Universität Regensburg
Mathematik

Beta-elements and divided congruences

Jens Hornbostel and Niko Naumann

Preprint Nr. 12/2005

Beta-elements and divided congruences

Jens Hornbostel and Niko Naumann

Abstract

The f-invariant is an injective homomorphism from the 2-line of the Adams-Novikov spectral sequence to a group which is closely related to divided congruences of elliptic modular forms. We compute the f-invariant for two infinite families of β-elements and explain the relation of the arithmetic of divided congruences with the Kervaire invariant one problem.

1. Introduction

One of the most successful tools for studying the stable homotopy groups of spheres is the Adams-Novikov spectral sequence (ANSS)

$$\mathrm{E}_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathrm{MU}_*\mathrm{MU}}(\mathrm{MU}_*, \mathrm{MU}_*) \Rightarrow \pi^s_{t-s}(\mathrm{S}^0).$$

The corresponding filtration on $\pi^s_* := \pi^s_*(\mathrm{S}^0)$ defines a succession of invariants of framed bordism, each being defined whenever all of its predecessors vanish, the first one of which is simply the degree

$$d : \mathrm{F}^{0,*}/\mathrm{F}^{1,*} = \pi^s_0 \longrightarrow \mathrm{E}_2^{0,0} = \mathbb{Z}$$

which is an isomorphism. The next invariant, defined for all $n > 0$, is the e-invariant

$$e : \pi^s_n = \mathrm{F}^{1,n+1} \longrightarrow \mathrm{E}_2^{1,n+1} \subseteq \mathbb{Q}/\mathbb{Z},$$

c.f. [Sw, Chapter 19]. Though defined purely homotopy-theoretic here, the e-invariant is well-known to encode subtle geometric information. For its relation to index theory via the η-invariant see [APS, Theorem 4.14]. The e-invariant vanishes for all even $n = 2k \geqslant 2$, thus giving rise to the f-invariant

$$f : \pi^s_{2k} = \mathrm{F}^{2,2k+2} \longrightarrow \mathrm{E}_2^{2,2k+2}.$$

The understanding of this invariant is fragmentary at the moment. In particular, there is no index-theoretic interpretation of it comparable to the one available for the e-invariant.
As a first step towards understanding the f-invariant, G. Laures [L1] showed how elliptic homology can be used to consider the f-invariant

2000 Mathematics Subject Classification 55Q10, 55T15, 55N34, 11F33
Keywords: beta elements, f-invariant, divided congruences

$$f : \pi^s_{2k} \longrightarrow \mathrm{E}_2^{2,2k+2} \hookrightarrow \underline{\underline{\mathrm{D}}}_{k+1} \otimes \mathbb{Q}/\mathbb{Z}$$

as taking values in a group which is closely related to divided congruences of modular forms. Note that this is similar to the role taken by complex K-theory in the study of the e-invariant. Strictly speaking, at this point we had better switched from MU to BP. In fact, we will always work locally at a fixed prime p in the following.

This surprising connection of stable homotopy theory with something as genuinely arithmetic as divided congruences certainly motivates to ask for a thorough understanding of how these are related by the f-invariant.

The main purpose of this paper is to make this relation explicit.

We also include a fairly self-contained review of G. Laures' above version of the f-invariant to help the reader who might be interested in making his own computations. We now review the individual sections in more detail.

In section 2, we first remind the reader of the β-elements which generate the 2-line of the ANSS (with a little exception at the prime 2). We then construct, for suitable Hopf algebroids, a complex which is quasi-isomorphic to the cobar complex and which will facilitate later computations. Finally, we show how to use elliptic homology to obtain the f-invariant as above.

In section 3, we recall some fundamental results of N. Katz on the arithmetic of divided congruences and point out an interesting relation between BP-theory and the mod p Igusa tower (Theorem 5). Next, we give some specific computations of modular forms and divided congruences for $\Gamma_1(3)$ which we will need to study the f-invariant of the Kervaire elements $\beta_{2^n,2^n} \in \mathrm{Ext}^{2,2^{n+2}}[\mathrm{BP}]$ at the prime $p = 2$.

