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## Instanton contributions to the $\tau$ decay widths

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### Abstract

Contrary to some previous claims, we find a sizable instanton contribution to the finite energy sum rule used to extract the value of the strong coupling from the measured  $\tau$  decay widths. It is of the same order of magnitude as standard nonperturbative corrections induced by vacuum quark and gluon condensates. Our result indicates that there might be no hierarchy of power corrections in finite energy sum rules at the scale of  $\tau$  mass. Therefore, the standard nonperturbative corrections do not necessarily improve the accuracy of the theoretical prediction, but can rather be used to estimate an intrinsic accuracy of the pure perturbative calculation, which turns out to be rather high on this evidence, of order one percent.

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1. In recent years the hadronic decay widths of the  $\tau$  lepton have attracted considerable attention [1, 2, 3, 4]. The main interest originates from the observation that the inclusive nature of the decay widths allows for a theoretical prediction, comparison of which with the data can fix the value of the strong coupling with an accuracy competitive to its determination at the  $Z^0$  pole. A direct measurement of the coupling at the  $\tau$  scale might provide most conclusive evidence for the perturbative running of the effective QCD coupling. This is important, since the values of the coupling extracted from deep inelastic scattering data and at the  $Z^0$  pole could indicate a slower evolution compared to the QCD renormalization group equations, a possibility that has already triggered some activity towards alternatives to the standard QCD evolution [5, 6, 7].

Due to momentum analyticity, all dynamical information on  $\tau$  decays is contained in the two-point correlation functions of flavour-changing vector and axial vector currents, which are analyzed in the framework of the SVZ operator product expansion [8]. In this approach the perturbative expansion of the correlation functions is complemented by nonperturbative corrections, which are proportional to the vacuum expectation values of local operators build of quark and gluon fields (condensates) and which are suppressed by increasing powers of the  $\tau$  mass  $m_\tau \sim 1.8$  GeV. The main evidence for the applicability of standard SVZ expansions comes from QCD sum rules, which are believed to work with the typical accuracy of 10–20%. The problem is that a determination of  $\alpha_s(m_\tau)$  with, say, 10% accuracy requires knowledge of the decay widths at the one percent level. It has been found that “standard” nonperturbative corrections to the  $\tau$  decay rate do not exceed a few percent [1]. This still leaves open the possibility of nonperturbative effects that are beyond the SVZ expansion, an issue that should be investigated separately.

In the recent paper [9] Nason and Porrati study the contribution to the total  $\tau$  hadronic width from small size instantons in the QCD vacuum. They find that even with the highest value of  $\alpha_s(m_\tau)$  allowed, the instanton contributions never grow beyond  $10^{-6} - 10^{-5}$ , and, therefore, are completely negligible. This smallness can essentially be traced to the well known fact [10] that the instanton density is proportional to the product of light quark masses  $m_u m_d m_s$  and instanton-induced transitions vanish for strictly massless quarks. This is only true, however, for one-instanton contributions in an empty (perturbative) vacuum, which is far from a physical reality. As explained at length in [11], the density of small instantons of size  $\rho$  in the background field of large-scale vacuum fluctuations is modified in such a way that the current quark mass is substituted by an effective mass,

$$m_q \rightarrow m_q - \frac{2}{3}\pi^2 \langle \bar{q}q \rangle \rho^2. \quad (1)$$

This effect of the vacuum “medium” is very large. To obtain a rough estimate, one may replace the product of current quark masses in the answer given in [9] by the product of constituent masses  $M_u \simeq M_d \sim 350$  MeV,  $M_s \sim 500$  MeV. This yields an increase by a factor  $\sim 10^4$ , and boosts the contribution obtained in [9] into the range of percent corrections, which are important. Thus, a more quantitative analysis of this effect is mandatory, and this is the subject of the present paper. More precisely, we calculate the instanton-induced contribution to the relevant two-point correlation functions, which is

related in the framework of the operator product expansion to an exponential correction to the coefficient function in front of six-quark operators  $(\bar{q}q)^3$ . Since the current quark masses on the right hand side of (1) are small compared to the second term, we shall omit them altogether in the following.

**2.** The correlation function of interest is the  $T$ -product of two flavour-changing vector or axial vector currents,

$$\Pi_{\mu\nu}^{ud}(q) = i \int d\Delta e^{iq\Delta} \langle 0|T \left( j_{\mu}^{\dagger}(x) j_{\nu}(y) \right) |0\rangle \quad \Delta \equiv x - y, \quad (2)$$

$$j_{\mu}(x) \equiv \bar{u}(x) \gamma_{\mu} \left\{ \begin{array}{c} 1 \\ \gamma_5 \end{array} \right\} d(x), \quad (3)$$

of two quark fields, which we denote by  $u$  and  $d$ , in a theory with  $n_f$  massless quarks. At large euclidean  $q^2$  the characteristic size  $\rho$  of the instanton scales as  $\rho^2 \sim q^{-2} \ll \Lambda^{-2}$ , where  $\Lambda$  is a typical QCD scale parameter. Therefore an effective Lagrangian approach is appropriate. To lowest order in  $(\rho\Lambda)^2$  an instanton of size  $\rho$  (in the singular gauge) induces an effective  $2n_f$ -quark vertex with the quark legs corresponding to the instanton zero modes. Since these zero modes possess definite chirality it is most convenient to use the two-component Weyl spinor notation: the euclidean quark fields and  $\gamma$ -matrices are written as

