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Biextensions of 1-Motives by 1-Motives

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Abstract

Let S be a scheme and let G_i $(i = 1, 2, 3)$ be an extension of an abelian S-scheme A_i by a S-torus $Y_i(1)$. We first sketch the proof of the following result: In the topos $\mathbf{T}_{\mathrm{fppf}}$, the category of biextensions of (G_1, G_2) by G_3 is equivalent to the category of biextensions of (A_1, A_2) by $Y_3(1)$. Using this result, we define the notion of biextension of 1-motives by 1-motives. If $\mathcal{M}(S)$ denotes what should be the Tannakian category generated by 1-motives over S (in a geometrical sense), as a candidate for the morphisms of $\mathcal{M}(S)$ from the tensor product of two 1-motives $M_1 \otimes M_2$ to another 1-motive M_3, we propose the isomorphism classes of biextensions of (M_1, M_2) by M_3.

Résumé

Soient S un schéma et G_i $(i = 1, 2, 3)$ une extension d'un S-schéma abélien A_i par un S-tore $Y_i(1)$. On esquisse la preuve du résultat suivant : dans le topos $\mathbf{T}_{\mathrm{fppf}}$, la catégrie des biextensions de (G_1, G_2) par G_3 est équivalente à la catégorie des biextensions de (A_1, A_2) par $Y_3(1)$. En utilisant ce résultat, on définit la notion de biextension de 1-motifs par des 1-motifs. Si $\mathcal{M}(S)$ désigne ce que devrait être la catégorie tannakienne engendrée par les 1-motifs sur S (en un sens géométrique), on propose comme candidat pour les morphismes de $\mathcal{M}(S)$ du produit tensoriel de deux 1-motifs $M_1 \otimes M_2$ vers un 1-motif M_3, les classes d'isomorphismes de biextensions de (M_1, M_2) par M_3.

Version française abrégée

Soient S un schéma et G_i $(i = 1, 2, 3)$ une extension d'un S-schéma abélien A_i par un S-tore $Y_i(1)$. Dans le topos $\mathbf{T}_{\mathrm{fppf}}$, la catégrie des biextensions de (G_1, G_2) par G_3 est équivalente à la catégorie des biextensions des S-schémas abéliens sous-jacents (A_1, A_2) par le S-tore sous-jacent $Y_3(1)$. En utilisant ce résultat, on définit la notion de biextension de 1-motifs par des 1-motifs. De plus, si $\mathcal{M}(S)$ désigne ce que devrait être la catégorie tannakienne engendrée par les 1-motifs sur S (en un sens géométrique), on

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propose comme candidat pour les morphismes de $\mathcal{M}(S)$ du produit tensoriel de deux 1-motifs $M_1 \otimes M_2$ vers un 1-motif M_3, les classes d'isomorphismes de biextensions de (M_1, M_2) par M_3. Dans le cas où $S = \mathrm{Spec}\,(k)$ avec k un corps de charactéristique nulle, cette définition est compatible avec la notion correspondante de morphisme dans la catégorie des systèmes de réalizations. En généralisant, on obtient, modulo isogénies, la notion de morphisme de $\mathcal{M}(S)$ d'un produit tensoriel fini de 1-motifs vers un autre 1-motif.

1. Introduction

Let S be a scheme and let G_i (for $i = 1, 2, 3$) be an extension of an abelian S-scheme A_i by a S-torus $Y_i(1)$. We first see that in the topos $\mathbf{T}_{\mathrm{fppf}}$, the category of biextensions of (G_1, G_2) by G_3 is equivalent to the category of biextensions of the underlying abelian S-schemes (A_1, A_2) by the underlying S-torus $Y_3(1)$. Using this result, we define the notion of biextension of 1-motives by 1-motives.

