytic semigroup with generator $\mathcal{A}$. □

The next lemma, which follows with classical arguments from Courant and Hilbert [CH68], gets together eigenvalues of $\mathcal{A}$ and properties of the bilinear form I. The lemma is essentially the same as in Section 3.2 of mean curvature flow, we just have to replace the L^2-inner product with $(.\,,.)_{-1}$.

Lemma 3.40. *Let*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$$

be the eigenvalues of $\mathcal{A}$ (taken multiplicity into account).

(i) For all $n \in \mathbb{N}$, the following description of the eigenvalues holds

$$\begin{aligned} \lambda_n &= \inf_{W \in \Sigma_{n-1}} \sup_{\rho \in W \setminus \{0\}} -\frac{I(\rho,\rho)}{(\rho,\rho)_{-1}}\,, \\ -\lambda_n &= \sup_{W \in \Sigma_{n-1}} \inf_{\rho \in W^\perp \setminus \{0\}} \frac{I(\rho,\rho)}{(\rho,\rho)_{-1}}\,, \end{aligned}$$

where Σ_n is the collection of n-dimensional subspaces of V and $W^\perp$ is the orthogonal complement with respect to the $(.\,,.)_{-1}$ inner product.

(ii) The eigenvalues λ_n depend continuously on $S(n^*, n^*)$ and $|\sigma^*|$ in the L^∞-norm.

Proof. The first part follows with the help Courant's maximum-minimum principle from [CH68] and the second part follows due to the structure of I,

$$I(\rho,\rho) = \int_{\Gamma^*} \left(|\nabla_{\Gamma^*}\rho|^2 - |\sigma^*|^2\rho^2 \right) d\mathcal{H}^n - \int_{\partial\Gamma^*} S(n^*, n^*)\rho^2 \, d\mathcal{H}^{n-1}\,,$$

from which the continuous dependence can be seen directly. □

As before in Section 3.2 of the mean curvature flow, we can describe the eigenvalue λ_1 in the above lemma more explicitly.

Remark 3.41. *For the largest eigenvalue λ_1 of $\mathcal{A}$ we have the description*

$$-\lambda_1 = \min_{\rho \in V \setminus \{0\}} \frac{I(\rho,\rho)}{(\rho,\rho)_{-1}}\,. \tag{3.64}$$

From Theorem 3.39 we have asymptotic stability of the linearized equation (3.52) if and only if $\lambda_1 < 0$. This leads to the following main conclusion.

Theorem 3.42. *The linearized equation (3.52) is asymptotically stable if and only if*

$$I(\rho,\rho) \quad > \quad 0$$

for all $\rho \in V\backslash\{0\}$, *where*

$$I(\rho,\rho) \quad = \quad \int_{\Gamma^*} \left(|\nabla_{\Gamma^*}\rho|^2 - |\sigma^*|^2\rho^2\right)\, \mathrm{d}\mathcal{H}^n - \int_{\partial\Gamma^*} S(n^*,n^*)\rho^2\, \mathrm{d}\mathcal{H}^{n-1}\,.$$

3.4.2 Some comments on nonlinear stability

In this short subsection we want to give some comments on nonlinear stability of surface diffusion equation with boundary contact (3.48). This is the task of deriving stability results directly for the highly nonlinear problem (3.49). Given a stationary solution Γ^* of (3.48), which is linearly stable in the sense of Theorem 3.42, we say that Γ^* is nonlinear stable provided that a solution $\Gamma(t)$ of (3.48) with starting configuration Γ_0 close to Γ^* in a suitable sense, converges to Γ^* also in a suitable sense.

This problem was considered in the curve case by Garcke, Ito and Kohsaka [GIK08] and for closed hypersurfaces in higher dimensions without outer boundary contact by Escher, Mayer and Simonett in [EMS98]. In [EMS98] the authors use the concept of central manifolds, which is hard to apply in our case due to the highly nonlinear boundary condition. In this direction we also mention the work of Huisken (among lots of others [Hui84] and [Hui86]), who considered mean curvature flow for closed hypersurfaces and the work of Stahl [Sta95] and [Sta96], who extended the results of Huisken to the case of outer fixed boundary contact. Although their work is primarily concerned with the description of arising singularities, they use formulas for evolution of, for example, mean curvature and the second fundamental form, that are closely related to our case. We just remark that in the last two cases the authors use the maximum principle, which is not available for surface diffusion.

Therefore we propose to handle the nonlinear stability by a generalization of [GIK08], which is based on strong a-priori estimates and semigroup theory. We will give here some of the extensions of [GIK08] to the higher dimensional case, which are interesting on their own.

With the help of the results from the previous Section 3.4, we can derive an evolution equation for mean curvature.

Lemma 3.43. *Let* Γ *be a smooth solution of surface diffusion with boundary contact (3.48) i.e. with the notations from Section 3.4 it holds* $\Gamma(t) \subset \Omega$, $\partial\Gamma(t) \subset \partial\Omega$ *and for* $t > 0$

$$V = -\Delta_{\Gamma(t)} H \qquad \text{in } \Gamma(t)$$

with boundary conditions

$$\begin{cases} \angle(\Gamma(t), \partial\Omega) = \frac{\pi}{2} & \text{on } \partial\Gamma(t)\,, \\ \nabla_{\Gamma(t)} H \cdot n_{\partial\Gamma(t)} = 0 & \text{on } \partial\Gamma(t)\,. \end{cases}$$

Then we have the following evolution of mean curvature for $t > 0$

$$\partial^\circ H = -\Delta^2_{\Gamma(t)} H - |\sigma|^2 \Delta_{\Gamma(t)} H \qquad \text{in } \Gamma(t)$$

and on the boundary

$$\begin{cases} (\partial_\mu - S(n,n))\,\Delta_{\Gamma(t)} H = 0 & \text{on } \partial\Gamma(t)\,, \\ \nabla_{\Gamma(t)} H \cdot n_{\partial\Gamma(t)} = 0 & \text{on } \partial\Gamma(t)\,. \end{cases}$$

Proof. The evolution of mean curvature is seen immediately with the result $\partial^\circ H = \Delta_{\Gamma(t)} V + |\sigma|^2 V$ in formula (2.3) and the evolution $V = -\Delta_{\Gamma(t)} H$.

For the second part we write the angle condition as

$$n(t,p) \cdot \mu(p) = 0 \qquad \text{on } \partial\Gamma(t)\,,$$

for all $t > 0$, where $n(t,p)$ is a unit normal to $\Gamma(t)$ at $p \in \Gamma(t)$ and μ is the outer unit normal of $\partial\Omega$. With the formula $\partial^\circ n = -\nabla_{\Gamma(t)} V$ from (2.4) and analogue calculations as in Lemma 2.38 we see

$$\begin{aligned} 0 &= \partial^\circ (n \cdot \mu) \\ &= \partial^\circ n \cdot \mu + n \cdot \partial^\circ \mu \\ &= -\nabla_{\Gamma(t)} V \cdot \mu + n \cdot V d_p\mu(n) \\ &= -\nabla_{\Gamma(t)} V \cdot \mu + V S(n,n) \\ &= (\partial_\mu - S(n,n))\,(-V) \\ &= (\partial_\mu - S(n,n))\,\Delta_{\Gamma(t)} H\,. \end{aligned}$$

Note that S is our notation for the second fundamental form of $\partial\Omega$ with respect to the inwards pointing unit normal $(-\mu)$ of Ω. $\square$

Now we derive basic evolution formulas for area, $\int_{\Gamma(t)} H^2$ and $\int_{\Gamma(t)} |\nabla_{\Gamma(t)} H|^2$.

Lemma 3.44. *Let Γ be a smooth solution of (3.48) as in the previous lemma. Then we have*

$$(i) \qquad \frac{d}{dt} \int_{\Gamma(t)} 1 = - \int_{\Gamma(t)} |\nabla_{\Gamma(t)} H|^2\,, \tag{3.65}$$

$$(ii) \qquad \frac{d}{dt} \frac{1}{2} \int_{\Gamma(t)} H^2 = \int_{\Gamma(t)} \left(\Delta_{\Gamma(t)} H - \frac{1}{2} H^3 + |\sigma|^2 \right) V - \int_{\partial\Gamma(t)} S(n,n)\, H\, V\,, \tag{3.66}$$

$$(iii) \qquad \frac{d}{dt} \int_{\Gamma(t)} \frac{1}{2} |\nabla_{\Gamma(t)} H|^2 = \int_{\Gamma(t)} \frac{1}{2} \partial^\circ \left(|\nabla_{\Gamma(t)} H|^2 \right) - \frac{1}{2} \int_{\Gamma(t)} |\nabla_{\Gamma(t)} H|^2 V H\,. \tag{3.67}$$

Proof. (i) follows directly from Lemma 2.46 and the surface diffusion $V = -\Delta_{\Gamma(t)} H$ with integration by parts

$$\frac{d}{dt} \int_{\Gamma(t)} 1 = - \int_{\Gamma(t)} V\, H = \int_{\Gamma(t)} \Delta_{\Gamma(t)} H \cdot H = - \int_{\Gamma(t)} |\nabla_{\Gamma(t)} H|^2\,.$$

For (ii) we use the Transport theorem 2.44 for $f = \frac{1}{2} H^2$, the formula for $V = -\Delta_{\Gamma(t)} H$ on the boundary from the previous Lemma 3.43 and the vanishing of the normal boundary velocity

$v_{\partial\Gamma} = 0$ due to the right angle condition as in the proof of Lemma 2.46.

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_{\Gamma(t)} H^2 &= \int_{\Gamma(t)} H\,\partial^\circ H - \frac{1}{2}\int_{\Gamma(t)} H^2\, V\, H + \frac{1}{2}\int_{\partial\Gamma(t)} H^2\, \underbrace{v_{\partial\Gamma(t)}}_{=0} \\
&= \int_{\Gamma(t)} H\left(\Delta_{\Gamma(t)}V + |\sigma|^2\, V\right) - \frac{1}{2}\int_{\Gamma(t)} V\, H^3 \\
&= -\int_{\Gamma(t)} \nabla_{\Gamma(t)}H \cdot \nabla_{\Gamma(t)}V + \int_{\partial\Gamma(t)} H\,\underbrace{(\nabla_{\Gamma(t)}V \cdot n_{\partial\Gamma(t)})}_{-S(n,n)\,V} \\
&\quad + \int_{\Gamma(t)} |\sigma|^2\, V\, H - \frac{1}{2}\int_{\Gamma(t)} V\, H^3 \\
&= \int_{\Gamma(t)} \Delta_{\Gamma(t)}H\, V - \int_{\partial\Gamma(t)} \underbrace{(\nabla_{\Gamma(t)}H \cdot n_{\partial\Gamma(t)})}_{=0}\, V - \int_{\partial\Gamma(t)} S(n,n)\, V\, H \\
&\quad + \int_{\Gamma(t)} |\sigma|^2\, V\, H - \frac{1}{2}\int_{\Gamma(t)} V\, H^3 \\
&= \int_{\Gamma(t)} \left(\Delta_{\Gamma(t)}H + |\sigma|^2\, H - \frac{1}{2}H^3\right) V - \int_{\partial\Gamma(t)} S(n,n)\, V\, H\,.
\end{aligned}$$

To see (iii), we just have to apply the Transport theorem 2.44 for $f = \frac{1}{2}|\nabla_{\Gamma(t)}H|^2$ to derive

$$\begin{aligned}
\frac{d}{dt}\int_{\Gamma(t)} \frac{1}{2}|\nabla_{\Gamma(t)}H|^2 &= \int_{\Gamma(t)} \frac{1}{2}\partial^\circ\left(\nabla_{\Gamma(t)}H \cdot \nabla_{\Gamma(t)}H\right) - \int_{\Gamma(t)} \frac{1}{2}|\nabla_{\Gamma(t)}H|^2 V H \\
&\quad + \int_{\partial\Gamma(t)} \frac{1}{2}|\nabla_{\Gamma(t)}H|^2\, v_{\partial\Gamma}\,.
\end{aligned}$$

As in the proof of Lemma 2.46 we get the vanishing of the normal boundary velocity $v_{\partial\Gamma} = 0$ due to the right angle condition. □

We give a remark concerning identity (3.67).

Remark 3.45. *Of course the term* $\int_{\Gamma(t)} \frac{1}{2}\partial^\circ\left(|\nabla_{\Gamma(t)}H|^2\right)$ *in the evolution formula for* $\int_{\Gamma(t)} \frac{1}{2}|\nabla_{\Gamma(t)}H|^2$ *is not satisfactory. We give some local calculations which lead at least to an estimate.*

The problem with the term $\frac{1}{2}\partial^\circ\left(|\nabla_{\Gamma(t)}H|^2\right) = \nabla_{\Gamma(t)}H \cdot \partial^\circ\nabla_{\Gamma(t)}H$ is that the normal time derivative and the surface gradient do not commute with each other, which leads to

$$\begin{aligned}
\nabla_{\Gamma(t)}H \cdot \partial^\circ\left(\nabla_{\Gamma(t)}H\right) &= \nabla_{\Gamma(t)}H \cdot \nabla_{\Gamma(t)}\left(\partial^\circ H\right) + \text{other terms} \\
&= \nabla_{\Gamma(t)}H \cdot \nabla_{\Gamma(t)}\left(\Delta_{\Gamma(t)}V + |\sigma|^2 V\right) + \text{other terms}\,.
\end{aligned}$$

Locally we can give the missing terms as follows. Let $v_1, \ldots, v_n$ be an orthonormal moving frame of $\Gamma(t)$, i.e. for all $p \in \Gamma(t)$ the vectors $v_1(p), \ldots, v_n(p)$ form an orthonormal basis of $T_p\Gamma(t)$ and therefore the vectors $(0, v_1(p)), \ldots, (0, v_n(p))$ form an orthonormal basis of $T_{(t,p)}\Gamma$, see Definition 2.13 from Chapter 2. With the normal time derivative ∂° given as directional derivative in direction of $(1, Vn) \in T_{(t,p)}\Gamma$ from Lemma 2.37 this gives us the possibility to write

$$\begin{aligned}
\partial^\circ \nabla_{\Gamma(t)} H &= \partial_{(1,Vn)} \left(\sum_{i=1}^n \left(\partial_{(0,v_i)} H\right) \, (0, v_i) \right) \\
&= \sum_{i=1}^n \left(\partial_{(1,Vn)} \partial_{(0,v_i)} H\right) \, (0, v_i) + \sum_{i=1}^n \left(\partial_{(0,v_i)} H\right) \, \partial_{(1,Vn)}(0, v_i) \\
&= \sum_{i=1}^n \left(\partial_{(0,v_i)} \partial_{(1,Vn)} H\right) \, (0, v_i) + \sum_{i=1}^n \left(\partial_{[(1,Vn),(0,v_i)]} H\right) \, (0, v_i) \\
&\quad + \sum_{i=1}^n \left(\partial_{(0,v_i)} H\right) \, \partial_{(1,Vn)}(0, v_i) \,.
\end{aligned}$$

Here we used the Lie derivative, which is given through

$$[v, w] := \partial_v w - \partial_w v$$

for tangent vector fields v, w, i.e. $v, w : \Gamma \to \mathbb{R} \times \mathbb{R}^{n+1}$ with $v(t,p), w(t,p) \in T_{(t,p)}\Gamma$.
Taking this term in the scalar product with $\nabla_{\Gamma(t)} H$, one can find inequalities for the terms involving the local basis and get at least an inequality for the evolution $\frac{d}{dt} \int_{\Gamma(t)} |\nabla_{\Gamma(t)} H|^2$.
Integration by parts yields then

$$\begin{aligned}
\int_{\Gamma(t)} \frac{1}{2} \partial^\circ |\nabla_{\Gamma(t)} H|^2 &= - \int_{\Gamma(t)} \Delta_{\Gamma(t)} H \, \Delta_{\Gamma(t)} V - \int_{\Gamma(t)} \Delta_{\Gamma(t)} H \, |\sigma|^2 \, V + \text{other terms} \\
&\quad + \int_{\partial\Gamma(t)} \underbrace{\nabla_{\Gamma(t)} H \cdot n_{\partial\Gamma(t)}}_{=0} \left(\Delta_{\Gamma(t)} V + |\sigma|^2 V\right) \\
&= \int_{\Gamma(t)} V \, \Delta_{\Gamma(t)} V + \int_{\Gamma(t)} |\sigma|^2 \, V^2 + \text{other terms} \\
&= - \int_{\Gamma(t)} |\nabla_{\Gamma(t)} V|^2 + \int_{\Gamma(t)} |\sigma|^2 \, V^2 + \text{other terms} \\
&\quad + \int_{\partial\Gamma(t)} V \, \underbrace{\left(\nabla_{\Gamma(t)} V \cdot n_{\partial\Gamma(t)}\right)}_{=-S(n,n)V} \,.
\end{aligned}$$

By collecting the terms we get

$$\begin{aligned}
\frac{d}{dt} \int_{\Gamma(t)} \frac{1}{2} |\nabla_{\Gamma(t)} H|^2 &= - \left(\int_{\Gamma(t)} |\nabla_{\Gamma(t)} V|^2 - \int_{\Gamma(t)} |\sigma|^2 \, V^2 + \int_{\partial\Gamma(t)} S(n,n) \, V^2 \right) \\
&\quad - \frac{1}{2} \int_{\Gamma(t)} |\nabla_{\Gamma(t)} H|^2 \, V \, H + \text{other terms} \,.
\end{aligned}$$

Of course here is a lot of work to be done to get an exact result. $(\Box)$

The next lemma assures uniqueness of a geometric problem for hypersurfaces Γ given by the parametrization of Section 3.1. In the case of fixed boundary, there is a similar proof in Grosse-Brauckmann [Gro96].

Lemma 3.46. *Let Γ^* be a stationary solution of (3.48) such that the bilinear form I from Theorem 3.42 is positive and let Γ be hypersurfaces given with the help of the parametrization from Section 3.1 through*

$$\Gamma = \Gamma^\rho = \{\Phi^\rho(q) \,|\, q \in \Gamma^*\}, \tag{3.68}$$

where $\rho : \Gamma^* \to \mathbb{R}$ is independent of time t.
Then there exists a C^2-neighbourhood of Γ^* such that $\rho \equiv 0$ (i.e. Γ^*) is the unique solution of the problem

$$H = \overline{H}\,, \quad \angle(\Gamma, \partial\Omega) = \frac{\pi}{2}\,, \quad Vol(\Gamma) = Vol(\Gamma^*)\,. \tag{3.69}$$

Here, $\overline{H} = ⨍_\Gamma H$ is the mean value of mean curvature and the volume $Vol(\Gamma)$ is calculated as in Section 2.4. Furthermore, C^2-neighbourhood of Γ^* means hypersurfaces Γ given as above with small $\|\rho\|_{C^2(\Gamma^*)}$-norm.

Proof. We want to make use of the following abstract implicit function theorem, see the book of Zeidler [Zeid86]. Suppose that

(*i*) X, Y, Z are real Banach spaces, $U = U(x_0, y_0) \subset X \times Y$ is an open neighbourhood of $(x_0, y_0) \in X \times Y$ and $F : U \to Z$ fulfills $F(x_0, y_0) = 0$.

(*ii*) F_y exists as partial Fréchet derivative on U and $F_y(x_0, y_0) : Y \to Z$ is bijective.

(*iii*) F and F_y are continuous at (x_0, y_0).

Then there exist $r_0, r > 0$, such that for every $x \in X$ satisfying $\|x - x_0\| < r_0$, there is exactly one $y(x) \in Y$ for which $\|y(x) - y_0\| < r$ and $F(x, y(x)) = 0$.
We use this theorem for

$$\begin{aligned}
X &:= \{\rho \in C^2(\Gamma^*) \,|\, \rho \equiv \text{const}\}\,,\\
Y &:= \left\{\rho \in C^2(\Gamma^*) \,|\, \int_{\Gamma^*} \rho = 0\right\}\,,\\
Z &:= \left\{\rho \in C^0(\Gamma^*) \,|\, \int_{\Gamma^*} \rho = 0\right\} \times C^0(\partial\Gamma^*) \text{ and}\\
F(m, u) &:= \left(H - \overline{H}\,, \angle(\partial\Omega, \Gamma) - \frac{\pi}{2}\right)\,,
\end{aligned}$$

where the mean curvature H and the average mean curvature $\overline{H}$ are computed for the hypersurface $\Gamma = \Gamma^\rho$ that we get from (3.68) with $\rho = u + m$. Also the expression $\angle(\partial\Omega, \Gamma)$ is the angle between the outer boundary and Γ. Since Γ^* corresponds to $\rho \equiv 0$ and is a stationary solution of (3.48), we have $F(0,0) = 0$. The partial derivative $\partial_u F(0,0) : Y \to Z$ is given by

$$\partial_u F(0,0)v = \frac{d}{d\varepsilon} F(0, \varepsilon v)\bigg|_{\varepsilon=0}$$

and can be computed with the methods from Subsection 3.4.1 as

$$\partial_u F(0,0)v \quad = \quad \left(\Delta_{\Gamma^*} v + |\sigma^*|^2 v - \frac{1}{|\Gamma^*|} \int_{\Gamma^*} \left(\Delta_{\Gamma^*} v + |\sigma^*|^2 v \right) \, , \, \partial_\mu v - S(n^*, n^*) v \right) .$$

Due to the fact that the bilinear form I from Theorem 3.42 is positive, Hölder regularity theory implies that $\partial_u F(0,0)$ is invertible. It is also true that F and $\partial_u F$ are continuous at $(0,0)$.

Hence, for $m \in X$ small, we find exactly one $u(m) \in Y$ such that

$$F(m, u(m)) = 0 \, .$$

Now we define

$$\rho_m = u(m) + m$$

for small m and let $\Gamma^m = \Gamma^{\rho_m}$ be the hypersurface from (3.68) given with the help of the parametrization from Section 3.1. With the derivative of volume from Section 2.4 we can conclude

$$\begin{aligned} \mathrm{Vol}(\Gamma^m) \quad &= \quad \mathrm{Vol}(\Gamma^*) + \int_{\Gamma^*} (u(m) + m) + o(\|u(m) + m\|_{C^2(\Gamma^*)}) \\ &= \quad \mathrm{Vol}(\Gamma^*) + m\,|\Gamma^*| + o(\|u(m) + m\|_{C^2(\Gamma^*)}) \, . \end{aligned}$$

By a contradiction argument we get therefore for $m \neq 0$

$$|\mathrm{Vol}(\Gamma^m) - \mathrm{Vol}(\Gamma^*)| \neq 0 \, , \tag{3.70}$$

if $\|(m, u(m))\|_{C^2(\Gamma^*)}$ is small enough.

To finish the proof, we assume the existence of a solution ρ of (3.69) with $\|\rho\|_{C^2(\Gamma^*)}$ small. With the splitting $\rho = u + m$ with $m = \overline{\rho}$ and $u = \rho - \overline{\rho}$, where

$$\overline{\rho} = \frac{1}{|\Gamma^*|} \int_{\Gamma^*} \rho \, ,$$

we see that $F(m,u) = 0$. Due to the volume-preserving property and (3.70), we obtain that $m = 0$ and $u \equiv 0$, which implies $\rho \equiv 0$ and proves the lemma. □

We want to give some comments on the remaining work to show an analogous result for nonlinear stability as in Garcke, Ito and Kohsaka [GIK08]. When imitating the steps from [GIK08], one has to prove at first a local existence result for the higher-dimensional case. Then one has to control higher derivatives $\int_{\Gamma(t)} |\nabla^m_{\Gamma(t)} H|^2$ to get the estimates of mean curvature in the adequate norm. This should lead to the strong a-priori estimates to show a unique global-in-time existence result. With the help of the above Lemma 3.46 a method similar to the one used in Elliott and Garcke [EG97] should yield nonlinear stability.

At last we mention that this approach contains of course a lot of work that is not done in this dissertation and is therefore left for future work.

3.5 Examples for stability

In this section, we consider explicit given situations for the geometry. This means we will specify a region Ω together with a hypersurface Γ^* lying inside Ω and touching the boundary at a right angle. Γ^* will be a stationary solution and we want to determine a characteristic behaviour concerning the linearized stability of Γ^*. Firstly we consider an example for surface diffusion with outer boundary contact.

For $a, c > 0$ we let

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 \,|\, \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} < 1\}$$

be surrounded by the ellipsoid

$$\partial\Omega = E = \{(x,y,z) \in \mathbb{R}^3 \,|\, \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1\}\,.$$

A parametrization of E is given by

$$f : [0,\pi] \times [0,2\pi] \longrightarrow E\,, \qquad f(u,v) = (a\sin u\cos v, a\sin u\sin v, c\cos u)\,.$$

We consider a stationary solution Γ^* of the surface diffusion equation (3.48) given by

$$\Gamma^* = \{(x,y,0) \in \mathbb{R}^3 \,|\, x^2 + y^2 \le a^2\}\,.$$

Γ^* is a circle in the (x,y)-plane lying inside the ellipsoid E with boundary

$$\begin{aligned} \partial\Gamma^* &= \{(x,y,0) \in \mathbb{R}^3 \,|\, x^2 + y^2 = a^2\} \\ &= \{f(\frac{\pi}{2}, v) \,|\, v \in [0,2\pi]\}\,, \end{aligned}$$

that touches E at a right angle.

To decide on linearized stability of Γ^*, we have to examine due to Theorem 3.42 the positivity of

$$I(\rho,\rho) = \int_{\Gamma^*} \left(|\nabla_{\Gamma^*}\rho|^2 - |\sigma^*|^2\rho^2\right) - \int_{\partial\Gamma^*} S(n^*,n^*)\rho^2$$

for all $\rho \in H^1(\Gamma^*)\backslash\{0\}$ with $\int_{\Gamma^*}\rho = 0$. Here, $|\sigma^*|^2$ is the squared norm of the second fundamental form σ^* of Γ^*, given by $|\sigma^*|^2 = (\kappa_1^*)^2 + (\kappa_2^*)^2$. Since Γ^* is a flat disc, we observe that the principal curvatures κ_i^* vanish and therefore $|\sigma^*|^2 = 0$. S is the second fundamental form of $\partial\Omega = E$ with respect to the inwards pointing unit normal $(-\mu)$ of Ω, which is given at $q = f(u,v) \in E$ with the help of the cross product through

$$\mu(q) = \mu(u,v) = \frac{\partial_u f \times \partial_v f}{|\partial_u f \times \partial_v f|} = \frac{1}{\sqrt{a^2\cos^2 u + c^2\sin^2 u}}\,(c\sin u\cos v, c\sin u\sin v, a\cos u)\ ,$$

where we used

$$\begin{aligned} \partial_u f(u,v) &= (a\cos u\cos v, a\cos u\sin v, -c\sin u) \quad \text{and} \\ \partial_v f(u,v) &= (-a\sin u\sin v, a\sin u\cos v, 0)\ . \end{aligned}$$

At points $q \in \partial\Gamma^* \cap E$, that is at $q = f(\frac{\pi}{2}, v)$, this leads to

$$\mu(q) = \mu(\frac{\pi}{2}, v) = (\cos v, \sin v, 0)\ .$$

The corresponding matrix representation of S, i.e. $h_{ij}(u,v) = (-\mu(u,v), \partial_{ij} f(u,v))$ for $i,j \in \{1,2\}$ is then given by

$$\begin{aligned}(h_{ij}(u,v))_{i,j=1}^2 &= -\frac{1}{\sqrt{a^2\cos^2 u + c^2 \sin^2 u}} \cdot \\ &\quad \begin{pmatrix} a\sin u(-c\sin u) + (a\cos u)(-c\cos u) & 0 \\ 0 & a\sin u(-c\sin u)\end{pmatrix}\ .\end{aligned}$$

which can be calculated for rotational surfaces as in the book of Kühnel [Kue06], for example. At points $q \in \partial\Gamma^* \cap E$, that is at $q = f(\frac{\pi}{2}, v)$, we get

$$\Big(h_{ij}(\frac{\pi}{2}, v)\Big)_{i,j=1}^2 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\ .$$

With the above formula for the normal to E, we see that the normal $n^* = (0,0,1)$ is orthogonal to μ and therefore $n^* \in T_qE$ for $q \in \partial\Gamma^* \cap E$. Writing $n^* = (a_1, a_2)$ in the basis $\big(\partial_u f(\frac{\pi}{2}, v), \partial_v f(\frac{\pi}{2}, v)\big)$ at a point $q = f(\frac{\pi}{2}, v)$ yields

$$n^* = -\frac{1}{c}\,\partial_u f(\frac{\pi}{2}, v) + 0\,\partial_v f(\frac{\pi}{2}, v)\,,$$

so that $a_1 = -\frac{1}{c}$ and $a_2 = 0$. Finally we can calculate at a point $q = f(\frac{\pi}{2}, v) \in \partial\Gamma^* \cap E$

$$S(n^*, n^*) = \sum_{i,j=1}^2 a_i a_j h_{ij}(\frac{\pi}{2}, v) = \frac{1}{c^2} h_{11}(\frac{\pi}{2}, v) = \frac{a}{c^2}\,.$$

With this results the bilinear form from Theorem 3.42 reduces to

$$I(\rho,\rho) = \int_{\Gamma^*} |\nabla_{\Gamma^*}\rho|^2 - \frac{a}{c^2}\int_{\partial\Gamma^*} \rho^2\,. \tag{3.71}$$

To determine the minimum of I we proceed in an analogue manner as in Courant and Hilbert [CH68]. By using the fact that Γ^* is a flat disc in $\mathbb{R}^3$ with radius $a > 0$, we can replace the bilinear form (3.71) by the following one.

$$I(\rho,\rho) = \int_{B_a(0)} |\nabla\rho|^2 - \frac{a}{c^2}\int_{\partial B_a(0)} \rho^2\,, \tag{3.72}$$

where $B_a(0)$ is the ball in $\mathbb{R}^2$ with center 0 and radius $a > 0$, and $\rho \in H^1(B_a(0))$ with $\int_{B_a(0)} \rho = 0$. Note that ∇ is the usual gradient in $\mathbb{R}^2$. We can simplify the bilinear form (3.72) further by introducing polar coordinates (r, ϑ) to get

$$I(\varphi,\varphi) = \int_0^{2\pi}\int_0^a \Big((\partial_r\varphi)^2 + \frac{1}{r^2}(\partial_\vartheta\varphi)^2\Big)\, r\,dr\,d\vartheta - \frac{a}{c^2}\int_0^{2\pi} (\varphi(a,\theta))^2\, a\,d\vartheta\,, \tag{3.73}$$

where $\varphi = \rho \circ \Pi$ for polar coordinates $\Pi(r, \vartheta)$ with $\varphi \in H^1((0, 2\pi) \times (0, a))$ and $\int_0^{2\pi} \int_0^a \varphi \, r = 0$. Here we used the transformation rule $|\nabla \rho|^2 = (\partial_r \varphi)^2 + \frac{1}{r^2}(\partial_\vartheta \varphi)^2$.

If we now want to solve the minimum problem

$$I(\varphi, \varphi) \longrightarrow \min\,, \quad \varphi \in H^1((0, 2\pi) \times (0, a)) \text{ and } \int_0^{2\pi} \int_0^a \varphi \, r = 0\,, \tag{3.74}$$

we can assume for φ a Fourier series expansion as

$$\varphi(r, \vartheta) \;=\; \frac{1}{2} f_0(r) + \sum_{n=1}^{\infty} \left[f_n(r) \cos(n\vartheta) + g_n(r) \sin(n\vartheta) \right]\,, \tag{3.75}$$

for functions f_0, f_n and g_n. Due to the volume constraint we observe that $\int_0^a f_0(r)\, r\, dr = 0$. At the boundary of $B_a(0)$, formula (3.75) gives for $r = a$

$$\varphi(a, \vartheta) \;=\; \frac{1}{2} f_0(a) + \sum_{n=1}^{\infty} \left[f_n(a) \cos(n\vartheta) + g_n(a) \sin(n\vartheta) \right]\,.$$

Differentiating (3.75) with respect to r and ϑ, inserting it into formula (3.73) for $I(\varphi, \varphi)$ and using the orthogonality of the trigonometric functions, we deduce the following expression for I.

$$\begin{aligned} I(\varphi, \varphi) \;=\; & \pi \int_0^a (f_0'(r))^2 \, r\, dr + \pi \sum_{n=1}^{\infty} \int_0^a \left((f_n'(r))^2 + \frac{n^2}{r^2} (f_n(r))^2 \right) r\, dr \\ & + \pi \sum_{n=1}^{\infty} \int_0^a \left((g_n'(r))^2 + \frac{n^2}{r^2} (g_n(r))^2 \right) r\, dr - \frac{a^2}{c^2} \pi (f_0(a))^2 \\ & - \frac{a^2}{c^2} \pi \sum_{n=1}^{\infty} (f_n(a))^2 - \frac{a^2}{c^2} \pi \sum_{n=1}^{\infty} (g_n(a))^2\,. \end{aligned} \tag{3.76}$$

Due to this structure we can minimize instead of I also the series of problems given by

$$\int_0^a (f_0'(r))^2 \, r\, dr - \frac{a^2}{c^2} (f_0(a))^2 \longrightarrow \min\,, \tag{3.77}$$

$$\int_0^a \left((f_n'(r))^2 + \frac{n^2}{r^2} (f_n(r))^2 \right) r\, dr - \frac{a^2}{c^2} (f_n(a))^2 \longrightarrow \min \text{ for } n \in \mathbb{N}\,, \tag{3.78}$$

$$\int_0^a \left((g_n'(r))^2 + \frac{n^2}{r^2} (g_n(r))^2 \right) r\, dr - \frac{a^2}{c^2} (g_n(a))^2 \longrightarrow \min \text{ for } n \in \mathbb{N}\,. \tag{3.79}$$

The first line (3.77) can be minimized at once by $f_0' = 0$, and therefore $f_0(r) \equiv c$ for some constant c. Due to the constraint $\int_0^a f_0(r)\, r\, dr = 0$, we observe $f_0(r) \equiv 0$ and in particular $f_0(a) = 0$. So the first line will yield the minimal value 0.