$$q = \begin{pmatrix} i\chi_{\dot{\alpha}}^k \\ \psi^{k\alpha} \end{pmatrix}, \quad \bar{q} = (\bar{\psi}_k^{\dot{\alpha}}, i\bar{\chi}_{k\alpha}), \quad \gamma_{\mu} = \begin{pmatrix} 0 & \bar{\sigma}_{\mu\dot{\alpha}\alpha} \\ \sigma_{\mu}^{\alpha\dot{\alpha}} & 0 \end{pmatrix}, \quad (4)$$

where  $k = 1, 2, \dots, N_c$  is a colour and  $\alpha, \dot{\alpha} = 1, 2$  are spinor indices. We use the notations  $\sigma_{\mu}^{\alpha\dot{\alpha}} = (1, -i\vec{\sigma})$ ,  $\bar{\sigma}_{\mu\dot{\alpha}\alpha} = (1, i\vec{\sigma})$ , where  $\vec{\sigma}$  are the Pauli matrices, and for vectors  $v_{\mu}$  define  $v \equiv v_{\mu} \sigma_{\mu}$ ,  $\bar{v} \equiv v_{\mu} \bar{\sigma}_{\mu}$ . Whenever a mixture of spinor and colour indices takes place, it is understood that spinor matrices act in the  $2 \times 2$  upper left corner of  $N_c \times N_c$  colour matrices. The instanton vertex is then described by 't Hooft's effective Lagrangian

$$L_{\psi}^I = (4\pi^2 \rho_I^3)^{n_f} O_I, \quad O_I = \prod_{i=1}^{n_f} (\bar{\chi}_i \varphi) (\bar{\kappa} \psi_i), \quad (5)$$

where  $\varphi^{k\alpha} = \epsilon^{\alpha\beta} U_{I\beta}^k$ ,  $\bar{\kappa}_{k\alpha} = -\epsilon_{\alpha\beta} (U_I^{\dagger})_k^{\beta}$ ,  $U_I$  is the  $SU(N_c)$ -matrix of the instanton orientation and  $\epsilon^{\alpha\beta}$  is the antisymmetric tensor with  $\epsilon^{12} = 1$ . Colour and spinor indices will be suppressed, whenever no confusion can arise.

Let us recall the derivation of this expression. To find the coefficient multiplying the operator  $O_I$ , one should compare the  $2n_f$ -quark Green function  $\prod_{i=1}^{n_f} \psi_i(x_i) \bar{\chi}_i(y_i)$ , evaluated in the instanton background in the near mass shell limit  $x_i, y_i \rightarrow \infty$ , with the result obtained from the effective Lagrangian (5). The  $2n_f$ -quark amplitude is simply obtained by substituting the zero modes for the quark fields, i.e. equals  $\prod_{i=1}^{n_f} \kappa_0(x_k) \bar{\varphi}_0(y_k)$ . Explicit expressions for the zero modes for the instanton with center at the origin are

$$\kappa_0(x) = \frac{\bar{x}\varphi}{2\pi^2x^4} \frac{2\pi\rho_I^{3/2}}{\Pi_x}, \quad \bar{\varphi}_0(x) = \frac{\bar{k}x}{2\pi^2x^4} \frac{2\pi\rho_I^{3/2}}{\Pi_x}, \quad (6)$$

where  $\Pi_x = 1 + \rho_I^2/x^2$ . For the instanton with center at  $x_0$ , the zero modes are obtained by the obvious substitution  $x \rightarrow x - x_0$ . The large distance behaviour of the zero modes coincides (in the singular gauge) with that of the perturbative propagator up to the factors  $2\pi\rho_I^{3/2}\varphi$ ,  $2\pi\rho_I^{3/2}\bar{k}$ , respectively. Therefore the leading term of an expansion of the zero modes in  $\rho_I^2/x^2$  is indeed reproduced by an insertion of the effective Lagrangian (5). Subsequent terms in this expansion correspond to the exchange of soft gluons, that is, to higher dimensional operators in the effective Lagrangian containing extra gluon fields.

Let us now return to the correlation functions. In two-component notation

$$j_\mu(x) = \bar{\psi}_u(x)\bar{\sigma}_\mu\psi_d(x) + \lambda\bar{\chi}_u(x)\sigma_\mu\chi_d(x), \quad \lambda = \begin{cases} -1 & \text{vector} \\ 1 & \text{axial vector} \end{cases}. \quad (7)$$