In section 4, we first compute the f-invariants of the infinite family of β-elements β_t for $t \geqslant 1$ not divisible by p (Theorem 10). Then we explain how to approach the problem of computing the Chern numbers determining the $\beta_{2^n,2^n}$ (see [L2, Corollary 4.2.5] for the case of β_1 at the prime 3). We do this by explicit computations in BP-theory for dimension 2 and 6 (Theorem 18). The computations get very complicated in higher dimensions. In order to use divided congruences, we then compute the f-invariants of the family $\beta_{s2^n,2^n}$ for $n \geqslant 0$ and $s \geqslant 1$ odd (Theorem 19). We hope that clever use of divided congruences will enable us to compute the Chern numbers determining the $\beta_{2^n,2^n}$ for all n. See Corollary 21 for a quick summary of what we can and cannot do at the moment.

Acknowledgements

The idea that it might be possible to project to the element $\beta_{2^n,2^n}$ using divided congruences - and consequently rephrase the Kervaire invariant one problem - was communicated to us by G. Laures.

2. The construction of the f-invariant

We remind the reader of the construction of β-elements in section 2.1. In section 2.2, we construct a complex which is quasi-isomorphic to the cobar complex and in which we will compute representatives for some of the β-elements. This is used in section 2.3 where we explain how to express the f-invariant of elements in $\mathrm{Ext}^{2,2k}[\mathrm{BP}]$ in terms of divided congruences.

2.1 Beta-elements in stable homotopy

We review some facts on Brown-Peterson homology BP at the prime p and β-elements. See [MRW] and [R] for more details. The Brown-Peterson spectrum BP has coefficient ring $\mathrm{BP}_* = \mathbb{Z}_{(p)}[v_1, v_2, ...]$ with v_i in dimension $2(p^i - 1)$. The universal p-typical formal group law is defined over this ring. The couple $(\mathrm{A}, \Gamma) := (\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$ becomes a Hopf algebroid in a standard way and we have $\mathrm{BP}_*\mathrm{BP} = \mathrm{BP}_* \otimes \mathbb{Z}_{(p)}[t_1, t_2, ...]$ such that the left unit η_L of the Hopf algebroid (A, Γ) is the standard inclusion. The right unit η_R is determined over $\mathbb{Q}$ by the formula in [R, Theorem A.2.1.27]. Choosing the Hazewinkel generators [R, A.2.2.1] for the v_i, a short computation yields $\eta_R(v_1) = v_1 + pt_1$ and $\eta_R(v_2) = v_2 + v_1 t_1^p - v_1^p t_1 \bmod p$.

We have the chromatic resolution of BP_* as a left $\mathrm{BP}_*\mathrm{BP}$-comodule

$$\mathrm{BP}_* \to M^0 \to M^1 \to ...$$

which gives rise to the chromatic spectral sequence

$$\mathrm{Ext}^{s,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, M^t) \Rightarrow \mathrm{Ext}^{s+t,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*).$$

This allows the construction of elements in $\mathrm{Ext}^{*,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*)$, the so called Greek-letter elements. Strictly speaking, these elements arise from comodule sequences $0 \to N^n \to M^n \to N^{n+1} \to 0$, but for our computations we will need the related comodule sequences (1) and (2) below. See [MRW, Lemma 3.7 and Remark 3.8] for the relationship between them. We abbreviate $\mathrm{H}^n(\cdot) := \mathrm{Ext}^{n,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \cdot)$ in the following.

To construct the β-elements [MRW, p. 476/477], choose integers $t, s, r \geqslant 1$ such that $(p^r, v_1^s, x_n^{t'}) \subseteq \mathrm{BP}_*$ is an invariant ideal where $t = p^n t'$, $(p, t') = 1$ and x_n is a homogeneous polynomial in v_1, v_2 and v_3 considered as an element of $v_2^{-1}\mathrm{BP}_*/(p^r, v_1^s)$ (see [R, p. 202] or [MRW, p. 476]), for example $x_0 = v_2$.

Consider the two short exact sequences of $\mathrm{BP}_*\mathrm{BP}$-comodules

$$0 \to \mathrm{BP}_* \xrightarrow{p^r} \mathrm{BP}_* \to \mathrm{BP}_*/(p^r) \to 0 \tag{1}$$

$$0 \to \Sigma^{2s(p-1)}\mathrm{BP}_*/(p^r) \xrightarrow{v_1^s} \mathrm{BP}_*/(p^r) \to \mathrm{BP}_*/(p^r, v_1^s) \to 0. \tag{2}$$

Using the induced boundary maps

$$\delta : \mathrm{H}^0(\mathrm{BP}_*/(p^r, v_1^s)) \longrightarrow \mathrm{H}^1(\mathrm{BP}_*/(p^r)) \text{ and}$$

$$\delta' : \mathrm{H}^1(\mathrm{BP}_*/(p^r)) \longrightarrow \mathrm{H}^2(\mathrm{BP}_*)$$

we define $\beta_{t,s,r} := \delta'\delta(x_n^{t'})$.