For $n > 0$, we must have $f_n(0) = 0$, otherwise the function $\frac{n^2}{r^2}(f_n(r))^2 r = \frac{n^2}{r}(f_n(r))^2$ from (3.76) would have a pole at $r = 0$ that is not integrable. Therefore we can write the integral in (3.78) as follows.

$$
\begin{aligned}
& \int_0^a \left((f_n'(r))^2 + \frac{n^2}{r^2}(f_n(r))^2 \right) r\,dr \\
= \; & \int_0^a \left(f_n' - \frac{n}{r} f_n \right)^2 r\,dr + \int_0^a 2n f_n\, f_n'\, dr \\
= \; & \int_0^a \left(f_n' - \frac{n}{r} f_n \right)^2 r\,dr + n(f_n(a))^2 ,
\end{aligned}
$$

so that the above minimum problem for f_n reads as

$$
\int_0^a \left(f_n' - \frac{n}{r} f_n \right)^2 r\,dr + \left(n - \frac{a^2}{c^2} \right) (f_n(a))^2 \longrightarrow \min \quad \text{for } n \in \mathbb{N} .
$$

The minimum is attained if $f_n' - \frac{n}{r} f_n = 0$, which gives $f_n(r) = c_n r^n$ for some constant c_n. The minimal value is then given by

$$
\left(n - \frac{a^2}{c^2} \right) (f_n(a))^2 .
$$

Analogous calculations for g_n yield finally the minimal value of I given by

$$
\pi \sum_{n=1}^{\infty} \left(n - \frac{a^2}{c^2} \right) \left((f_n(a))^2 + (g_n(a))^2 \right) . \tag{3.80}
$$

With this minimal value we can give the following result about linear stability of Γ^*.

Lemma 3.47. *With the above notations we get the following result for Γ^*.*

(i) If $c > a$, Γ^* is linearly asymptotically stable.

(ii) If $c < a$, Γ^* is linearly asymptotically instable.

Proof. Due to Theorem 3.42 we have asymptotic stability of Γ^* if and only if $I(\rho, \rho) > 0$ for all $\rho \in H^1(\Gamma^*)$ with $\int_{\Gamma^*} \rho = 0$, where I is given here as in (3.71). With the help of the above remarks, we calculated the minimum of I in (3.80), where $f_n(a)$ and $g_n(a)$ are arbitrary values. If $c > a$, we see that $\left(n - \frac{a^2}{c^2} \right) \geq \left(1 - \frac{a^2}{c^2} \right) > 0$ and the above minimal value is positive.

If on the other hand $c < a$, we choose $f_1(a) = g_1(a) = \frac{1}{2}$ and $f_n(a) = g_n(a) = 0$ for $n > 1$, so that the above minimal value simplifies to

$$
\pi \left(1 - \frac{a^2}{c^2} \right) < 0 .
$$

This yields the proof. $\square$

Lemma 3.48. *When we consider with the above notations the flat disc Γ^* lying inside of an hyperboloid instead of an ellipsoid, we remark briefly that in this case an inequality*

$$S(n^*, n^*) \leq c_0 < 0$$

holds for some constant $c_0 < 0$. Together with $|\sigma^*|^2 = 0$, this gives for the bilinear form

$$\begin{aligned} I(\rho, \rho) &= \int_{\Gamma^*} |\nabla_{\Gamma^*}|^2 - \int_{\partial\Gamma^*} S(n^*, n^*)\rho^2 \\ &\geq \int_{\Gamma^*} |\nabla_{\Gamma^*}|^2 + |c_0| \underbrace{\int_{\partial\Gamma^*} \rho^2}_{\geq 0} \\ &\geq C\|\rho\|^2_{H^1(\Gamma^*)}, \end{aligned}$$

where the last inequality holds due to the Poincaré inequality on $H^1(\Gamma^*) \cap \{\rho \mid \int_{\Gamma^*} \rho = 0\}$. Therefore in this case linearized stability for Γ^* holds without any conditions as in the previous Lemma 3.47.

Lemma 3.49. *For the mean curvature flow we want to give the following abstract short example. When we have a bound of the form*

$$S(n^*, n^*) \geq c_0 > 0\,,$$

which is connected to strict convexity of Ω, we use the lack of the integral constraint in Theorem 3.17, and insert constant functions $\rho \equiv c$ into the bilinear form I to get

$$\begin{aligned} I(c, c) &= -c^2 \left(\underbrace{\int_{\Gamma^*} |\sigma^*|^2}_{\geq 0} + \int_{\partial\Gamma^*} \underbrace{S(n^*, n^*)}_{\geq c_0} \right) \\ &< 0\,. \end{aligned}$$

So in this case we have linear instability for Γ^*.

The last lemma is already known in the literature and can be found for example in the paper of Ei, Sato and Yanagida [ESY96].

Chapter 4

Triple Lines with Boundary Contact

In this chapter we will extend the previous one by considering instead of one now three evolving hypersurfaces, which lie inside a fixed region, meet the outer boundary at a right angle and get together at a triple line, where also some angle conditions have to be fulfilled. The hypersurfaces will evolve due to the mean curvature and the surface diffusion flow, each flow considered in one section of the chapter. As before we give some basic geometric properties of the regarded flows concerning the evolution of area and volume and the properties of stationary states.

In analogy to the previous chapter the main goal here is to do linearized stability analysis. To this end we extend the work of Garcke, Ito and Kohsaka [GIK10], where the authors consider surface diffusion with triple junctions for curves in the plane, to the present case of hypersurfaces in $\mathbb{R}^{n+1}$. A similar approach was considered by the authors in [GIK09], where they derived the linearized problem through the second variation of an energy functional, but we will stick to the style in [GIK10].

As in the previous chapter we first have to introduce a setting that allows us to formulate the geometric evolution laws as partial differential equations for functions defined on fixed reference hypersurfaces, which will be stationary solutions. Then we linearize the arising equations and with the help of spectral theory we formulate a criterion for asymptotic stability through the positivity of some bilinear form. As in the previous chapter it will be important to identify the linearized equations as gradient flows.

In the first Section 4.1 we consider the mean curvature flow with outer boundary contact, which means that three hypersurfaces evolve due to the mean curvature flow, meet each other at a triple line with some prescribed angles and touch the outer boundary at a right angle. For this situation we use a parametrization that is a coupling of the one from Section 3.1 for an evolution equation for one hypersurface near the outer boundary and a more explicit one near the triple line. More precisely, this mapping near the triple line will depend on two parameters where one is responsible for a normal direction and the other one for a tangential movement.

In the second Section 4.2 of this chapter we consider the surface diffusion flow with outer boundary contact and a triple line for three evolving hypersurfaces. Since this is a fourth order flow, we get more boundary conditions than in the previous section but for the formulation of the resulting partial differential equations for unknown functions, we can use the same parametrization as in Section 4.1.

Although the linearized stability analysis in both sections gets more complicated than in Chapter 3 for one evolving hypersurface, it is strongly influenced by it. This is done in the same way as the paper [GIK10] depends on the stability analysis in [GIK05].

4.1 Mean curvature flow

In this first section of the chapter about triple lines with outer boundary contact we consider mean curvature flow with prescribed angle conditions at the triple line and a right angle condition at the outer boundary. We formulate the problem in detail, give some geometric properties and derive with the help of a more explicit parametrization than in Chapter 3 near the triple line the linearized problem for three functions ρ_i, $i = 1, 2, 3$. Then we proceed with stability analysis for this linearized problem, where we have to take care of the three different hypersurfaces in this case, of course.

So we consider here the following problem. Seek for three evolving hypersurfaces

$$\Gamma_i = \bigcup_{t \in [0,T)} \{t\} \times \Gamma_i(t) \quad \text{with} \quad \Gamma_i(t) \subset \mathbb{R}^{n+1} , \tag{4.1}$$

as in Definition 2.31, moving due to the mean curvature flow weighted with surface energy densities $\gamma_i > 0$, $i = 1, 2, 3$, such that $\Gamma_i(t)$ lies in a fixed bounded region $\Omega \subset \mathbb{R}^{n+1}$ and the following decomposition is fulfilled. The boundary $\partial\Gamma_i(t)$ shall be a disjoint union of two parts

$$\partial\Gamma_i(t) \quad = \quad L_i(t) \cup S_i(t) , \tag{4.2}$$

such that

$$L(t) = L_1(t) = L_2(t) = L_3(t) \tag{4.3}$$

is an $(n-1)$-dimensional manifold, called triple line, and the other parts represent the sections with the outer fixed boundary $\partial\Omega$, i.e.

$$S_i(t) = \partial\Gamma_i(t) \cap \partial\Omega . \tag{4.4}$$

Note our implicit assumption that $L(t)$ does not intersect $\partial\Omega$.

In formulas, we have to find hypersurfaces as in (4.1)-(4.4), such that

$$\left\{\begin{array}{rclllr} V_i &=& \gamma_i H_i & \text{in } \Gamma_i(t) & \text{for all } t > 0 & \text{for } i = 1,2,3 , \\ \angle(\Gamma_i(t), \partial\Omega) &=& \frac{\pi}{2} & \text{on } S_i(t) & \text{for all } t > 0 , i = 1,2,3 , & \\ \angle(\Gamma_1(t), \Gamma_2(t)) &=& \theta_3 & \text{on } L(t) & \text{for all } t > 0 , & \\ \angle(\Gamma_2(t), \Gamma_3(t)) &=& \theta_1 & \text{on } L(t) & \text{for all } t > 0 , & \\ \angle(\Gamma_3(t), \Gamma_1(t)) &=& \theta_2 & \text{on } L(t) & \text{for all } t > 0 , & \\ \Gamma_i(0) &=& \Gamma_i^0 & & & \text{for } i = 1,2,3 . \end{array}\right. \tag{4.5}$$

Here Γ_i^0, $i = 1, 2, 3$ are given starting hypersurfaces, which fulfill (4.2)-(4.4) and the angle conditions from (4.5). V_i and H_i are the normal velocity and mean curvature of Γ_i as defined in

Chapter 2. θ_1, θ_2 and θ_3 are given contact angles with $0 < \theta_i < \pi$, which fulfill $\theta_1 + \theta_2 + \theta_3 = 2\pi$ and Young's law

$$\frac{\sin\theta_1}{\gamma_1} = \frac{\sin\theta_2}{\gamma_2} = \frac{\sin\theta_3}{\gamma_3} \,. \tag{4.6}$$

With the help of the outer unit conormals $n_{\partial\Gamma_i(t)}$ of $\Gamma_i(t)$ at $\partial\Gamma_i(t)$ we can write the angle conditions at the triple line through the requirement that

$$\begin{cases} n_{\partial\Gamma_1(t)} \cdot n_{\partial\Gamma_2(t)} &= \cos\theta_3 \,, \\ n_{\partial\Gamma_2(t)} \cdot n_{\partial\Gamma_3(t)} &= \cos\theta_1 \,, \\ n_{\partial\Gamma_3(t)} \cdot n_{\partial\Gamma_1(t)} &= \cos\theta_2 \,. \end{cases} \tag{4.7}$$

Due to the condition $\theta_1 + \theta_2 + \theta_3 = 2\pi$ and to the claim (4.3) that the three hypersurfaces meet at one triple line, two of the above angle conditions already imply the third one.

4.1.1 Geometric properties of the flow

In this subsection we want to give some basic properties of the flow (4.5). These properties will be an equivalence between Young's law and a balance of forces, a property of the normals and conormals of the arising hypersurfaces, decreasing of area and properties of stationary states.

The first point is to show an equivalence of the angle conditions (4.7) and Young's law (4.6) to a balance of forces given by

$$\gamma_1 n_{\partial\Gamma_1(t)} + \gamma_2 n_{\partial\Gamma_2(t)} + \gamma_3 n_{\partial\Gamma_3(t)} = 0 \qquad \text{on } L(t) \,. \tag{4.8}$$

Therefore it is important to observe that the three vectors $n_{\partial\Gamma_1(t)}$, $n_{\partial\Gamma_2(t)}$ and $n_{\partial\Gamma_3(t)}$ all lie in a two-dimensional space, namely the orthogonal complement of the tangent space of $L(t)$, i.e.

$$n_{\partial\Gamma_i(t)}(p) \in (T_pL(t))^{\perp} \,.$$

Since $L(t)$ is an $(n-1)$-dimensional submanifold of $\mathbb{R}^{n+1}$, this orthogonal complement is in fact a two-dimensional space.

Lemma 4.1. *Let θ_1, θ_2 and θ_3 be given contact angles for the conormals $n_{\partial\Gamma_i(t)}$ as in (4.7) with $0 < \theta_i < \pi$ and $\theta_1 + \theta_2 + \theta_3 = 2\pi$. Then there holds an equivalence between Young's law (4.6) and the balance of forces (4.8).*

Proof. First, let Young's law be fulfilled. From the angle condition $n_{\partial\Gamma_1(t)} \cdot n_{\partial\Gamma_2(t)} = \cos\theta_3 \neq \pm 1$, we see that $n_{\partial\Gamma_1(t)}$ and $n_{\partial\Gamma_2(t)}$ are linearly independent vectors and hence are a basis in the two-dimensional space $(T_pL(t))^{\perp}$. Since the three vectors $n_{\partial\Gamma_i(t)}$, $i = 1, 2, 3$ all lie in this space, we can show instead of the identity (4.8) the two scalar identities

$$\begin{cases} \gamma_1 \left(n_{\partial\Gamma_1(t)} \cdot n_{\partial\Gamma_1(t)}\right) + \gamma_2 \left(n_{\partial\Gamma_2(t)} \cdot n_{\partial\Gamma_1(t)}\right) + \gamma_3 \left(n_{\partial\Gamma_3(t)} \cdot n_{\partial\Gamma_1(t)}\right) &= 0 \quad \text{and} \\ \gamma_1 \left(n_{\partial\Gamma_1(t)} \cdot n_{\partial\Gamma_2(t)}\right) + \gamma_2 \left(n_{\partial\Gamma_2(t)} \cdot n_{\partial\Gamma_2(t)}\right) + \gamma_3 \left(n_{\partial\Gamma_3(t)} \cdot n_{\partial\Gamma_2(t)}\right) &= 0 \,. \end{cases}$$

With the help of the angle conditions (4.7) this reads also as

$$\begin{cases} \left(-\frac{\gamma_1}{\gamma_3}\right) + \left(-\frac{\gamma_2}{\gamma_3}\right)\cos\theta_3 & = & \cos\theta_2 \quad \text{and} \\ \left(-\frac{\gamma_1}{\gamma_3}\right)\cos\theta_3 + \left(-\frac{\gamma_2}{\gamma_3}\right) & = & \cos\theta_1 \, . \end{cases}$$

Now Young's law gives the identities $\frac{\gamma_1}{\gamma_3} = \frac{\sin\theta_1}{\sin\theta_3}$ and $\frac{\gamma_2}{\gamma_3} = \frac{\sin\theta_2}{\sin\theta_3}$, so that we have to show

$$\begin{cases} \left(-\frac{\sin\theta_1}{\sin\theta_3}\right) + \left(-\frac{\sin\theta_2}{\sin\theta_3}\right)\cos\theta_3 & = & \cos\theta_2 \quad \text{and} \\ \left(-\frac{\sin\theta_1}{\sin\theta_3}\right)\cos\theta_3 + \left(-\frac{\sin\theta_2}{\sin\theta_3}\right) & = & \cos\theta_1 \, . \end{cases}$$

Multiplying with $\sin\theta_3$ and rearranging the terms gives

$$\begin{cases} -\sin\theta_1 & = & \cos\theta_2\sin\theta_3 + \sin\theta_2\cos\theta_3 \quad \text{and} \\ -\sin\theta_2 & = & \cos\theta_1\sin\theta_3 + \sin\theta_1\cos\theta_3 \, . \end{cases}$$

Finally, we observe $-\sin\theta_1 = -\sin(2\pi - \theta_2 - \theta_3) = -\sin(-(\theta_2+\theta_3)) = \sin(\theta_2+\theta_3)$ (analogously for $-\sin\theta_2$), and the above identities are addition theorems, which hold true.
On the other hand, let the balance of forces (4.8) be true. Since the three vectors $n_{\partial\Gamma_i(t)}$, $i = 1, 2, 3$ lie in a two-dimensional space, (4.8) implies that the three vectors $\gamma_1 n_{\partial\Gamma_1(t)}$, $\gamma_2 n_{\partial\Gamma_2(t)}$ and $\gamma_3 n_{\partial\Gamma_3(t)}$ can be arranged to give a triangle in this two-dimensional space. The angles in this triangle are labelled through (4.7) and the law of sines gives exactly Young's law (4.6). □

With the help of the following construction we choose a direction of the normals $n_i(t)$ of $\Gamma_i(t)$.

Remark 4.2. *With the same argument as above for the outer unit conormals, we see that at the triple line $L(t)$ even the vectors*

$$\pm n_1(t)\, ,\ n_{\partial\Gamma_1(t)}\, ,\ \pm n_2(t)\, ,\ n_{\partial\Gamma_2(t)}\, ,\ \pm n_3(t)\, ,\ n_{\partial\Gamma_3(t)}$$

all lie in a two-dimensional space, namely the orthogonal complement of the tangent space of $L(t)$.

We choose unit normals $n_j(t)$ of $\Gamma_j(t)$ in an appropriate direction through the requirement that the angle between $n_{\partial\Gamma_i(t)}$ and $n_j(t)$ increases by $\pi/2$ compared to the angle between $n_{\partial\Gamma_i(t)}$ and $n_{\partial\Gamma_j(t)}$, i.e. we have the following formulas

$$n_i(t)\cdot n_j(t) = \cos\theta_k\, , \tag{4.9}$$

$$n_{\partial\Gamma_i(t)}\cdot n_{\partial\Gamma_j(t)} = \cos\theta_k\, , \tag{4.10}$$

$$n_{\partial\Gamma_i(t)}\cdot n_j(t) = \cos(\theta_k + \frac{\pi}{2}) = -\sin\theta_k\, , \tag{4.11}$$

each on $L(t)$ and for $(i, j, k) = (1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$. To be precise we require formula (4.11) at a fixed point of $L(t)$, extend the normals by continuity to all of $\Gamma_j(t)$ and observe

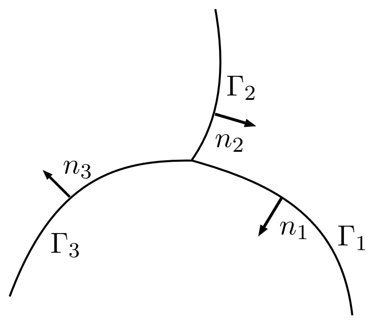


Figure 4.1: The choice of the normals.

the validity of (4.11) on all of $L(t)$ again by continuity. See Figure 4.1 for a sketch in the two-dimensional situation for curves near the triple line.

With an analogue version of Lemma 4.1 we can write instead of (4.8) also

$$\gamma_1 n_1(t) + \gamma_2 n_2(t) + \gamma_3 n_3(t) = 0 \qquad \text{on } L(t)\,. \tag{4.12}$$

In the next lemma we show a decreasing of the weighted total area.

Lemma 4.3. *For evolving hypersurfaces which move according to weighted mean curvature flow and fulfill the angle conditions from (4.5), the weighted total area*

$$A(t) \;=\; \sum_{i=1}^{3} \gamma_i A_i(t)\,, \tag{4.13}$$

with $A_i(t) = \int\limits_{\Gamma_i(t)} 1\, d\mathcal{H}^n$, *decreases in time* t.

Proof. With the proof of the formula for the derivative of area from Lemma 2.46, we see

$$\frac{d}{dt} A_i(t) \;=\; -\int_{\Gamma_i(t)} V_i\, H_i\, d\mathcal{H}^n + \int_{S_i(t)} \underbrace{v_{\partial\Gamma_i}}_{=0}\, d\mathcal{H}^{n-1} + \int_{L(t)} v_{\partial\Gamma_i}\, d\mathcal{H}^{n-1}\,,$$

where the normal boundary velocity $v_{\partial\Gamma_i}$ at the outer boundary $S_i(t)$ vanishes due to the right angle condition as in Lemma 2.46. Therefore we observe for the weighted total area

$$\frac{d}{dt} A(t) \;=\; -\sum_{i=1}^{3} \gamma_i \int_{\Gamma_i(t)} V_i\, H_i\, d\mathcal{H}^n + \sum_{i=1}^{3} \gamma_i \int_{L(t)} v_{\partial\Gamma_i}\, d\mathcal{H}^{n-1}\,.$$

For the normal boundary velocities at the triple line, we observe

$$v_{\partial\Gamma_i}(t,p) = n_{\partial\Gamma_i}(t,p) \cdot \frac{d}{d\tau} c_i(\tau)\Big|_{\tau=t} \qquad \text{for } p \in \partial\Gamma_i(t)\,,$$

where $c_i : (t-\varepsilon, t+\varepsilon) \to \mathbb{R}^{n+1}$ are curves with $c_i(\tau) \in \partial\Gamma_i(\tau)$ and $c_i(t) = p$. Since we require $L(t) = \partial\Gamma_1(t) = \partial\Gamma_2(t) = \partial\Gamma_3(t)$, and since the definition of normal boundary velocity is independent of the curve, we can take one curve for all three hypersurfaces Γ_i, that is we get

$$v_{\partial\Gamma_i}(t,p) = n_{\partial\Gamma_i}(t,p) \cdot \frac{d}{d\tau} c(\tau)\bigg|_{\tau=t} .$$

Plugging this into the derivative of the weighted total area and using $V_i = \gamma_i H_i$, we can deduce

$$\begin{aligned} \frac{d}{dt} A(t) &= -\sum_{i=1}^{3} \gamma_i^2 \int_{\Gamma_i(t)} H_i^2 \, d\mathcal{H}^n + \int_{L(t)} \underbrace{\sum_{i=1}^{3} \gamma_i n_{\partial\Gamma_i(t)}}_{=0 \text{ from } (4.8)} \cdot \frac{d}{d\tau} c(\tau)\bigg|_{\tau=t} d\mathcal{H}^{n-1} \\ &= -\sum_{i=1}^{3} \gamma_i^2 \int_{\Gamma_i(t)} H_i^2 \, d\mathcal{H}^n \\ &\leq 0 . \end{aligned}$$

This shows the lemma. □

As in the case for one hypersurface we want to describe $\Gamma_i(t)$ with the help of functions $\rho_i : [0,T) \times \Gamma_i^* \to \mathbb{R}$ as graphs over some fixed stationary solution of (4.5). This means we fix three hypersurfaces Γ_i^*, which lie in Ω, and the boundary has a decomposition $\partial\Gamma_i^* = L_i^* \cup S_i^*$, such that the three hypersurfaces meet at a triple line $L^* = L_1^* = L_2^* = L_3^*$ and the other parts are sections with the outer fixed boundary, i.e. $S_i^* = \partial\Gamma_i^* \cap \partial\Omega$.

These hypersurfaces shall fulfill the angle conditions and the mean curvature equations from (4.5) with $V_i \equiv 0$. As above we can show that the outer unit conormals $n_{\partial\Gamma_i^*}$ of Γ_i^* at $\partial\Gamma_i^*$ fulfill

$$\gamma_1 n_{\partial\Gamma_1^*} + \gamma_2 n_{\partial\Gamma_2^*} + \gamma_3 n_{\partial\Gamma_3^*} = 0 \qquad \text{on } L^* ,$$

and we introduce notation, such that the unit normals n_i^* of Γ_i^* emerge from $n_{\partial\Gamma_i^*}$ by the requirement that $n_{\partial\Gamma_i^*} \cdot n_j^* = \cos(\theta_k + \pi/2)$ and fulfill

$$\gamma_1 n_1^* + \gamma_2 n_2^* + \gamma_3 n_3^* = 0 \qquad \text{on } L^* .$$

For these stationary solutions the following lemma holds.

Lemma 4.4. *Stationary hypersurfaces as above are minimal hypersurfaces, i.e. they fulfill $H_i^* \equiv 0$, and the identity*

$$\gamma_1 \kappa_{n_{\partial\Gamma_1^*}} + \gamma_2 \kappa_{n_{\partial\Gamma_2^*}} + \gamma_3 \kappa_{n_{\partial\Gamma_3^*}} = 0 \qquad \text{on } L^*$$

holds true, where $\kappa_{n_{\partial\Gamma_i^*}} = \sigma_i^*(n_{\partial\Gamma_i^*}, n_{\partial\Gamma_i^*})$ is the normal curvature of Γ_i^* in direction of $n_{\partial\Gamma_i^*}$. Remember that σ_i^* is our notation for the second fundamental form of Γ_i^* with respect to the unit normal n_i^*.

Proof. The fact $H_i^* \equiv 0$ follows directly from the mean curvature equations with $V_i \equiv 0$. For $q \in L^*$, we can decompose the tangent space $T_q\Gamma_i^*$ with the help of the outer unit conormal $n_{\partial\Gamma_i^*}$ of Γ_i^* at L^* into

$$T_q\Gamma_i^* \quad = \quad T_qL^* \cup \operatorname{span}\{n_{\partial\Gamma_i^*}\}\,.$$

Therefore we can complete $n_{\partial\Gamma_i^*}$ to an orthonormal basis $\{n_{\partial\Gamma_i^*}, t_1, \ldots, t_{n-1}\}$ of $T_q\Gamma_i^*$ with the help of suitable vectors $t_1, \ldots, t_{n-1} \in T_qL^*$. Note that we choose for every $i = 1, 2, 3$ the same set of vectors $t_1, \ldots, t_{n-1}$. Since the mean curvature H_i^* is the trace of the Weingarten map, see Definition 2.19, we obtain the identity

$$\gamma_i H_i^* \quad = \quad \gamma_i \sigma_i^*(n_{\partial\Gamma_i^*}, n_{\partial\Gamma_i^*}) + \gamma_i \sum_{j=1}^{n-1} \sigma_i^*(t_j, t_j)\,. \tag{4.14}$$

With the above result about Γ_i^* being a minimal hypersurface and with our notation of normal curvature, we can write this as

$$0 \quad = \quad \gamma_i \kappa_{n_{\partial\Gamma_i^*}} + \gamma_i \sum_{j=1}^{n-1} \sigma_i^*(t_j, t_j)\ .$$

Summing over $i = 1, 2, 3$ gives for the second term

$$\sum_{i=1}^{3} \gamma_i \sum_{j=1}^{n-1} \sigma_i^*(t_j, t_j) = -\sum_{j=1}^{n-1} \sum_{i=1}^{3} \gamma_i \partial_{t_j} n_i^* \cdot t_j = -\sum_{j=1}^{n-1} \partial_{t_j} \underbrace{\left(\sum_{i=1}^{3} \gamma_i n_i^* \right)}_{=0 \text{ on } L^*} \cdot t_j = 0\ ,$$

where the last zero appears due to the fact that t_j is a tangent vector of L^*. For the normal curvatures in direction $n_{\partial\Gamma_i^*}$ this gives finally

$$\gamma_1 \kappa_{n_{\partial\Gamma_1^*}} + \gamma_2 \kappa_{n_{\partial\Gamma_2^*}} + \gamma_3 \kappa_{n_{\partial\Gamma_3^*}} = 0 \qquad \text{on } L^*$$

and we finished the proof. □

4.1.2 Parametrization and resulting partial differential equations

In this subsection we want to introduce the considered parametrization, which is more explicit near the triple line than in the previous Chapter 3. We will describe the considered evolving hypersurfaces as graphs over fixed stationary reference hypersurfaces and give a remark about our formulation for the condition that the arising evolving hypersurfaces meet at a triple line. Finally we formulate the emerging equations for the unknown functions, that will be linearized in the next part.

To describe the considered hypersurfaces $\Gamma_i(t)$, we will use the representation from Section 3.1 near the fixed boundary $\partial\Omega$ and an explicit mapping near the triple line L^*, and finally compose them with the help of a cut-off function.

So for $i = 1, 2, 3$ and small $\varepsilon, \delta > 0$ let

$$\Psi_i : \Gamma_i^* \times (-\varepsilon, \varepsilon) \longrightarrow \Omega\,, \qquad (q, w) \mapsto \Psi_i(q, w) \tag{4.15}$$

be a mapping from Section 3.1 with $\Psi_i(q, 0) = q$ for all $q \in \Gamma_i^*$, $\Psi_i(q, w) \in \partial\Omega$ for all $q \in \partial\Gamma_i^* \cap \partial\Omega = S_i^*$ and $\partial_w \Psi_i(q, 0) \cdot n_i^*(q) = 1$ for all $q \in \Gamma_i^*$.

Also let Z_i be a mapping given through

$$\begin{aligned} Z_i : \Gamma_i^* \times (-\varepsilon, \varepsilon) \times (-\delta, \delta) &\longrightarrow \mathbb{R}^{n+1}\,, \\ (q, w, s) &\mapsto Z_i(q, w, s) := q + w\, n_i^*(q) + s\, t_i^*(q)\,, \end{aligned} \tag{4.16}$$

where $i = 1, 2, 3$ and t_i^* is a tangent vector field on Γ_i^* with support in a neighbourhood of L_i^*, which equals the outer unit conormal $n_{\partial\Gamma_i^*}$ at L_i^*. More precisely we choose an open set $U \subset \mathbb{R}^{n+1}$, such that U is a neighbourhood of the triple line L^* and set $U_i := U \cap \Gamma_i^*$. Then we require for t_i^* that

$$t_i^*(q) = \begin{cases} 0 & \text{for } q \in \Gamma_i^* \backslash \overline{U_i}\,, \\ \in T_q\Gamma_i^* & \text{for } q \in U_i\,, \\ n_{\partial\Gamma_i^*}(q) & \text{for } q \in L_i^*\,. \end{cases} \tag{4.17}$$

Now we choose a neighbourhood of L^* given by some small tube $B_{2\tau}(L^*)$ around L^*, where $2\tau > 0$ such that $B_{2\tau}(L^*)$ is compactly included in Ω, i.e. $\overline{B_{2\tau}(L^*)} \subset \Omega$. Since our decomposition of $\partial\Gamma_i^*$ assured that $L^* \subset \Omega$, such a neighbourhood can be found.

An additional assumption is now that the evolution of the triple line shall always stay inside the neighbourhood $B_{2\tau}(L^*)$, in particular the triple line will never touch the outer fixed boundary $\partial\Omega$. To this end, we choose a smooth cut-off function $\eta \in C^\infty(\Omega)$, such that

$$\eta(x) = \begin{cases} 1 & , \quad x \in B_\tau(L^*)\,, \\ 0 & , \quad x \in \Omega \backslash B_{2\tau}(L^*)\,. \end{cases}$$

For $i = 1, 2, 3$ and functions

$$\begin{aligned} &\rho_i : [0, T) \times \Gamma_i^* \longrightarrow \mathbb{R} \ \text{ and} \\ &\mu_i : [0, T) \times L^* \longrightarrow \mathbb{R} \end{aligned}$$

with $|\rho_i| < \varepsilon$ and $|\mu_i| < \delta$, we define the mappings $\Phi_i = \Phi_i^{\rho_i, \mu_i}$ (we often omit the superscript (ρ_i, μ_i) for shortness) for $i = 1, 2, 3$ through

$$\begin{aligned} &\Phi_i : [0, T) \times \Gamma_i^* \longrightarrow \Omega\,, \\ &\Phi_i(t, q) := \eta(q)\, Z_i(q, \rho_i(t, q), \mu_i(t, \mathrm{pr}_i(q))) + (1 - \eta(q))\, \Psi_i(q, \rho_i(t, q)) \end{aligned} \tag{4.18}$$

Here $\mathrm{pr}_i : \Gamma_i^* \to L_i^*$ is some kind of projection on L_i^*, which we define as follows. We let $V \subset \mathbb{R}^{n+1}$ be an open set such that U from the above definition of the tangent vector field t_i^* is compactly embedded in V, i.e. $U \subset\subset V$ and set $V_i := V \cap \Gamma_i^*$. If V is a small enough neighbourhood of L^*, we define the projection pr_i through

$$\mathrm{pr}_i(q) = \begin{cases} u & \text{for } q \in V_i\,, \\ q_0 & \text{for } q \in \Gamma_i^* \backslash \overline{V_i}\,. \end{cases} \tag{4.19}$$

Here q_0 is some fixed point on L_i^* and $u = \mathrm{pr}_i(q)$ is the unique point on L_i^*, that is mapped to q with the geodesic line $\alpha_i(s)$ on Γ_i^* with

$$\alpha_i(0) = u \quad \text{and} \quad \alpha_i'(0) = n_{\partial\Gamma_i^*}(q)\,.$$

Note that we need this projection just inside of the small neighbourhood V of L^*, because it is used in the product $\mu_i(t, \mathrm{pr}_i(q))\, t_i^*(q)$, where the second term is 0 outside of the even smaller neighbourhood U of L^*.

We also set for fixed t as above

$$(\Phi_i)_t : \Gamma_i^* \longrightarrow \mathbb{R}^{n+1}\,, \qquad (\Phi_i)_t(q) := \Phi_i(t,q)\,,$$

which is a diffeomorphism onto its image if ε and δ are small enough. Finally we define new hypersurfaces through

$$\Gamma_{\rho_i,\,\mu_i}(t) \quad := \quad \{(\Phi_i)_t(q)\,|\, q \in \Gamma_i^*\}\,. \tag{4.20}$$

We observe that for $\rho_i \equiv 0$ and $\mu_i \equiv 0$ the resulting hypersurface is simply $\Gamma_{\rho_i\equiv 0,\,\mu_i\equiv 0}(t) = \Gamma_i^*$ for every t.

The condition that the new hypersurfaces meet in one triple line $L(t)$, can now be formulated through

$$\Phi_1(t,q) = \Phi_2(t,q) = \Phi_3(t,q) \qquad \text{for } \; q \in L^* (= L_1^* = L_2^* = L_3^*) \tag{4.21}$$

for all $t > 0$.

For the new hypersurfaces $\Gamma_i(t) := \Gamma_{\rho_i,\mu_i}(t)$ there exists also a decomposition of the boundary $\partial\Gamma_i(t)$ through

$$\partial\Gamma_i(t) = L_i(t) \cup S_i(t)\,,$$

where $S_i(t) = \partial\Gamma_i(t) \cap \partial\Omega$ and from (4.21) we can identify the other parts $L_i(t) = \partial\Gamma_i(t)\backslash S_i(t)$ to one compact $(n-1)$-dimensional submanifold

$$L(t) = L_1(t) = L_2(t) = L_3(t)\,.$$

Note that (4.21) can be formulated as

$$Z_1(t, \rho_1(t,q), \mu_1(t,q)) = Z_2(t, \rho_2(t,q), \mu_2(t,q)) = Z_3(t, \rho_3(t,q), \mu_3(t,q)) \qquad \text{for } \; q \in L^*\,,$$

since the cut-off function η equals 1 at the triple line L^* and the projections give $\mathrm{pr}_i(q) = q$. The last identity can also be written as

$$\rho_1 n_1^* + \mu_1 n_{\partial\Gamma_1^*} = \rho_2 n_2^* + \mu_2 n_{\partial\Gamma_2^*} = \rho_3 n_3^* + \mu_3 n_{\partial\Gamma_3^*} \qquad \text{on } \; L^*\,.$$

Since on L^* the six vectors

$$n_1^*\,,\; n_{\partial\Gamma_1^*}\,,\; n_2^*\,,\; n_{\partial\Gamma_2^*}\,,\; n_3^*\,,\; n_{\partial\Gamma_3^*}$$

lie in the two-dimensional space $(T_q L^*)^{\perp}$, the equations

$$\Phi_1 = \Phi_2 \quad \text{and} \quad \Phi_2 = \Phi_3 \quad \text{on } L^*$$

(the third one is then automatically fulfilled) lead to 4 conditions, namely 2 in each case. Therefore it is reasonable to try to find 4 equivalent conditions to (4.21), which is done in the next lemma.