In order to find the one-instanton contribution to the coefficient function of the operator of lowest dimension,  $O_I$ , one inserts the current product into a  $2n_f$ -point Green function, evaluates it in the instanton background and takes the large distance asymptotics. The relevant diagrams for all possible contractions of

$$\left[ \prod_{i=1}^{n_f} \psi(z_i)\bar{\chi}(z'_i) \right] j_\mu^\dagger(x)j_\nu(y) \quad z_i, z'_i \rightarrow \infty \quad (8)$$

are shown in Fig.1, where the instanton is depicted as its effective  $2n_f$ -quark vertex. The solid line denotes the quark zero modes and the dashed line represents the quark propagator over the nonzero modes in the instanton background. Its explicit form is known from [12]:

$$\begin{aligned} S_I(x, y) \equiv -i\langle\psi(x)\bar{\psi}(y)\rangle_I &= \frac{1}{\sqrt{\Pi_x\Pi_y}} \left\{ \frac{\Delta}{2\pi^2\Delta^4} (1 - U_I P U_I^\dagger) \right. \\ &\quad \left. + \frac{\Delta}{2\pi^2\Delta^4} \left( 1 + \rho_I^2 \frac{U_I x \bar{y} U_I^\dagger}{x^2 y^2} \right) + \frac{\sigma_\mu \rho_I^2}{4\pi^2 \Delta^2} \frac{U_I x \bar{\Delta} \sigma_\mu \bar{y} U_I^\dagger}{x^2 y^2 (y^2 + \rho_I^2)} \right\}, \\ \bar{S}_I(x, y) \equiv i\langle\chi(x)\bar{\chi}(y)\rangle_I &= \frac{1}{\sqrt{\Pi_x\Pi_y}} \left\{ \frac{\bar{\Delta}}{2\pi^2\Delta^4} (1 - U_I P U_I^\dagger) \right. \\ &\quad \left. + \frac{\bar{\Delta}}{2\pi^2\Delta^4} \left( 1 + \rho_I^2 \frac{U_I x \bar{y} U_I^\dagger}{x^2 y^2} \right) + \frac{\bar{\sigma}_\mu \rho_I^2}{4\pi^2 \Delta^2} \frac{U_I x \bar{\sigma}_\mu \Delta \bar{y} U_I^\dagger}{x^2 y^2 (x^2 + \rho_I^2)} \right\}, \end{aligned} \quad (9)$$

where  $P$  is the projector onto the  $2 \times 2$  upper left corner of the  $N_c \times N_c$  matrix of the instanton orientation. Just as in the case of the zero modes the near mass shell asymptotics of these propagators coincides with the perturbative propagators up to factors  $1/\sqrt{\Pi}$ :

$$S_I(x, z) \xrightarrow{z \rightarrow \infty} \frac{1}{\sqrt{\Pi_x}} \frac{-z}{2\pi^2 z^4} \quad \bar{S}_I(z, x) \xrightarrow{z \rightarrow \infty} \frac{1}{\sqrt{\Pi_x}} \frac{\bar{z}}{2\pi^2 z^4} \quad (10)$$

Thus amputation of the external legs in the near mass shell limit, denoted by a cross in the diagrams in Fig.1, leaves the spinors  $\kappa$  and  $\bar{\varphi}$  for the zero mode legs, see eq.(6), and a factor  $1/\sqrt{\Pi}$  for the propagator legs from (10). The leading term in the operator product expansion of the current product in the instanton background is then given by

$$\begin{aligned} & \int d\Delta e^{iq\Delta} \langle j_\mu(x) j_\nu(y) \rangle_{(\bar{\psi}\psi)^{n_f}}^I = \int d\Delta e^{iq\Delta} \int \frac{d\rho}{\rho^5} d(x_0) dU (4\pi^2 \rho^3)^{n_f-1} \prod_{i \neq u, d} [(\bar{\chi}_i \varphi)(\bar{\kappa} \psi_i)] \\ & \times \left\{ (4\pi^2 \rho^3) \left[ \text{tr} (\bar{\sigma}_\mu S_I(x, y) \bar{\sigma}_\nu S_I(y, x)) + \text{tr} (\sigma_\mu \bar{S}_I(x, y) \sigma_\nu \bar{S}_I(y, x)) \right] (\bar{\chi}_u \varphi)(\bar{\kappa} \psi_u)(\bar{\chi}_d \varphi)(\bar{\kappa} \psi_d) \right. \\ & - 2\pi \rho^{3/2} \frac{1}{\sqrt{\Pi_x}} \left[ (\bar{\chi}_u \varphi)(\bar{\varphi}_0(y) \bar{\sigma}_\nu S_I(y, x) \bar{\sigma}_\mu \psi_u)(\bar{\chi}_d \varphi)(\bar{\kappa} \psi_d) \right. \\ & \left. \left. + (\bar{\chi}_u \varphi)(\bar{\kappa} \psi_u)(\bar{\chi}_d \sigma_\mu \bar{S}_I(x, y) \sigma_\nu \bar{\kappa}_0(y))(\bar{\kappa} \psi_d) \right] + (u \leftrightarrow d, \mu \leftrightarrow \nu, x \leftrightarrow y) \right. \\ & \left. + \frac{1}{\sqrt{\Pi_x \Pi_y}} \left[ \left\{ (\bar{\chi}_u \varphi)(\bar{\varphi}_0(x) \bar{\sigma}_\mu \psi_u)(\bar{\chi}_d \varphi)(\bar{\varphi}_0(y) \bar{\sigma}_\nu \psi_d) + (\bar{\chi}_u \sigma_\nu \kappa_0(y))(\bar{\kappa} \psi_u)(\bar{\chi}_d \sigma_\mu \kappa_0(x))(\bar{\kappa} \psi_d) \right\} \right. \right. \\ & \left. \left. + \lambda \left\{ (\bar{\chi}_u \sigma_\nu \kappa_0(y))(\bar{\varphi}_0(x) \bar{\sigma}_\mu \psi_u)(\bar{\chi}_u \varphi)(\bar{\kappa} \psi_u) + (\bar{\chi}_d \sigma_\mu \kappa_0(x))(\bar{\varphi}_0(y) \bar{\sigma}_\nu \psi_d)(\bar{\chi}_d \varphi)(\bar{\kappa} \psi_d) \right\} \right] \right\}. \end{aligned} \quad (11)$$