Let $\mathcal{M}(S)$ be what should be the Tannakian category generated by 1-motives over S in a geometrical sense: objects are sub-quotients of sums of finite tensor products of 1-motives and their duals. We know very little about this category $\mathcal{M}(S)$. If $S = \mathrm{Spec}\,(k)$ with k a field of characteristic 0 embeddable in $\mathbb{C}$, we know something more about $\mathcal{M}(k)$: in fact, identifying 1-motives with their mixed realizations, we can identify $\mathcal{M}(k)$ with a Tannakian sub-category of an "appropriate" Tannakian category $\mathcal{MR}(k)$ of mixed realizations (I 2.1 [8] or 1.10 [6]). However this identification furnishes no new concrete information about the geometrical description of objects and morphisms of $\mathcal{M}(k)$!

Using our definition of biextensions of 1-motives by 1-motives, we propose as *a candidate* for the morphisms of $\mathcal{M}(S)$ from the tensor product of two 1-motives $M_1 \otimes M_2$ to a third one M_3, the isomorphism classes of biextensions of (M_1, M_2) by M_3:

$$\mathrm{Hom}_{\mathcal{M}(S)}(M_1 \otimes M_2, M_3) = \mathrm{Biext}^1(M_1, M_2; M_3) \qquad (1)$$

If $S = \mathrm{Spec}\,(k)$ with k a field of characteristic 0, our definition is compatible with the corresponding notion of morphisms in the category $\mathcal{MR}(k)$ of mixed realizations (theorem 10).

Following Deligne's philosophy of motives described in 1.11 [6], our definition (1) furnishes *the geometrical origin* of the morphisms of $\mathcal{MR}(k)$ from the tensor product of the realizations of two 1-motives to the realization of another 1-motive. For example, if M is a 1-motive over k, the valuation map $ev_M : M \otimes M^\vee \longrightarrow \mathbb{Z}(0)$ of M (which expresses the duality between M and its dual $M^\vee$ as objects of $\mathcal{M}(k)$) is the twist by $\mathbb{Z}(-1)$ of the Poincaré biextension $\mathcal{P}_M$ of M, which we see as morphism $M \otimes M^* \longrightarrow \mathbb{Z}(1)$ of $\mathcal{M}(k)$. Therefore $ev_M = \mathcal{P}_M \otimes \mathbb{Z}(-1) : M \otimes M^\vee \longrightarrow \mathbb{Z}(0)$ is the geometrical origin of the corresponding morphism $\mathrm{T}(M) \otimes \mathrm{T}(M^\vee) \longrightarrow \mathrm{T}(\mathbb{Z}(0))$ in $\mathcal{MR}(k)$ which can therefore be called a *motivic morphism* (cf. 1.11 [6]).

It would be great to have a description of the Tannakian category $\mathcal{M}(S)$ like the one given in [7] for the Tannakian category of mixed Tate motives, but it seems to the author that for the moments we don't have enough mathematical results for such a construction. We recall that if k is a field of characteristic 0, in [4] Brylinski has constructed a Tannakian category generated by 1-motives in term of (absolute) Hodge cycles.

The idea of defining morphisms through biextensions goes back to Grothendieck, who defines pairings from biextensions (cf. [10] Exposé VIII §2). Generalizing Grothendieck's work, in [5] §10.2 Deligne defines the notion of biextension of two complexes of abelian groups concentrated in degree 0 and -1 (over any topos $\mathbf{T}$) by an abelian group. Applying this definition to two 1-motives M_1, M_2 over $\mathbb{C}$ and to $\mathbb{G}_m$, he associates to each isomorphism class of such biextensions, a pairing $\mathrm{T}(M_1) \otimes \mathrm{T}(M_2) \longrightarrow \mathrm{T}(\mathbb{G}_m)$ in the category $\mathcal{MR}(\mathbb{C})$.

We can extend definition (1) to a finite tensor product of 1-motives: modulo isogeny, a morphism from a finite tensor product of 1-motives to a 1-motive can be described geometrically as a sum of isomorphism classes of biextensions of 1-motives by 1-motives (Theorem 11).

A special case of definition (1) was already used in the computation of the unipotent radical of the Lie algebra of the motivic Galois group of a 1-motive defined over a field k of characteristic 0 (cf. [1]).

In this paper S is a scheme.