Lemma 4.5. *Equivalent to the equations*

$$\Phi_1 = \Phi_2 \quad \textit{and} \quad \Phi_2 = \Phi_3 \quad \textit{on } L^* \tag{4.22}$$

are the following conditions, which describe an identity for the weighted sum of the ρ_i and a linear dependence of μ_i to all of the ρ_i on L^* given through

$$\begin{cases} (i) & \gamma_1\rho_1 + \gamma_2\rho_2 + \gamma_3\rho_3 = 0 \quad \textit{on } L^*\,, \\ (ii) & \mu_i = \frac{1}{s_i}\left(c_j\rho_j - c_k\rho_k\right) \quad \textit{on } L^*\,. \end{cases} \tag{4.23}$$

for $(i,j,k) = (1,2,3), (2,3,1)$ and $(3,1,2)$ and where $s_i = \sin\theta_i$ and $c_i = \cos\theta_i$.

Proof. At first let (4.22) be fulfilled. We omit the variables and remark that due to (4.22) also the condition $\Phi_3 = \Phi_1$ is fulfilled on L^*, which leads then to the identities

$$\rho_i n_i^* + \mu_i n_{\partial\Gamma_i^*} = \rho_j n_j^* + \mu_j n_{\partial\Gamma_j^*} \quad \text{on } L^* \tag{4.24}$$

for $(i,j) = (1,2), (2,3)$ and $(3,1)$.

Putting a function α on L^* through

$$\alpha := \rho_1 n_1^* + \mu_1 n_{\partial\Gamma_1^*} = \rho_2 n_2^* + \mu_2 n_{\partial\Gamma_2^*} = \rho_3 n_3^* + \mu_3 n_{\partial\Gamma_3^*}$$

we obtain $\alpha \cdot n_i^* = \rho_i$ for $i = 1,2,3$. Thus Young's law (4.6) respectively the balance of forces (4.8) for the stationary hypersurfaces Γ_i^* gives on L^*

$$\sum_{i=1}^{3} \gamma_i\rho_i = \sum_{i=1}^{3} \gamma_i\left(\alpha \cdot n_i^*\right) = \alpha \cdot \underbrace{\sum_{i=1}^{3} \gamma_i n_i^*}_{=0} = 0\,.$$

To derive (ii), we take the scalar product with $n_{\partial\Gamma_i^*}$ in (4.24) to get

$$\rho_i \underbrace{\left(n_i^* \cdot n_{\partial\Gamma_i^*}\right)}_{=0} + \mu_i \underbrace{\left(n_{\partial\Gamma_i^*} \cdot n_{\partial\Gamma_i^*}\right)}_{=1} \quad = \quad \rho_j \underbrace{\left(n_j^* \cdot n_{\partial\Gamma_i^*}\right)}_{=-\sin\theta_k} + \mu_j \underbrace{\left(n_{\partial\Gamma_j^*} \cdot n_{\partial\Gamma_i^*}\right)}_{=\cos\theta_k}$$

for triples $(i,j,k) = (1,2,3), (2,3,1)$ and $(3,1,2)$, where we used the angle conditions. With the abbreviation $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$ this leads to the three equations

$$\begin{aligned} \mu_1 &= \mu_2\, c_3 - \rho_2\, s_3\,, \\ \mu_2 &= \mu_3\, c_1 - \rho_3\, s_1\,, \\ \mu_3 &= \mu_1\, c_2 - \rho_1\, s_2\,. \end{aligned}$$

Solving this linear equations with respect to μ_i leads to the following dependence

$$(1 - c_1c_2c_3)\mu_i = -(c_kc_is_j\rho_i + s_k\rho_j + c_ks_i\rho_k)$$

for $(i, j, k) = (1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$. Further, (i) and Young's law (4.6) imply

$$(1 - c_1c_2c_3)\mu_i = -\frac{1}{s_i}\left(\left(s_ks_i - c_kc_i(s_j)^2\right)\rho_j + \left(c_k(s_i)^2 - c_kc_is_js_k\right)\rho_k\right).$$

With the following observation from the addition theorems for the angle functions

$$s_ks_i - c_kc_i(s_j)^2 = -c_j(1 - c_ic_jc_k) \quad \text{and} \quad c_k(s_i)^2 - c_kc_is_js_k = c_k(1 - c_ic_jc_k)\,,$$

we are led to (ii).

To derive the remaining part of the lemma, some linear algebra is needed. We fix $p \in L^*$ and formulate (4.22) with the help of the matrix

$$A = \begin{pmatrix} n_1^* & -n_2^* & 0 & n_{\partial\Gamma_1^*} & -n_{\partial\Gamma_2^*} & 0 \\ 0 & n_2^* & -n_3^* & 0 & n_{\partial\Gamma_2^*} & -n_{\partial\Gamma_3^*} \end{pmatrix}$$

and the vector $(\rho, \mu) = (\rho_1, \rho_2, \rho_3, \mu_1, \mu_2, \mu_3)$ through

$$(\rho, \mu) \text{ fulfill } (4.22) \Longleftrightarrow A\begin{pmatrix}\rho\\ \mu\end{pmatrix} = 0 \Longleftrightarrow (\rho, \mu) \in \ker A\,.$$

Since $\Phi_1 = \Phi_2$ and $\Phi_2 = \Phi_3$ on L^* are each identities for linear combinations of the vectors $n_1^*, n_2^*, n_3^*, n_{\partial\Gamma_1^*}, n_{\partial\Gamma_2^*}, n_{\partial\Gamma_3^*}$, which lie in a two-dimensional space, the image of A has at most dimension four. From the fact that the first, the third, the fourth and the sixth column in A are linearly independent, we see that in fact $\dim(\text{im}A) = 4$. This leads to $\dim(\ker A) = 6 - 4 = 2$. Now we observe that (4.23) can be written with the help of the matrix

$$B = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & 0 & 0 & 0 \\ 0 & -\frac{c_2}{s_1} & -\frac{c_3}{s_1} & 1 & 0 & 0 \\ -\frac{c_1}{s_2} & 0 & -\frac{c_3}{s_2} & 0 & 1 & 0 \\ -\frac{c_1}{s_3} & -\frac{c_2}{s_3} & 0 & 0 & 0 & 1 \end{pmatrix}$$

through

$$(\rho, \mu) \text{ fulfill } (4.23) \Longleftrightarrow B\begin{pmatrix}\rho\\ \mu\end{pmatrix} = 0 \Longleftrightarrow (\rho, \mu) \in \ker B\,.$$

Since the third, the fourth, the fifth and the sixth column of B are linearly independent, we see that the rank of B, i.e. the dimension of the image of B, is four. The rank formula leads to $\dim(\ker B) = 6 - 4 = 2$.

With the above calculations we showed $\ker A \subset \ker B$, and since both kernels have dimension two, we conclude $\ker A = \ker B$, which gives the desired equivalence of the lemma. $\square$

Remark 4.6. *With analogue calculations as in the last proof, i.e. taking the scalar product of α with $n_{\partial\Gamma_i^*}$ and of (4.24) with n_i^*, we get the following equations*

$$\begin{cases} (i) & \gamma_1\mu_1 + \gamma_2\mu_2 + \gamma_3\mu_3 = 0 \quad \text{on } L^*\,, \\ (ii) & \rho_i = \frac{1}{s_i}\left(c_j\mu_j - c_k\mu_k\right) \quad \text{on } L^*\,. \end{cases} \tag{4.25}$$

for $(i,j,k) = (1,2,3), (2,3,1)$ *and* $(3,1,2)$.

From now on, we always assume condition (4.21) and write equation (4.5) over the fixed stationary hypersurfaces Γ_1^*, Γ_2^* and Γ_3^* as partial differential equations for μ_i and ρ_i as follows.

$$\begin{cases} V_i(\Phi_i(t,q)) = H_i(\Phi_i(t,q)) & \text{in } \Gamma_i^* & \text{for all } t>0\,, & i=1,2,3\,, \\ (n_i\cdot\mu)\,(\Phi_i(t,q)) = 0 & \text{on } S_i^* & \text{for all } t>0\,, & i=1,2,3\,, \\ n_1(\Phi_1(t,q))\cdot n_2(\Phi_2(t,q)) = \cos\theta_3 & \text{on } L^* & \text{for all } t>0\,, & \\ n_2(\Phi_2(t,q)\cdot n_3(\Phi_3(t,q)) = \cos\theta_1 & \text{on } L^* & \text{for all } t>0\,, & \\ (\rho_i(0,q),\mu_i(0,q)) = (\rho_i^0,\mu_i^0) & \text{in } \Gamma_i^*\,, & & i=1,2,3\,, \end{cases} \tag{4.26}$$

where $n_i(\Phi_i(t,q))$ are the normals of $\Gamma_{\rho_i,\mu_i}(t)$ at $\Phi_i(t,q)$, μ is the outer unit normal of Ω at $\partial\Omega$ and we assume that the surfaces Γ_i^0 from (4.5) are given through

$$\Gamma_i^0 = \{\Psi_i(q,\rho_i^0(q),\mu_i^0(\mathrm{pr}_i(q))) \mid q\in\Gamma_i^*\}\,. \tag{4.27}$$

As explained in (2.12), we use the abbreviation $V_i(\Phi_i(t,q)) = V_i(t,\Phi_i(t,q))$ and analogously for H_i and n_i.

Due to the condition $\theta_1+\theta_2+\theta_3 = 2\pi$ and the fact, that the hypersurfaces all meet at a triple line at their boundary, which follows from (4.21), the third angle condition

$$(n_3\circ\Phi_3)\cdot(n_1\circ\Phi_1)\,(t,q) \;=\; \cos\theta_2 \quad \text{on } L^* \tag{4.28}$$

is automatically fulfilled and we omit it from now on. The equation (4.26) gives a second order system of partial differential equations for the functions $(\rho_1,\mu_1,\rho_2,\mu_2,\rho_3,\mu_3)$.

4.1.3 Linearization around a stationary state

As in the previous Chapter 3 we mean by the linearization of mean curvature flow (4.5) around stationary hypersurfaces Γ_1^*, Γ_2^*, Γ_3^* always the linearization of (4.26) around $(\rho_i,\mu_i)\equiv(0,0)$ for $i=1,2,3$. To obtain this linearization we consider the terms separately, write $\varepsilon\rho_i$ and $\varepsilon\mu_i$ instead of ρ_i and μ_i for $i=1,2,3$, differentiate with respect to ε and set $\varepsilon=0$ in the resulting equations. In this way, we get a system of three linear partial differential equations for the functions $(\rho_1,\mu_1,\rho_2,\mu_2,\rho_3,\mu_3)$, which are coupled through the boundary conditions.

Remark 4.7. *As in Remark 3.3 from Chapter 3 we just state that a formally correct description of the linearization is given with the help of the first variation for each term in (4.26). Therefore we consider each of the terms in the first line in (4.26) as operator*

$$F_i : C^\infty(\Gamma_i^*)\times C^\infty(\partial\Gamma_i^*) \to C^\infty(\Gamma_i^*)\,,$$

(omit the t-variable) and define the first variation of F at $(\rho_i,\mu_i)\equiv(0,0)$ as

$$\delta F(\rho_i,\mu_i) := \frac{\partial F}{\partial(\rho_i,\mu_i)}(0,0)(\rho_i,\mu_i) = \frac{d}{d\varepsilon}F(\varepsilon\rho_i,\varepsilon\mu_i)\Big|_{\varepsilon=0}\,.$$

For the linearization of the boundary conditions, which make sure a coupling of the equations, we would have to define the mapping as

$$F_{i,j} : C^\infty(\Gamma_i^*)\times C^\infty(\partial\Gamma_i^*)\times C^\infty(\Gamma_j^*)\times C^\infty(\partial\Gamma_j^*)\to C^\infty(L^*)\,,$$

for $(i,j)=(1,2)$ and $(2,3)$ and define an analogue first variation.

For building the linearization of each term in (4.26), we can use results of the previous Chapter 3 with the exception of the angle conditions at the triple line.

Lemma 4.8. *The linearization of the mean curvature equations*

$$V_i(t,\Phi_i(t,q)) = \gamma_i H_i(t,\Phi_i(t,q)) \qquad \text{in } \Gamma_i^*$$

around the stationary state represented through $(\rho_i,\mu_i)\equiv(0,0)$ are given by

$$\partial_t\rho_i = \gamma_i\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right) \qquad \text{in } \Gamma_i^*\,.$$

Proof. The linearization of the normal velocities

$$\frac{d}{d\varepsilon}V_i(t,\Psi_i(q,\varepsilon\rho_i(t,q),\varepsilon\mu_i(t,\mathrm{pr}_i(q)))\Big|_{\varepsilon=0} \;=\; \partial_t\rho_i(t,q)$$

follows from Lemma 3.4 in the previous chapter. Since we have a special parametrization in this part, we can also use the following observations.
We calculate for the normal velocities with the help of Lemma 2.40

$$\begin{aligned}
V_i(t,\Phi_i(t,q)) &= n_i(t,\Phi_i(t,q))\cdot\partial_t\Phi_i(t,q)\\
&= n_i(t,\Phi_i(t,q))\cdot\partial_t\left(q+\rho_i(t,q)\,n_i^*(q)+\mu_i(t,\mathrm{pr}_i(q))\,t_i^*(q)\right)\\
&= n_i(t,\Phi_i(t,q))\cdot n_i^*(q)\,\partial_t\rho_i(t,q) + n_i(t,\Phi_i(t,q))\cdot t_i^*(q)\,\partial_t\mu_i(t,\mathrm{pr}_i(q))\,.
\end{aligned}$$

For the linearization, this gives

$$\begin{aligned}
\frac{d}{d\varepsilon}V_i(t,\Phi_i^{\varepsilon\rho_i,\varepsilon\mu_i}(t,q))\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}V_i\left(t,q+\varepsilon\rho_i(t,q)\,n_i^*(q)+\varepsilon\mu_i(t,\mathrm{pr}_i(q))\,t_i^*(q)\right)\Big|_{\varepsilon=0}\\
&= \underbrace{n_i(t,q)\cdot n_i^*(q)}_{=1}\,\partial_t\rho_i(t,q) + \underbrace{n_i(t,q)\cdot t_i^*(q)}_{=0}\,\partial_t\mu_i(t,\mathrm{pr}_i(q))\\
&= \partial_t\rho_i(t,q)\,,
\end{aligned}$$

where we used the fact that for $q\in\Gamma_i^*$ it holds $n_i(t,q)=n_i^*(q)$.

For the linearization of mean curvature H_i we know from Lemma 3.5 that

$$\frac{d}{d\varepsilon}H_i(t,\Phi_i^{\varepsilon\rho_i,\varepsilon\mu_i}(t,q))\bigg|_{\varepsilon=0} = \Delta_{\Gamma_i^*}\rho_i(t,q) + |\sigma_i^*|^2(t,q)\rho_i(t,q)\,.$$

To be precise, we have to mention that in this case we consider two functions (ρ_i,μ_i) instead of one as in the parametrization for Lemma 3.5. But anyhow the calculations generalize directly, the only thing that could cause problems is the calculation of the normal velocity for the considered evolving hypersurface $\widetilde{\Gamma}_i$ with parameter ε instead of t given by

$$\widetilde{\Gamma}_i(\varepsilon) := \left\{ \left(\Phi_i^{\varepsilon\rho_i,\varepsilon\mu_i}\right)_t(q) \,|\, q \in \Gamma_i^* \right\}\,.$$

Therefore we do this detail explicitly.

$$\begin{aligned}\widetilde{V}\left(\varepsilon,\left(\Phi_i^{\varepsilon\rho_i,\varepsilon\mu_i}\right)_t(q)\right) &= \widetilde{n}\left(\varepsilon,\left(\Phi_i^{\varepsilon\rho_i,\varepsilon\mu_i}\right)_t(q)\right)\cdot\frac{d}{d\varepsilon}\left(\Phi_i^{\varepsilon\rho_i,\varepsilon\mu_i}\right)_t(q)\\ &= \widetilde{n}\left(\varepsilon,\left(\Phi_i^{\varepsilon\rho_i,\varepsilon\mu_i}\right)_t(q)\right)\cdot\left(\rho_i(t,q)\,n_i^*(q)+\mu_i(t,\mathrm{pr}_i(q))\,t_i^*(q)\right)\,,\end{aligned}$$

which gives for $\varepsilon = 0$

$$\begin{aligned}\widetilde{V}(0,q) &= \widetilde{n}(0,q)\cdot\left(\rho_i(t,q)\,n_i^*(q)+\mu_i(t,\mathrm{pr}_i(q))\,t_i^*(q)\right)\\ &= n_i^*(q)\cdot\left(\rho_i(t,q)\,n_i^*(q)+\mu_i(t,\mathrm{pr}_i(q))\,t_i^*(q)\right)\\ &= \rho_i(t,q)\,.\end{aligned}$$

Since this coincides with the result from Lemma 3.5, we finished the proof. □

Remark 4.9. *For the linearization of mean curvature we get in fact from the calculations in Lemma 3.5 from Chapter 3 the identity*

$$\begin{aligned}\frac{d}{d\varepsilon}H_i(t,\Phi_i^{\varepsilon\rho_i,\varepsilon\mu_i}(t,q))\bigg|_{\varepsilon=0} &= \Delta_{\Gamma_i^*}\rho_i(t,q) + |\sigma_i^*|^2(t,q)\rho_i(t,q)\\ &\quad +\nabla_{\Gamma_i^*}H_i(q)\cdot\left(\frac{d}{d\varepsilon}\Phi_i^{\varepsilon\rho_i,\varepsilon\mu_i}(t,q)\bigg|_{\varepsilon=0}\right)^T\,.\end{aligned}$$

If we would consider reference hypersurfaces Γ_i^*, which are not necessary stationary, the last term would not vanish because the mean curvature is then in general neither zero nor constant. In any case, with the special parametrization Φ_i of this chapter, we can even calculate the last term through

$$\begin{aligned}\left(\frac{d}{d\varepsilon}\Phi_i^{\varepsilon\rho_i,\varepsilon\mu_i}(t,q)\bigg|_{\varepsilon=0}\right)^T &= \left(\frac{d}{d\varepsilon}\left(q+\varepsilon\rho_i(t,q)\,n_i^*(q)+\varepsilon\mu_i(t,\mathrm{pr}_i(q))\,t_i^*(q)\right)\bigg|_{\varepsilon=0}\right)^T\\ &= \left(\rho_i(t,q)\,n_i^*(q)+\mu_i(t,\mathrm{pr}_i(q))\,t_i^*(q)\right)^T\\ &= \mu_i(t,\mathrm{pr}_i(q))\,t_i^*(q)\,,\end{aligned}$$

so that we get

$$\frac{d}{d\varepsilon} H_i(t, \Phi_i^{\varepsilon\rho_i,\varepsilon\mu_i}(t,q))\bigg|_{\varepsilon=0} = \Delta_{\Gamma_i^*}\rho_i(t,q) + |\sigma_i^*|^2(t,q)\rho_i(t,q) + \nabla_{\Gamma_i^*} H_i(q)\cdot\mu_i(t,\mathrm{pr}_i(q))\, t_i^*(q)\,.$$

Note that in this case the linearization of mean curvature also depends on the functions μ_i. This observation will be important in future work for a local existence result.

The next step is to linearize the angle conditions from (4.26), which were given through

$$n_i(t,\Phi_i^{\rho_i,\mu_i}(t,q))\cdot n_j(t,\Phi_j^{\rho_j,\mu_j}(t,q)) = \cos\theta_k \qquad \text{on } L^* \tag{4.29}$$

for all $t>0$, where $(i,j,k) = (1,2,3)$ or $(2,3,1)$. To calculate the linearization at a fixed point $q_0 \in L^*(= L_1^* = L_2^* = L_3^*)$ for $t > 0$, we choose as in the linearization of the angle condition (3.19) from Section 3.2 a suitable local parametrization F_i of L_i^* as in (3.20) with nice properties at a fixed point. So we are able to claim for a local parametrization

$$F_i : D_i \longrightarrow \Gamma_i^*\,, \qquad x\mapsto F_i(x) \tag{4.30}$$

with $F_i(x_0^i) = q_0$ for some $x_0^i \in \partial D_i$ the following assumptions.

(A) $\partial_1 F_i(x_0^i), \dots, \partial_n F_i(x_0^i)$ is an orthonormal basis of $T_{q_0}\Gamma_i^*$,

(B) $\partial_1 F_i(x_0^i) = n_{\partial\Gamma_i^*}(q_0)$, where $n_{\partial\Gamma_i^*}$ is the outer unit conormal of Γ_i^* at L_i^* and

(C) $(\partial_1 F_i \times \ldots \times \partial_n F_i)(x_0^i) = n_i^*(F_i(x_0^i))$, where we just fix the sign.

These properties are the same as for the parametrization (3.20) in Section 3.2, where we calculated the linearization of the right angle condition, and can always be achieved at a fixed point.

To calculate the linearization of the boundary conditions (4.29), we need the following properties.

Lemma 4.10. *With the help of the parametrizations F_i it holds for $F_i(x) = q \in \Gamma_i^*$*

(i) $\Psi_i(F_i(x),0,0) = F_i(x)$,

(ii) $\partial_j\Psi_i(F_i(x),0,0) = \partial_j F_i(x)$, $\partial_w\Psi_i(F_i(x),0,0) = n_i^*(F_i(x))$,
$\partial_s\Psi_i(F_i(x),0,0) = t_i^*(F_i(x))$.

Additionally, for the fixed point $F_i(x_0^i) = q_0 \in L^*$ it holds

(iii) $$\left(\partial_1 F_i \times \ldots \times \overbrace{n_i^*\circ F_i}^{l\text{-th pos.}} \times \ldots \times \partial_n F_i\right)(x_0^i) = (-1)\partial_l F_i(x_0^i),$$

(iv) $$\left(\partial_1 F_i \times \ldots \times \partial_l\, \overbrace{(n_i^*\circ F_i)}^{l\text{-th pos.}} \times \ldots \times \partial_n F_i\right)(x_0^i)$$
$$= \Big(\partial_l\,(n_i^*\circ F_i)\cdot\partial_l F_i\Big)(x_0^i)\ \, (n_i^*\circ F_i)\,(x_0^i),$$

$$(v)\quad \Big(\partial_1 F_i \times \ldots \times \overbrace{t_i^* \circ F_i}^{l\text{-}th\ pos.} \times \ldots \times \partial_n F_i\Big)(x_0^i) = \Big((t_i^* \circ F_i)\cdot \partial_l F_i\Big)(x_0^i)\ (n_i^* \circ F_i)(x_0^i),$$

$$(vi)\quad \Big(\partial_1 F_i \times \ldots \times \partial_l\,\overbrace{(t_i^* \circ F_i)}^{l\text{-}th\ pos.} \times \ldots \times \partial_n F_i\Big)(x_0^i)$$
$$= \Big(\partial_l\,(t_i^* \circ F_i)\cdot \partial_l F_i\Big)(x_0^i)\ (n_i^* \circ F_i)(x_0^i) - \Big(\partial_l\,(t_i^* \circ F_i)\cdot n_i^*\Big)(x_0^i)\ \partial_l F_i(x_0^i).$$

Proof. Parts (i) and (ii) follow directly from the definition of Ψ_i, which was given by

$$\Psi_i(q,w,s) = q + w\cdot n_i^*(q) + s\cdot t_i^*(q)\,.$$

Part (iii) now follows from assumption (C) for F_i and Lemma 5.6 in the appendix.
For the remaining parts we observe at the fixed point

$$t_i^*(q_0)\cdot n_i^*(q_0) = 0\,,$$
$$\left(\partial_l\,(n_i^* \circ F_i)\cdot (n_i^* \circ F_i)\right)(x_0^i) = 0\,,$$

and that the vectors $\partial_1 F_i(x_0^i),\ldots,\partial_n F_i(x_0^i), n_i^*(q_0)$ form an orthonormal basis of $\mathbb{R}^{n+1}$. Therefore we have the following representations

$$\begin{aligned}
t_i^*(q_0) &= (t_i^* \circ F_i)(x_0^i) = \sum_{k=1}^{n}\Big((t_i^* \circ F_i)\cdot \partial_k F_i\Big)(x_0^i)\ \partial_k F_i(x_0^i)\,,\\
\partial_l n_i^*(q_0) &= \partial_l\,(n_i^* \circ F_i)(x_0^i) = \sum_{k=1}^{n}\Big(\partial_l\,(n_i^* \circ F_i)\cdot \partial_k F_i\Big)(x_0^i)\ \partial_k F_i(x_0^i)\quad\text{and}\\
\partial_l t_i^*(q_0) &= \partial_l\,(t_i^* \circ F_i)(x_0^i)\\
&= \sum_{k=1}^{n}\Big(\partial_l\,(t_i^* \circ F_i)\cdot \partial_k F_i\Big)(x_0^i)\ \partial_k F_i(x_0^i)\\
&\quad + \Big(\partial_l\,(t_i^* \circ F_i)\cdot (n_i^* \circ F_i)\Big)(x_0^i)\ (n_i^* \circ F_i)(x_0^i)\,.
\end{aligned}$$

To see (iv), we use the above representations and the properties of the vector product, which are summarized in the appendix to get

$$\begin{aligned}
&\Big(\partial_1 F_i \times \ldots \times \partial_l\,\overbrace{(n_i^* \circ F_i)}^{\text{l-th pos.}} \times \ldots \times \partial_n F_i\Big)(x_0^i)\\
=\ & \sum_{k=1}^{n}\Big(\partial_l\,(n_i^* \circ F_i)\cdot \partial_k F_i\Big)(x_0^i)\ \underbrace{\Big(\partial_1 F_i \times \ldots \times \overbrace{\partial_k F_i}^{\text{l-th pos.}} \times \ldots \times \partial_n F_i\Big)}_{=\delta_{kl}(n_i^* \circ F_i)}(x_0^i)\\
=\ & \Big(\partial_l\,(n_i^* \circ F_i)\cdot \partial_l F_i\Big)(x_0^i)\ (n_i^* \circ F_i)(x_0^i)\,.
\end{aligned}$$

For (v), we proceed analogously to get

$$
\begin{aligned}
&\Big(\partial_1 F_i \times \ldots \times \overbrace{t_i^* \circ F_i}^{\text{l-th pos.}} \times \ldots \times \partial_n F_i\Big)(x_0^i) \\
= \;& \sum_{k=1}^n \Big((t_i^* \circ F_i)\cdot \partial_k F_i\Big)(x_0^i)\; \underbrace{\Big(\partial_1 F_i \times \ldots \times \overbrace{\partial_k F_i}^{\text{l-th pos.}} \times \ldots \times \partial_n F_i\Big)}_{=\delta_{kl}(n_i^* \circ F_i)}(x_0^i) \\
= \;& \Big((t_i^* \circ F_i)\cdot \partial_l F_i\Big)(x_0^i)\; (n_i^* \circ F_i)(x_0^i)\,.
\end{aligned}
$$

Finally, we have the following identity, which shows (vi).

$$
\begin{aligned}
&\Big(\partial_1 F_i \times \ldots \times \partial_l \overbrace{(t_i^* \circ F_i)}^{\text{l-th pos.}} \times \ldots \times \partial_n F_i\Big)(x_0^i) \\
= \;& \sum_{k=1}^n \Big(\partial_l\,(t_i^* \circ F_i)\cdot \partial_k F_i\Big)(x_0^i)\; \underbrace{\Big(\partial_1 F_i \times \ldots \times \overbrace{\partial_k F_i}^{\text{l-th pos.}} \times \ldots \times \partial_n F_i\Big)}_{=\delta_{kl}(n_i^* \circ F_i)}(x_0^i) \\
&+ \Big(\partial_l\,(t_i^* \circ F_i)\cdot (n_i^* \circ F_i)\Big)(x_0^i)\; \underbrace{\Big(\partial_1 F_i \times \ldots \times \overbrace{(n_i^* \circ F_i)}^{\text{l-th pos.}} \times \ldots \times \partial_n F_i\Big)}_{(-1)\partial_l F_i}(x_0^i) \\
= \;& \Big(\partial_l\,(t_i^* \circ F_i)\cdot \partial_l F_i\Big)(x_0^i)\; (n_i^* \circ F_i)(x_0^i) - \Big(\partial_l\,(t_i^* \circ F_i)\cdot n_i^*\Big)(x_0^i)\; \partial_l F_i(x_0^i)\,.
\end{aligned}
$$

□

Now we are in a position to derive the linearization of the angle condition (4.29) at the triple junction.

Lemma 4.11. *The linearization of*

$$n_i(t, \Phi_i^{\rho_i,\mu_i}(t,q)) \cdot n_j(t, \Phi_j^{\rho_j,\mu_j}(t,q)) = \cos\theta_k \quad \text{on } L^*$$

around $(\rho,\mu) = (0,0)$, where $\rho = (\rho_1,\rho_2,\rho_3)$ and $\mu = (\mu_1,\mu_2,\mu_3)$, is given through

$$\partial_{n_{\partial\Gamma_i^*}}\rho_i + \kappa_{n_{\partial\Gamma_i^*}}\mu_i = \partial_{n_{\partial\Gamma_j^*}}\rho_j + \kappa_{n_{\partial\Gamma_i^*}}\mu_j \quad \text{on } L^*\,, \tag{4.31}$$

where $\kappa_{n_{\partial\Gamma_i^*}} = \sigma_i^*(n_{\partial\Gamma_i^*}, n_{\partial\Gamma_i^*})$ is the normal curvature of Γ_i^* in direction $n_{\partial\Gamma_i^*}$. Equivalently, we can write this equation as

$$\partial_{n_{\partial\Gamma_i^*}}\rho_i + \frac{1}{s_i}\Big(c_j\kappa_{n_{\partial\Gamma_j^*}} - c_k\kappa_{n_{\partial\Gamma_k^*}}\Big)\rho_i = \partial_{n_{\partial\Gamma_j^*}}\rho_j + \frac{1}{s_j}\Big(c_k\kappa_{n_{\partial\Gamma_k^*}} - c_i\kappa_{n_{\partial\Gamma_i^*}}\Big)\rho_j \quad \text{on } L^*\,, \tag{4.32}$$

where $(i,j,k) = (1,2,3)$, $(2,3,1)$ or $(3,1,2)$, $s_i = \sin\theta_i$ and $c_i = \cos\theta_i$.

Proof. We show the linearization at a fixed point $q_0 \in L^*$ for $t_0 > 0$ and choose parametrizations F_i as in (4.30) with properties (A)-(C) at the fixed point $F_i(x_0^i) = q_0$.
Using the diffeomorphism $(\Phi_i)_t : \Gamma_i^* \to \Gamma_{\rho_i,\mu_i}(t)$ we also get a parametrization of $\Gamma_{\rho_i,\,\mu_i}(t)$, which we denote by

$$G_i^t : D_i \longrightarrow \Gamma_{\rho_i,\mu_i}(t)\,, \qquad G_i^t(x) := \Phi_i(t, F_i(x))\,.$$

Then the normal n_i of $\Gamma_{\rho_i,\mu_i}(t)$ at $p = \Phi_i(t,q) \in \Gamma_{\rho_i,\,\mu_i}(t)$ for some $q \in \Gamma_i^*$, is given with the help of the cross product of n vectors in $\mathbb{R}^{n+1}$ through

$$n_i(t,p) = n_i(t, \Phi_i(t,q)) = n_i(t, G_i(x)) = \frac{\partial_1 G_i^t(x) \times \ldots \times \partial_n G_i^t(x)}{|\partial_1 G_i^t(x) \times \ldots \times \partial_n G_i^t(x)|}\,. \tag{4.33}$$

For some properties of the vector product, we refer to the appendix.
A calculation of the partial derivative $\partial_l G_i^t(x)$ gives

$$\begin{aligned}
\partial_l G_i^t(x) &= \partial_l F_i(x) + \partial_l \rho_i(t, F_i(x))\, n_i^*(F_i(x)) + \rho_i(t, F_i(x))\, \partial_l n_i^*(F_i(x)) \\
&\quad + \partial_l \mu_i(t, F_i(x))\, t_i^*(F_i(x)) + \rho_i(t, F_i(x))\, \partial_l t_i^*(F_i(x)) \\
&= \partial_l F_i + \partial_l \rho_i\, n_i^* + \rho_i\, \partial_l n_i^* + \partial_l \mu_i\, t_i^* + \rho_i\, \partial_l t_i^*\,,
\end{aligned}$$

where we omitted variables for reasons of shortness.
Now we consider the numerator of $n_i(t, G_i^t(x))$ from (4.33).

$$\begin{aligned}
\partial_1 G_i^t \times \ldots \times \partial_n G_i^t &= \bigtimes_{l=1}^{n} \left(\partial_l F_i + \partial_l \rho_i\, n_i^* + \rho_i \partial_l n_i^* + \partial_l \mu_i\, t_i^* + \mu_i\, \partial_l t_i^*\right) \\
&= (\partial_1 F_i \times \ldots \times \partial_n F_i) + \sum_{l=1}^{n} \partial_l \rho_i \left(\partial_1 F_i \times \ldots \times \overset{\text{l-th pos.}}{\widehat{n_i^*}} \times \ldots \times \partial_n F_i\right) \\
&\quad + \sum_{l=1}^{n} \rho_i \left(\partial_1 F_i \times \ldots \times \overset{\text{l-th pos.}}{\widehat{\partial_l n_i^*}} \times \ldots \times \partial_n F_i\right) \\
&\quad + \sum_{l=1}^{n} \partial_l \mu_i \left(\partial_1 F_i \times \ldots \times \overset{\text{l-th pos.}}{\widehat{t_i^*}} \times \ldots \times \partial_n F_i\right) \\
&\quad + \sum_{l=1}^{n} \mu_i \left(\partial_1 F_i \times \ldots \times \overset{\text{l-th pos.}}{\widehat{\partial_l t_i^*}} \times \ldots \times \partial_n F_i\right) \\
&\quad + \text{quadratic terms in } \rho_i \text{ and } \mu_i\,,
\end{aligned}$$

where the quadratic terms are not written down explicitly, because they will not give a contribution to the linearization. Cubic or higher order terms in ρ_i and μ_i do not appear, because the vector product will always vanish for such expressions.