Only the last term coming from the last diagram of Fig.1 differs by its sign for the vector and axial vector currents. The integration over the instanton size is performed with the instanton density [13]

$$d(\rho) = \frac{c_1}{(N_c - 1)!(N_c - 2)!} e^{-N_c c_2 + n_f c_3} \left( \frac{2\pi}{\alpha(\rho)} \right)^{2N_c} e^{-2\pi/\alpha(\rho)}, \quad (12)$$

where  $c_1 = 0.466$  and the constants  $c_2, c_3$  take the values  $c_2 = 1.54, c_3 = 0.153$  in the  $\overline{MS}$  scheme. Now it is only a matter of patience to integrate over the instanton position and size and to take the Fourier transform. A typical  $\rho$ -integral has the structure

$$\int_0^\infty \frac{d\rho}{\rho^5} \rho^{2B+1} \ln^E \frac{1}{\rho^2 \Lambda^2} K_1(\rho q), \quad (13)$$

where  $K_1$  is a modified Bessel function and  $B, E$  are some numbers. The logarithmic size-dependence stems from three sources: from the two-loop running of  $\alpha(\rho)$  in the exponent of the instanton density, from the preexponential factor  $\alpha(\rho)^{-2N_c}$  and from the anomalous dimension of the operators in the effective Lagrangian which are normalized at the scale  $\mu = 1/\rho$ .<sup>†</sup> To perform this integral, we first calculate a similar integral

<sup>†</sup> For consistency, we adjust the values of the one-loop and the two-loop  $\Lambda$  to reproduce the same value of  $\alpha(m_\tau)$ , which is the only parameter appearing in the final formulas.

with the logarithmic factor substituted by a power-like one  $(\rho\Lambda)^{-2\epsilon}$ , and introduce the projection operator

$$\mathcal{P}_E^\epsilon \{F(\epsilon)\} \equiv \frac{\Gamma(E+1)}{2\pi i} \int_C \frac{d\epsilon}{\epsilon^{E+1}} F(\epsilon), \quad (14)$$

where the contour  $C$  wraps around the negative real axis. For example, for integer  $E$ , or general  $E$  and  $\ln x > 0$ , one has  $\mathcal{P}_E^\epsilon \{x^\epsilon\} = \ln^E x$ . For arbitrary real  $E$  this operation produces an asymptotic series

$$\mathcal{P}_E^\epsilon \left\{ \left( \frac{q^2}{\Lambda^2} \right)^\epsilon Z(\epsilon) \right\} = \ln^E \left( \frac{q^2}{\Lambda^2} \right) \sum_{n=0}^{\infty} \frac{\Gamma(E+1)}{\Gamma(E+1-n)} \frac{z_n}{\ln^n(q^2/\Lambda^2)} \quad (15)$$

with  $Z(\epsilon) = \sum z_n \epsilon^n$  being the result of the  $\rho$ -integration. For natural  $E$  the series terminates and yields an exact answer for integrals of the type (13). For general real  $E$  the asymptotic expansion in (15) yields a good approximation provided  $\ln(q^2/\Lambda^2)$  is sufficiently large, and provided the integral is dominated by  $\rho < \Lambda^{-1}$ .

Using the projection operator defined in (14), we can write down the instanton contribution to the two-point correlation function of (axial) vector currents in the background of vacuum fermion fields in the compact form