2. Biextensions of some extensions

Let $\mathbf{T}_{\mathrm{fppf}}$ be the topos associated to the site of locally of finite presentation S-schemes, endowed with the fppf topology. If P, Q and G are S-group schemes in the topos $\mathbf{T}_{\mathrm{fppf}}$, $\mathbf{Biext}(P,Q;G)$ (resp. $\mathrm{Biext}^0(P,Q;G)$ and $\mathrm{Biext}^1(P,Q;G)$) denotes the category of biextensions of (P,Q) by G (resp. the group of automorphisms of any biextension of (P,Q) by G and the group of isomorphism classes of biextensions of (P,Q) by G).

Theorem 1 *Let G_i (for $i = 1,2,3$) be a an extension of an abelian S-scheme A_i by a S-torus $Y_i(1)$ with cocharacter group $Y_i = \underline{\mathrm{Hom}}_S(\mathbb{G}_m, Y_3(1))$. We have the following equivalence of categories*

$$\mathbf{Biext}(G_1, G_2; G_3) \cong \mathbf{Biext}(A_1, A_2; Y_3(1))$$

Proof: We will prove the following equivalences of categories:

$$\begin{aligned} \mathbf{Biext}(G_1, G_2; Y_3(1)) &\cong \mathbf{Biext}(A_1, A_2; Y_3(1)) \\ \mathbf{Biext}(G_1, G_2; G_3) &\cong \mathbf{Biext}(G_1, G_2; Y_3(1)) \end{aligned} \tag{2}$$

For the first equivalence, remark that by [9] Exposé X Corollary 4.5, we can suppose that $Y_3(1)$ is $\mathbb{G}_m^{\mathrm{rk} Y_3}$ (if necessary we localize over S for the étale topology). Since the categories $\mathbf{Biext}(G_1, G_2; \mathbb{G}_m)$ and $\mathbf{Biext}(A_1, A_2; \mathbb{G}_m)$ are additive categories in the variable $\mathbb{G}_m$ (cf. [10] I Exposé VII 2.4), it suffices to prove that $\mathbf{Biext}(G_1, G_2; \mathbb{G}_m) \cong \mathbf{Biext}(A_1, A_2; \mathbb{G}_m)$ and this is done in [10] Exposé VIII (3.6.1).
According to the homological interpretation of the group Biext^i ($i = 0, 1$) (cf. [10] Exposé VII 3.6.5 and (3.7.4)), from the exact sequence $0 \to Y_3(1) \to G_3 \to A_3 \to 0$, we have the long exact sequence

$$0 \to \mathrm{Biext}^0(G_1, G_2; Y_3(1)) \to \mathrm{Biext}^0(G_1, G_2; G_3) \to \mathrm{Biext}^0(G_1, G_2; A_3) \to$$

$$\to \mathrm{Biext}^1(G_1, G_2; Y_3(1)) \to \mathrm{Biext}^1(G_1, G_2; G_3) \to \mathrm{Biext}^1(G_1, G_2; A_3) \to .....$$

Hence in order to prove the second equivalence of categories of (2), it is enough to show that $\mathbf{Biext}(G_1, G_2; A_3) = 0$ and this is done in [2].

Remark 2 *Let B be a biextension of (A_1, A_2) by $Y_3(1)$. The biextension $\mathcal{B}$ of (G_1, G_2) by G_3 corresponding to B, through the equivalences of categories (2), is ${\iota_3}_*(\pi_1, \pi_2)^* B$ where, for $i = 1,2,3$, $\pi_i : G_i \longrightarrow A_i$ is the projection of G_i over A_i and $\iota_i : Y_i(1) \longrightarrow G_i$ is the inclusion of $Y_i(1)$ over G_i.*

3. The category of biextensions of 1-motives by 1-motives

In [5] (10.1.10), P. Deligne defines a smooth **1-motive** $M = [X \stackrel{u}{\longrightarrow} G]$ over a scheme S as

(i) a S-group scheme X which is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module,

(ii) an extension G of an abelian S-scheme A by a S-torus $Y(1)$, with cocharacter group Y,

(iii) a morphism $u : X \longrightarrow G$ of S-group schemes.