With the help of the results from Lemma 4.10 for the parametrization, we can proceed at the fixed point $q_0 \in L^*$ for $t_0 > 0$ as follows.

$$\begin{aligned}
& \partial_1 G_i \times \ldots \times \partial_n G_i - \text{ quadratic terms from above} \\
= \quad & n_i^* - \sum_{l=1}^n \partial_l \rho_i \, \partial_l F_i + \sum_{l=1}^n \rho_i \left(\partial_l n_i^* \cdot \partial_l F_i\right) \, n_i^* + \sum_{l=1}^n \partial_l \mu_i \left(t_i^* \cdot \partial_l F_i\right) \, n_i^* \\
& - \sum_{l=1}^n \mu_i \left(\partial_l t_i^* \cdot n_i^*\right) \, \partial_l F_i \\
= \quad & \left(1 + \sum_{l=1}^n \rho_i \left(\partial_l n_i^* \cdot \partial_l F_i\right) + \sum_{l=1}^n \partial_l \mu_i \left(t_i^* \cdot \partial_l F_i\right) + \sum_{l=1}^n \mu_i \left(\partial_l t_i^* \cdot \partial_l F_i\right)\right) n_i^* \\
& - \sum_{l=1}^n \partial_l \rho_i \, \partial_l F_i - \sum_{l=1}^n \mu_i \left(\partial_l t_i^* \cdot n_i^*\right) \, \partial_l F_i \\
=: \quad & R_i(\rho_i, \mu_i) \, ,
\end{aligned}$$

where we use the abbreviation R_i to get a better view for the linearization.
So we want to linearize the relation

$$\frac{R_i(\rho_i, \mu_i)}{|R_i(\rho_i, \mu_i)|} \cdot \frac{R_j(\rho_j, \mu_j)}{|R_j(\rho_j, \mu_j)|} \quad = \quad \cos \theta_k \tag{4.34}$$

around $(\rho_i, \mu_i) \equiv (0, 0)$.
Replacing ρ_i and μ_i by $\varepsilon \rho_i$ and $\varepsilon \mu_i$ and setting

$$Q_i(\varepsilon) := R_i(\varepsilon \rho_i, \varepsilon \mu_i)$$

we have to compute the term

$$\frac{d}{d\varepsilon} \left(\frac{Q_i(\varepsilon)}{|Q_i(\varepsilon)|} \cdot \frac{Q_j(\varepsilon)}{|Q_j(\varepsilon)|} \right) \Big|_{\varepsilon = 0} .$$

We see the identity

$$Q_i(0) = R_i(0\rho_i, 0\mu_i) = R_i(0, 0) = n_i^*$$

and can therefore calculate abstractly

$$\begin{aligned}
\frac{d}{d\varepsilon} \left(\frac{Q_i(\varepsilon)}{|Q_i(\varepsilon)|} \right) \Big|_{\varepsilon = 0} \quad & = \quad \frac{|Q_i(0)| \, Q_i'(0) - Q_i(0) \, \frac{d}{d\varepsilon} \left(|Q_i(\varepsilon)|\right) \big|_{\varepsilon=0}}{|Q_i(0)|^2} \\
& = \quad Q_i'(0) - Q_i(0) \, \frac{Q_i(0) \cdot Q_i'(0)}{|Q_i(0)|} \\
& = \quad Q_i'(0) - n_i^* \, \left(Q_i'(0) \cdot n_i^*\right) \\
& = \quad \left(Q_i'(0)\right)^T ,
\end{aligned}$$

where we used the projection on the tangent space of Γ_i^* given by $(y)^T = y - (y \cdot n_i^*)\, n_i^*$. With the relation $Q_i'(0) = \frac{d}{d\varepsilon} R_i(\varepsilon\rho_i, \varepsilon\mu_i)\big|_{\varepsilon=0}$ and with the definition of R we see

$$\left(Q_i'(0)\right)^T = \left(\frac{d}{d\varepsilon} R_i(\varepsilon\rho_i, \varepsilon\mu_i)\bigg|_{\varepsilon=0} \right)^T \quad = \quad -\sum_{l=1}^{n} \partial_l \rho_i \, \partial_l F_i - \mu_i \sum_{l=1}^{n} \left(\partial_l t_i^* \cdot n_i^*\right) \partial_l F_i \, .$$

Therefore, we get

$$\begin{aligned}
& \frac{d}{d\varepsilon} \left(\frac{Q_i(\varepsilon)}{|Q_i(\varepsilon)|} \cdot \frac{Q_j(\varepsilon)}{|Q_j(\varepsilon)|} \right)\bigg|_{\varepsilon=0} \\
= \quad & \left(Q_i'(0)\right)^T \cdot \frac{Q_j(0)}{|Q_j(0)|} + \frac{Q_i(0)}{|Q_i(0)|} \cdot \left(Q_j'(0)\right)^T \\
= \quad & \left(-\sum_{l=1}^{n} \partial_l \rho_i \, \partial_l F_i - \mu_i \sum_{l=1}^{n} \left(\partial_l t_i^* \cdot n_i^*\right) \partial_l F_i \right) \cdot n_j^* \\
& + n_i^* \cdot \left(-\sum_{l=1}^{n} \partial_l \rho_j \, \partial_l F_j - \mu_j \sum_{l=1}^{n} \left(\partial_l t_j^* \cdot n_j^*\right) \partial_l F_j \right) .
\end{aligned}$$

At this moment we use the assumption (B) that $\partial_1 F_i$ equals the outer unit conormal $n_{\partial\Gamma_i^*}$ at the fixed point x_0^i. Because of the orthogonality of $\partial_1 F_i, \ldots, \partial_n F_i$ from assumption (B), we can conclude that the tangent vectors $\partial_2 F_i, \ldots, \partial_n F_i$ are all perpendicular to $n_{\partial\Gamma_i^*}$. Of course, they are also perpendicular to the normal n_i^*, everything at the fixed point $q_0 = F(x_0^i) \in L^*$, which means

$$\partial_2 F_i, \ldots, \partial_n F_i \begin{cases} \perp n_i^* \text{ and} \\ \perp n_{\partial\Gamma_i^*} \, . \end{cases}$$

From Remark 4.2 we also know that the vectors

$$n_1^* \, , \; n_{\partial\Gamma_1^*} \, , \; n_2^* \, , \; n_{\partial\Gamma_2^*} \, , \; n_3^* \, , \; n_{\partial\Gamma_3^*}$$

all lie in a two-dimensional space, namely the space which is orthogonal to the tangent space of L^*. So we can write n_j^* as a linear combination of n_i^* and $n_{\partial\Gamma_i^*}$, that is

$$n_j^* \quad \in \quad \operatorname{span}\{n_i^*, n_{\partial\Gamma_i^*}\} \, .$$

Therefore in the above linearization of the angle conditions the scalar products involving $\partial_2 F_i, \ldots,$ $\partial_n F_i$ and also $\partial_2 F_j, \ldots, \partial_n F_j$ all cancel out and the following terms remain

$$\begin{aligned}
& -\frac{d}{d\varepsilon} \left(\frac{Q_i(\varepsilon)}{|Q_i(\varepsilon)|} \cdot \frac{Q_j(\varepsilon)}{|Q_j(\varepsilon)|} \right)\bigg|_{\varepsilon=0} \\
= \quad & \Big(\partial_1 \rho_i \, \partial_1 F_i + \mu_i \left(\partial_1 t_i^* \cdot n_i^*\right) \partial_1 F_i \Big) \cdot n_j^* + n_i^* \cdot \Big(\partial_1 \rho_j \, \partial_1 F_j + \mu_j \left(\partial_1 t_j^* \cdot n_j^*\right) \partial_1 F_j \Big) \\
= \quad & \Big(\partial_1 \rho_i \, n_{\partial\Gamma_i^*} + \mu_i \left(\partial_1 t_i^* \cdot n_i^*\right) n_{\partial\Gamma_i^*} \Big) \cdot n_j^* + n_i^* \cdot \Big(\partial_1 \rho_j \, n_{\partial\Gamma_j^*} + \mu_j \left(\partial_1 t_j^* \cdot n_j^*\right) n_{\partial\Gamma_i^*} \Big) \\
= \quad & \left(\partial_1 \rho_i + \mu_i \left(\partial_1 t_i^* \cdot n_i^*\right)\right) \left(n_{\partial\Gamma_i^*} \cdot n_j^*\right) + \left(\partial_1 \rho_j + \mu_j \left(\partial_1 t_j^* \cdot n_j^*\right)\right) \Big(n_{\partial\Gamma_j^*} \cdot n_i^*\Big) \, .
\end{aligned}$$

Due to the angle conditions for the stationary reference hypersurfaces Γ_i^*, it holds that one of the terms $\left(n_{\partial\Gamma_i^*}\cdot n_j^*\right)$ and $\left(n_{\partial\Gamma_j^*}\cdot n_i^*\right)$ is $\sin\theta_k$ and the other one is $-\sin\theta_k$. Since $\sin\theta_k \neq 0$, cancelling provides for the linearization of the angle condition

$$\partial_1\rho_i + \mu_i\left(\partial_1 t_i^*\cdot n_i^*\right) \quad = \quad \partial_1\rho_j + \mu_j\left(\partial_1 t_j^*\cdot n_j^*\right)$$

for $(i,j) = (1,2)$ and $(2,3)$.
In geometric terms, the derivative ∂_1 here is a directional derivative in direction of the conormal, which follows from assumption (B), so we get

$$\begin{aligned}
&\partial_1\rho_i = \partial_{n_{\partial\Gamma_i^*}}\rho_i = \nabla_{\Gamma_i^*}\rho_i\cdot n_{\partial\Gamma_i^*} \quad \text{and}\\
&\left(\partial_1 t_i^*\cdot n_i^*\right) = \left(\partial_{n_{\partial\Gamma_i^*}} n_{\partial\Gamma_i^*}\cdot n_i^*\right) = -n_{\partial\Gamma_i^*}\cdot\partial_{n_{\partial\Gamma_i^*}} n_i^* = \sigma_i^*(n_{\partial\Gamma_i^*}, n_{\partial\Gamma_i^*}) = \kappa_{n_{\partial\Gamma_i^*}}\,,
\end{aligned}$$

where σ_i^* is the second fundamental form of Γ_i^* with respect to n_i^* and $\kappa_{n_{\partial\Gamma_i^*}}$ is the normal curvature of Γ_i^* in direction of the conormal $n_{\partial\Gamma_i^*}$.
The linearization of the angle condition then reads as follows

$$\partial_{n_{\partial\Gamma_i^*}}\rho_i + \kappa_{n_{\partial\Gamma_i^*}}\mu_i \quad = \quad \partial_{n_{\partial\Gamma_j^*}}\rho_j + \kappa_{n_{\partial\Gamma_j^*}}\mu_j\,,$$

for $(i,j) = (1,2)$ and $(2,3)$.
To derive (4.32) from this identity, we use (4.23),(ii) from Lemma 4.5 to get

$$\partial_{n_{\partial\Gamma_i^*}}\rho_i + \kappa_{n_{\partial\Gamma_i^*}}\frac{1}{s_i}\left(c_j\rho_j - c_k\rho_k\right) \quad = \quad \partial_{n_{\partial\Gamma_j^*}}\rho_j + \kappa_{n_{\partial\Gamma_j^*}}\frac{1}{s_j}\left(c_k\rho_k - c_i\rho_i\right)$$

for $(i,j,k) = (1,2,3)$, $(2,3,1)$ and $(3,1,2)$. Now we take into account Young's law (4.6) and (4.23),(i) from Lemma 4.5 to observe

$$\begin{aligned}
\frac{1}{s_i}\left(c_j\rho_j - c_k\rho_k\right) &= \frac{1}{\gamma_i}\left(\frac{c_j}{s_j}\gamma_j\rho_j - \frac{c_k}{s_k}\gamma_k\rho_k\right)\\
&= \frac{1}{\gamma_i}\left(\frac{c_j}{s_j}\gamma_j\rho_j + \frac{c_k}{s_k}\left(\gamma_i\rho_i + \gamma_j\rho_j\right)\right)\\
&= \frac{1}{\gamma_i}\left(\frac{c_k}{s_k}\gamma_i\rho_i + \left(\frac{c_j}{s_j} + \frac{c_k}{s_k}\right)\gamma_j\rho_j\right)\\
&= \frac{1}{s_k}c_k\rho_i + \frac{1}{s_k}\underbrace{s_k\frac{s_j}{s_i}\left(\frac{c_j}{s_j} + \frac{c_k}{s_k}\right)}_{=(*)}\rho_j\\
&= \frac{1}{s_k}\left(c_k\rho_i - \rho_j\right)\,,
\end{aligned}$$

where we used $\theta_1 + \theta_2 + \theta_3 = 2\pi$ and an addition theorem to calculate

$$\begin{aligned}
(*) \quad &= \quad \frac{s_k s_i\left(c_j s_k + c_k s_j\right)}{s_i s_j s_k} = \frac{\cos\theta_j\sin\theta_k + \cos\theta_k\sin\theta_j}{\sin\theta_i} = \frac{\sin(\theta_j + \theta_k)}{\sin\theta_i} = \frac{\sin(2\pi - \theta_i)}{\sin\theta_i}\\
&= \quad -1\,.
\end{aligned}$$

By an analogous argument, we also have

$$\frac{1}{s_j}\left(c_k\rho_k - c_i\rho_i\right) = \frac{1}{s_k}\left(\rho_i - c_k\rho_j\right) .$$

Plugging these two identities into the linearization of the angle condition gives

$$\partial_{n_{\partial\Gamma_i^*}}\rho_i + \kappa_{n_{\partial\Gamma_i^*}}\frac{1}{s_k}\left(c_k\rho_i - \rho_j\right) \quad = \quad \partial_{n_{\partial\Gamma_j^*}}\rho_j + \kappa_{n_{\partial\Gamma_j^*}}\frac{1}{s_k}\left(\rho_i - c_k\rho_j\right) .$$

Arranging the terms of ρ_i on one side and these of ρ_j on the other side, finally leads to

$$\partial_{n_{\partial\Gamma_i^*}}\rho_i + \frac{1}{s_k}\left(c_k\kappa_{n_{\partial\Gamma_i^*}} - \kappa_{n_{\partial\Gamma_j^*}}\right)\rho_i \quad = \quad \partial_{n_{\partial\Gamma_j^*}}\rho_j + \frac{1}{s_k}\left(\kappa_{n_{\partial\Gamma_i^*}} - c_k\kappa_{n_{\partial\Gamma_j^*}}\right)\rho_j .$$

Using Lemma 4.4 for the stationary hypersurfaces and Young's law (4.6), we derive

$$\begin{aligned}
\frac{1}{s_k}\left(c_k\kappa_{n_{\partial\Gamma_i^*}} - \kappa_{n_{\partial\Gamma_j^*}}\right) &= -\frac{c_k}{\gamma_i s_k}\left(\gamma_j\kappa_{n_{\partial\Gamma_j^*}} + \gamma_k\kappa_{n_{\partial\Gamma_k^*}}\right) - \frac{1}{s_k}\kappa_{n_{\partial\Gamma_j^*}} \\
&= -\frac{c_k}{\gamma_k s_i}\left(\gamma_j\kappa_{n_{\partial\Gamma_j^*}} + \gamma_k\kappa_{n_{\partial\Gamma_k^*}}\right) - \frac{1}{s_k}\kappa_{n_{\partial\Gamma_j^*}} \\
&= -\frac{1}{s_k s_i}\left(c_k s_j + s_i\right)\kappa_{n_{\partial\Gamma_j^*}} - \frac{c_k}{s_i}\kappa_{n_{\partial\Gamma_k^*}} \\
&= -\frac{1}{s_k s_i}\left(c_k s_j - \left(s_j c_k + s_k c_j\right)\right)\kappa_{n_{\partial\Gamma_j^*}} - \frac{c_k}{s_i}\kappa_{n_{\partial\Gamma_k^*}} \\
&= \frac{1}{s_i}\left(c_j\kappa_{n_{\partial\Gamma_j^*}} - c_k\kappa_{n_{\partial\Gamma_k^*}}\right) ,
\end{aligned}$$

where we used the addition theorem $s_i = -\left(s_j c_k + s_k c_j\right)$. An analogous calculation gives

$$\frac{1}{s_k}\left(\kappa_{n_{\partial\Gamma_i^*}} - c_k\kappa_{n_{\partial\Gamma_j^*}}\right) \quad = \quad \frac{1}{s_j}\left(c_k\kappa_{n_{\partial\Gamma_k^*}} - c_i\kappa_{n_{\partial\Gamma_i^*}}\right) .$$

Plugging this into the above equation leads to

$$\partial_{n_{\partial\Gamma_i^*}}\rho_i + \frac{1}{s_i}\left(c_j\kappa_{n_{\partial\Gamma_j^*}} - c_k\kappa_{n_{\partial\Gamma_k^*}}\right)\rho_i \quad = \quad \partial_{n_{\partial\Gamma_j^*}}\rho_j + \frac{1}{s_j}\left(c_k\kappa_{n_{\partial\Gamma_k^*}} - c_i\kappa_{n_{\partial\Gamma_i^*}}\right)\rho_j ,$$

which is the assertion (4.32). □

To proceed, we abbreviate for reasons of shortness the following terms on L^*.

$$a_1 := \frac{1}{s_1}\left(c_2\,\kappa_{n_{\partial\Gamma_2^*}} - c_3\,\kappa_{n_{\partial\Gamma_3^*}}\right) , \tag{4.35}$$

$$a_2 := \frac{1}{s_2}\left(c_3\,\kappa_{n_{\partial\Gamma_3^*}} - c_1\,\kappa_{n_{\partial\Gamma_1^*}}\right) \quad \text{and} \tag{4.36}$$

$$a_3 := \frac{1}{s_3}\left(c_1\,\kappa_{n_{\partial\Gamma_1^*}} - c_1\,\kappa_{n_{\partial\Gamma_1^*}}\right) . \tag{4.37}$$

We remind the Definition 2.17 of normal curvature $\kappa_{n_{\partial\Gamma_i^*}}$ through $\kappa_{n_{\partial\Gamma_i^*}} = \sigma_i^*(n_{\partial\Gamma_i^*}, n_{\partial\Gamma_i^*})$, where σ_i^* is the second fundamental form of Γ_i^* with respect to n_i^*.

When we now consider two angle conditions for $(i,j,k) = (1,2,3)$ and $(2,3,1)$ in the previous Lemma 4.11, we get the following short identities for the linearization on L^*.

$$\partial_{n_{\partial\Gamma_1^*}}\rho_1 + a_1\,\rho_1 = \partial_{n_{\partial\Gamma_2^*}}\rho_2 + a_2\,\rho_2 = \partial_{n_{\partial\Gamma_3^*}}\rho_3 + a_3\,\rho_3\,. \tag{4.38}$$

Finally the linearization of the right angle condition at the outer boundary S_i^* can be adressed directly to Lemma 3.7 because on S_i^* the parametrization fulfills

$$\Phi_i^{\rho_i,\mu_i}(t,q) = \Psi_i(t,\rho_i(t,q))$$

and equals therefore the curvilinear coordinate system from Chapter 3.

Altogether, we get for the linearization of (4.26) the following linear system of partial differential equations for (ρ_i,μ_i), $i = 1,2,3$, which fulfill (4.21).

$$\partial_t\rho_i = \gamma_i\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right) \qquad \text{in } \Gamma_i^*\,, \tag{4.39}$$

with boundary conditions on S_i^* given through

$$\left(\partial_\mu - S(n_i^*,n_i^*)\right)\rho_i \;=\; 0 \tag{4.40}$$

and boundary conditions on L^* given through

$$\left\{\begin{array}{l} \gamma_1\rho_1 + \gamma_2\rho_2 + \gamma_3\rho_3 = 0\,, \\ \partial_{n_{\partial\Gamma_1^*}}\rho_1 + a_1\,\rho_1 = \partial_{n_{\partial\Gamma_2^*}}\rho_2 + a_2\,\rho_2 = \partial_{n_{\partial\Gamma_3^*}}\rho_3 + a_3\,\rho_3\,. \end{array}\right. \tag{4.41}$$

Note that the functions μ_i do not appear in this partial differential equation and can be calculated through the algebraic equations from Lemma 4.5.

4.1.4 Conditions for linearized stability

In this section we want to give a condition for the linearized stability of the mean curvature flow (4.5) with a triple line and outer boundary contact around a stationary state Γ^*. With our choice of parametrization this means that we consider the linearized equation (4.39) together with the linearized boundary conditions (4.40) and (4.41) and examine the stability of the zero solution. To this end, we use the ideas of Garcke, Ito and Kohsaka [GIK10], where they considered surface diffusion with triple lines and outer boundary contact for curves in the plane. We modify this work to the present case of mean curvature flow with triple junction with outer boundary contact for hypersurfaces in $\mathbb{R}^{n+1}$. The necessary steps are similar to the linearized stability analysis from Section 3.2 but we have to take care of three different hypersurfaces and the equations on the triple line.

In generalization of Section 3.2 for the case of one hypersurface with outer boundary contact we get the following equivalence.

$$\Gamma^* \text{ is linearly asymptotically stable}$$
$$\Longleftrightarrow \begin{cases} I(\rho,\rho) := \sum_{i=1}^3 \gamma_i \int_{\Gamma_i^*} \left(|\nabla_{\Gamma_i^*}\rho_i|^2 - |\sigma_i^*|^2\rho_i^2\right) - \sum_{i=1}^3 \gamma_i \int_{S_i^*} S(n_i^*, n_i^*)\,\rho_i^2 \\ \qquad\qquad + \sum_{i=1}^3 \gamma_i \int_{L^*} a_i\,\rho_i^2 \\ \text{is positive for all } 0 \neq \rho = (\rho_1,\rho_2,\rho_3) \text{ with } \rho_i \in H^1(\Gamma_i^*) \\ \text{and } \gamma_1\rho_1 + \gamma_2\rho_2 + \gamma_3\rho_3 = 0 \text{ at } L^*\,. \end{cases}$$

To achieve this goal, we proceed analogously as in Section 3.2 by describing problem (4.39)-(4.41) as the L^2-gradient flow of an energy defined with the help of a bilinear form I from Definition 4.12. Then we analyze the spectrum of the linearized operator corresponding to (4.39)-(4.41) and get a connection between the eigenvalues of $\mathcal{A}$ and the bilinear form I.

Definition 4.12. *We use the following abbreviations for function spaces. For $k \in \mathbb{N}_0$ we set (omit the integrability value $p = 2$)*

$$\begin{aligned} \mathcal{H}^k &:= H^k(\Gamma_1^*) \times H^k(\Gamma_2^*) \times H^k(\Gamma_3^*)\,, \\ \mathcal{W} &:= \left\{\xi = (\xi_1,\xi_2,\xi_3) \in \mathcal{H}^1 \,|\, \xi_1 + \xi_2 + \xi_3 = 0 \;\text{ at } L^*\right\} \;\text{ and} \\ \mathcal{Z} &:= \left\{\xi = (\xi_1,\xi_2,\xi_3) \in \mathcal{H}^1 \,|\, \gamma_1\xi_1 + \gamma_2\xi_2 + \gamma_3\xi_3 = 0 \;\text{ at } L^*\right\}\,. \end{aligned}$$

Observe that we assume in this section $\partial\Gamma_i^* = L^*$ and that for $k = 0$ the convention is $\mathcal{H}^0 = L^2(\Gamma_1^*) \times L^2(\Gamma_2^*) \times L^2(\Gamma_3^*)$.
We define a bilinear form for $\rho = (\rho_1,\rho_2,\rho_3)$ and $\eta = (\eta_1,\eta_2,\eta_3)$ in $\mathcal{H}^1$ through

$$\begin{aligned} I(\rho,\eta) &:= \sum_{i=1}^3 \gamma_i \int_{\Gamma_i^*} \left(\nabla_{\Gamma_i^*}\rho_i \cdot \nabla_{\Gamma_i^*}\eta_i - |\sigma_i^*|^2\rho_i\eta_i\right) \mathrm{d}\mathcal{H}^n - \sum_{i=1}^3 \gamma_i \int_{S_i^*} S(n_i^*, n_i^*)\,\rho_i\eta_i\,\mathrm{d}\mathcal{H}^{n-1} \\ &\quad + \sum_{i=1}^3 \gamma_i \int_{L^*} a_i\rho_i\eta_i\,\mathrm{d}\mathcal{H}^{n-1}\,, \end{aligned}$$

and the associated energy for $\rho \in \mathcal{H}^1$ by

$$E(\rho) := \frac{1}{2} I(\rho,\rho)\,.$$

Here the a_i are given by (4.35)-(4.37). We say that a time dependent function ρ with values in $\mathcal{H}^1$ is a solution of the L^2-gradient flow equation to E if and only if

$$(\partial_t\rho(t),\xi)_{L^2} = -\partial E(\rho(t))(\xi)$$

for all $\xi \in \mathcal{H}^1$ and all $t > 0$. Here we use the L^2-inner product component-wise, i.e.

$$(\partial_t\rho(t),\xi)_{L^2} = \sum_{i=1}^3 (\partial_t\rho_i(t),\xi_i)_{L^2}\,,$$

and we observe that

$$\partial E(\rho(t))(\xi) \quad = \quad I(\rho(t), \xi)\,.$$

The next lemma shows that the linearized problem (4.39) and (4.41) is the L^2-gradient flow of E.

Lemma 4.13. *The function $\rho = (\rho_1, \rho_2, \rho_3) \in L^2(0,T;\mathcal{H}^2) \cap H^1(0,T;\mathcal{H}^0)$ is a solution to (4.39) with boundary conditions (4.40) at the outer boundary S_i^* and (4.41) at the triple line L^* for all $t > 0$, if and only if $\rho \in L^2(0,T;\mathcal{Z}) \cap H^1(0,T;\mathcal{H}^0)$ and*

$$(\partial_t \rho(t), \xi)_{L^2} \quad = \quad -I(\rho(t), \xi)$$

for all $\xi \in \mathcal{Z}$ and all $t > 0$.

Proof. If $\rho \in \mathcal{H}^2$ is a solution of (4.39) -(4.41), we omit the time variable t and test with a $\xi = (\xi_1, \xi_2, \xi_3) \in \mathcal{Z}$ to get with integration by parts and (4.39)

$$\begin{aligned}
(\partial_t \rho, \xi)_{L^2} &= \sum_{i=1}^{3} (\partial_t \rho_i, \xi_i)_{L^2} \\
&= \sum_{i=1}^{3} \gamma_i \int_{\Gamma_i^*} \left(\Delta_{\Gamma_i^*} \rho_i \xi_i + |\sigma_i^*|^2 \rho_i \xi_i\right) \\
&= -\sum_{i=1}^{3} \gamma_i \int_{\Gamma_i^*} \left(\nabla_{\Gamma_i^*} \rho_i \cdot \nabla_{\Gamma_i^*} \xi_i - |\sigma_i^*|^2 \rho_i \xi_i\right) + \sum_{i=1}^{3} \gamma_i \int_{\partial \Gamma_i^*} \left(\nabla_{\Gamma_i^*} \rho_i \cdot n_{\partial \Gamma_i^*}\right) \xi_i \\
&= -\sum_{i=1}^{3} \gamma_i \int_{\Gamma_i^*} \left(\nabla_{\Gamma_i^*} \rho_i \cdot \nabla_{\Gamma_i^*} \xi_i - |\sigma_i^*|^2 \rho_i \xi_i\right) + \underbrace{\sum_{i=1}^{3} \gamma_i \int_{L^*} \left(\nabla_{\Gamma_i^*} \rho_i \cdot n_{\partial \Gamma_i^*} + a_i \rho_i\right) \xi_i}_{(*)} \\
&\quad - \sum_{i=1}^{3} \gamma_i \int_{L^*} a_i \rho_i \xi_i + \sum_{i=1}^{3} \gamma_i \int_{S_i^*} \underbrace{\left(\nabla_{\Gamma_i^*} \rho_i \cdot \mu\right)}_{= S(n_i^*, n_i^*) \rho_i} \xi_i \\
&= -I(\rho, \xi)\,.
\end{aligned}$$

For $(*)$ we use the boundary condition (4.41) and $\sum_{i=1}^{3} \gamma_i \xi_i = 0$ for $\xi \in \mathcal{Z}$ at L^* to achieve

$$(*) = \sum_{i=1}^{3} \gamma_i \int_{L^*} \left(\nabla_{\Gamma_i^*} \rho_i \cdot n_{\partial \Gamma_i^*} + a_i \rho_i\right) \xi_i = \int_{L^*} \left(\nabla_{\Gamma_1^*} \rho_1 \cdot n_{\partial \Gamma_1^*} + a_1 \rho_1\right) \underbrace{\sum_{i=1}^{3} \gamma_i \xi_i}_{=0} \quad = \quad 0\,.$$

If on the other hand $\rho \in L^2(0,T;\mathcal{Z}) \cap H^1(0,T;\mathcal{H}^0)$ fulfills $(\partial_t \rho(t), \xi)_{L^2} = -I(\rho(t), \xi)$ for all $\xi \in \mathcal{Z}$, regularity theory gives us $\partial_t \rho \in L^(0,T;\mathcal{H}^0)$ and $\rho \in L^2(0,T;\mathcal{H}^2)$. Therefore we can do the above calculation backwards and use the fundamental lemma to get that ρ is indeed a solution of (4.39) with the boundary conditions (4.40) at the outer boundary S_i^* and (4.41) at the triple line L^*. $\square$

Let us now define the corresponding linearized operator to (4.39) and (4.41) through

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{H}^0$$

with

$$\mathcal{D}(\mathcal{A}) \quad = \quad \left\{\rho \in \mathcal{H}^2 \,|\, \rho \text{ satisfies (4.40) at } S_i^* \text{ and (4.41) at } L^*\right\} \tag{4.42}$$

by

$$\mathcal{A}\rho := \left((\mathcal{A}\rho)_1 , (\mathcal{A}\rho)_2 , (\mathcal{A}\rho)_3\right) , \qquad \text{where} \;\; (\mathcal{A}\rho)_i := \gamma_i \left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right) \tag{4.43}$$

for all $\rho = (\rho_1, \rho_2, \rho_3) \in \mathcal{D}(\mathcal{A})$.

The linearized problem (4.39) and (4.41) is then related to the problem in finding a time dependent function $\rho \in L^2(0, T; \mathcal{D}(\mathcal{A})) \cap H^1(0, T; \mathcal{H}^0)$ with

$$\partial_t\rho \quad = \quad \mathcal{A}\rho\,.$$

From now on we skip the variable t, so that by Definition 4.12 and Lemma 4.13 we also have for all $\rho \in \mathcal{D}(\mathcal{A})$ and $\xi \in \mathcal{Z}$ the identity

$$(\mathcal{A}\rho, \xi)_{L^2} \quad = \quad -I(\rho, \xi)\,.$$

This gives us the opportunity to show symmetry of $\mathcal{A}$ in a simple way as before in Section 3.2.

Lemma 4.14. *The operator $\mathcal{A}$ is symmetric with respect to the L^2-inner product. Therefore it has real eigenvalues.*

Proof. Exactly the same as in Lemma 3.10. $\square$

The next point is to describe the spectrum of $\mathcal{A}$. Therefore we have to generalize the inequality from Lemma 3.11 to the present case of a triple line, so that we get as a corollary an upper bound for the eigenvalues of $\mathcal{A}$. We introduce the following notation for $\rho = (\rho_1, \rho_2, \rho_3) \in \mathcal{H}^k$.

$$\|\rho\|_{\mathcal{H}^k} \quad := \quad \left(\sum_{i=1}^{3} \|\rho_i\|_{H^k}^2\right)^{\frac{1}{2}}\,, \tag{4.44}$$

which denotes a norm on $\mathcal{H}^k$.

Lemma 4.15. *There exist positive constants C_1 and C_2 such that*

$$\|\rho\|_{\mathcal{H}^1}^2 \quad \leq \quad C_1\|\rho\|_{\mathcal{H}^0}^2 + C_2 I(\rho, \rho) \tag{4.45}$$

for all $\rho \in \mathcal{H}^1$.

Proof. We will use inequality (3.33) from Lemma 3.11, which in this case with three hypersurfaces reads as follows. For all $\delta > 0$ there exists a $C_\delta > 0$, such that

$$\|\rho_i\|^2_{L^2(\partial\Gamma_i^*)} \leq \delta\|\nabla_{\Gamma_i^*}\rho_i\|^2_{L^2(\Gamma_i^*)} + C_\delta\|\rho_i\|^2_{L^2(\Gamma_i^*)} \tag{4.46}$$

for all $\rho_i \in H^1(\Gamma_i^*)$, $i = 1,2,3$. In fact, we get tree different constants C_δ^i, but we can take the largest one and call it C_δ.

Now we proceed with the estimate

$$\begin{aligned} I(\rho,\rho) &= \sum_{i=1}^{3}\gamma_i\int_{\Gamma_i^*}\left(|\nabla_{\Gamma_i^*}\rho_i|^2 - |\sigma_i^*|^2\rho_i^2\right) + \sum_{i=1}^{3}\gamma_i\int_{L^*} a_i\rho_i^2 - \sum_{i=1}^{3}\gamma_i\int_{S_i^*} S(n_i^*, n_i^*)\rho_i^2 \\ &\geq \gamma\sum_{i=1}^{3}\|\nabla_{\Gamma_i^*}\rho_i\|^2_{L^2(\Gamma_i^*)} - m\sum_{i=1}^{3}\|\rho_i\|^2_{L^2(\Gamma_i^*)} - M\sum_{i=1}^{3}\|\rho_i\|^2_{L^2(\partial\Gamma_i^*)}, \end{aligned}$$

where we set

$$\begin{aligned} \gamma &:= \min\{\gamma_i \,|\, i = 1,2,3\}, \\ m &:= \max\{\||\sigma_i^*|^2\|_{L^\infty(\Gamma_i^*)} \,|\, i = 1,2,3\} \text{ and} \\ M &:= \max\{\|\gamma_i a_i\|_{L^\infty(\partial\Gamma_i^*)}, \|\gamma_i S(n_i^*, n_i^*)\|_{L^\infty(\partial\Gamma_i^*)} \,|\, i = 1,2,3\}. \end{aligned}$$

We use (4.46) to get

$$I(\rho,\rho) \geq (\gamma - \delta M)\|\nabla_{\Gamma_i^*}\rho_i\|^2_{L^2(\Gamma_i^*)} - (m + C_\delta M)\|\rho_i\|^2_{L^2(\Gamma_i^*)}.$$

By choosing $\delta > 0$ so small that $(\gamma - \delta M) > 0$, we derive for some constants $c, C > 0$ the inequality

$$c\|\rho_i\|^2_{L^2(\Gamma_i^*)} + CI(\rho,\rho) \geq \|\nabla_{\Gamma_i^*}\rho_i\|^2_{L^2(\Gamma_i^*)}.$$

Adding $\|\rho_i\|^2_{L^2(\Gamma_i^*)}$ to both sides and summing over $i = 1,2,3$ gives the assertion with some positive constants C_1 and C_2. ☐

With the help of the previous lemma we are able to show boundedness from above for the eigenvalues of $\mathcal{A}$.