$$\begin{aligned} \int d\Delta e^{iq\Delta} \langle j_\mu(x) j_\nu(y) \rangle_{(\bar{\psi}\psi)^{n_f}}^I &= 4^{B-2} (4\pi^2)^{n_f} \frac{c_1}{(N_c-1)!(N_c-2)!} e^{-N_c c_2 + n_f c_3} \\ &\times \left( \frac{1}{q^2} \right)^{3n_f/2} \left( \frac{2\pi}{\alpha(q)} \right)^{2N_c} e^{-2\pi/\alpha(q)} \int dU \prod_{i \neq u, d} (\bar{\chi}_i \varphi) (\bar{\kappa} \psi_i) (-\beta_0 \alpha(q))^E \\ &\times \mathcal{P}_E^\epsilon \left\{ e^{-\epsilon/(\beta_0 \alpha(q))} F(\epsilon) \left[ (\delta_{\mu\nu} q^2 - q_\mu q_\nu) \frac{2}{2B-1-2\epsilon} (\bar{\chi}_u \varphi) (\bar{\kappa} \psi_u) (\bar{\chi}_d \varphi) (\bar{\kappa} \psi_d) \right. \right. \\ &\quad + (\bar{\chi}_u \varphi) (\bar{\kappa} [q\bar{\sigma}_\mu - q_\mu] \psi_u) (\bar{\chi}_d \varphi) (\bar{\kappa} [q\bar{\sigma}_\nu - q_\nu] \psi_d) \\ &\quad + (\bar{\chi}_u [\sigma_\nu \bar{q} - q_\nu] \varphi) (\bar{\kappa} \psi_u) (\bar{\chi}_d [\sigma_\mu \bar{q} - q_\mu] \varphi) (\bar{\kappa} \psi_d) \\ &\quad + \lambda (\bar{\chi}_u [\sigma_\nu \bar{q} - q_\nu] \varphi) (\bar{\kappa} \psi_u) (\bar{\chi}_u \varphi) (\bar{\kappa} [q\bar{\sigma}_\mu - q_\mu] \psi_u) \\ &\quad \left. \left. + \lambda (\bar{\chi}_d [\sigma_\mu \bar{q} - q_\mu] \varphi) (\bar{\kappa} \psi_d) (\bar{\chi}_d \varphi) (\bar{\kappa} [q\bar{\sigma}_\nu - q_\nu] \psi_d) \right] \right\}. \quad (16) \end{aligned}$$

Here we defined

$$B = \frac{b+3n_f}{2}, \quad b = -4\pi\beta_0, \quad E = 2N_c - \frac{2\pi\beta_1}{\beta_0} + \gamma, \quad (17)$$

$$F(\epsilon) \equiv 4^{-\epsilon} \frac{\Gamma(B-\epsilon)\Gamma(B-2-\epsilon)\Gamma(B-1-\epsilon)^2}{\Gamma(2B-2-2\epsilon)}, \quad (18)$$

where  $\beta_0, \beta_1$  are the first two coefficients of the  $\beta$ -function,  $\beta(\alpha) \equiv \mu^2 \partial / \partial \mu^2 \alpha = \sum \beta_n \alpha^{n+2}$ . The quantity  $\gamma$  stands schematically for the anomalous dimensions of the

six-quark operators – that should be normalized at  $q$  – appearing in the square brackets in eq.(16). An overall factor of two, which accounts for an equal contribution in the background of an antiinstanton, is included in (16).

A comment is in order, concerning the integration over the instanton size. The integration over  $\rho$  contains a divergent term  $\sim (\delta_{\mu\nu}q^2 - q_\mu q_\nu) 1/q^4 \int d\rho \rho^{2B-5}$ . This term comes from large distances and has to be identified with an instanton contribution to the matrix element of the operator  $\alpha G_{\mu\nu}G_{\mu\nu}$ . It will be neglected in the following. The remaining terms are convergent and represent the instanton contribution to the coefficient functions of the  $2n_f$ -quark operators shown in (16). The distinction between contributions to the matrix elements and to the coefficient functions is unambiguous. The relevant terms can easily be distinguished by their  $q^2$ -dependence.

The answer in (16) has to be averaged over the physical vacuum. For numerical estimation, we assume factorization of the vacuum expectation values of the relevant six-quark operators, which allows to express them in terms of the quark condensate  $\langle \bar{q}q \rangle$ . Upon factorization, the second and the third term in the square brackets in (16) vanish. Since we are working in the chiral limit  $m_u = m_d = m_s = 0$ , we use the  $SU(3)$ -flavour symmetry  $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \langle \bar{s}s \rangle$ , and take the numerical value  $\langle \bar{q}q \rangle = -(240 \text{ MeV})^3$  at the scale 1 GeV. Extracting the Lorentz structure,  $\Pi_{\mu\nu}^{ud}(q) = (q_\mu q_\nu - \delta_{\mu\nu}q^2) \Pi^{ud}(q^2)$ , and taking the trivial integration over the instanton orientation, we arrive at

$$\begin{aligned} \Pi_{inst}^{ud}(q^2) &= -2 \cdot 4^{B-2} \left( \frac{2\pi^2}{N_c} \right)^{n_f} (2\pi)^{2N_c} \frac{c_1}{(N_c - 1)!(N_c - 2)!} e^{-N_c c_2 + n_f c_3} \\ &\times \prod_{i=1}^{n_f} \left[ \frac{\langle \bar{q}_i q_i \rangle(\mu)}{q^3} \alpha(\mu)^{\gamma/n_f} \right] e^{-2\pi/\alpha(q)} \left( \frac{1}{\alpha(q)} \right)^{2N_c + \gamma} \\ &\times (-\beta_0 \alpha(q))^E \mathcal{P}_E^\epsilon \left\{ e^{-\epsilon/(\beta_0 \alpha(q))} F(\epsilon) \left[ \frac{1}{2B - 1 - 2\epsilon} - \lambda \right] \right\}. \end{aligned} \quad (19)$$