It is known [MRW, Theorem 2.6], [Sh] for which indices (t, s, r) the elements $\beta_{t,s,r}$ are non-zero in $\mathrm{H}^2(BP_*)$. In this case the order of $\beta_{t,s,r}$ is p^r. By construction, we have $\beta_{t,s,r} \in \mathrm{H}^{2,2t(p^2-1)-2s(p-1)}(\mathrm{BP}_*)$.

Example 1. *For $p = 2$ and $n \geqslant 1$, the element $\beta_{2^n,2^n} := \beta_{2^n,2^n,1} \in \mathrm{Ext}^{2,2^{n+2}}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*)$ is called the Kervaire element. It is mapped via the Thom reduction [R, Theorem 5.4.6] to the element $h_{n+1}^2 \in \mathrm{Ext}^{2,2^{n+2}}_{\mathrm{H}\mathbb{Z}/2_*\mathrm{H}\mathbb{Z}/2}(\mathrm{H}\mathbb{Z}/2_*, \mathrm{H}\mathbb{Z}/2_*)$. The latter element survives to a non-zero element of $\pi_{2^{n+2}-2}(\mathrm{S}^0)$ in the Adams spectral sequence if and only if (as W. Browder [B, Theorem 7.1] has shown) there exists* *a framed manifold of dimension $2^{n+2}-2$ with non-vanishing Kervaire invariant. Whether or not this is the case is unknown for $n \geqslant 5$, for $n \leqslant 4$ see [BJM], [KM].*

2.2 The rationalised cobar complex

The standard way of displaying elements in $\mathrm{Ext}^n := \mathrm{Ext}^n_\Gamma(\mathrm{A},\mathrm{A})$ of a flat Hopf algebroid (A,Γ) is by means of the cobar complex. In this section we shall give another description of this Ext group as a subquotient of $(\mathrm{A}\otimes\mathbb{Q})^{\otimes n}$ needed to compute f-invariants (Proposition 3). The results of this section are a more algebraic version of [L1, Section 3.1].

Let (A,Γ) be a Hopf algebroid with structure maps η_L, η_R, ϵ and Δ. This determines a cosimplicial abelian group $\Gamma^{\cdot}$ as follows: Set $\Gamma^n := \Gamma^{\otimes_{\mathrm{A}} n}$ with cofaces $\partial^i : \Gamma^n \longrightarrow \Gamma^{n+1}$; $\partial^0(\gamma_1\otimes\ldots\otimes\gamma_n) := 1\otimes\gamma_1\otimes\ldots\otimes\gamma_n$; $\partial^i(\gamma_1\otimes\ldots\otimes\gamma_n) := \gamma_1\otimes\ldots\otimes\Delta(\gamma_i)\otimes\ldots\otimes\gamma_n$ $(1\leqslant i\leqslant n)$ and $\partial^{n+1}(\gamma_1\otimes\ldots\otimes\gamma_n) := \gamma_1\otimes\ldots\otimes\gamma_n\otimes 1$ for $n\geqslant 1$ and $\partial^0 := \eta_R$, $\partial^1 := \eta_L$ for $n=0$ and codegeneracies $\sigma^i : \Gamma^{n+1}\longrightarrow\Gamma^n$, $\sigma^i(\gamma_0\otimes\ldots\otimes\gamma_n) := \gamma_0\otimes\ldots\otimes\epsilon(\gamma_i)\otimes\ldots\otimes\gamma_n$. We also denote by $\Gamma^{\cdot}$ the associated cochain complex. Following [R, Definition A.1.2.11], we define the reduced cobar complex (usually denoted as $\mathrm{C}^{\cdot}_\Gamma(\mathrm{A},\mathrm{A})$) as being the subcomplex $\overline{\Gamma}^{\cdot}\subseteq\Gamma^{\cdot}$ with $\overline{\Gamma}^n := \overline{\Gamma}^{\otimes_{\mathrm{A}} n}$ for $n\geqslant 1$ where $\overline{\Gamma} := \ker(\epsilon)$ and $\overline{\Gamma}^0 := \mathrm{A}$. This is a subcomplex because $\Delta(\gamma)-\gamma\otimes 1-1\otimes\gamma\in\overline{\Gamma}\otimes_{\mathrm{A}}\overline{\Gamma}$ for any $\gamma\in\overline{\Gamma}$.