Lemma 4.16. *Let λ be an eigenvalue from $\mathcal{A}$. Then the following inequality holds.*

$$\lambda \leq \frac{C_1}{C_2},$$

where C_1 and C_2 are the positive constants of the previous Lemma 4.15.

Proof. With the help of the previous Lemma 4.15 the proof is exactly the same as in Lemma 3.12. ☐

The next step is to show that $\mathcal{A}$ is self-adjoint with respect to the L^2-inner product, which will be done as in the previous chapter in Lemma 3.13 with the help of a property that implies equivalence of symmetry and self-adjointness.

Lemma 4.17. *The operator $\mathcal{A}$ is self-adjoint with respect to the L^2-inner product.*

Proof. We use the following theorem of operator theory. If there exists an $\omega \in \mathbb{R}$, such that

$$\operatorname{im}(\omega Id - \mathcal{A}) \quad = \quad \mathcal{H}^0 ,$$

then the properties symmetry and self-adjointness are equivalent, see for example the book of Weidmann [Weid76].

So we have to show that there exists an $\omega \in \mathbb{R}$ such that for given $f \in \mathcal{H}^0$ there exists a $\rho \in \mathcal{D}(\mathcal{A})$ with

$$\omega\rho - \mathcal{A}\rho \quad = \quad f .$$

In detail, this means

$$\begin{cases} -\gamma_i \left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right) + \omega\rho_i = f_i & \text{in } \Gamma_i^* ,\ i = 1,2,3 , \\ \left(\partial_\mu + S(n_i^*, n_i^*)\right)\rho_i = 0 & \text{on } \S_i^* ,\ i = 1,2,3 , \\ \gamma_1\rho_2 + \gamma_2\rho_2 + \gamma_3\rho_3 = 0 & \text{on } L^* , \\ \left(\nabla_{\Gamma_1^*}\rho_1 \cdot n_{\partial\Gamma_1^*}\right) + a_1\rho_1 = \left(\nabla_{\Gamma_2^*}\rho_2 \cdot n_{\partial\Gamma_2^*}\right) + a_2\rho_2 & \\ = \left(\nabla_{\Gamma_3^*}\rho_3 \cdot n_{\partial\Gamma_3^*}\right) + a_3\rho_3 & \text{on } L^* . \end{cases} \tag{4.47}$$

With analogue calculations as in Lemma 4.13 we get the weak formulation of (4.47) through the following problem. For given $f \in \mathcal{H}^0$ find a $\rho \in \mathcal{Z}$, i.e. $\rho \in \mathcal{H}^1$ and $\gamma_1\rho_1 + \gamma_2\rho_2 + \gamma_3\rho_3 = 0$ on L^*, such that

$$I(\rho, \psi) + \omega\,(\rho, \psi)_{L^2} \quad = \quad (f, \psi)_{L^2}$$

for all $\psi \in \mathcal{W}$, i.e. $\psi \in \mathcal{H}^1$ and $\psi_1 + \psi_2 + \psi_3 = 0$ on L^*. With the help of Lemma 4.15 and 4.16 we can show with the same calculation as in Lemma 3.13 that the left side defines a coercive bilinear form for large $\omega \in \mathbb{R}$.

The fundamental theorem of Lax-Milgram gives then a unique weak solution $\overline{\rho} \in \mathcal{Z}$. Since $f \in \mathcal{H}^0$, regularity theory leads to $\overline{\rho} \in \mathcal{H}^2$ and to the remaining boundary condition in (4.47). So we found an $\omega \in \mathbb{R}$ and a $\overline{\rho} \in \mathcal{D}(\mathcal{A})$, such that

$$\omega\overline{\rho} - \mathcal{A}\overline{\rho} \quad = \quad f .$$

Therefore we can apply the above theorem from operator theory and conclude from the symmetry of Lemma 4.14 even the self-adjointness of $\mathcal{A}$. □

With the help of the previous results we are able to apply standard theory of self-adjoint operators and the theory of semigroups to get the following theorem.

Theorem 4.18.

(*i*) *The spectrum of $\mathcal{A}$ consists of countable many real eigenvalues.*

(*ii*) *The initial value problem (4.39)-(4.41) is solvable for given initial data in $\mathcal{H}^0$.*

(iii) The zero solution of the linearized problem (4.39)-(4.41) is asymptotically stable if and only if the largest eigenvalue of $\mathcal{A}$ is negative.

Proof. With the same abstract arguments as in the proof of Lemma 3.14 we can show the assertions with the help of Lemma 4.16 and Lemma 4.17. $\square$.

The next lemma relates eigenvalues of $\mathcal{A}$ to the bilinear form I, so that we can formulate our linearized stability criterion.

Lemma 4.19. *Let*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$$

be the eigenvalues of $\mathcal{A}$ (taken into account the multiplicity).

(i) For all $n \in \mathbb{N}$, the following description of the eigenvalues of $\mathcal{A}$ holds.

$$\begin{aligned} \lambda_n &= \inf_{W \in \Sigma_{n-1}} \sup_{\rho \in W \setminus \{0\}} -\frac{I(\rho,\rho)}{(\rho,\rho)_{L^2}}, \\ -\lambda_n &= \sup_{W \in \Sigma_{n-1}} \inf_{\rho \in W^\perp \setminus \{0\}} \frac{I(\rho,\rho)}{(\rho,\rho)_{L^2}}, \end{aligned}$$

where Σ_n is the collection of n-dimensional subspaces of $\mathcal{Z}$ and $W^\perp$ is the orthogonal complement with respect to the L^2-inner product.

(ii) The eigenvalues λ_n depend continuously on $\kappa_{n_{\partial\Gamma_i^*}}$ and $|\sigma_i^*|$ in the L^∞-norm.

Proof. As in Lemma 3.15, for the first part we just refer to the classical work of Courant and Hilbert [CH68]. The second part follows directly from the structure of I. $\square$

In the next remark the largest eigenvalue of $\mathcal{A}$ is described more explicitly.

Remark 4.20. *For the largest eigenvalue λ_1 of $\mathcal{A}$ we have the description*

$$-\lambda_1 = \min_{\rho \in \mathcal{Z} \setminus \{0\}} \frac{I(\rho,\rho)}{(\rho,\rho)_{L^2}}, \tag{4.48}$$

which can be seen directly from the second description of λ_1 in Lemma 4.19 through $-\lambda_1 = \sup_{W \in \Sigma_0} \inf_{\rho \in W^\perp \setminus \{0\}} \frac{I(\rho,\rho)}{(\rho,\rho)_{L^2}}$ and $\Sigma_0 = \emptyset$ and therefore $W^\perp = \mathcal{Z}$. The fact that the minimum is attained also follows from the classical work of Courant and Hilbert [CH68].

From Theorem 4.18 we have asymptotic stability of the linearized problem (4.39)-(4.41) if and only if $\lambda_1 < 0$. This leads to the following main conclusion.

Theorem 4.21. *The linearized problem (4.39)-(4.41) is asymptotically stable if and only if*

$$I(\rho,\rho) \;>\; 0$$

for all $\rho \in \mathcal{Z}\backslash\{0\}$*, where*

$$\begin{aligned} I(\rho,\rho) \;=\;& \sum_{i=1}^{3}\gamma_i \int_{\Gamma_i^*}\left(|\nabla_{\Gamma_i^*}\rho_i|^2 - |\sigma_i^*|^2\rho_i^2\right)\mathrm{d}\mathcal{H}^n - \sum_{i=1}^{3}\gamma_i\int_{S_i^*} S(n_i^*,n_i^*)\rho_i^2\mathrm{d}\mathcal{H}^{n-1} \\ &+\sum_{i=1}^{3}\gamma_i\int_{L^*} a_i\rho_i^2\mathrm{d}\mathcal{H}^{n-1}\,. \end{aligned}$$

4.2 Surface diffusion flow

In this last section we want to complete our considerations with the surface diffusion flow with triple lines and outer boundary contact with a fixed bounded region. As in the previous Section 4.1 we formulate the problem in detail, give some geometric properties and derive the linearized problem for functions ρ_i, $i=1,2,3$. We use the same parametrization from (4.18) as in the previous section. Then we proceed with stability analysis as in Section 3.4 for the emerging linear fourth-order system of partial differential equations.

So we want to find three evolving hypersurfaces $\Gamma_i = \bigcup_{t\in[0,T)}\{t\}\times\Gamma_i(t)$ with $\Gamma_i(t)\subset\mathbb{R}^{n+1}$ as in Definition 2.31, moving due to the surface diffusion flow, such that $\Gamma_i(t)$ lies in a fixed bounded region $\Omega\subset\mathbb{R}^{n+1}$ and the decomposition (4.2)-(4.4) is fulfilled. This means that the boundary can be seperated disjointly into $\partial\Gamma_i(t) = L_i(t)\cup S_i(t)$, such that $L(t)=L_1(t)=L_2(t)=L_3(t)$ is a triple line and the other parts $S_i(t)=\partial\Gamma_i(t)\cap\partial\Omega$ represent the sections with the outer fixed boundary. Note our implicit assumption that $L(t)$ does not intersect $\partial\Omega$.

In formulas, we have to find hypersurfaces as in (4.1)-(4.4) which fulfill the following surface diffusion equation in $\Gamma_i(t)$

$$V_i \;=\; -m_i\gamma_i\Delta_{\Gamma_i(t)}H_i\,, \tag{4.49}$$

where the positive constants γ_i and m_i are the surface energy density and the mobility of the interface $\Gamma_i(t)$.

At the outer boundary $S_i(t)$, we require as in the case of one hypersurface in Section 3.4 the following right angle and natural boundary conditions.

$$\left\{\begin{aligned} \angle(\Gamma_i(t),\partial\Omega) \;&=\; \tfrac{\pi}{2}\,,\\ \nabla_{\Gamma_i(t)}H_i\cdot n_{\partial\Gamma_i(t)} \;&=\; 0\,. \end{aligned}\right. \tag{4.50}$$

At the triple line $L(t)$, we require the following conditions

$$\left\{\begin{array}{l} \angle(\Gamma_1(t),\Gamma_2(t))=\theta_3\,,\;\; \angle(\Gamma_2(t),\Gamma_3(t))=\theta_1\,,\;\; \angle(\Gamma_3(t),\Gamma_1(t))=\theta_2\,,\\ \gamma_1H_1+\gamma_2H_2+\gamma_3H_3=0\,,\\ m_1\gamma_1\nabla_{\Gamma_1(t)}H_1\cdot n_{\partial\Gamma_1(t)} = m_2\gamma_2\nabla_{\Gamma_2(t)}H_2\cdot n_{\partial\Gamma_2(t)} = m_3\gamma_3\nabla_{\Gamma_3(t)}H_3\cdot n_{\partial\Gamma_3(t)}\,. \end{array}\right. \tag{4.51}$$

The first condition denotes angles between the hypersurfaces at the triple line, where we require as in the previous section $0<\theta_i<\pi$, $\theta_1+\theta_2+\theta_3=2\pi$ and Young's law (4.6). The second condition follows from the continuity of the chemical potentials and the third conditions are the flux balance at the triple junction. We refer to Garcke and Novick-Cohen [GN00] for the derivation of the above model in the planar case. For reasons of shortness we will omit the starting configuration in this section.

As in Lemma 4.1 of the previous section, we can show the following balance of forces at the triple line due to the angle conditions and Young's law,

$$\gamma_1 n_{\partial\Gamma_1(t)} + \gamma_2 n_{\partial\Gamma_2(t)} + \gamma_3 n_{\partial\Gamma_3(t)} = 0\,.$$

Also Remark 4.2 and the corresponding equations (4.9)-(4.12) are fulfilled, see Figure 4.2 for a sketch in the curve case.

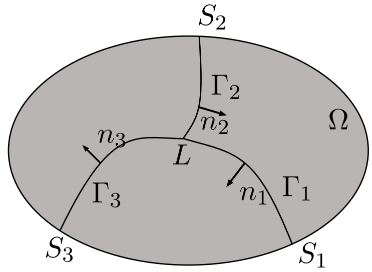


Figure 4.2: The choice of the normals.

4.2.1 Geometric properties of the flow

Here we want to show the properties area decreasing and volume preserving for surface diffusion (4.49)-(4.51). Therefore we have to generalize the calculations from Lemma 2.46 about evolution of area and volume in case of one hypersurface lying in a fixed bounded region to the present situation for three hypersurfaces that get together at a triple line. After this we will give properties for stationary states of (4.49)-(4.51).

Lemma 4.22. *Solutions $\Gamma_i(t)$ of the surface diffusion equation (4.49) which fulfill the boundary conditions (4.50) and (4.51), decrease the weighted total area*

$$A(t) = \sum_{i=1}^{3} \gamma_i \int_{\Gamma_i(t)} 1 \, \mathrm{d}\mathcal{H}^n$$

and preserve the enclosed volumes. With $\Omega_{ij}(t)$ defined as the region in Ω bounded by $\Gamma_i(t)$, $\Gamma_j(t)$ and $\partial\Omega$ for $(i,j) = (1,2)$, $(2,3)$, $(3,1)$, see Figure 4.3, this means in detail

$$\frac{d}{dt} \int_{\Omega_{ij}(t)} 1 \, \mathrm{d}x \quad = \quad 0\,.$$

Proof. Similar calculations as in Lemma 4.3 from the previous section lead to

$$\frac{d}{dt} A(t) = -\sum_{i=1}^{3} \gamma_i \int_{\Gamma_i(t)} V_i \, H_i \, \mathrm{d}\mathcal{H}^n + \int_{L(t)} \sum_{i=1}^{3} \gamma_i v_{\partial\Gamma_i} \, \mathrm{d}\mathcal{H}^{n-1} + \sum_{i=1}^{3} \gamma_i \int_{S_i(t)} v_{\partial\Gamma_i} \, \mathrm{d}\mathcal{H}^{n-1}\,.$$

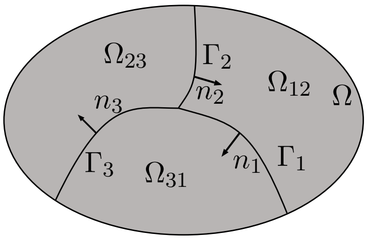


Figure 4.3: The definition of Ω_{ij}.

The second term vanishes as in the proof of Lemma 4.3 due to Young's law and the angle conditions and for the third term we can argue in the same way as in Lemma 2.46 in Chapter 2, where we saw $v_{\partial\Gamma_i} = 0$ due to the right angle at the fixed boundary. So we get by using the surface diffusion equation $V_i = -\gamma_i m_i \Delta_{\Gamma_i(t)} H_i$

$$\begin{aligned}\frac{d}{dt}A(t) &= -\sum_{i=1}^{3}\int_{\Gamma_i(t)} -\gamma_i^2 m_i \Delta_{\Gamma_i(t)} H_i \cdot H_i \,\mathrm{d}\mathcal{H}^n \\ &= -\sum_{i=1}^{3}\gamma_i^2 m_i \int_{\Gamma_i(t)} |\nabla_{\Gamma_i(t)} H_i|^2 \,\mathrm{d}\mathcal{H}^n + \sum_{i=1}^{3}\gamma_i m_i \int_{\partial\Gamma_i(t)} \left(\nabla_{\Gamma_i(t)} H_i \cdot n_{\partial\Gamma_i(t)}\right) H_i \,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned}$$

For the second term, we observe

$$\begin{aligned}&\sum_{i=1}^{3}\gamma_i m_i \int_{\partial\Gamma_i(t)} \left(\nabla_{\Gamma_i(t)} H_i \cdot n_{\partial\Gamma_i(t)}\right) H_i \,\mathrm{d}\mathcal{H}^{n-1} \\ =\;& \sum_{i=1}^{3}\int_{L(t)} m_i \left(\nabla_{\Gamma_i(t)} H_i \cdot n_{\partial\Gamma_i(t)}\right) \gamma_i H_i \,\mathrm{d}\mathcal{H}^{n-1} + \sum_{i=1}^{3}\gamma_i m_i \int_{S_i(t)} \underbrace{\left(\nabla_{\Gamma_i(t)} H_i \cdot n_{\partial\Gamma_i(t)}\right)}_{=0} H_i \,\mathrm{d}\mathcal{H}^{n-1} \\ =\;& \int_{L(t)} m_1 \left(\nabla_{\Gamma_1(t)} H_1 \cdot n_{\partial\Gamma_1(t)}\right) \underbrace{\sum_{i=1}^{3}\gamma_i H_i}_{=0} \,\mathrm{d}\mathcal{H}^{n-1} \\ =\;& 0\,,\end{aligned}$$

where we used the boundary conditions (4.50) and (4.51). Therefore it holds

$$\begin{aligned}\frac{d}{dt}A(t) &= -\sum_{i=1}^{3}\gamma_i^2 m_i \int_{\Gamma_i(t)} |\nabla_{\Gamma_i(t)} H_i|^2 \,\mathrm{d}\mathcal{H}^{n-1} \\ &\leq 0\,.\end{aligned}$$

For the volume preservation we first claim that

$$\frac{d}{dt}\int_{\Omega_{ij}(t)} 1 \,\mathrm{d}x = -\int_{\Gamma_i(t)} V_i \,\mathrm{d}\mathcal{H}^n + \int_{\Gamma_j(t)} V_j \,\mathrm{d}\mathcal{H}^n\,. \tag{4.52}$$

To see this identity, we proceed analogously to Lemma 2.46 from Chapter 2 to get

$$(n+1)\int_{\Omega_{ij}(t)} 1\,\mathrm{d}x = \int_{\Omega_{ij}(t)} \operatorname{div} id\,\mathrm{d}x = \int_{\partial\Omega_{ij}(t)} p\cdot\nu(t,p)\,\mathrm{d}\mathcal{H}^n\,.$$

Here ν is the outer unit normal of $\Omega_{ij}(t)$ and with our choice of the normals of the hypersurfaces it equals $\nu=\mu$ on $\Lambda(t)$, which describes as in Lemma 2.46 the part of the boundary of Ω, that coincides with $\partial\Omega_{ij}(t)$, $\nu=-n_i$ on $\Gamma_i(t)$ and $\nu=n_j$ on $\Gamma_j(t)$. Therefore we can decompose the above boundary integral into

$$\int_{\partial\Omega_{ij}(t)} p\cdot\nu(t,p) \;=\; \underbrace{\int_{\Lambda(t)} p\cdot\mu(p)\,\mathrm{d}\mathcal{H}^n}_{(1)} - \underbrace{\int_{\Gamma_i(t)} p\cdot n_i(t,p)\,\mathrm{d}\mathcal{H}^n}_{(2)} + \underbrace{\int_{\Gamma_j(t)} p\cdot n_j(t,p)\,\mathrm{d}\mathcal{H}^n}_{(3)}\,.$$

As in Lemma 2.46 we get for the first term

$$\frac{d}{dt}\int_{\Lambda(t)} p\cdot\mu(p)\,\mathrm{d}\mathcal{H}^n = \int_{\partial\Lambda(t)} (p\cdot\mu(p))\;v_{\partial\Lambda}\,\mathrm{d}\mathcal{H}^{n-1}\,,$$

where the other terms from the Transport theorem 2.44 vanish due to the fact that the normal velocity of Λ is zero. For the second and third term (2) and (3) from above can also use the calculations from Lemma 2.46 to get for $l=i,j$

$$\begin{aligned}\frac{d}{dt}\int_{\Gamma_l(t)} p\cdot n_l(t,p)\,\mathrm{d}\mathcal{H}^n \;&=\; (n+1)\int_{\Gamma_l(t)} V_l\,\mathrm{d}\mathcal{H}^n + \int_{\partial\Gamma_l(t)} (p\cdot n_l(t,p))\,v_{\partial\Gamma_l}(t,p)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\quad - \int_{\partial\Gamma_l(t)} (p\cdot n_{\partial\Gamma_l}(t,p))\,V_l(t,p)\,\mathrm{d}\mathcal{H}^{n-1}\,.\end{aligned}$$

With the decompositions

$$\begin{aligned}\partial\Lambda(t) \;&=\; S_i(t)\cup S_j(t)\,,\\ \partial\Gamma_i(t) \;&=\; S_i(t)\cup L(t) \text{ and}\\ \partial\Gamma_j(t) \;&=\; S_j(t)\cup L(t)\end{aligned}$$

we get then for the derivative of the volume the following formula

$$\begin{aligned}(n+1)\frac{d}{dt}\int_{\Omega_{ij}(t)} 1\,\mathrm{d}x \;&=\; -(n+1)\int_{\Gamma_i(t)} V_i\,\mathrm{d}\mathcal{H}^n + (n+1)\int_{\Gamma_j(t)} V_j\,\mathrm{d}\mathcal{H}^n\\ &\quad + \int_{S_i(t)} (p\cdot\mu(p))\,v_{\partial\Lambda}(t,p)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{S_j(t)} (p\cdot\mu(p))\,v_{\partial\Lambda}(t,p)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\quad + \int_{S_i(t)} (p\cdot n_{\partial\Gamma_i}(t,p))\,V_i(t,p)\,\mathrm{d}\mathcal{H}^{n-1} - \int_{S_j(t)} \big(p\cdot n_{\partial\Gamma_j}(t,p)\big)\,V_j(t,p)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\quad + \int_{L(t)} (p\cdot n_{\partial\Gamma_i}(t,p))\,V_i(t,p)\,\mathrm{d}\mathcal{H}^{n-1} - \int_{L(t)} \big(p\cdot n_{\partial\Gamma_j}(t,p)\big)\,V_j(t,p)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\quad - \int_{L(t)} (p\cdot n_i(t,p))\,v_{\partial\Gamma_i}(t,p)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{L(t)} (p\cdot n_j(t,p))\,v_{\partial\Gamma_j}(t,p)\,\mathrm{d}\mathcal{H}^{n-1}\,.\end{aligned}$$

Due to the choice of the normals n_i, on $S_i(t)$ there holds $v_{\partial\Lambda} = -V_i$ and we can proceed as in Lemma 2.46 to see that the integrals over $S_i(t)$ and $S_j(t)$ vanish. This leads us to

$$\begin{aligned}(n+1)\frac{d}{dt}\int_{\Omega_{ij}(t)} 1\,\mathrm{d}x &= -(n+1)\int_{\Gamma_i(t)} V_i\,\mathrm{d}\mathcal{H}^n + (n+1)\int_{\Gamma_j(t)} V_j\,\mathrm{d}\mathcal{H}^n \\ &\quad + \int_{L(t)} (p\cdot n_{\partial\Gamma_i}(t,p))\,V_i(t,p)\,\mathrm{d}\mathcal{H}^{n-1} - \int_{L(t)} \big(p\cdot n_{\partial\Gamma_j}(t,p)\big)\,V_j(t,p)\,\mathrm{d}\mathcal{H}^{n-1} \\ &\quad - \int_{L(t)} (p\cdot n_i(t,p))\,v_{\partial\Gamma_i}(t,p)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{L(t)} (p\cdot n_j(t,p))\,v_{\partial\Gamma_j}(t,p)\,\mathrm{d}\mathcal{H}^{n-1}\,.\end{aligned}$$

To see that the integrals over the triple line $L(t)$ vanish, we first recall from Remark 4.2 that the vectors $n_i, n_j, n_{\partial\Gamma_i}, n_{\partial\Gamma_j}$ at the point (t,p) all lie in the two-dimensional subspace $U = (T_pL(t))^\perp$. Then let D be the orthogonal matrix that assigns the chosen unit normal to the outer unit conormal, that is at $L(t)$ the following formulas hold

$$n_i = Dn_{\partial\Gamma_i}\,,\ n_j = Dn_{\partial\Gamma_j}\,,\ -n_{\partial\Gamma_i} = Dn_i \ \text{ and } \ -n_{\partial\Gamma_j} = Dn_j\,.$$

Due to the orthogonality condition $D^T = D^{-1}$ we can conclude

$$\begin{aligned}&p\cdot n_i = p\cdot (Dn_{\partial\Gamma_i}) = \big(D^{-1}p\big)\cdot n_{\partial\Gamma_i}\,,\\ &p\cdot n_{\partial\Gamma_i} = p\cdot(-Dn_i) = -\big(D^{-1}p\big)\cdot n_i\,,\\ &\text{and analogously}\\ &p\cdot n_j = \big(D^{-1}p\big)\cdot n_{\partial\Gamma_j} \ \text{ and } \ p\cdot n_{\partial\Gamma_j} = -\big(D^{-1}p\big)\cdot n_j\,.\end{aligned}$$

This gives for the considered integrals over the triple line $L(t)$

$$\frac{1}{n+1}\int_{L(t)} \big(D^{-1}p\big)\cdot\Big(\big(n_i\,V_i + n_{\partial\Gamma_i}\,v_{\partial\Gamma_i}\big) - \big(n_j\,V_j + n_{\partial\Gamma_j}\,v_{\partial\Gamma_j}\big)\Big)\,\mathrm{d}\mathcal{H}^{n-1}\,.$$

As in the proof of Lemma 2.46 we can calculate the velocities with the help of the same curve $c(\tau) \in L(\tau)$ with $c(t) = p \in L(t)$ through

$$V_i = n_i\cdot c'(t)\,,\ V_j = n_j\cdot c'(t)\,,\ v_{\partial\Gamma_i} = n_{\partial\Gamma_i}\cdot c'(t) \ \text{ and } \ v_{\partial\Gamma_j} = n_{\partial\Gamma_j}\cdot c'(t)\,.$$

Therefore we can write

$$\begin{aligned}n_i\,V_i + n_{\partial\Gamma_i}\,v_{\partial\Gamma_i} = n_i\cdot\big(n_i\cdot c'(t)\big) + n_{\partial\Gamma_i}\big(n_{\partial\Gamma_i}\cdot c'(t)\big) &= Pc'(t) \ \text{ and}\\ n_j\,V_j + n_{\partial\Gamma_j}\,v_{\partial\Gamma_j} = n_j\cdot\big(n_j\cdot c'(t)\big) + n_{\partial\Gamma_j}\big(n_{\partial\Gamma_j}\cdot c'(t)\big) &= Pc'(t)\,,\end{aligned}$$

where P denotes the projection onto the two-dimensional subspace U. With this observation the above integral over $L(t)$ vanishes and we proved the formula (4.52).

With equation (4.49) and integration by parts we can conclude further from (4.52)

$$
\begin{aligned}
\frac{d}{dt}\int_{\Omega_{ij}(t)} 1\,\mathrm{d}x &= \int_{\Gamma_i(t)} m_i\gamma_i\Delta_{\Gamma_i(t)}H_i\,\mathrm{d}\mathcal{H}^n - \int_{\Gamma_j(t)} m_j\gamma_j\Delta_{\Gamma_j(t)}H_j\,\mathrm{d}\mathcal{H}^n \\
&= \int_{\partial\Gamma_i(t)} m_i\gamma_i\nabla_{\Gamma_i(t)}H_i\cdot n_{\partial\Gamma_i}\,\mathrm{d}\mathcal{H}^{n-1} - \int_{\partial\Gamma_j(t)} m_j\gamma_j\nabla_{\Gamma_j(t)}H_j\cdot n_{\partial\Gamma_j}\,\mathrm{d}\mathcal{H}^{n-1} \\
&= \int_{S_i(t)} m_i\gamma_i\underbrace{\nabla_{\Gamma_i(t)}H_i\cdot n_{\partial\Gamma_i}}_{=0}\,\mathrm{d}\mathcal{H}^{n-1} - \int_{S_j(t)} m_j\gamma_j\underbrace{\nabla_{\Gamma_j(t)}H_j\cdot n_{\partial\Gamma_j}}_{=0}\,\mathrm{d}\mathcal{H}^{n-1} \\
&\quad + \int_{L(t)}\underbrace{\left(m_i\gamma_i\nabla_{\Gamma_i(t)}H_i\cdot n_{\partial\Gamma_i} - m_j\gamma_j\nabla_{\Gamma_j(t)}H_j\cdot n_{\partial\Gamma_j}\right)}_{=0}\,\mathrm{d}\mathcal{H}^{n-1} \\
&= 0\,,
\end{aligned}
$$

where we used the boundary conditions (4.50) on $S_i(t)$ and $S_j(t)$ and (4.51) on $L(t)$. $\square$

As in the previous parts of this work we fix a stationary solution of the above problem (4.49)-(4.51). This means we consider three hypersurfaces Γ_i^*, which lie in Ω, and the boundary has a decomposition $\partial\Gamma_i^* = L_i^* \cup S_i^*$, such that the three hypersurfaces meet at a triple line $L^* = L_1^* = L_2^* = L_3^*$ and the other parts are sections with the outer fixed boundary, i.e. $S_i^* = \partial\Gamma_i^* \cap \partial\Omega$. Γ_i^* shall fulfill the surface diffusion equation (4.49) with $V_i = 0$, the conditions (4.50) at S_i^* and (4.51) at the triple line L^*. We choose the normals n_i^* of Γ_i^* so that $\gamma_1 n_1^* + \gamma_2 n_2^* + \gamma_3 n_2^* = 0$, as we did in the previous section. We can show similar to the surface diffusion equation for one hypersurface from Section 3.4 that Γ_i^* has constant mean curvature.

Lemma 4.23. *Stationary solutions as described above have constant mean curvature and fulfill the identity*

$$\gamma_1\kappa_{n_{\partial\Gamma_1^*}} + \gamma_2\kappa_{n_{\partial\Gamma_2^*}} + \gamma_3\kappa_{n_{\partial\Gamma_3^*}} = 0 \quad \textit{on} \quad L^*$$

as in Lemma 4.4.

Proof. We proceed as in Lemma 3.25 to get from $\Delta_{\Gamma_i^*}H_i^* = 0$, where H_i^* is the mean curvature of Γ_i^*, that

$$0 = \int_{\Gamma_i^*}\Delta_{\Gamma_i^*}H_i^*\cdot H_i^* = -\int_{\Gamma_i^*}|\nabla_{\Gamma_i^*}H_i^*|^2 + \int_{\partial\Gamma_i^*}\left(\nabla_{\Gamma_i^*}H_i^*\cdot n_{\partial\Gamma_i^*}\right)H_i^*\,.$$

Multiplying this equation with $m_i\gamma_i^2 > 0$ and summing gives

$$\begin{aligned} 0 &= -\sum_{i=1}^{3} m_i\gamma_i^2 \int_{\Gamma_i^*} |\nabla_{\Gamma_i^*} H_i^*|^2 + \sum_{i=1}^{3} m_i\gamma_i^2 \int_{L^*} \left(\nabla_{\Gamma_i^*} H_i^* \cdot n_{\partial\Gamma_i^*}\right) H_i^* \\ &\quad + \sum_{i=1}^{3} m_i\gamma_i^2 \int_{S_i^*} \underbrace{\left(\nabla_{\Gamma_i^*} H_i^* \cdot n_{\partial\Gamma_i^*}\right)}_{=0} H_i^* \\ &= -\sum_{i=1}^{3} m_i\gamma_i^2 \int_{\Gamma_i^*} |\nabla_{\Gamma_i^*} H_i^*|^2 + \sum_{i=1}^{3} \int_{L^*} m_i\gamma_i \left(\nabla_{\Gamma_i^*} H_i^* \cdot n_{\partial\Gamma_i^*}\right) \gamma_i H_i^* \\ &= -\sum_{i=1}^{3} m_i\gamma_i^2 \int_{\Gamma_i^*} |\nabla_{\Gamma_i^*} H_i^*|^2 + \int_{L^*} m_1\gamma_1 \left(\nabla_{\Gamma_1^*} H_1^* \cdot n_{\partial\Gamma_1^*}\right) \underbrace{\sum_{i=1}^{3} \gamma_i H_i^*}_{=0} \\ &= -\sum_{i=1}^{3} m_i\gamma_i^2 \int_{\Gamma_i^*} |\nabla_{\Gamma_i^*} H_i^*|^2 . \end{aligned}$$

Therefore we get the equality $\nabla_{\Gamma_i^*} H_i^* = 0$ on Γ_i^* and the same argumentation as in Lemma 3.25 applies to give the claim of constant mean curvature.

To show the identity for the normal curvatures, we use the same notations and calculations as in Lemma 4.4 to get

$$\gamma_i H_i^* = \gamma_i \kappa_{n_{\partial\Gamma_i^*}} + \gamma_i \sum_{j=1}^{n-1} \sigma_i^*(t_j, t_j) .$$

Now we use the second equation $\gamma_1 H_1^* + \gamma_2 H_2^* + \gamma_3 H_3^* = 0$ on L^* from (4.51) for the stationary hypersurfaces to get

$$0 = \sum_{i=1}^{3} \gamma_i \kappa_{n_{\partial\Gamma_i^*}} + \sum_{i=1}^{3} \gamma_i \sum_{j=1}^{n-1} \sigma_i^*(t_j, t_j) .$$

As in the proof of Lemma 4.4 we can show that the second term is zero and therefore get the claim $\gamma_1 \kappa_{n_{\partial\Gamma_1^*}} + \gamma_2 \kappa_{n_{\partial\Gamma_2^*}} + \gamma_3 \kappa_{n_{\partial\Gamma_3^*}} = 0$. □

4.2.2 Parametrization and resulting partial differential equations

To formulate partial differential equations from the geometric evolution equation (4.49)-(4.51), we use the same parametrization of the hypersurfaces $\Gamma_i(t)$ over some fixed stationary state Γ^* of (4.49)-(4.51) as in the previous section. This means that for $i = 1, 2, 3$ and functions

$$\rho_i : [0, T) \times \Gamma_i^* \longrightarrow \mathbb{R} \text{ and}$$
$$\mu_i : [0, T) \times L^* \longrightarrow \mathbb{R}$$

with $|\rho_i| < \varepsilon$ and $|\mu_i| < \delta$, we define the mappings $\Phi_i = \Phi_i^{\rho_i,\mu_i}$ (we often omit the superscript (ρ_i, μ_i) for shortness) through

$$\begin{aligned}
&\Phi_i : [0,T) \times \Gamma_i^* \longrightarrow \Omega\,,\\
&\Phi_i(t,q) := \eta(q)\, Z_i(q, \rho_i(t,q), \mu_i(t, \mathrm{pr}_i(q))) + (1-\eta(q))\, \Psi_i(q, \rho_i(t,q))
\end{aligned}$$

for $i = 1,2,3$, where $\mathrm{pr}_i : \Gamma_i^* \to \partial\Gamma_i^*$ is the projection as in the previous section given through the construction in (4.19). As in the previous Section 4.1 this projection will be used just in a small neighbourhood around the triple line L^*. The mappings Z_i and Ψ_i are defined in (4.15) and (4.16).