For the anomalous dimensions of the six-quark operators factorization amounts to taking  $\gamma = 4n_f/b$ , where  $4/b$  is the anomalous dimension of the quark condensate. However, since factorization is known not to be consistent with the renormalization group, we should rather allow  $\gamma$  to vary in a certain range, ascribing the variation of  $\Pi_{inst}^{ud}$  to the uncertainty inherent to the factorization hypothesis.<sup>‡</sup>

Corrections to eq.(19) come from (i) higher-dimensional operators, suppressed by powers of  $(\rho\Lambda)^2$  and (ii) exchange of hard particles, suppressed by powers of  $\alpha(\rho_*)$ , where  $\rho_*$  is the average size of instantons that contribute to the correlation functions. From now on we shall discuss the case  $n_f = N_c = 3$ ,  $q = m_\tau$ , which is the case of physical interest in  $\tau$  decays, and regard  $\Pi_{inst}^{ud}$  as a function of  $\alpha(m_\tau)$ . Then  $b = B = 9$  and

<sup>‡</sup> Strictly speaking, in presence of instantons, the renormalization group equations for  $(2n_f)$ -quark operators are more complicated and include mixing with the unity operator. This can be checked by a direct calculation of the (anti)instanton contribution to the vacuum expectation value  $\langle (\bar{q}q)^{n_f} \rangle$ , which proves to contain a logarithm of the renormalization scale. Taking into account this mixing corresponds to the calculation of correlation functions in the instanton-antiinstanton background in the spirit of [14], but does not seem appropriate for our present purposes since the size of the antiinstanton in this calculation turns out to be unacceptably large.

$E = 34/9$ . To estimate the value of  $\alpha(m_\tau)$ , at which the above mentioned corrections become of the same order as the contribution kept in (19), we should first find  $\rho_*$ . The distribution of instanton sizes relevant to the term proportional to  $\lambda$  in (19) is given by the integral

$$\int \frac{d\rho}{\rho^5} \rho^{2B+1} \ln^E \frac{1}{\rho^2 \Lambda^2} \int_0^1 du \left( \frac{2}{\sqrt{u\bar{u}}} K_1 \left( \frac{\rho m_\tau}{\sqrt{u\bar{u}}} \right) + \frac{\rho m_\tau}{u\bar{u}} K_0 \left( \frac{\rho m_\tau}{\sqrt{u\bar{u}}} \right) \right) \quad (20)$$

with  $\bar{u} = 1 - u$ . The  $u$ -integration is sharply peaked at  $u = 1/2$ , which allows to put  $u = 1/2$  for examination of the  $\rho$ -distribution. We plot the real part of this distribution in Fig.2 as a function of  $\alpha(m_\tau)$  and use  $\alpha(m_\tau) = (-\beta_0 \ln m_\tau^2/\Lambda^2)^{-1}$  to relate  $\Lambda$  and  $\alpha(m_\tau)$ .

The distribution clearly shows two peaks. The first one is located at values  $\rho \sim (4 - 5)/m_\tau$ , the precise coefficient depending weakly on  $E$  and  $\alpha(m_\tau)$ , and corresponds to the effective value of the instanton size  $\rho_* \sim 400$  MeV. The second one corresponds to the integration region  $\rho > 1/\Lambda$ , and its contribution should be small enough (say, by factor 5) in order that the calculation makes sense. Combined with the requirement that the value of the coupling at the effective scale is not too large, say  $\alpha(\rho_*) < 1$ , this restriction yields the critical value of  $\alpha(m_\tau)$ , above which the instanton contribution is ill-defined

$$\alpha(m_\tau)_{cr} \approx 0.32, \quad (21)$$

even if  $\Pi_{inst}^{ud}$  is still small. This value of the coupling lies well within the interval under discussion  $\alpha(m_\tau) \sim 0.28 - 0.38$  [4]. Note that since  $\rho_* m_\tau \sim 4 - 5$  is a large number, the effect of including the logarithms  $\ln 1/(\rho^2 \Lambda^2)$  into the integration is important, and yields a suppression by roughly a factor

$$\left( \frac{\ln(1/\rho_*^2 \Lambda^2)}{\ln(m_\tau^2/\Lambda^2)} \right)^E \sim \left( \frac{\alpha(m_\tau)}{\alpha(\rho_*)} \right)^E \sim 10^{-2} \quad (22)$$

for typical values of  $E$  and  $\alpha(m_\tau)$ . Technically, this effect comes from summation of higher-order terms in the expansion in (15).

**3.** We are now in the position to calculate the instanton contribution to the  $\tau$  decay widths. As usual, they are normalized as

$$R_\tau \equiv \frac{\Gamma(\tau^- \rightarrow \nu_\tau + \text{hadrons})}{\Gamma(\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e)}. \quad (23)$$