1-motives are mixed motives of niveau ≤ 1: the weight filtration W_* on $M = [X \stackrel{u}{\longrightarrow} G]$ is $\mathrm{W}_i(M) = M$ for each $i \geq 0, \mathrm{W}_{-1}(M) = [0 \longrightarrow G], \mathrm{W}_{-2}(M) = [0 \longrightarrow Y(1)], \mathrm{W}_j(M) = 0$ for each $j \leq -3$. In particular, we have $\mathrm{Gr}_0^{\mathrm{W}}(M) = [X \longrightarrow 0], \mathrm{Gr}_{-1}^{\mathrm{W}}(M) = [0 \longrightarrow A]$ and $\mathrm{Gr}_{-2}^{\mathrm{W}}(M) = [0 \longrightarrow Y(1)]$.

Let $M_i = [X_i \stackrel{u_i}{\longrightarrow} G_i]$ $(i = 1, 2, 3)$ be a 1-motive over S. The following definition of biextension of (M_1, M_2) by M_3, is a generalization of Deligne's definition [5] (10.2):

Definition 3 *A* ***biextension*** $\mathcal{B} = (\mathcal{B}, \Psi_1, \Psi_2, \Psi, \lambda)$ ***of*** (M_1, M_2) ***by*** M_3 *consists of*

(i) a biextension of $\mathcal{B}$ of (G_1, G_2) by G_3;

(ii) a trivialization (= biaddictive section) Ψ_1 (resp. Ψ_2) of the biextension $(u_1, id_{G_2})^*\mathcal{B}$ (resp. $(id_{G_1}, u_2)^*\mathcal{B}$) of (X_1, G_2) by G_3 (resp. (G_1, X_2) by G_3) obtained as pull-back of the biextension $\mathcal{B}$ via (u_1, id_{G_2}) (resp. (id_{G_1}, u_2));

(iii) a trivialization Ψ of the biextension $(u_1, u_2)^*\mathcal{B}$ of (X_1, X_2) by G_3 obtained as pull-back of the biextension $\mathcal{B}$ by (u_1, u_2), which coincides with the trivializations induced by Ψ_1 and Ψ_2 over $X_1 \times X_2$ (i.e. $(u_1, id_{X_2})^*\Psi_2 = \Psi = (id_{X_1}, u_2)^*\Psi_1$);

(iv) a morphism $\lambda : X_1 \times X_2 \longrightarrow X_3$ of S-group schemes such that $u_3 \circ \lambda : X_1 \times X_2 \longrightarrow G_3$ is compatible with the trivialization Ψ of the biextension $(u_1, u_2)^*\mathcal{B}$ of (X_1, X_2) by G_3.

Let $M_i = [X_i \stackrel{u_i}{\longrightarrow} G_i]$ and $M'_i = [X'_i \stackrel{u'_i}{\longrightarrow} G'_i]$ $(i = 1, 2, 3)$ be 1-motives over S. Moreover let $(\mathcal{B}, \Psi_1, \Psi_2, \lambda)$ be a biextension of (M_1, M_2) by M_3 and let $(\mathcal{B}', \Psi'_1, \Psi'_2, \lambda')$ be a biextension of (M'_1, M'_2) by M'_3.

Definition 4 *A* ***morphism of biextensions*** $(F, \Upsilon_1, \Upsilon_2, \Upsilon, g_3) : (\mathcal{B}, \Psi_1, \Psi_2, \lambda) \longrightarrow (\mathcal{B}', \Psi'_1, \Psi'_2, \lambda')$ *consists of*

(i) a morphism $F = (F, f_1, f_2, f_3) : \mathcal{B} \longrightarrow \mathcal{B}'$ from the biextension $\mathcal{B}$ to the biextension $\mathcal{B}'$. In particular, $f_1 : G_1 \longrightarrow G'_1, f_2 : G_2 \longrightarrow G'_3$ and $f_3 : G_3 \longrightarrow G'_3$ are morphisms of groups S-schemes.