We also set for fixed t as in the previous parts the diffeomorphisms onto their images

$$(\Phi_i)_t : \Gamma_i^* \longrightarrow \Omega\,,\ (\Phi_i)_t(q) := \Phi_i(t,q)\,, \tag{4.53}$$

and finally define new surfaces through

$$\Gamma_{\rho_i,\mu_i}(t) \quad := \quad \{(\Phi_i)_t(q)\,|\, q \in \Gamma_i^*\}\,. \tag{4.54}$$

As in the previous section we formulate the condition, that the evolution of the triple line L^* results still in a triple line, through the requirement that

$$\Phi_1(t,q) = \Phi_2(t,q) = \Phi_3(t,q) \qquad \text{for } \ q \in L^*(= L_1^* = L_2^* = L_3^*)\,. \tag{4.55}$$

For the new hypersurfaces $\Gamma_i(t) := \Gamma_{\rho_i,\mu_i}(t)$ there exists also a decomposition of the boundary $\partial\Gamma_i(t)$ through $\partial\Gamma_i(t) = L_i(t) \cup S_i(t)$, where $S_i(t) = \partial\Gamma_i(t) \cap \partial\Omega$ and from (4.55) we can identify the other parts $L_i(t) = \partial\Gamma_i(t)\backslash S_i(t)$ to one compact $(n-1)$-dimensional submanifold $L(t) = L_1(t) = L_2(t) = L_3(t)$.

From now on, we always assume condition (4.55) and write the surface diffusion equation (4.49) and the boundary conditions (4.50) and (4.51) over the fixed stationary hypersurfaces Γ_i^* to get partial differential equations for ρ_i and μ_i, $i = 1,2,3$. This gives for the surface diffusion equations in Γ_i^*

$$V_i(\Phi_i(t,q)) \quad = \quad -m_i\gamma_i\Delta_{\Gamma_i(t)} H_i(\Phi(t,q))\,, \tag{4.56}$$

for the boundary equations on S_i^*

$$\left\{\begin{aligned}
(n_i \cdot \mu)\,(\Phi_i(t,q)) \quad &= \quad 0\,,\\
\nabla_{\Gamma_i(t)} H_i(\Phi_i(t,q)) \cdot n_{\partial\Gamma_i(t)}(\Phi_i(t,q)) \quad &= \quad 0\,,
\end{aligned}\right. \tag{4.57}$$

and for the boundary equations at the triple line L^*

$$\left\{\begin{aligned}
&n_1(\Phi_1(t,q)) \cdot n_2(\Phi_2(t,q)) = \cos\theta_3\,,\\
&n_2(\Phi_2(t,q)) \cdot n_3(\Phi_3(t,q)) = \cos\theta_2\,,\\
&\gamma_1 H_1(\Phi_1(t,q)) + \gamma_2 H_2(\Phi_2(t,q)) + \gamma_3 H_3(\Phi_3(t,q)) = 0\,,\\
&m_1\gamma_1\nabla_{\Gamma_1(t)} H_1(\Phi_1(t,q)) \cdot n_{\partial\Gamma_1(t)}(\Phi_1(t,q))\\
&\ = m_2\gamma_2\nabla_{\Gamma_2(t)} H_2(\Phi_2(t,q)) \cdot n_{\partial\Gamma_2(t)}(\Phi_2(t,q))\\
&\ = m_3\gamma_3\nabla_{\Gamma_3(t)} H_3(\Phi_3(t,q)) \cdot n_{\partial\Gamma_3(t)}(\Phi_3(t,q))\,.
\end{aligned}\right. \tag{4.58}$$

As in the previous section, see (4.27), we also match the starting condition and use the abbreviation $V_i(\Phi_i(t,q)) = V_i(t,\Phi_i(t,q))$, analogously for H_i and n_i as explained in (2.12). We omitted the third angle condition because under our assumptions it is automatically fulfilled, compare (4.28).

4.2.3 Linearization around a stationary state

The next step is to give the linearization of (4.56), (4.57) and (4.58) around $(\rho_i,\mu_i)\equiv(0,0)$, which is our interpretation of the linearization of (4.49)-(4.51) around a stationary state given by Γ_1^*, Γ_2^* and Γ_3^*. Therefore we consider the terms separately and observe that most of the necessary work is already done.

In fact, by putting together the calculations of Lemma 3.4 and of Lemma 4.8 for the normal velocity, we get as expected

$$\frac{d}{d\varepsilon}V_i(\Phi_i^{\varepsilon\rho_i,\varepsilon\mu_i}(t,q))\bigg|_{\varepsilon=0} \;=\; \partial_t\rho_i(t,q)\,.$$

Furthermore the geometric argumentation from Lemma 3.5 and Lemma 3.26 also applies in this case and gives us the linearization for the Laplace-Beltrami operator of mean curvature.

To deal with the linearization of the boundary conditions in (4.57), we just have to observe that on S_i^* the parametrization fulfills

$$\Phi_i^{\rho_i,\mu_i}(t,q) = \Psi_i(t,\rho_i(t,q)) \tag{4.59}$$

and equals therefore the curvilinear coordinate system from Chapter 3. Therefore it is possible to apply Lemma 3.7 and Lemma 3.27, which gives us the desired linearization.

In fact, for Lemma 3.27 we do not need the identity (4.59), the calculations there hold also for the parametrization in this section. We just have to replace the outer unit normal μ of the fixed region Ω with the outer unit conormal $n_{\partial\Gamma_i^*}$ of Γ_i^*. These two were the same in that case due to the right angle between Γ_i^* and the outer fixed boundary. So we get the linearization of the third condition in (4.58). Together with Lemma 4.11 about the linearization of the angle conditions at the triple line and with Lemma 4.8 for the mean curvature we get the linearization of all terms in the boundary conditions of (4.58) at the triple line L^*.

Altogether we obtain the linearized problem for $i=1,2,3$ and $t>0$

$$\partial_t\rho_i = -m_i\gamma_i\Delta_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right) \qquad \text{in } \Gamma_i^* \tag{4.60}$$

with the boundary conditions on S_i^*

$$\left\{\begin{array}{rcl} \left(\partial_\mu - S(n_i^*,n_i^*)\right)\rho_i &=& 0\,,\\ \partial_\mu\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right) &=& 0\,, \end{array}\right. \tag{4.61}$$

and the boundary conditions on the triple line L^*

$$\left\{\begin{array}{l} \gamma_1\rho_1+\gamma_2\rho_2+\gamma_3\rho_3 = 0\,,\\ \partial_{n_{\partial\Gamma_1^*}}\rho_1 + a_1\rho_1 = \partial_{n_{\partial\Gamma_2^*}}\rho_2 + a_2\,\rho_2 = \partial_{n_{\partial\Gamma_3^*}}\rho_3 + a_3\,\rho_3\,,\\ \gamma_1\left(\Delta_{\Gamma_1^*}\rho_1 + |\sigma_1^*|^2\rho_1\right) + \gamma_2\left(\Delta_{\Gamma_2^*}\rho_2 + |\sigma_2^*|^2\rho_2\right) + \gamma_3\left(\Delta_{\Gamma_3^*}\rho_3 + |\sigma_3^*|^2\rho_3\right) = 0\,,\\ m_1\gamma_1\partial_{n_{\partial\Gamma_1^*}}\left(\Delta_{\Gamma_1^*}\rho_1 + |\sigma_1^*|^2\rho_1\right) = m_2\gamma_2\partial_{n_{\partial\Gamma_2^*}}\left(\Delta_{\Gamma_2^*}\rho_2 + |\sigma_2^*|^2\rho_2\right)\\ \quad = m_3\gamma_3\partial_{n_{\partial\Gamma_3^*}}\left(\Delta_{\Gamma_3^*}\rho_3 + |\sigma_3^*|^2\rho_3\right)\,. \end{array}\right. \tag{4.62}$$

We recall some notations. σ_i^* is the second fundamental form of Γ_i^* with respect to n_i^* and $|\sigma_i^*|^2 = \sum_{i=1}^n \kappa_i^2$ is its squared norm given through the squared sum of the principal curvatures. μ is the outer unit normal of Ω and S is the second fundamental form of $\partial\Omega$ with respect to the inwards pointing normal $(-\mu)$ of Ω. Due to the right angle condition at the outer fixed boundary the normal n_i^* of Γ_i^* lies in the tangent space of $\partial\Omega$ and can therefore be inserted in S. Finally, the terms a_i are the abbreviations given by (4.35), (4.36) and (4.37).

Remark 4.24. *The above system of partial differential equations is just a problem for ρ_i. The functions μ_i are then given by the linear dependence from Lemma 4.5 through*

$$\mu_i \;=\; \frac{1}{s_i}\left(c_j\rho_j - c_k\rho_k\right) \quad on \;\; L^*\,.$$

We remark that this is the case because we linearize around a stationary solution and would not be true any more if the linearization is around an arbitrary hypersurface satisfying the boundary conditions.

As in Section 3.4 for surface diffusion with boundary contact for one hypersurface without triple junction there holds a solvability condition.

Lemma 4.25. *Solutions of the linearized problem (4.60) with boundary conditions (4.61) and (4.62) fulfill*

$$\int_{\Gamma_1^*}\rho_1 = \int_{\Gamma_2^*}\rho_2 = \int_{\Gamma_3^*}\rho_3\,.$$

Proof. Observe that through a linearization around $(\rho_i,\mu_i)\equiv(0,0)$, the starting condition for the linearized problem, that we omitted for reasons of shortness, is $(\rho_i)_{t=0}=0$. So we can do the analogue calculation as in Lemma 3.28 to get

$$\begin{aligned}
\int_{\Gamma_i^*}\rho_i &= \int_0^t\int_{\Gamma_i^*}\partial_t\rho_i = \int_0^t\int_{\Gamma_i^*} -m_i\gamma_i\Delta_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right)\\
&= \int_0^t\int_{\partial\Gamma_i^*} -m_i\gamma_i\left(\nabla_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\rho_i+|\sigma_i^*|^2\rho_i\right)\cdot n_{\partial\Gamma_i^*}\right)\\
&= \int_0^t\int_{S_i^*} -m_i\gamma_i\underbrace{\left(\nabla_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\rho_i+|\sigma_i^*|^2\rho_i\right)\cdot\mu\right)}_{=0 \text{ due to } (4.61)} + \int_0^t\int_{L^*} -m_i\gamma_i\left(\nabla_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\rho_i+|\sigma_i^*|^2\rho_i\right)\cdot n_{\partial\Gamma_i^*}\right)\\
&= \int_0^t\int_{L^*} -m_1\gamma_1\left(\nabla_{\Gamma_1^*}\left(\Delta_{\Gamma_1^*}\rho_1+|\sigma_1^*|^2\rho_1\right)\cdot n_{\partial\Gamma_1^*}\right)\,,
\end{aligned}$$

where the last identity follows with the help of the boundary condition (4.62). This gives the same value for $\int_{\Gamma_i^*}\rho_i$ for all $i=1,2,3$ and the claim is shown. □

4.2.4 Conditions for linearized stability

As in Section 3.4 for one hypersurface we want to describe linearized stability of the stationary solution Γ^* of (4.49)-(4.51), i.e. with our parametrization stability of the zero solution of the linearized problem (4.60)-(4.62) through the requirement that some bilinear form is positive. Garcke, Ito and Kohsaka [GIK10] generalized their own paper [GIK05] for surface diffusion flow for one curve to the case of three curves meeting at a triple point. In an analogue way we will generalize here the case of surface diffusion flow for one hypersurface from Section 3.4 to the case of three hypersurfaces meeting at a triple line.

To recall the approach, we will show that the linearized problem (4.60) - (4.62) can be interpreted as a gradient flow with respect to an energy E given by a symmetric bilinear form I. Then we relate the eigenvalues of the solution operator corresponding to the linearized problem to the positivity of the bilinear form I to achieve the following result for a stationary solution $\Gamma^* = \bigcup_{i=1}^{3} \Gamma_i^*$ of problem (4.60)-(4.62).

$$\Gamma^* \text{ is linearly asymptotically stable}$$
$$\Longleftrightarrow \left\{ \begin{array}{lcl} I(\rho,\rho) & := & \sum_{i=1}^{3} \gamma_i \int_{\Gamma_i^*} \left(|\nabla_{\Gamma_i^*} \rho_i|^2 - |\sigma_i^*|^2 \rho_i^2 \right) - \sum_{i=1}^{3} \gamma_i \int_{S_i^*} S(n_i^*, n_i^*)\, \rho_i^2 \\ & & + \sum_{i=1}^{3} \gamma_i \int_{L^*} a_i\, \rho_i^2 \\ \multicolumn{3}{l}{\text{is positive for all } 0 \neq \rho = (\rho_1, \rho_2, \rho_3) \text{ with } \rho_i \in H^1(\Gamma_i^*) \text{ such that}} \\ \multicolumn{3}{l}{\int_{\Gamma_1^*} \rho_1 = \int_{\Gamma_2^*} \rho_2 = \int_{\Gamma_3^*} \rho_3 \text{ and } \gamma_1\rho_1 + \gamma_2\rho_2 + \gamma_3\rho_3 = 0 \text{ on } L^*.} \end{array} \right.$$

The following abbreviations for function spaces resp. dual spaces will be useful. For $k \in \mathbb{N}$, we set (omit the integrability value $p = 2$)

$$\begin{aligned} \mathcal{H}^k &:= H^k(\Gamma_1^*) \times H^k(\Gamma_2^*) \times H^k(\Gamma_3^*)\,, \\ \left(\mathcal{H}^k\right)' &:= \left(H^k(\Gamma_1^*)\right)' \times \left(H^k(\Gamma_2^*)\right)' \times \left(H^k(\Gamma_3^*)\right)'\,, \\ \mathcal{Y} &:= \left\{ (\xi_1, \xi_2, \xi_3) \in \mathcal{H}^1 \mid \xi_1 + \xi_2 + \xi_3 = 0 \text{ on } L^* \text{ and } \int_{\Gamma_1^*} \xi_1 = \int_{\Gamma_2^*} \xi_2 = \int_{\Gamma_3^*} \xi_3 \right\}, \\ \widetilde{\mathcal{Y}} &:= \left\{ (\xi_1, \xi_2, \xi_3) \in \mathcal{H}^1 \mid \xi_1 + \xi_2 + \xi_3 = 0 \text{ on } L^* \right\}, \\ \mathcal{E} &:= \left\{ (v_1, v_2, v_3) \in \mathcal{H}^1 \mid \gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3 = 0 \text{ on } L^* \text{ and } \int_{\Gamma_1^*} v_1 = \int_{\Gamma_2^*} v_2 = \int_{\Gamma_3^*} v_3 \right\}, \\ \mathcal{H}^{-1} &:= \left\{ (w_1, w_2, w_3) \in \left(\mathcal{H}^1\right)' \mid \langle w_1, 1\rangle = \langle w_2, 1\rangle = \langle w_3, 1\rangle \right\}. \end{aligned}$$

Here $\langle . , . \rangle$ is the duality pairing between the dual space $\left(H^1(\Gamma_i^*)\right)'$ and the Sobolev space $H^1(\Gamma_i^*)$. We will also denote the duality pairing between $w = (w_1, w_2, w_3) \in \mathcal{H}^{-1}$ and $u = (u_1, u_2, u_3) \in \mathcal{H}^1$ with the same symbol, i.e.

$$\langle w, u\rangle = \langle w_1, u_2\rangle + \langle w_2, u_2\rangle + \langle w_3, u_3\rangle\ .$$

We will show that the linearized problem (4.60) - (4.62) is a gradient flow with respect to the $\mathcal{H}^{-1}$ inner product. Therefore we have to generalize Definition 3.29 from Section 3.4 to the present case of three hypersurfaces.

Definition 4.26. *We say that $u^w = (u_1^w, u_2^w, u_3^w) \in \mathcal{Y}$ for a given $w = (w_1, w_2, w_3) \in \mathcal{H}^{-1}$ is a weak solution of*

$$\begin{cases} -m_i \Delta_{\Gamma_i^*} u_i^w = w_i & \text{in } \Gamma_i^* \quad (i = 1, 2, 3)\,, \\ u_1^w + u_2^w + u_3^w = 0 & \text{on } L^*\,, \\ m_1 \nabla_{\Gamma_1^*} u_1^w \cdot n_{\partial\Gamma_1^*} = m_2 \nabla_{\Gamma_2^*} u_2^w \cdot n_{\partial\Gamma_2^*} = m_3 \nabla_{\Gamma_3^*} u_3^w \cdot n_{\partial\Gamma_3^*} & \text{on } L^*\,, \\ \nabla_{\Gamma_i^*} u_i^w \cdot n_{\partial\Gamma_i^*} = 0 & \text{on } S_i^* \quad (i = 1, 2, 3)\,, \end{cases} \tag{4.63}$$

if and only if $u^w \in \mathcal{Y}$ satisfies

$$\langle w, \xi \rangle \;=\; \sum_{i=1}^{3} m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} u_i^w \cdot \nabla_{\Gamma_i^*} \xi_i \tag{4.64}$$

for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathcal{Y}$.

For later use we show in the next lemma that the above weak formulation (4.64) can also be written with the help of testfunctions from the larger space $\widetilde{\mathcal{Y}}$ instead of $\mathcal{Y}$.

Lemma 4.27. *Equation (4.64) can be written equivalently with testfunctions $\xi \in \widetilde{\mathcal{Y}}$ instead of $\mathcal{Y}$. In detail this means for $w \in \mathcal{H}^{-1}$ and $u^w \in \mathcal{Y}$ the equivalence between the following two equations*

(i) $\langle w, \xi \rangle = \sum_{i=1}^{3} m_i \nabla_{\Gamma_i^*} u_i^w \cdot \nabla_{\Gamma_i^*} \xi_i$ *for all $\xi \in \mathcal{Y}$ and*

(ii) $\left\langle w, \widetilde{\xi} \right\rangle = \sum_{i=1}^{3} m_i \nabla_{\Gamma_i^*} u_i^w \cdot \nabla_{\Gamma_i^*} \widetilde{\xi}_i$ *for all $\widetilde{\xi} \in \widetilde{\mathcal{Y}}$.*

Proof. The inclusion $\mathcal{Y} \subset \widetilde{\mathcal{Y}}$ leads to the implication $(ii) \Rightarrow (i)$.
For the other implication let $\widetilde{\xi} = (\widetilde{\xi}_1, \widetilde{\xi}_2, \widetilde{\xi}_3) \in \widetilde{\mathcal{Y}}$ be given, i.e. $\widetilde{\xi}_i \in H^1(\Gamma_i^*)$ and $\widetilde{\xi}_1 + \widetilde{\xi}_2 + \widetilde{\xi}_3 = 0$ on L^*. We want to find constants (c_1, c_2, c_3), such that

$$\xi := (\widetilde{\xi} - c) := (\widetilde{\xi}_1 - c_1, \widetilde{\xi}_2 - c_2, \widetilde{\xi}_3 - c_3) \quad \in \quad \mathcal{Y}\,.$$

This means, we have to find constants $c = (c_1, c_2, c_3)$ such that

$$c_1 + c_2 + c_3 = 0 \quad \text{ and }$$
$$\int_{\Gamma_1^*} \left(\widetilde{\xi}_1 - c_1\right) = \int_{\Gamma_2^*} \left(\widetilde{\xi}_2 - c_2\right) = \int_{\Gamma_3^*} \left(\widetilde{\xi}_3 - c_3\right)\,.$$

We formulate these conditions as a linear system of three equations for the unknowns (c_1, c_2, c_3) and observe that the corresponding matrix

$$M \;:=\; \begin{pmatrix} 1 & 1 & 1 \\ -|\Gamma_1^*| & |\Gamma_2^*| & 0 \\ 0 & -|\Gamma_2^*| & |\Gamma_3^*| \end{pmatrix}$$

is invertible due to

$$\det M = |\Gamma_2^*| \cdot |\Gamma_3^*| + |\Gamma_1^*| \cdot |\Gamma_2^*| + |\Gamma_1^*| \cdot |\Gamma_3^*| \quad > \quad 0\,.$$

Therefore we can find c with the above properties and $\xi = \widetilde{\xi} - c$ fulfills $\xi \in \mathcal{Y}$ and can be used as a testfunction in (i) to get

$$\left\langle w, \widetilde{\xi} - c \right\rangle \quad = \quad \sum_{i=1}^{3} m_i \nabla_{\Gamma_i^*} u_i^w \cdot \nabla_{\Gamma_i^*} \widetilde{\xi}_i\,,$$

where the constant on the right side has vanished. Due to $\langle w_1, 1\rangle = \langle w_2, 1\rangle = \langle w_3, 1\rangle$ the left side can be written as

$$\left\langle w, \widetilde{\xi} - c \right\rangle = \left\langle w, \widetilde{\xi} \right\rangle - \sum_{i=1}^{3} \langle w_i, c_i \rangle = \left\langle w, \widetilde{\xi} \right\rangle - \langle w_1, 1\rangle \underbrace{\sum_{i=1}^{3} c_i}_{=0} = \left\langle w, \widetilde{\xi} \right\rangle$$

and we proved (ii). $\square$

Since the problem (4.63) is a bit unusual due to the different domains of definition Γ_i^*, we want to show equivalence of strong and weak solutions in the smooth case.

Lemma 4.28. *Let $w \in \mathcal{H}^{-1}$ be smooth, so that we can assume $\langle w, \xi \rangle = \sum_{i=1}^{3} \int_{\Gamma_i^*} w_i\, \xi_i$ for the duality pairing. Then $u^w \in \mathcal{Y}$ is a smooth solution of (4.63) then and only then when $u^w \in \mathcal{Y}$ is smooth and fulfills (4.64).*

Proof. Let $u^w \in \mathcal{Y}$ be a smooth solution of (4.63). By testing with $\xi \in \mathcal{Y}$, we get with the help of integration by parts

$$\begin{aligned}
\langle w, \xi \rangle \quad &= \quad \sum_{i=1}^{3} \int_{\Gamma_i^*} w_i\, \xi_i \\
&= \quad \sum_{i=1}^{3} \int_{\Gamma_i^*} (-m_i \Delta_{\Gamma_i^*} u_i^w)\, \xi_i \\
&= \quad \sum_{i=1}^{3} m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} u_i^w \cdot \nabla_{\Gamma_i^*} \xi_i - \sum_{i=1}^{3} m_i \int_{S_i^*} \underbrace{\left(\nabla_{\Gamma_i^*} u_i^w \cdot n_{\partial \Gamma_i^*}\right)}_{=0} \xi_i \\
&\quad\; - \sum_{i=1}^{3} m_i \int_{L^*} \left(\nabla_{\Gamma_i^*} u_i^w \cdot n_{\partial \Gamma_i^*}\right) \xi_i \\
&= \quad \sum_{i=1}^{3} m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} u_i^w \cdot \nabla_{\Gamma_i^*} \xi_i - \int_{L^*} m_1 \left(\nabla_{\Gamma_1^*} u_1^w \cdot n_{\partial \Gamma_1^*}\right) \underbrace{\sum_{i=1}^{3} \xi_i}_{=0} \\
&= \quad \sum_{i=1}^{3} m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} u_i^w \cdot \nabla_{\Gamma_i^*} \xi_i\,.
\end{aligned}$$

Conversely, let $u^w \in \mathcal{Y}$ be smooth and fulfill (4.64) for testfunctions $\xi \in \widetilde{\mathcal{Y}}$, which is possible due to Lemma 4.27. Integration by parts gives then

$$\begin{aligned}
\sum_{i=1}^{3} \int_{\Gamma_i^*} w_i\, \xi_i &= \sum_{i=1}^{3} m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} u_i^w \cdot \nabla_{\Gamma_i^*} \xi_i \\
&= -\sum_{i=1}^{3} m_i \int_{\Gamma_i^*} \Delta_{\Gamma_i^*} u_i^w\, \xi_i + \sum_{i=1}^{3} m_i \int_{L^*} \left(\nabla_{\Gamma_i^*} u_i^w \cdot n_{\partial\Gamma_i^*}\right) \xi_i \\
&\quad + \sum_{i=1}^{3} m_i \int_{S_i^*} \left(\nabla_{\Gamma_i^*} u_i^w \cdot n_{\partial\Gamma_i^*}\right) \xi_i \,.
\end{aligned}$$

Therefore it holds

$$\begin{aligned}
0 &= \sum_{i=1}^{3} \int_{\Gamma_i^*} \left(w_i + m_i \Delta_{\Gamma_i^*} u_i^w\right) \xi_i \\
&\quad + \sum_{i=1}^{3} \int_{L^*} m_i \left(\nabla_{\Gamma_i^*} u_i^w \cdot n_{\partial\Gamma_i^*}\right) \xi_i + \sum_{i=1}^{3} \int_{S_i^*} m_i \left(\nabla_{\Gamma_i^*} u_i^w \cdot n_{\partial\Gamma_i^*}\right) \xi_i
\end{aligned}$$

for all $\xi_i \in H^1(\Gamma_i^*)$ with $\xi_1 + \xi_2 + \xi_3 = 0$ on L^*.
By setting two of the ξ_i constantly equal to zero and using zero boundary conditions for the remaining one, we get with the help of the fundamental lemma $w_i = -m_i \Delta_{\Gamma_i^*} u_i^w$ on Γ_i^*. Since ξ_i is arbitrary at S_i^*, we also get the boundary condition $\nabla_{\Gamma_i^*} u_i^w \cdot n_{\partial\Gamma_i^*} = 0$ at S_i^*, remaining with the identity

$$0 = \sum_{i=1}^{3} \int_{L^*} m_i \left(\nabla_{\Gamma_i^*} u_i^w \cdot n_{\partial\Gamma_i^*}\right) \xi_i \,.$$

Here we use $\xi_1 + \xi_2 + \xi_3 = 0$ at L^* to get

$$m_1 \nabla_{\Gamma_1^*} u_1^w \cdot n_{\partial\Gamma_1^*} = m_2 \nabla_{\Gamma_2^*} u_2^w \cdot n_{\partial\Gamma_2^*} = m_3 \nabla_{\Gamma_3^*} u_3^w \cdot n_{\partial\Gamma_3^*} \text{ at } L^* \,.$$

Altogether we showed that u^w is a strong solution of (4.63). □

The next step is to show a Poincaré-type inequality for functions in $\mathcal{E}$ resp. in $\mathcal{Y}$. Therefore we use the notation for $\rho = (\rho_1, \rho_2, \rho_3)$

$$\|\rho\| := \left(\sum_{i=1}^{3} \|\rho_i\|^2_{L^2(\Gamma_i^*)}\right)^{1/2} \quad \text{and} \quad \|\nabla_{\Gamma^*} \rho\| := \left(\sum_{i=1}^{3} \|\nabla_{\Gamma_i^*} \rho_i\|^2_{L^2(\Gamma_i^*)}\right)^{1/2} . \tag{4.65}$$

Lemma 4.29. *There exists a constant $C > 0$, such that*

$$\|\rho\| \le C \, \|\nabla_{\Gamma^*} \rho\|$$

holds for all $\rho = (\rho_1, \rho_2, \rho_3) \in \mathcal{E}$.
The statement is also true for functions $\rho = (\rho_1, \rho_2, \rho_3) \in \mathcal{Y}$.

Proof. We argument by contradiction and assume that we can find a sequence $(\tilde{\rho}^n)_{n\in\mathbb{N}} \in \mathcal{E}$, such that

$$\|\tilde{\rho}^n\| > n\,\|\nabla_{\Gamma^*}\tilde{\rho}^n\|\,.$$

In particular, this gives $\|\tilde{\rho}^n\| > 0$ and normalizing $\rho^n := \frac{\tilde{\rho}^n}{\|\tilde{\rho}^n\|}$ leads to a sequence $\rho^n \in \mathcal{E}$ with $\|\rho^n\| = 1$ and

$$1 > n\,\|\nabla_{\Gamma^*}\rho^n\|\,.$$

For the components, we get the bound

$$\|\rho_i^n\|_{L^2(\Gamma_i^*)} \le \sum_{j=1}^{3}\|\rho_j^n\|_{L^2(\Gamma_j^*)} \le \sqrt{3}\left(\sum_{j=1}^{3}\|\rho_j^n\|_{L^2(\Gamma_j^*)}^2\right)^{\frac{1}{2}} = \sqrt{3}\|\rho^n\| = \sqrt{3}\,.$$

For the surface gradient of the components, we observe the convergence

$$\|\nabla_{\Gamma_i^*}\rho_i^n\|_{L^2(\Gamma_i^*)} \le \sqrt{3}\,\|\nabla_{\Gamma^*}\rho^n\| \le \frac{\sqrt{3}}{n} \longrightarrow 0 \quad \text{for } n\to\infty\,.$$

Therefore, we can deduce the weak convergence $\rho_i^n \rightharpoonup C_i$ in $H^1(\Gamma_i^*)$ for constants $C_i \in \mathbb{R}$. The Rellich embedding theorem gives

$$\rho_i^n \longrightarrow C_i \text{ in } L^2(\Gamma_i^*) \quad \text{for } n\to\infty\,.$$

Furthermore, the integral condition $\int_{\Gamma_1^*}\rho_1 = \int_{\Gamma_2^*}\rho_2 = \int_{\Gamma_3^*}\rho_3$ leads to $|\Gamma_1^*|\cdot C_1 = |\Gamma_2^*|\cdot C_2 = |\Gamma_3^*|\cdot C_3$, so that we can conclude that the constants C_i all have the same sign.
Finally, the boundary condition $\gamma_1\rho_1^n + \gamma_2\rho_2^n + \gamma_3\rho_3^n = 0$ on L^* gives $\gamma_1 C_1 + \gamma_2 C_2 + \gamma_3 C_3 = 0$ and therefore $C_1 = C_2 = C_3 = 0$. More precisely, we have to use the compact embedding $H^1(\Gamma_i^*) \hookrightarrow L^2(\partial\Gamma_i^*)$ here.
But this is a contradiction to $\|\rho^n\| = 1$ for all $n \in \mathbb{N}$. ☐

Now we can show unique existence of a weak solution from problem (4.63).

Lemma 4.30. *For each $w \in \mathcal{H}^{-1}$, there exists a unique weak solution $u^w \in \mathcal{Y}$ of problem (4.63).*

Proof. We set

$$B(u,\xi) := \sum_{i=1}^{3} m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*}u_i \cdot \nabla_{\Gamma_i^*}\xi_i$$

for $u, \xi \in \mathcal{Y}$. Since from Lemma 4.29 a Poincaré-type inequality holds for all $u \in \mathcal{Y}$, the bilinear form B is continuous and coercive on $\mathcal{Y}$. From the Lax-Milgram theorem we get then for a given $w \in \mathcal{H}^{-1}$ the existence of a unique $u^w \in \mathcal{Y}$ such that

$$B(u^w,\xi) = \langle w,\xi\rangle\ .$$

This means that $u^w \in \mathcal{Y}$ is a weak solution of (4.63). $\square$

Now we are able to define the $\mathcal{H}^{-1}$-inner product.

Definition 4.31. *For $v, w \in \mathcal{H}^{-1}$ we define the inner product*

$$(v,w)_{-1} \quad := \quad \sum_{i=1}^{3} m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} u_i^v \cdot \nabla_{\Gamma_i^*} u_i^w \,,$$

where $u^v = (u_1^v, u_2^v, u_3^v)$, $u^w = (u_1^w, u_2^w, u_3^w) \in \mathcal{Y}$ are the weak solutions of (4.63) for given $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3) \in \mathcal{H}^{-1}$.

We also define the corresponding norm

$$\|v\|_{-1} \quad := \quad \sqrt{(v,v)_{-1}}$$

for $v \in \mathcal{H}^{-1}$ and remark that the identity

$$(v,w)_{-1} \quad = \quad \langle v, u^w \rangle$$

holds for all $u, w \in \mathcal{H}^{-1}$.

In an analogous manner as in the previous chapter we define a symmetric bilinear form and an energy on $\mathcal{H}^1$.

Definition 4.32. *For $\rho = (\rho_1, \rho_2, \rho_3)$ and $\eta = (\eta_1, \eta_2, \eta_3)$ in $\mathcal{H}^1$ we define*

$$\begin{aligned} I(\rho,\eta) \quad := \quad & \sum_{i=1}^{3} \gamma_i \int_{\Gamma_i^*} \left(\nabla_{\Gamma_i^*} \rho_i \, \nabla_{\Gamma_i^*} \eta_i - |\sigma_i^*|^2 \rho_i \, \eta_i \right) \mathrm{d}\mathcal{H}^n - \sum_{i=1}^{3} \int_{S_i^*} \gamma_i S(n_i^*, n_i^*) \rho_i \, \eta_i \, \mathrm{d}\mathcal{H}^{n-1} \\ & + \sum_{i=1}^{3} \int_{L^*} \gamma_i a_i \rho_i \, \eta_i \, \mathrm{d}\mathcal{H}^{n-1} \end{aligned}$$

and the associated energy for $\rho \in \mathcal{H}^1$ by

$$E(\rho) \quad := \quad \frac{1}{2} I(\rho,\rho)\,.$$

We remind that a_i are the abbreviations from (4.35)-(4.37).

Now we want to show that the linearized problem (4.60) - (4.62) is the gradient flow of E with respect to the $\mathcal{H}^{-1}$ inner product $(.\,,.)_{-1}$. Analogously as in the previous chapter this means that a solution ρ of the linearized problem fulfills

$$(\partial_t \rho, \xi)_{-1} \quad = \quad -I(\rho,\xi)$$

for all $\xi \in \mathcal{E}$.

We introduce the following time independent problem.