Decomposing the correlation functions as

$$\Pi_{\mu\nu, V/A}^{ij}(q) = (q_\mu q_\nu - \delta_{\mu\nu} q^2) \Pi_{(1), V/A}^{ij}(q^2) + q_\mu q_\nu \Pi_{(0), V/A}^{ij}(q^2) \quad (24)$$



and using analyticity in the cut  $q^2$ -plane, one can express the total hadronic width  $R_\tau$  as

$$R_\tau = 6\pi i \int_{|s|=m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[ \left(1 + 2\frac{s}{m_\tau^2}\right) \Pi_{(1)}(s) + \Pi_{(0)}(s) \right], \quad (25)$$

where  $s = -q^2$  and

$$\Pi_{(J)}(s) \equiv |V_{ud}|^2 \left( \Pi_{(J),V}^{ud}(s) + \Pi_{(J),A}^{ud}(s) \right) + |V_{us}|^2 \left( \Pi_{(J),V}^{us}(s) + \Pi_{(J),A}^{us}(s) \right) \quad (26)$$

with the CKM matrix elements  $V_{ud}, V_{us}$ . Experimentally the contribution from the strange current,  $R_{\tau,S}$ , can be separated according to the net strangeness of the final state. The nonstrange contribution can be further resolved into vector and axial vector pieces,  $R_{\tau,V}$  and  $R_{\tau,A}$ , according to whether the final state contains an even or odd number of pions. Thus, one defines

$$R_\tau = R_{\tau,V} + R_{\tau,A} + R_{\tau,S} \equiv 3 \left( |V_{ud}|^2 + |V_{us}|^2 \right) (1 + \delta), \quad (27)$$

$$R_{\tau,V/A} \equiv \frac{3}{2} |V_{ud}|^2 (1 + \delta_{V/A}), \quad (28)$$

separating the ‘‘Born’’ term from the QCD corrections  $\delta$  (a small multiplicative electroweak correction is understood), that are of the order of 20%. We will now determine the instanton contribution  $\delta^{inst}, \delta_{V/A}^{inst}$  to the QCD corrections. In the massless limit the correlation functions are transverse, see eq.(19), i.e.  $\Pi_{(0)}^{inst}(q^2) \equiv 0$ . The contour integral in (25) is then taken including logarithms of  $s/\Lambda^2$  exactly, using the same trick as above for the integration over the instanton size. This produces immediately

$$\begin{aligned} \delta_{V/A}^{inst} &= -\pi \cdot 4^B \left( \frac{2\pi^2}{3} \right)^3 (2\pi)^6 c_1 e^{-3c_2+3c_3} \prod_{i=1}^3 \left[ \frac{\langle \bar{q}_i q_i \rangle(\mu)}{m_\tau^3} \alpha(\mu)^{\gamma/3} \right] e^{-2\pi/\alpha(m_\tau)} \\ &\times \left( \frac{1}{\alpha(m_\tau)} \right)^{6+\gamma} (-\beta_0 \alpha(m_\tau))^E \mathcal{P}_E^\epsilon \left\{ e^{-\epsilon/(\beta_0 \alpha(q))} F(\epsilon) H(\epsilon) \left[ \frac{1}{2B-1-2\epsilon} \pm 1 \right] \right\}, \end{aligned} \quad (29)$$

which is our final result. The upper sign holds for vector currents, the lower sign for axial vector currents. The effect of the contour integration – up to constant factors – is contained in the function  $H(\epsilon)$ ,

$$H(\epsilon) = \sin \pi(B - \epsilon) \left[ \frac{1}{1 - B + \epsilon} - \frac{3}{3 - B - \epsilon} + \frac{2}{4 - B + \epsilon} \right]. \quad (30)$$

Note that the expansion of  $H(\epsilon)$  starts with order  $\epsilon$ , since  $B$  is integer. Therefore the transition from the correlation functions to the decay widths suppresses the instanton

contribution by one power of  $\alpha(m_\tau)$ . With  $\langle \bar{s}s \rangle = \langle \bar{u}u \rangle$  the instanton contribution to the strange decays equals that to the nonstrange decays. Then

$$\delta^{inst} = \frac{1}{2} (\delta_V^{inst} + \delta_A^{inst}) \simeq \frac{1}{20} \delta_V^{inst}. \quad (31)$$

The term differing in sign for the vector and axial vector widths in (29) cancels in the total width, leaving only the first term with the small coefficient  $\sim 1/(2B-1)$ . Thus there is a strong cancellation in the sum of vector and axial vector contributions, which is similar in effect to the cancellation in the contribution from four-quark operators in the standard SVZ expansion [1]. However, this cancellation relies strongly on the factorization approximation, which eliminates the second and third term in (16) having much larger coefficients. For this reason, the instanton contribution to the total width is much more sensitive to deviations from factorization and therefore less reliable numerically. Since  $\delta_V^{inst} = -\delta_A^{inst} + 2\delta^{inst}$ , the axial vector contribution is essentially of the same magnitude as for the vector current but with an opposite sign. In Fig.3 we plot  $\delta_V^{inst}$  and  $\delta^{inst}$  as a function of  $\alpha(m_\tau)$  for three choices of the anomalous dimension of the  $(\bar{q}q)^3$  operator  $\gamma = 2/3, 4/3$ , and  $6/3$ , corresponding to  $E = 28/9, 34/9$ , and  $40/9$ , respectively. (The middle value corresponds to factorization.) At  $\alpha(m_\tau) = 0.32$  we get

$$\delta_V^{inst} \simeq -\delta_A^{inst} = 0.03 - 0.05, \quad \delta^{inst} \simeq 0.002 - 0.003. \quad (32)$$

At larger values of the coupling  $\alpha(m_\tau) \sim 0.34 - 0.36$  the instanton contribution blows up, but our calculation is no longer under control. Thus the instanton contribution is essentially of the same size as the contribution from dimension-6 operators  $\sim \langle \bar{q}q \rangle^2 / q^6$  [1],

$$\delta_V^{D=6} = 0.024 \pm 0.013, \quad \delta_A^{D=6} = -(0.038 \pm 0.020), \quad \delta^{D=6} = -(0.007 \pm 0.004), \quad (33)$$

which are the largest power corrections in the standard approach.