(ii) a morphism of biextensions $\Upsilon_1 = (\Upsilon_1, g_1, f_2, f_3) : (u_1, id_{G_2})^*\mathcal{B} \longrightarrow (u'_1, id_{G'_2})^*\mathcal{B}'$ compatible with the morphism $F = (F, f_1, f_2, f_3)$ and with the trivializations Ψ_1 and Ψ'_1, and a morphism of biextensions $\Upsilon_2 = (\Upsilon_2, f_1, g_2, f_3) : (id_{G_1}, u_2)^*\mathcal{B} \longrightarrow (id_{G'_1}, u'_2)^*\mathcal{B}'$ compatible with the morphism $F = (F, f_1, f_2, f_3)$ and with the trivializations Ψ_2 and Ψ'_2. In particular $g_1 : X_1 \longrightarrow X'_1$ and $g_2 : X_2 \longrightarrow X'_2$ are morphisms of groups S-schemes.

(iii) a morphism of biextensions $\Upsilon = (\Upsilon, g_1, g_2, f_3) : (u_1, u_2)^*\mathcal{B} \longrightarrow (u'_1, u'_2)^*\mathcal{B}'$ compatible with the morphism $F = (F, f_1, f_2, f_3)$ and with the trivializations Ψ and Ψ'.

(iv) a morphism $g_3 : X_3 \longrightarrow X'_3$ of S-group schemes compatible with u_3 and u'_3 (i.e. $u'_3 \circ g_3 = f_3 \circ u_3$) and such that $\lambda' \circ (g_1 \times g_2) = g_3 \circ \lambda$.

Remark 5 *The pair (g_i, f_i) $(i = 1, 2, 3)$ defines a morphism from M_i to M'_i.*

We denote by $\mathbf{Biext}(M_1, M_2; M_3)$ the category of biextensions of (M_1, M_2) by M_3. It is a Picard category (we can make the sum of two objects). Let $\mathrm{Biext}^1(M_1, M_2; M_3)$ be the group of isomorphisms classes of biextensions of (M_1, M_2) by M_3.

Proposition [5] (10.2.14) furnishes a more symmetric description of 1-motives: consider the 7-uplet $(X, Y^\vee, A, A^*, v, v^*, \psi)$ where X and $Y^\vee$ are two S-group schemes which are locally for the étale topology constant group schemes defined by finitely generated free $\mathbb{Z}$-modules; A and A^* are two abelian S-schemes dual to each other; $v : X \longrightarrow A$ and $v^* : Y^\vee \longrightarrow A^*$ are two morphisms of S-group schemes; and ψ is a trivialization of the pull-back $(v, v^*)^*\mathcal{P}_A$ via (v, v^*) of the Poincaré biextension $\mathcal{P}_A$ of (A, A^*).

The pull-back $(v, v^*)^*\mathcal{P}_A$ by (v, v^*) of the Poincaré biextension $\mathcal{P}_A$ of (A, A^*) is a biextension of $(X, Y^\vee)$ by $\mathbb{G}_m$. We denote by $((v, v^*)^*\mathcal{P}_A) \otimes Y$ the biextension of $(X, \mathbb{Z})$ by $Y(1)$ corresponding to the biextension

$(v,v^*)^*\mathcal{P}_A$ through the equivalence of categories $\mathbf{Biext}(X,Y^\vee;\mathbb{G}_m) \cong \mathbf{Biext}(X,\mathbb{Z};Y(1))$ ([10] Exposé VII 2.4). The trivialization ψ of $(v,v^*)^*\mathcal{P}_A$ defines a trivialization $\psi \otimes Y$ of $((v,v^*)^*\mathcal{P}_A)\otimes Y$, and vice versa.