Definition 4.33. *For a given $w = (w_1, w_2, w_3) \in \mathcal{H}^{-1}$ we say that $\rho = (\rho_1, \rho_2, \rho_3) \in \mathcal{H}^3$ with $\int_{\Gamma_1^*} \rho_1 = \int_{\Gamma_2^*} \rho_2 = \int_{\Gamma_3^*} \rho_3$ is a weak solution of the boundary value problem*

$$w_i \quad = \quad -m_i \gamma_i \Delta_{\Gamma_i^*} \left(\Delta_{\Gamma_i^*} \rho_i + |\sigma_i^*|^2 \rho_i \right) \quad \text{in } \Gamma_i^*, \tag{4.66}$$

with the boundary conditions (4.61) on S_i^* and the boundary conditions (4.62) on the triple line L^*, if and only if ρ satisfies

$$\langle w, \xi \rangle \quad = \quad \sum_{i=1}^{3} m_i \gamma_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} \left(\Delta_{\Gamma_i^*} \rho_i + |\sigma_i^*|^2 \rho_i \right) \cdot \nabla_{\Gamma_i^*} \xi_i \tag{4.67}$$

for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathcal{Y}$ and fulfills the boundary conditions

$$\left(\partial_\mu - S(n_i^*, n_i^*) \right) \rho_i \quad = \quad 0 \tag{4.68}$$

on S_i^* and

$$\left\{ \begin{array}{l} \gamma_1 \rho_1 + \gamma_2 \rho_2 + \gamma_3 \rho_3 = 0 \,, \\ \partial_{n_{\partial \Gamma_1^*}} \rho_1 + a_1 \rho_1 = \partial_{n_{\partial \Gamma_2^*}} \rho_2 + a_2 \, \rho_2 = \partial_{n_{\partial \Gamma_3^*}} \rho_3 + a_3 \, \rho_3 \,, \\ \gamma_1 \left(\Delta_{\Gamma_1^*} \rho_1 + |\sigma_1^*|^2 \rho_1 \right) + \gamma_2 \left(\Delta_{\Gamma_2^*} \rho_2 + |\sigma_2^*|^2 \rho_2 \right) + \gamma_3 \left(\Delta_{\Gamma_3^*} \rho_3 + |\sigma_3^*|^2 \rho_3 \right) = 0 \end{array} \right. \tag{4.69}$$

on the triple line L^*.

The next lemma shows the above claim regarding the gradient flow structure.

Lemma 4.34. *Let $w = (w_1, w_2, w_3) \in \mathcal{H}^{-1}$ and $\rho = (\rho_1, \rho_2, \rho_3) \in \mathcal{E}$ be given. Then ρ is a weak solution of (4.66) if and only if*

$$(w, \xi)_{-1} \quad = \quad -I(\rho, \xi)$$

for all $\xi \in \mathcal{E}$.

Proof. Let $\rho \in \mathcal{E}$ be a weak solution of (4.66). Due to $\xi \in \mathcal{E} \subset \mathcal{H}^{-1}$ through $\langle \xi, u \rangle = \sum_{i=1}^{3} \int_{\Gamma_i^*} \xi_i \, u_i$ for $u \in \mathcal{H}^1$ we get from Definition 4.31

$$(w, \xi)_{-1} \quad = \quad \left\langle w, u^\xi \right\rangle \,.$$

Using $u^\xi \in \mathcal{Y}$ as a testfunction in the weak formulation of (4.66), we observe

$$\left\langle w, u^\xi \right\rangle = \sum_{i=1}^{3} m_i \gamma_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} \left(\Delta_{\Gamma_i^*} \rho_i + |\sigma_i^*|^2 \rho_i \right) \cdot \nabla_{\Gamma_i^*} u_i^\xi = \sum_{i=1}^{3} m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} \Theta_i \cdot \nabla_{\Gamma_i^*} u_i^\xi \,,$$

where we defined for shortness $\Theta_i = \gamma_i \left(\Delta_{\Gamma_i^*} \rho_i + |\sigma_i^*|^2 \rho_i \right)$. The third boundary condition on L^* from problem (4.66) yields $\Theta_1 + \Theta_2 + \Theta_3 = 0$ on L^*. Due to Lemma 4.27 we can use $\Theta = (\Theta_1, \Theta_2, \Theta_3)$ as a testfunction in (4.64) to get

$$\sum_{i=1}^{3} \int_{\Gamma_i^*} \xi_i \cdot \Theta_i \quad = \quad \sum_{i=1}^{3} m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} \Theta_i \cdot \nabla_{\Gamma_i^*} u_i^\xi \,.$$

Here we used the inclusion $\xi \in \mathcal{E} \subset \mathcal{H}^{-1}$ through $\langle \xi, \Theta \rangle = \sum_{i=1}^{3} \int_{\Gamma_i^*} \xi_i \, \Theta_i$.
Thus we can conclude with integration by parts

$$
\begin{aligned}
(w,\xi)_{-1} &= \sum_{i=1}^{3} \int_{\Gamma_i^*} \xi_i \, \gamma_i \left(\Delta_{\Gamma_i^*} \rho_i + |\sigma_i^*|^2 \rho_i \right) \\
&= -\sum_{i=1}^{3} \gamma_i \int_{\Gamma_i^*} \left(\nabla_{\Gamma_i^*} \xi_i \cdot \nabla_{\Gamma_i^*} \rho_i - |\sigma_i^*|^2 \xi_i \, \rho_i \right) + \sum_{i=1}^{3} \gamma_i \int_{\partial \Gamma_i^*} \xi_i \, \left(\nabla_{\Gamma_i^*} \rho_i \cdot n_{\partial \Gamma_i^*} \right) \\
&= -\sum_{i=1}^{3} \gamma_i \int_{\Gamma_i^*} \left(\nabla_{\Gamma_i^*} \xi_i \cdot \nabla_{\Gamma_i^*} \rho_i - |\sigma_i^*|^2 \xi_i \, \rho_i \right) + \sum_{i=1}^{3} \gamma_i \int_{L^*} \xi_i \, \partial_{n_{\partial \Gamma_i^*}} \rho_i \\
&\quad + \sum_{i=1}^{3} \gamma_i \int_{S_i^*} \xi_i \, \partial_\mu \rho_i \, .
\end{aligned}
$$

Using $\gamma_1 \xi_1 + \gamma_2 \xi_2 + \gamma_3 \xi_3 = 0$ at L^* for $\xi \in \mathcal{E}$ and the third boundary condition on L^* for the weak solution ρ of (4.66), we get

$$
\begin{aligned}
\sum_{i=1}^{3} \gamma_i \int_{L^*} \xi_i \, \partial_{n_{\partial \Gamma_i^*}} \rho_i &= \sum_{i=1}^{3} \gamma_i \int_{L^*} \xi_i \, \Big(\partial_{n_{\partial \Gamma_i^*}} \rho_i + a_i \rho_i \Big) - \sum_{i=1}^{3} \gamma_i \int_{L^*} a_i \, \xi_i \, \rho_i \\
&= \int_{L^*} \Big(\partial_{n_{\partial \Gamma_1^*}} \rho_1 + a_1 \rho_1 \Big) \underbrace{\sum_{i=1}^{3} \gamma_i \xi_i}_{=0} - \sum_{i=1}^{3} \gamma_i \int_{L^*} a_i \, \xi_i \, \rho_i \\
&= -\sum_{i=1}^{3} \gamma_i \int_{L^*} a_i \, \xi_i \, \rho_i \, .
\end{aligned}
$$

From the first boundary condition on S_i^* for the weak solution ρ of (4.66) we get

$$
\sum_{i=1}^{3} \gamma_i \int_{S_i^*} \xi_i \cdot \partial_\mu \rho_i = \sum_{i=1}^{3} \gamma_i \int_{S_i^*} \xi_i \cdot S(n_i^*, n_i^*) \rho_i \, .
$$

Altogether, we arrive at

$$
(w,\xi)_{-1} = -I(\rho, \xi)
$$

for all $\xi \in \mathcal{E}$.

Conversely, assume that $\rho \in \mathcal{E}$ satisfies $(w,\xi)_{-1} = -I(\rho,\xi)$ for all $\xi \in \mathcal{E}$. Now let $\zeta \in \mathcal{H}^3 \cap \mathcal{Y}$ be a given function with

$$
m_1 \left(\nabla_{\Gamma_1^*} \zeta_1 \cdot n_{\partial \Gamma_1^*} \right) = m_2 \left(\nabla_{\Gamma_2^*} \zeta_2 \cdot n_{\partial \Gamma_2^*} \right) = m_3 \left(\nabla_{\Gamma_3^*} \zeta_3 \cdot n_{\partial \Gamma_3^*} \right) \quad \text{on } L^* \, , \tag{4.70}
$$

$$
\left(\nabla_{\Gamma_i^*} \zeta_i \cdot n_{\partial \Gamma_i^*} \right) = 0 \quad \text{on } S_i^* \text{ and} \tag{4.71}
$$

$$
\gamma_1 m_1 \Delta_{\Gamma_1^*} \zeta_1 + \gamma_2 m_2 \Delta_{\Gamma_2^*} \zeta_2 + \gamma_3 m_3 \Delta_{\Gamma_3^*} \zeta_3 = 0 \quad \text{on } L^* \, . \tag{4.72}
$$

With the help the abbreviation $m\Delta_{\Gamma^*}\zeta = \big(m_1\Delta_{\Gamma_1^*}\zeta_1, m_2\Delta_{\Gamma_2^*}\zeta_2, m_3\Delta_{\Gamma_3^*}\zeta_3\big)$ we set $\xi := m\Delta_{\Gamma^*}\zeta$. Then we get through condition (4.72) and an observation analogously to Lemma 4.25 the property $\xi \in \mathcal{E}$ for ξ, so that we can plug it into the assumption in this part of the proof. Since ζ is a solution of problem (4.63) for the right side ξ, we see with our above notation that $\zeta = u^\xi$ and from Definition 4.31 we get $-I(\rho,\xi) = (w,\xi)_{-1} = \langle w,\zeta\rangle$. This leads to the following equation

$$\begin{aligned}
\langle w,\zeta\rangle &= -I(\rho,\xi)\\
&= I(\rho, m\Delta_{\Gamma^*}\zeta)\\
&= \sum_{i=1}^{3} m_i\gamma_i \int_{\Gamma_i^*} \left(\nabla_{\Gamma_i^*}\rho_i\cdot\nabla_{\Gamma_i^*}\Delta_{\Gamma_i^*}\zeta_i - |\sigma_i^*|^2\rho_i\,\Delta_{\Gamma_i^*}\zeta_i\right)\\
&\quad -\sum_{i=1}^{3} m_i\gamma_i \int_{S_i^*} S(n_i^*,n_i^*)\,\rho_i\,\Delta_{\Gamma_i^*}\zeta_i + \sum_{i=1}^{3} m_i\gamma_i\int_{L^*} a_i\,\rho_i\,\Delta_{\Gamma_i^*}\zeta_i\,.
\end{aligned}$$

Since $w \in \mathcal{H}^{-1}$, we obtain from regularity theory that $\rho \in \mathcal{H}^3$. Then we can integrate by parts to see

$$\begin{aligned}
\langle w,\zeta\rangle &= -\sum_{i=1}^{3} m_i\gamma_i \int_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\rho_i\,\Delta_{\Gamma_i^*}\zeta_i - \nabla_{\Gamma_i^*}\left(|\sigma_i^*|^2\rho_i\right)\cdot\nabla_{\Gamma_i^*}\zeta_i\right)\\
&\quad +\sum_{i=1}^{3} m_i\gamma_i \int_{\partial\Gamma_i^*}\left(\left(\nabla_{\Gamma_i^*}\rho_i\cdot n_{\partial\Gamma_i^*}\right)\,\Delta_{\Gamma_i^*}\zeta_i - |\sigma_i^*|^2\rho_i\,\left(\nabla_{\Gamma_i^*}\zeta_i\cdot n_{\partial\Gamma_i^*}\right)\right)\\
&\quad -\sum_{i=1}^{3} m_i\gamma_i \int_{S_i^*} S(n_i^*,n_i^*)\,\rho_i\,\Delta_{\Gamma_i^*}\zeta_i + \int_{L^*}\sum_{i=1}^{3} m_i\,\gamma_i\,a_i\,\rho_i\,\Delta_{\Gamma_i^*}\zeta_i\\
&= -\sum_{i=1}^{3} m_i\gamma_i \int_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\rho_i\,\Delta_{\Gamma_i^*}\zeta_i - \nabla_{\Gamma_i^*}\left(|\sigma_i^*|^2\rho_i\right)\cdot\nabla_{\Gamma_i^*}\zeta_i\right)\\
&\quad +\sum_{i=1}^{3} m_i\gamma_i \int_{S_i^*}\left(\partial_\mu\rho_i - S(n_i^*,n_i^*)\,\rho_i\right)\,\Delta_{\Gamma_i^*}\zeta_i\\
&\quad +\sum_{i=1}^{3} m_i\gamma_i \int_{L^*}\left(\partial_{n_{\partial\Gamma_i^*}}\rho_i + a_i\,\rho_i\right)\,\Delta_{\Gamma_i^*}\zeta_i - \sum_{i=1}^{3} m_i\gamma_i\int_{\partial\Gamma_i^*}|\sigma_i^*|^2\rho_i\,\underbrace{\left(\nabla_{\Gamma_i^*}\zeta_i\cdot n_{\partial\Gamma_i^*}\right)}_{=0\text{ on }S_i^*}\\
&= \sum_{i=1}^{3} m_i\gamma_i \int_{\Gamma_i^*}\nabla_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right)\cdot\nabla_{\Gamma_i^*}\zeta_i\\
&\quad +\sum_{i=1}^{3} m_i\gamma_i \int_{S_i^*}\left(\partial_\mu\rho_i - S(n_i^*,n_i^*)\,\rho_i\right)\,\Delta_{\Gamma_i^*}\zeta_i\\
&\quad +\sum_{i=1}^{3} m_i\gamma_i \int_{L^*}\left(\partial_{n_{\partial\Gamma_i^*}}\rho_i + a_i\,\rho_i\right)\,\Delta_{\Gamma_i^*}\zeta_i\\
&\quad -\sum_{i=1}^{3} m_i\gamma_i \int_{L_i^*}\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right)\,\left(\nabla_{\Gamma_i^*}\zeta_i\cdot n_{\partial\Gamma_i^*}\right)\,.
\end{aligned}$$

For a given testfunction $\eta \in \mathcal{Y}$ we choose a sequence $\eta^n = (\eta_1^n, \eta_2^n, \eta_3^n) \in \mathcal{H}^3 \cap \mathcal{Y}$ with

$$\begin{aligned} \nabla_{\Gamma_i^*}\eta_i^n \cdot n_{\partial\Gamma_i^*} = 0 &\quad \text{on } \partial\Gamma_i^*\,, \\ \Delta_{\Gamma_i^*}\eta_i^n = 0 &\quad \text{on } \partial\Gamma_i^* \text{ and} \\ \eta^n \to \eta &\quad \text{in } \mathcal{H}^1\,. \end{aligned}$$

From the last equation, with $\zeta = \eta^n$ we observe

$$\langle w, \eta^n\rangle = \sum_{i=1}^3 m_i\gamma_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right) \cdot \nabla_{\Gamma_i^*}\eta_i^n\,.$$

Due to the convergence $\eta^n \to \eta$ in $\mathcal{H}^1$, we get

$$\langle w, \eta\rangle = \sum_{i=1}^3 m_i\gamma_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right) \cdot \nabla_{\Gamma_i^*}\eta_i$$

for arbitrary $\eta \in \mathcal{Y}$. Inserting this into the last equation for ζ leads to the following boundary integrals

$$\begin{aligned} 0 = &\sum_{i=1}^3 m_i\gamma_i \int_{S_i^*} \left(\partial_\mu\rho_i - S(n_i^*, n_i^*)\,\rho_i\right)\,\Delta_{\Gamma_i^*}\zeta_i + \sum_{i=1}^3 m_i\gamma_i \int_{L^*}\left(\partial_{n_{\partial\Gamma_i^*}}\rho_i + a_i\,\rho_i\right)\,\Delta_{\Gamma_i^*}\zeta_i \\ &- \sum_{i=1}^3 m_i\gamma_i \int_{L_i^*}\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right)\,\left(\nabla_{\Gamma_i^*}\zeta_i \cdot n_{\partial\Gamma_i^*}\right) \end{aligned}$$

for all ζ with the properties (4.70)-(4.72).
Since $\Delta_{\Gamma_i^*}\zeta_i$ is arbitrary at S_i^*, we get $\partial_\mu\rho_i - S(n_i^*, n_i^*)\rho_i = 0$ at S_i^*. Moreover, from the boundary conditions (4.70) and (4.72) at L^* for the testfunctions ζ, we are led to

$$\begin{gathered} \gamma_1\left(\Delta_{\Gamma_1^*}\rho_1 + |\sigma_1^*|^2\rho_1\right) + \gamma_2\left(\Delta_{\Gamma_2^*}\rho_2 + |\sigma_2^*|^2\rho_2\right) + \gamma_3\left(\Delta_{\Gamma_3^*}\rho_3 + |\sigma_3^*|^2\rho_3\right) = 0 \quad \text{and} \\ \partial_{n_{\partial\Gamma_1^*}}\rho_1 + a_1\rho_1 = \partial_{n_{\partial\Gamma_2^*}}\rho_2 + a_2\,\rho_2 = \partial_{n_{\partial\Gamma_3^*}}\rho_3 + a_3\,\rho_3 \end{gathered}$$

at the triple line L^*.
Altogether we showed that $\rho \in \mathcal{E}$ is a weak solution of problem (4.66). □

We define the linearized operator corresponding to the linearized problem (4.60) - (4.62) through

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{H}^{-1}$$

with

$$\begin{aligned} \mathcal{D}(\mathcal{A}) = \{\rho = (\rho_1, \rho_2, \rho_3) \in \mathcal{H}^3 \quad | \quad &\rho \text{ satisfies (4.68) on } S_i^* \text{ and (4.69) on } L^*\,, \\ &\text{and} \int_{\Gamma_1^*}\rho_1 = \int_{\Gamma_2^*}\rho_2 = \int_{\Gamma_3^*}\rho_3\} \end{aligned} \tag{4.73}$$

by

$$\langle \mathcal{A}\rho, \xi \rangle = \sum_{i=1}^{3} m_i \gamma_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} \left(\Delta_{\Gamma_i^*} \rho_i + |\sigma_i^*|^2 \rho_i \right) \cdot \nabla_{\Gamma_i^*} \xi_i \tag{4.74}$$

for all $\rho \in \mathcal{D}(\mathcal{A})$ and $\xi \in \mathcal{H}^1$.

The boundary value problem (4.66) is then related to the problem in finding a $\rho \in \mathcal{D}(\mathcal{A})$ with

$$\mathcal{A}\rho = w\,.$$

By Lemma 4.34, we observe for all $\xi \in \mathcal{E}$ the identity

$$(\mathcal{A}\rho, \xi)_{-1} = -I(\rho, \xi)\,.$$

With this property we can show symmetry of $\mathcal{A}$.

Lemma 4.35. *The operator $\mathcal{A}$ is symmetric with respect to the inner product $(.\,,.)_{-1}$.*

Proof. Exactly the same as in Lemma 3.34. □

To study the spectrum of $\mathcal{A}$ as in the previous chapter, we need the analogue inequalities of Lemmata 3.35 and 3.36 to get as a corollary an upper bound for the eigenvalues of $\mathcal{A}$.

Lemma 4.36. *For all $\delta > 0$ there exists a $C_\delta > 0$, such that for all $\rho = (\rho_1, \rho_2, \rho_3) \in \mathcal{E}$ and each $i = 1, 2, 3$ the inequality*

$$\|\rho_i\|^2_{L^2(\partial\Gamma_i^*)} \leq \delta \|\nabla_{\Gamma^*}\rho\|^2 + C_\delta \|\rho\|^2_{-1}\,,$$

holds, where we used the $\|.\|_{-1}$-norm on $\mathcal{H}^{-1}$ from Definition 4.32 and the Definition of $\|\nabla_{\Gamma^*}\rho\|$ from (4.65).

Proof. With the help of the Poincaré-type inequality from Lemma 4.29 we can apply a similar argument as in the proof of Lemma 3.35 for the case of one hypersurface without a triple line. Thus we omit it. □

Lemma 4.37. *There exist positive constants C_1 and C_2, such that*

$$\|\nabla_{\Gamma^*}\rho\|^2 \leq C_1 \|\rho\|^2_{-1} + C_2\, I(\rho, \rho)$$

for all $\rho \in \mathcal{E}$.

Proof. Using the previous Lemma 4.36 and the Poincaré-type inequality from Lemma 4.29 we again just refer to a similar argument in the proof of Lemma 3.35. □

Lemma 4.38. *The largest eigenvalue of $\mathcal{A}$ is bounded from above by $\frac{C_1}{C_2}$, where C_1 and C_2 are the positive constants from Lemma 4.37.*

Proof. Exactly the same as in Lemma 3.37. □

Now we are able to show self-adjointness of $\mathcal{A}$. As in Lemma 3.38 of Section 3.4 we show a property that implies equivalence of symmetry and self-adjointness and then cite Lemma 4.35.

Lemma 4.39. *The operator $\mathcal{A}$ is self-adjoint with respect to the $(.\,,.)_{-1}$ inner product.*

Proof. The proof is formally the same as in Lemma 3.38, we just have to take care of the additional boundary conditions at the triple line, which will be done in the next steps.

We use the following theorem of operator theory from the book of Weidmann [Weid76]. If there exists an $\omega \in \mathbb{R}$, such that

$$\operatorname{im}(\omega Id - \mathcal{A}) \quad = \quad \mathcal{H}^{-1}\,,$$

then the properties symmetry and self-adjointness of $\mathcal{A}$ are equivalent.

So we have to show that there exists an $\omega \in \mathbb{R}$ such that for a given $f \in \mathcal{H}^{-1}$ there exists a $\rho \in \mathcal{D}(\mathcal{A})$ with

$$\omega\rho - \mathcal{A}\rho \quad = \quad f\,.$$

This means that ρ is a weak solution of the boundary value problem

$$\begin{cases} \Delta_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right) + \omega\rho_i = f & \text{in } \Gamma_i^*\,, \\ \rho \text{ satisfies (4.61)} & \text{on } S_i^*\,, \\ \rho \text{ satisfies (4.62)} & \text{on } L^*\,. \end{cases}$$

In detail the weak solution consists in finding a $\rho \in \mathcal{H}^3$ with the boundary condition (4.68) on S_i^* and (4.69) on the triple line L^* such that

$$-\sum_{i=1}^{3}\left(m_i\gamma_i\int_{\Gamma_i^*}\nabla_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right)\cdot\nabla_{\Gamma_i^*}\xi_i - \omega\int_{\Gamma_i^*}\rho_i\,\xi_i\right) \quad = \quad \langle f,\xi\rangle$$

holds for all $\xi \in \mathcal{Y}$. As in Lemma 4.25 such a weak solution fulfills $\int_{\Gamma_1^*}\rho_1 = \int_{\Gamma_2^*}\rho_2 = \int_{\Gamma_3^*}\rho_3$, so that $\rho \in \mathcal{D}(\mathcal{A})$.

To get a solution, we use the minimizing problem

$$F(\rho) := \frac{1}{2}\left(I(\rho,\rho) + \omega\|\rho\|_{-1}^2\right) - \sum_{i=1}^{3}\int_{\Gamma_i^*} u_i^f\rho_i \;\longrightarrow \min$$

for all $\rho \in \mathcal{E}$, where $u^f \in \mathcal{Y}$ is the weak solution of (4.63) with respect to $f \in \mathcal{H}^{-1}$. With the help of Lemma 4.37 we can show with an analogue argumentation as in the proof of Lemma 3.38 that F is coercive on $\mathcal{E}$ for large ω, so that the minimizing problem has a solution $\overline{\rho} = (\overline{\rho}_1, \overline{\rho}_2, \overline{\rho}_3) \in \mathcal{E}$,

when ω is large enough.
Taking the first variation of F we get

$$I(\overline{\rho}, v) + \omega\,(\overline{\rho}, v)_{-1} \;=\; \sum_{i=1}^{3} \int_{\Gamma_i^*} u_i^f\, v_i$$

for all $v \in \mathcal{E}$. By the Definition of $u^{\overline{\rho}} \in \mathcal{Y}$ as weak solution of (4.63) with respect to $\overline{\rho} \in \mathcal{E} \subset \mathcal{H}^{-1}$ and Definition 4.31 we observe that

$$\omega\,(\overline{\rho}, v)_{-1} = \omega \left\langle v, u^{\overline{\rho}} \right\rangle = \sum_{i=1}^{3} \int_{\Gamma_i^*} u_i^{\overline{\rho}}\, v_i$$

for all $v \in \mathcal{E}$.
So the above first variation is the weak version of the boundary value problem

$$\begin{cases} -\gamma_i \left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right) + \omega u_i^{\overline{\rho}} + c_i = u_i^f & \text{in } \Gamma_i^*\,, \\ \rho \text{ satisfies the first condition in (4.61)} & \text{on } S_i^*\,, \\ \rho \text{ satisfies the first and second condition in (4.62)} & \text{on } L^*\,. \end{cases} \tag{4.75}$$

Here c_i are constants as in the proof of Lemma 4.27 that appear due to the condition $\int_{\Gamma_1^*} v_1 = \int_{\Gamma_2^*} v_2 = \int_{\Gamma_2^*} v_2$ for the testfunctions. This constants are generalized Lagrange-multipliers when compared to the proof of Lemma 3.38.
Since $u^{\overline{\rho}}$ and u^f lie in $\mathcal{H}^1$, regularity theory gives us $\overline{\rho} \in \mathcal{H}^3$ and the fact that the identities in (4.75) hold pointwise. Summing the first line in (4.75) leads to the third condition in (4.62), since $\sum_{i=1}^3 c_i = 0$, $\sum_{i=1}^3 u_i^{\overline{\rho}} = 0$ and $\sum_{i=1}^3 u_i^f = 0$, where the last two identities hold on L^* due to $u^{\overline{\rho}}$, $u^f \in \mathcal{Y}$. We arrive at

$$\begin{aligned} & -\sum_{i=1}^{3} m_i\gamma_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\overline{\rho}_i + |\sigma_i^*|^2\overline{\rho}_i\right)\cdot\nabla_{\Gamma_i^*}\xi_i + \sum_{i=1}^{3}\omega\, m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} u_i^{\overline{\rho}}\cdot\nabla_{\Gamma_i^*}\xi_i \\ = \;& \sum_{i=1}^{3} m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} u_i^f \cdot \nabla_{\Gamma_i^*}\xi_i\,, \end{aligned}$$

where we differentiated the first line in (4.75) and tested with $m_i\nabla_{\Gamma_i^*}\xi_i$ for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathcal{Y}$. With the Definition (4.64) of the weak solutions $u^{\overline{\rho}}$ and u^f we can rewrite the last equation to

$$-\sum_{i=1}^{3} m_i\gamma_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\overline{\rho}_i + |\sigma_i^*|^2\overline{\rho}_i\right)\cdot\nabla_{\Gamma_i^*}\xi_i + \sum_{i=1}^{3}\omega \int_{\Gamma_i^*} \overline{\rho}_i\,\xi_i \;=\; \underbrace{\sum_{i=1}^{3} \int_{\Gamma_i^*} \langle f_i, \xi_i\rangle}_{=\langle f,\xi\rangle}$$

for all $\xi \in \mathcal{Y}$. So we found a $\overline{\rho} \in \mathcal{D}(\mathcal{A})$ with

$$\omega\overline{\rho} - \mathcal{A}\overline{\rho} \;=\; f$$

for ω large enough, which was remaining to get the assertion. $\square$

With the help of the previous results we are able to apply standard theory of self-adjoint operators and the theory of semigroups to get the following theorem.

Theorem 4.40.

(i) The spectrum of $\mathcal{A}$ consists of countable many real eigenvalues.

(ii) The initial value problem (4.60) - (4.62) is solvable for given initial data in $\mathcal{H}^{-1}$.

(iii) The zero solution of the linearized problem (4.60) - (4.62) is asymptotically stable if and only if the largest eigenvalue of $\mathcal{A}$ is negative.

Proof. With the same abstract arguments as in the proof of Lemma 3.39 we can show the assertions with the help of Lemma 4.38 and Lemma 4.39. $\square$.

The next lemma relates eigenvalues of $\mathcal{A}$ to the bilinear form I, so that we can formulate our linearized stability criterion.

Lemma 4.41. *Let*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$$

be the eigenvalues of $\mathcal{A}$ (taken into account the multiplicity).

(i) For all $n \in \mathbb{N}$, the following description of the eigenvalues of $\mathcal{A}$ holds.

$$\begin{aligned} \lambda_n &= \inf_{W \in \Sigma_{n-1}} \sup_{\rho \in W \setminus \{0\}} -\frac{I(\rho,\rho)}{(\rho,\rho)_{-1}}, \\ -\lambda_n &= \sup_{W \in \Sigma_{n-1}} \inf_{\rho \in W^\perp \setminus \{0\}} \frac{I(\rho,\rho)}{(\rho,\rho)_{-1}}, \end{aligned}$$

where Σ_n is the collection of n-dimensional subspaces of $\mathcal{E}$ and $W^\perp$ is the orthogonal complement with respect to the $(.,.)_{-1}$- inner product.

(ii) The eigenvalues λ_n depend continuously on $S(n_i^*, n_i^*)$, $\kappa_{n_{\partial\Gamma_i^*}}$ and $|\sigma_i^*|$ in the L^∞-norm.

Proof. As in Lemma 3.40, for the first part we just refer to the classical work of Courant and Hilbert [CH68]. The second part follows directly from the structure of I. $\square$

As in the previous parts of this work, we describe the largest eigenvalue in the above lemma more explicitly.

Lemma 4.42. *For the largest eigenvalue λ_1 of $\mathcal{A}$ we have the description*

$$-\lambda_1 \;=\; \min_{\rho\in\mathcal{E}\setminus\{0\}} \frac{I(\rho,\rho)}{(\rho,\rho)_{-1}}\,, \tag{4.76}$$

which can be seen directly from the second description of λ_1 in Lemma 4.41 through $-\lambda_1 = \sup_{W\in\Sigma_0}\inf_{\rho\in W^\perp\setminus\{0\}}\frac{I(\rho,\rho)}{(\rho,\rho)_{-1}}$ and $\Sigma_0=\emptyset$ and therefore $W^\perp=\mathcal{E}$. The fact that the minimum is attained also follows from the classical work of Courant and Hilbert [CH68].

From Theorem 4.40 we have asymptotic stability of the linearized problem (4.60) - (4.62) if and only if $\lambda_1 < 0$. This leads to the following main conclusion of the final section.

Theorem 4.43. *The linearized problem (4.60) - (4.62) is asymptotically stable if and only if*

$$I(\rho,\rho) \;>\; 0$$

for all $\rho\in\mathcal{E}\setminus\{0\}$, where

$$\begin{aligned} I(\rho,\rho) \;:=\;& \sum_{i=1}^{3}\gamma_i\int_{\Gamma_i^*}\left(|\nabla_{\Gamma_i^*}\rho_i|^2-|\sigma_i^*|^2\rho_i^2\right)\mathrm{d}\mathcal{H}^n-\sum_{i=1}^{3}\gamma_i\int_{S_i^*}S(n_i^*,n_i^*)\rho_i^2\,\mathrm{d}\mathcal{H}^{n-1}\\ &+\int_{L^*}\frac{\gamma_1}{s_1}\left(c_2\kappa_{n_{\partial\Gamma_2^*}}-c_3\kappa_{n_{\partial\Gamma_3^*}}\right)\rho_1^2\,\mathrm{d}\mathcal{H}^{n-1}+\int_{L^*}\frac{\gamma_2}{s_2}\left(c_3\kappa_{n_{\partial\Gamma_3^*}}-c_1\kappa_{n_{\partial\Gamma_1^*}}\right)\rho_2^2\,\mathrm{d}\mathcal{H}^{n-1}\\ &+\int_{L^*}\frac{\gamma_3}{s_3}\left(c_1\kappa_{n_{\partial\Gamma_1^*}}-c_2\kappa_{n_{\partial\Gamma_2^*}}\right)\rho_3^2\,\mathrm{d}\mathcal{H}^{n-1}. \end{aligned}$$

For this time we wrote out the corresponding terms for the abbreviations a_i.

To complete the considered evolution laws, we want to describe in the following extensive corollary the linearized stability of volume preserving mean curvature flow with outer boundary contact and triple lines.

Corollary 4.44. *We consider the problem of finding three evolving hypersurfaces as in (4.1) which lie inside a fixed bounded region $\Omega\subset\mathbb{R}^{n+1}$ and whose boundaries allow the decomposition (4.2)-(4.4) in a triple line $L(t)$ and in an outer part $S_i(t)$. The hypersurfaces shall evolve due to weighted volume preserving mean curvature flow in $G_i(t)$*

$$V_i \;=\; \gamma_i\left(H_i-\overline{H}_i\right)\,,$$

with the following right angle condition at the outer boundary $S_i(t)$

$$\angle(\Gamma_i(t),\partial\Omega) \;=\; \frac{\pi}{2}$$

and the following angle conditions at the triple line $L(t)$ for angles θ_i as above

$$\angle(\Gamma_1(t),\Gamma_2(t)=\theta_3\,,\;\angle(\Gamma_2(t),\Gamma_3(t)=\theta_1\,,\;\angle(\Gamma_3(t),\Gamma_1(t)=\theta_2\,.$$

With the help of the parametrization of this section we can formulate the above evolution law as partial differential equations for functions ρ_i *on* Γ_i^*, *where* Γ_i^* *are stationary solutions. Linearization of these equations leads to the following linear problem.*

$$\begin{cases} \partial_t \rho_i = \big(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\big) - \fint_{\Gamma_i^*}\big(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\big) & \text{in } \Gamma_i^*\,, \\ 0 = \partial_\mu \rho_i - S(n_i^*, n_i^*)\rho_i & \text{on } S_i^*\,, \\ 0 = \gamma_1\rho_1 + \gamma_2\rho_2 + \gamma_3\rho_3 & \text{on } L^*\,, \\ \partial_{n_{\partial\Gamma_1^*}}\rho_1 + a_1\,\rho_1 = \partial_{n_{\partial\Gamma_2^*}}\rho_2 + a_2\,\rho_2 = \partial_{n_{\partial\Gamma_3^*}}\rho_3 + a_3\,\rho_3 & \text{on } L^*\,. \end{cases}$$

Here we use the abbreviations a_i *from (4.35)-(4.37) and we observe that a sovability condition as in Lemma 4.25 gives here the conditions* $\int_{\Gamma_i^*}\rho_i = 0$. *Stability analysis as in this and the previous section leads to the following linearized stability criterion for the stationary solution* $\Gamma^* = \bigcup_{i=1}^3 \Gamma_i^*$.

$$\begin{array}{ll} & \Gamma^* \text{ is linearly asymptotically stable} \\ \Longleftrightarrow & \left\{\begin{array}{lll} I(\rho,\rho) & := & \sum_{i=1}^3 \gamma_i \int_{\Gamma_i^*}\big(|\nabla_{\Gamma_i^*}\rho_i|^2 - |\sigma_i^*|^2\rho_i^2\big) - \sum_{i=1}^3 \gamma_i \int_{S_i^*} S(n_i^*, n_i^*)\,\rho_i^2 \\ & & + \sum_{i=1}^3 \gamma_i \int_{L^*} a_i\,\rho_i^2 \\ \multicolumn{3}{l}{\text{is positive for all } 0 \neq \rho = (\rho_1,\rho_2,\rho_3) \text{ with } \rho_i \in H^1(\Gamma_i^*) \text{ such that}} \\ \multicolumn{3}{l}{\int_{\Gamma_i^*}\rho_i = 0 \text{ and } \gamma_1\rho_1 + \gamma_2\rho_2 + \gamma_3\rho_3 = 0 \text{ on } L^*\,.} \end{array}\right. \end{array}$$

Remark 4.45. *As an example for stability results with the help of the above bilinear form, we want to refer to the well-known proof of the double bubble conjecture from Hutchings, Morgan, Ritoré and Ros [HMRR02], where the authors used the above bilinear form without the outer boundary part to show instability of the so-called nonstandard double bubble.*

Chapter 5

Appendix

In the appendix of this work, we present detailed proofs of the normal time derivative of mean curvature and the Gauß mapping, i.e. the normal for an evolving hypersurface. Although these formulas are known in the literature, for example in Ecker [Eck04] or Rosenberg [Ros93], the authors consider mainly the case when the evolving hypersurfaces are given with the help of a fixed reference hypersurface as in Section 3.1. Since we want to use these formulas also in the calculation of the evolution of area and volume in Section 2.4, we have to consider the general case for an arbitrary evolving hypersurface.