**4.** We performed an explicit calculation of the instanton contribution to the  $\tau$ -decay widths, which corresponds to the exponential correction to the coefficient function in front of the six-quark operators in the operator product expansion of the relevant correlation functions, and which is distinguished by the chiral properties of the instanton-induced effective vertex. Contrary to previous claims, we find a sizable instanton contribution, which for  $\alpha(m_\tau) \approx 0.32$  is of the same order of magnitude as standard nonperturbative corrections induced by nonzero vacuum expectation values of the operators of lowest dimension. An inherent uncertainty of our calculation is about a factor of two for vector and axial vector channels, and maybe larger for the total width owing to strong cancellations. For still higher values of  $\alpha(m_\tau)$  we cannot make quantitative statements, but there is no reason to expect that the instanton contribution becomes small.

Unfortunately, as far as numerical values are concerned, our result is of limited practical importance, since the effective instanton size has proved to be large,  $\rho_* \sim 400$  MeV. At such low scales, the instanton density is modified strongly by external gluon fields. The effect is conceptually quite similar to the effect of the quark condensate in (1), and the lowest-order correction reads [11]

$$d_{eff}(\rho) = d(\rho) \left[ 1 + \frac{\pi^4 \rho^4}{8\alpha^2(\rho)} \left\langle \frac{\alpha}{\pi} G^2 \right\rangle \right], \quad (34)$$

where  $\langle \alpha/\pi G^2 \rangle \simeq 0.012 \text{ GeV}^4$  is the gluon condensate. With the effective instanton size 400 MeV, the correction term in brackets in (34) is several times larger than unity, indicating that the effect is important. For the case of  $\tau$ -decay, this means that instanton contributions to the coefficient functions of operators of higher dimension, including gluon fields in addition to six quark fields, are likely to be larger than the one considered in this paper, so that the expansion in powers of  $1/m_\tau$  fails. One may speculate that this expansion is at best an asymptotical one, and the contributions  $\sim 1/m_\tau^{18}$  under discussion are already in the region where the series starts to diverge. Anyhow, in agreement with the old wisdom [11], we find that instanton calculations are hardly useful for quantitative estimates of nonperturbative effects, but rather can be used to indicate a scale at which the power expansion breaks down.

Thus, our conclusion is partly pessimistic. On the one hand, we find it difficult to justify the use of the operator product expansion in finite energy sum rules at the scale of the  $\tau$  mass, since contributions of higher orders are of the same order as leading power corrections. Our calculation essentially supports the old philosophy of SVZ [8], who introduced Borel sum rules in which higher-order power corrections are suppressed by factorials. In our case Borel transformation would introduce a factor  $1/8! \simeq 2.5 \cdot 10^{-5}$  and render the instanton contribution completely negligible. In the practice of the numerous QCD sum rule calculations this old argumentation has partly been forgotten. The reason is that higher-order corrections have never been calculated explicitly, and as far as leading-order power corrections are concerned, Borel and finite energy sum rules give very similar results, within the typical 10% accuracy. It is the requirement of a very high precision, which makes mandatory to reconsider the theoretical accuracy of the sum rule program for  $\tau$ -decays. In the light of our result a simultaneous extraction of  $\alpha(m_\tau)$  and nonperturbative parameters from  $\tau$  decays [4] is hard to justify.

On the other hand, all nonperturbative contributions to the total  $\tau$ -decay width prove to be small, less or of order 1%. Although the leading power corrections cannot be guaranteed to improve the accuracy, they can well indicate an intrinsic uncertainty of the theoretical prediction. Thus, it is plausible to expect an accuracy of the perturbative prediction for the  $\tau$  hadronic width of order 1%, but the situation could be substantially worse for the exclusive vector and axial vector channels. This allows for a determination of the QCD coupling within 10% accuracy, which is competitive to the current accuracy of the determination of  $\alpha$  at the scale of the  $Z$ -boson mass, but, in difference to the latter, cannot be substantially improved.

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## Figure Captions

**Fig.1** The four types of diagrams that contribute to the Wilson coefficient of six-quark operators. The instanton is depicted as a  $2n_f$ -quark vertex with only the flavours  $u$  and  $d$  shown explicitly. A circled cross denotes the insertion of the current.

**Fig.2** Distribution of instanton sizes as a function of  $\alpha(m_\tau)$ . The vertical scale is arbitrary.

**Fig.3** Instanton contribution to the vector decay channel (a) and the total width (b) for three values of “anomalous dimensions”,  $E = 28/9, 34/9, 40/9$ , as a function of  $\alpha(m_\tau)$ .