Using Theorem 1 and the more symmetrical definition of 1-motives, we can now give a more useful definition of a biextension of two 1-motives by a third one:

Proposition 6 *Let $M_i = (X_i, Y_i^\vee, A_i, A_i^*, v_i, v_i^*, \psi_i)$ $(i=1,2,3)$ be a 1-motive. A biextension $B = (B, \Psi_1', \Psi_2', \Psi', \Lambda)$ of (M_1,M_2) by M_3 consists of*

(i) a biextension of B of (A_1,A_2) by $Y_3(1)$;

(ii) a trivialization Ψ_1' (resp. Ψ_2') of the biextension $(v_1, id_{A_2})^*B$ (resp. $(id_{A_1}, v_2)^*B$) of (X_1,A_2) by $Y_3(1)$ (resp. of (A_1,X_2) by $Y_3(1)$) obtained as pull-back of the biextension B via (v_1,id_{A_2}) (resp. via (id_{A_1},v_2));

(iii) a trivialization Ψ' of the biextension $(v_1,v_2)^*B$ of (X_1,X_2) by $Y_3(1)$ obtained as pull-back of the biextension B via (v_1,v_2), which coincides with the trivializations induced by Ψ_1' and Ψ_2' over $X_1\times X_2$ (i.e. $(v_1,id_{X_2})^*\Psi_2' = \Psi' = (id_{X_1},v_2)^*\Psi_1'$);

(iv) a morphism $\Lambda : (v_1,v_2)^*B \longrightarrow ((v_3,v_3^*)^*\mathcal{P}_{A_3})\otimes Y_3$ of trivial biextensions, with $\Lambda_{|Y_3(1)}$ equal to the the identity, such that the following diagram is commutative

$$\begin{array}{ccc} Y_3(1) & = & Y_3(1) \\ | & & | \\ (v_1,v_2)^*B & \longrightarrow & ((v_3,v_3^*)^*\mathcal{P}_{A_3})\otimes Y_3 \\ {\scriptstyle \Psi'}\uparrow\downarrow & & \downarrow\uparrow {\scriptstyle \psi_3\otimes Y_3} \\ X_1\times X_2 & \longrightarrow & X_3\times\mathbb{Z}. \end{array} \tag{3}$$

4. Some morphisms of 1-motives

Definition 7 *In the category $\mathcal{M}(S)$, **the morphism** $M_1\otimes M_2 \longrightarrow M_3$ **from the tensor product of two 1-motives to a third 1-motive** is an isomorphism class of biextensions of (M_1,M_2) by M_3. We define* $\mathrm{Hom}_{\mathcal{M}(S)}(M_1\otimes M_2, M_3) = \mathrm{Biext}^1(M_1,M_2;M_3)$.

Remark 8 *Theorem 1 and more in particular the equicalences of categories (2) mean that biextensions satisfy the main property of morphisms of motives: they respect weights.*

Remark 9 *The definition 4 of morphisms of biextensions of 1-motives by 1-motives allows us to define a morphism between the morphisms of $\mathcal{M}(S)$ corresponding to such biextensions. More precisely, let M_i and M_i' (for $i=1,2,3$) be 1-motives over S. If we denote b the morphism $M_1\otimes M_2 \longrightarrow M_3$ corresponding to a biextension $(\mathcal{B},\Psi_1,\Psi_2,\lambda)$ of (M_1,M_2) by M_3 and by b' the morphism $M_1'\otimes M_2' \longrightarrow M_3'$ corresponding to a biextension $(\mathcal{B}',\Psi_1',\Psi_2',\lambda')$ of (M_1',M_2') by M_3', a morphism $(F,\Upsilon_1,\Upsilon_2,\Upsilon,g_3) : (\mathcal{B},\Psi_1,\Psi_2,\lambda) \longrightarrow (\mathcal{B}',\Psi_1',\Psi_2',\lambda')$ of biextensions defines the vertical arrows of the following diagram of morphisms of $\mathcal{M}(S)$*

$$\begin{array}{ccc} M_1\otimes M_2 & \xrightarrow{b} & M_3 \\ \downarrow & & \downarrow \\ M_1'\otimes M_2' & \xrightarrow{b'} & M_3'. \end{array}$$

It is clear now why from the data $(F,\Upsilon_1,\Upsilon_2,\Upsilon,g_3)$ we get a morphism of $\mathcal{M}(S)$ from M_3 to M_3' as remarked in 5. Moreover since M_i and M_i' (for $i=1,2$) are sub-1-motives of the motives $M_1\otimes M_2$ and