Finally we want to define the vector product in $\mathbb{R}^{n+1}$, that is used for the linearization of the angle conditions, and give some important properties.

5.1 Normal time derivative of mean curvature

Although the following formula is known, we give a proof of the normal time derivative of mean curvature of an evolving hypersurface. The literature is very short in this case and for the convenience of the reader here are the details.

Lemma 5.1. *With the notation of Chapter 2 the following formula for the normal time derivative of mean curvature for an evolving hypersurface holds.*

$$\partial^\circ H(t,p) \quad = \quad \Delta_{\Gamma(t)} V(t,p) + V(t,p) \sum_{i=1}^{n} \kappa_i^2(t,p)\,, \tag{5.1}$$

where the sum is often written as $|\sigma|^2$, the square of the second fundamental form.

Proof. We show the assertion at a fixed point $(t_0, p_0) \in \Gamma$ and choose coordinates with nice properties at this point. Firstly, we can always achieve with the help of a suitable rotation that the normal equals the direction e_{n+1}, that is

$$n(t_0, p_0) \quad = \quad (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}\,.$$

Note that all relevant terms like H, V and κ_i will not change values for the rotated surface. Locally around $(t_0, p_0) \in \Gamma$ we describe Γ as a graph, that is, there exists an open neighbourhood $W \subset \mathbb{R} \times \mathbb{R}^{n+1}$ of (t_0, p_0) and an open set $I \times A \subset \mathbb{R} \times \mathbb{R}^n$, such that with a function

$$\varphi : I \times A \longrightarrow \mathbb{R}\,, \qquad (t,x) \mapsto \varphi(t,x)$$

we have a local parametrization of Γ given through

$$\Phi : I \times A \longrightarrow \Gamma \cap W\,, \qquad (t,x) \mapsto (t, x, \varphi(t,x))\,.$$

Because of point (ii) of the Definition 2.31 of an evolving hypersurface it also holds that for fixed $t \in I$ the mapping

$$\Phi_t : A \longrightarrow \Gamma(t) \cap U\,, \qquad x \mapsto (x, \varphi(t,x))$$

is a local parametrization of $\Gamma(t)$ for some neighbourhood $U \subset \mathbb{R}^{n+1}$ of p_0.
With the help of this graph representation there holds the formula for the normal at $(t,p) = \Phi(t,x) \in \Gamma \cap W$ through

$$n(t,p) = \frac{1}{\sqrt{1 + |\nabla_x \varphi(t,x)|^2}} \left(-\nabla_x \varphi(t,x), 1\right) =: \beta(t,x) \left(-\nabla_x \varphi(t,x), 1\right)\,,$$

because $(\partial_1 \Phi_t(x), \ldots, \partial_n \Phi_t(x))$ is a basis of $T_p\Gamma(t)$ and

$$\partial_j \Phi_t(x) \cdot (-\nabla_x \varphi(t,x), 1) \;=\; (e_j, \partial_j \varphi(t,x)) \cdot (-\nabla_x \varphi(t,x), 1) = -\partial_j \varphi(t,x) + \partial_j \varphi(t,x) = 0$$

for $j = 1, \ldots, n$ and $(e_1, \ldots, e_n)$ the standard basis of $\mathbb{R}^n$.
Therefore at the fixed point $(t_0, p_0) = \Phi(t_0, x_0)$ the following identity

$$(0, \ldots, 0, 1) = n(t_0, p_0) = \beta(t_0, x_0)\,(-\nabla_x \varphi(t_0, x_0), 1)$$

holds, so that we see

$$\nabla_x \varphi(t_0, x_0) \;=\; 0\,. \tag{5.2}$$

From this last equation we also observe

$$\beta(t_0, x_0) = 1\,, \quad \partial_t \beta(t_0, x_0) = 0 \;\text{ and }\; \nabla_x \beta(t_0, x_0) = 0\,, \tag{5.3}$$

where the last two identities arise through a direct calculation of $\partial_t \beta$ and $\nabla_x \beta$. For fixed t, we can write the first fundamental form of $\Gamma(t)$ locally as

$$(g_{ij})_t\,(x) \;=\; (\partial_i \Phi_t(x), \partial_j \Phi_t(t,x)) = \delta_{ij} + \partial_i \varphi(t,x)\,\partial_j \varphi(t,x)\,,$$

where $\Phi_t(x) = p \in \Gamma(t)$. At the point x_0, we get therefore with (5.2)

$$(g_{ij})_{t_0}\,(x_0) \;=\; \delta_{ij}\,.$$

With the calculation

$$\partial_k \left(g_{ij}\right)_t (x) \quad = \quad \partial_k \partial_i \varphi(t,x)\, \partial_j \varphi(t,x) + \partial_i \varphi(t,x)\, \partial_k \partial_j \varphi(t,x)$$

we also see the vanishing of the Christoffel symbols from Remark 2.26 at the point x_0

$$\left(\Gamma^k_{ij}\right)_{t_0} (x_0) \quad = \quad 0\,,$$

where we used equation (5.2).
For an arbitrary smooth function $f : \Gamma(t) \to \mathbb{R}$ we set $\overline{f} : A \to \mathbb{R}$ defined through $\overline{f}(x) := f(\Phi_t(x))$ and conclude at the point $p_0 = \Phi_{t_0}(x_0)$ with the local representation of the Laplace-Beltrami operator from Remark 2.26 it holds that

$$\begin{aligned}
\Delta_{\Gamma(t)} f(p_0) \quad &= \quad \sum_{i,j=1}^{n} \underbrace{g^{ij}(x_0)}_{=\,\delta_{ij}} \left(\partial_i \partial_j (f \circ \Phi_{t_0})(x_0) - \sum_{k=1}^{n} \underbrace{\Gamma^k_{ij}(x_0)}_{=\,0}\, \partial_k (f \circ \Phi_{t_0})(x_0) \right) \\
&= \quad \sum_{i=1}^{n} \partial_i \partial_i \overline{f}(x_0) \\
&= \quad \Delta_x \overline{f}(x_0)\,,
\end{aligned}$$

where the Laplace operator in the last term is the usual one in euclidian space. In particular, we get for the normal velocity with the same notation as for f

$$\Delta_{\Gamma(t)} V(t,p) \quad = \quad \Delta_x \overline{V}(t,x)\,. \tag{5.4}$$

Now we use the following representation for mean curvature, when the surface is given as a graph which can be found for example in the book of Gilbarg and Trudinger [GT98].

$$\begin{aligned}
H(t,p) \quad &= \quad H(\Phi(t,x)) \\
&= \quad \sum_{i=1}^{n} \partial_i \left(\frac{\partial_i \varphi(t,x)}{\sqrt{1 + |\nabla \varphi(t,x)|^2}} \right) \\
&= \quad \sum_{i=1}^{n} \partial_i (\partial_i \varphi(t,x)\, \beta(t,x)) \qquad (5.5) \\
&= \quad \Delta_x \varphi(t,x)\, \beta(t,x) + \nabla_x \varphi(t,x) \cdot \nabla_x \beta(t,x)\,,
\end{aligned}$$

where $(t,p) = \Phi(t,x) \in \Gamma$.
For the normal velocity we calculate with the help of Lemma 2.40

$$\begin{aligned}
V(t,p) \quad &= \quad V(\Phi(t,x)) \\
&= \quad \partial_t \Phi_t(x) \cdot n(t,p) \\
&= \quad \partial_t \left(x, \varphi(t,x)\right) \cdot (-\nabla_x \varphi(t,x), 1)\ \beta(t,x) \\
&= \quad (0, \dots, 0, \partial_t \varphi(t,x)) \cdot (-\nabla_x \varphi(t,x), 1)\ \beta(t,x) \\
&= \quad \partial_t \varphi(t,x)\, \beta(t,x)\,,
\end{aligned}$$

where $p = \Phi_t(x) \in \Gamma(t)$ and the last term denotes the function $\overline{V}(t,x)$ from above.
To derive $\partial^\circ H(t_0, p_0)$, let $y : (t_0 - \varepsilon, t_0 + \varepsilon) \to \mathbb{R}^{n+1}$ be a curve with $y(t_0) = p_0$ and

$$\begin{cases} y(\tau) & \in & \Gamma(\tau)\,, \\ y'(\tau) & = & V(\tau, y(\tau))\, n(\tau, y(\tau))\,. \end{cases}$$

With the above formulas for the normal and normal velocity we see

$$y'(\tau) \;=\; \partial_t\varphi(\tau, c(\tau))\,\beta^2(\tau, c(\tau))\,(-\nabla_x\varphi(\tau, c(\tau)), 1)\,.$$

On the other hand it has to hold, due to the graph representation,

$$y(\tau) \;=\; \Phi_\tau(c(\tau)) = (c(\tau), \varphi(\tau, c(\tau)))$$

with a curve $c : (t_0 - \varepsilon, t_0 + \varepsilon) \to A$, $c(t_0) = x_0$ with $\Phi_{t_0}(x_0) = p_0$, so that

$$y'(\tau) \;=\; \left(c'(\tau), \partial_t\varphi(t, c(\tau)) + \nabla_x\varphi(t, c(\tau)) \cdot c'(\tau)\right)\,.$$

Comparing the two representations of $y'(\tau)$ at the point $\tau = t_0$ together with (5.2) yields

$$c'(t_0) \;=\; 0\,. \tag{5.6}$$

Finally we get the following identity for the normal-time derivative of mean curvature

$$\begin{aligned} \partial^\circ H(t_0, p_0) &= \frac{d}{d\tau} H(\tau, y(\tau))\Big|_{\tau=t_0} \\ &\overset{(5.5)}{=} \frac{d}{d\tau}\left[\beta(\tau, c(\tau))\,\Delta_x\varphi(\tau, c(\tau)) + \nabla_x\beta(\tau, c(\tau)) \cdot \nabla_x\varphi(\tau, c(\tau))\right]\Big|_{\tau=t_0} \\ &= \partial_t\beta(t_0, x_0)\,\Delta_x\varphi(t_0, x_0) + \nabla_x\beta(t_0, x_0) \cdot c'(t_0)\,\Delta\varphi(t_0, x_0) \\ &\quad + \beta(t_0, x_0)\,\partial_t\Delta_x\varphi(t_0, x_0) + \beta(t_0, x_0)\,\nabla_x\Delta_x\varphi(t_0, x_0) \cdot c'(t_0) \\ &\quad + \partial_t\nabla_x\beta(t_0, x_0) \cdot \nabla_x\varphi(t_0, x_0) + \mathrm{hess}_x\,\beta(t_0, x_0)(c'(t_0)) \cdot \nabla_x\varphi(t_0, x_0) \\ &\quad + \nabla_x\beta(t_0, x_0) \cdot \partial_t\nabla_x\varphi(t_0, x_0) + \nabla_x\beta(t_0, x_0) \cdot \mathrm{hess}_x\,\varphi(t_0, x_0)(c'(t_0)) \\ &= \partial_t\Delta_x\varphi(t_0, x_0)\,, \end{aligned}$$

because all terms except one vanish due to (5.2), (5.3) and (5.6).
Another calculation for the normal velocity gives with the help of (5.4) and (5.2)

$$\begin{aligned} \Delta_{\Gamma(t)}V(t_0, p_0) &= \Delta_x\left(\beta(t_0, x_0)\,\partial_t\varphi(t_0, x_0)\right) \\ &= \Delta_x\beta(t_0, x_0)\,\partial_t\varphi(t_0, x_0) + 2\nabla_x\beta(t_0, x_0) \cdot \nabla_x\partial_t\varphi(t_0, x_0) \\ &\quad + \beta(t_0, x_0)\,\Delta_x\partial_t\varphi(t_0, x_0) \\ &= \Delta_x\beta(t_0, x_0)\,\partial_t\varphi(t_0, x_0) + \Delta_x\partial_t\varphi(t_0, x_0)\,. \end{aligned}$$

The last two expressions together with $V(t_0, p_0) = \partial_t\varphi(t_0, x_0)$ lead to

$$\partial^\circ H(t_0, p_0) \;=\; \Delta_{\Gamma(t)}V(t_0, p_0) - \Delta_x\beta(t_0, x_0)\,V(t_0, p_0)\,, \tag{5.7}$$

so that the remaining work is to find the right expression for $\Delta_x \beta(t_0, x_0)$. To this end, a direct calculation gives

$$\begin{aligned}
\partial_i \beta(t,x) &= -\frac{1}{2}\left(1+|\nabla_x \varphi(t,x)|^2\right)^{-\frac{3}{2}} \partial_i \left(1+|\nabla_x \varphi(t,x)|^2\right) \\
&= -\beta^3(t,x)\,\nabla_x \varphi(t,x)\cdot \nabla_x \partial_i \varphi(t,x)
\end{aligned}$$

and

$$\begin{aligned}
\partial^2_{ii} \beta(t,x) &= -3\beta^2(t,x)\,\partial_i \beta(t,x)\,\nabla_x \varphi(t,x)\cdot \nabla_x \partial_i \varphi(t,x) \\
&\quad -\beta^3(t,x)\left(\partial_i \nabla_x \varphi(t,x)\cdot \nabla_x \partial_i \varphi(t,x) + \nabla_x \varphi(t,x)\cdot \partial_i \nabla_x \partial_i \varphi(t,x)\right)\,,
\end{aligned}$$

so that at the point (t_0, x_0) we can conclude with (5.3) and (5.2)

$$\partial^2_{ii} \beta(t_0,x_0) = -|\nabla_x \partial_i \varphi|^2(t_0,x_0)\,.$$

This gives eventually

$$\Delta_x \beta(t_0,x_0) = \sum_{i=1}^{n} \partial^2_{ii} \beta(t_0,x_0) = -\sum_{i,j=1}^{n} \left(\partial^2_{ij}\varphi\right)^2 (t_0,x_0)\,.$$

Without loss of generality we can now choose suitable coordinates in the tangent space, that is we write the hypersurface as a graph with respect to a basis $(w_1, \ldots, w_n, n)$, where $(w_1, \ldots, w_n)$ is a basis of the tangent space $T_{p_0}\Gamma(t_0)$ and $n(t_0, p_0) = (0, \ldots, 0, 1)$ as in the beginning of the proof. This gives us the possibility to denote the hessian $\operatorname{hess}_x \varphi(t_0, x_0)$ as a diagonal matrix, at least at the fixed point (t_0, x_0).
Then we get

$$\begin{aligned}
\Delta_x \beta(t_0,x_0) &= -\sum_{i=1}^{n} \left(\partial^2_{ii}\varphi(t_0,x_0)\right)^2 \\
&= \sum_{i=1}^{n} \kappa_i^2(t_0,x_0)\,,
\end{aligned}$$

where the last identity can be seen for example in [GT98].
Putting together the last equation with (5.7) we arrive at

$$\partial^\circ H(t_0,p_0) = \Delta_{\Gamma(t)} V(t_0,p_0) + V(t_0,p_0)\sum_{i=1}^{n} \kappa_i^2(t_0,p_0)$$

and we proved the lemma. □

5.2 Normal time derivative of the normal

Lemma 5.2. *With the notation from Chapter 2 the following formula for the normal time derivative of the normal for an evolving hypersurface holds*

$$\partial^\circ n(t,p) \quad = \quad -\nabla_{\Gamma(t)} V(t,p)\,. \tag{5.8}$$

Proof. As in the above Lemma 5.1, we show the claim at a fixed point $(t_0, p_0) \in \Gamma$ and use the same coordinates. Firstly, we do a transformation of the gradient as we did for the Laplace-Beltrami operator in the previous proof. So let $f : \Gamma(t) \to \mathbb{R}$ be an arbitrary smooth function and define $\overline{f} : A \to \mathbb{R}$ through $\overline{f}(x) := f(\Phi_t(x))$. Then we see with the local representation of the gradient from Remark 2.22 that

$$\begin{aligned}
\nabla_{\Gamma(t)} f(p) &= \sum_{i,j=1}^{n} \underbrace{g^{ij}(x)}_{=\delta_{ij}} \partial_i \left(f \circ \Phi_t\right)(x)\, \partial_j \Phi_t(x) \\
&= \sum_{i=1}^{n} \partial_i \left(f \circ \Phi_t\right)(x)\, \partial_i \Phi_t(x) \\
&= \sum_{i=1}^{n} \partial_i \overline{f}(x)\ \left(e_i, \partial_i \varphi(t,x)\right) \\
&= \left(\nabla_x \overline{f}(x), \sum_{i=1}^{n} \partial_i \overline{f}(x)\, \partial_i \varphi(t,x)\right) .
\end{aligned}$$

At the point $(t_0, p_0) = \Phi(t_0, x_0)$, we get with the help of (5.2)

$$\nabla_{\Gamma(t_0)} f(p_0) \quad = \quad \left(\nabla_x \overline{f}(x_0), 0\right) .$$

This gives for the normal velocity $V(t,p) = \beta(t,x)\, \partial_t \varphi(t,x)$ at the point $(t_0, p_0) = \Phi(t_0, x_0)$

$$\begin{aligned}
\nabla_{\Gamma(t)} V(t_0, p_0) &= \left(\nabla_x \left(\beta\, \partial_t \varphi\right)(t_0, x_0), 0\right) \\
&= \left(\nabla_x \beta(t_0, x_0)\, \partial_t \varphi(t_0, x_0) + \beta(t_0, x_0)\, \partial_t \nabla_x \varphi(t_0, x_0), 0\right) \\
&= \left(\partial_t \nabla_x \varphi(t_0, x_0), 0\right) ,
\end{aligned}$$

which is seen with the help of (5.3).
To calculate the normal-time derivative of the normal, we consider a curve as in the previous proof and get with the formula $n(t,p) = \beta(t,x)\ (-\nabla_x \varphi(t,x), 1)$ the following identities.

$$\begin{aligned}
\partial^\circ n(t_0, p_0) &= \frac{d}{d\tau} \left[\beta(\tau, c(\tau))\ (-\nabla_x \varphi(\tau, c(\tau)), 1)\right]\Big|_{\tau = t_0} \\
&= \partial_t \beta(t_0, x_0)\ (-\nabla_x \varphi(t_0, x_0), 1) + \nabla_x \beta(t_0, x_0) \cdot c'(t_0)\ (-\nabla_x \varphi(t_0, x_0), 1) \\
&\quad + \beta(t_0, x_0)\ (-\partial_t \nabla_x \varphi(t_0, x_0), 0) + \beta(t_0, x_0) \left(\sum_{i=1}^{n} \partial_i \nabla_x \varphi(t_0, x_0) \cdot c'(t_0)\, e_i, 0\right) \\
&= \left(-\nabla_x \partial_t \varphi(t_0, x_0), 0\right) ,
\end{aligned}$$

where all terms except one vanish due to (5.3) and (5.6). Putting together the above two formulas gives the desired claim. $\square$

5.3 Facts about the vector product

The vector product in $\mathbb{R}^{n+1}$ extends the cross product of two vectors in $\mathbb{R}^3$. It assigns n vectors $v_1, \ldots, v_n$ in $\mathbb{R}^{n+1}$ a new vector $v_1 \times \ldots \times v_n$, which is perpendicular to each v_i and its length equals the volume of the parallelotope spanned by $v_1, \ldots, v_n$ in $\mathbb{R}^{n+1}$. Since the vector product in $\mathbb{R}^{n+1}$ isn't that common in literature, we gather together some facts that are used in this work. Of course, one can find them also in some textbooks on analysis and algebra, for example [Fi02] and [Koe03].

At first, we give two definitions of the vector product of n vectors in $\mathbb{R}^{n+1}$, which are, despite the fact they are equivalent, good to know.

Definition 5.3 (Vector product, version 1)**.** *Let $v_1, \ldots, v_n \in \mathbb{R}^{n+1}$. Then we define the* ***vector product*** *as*

$$v_1 \times \ldots \times v_n \quad := \quad \sum_{i=1}^{n+1} (-1)^{i+1} \det(A_i)\, e_i\,, \tag{5.9}$$

where $e_1, \ldots, e_{n+1}$ is the standard basis of $\mathbb{R}^{n+1}$, $A \in M(n \times (n+1), \mathbb{R})$ is the matrix, which consists of the vectors $v_1, \ldots, v_n$ as columns,

$$A \quad = \quad \left.\overbrace{\left(v_1 \,\middle|\, v_2 \,\middle|\, \ldots \,\middle|\, v_n\right)}^{n \text{ columns}}\right\} n+1 \text{ rows}$$

and A_i results from A by deleting the i-th row. So $v_1 \times \ldots \times v_n$ is a formal development of

$$\det \left(\begin{array}{c|c|c|c|c} e_1 & & & & \\ e_2 & & & & \\ & v_1 & v_2 & \ldots & v_n \\ \vdots & & & & \\ e_{n+1} & & & & \end{array} \right)$$

with respect to the first column.

An equivalent definition is the following one.

Definition 5.4 (Vector product, version 2)**.** *Let $v_1, \ldots, v_n \in \mathbb{R}^{n+1}$. Define the function $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$ through*

$$\varphi(w) \quad := \quad \det \left(v_1 \,\middle|\, \ldots \,\middle|\, v_n \,\middle|\, w \right) .$$

Then φ is linear and therefore there exists a unique vector $z \in \mathbb{R}^{n+1}$, such that

$$\langle w, z\rangle_{\mathbb{R}^{n+1}} = \varphi(w) \quad \textit{for all } w \in \mathbb{R}^{n+1}.$$

This z is then called **vector product** *of $v_1, \ldots, v_n$ and denoted by $v_1 \times \ldots \times v_n$.*

In the next lemma, we quote the used properties of the vector product.

Lemma 5.5 (Properties of the vector product)**.**

(*i*) *The vector product is linear in each component and alternating.*

(*ii*) *$v_1 \times \ldots \times v_n$ is orthogonal to each of the vectors $v_1, \ldots, v_n$.*

(*iii*) *$v_1 \times \ldots \times v_n = 0 \Leftrightarrow v_1, \ldots, v_n$ are linearly dependent.*

(*iv*) *If $v_1, \ldots, v_n$ are linearly independent, then $(v_1, \ldots, v_n, v_1 \times \ldots \times v_n)$ has positive orientation.*

(*v*) *$|v_1 \times \ldots \times v_n| = \sqrt{\det(g_{ij})}$ with $g_{ij} = v_i \cdot v_j$. This means that $|v_1 \times \ldots \times v_n|$ is the volume of the parallelotope spanned by $v_1, \ldots, v_n$.*

The above definitions and the statements from the lemma can be found in [Fi02] or [Koe03], whereas the next statement, that we use in the linearization of the angle condition, would be a small exercise, that we give with proof.

Lemma 5.6. *Let $v_1, \ldots, v_n$ be an orthonormal system in $\mathbb{R}^{n+1}$, that is $v_1, \ldots, v_n$ are linearly independent and $v_i \cdot v_j = \delta_{ij}$. With $z := v_1 \times \ldots \times v_n$ it holds that*

$$v_1 \times \ldots \times \overset{\textit{(l-th pos.)}}{z} \times \ldots \times v_n = (-1) v_l . \tag{5.10}$$

Proof. Since $v_1, \ldots, v_n, z$ form an orthonormal basis of $\mathbb{R}^{n+1}$, we see with properties from the above lemma that

$$v_1 \times \ldots \times \overset{\text{(l-th pos.)}}{z} \times \ldots \times v_n = \alpha v_l ,$$

where $\alpha = \pm 1$. Now we calculate

$$\alpha = \alpha(v_l \cdot v_l) = v_l \cdot z \overset{\text{def. 5.4}}{=} \det\left(v_1 \middle| \ldots \middle| z \middle| \ldots \middle| v_n \middle| v_l\right) = (-1)\det\left(v_1 \middle| \ldots \middle| v_l \middle| \ldots \middle| v_n \middle| z\right) ,$$

where the last determinant is greater than 0 due to Lemma 5.5, (*iv*). Therefore we conclude

$$\alpha = (-1)$$

and finished the proof. □

Bibliography

[Alt04] Alt H.W., *Partielle Differentialgleichungen IV*, Lecture notes, University of Bonn, summer term 2004.

[AE09] Amann H., Escher J., *Analysis III*, Birkhäuser Verlag, 2009.

[Aub82] Aubin T., *Nonlinear Analysis on Manifolds; Monge-Ampere Equations*, Springer Verlag, 1982.

[BdoC84] Barbosa, J. L., do Carmo M., *Stability of Hypersurfaces with Constant Mean Curvature*, Math. Z. 185 (1984), p.339-353.

[BGN07] Barrett J. W., Garcke H., Nürnberg R., *On the Variational Approximation of Combined Second and Fourth Order Geometric Evolution Equations*, SIAM J.Scientific Comput. 29, Issue 3 (2007), p.1006-1041.

[BGN09] Barrett J. W., Garcke H., Nürnberg R., *Parametric Approximation of Surface Clusters driven by Isotropic and Anisotropic Surface Energies*, Interf. and Free Bound., to appear.

[Bra78] Brakke K., *The motion by a surface by its mean curvature*, Princeton Univ. Press, 1978.

[Broe92] Bröcker T., *Analysis III*, Spektrum Akademischer Verlag, 1992.

[BW95] Bronsard L., Wetton B. T. R., *A numerical method for tracking curve networks moving with curvature motion*, J. Comput. Phy., vol 120 (1995), no. 1, p.66-87.

[CEN96] Cahn J. W., Elliott C. M., Novick-Cohen A., *The Cahn-Hilliard equation with a concentration dependent mobility: Motion by minus the Laplacian of mean curvature*, European J. Appl. Math., vol. 7 (1996), p.287-301.

[CH68] Courant R., Hilbert D., *Methods of mathematical physics*, vol. I, Interscience, New York, 1968.

[CT94] Cahn J. W., Taylor J. E., *Linking anisotropic sharp and diffuse surface motion laws via gradient flows*, J. Statist. Phys., vol. 77 (1994), p.183-197.

[DDE05] Deckelnick K., Dziuk G., Elliott C. M., *Computation of geometric partial differential equations and mean curvature flow*, Acta Num., vol. 14 (2005), p.139-232.

[DE98] Deckelnick K., Elliott C. M., *Finite element error bounds for a curve shrinking with prescribed normal contact to a fixed boundary*, IMA J. Numer. Anal., vol 18 (1998), no. 4, p.635-654.

[DG90] Davi F., Gurtin M., *On the motion of a phase interface by surface diffusion*, Z. Angew. Math. Phys., vol. 41 (1990), p.782-811.

[Eck04] Ecker K., *Regularity Theory for Mean Curvature Flow*, Birkhäuser Verlag, 2004.

[EG97] Elliott C. M., Garcke H., *Existence results for diffusive surface motion laws*, Adv. Math. Sci. Appl., vol. 7, (1997), no.1, p.465-488.

[EGI03] Escher J., Garcke H., Ito K. *Exponential stability for a mirror-symmetric three phase boundary motion by surface diffusion*, Math. Nach., vol. 257 (2003), p.3-15.

[EJ07] Eschenburg J., Jost J., *Differentialgeometrie und Minimalflächen*, Springer Verlag, 2007.

[EMS98] Escher J., Mayer U. F., Simonett G., *The Surface Diffusion Flow for Immersed Hypersurfaces*, SIAM J.Math.Anal., vol. 29 (1998), no. 6, p.1419-1433.

[ES98] Escher J., Simonett G., *The volume preserving mean curvature flow near spheres*, Proc. Amer. Math. Soc., vol. 126 (1998), no. 9, p.2789-2796.

[ESY96] Ei S.-I., Sato M.-H., Yanagida E., *Stability of stationary interfaces with contact in a generalized mean curvature flow*, Amer. J. Math, vol. 118 (1996), p.653-687.

[EY93] Ei S.-I., Yanagida E., *Stability of stationary interfaces in a generalized mean curvature flow*, J. Fac. Csi. Uni. Tokyo Sect. IA Math, vol. 40 (1993), p.651-661.

[Fi02] Fischer G. *Lineare Algebra*, 13th ed., Vieweg Verlag, 2002.

[GH86] Gage M., Hamilton R. S., *The heat equation shrinking convex plane curves*, J. Diff. Geom., vol. 23 (1986), no. 1, p.69-95.

[GIK05] Garcke H., Ito K., Kohsaka Y., *Linearized Stability Analysis of Stationary Solutions for Surface Diffusion with Boundary Conditions*, SIAM J.Math.Anal., vol.36 (2005), no. 4, p.1031-1056.

[GIK08] Garcke H., Ito K., Kohsaka Y., *Nonlinear Stability of Stationary Solutions for Surface Diffusion with Boundary Conditions*, SIAM J.Math.Anal., vol.40 (2008), no.2, p.491-515.

[GIK09] Garcke H., Ito K., Kohsaka Y., *Stability analysis of phase boundary motion by surface diffusion with triple junction*, Banach Center Publications 86 (2009), p.83-101.

[GIK10] Garcke H., Ito K., Kohsaka Y., *Surface diffusion with triple junctions: A stability criterion for stationary solutions*, Adv. Diff. Equ. 15, no.(5-6) (2010), p.437-472.

[GKS09] Garcke H., Kohsaka Y., Ševčovič D., *Nonlinear stability of stationary solutions for curvature flow with triple junction*, Hokkaido Math. J., vol.38 (2009), p.731-769.

[GN00] Garcke H., Novick-Cohen A., *A singular limit for a system of degenerate Cahn-Hilliard equations*, Adv. in Diff. Equ. 5, no. (4-6) (2000), p.401-434.

[Gray87] Grayson M.A., *The heat equation shrinks embedded plane curves to round points*, J. Diff. Geom., vol. 26 (1987), no. 2, p.285-314.

[Gro96] Grosse-Brauckmann K., *Stable constant mean curvature surfaces minimize area*, Pac. J. Math., vol.175 (1996), no. 2, p.527-534.

[GT98] Gilbarg D, Trudinger N. S., *Elliptic Partial Equations of Second Order*, Springer Verlag, 3.edition, 1998.

[Gur93] Gurtin M. E., *Thermomechanics of Evolving Phase Boundaries in the plane*, Oxford: Clarendon Press, 1993.

[GW06] Garcke H., Wieland S., *Surfactant Spreading on Thin Viscous Films: Nonnegative Solutions of a Coupled Degenerate System*, SIAM J.Math.Anal., vol.37 (2006), no. 6, p.2025-2048.

[Hil03] Hildebrandt S., *Analysis 2*, Springer Verlag, 2003.

[HMRR02] Hutchings M., Morgan F., Ritoré M., Ros A., *Proof of the Double Bubble Conjecture*, Ann. Math., vol.155 (2002), p.459-489.

[Hui84] Huisken G., *Flow by mean curvature of convex surfaces into spheres*, J.Diff.Geom., vol. 20 (1984), p.237-266.

[Hui86] Huisken G., *Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature*, Invent.math., vol. 84 (1986), p.463-480.

[Hui87] Huisken G., *The volume preserving mean curvature flow*, J. Reine Angew. Math., vol. 382 (1987), p.35-48.

[IK01a] Ito K., Kohsaka Y., *Three phase boundary motion by surface diffusion: Stability of a mirror symmetric stationary solutions*, Interfaces Free Bound., vol 3 (2001), p.45-80.

[IK01b] Ito K., Kohsaka Y., *Three phase boundary motion by surface diffusion in triangular domain*, Adv. Math. Sci. Appl., vol 11 (2001), p.753-779.

[IY03] Ikota R., Yanagida E., *A stability criterion for stationary curves to the curvature-driven motion with a triple junction*, Diff.Int.Equ., vol. 16 (2003), no. 6, p.707-726.

[Jae01] Jänich K. *Vector analysis*, Springer Verlag, 2001.

[Jo98] Jost J. *Partielle Differentialgleichungen*, Springer Verlag, 1998.

[Kat95] Kato T., *Perturbation theory for linear operators*, Springer Verlag, 1995.

[Koe03] Königsberger K. *Analysis 2*, 5th ed., Springer Verlag, 2003.

[Kue06] Kühnel W., *Differential Geometry*, 2nd edition, American Mathematical Society, 2006.

[Lun95] Lunardi A., *Analytic semigroups and optimal regularity in parabolic problems*, Birkhäuser Verlag, 1995.

[Mu57] Mullins W. W., *Theory of thermal grooving*, J. Appl. Phys., vol. 28 (1957), p.333-339.

[Ros93] Rosenberg H., *Hypersurfaces of constant curvature in space forms*, Bull.Sc.Math., vol. 117 (1993), no. 2, p.211-239.

[RS97] Ros A., Souam R., *On Stability of Capillary Surfaces in a Ball*, Pac. J. Math., vol.178 (1997), no. 2, p.345-361.

[RV95] Ros A., Vergasta E., *Stability for Hypersurfaces of Constant Mean Curvature with Free Boundary*, Geom.Dedic., vol.56 (1995), p.19-33.

[Spi65] Spivak M., *Calculus on manifolds*, Harper-Collins, 1965.

[Sta95] Stahl A., *Mean Curvature Flow with Neumann Boundary Conditions on Smooth Hypersurfaces*, Preprint SFB 382 Nr.12, Uni.Tübingen, (1995).

[Sta96] Stahl A., *Convergence of solutions to the mean curvature flow with a Neumann boundary condition*, Calc.Var.Part.Diff.Equ., vol. 4 (1996), p.421-441.

[Vog00] Vogel, T. I., *Sufficient Conditions for Capillary Surfaces to be Energy Minima*, Pac. J. Math., vol.194 (2000), no. 2, p.469-489.

[Weid76] Weidmann J., *Linear operators in Hilbert spaces*, Springer Verlag, 1976.

[Zeid86] Zeidler E., *Nonlinear Functional Analysis and its Applications I*, Springer Verlag, 1986.