$M_1' \otimes M_2'$, *it is clear that from the data* $(F, \Upsilon_1, \Upsilon_2, \Upsilon, g_3)$ *we get also morphisms from* M_1 *to* M_1' *and from* M_2 *to* M_2' *as remarked in 5.*

Theorem 10 *Let* M_i *(for* $i = 1, 2, 3$*) be a 1-motive over a field* k *of characteristic 0 embeddable in* $\mathbb{C}$*. We have that* $\mathrm{Biext}^1(M_1, M_2; M_3) = \mathrm{Hom}_{\mathcal{MR}(k)}(\mathrm{T}(M_1) \otimes \mathrm{T}(M_2), \mathrm{T}(M_3))$.

Proof: see §10.2 [5] and [3].

We will denote by $\mathcal{M}^{\mathrm{iso}}(S)$ the Tannakian category generated by iso-1-motives, i.e. by 1-motives modulo isogenies.

Theorem 11 *Let* M *and* $M_1, \ldots, M_l$ *be 1-motives over* S*. In the category* $\mathcal{M}^{\mathrm{iso}}(S)$*, the morphism* $\otimes_{j=1}^{l} M_j \longrightarrow M$ *from a finite tensor product of 1-motives to a 1-motive is the sum of copies of isomorphism classes of biextensions of* (M_i, M_j) *by* M *for* $i, j = 1, \ldots l$ *and* $i \neq j$*. We have that*

$$\mathrm{Hom}_{\mathcal{M}^{\mathrm{iso}}(S)}(\otimes_{j=1}^{l} M_j, M) = \sum_{\substack{i,j \in \{1,\ldots,l\} \\ i \neq j}} \mathrm{Biext}^1(M_i, M_j; M).$$

Proof: Because morphisms between motives have to respect weights, the non trivial components of the morphism $\otimes_{j=1}^{l} M_j \longrightarrow M$ are the one of the morphism $\otimes_{j=1}^{l} M_j \big/ \mathrm{W}_{-3}(\otimes_{j=1}^{l} M_j) \longrightarrow M$. Since "to tensorize a motive by a motive of weight 0" means to take a certain number of copies of the motive itself, using the following Lemma we can conclude.

Lemma 12 *Let* l *and* i *be positive integers and let* $M_j = [X_j \stackrel{u_j}{\longrightarrow} G_j]$ *(for* $j = 1, \ldots, l$*) be a 1-motive defined over* S*. Denote by* M_0 *or* X_0 *the 1-motive* $\mathbb{Z}(0) = [\mathbb{Z} \longrightarrow 0]$*. If* $i \geq 1$ *and* $l + 1 \geq i$*, the motive* $\otimes_{j=1}^{l} M_j / \mathrm{W}_{-i}(\otimes_{j=1}^{l} M_j)$ *is isogeneous to the motive*

$$\sum \Big(\otimes_{k \in \{\nu_0, \ldots, \nu_{l-i+1}\}} X_k \Big) \bigotimes \Big(\otimes_{j \in \{\iota_0, \ldots, \iota_{i-1}\}} M_j / \mathrm{W}_{-i}(\otimes_{j \in \{\iota_0, \ldots, \iota_{i-1}\}} M_j) \Big) \tag{4}$$

where the sum is taken over all the $(l-i+1)$*-uplets* $\{\nu_0, \ldots, \nu_{l-i+1}\}$ *and all the* $(i-1)$*-uplets* $\{\iota_0, \ldots, \iota_{i-1}\}$ *of* $\{0, 1, \cdots, l\}$ *such that* $\{\nu_0, \ldots, \nu_{l-i+1}\} \cap \{\iota_0, \ldots, \iota_{i-1}\} = \emptyset$*,* $\nu_0 < \nu_1 < \ldots < \nu_{l-i+1}$*,* $\iota_0 < \iota_1 < \ldots < \iota_{i-1}$*,* $\nu_a \neq \nu_b$ *and* $\iota_c \neq \iota_d$*, for all* $a, b \in \{0, \ldots, l-i+1\}, a \neq b$ *and* $c, d \in \{0, \ldots, i-1\}, c \neq d$*.*

Proof: see [2].

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