We now assume that (A,Γ) is a flat Hopf algebroid such that

i) A (and hence Γ) is torsion free.

ii) The map $\mathrm{A}_{\mathbb{Q}}^{\otimes 2} := (\mathrm{A}\otimes\mathbb{Q})^{\otimes 2}\xrightarrow{\phi}\Gamma_{\mathbb{Q}} := \Gamma\otimes\mathbb{Q}$, $a\otimes b\mapsto a.\eta_R(b)$ is an isomorphism (η_L is suppressed from notation).

iii) The $\mathbb{Q}$-algebra $\mathrm{A}_{\mathbb{Q}}$ is augmented by some $\tau : \mathrm{A}_{\mathbb{Q}}\longrightarrow\mathbb{Q}$.

Remark 2. *The above assumptions i)-iii) are fulfilled by the flat Hopf algebroid* $(\mathrm{BP}_*,\mathrm{BP}_*\mathrm{BP})$*: The proof of ii) follows from the fact that over any $\mathbb{Q}$-algebra any two (p-typical) formal group laws are isomorphic via a unique strict isomorphism. If* $\mathrm{BP}_*\longrightarrow\mathrm{E}_*$ *is a non-zero Landweber exact algebra [HS], then* $(\mathrm{E}_*,\mathrm{E}_*\mathrm{E})$ *also fulfils all the above assumptions:* E_* *is a $\mathbb{Z}_{(p)}$-algebra on which p is not a zero divisor, hence* E_* *is torsion free. Conditions ii), iii) and the flatness of of* $(\mathrm{E}_*,\mathrm{E}_*\mathrm{E})$ *are inherited from* $(\mathrm{BP}_*,\mathrm{BP}_*\mathrm{BP})$ *because* $\mathrm{E}_*\mathrm{E}\simeq\mathrm{E}_*\otimes_{\mathrm{BP}_*}\mathrm{BP}_*\mathrm{BP}\otimes_{\mathrm{BP}_*}\mathrm{E}_*$ *, c.f. [Na, Proposition 10] for the flatness.*

We define the cosimplicial abelian group $\mathrm{D}^{\cdot}$ by $\mathrm{D}^n := \mathrm{A}_{\mathbb{Q}}^{\otimes(n+1)}$ $(n\geqslant 0)$ with cofaces $\partial^i : \mathrm{D}^n\longrightarrow\mathrm{D}^{n+1}$ to be given by $\partial^i(a_0\otimes\ldots\otimes a_n) := a_0\otimes\ldots\otimes 1\otimes\ldots\otimes a_n$ with the 1 in the $(i+1)^{st}$ position $(i=0,\ldots,n+1)$ and codegeneracies $\sigma^i : \mathrm{D}^n\longrightarrow\mathrm{D}^{n-1}$ $(i=0,\ldots,,n-1)$ defined by $\sigma^i(a_0\otimes\ldots\otimes a_n) := a_0\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes a_n$. By ii) above we have for any $n\geqslant 0$ an isomorphism

$$\phi^n : \mathrm{D}^n = \mathrm{A}_{\mathbb{Q}}^{\otimes(n+1)}\simeq(\mathrm{A}_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathrm{A}_{\mathbb{Q}})^{\otimes_{\mathrm{A}_{\mathbb{Q}}} n}\xrightarrow{\phi^{\otimes n}}\Gamma_{\mathbb{Q}}^{\otimes_{\mathrm{A}_{\mathbb{Q}}} n} = \Gamma^n\otimes\mathbb{Q} \tag{3}$$

which maps $a_0\otimes\ldots\otimes a_n\mapsto a_0\otimes\ldots\otimes a_{n-2}\otimes a_{n-1}.\eta_R(a_n)$ and one checks that $\phi^{\cdot}$ is an isomorphism of cosimplicial groups and hence of cochain complexes.

We have $\mathrm{H}^{\cdot}(\mathrm{D}^{\cdot}) = \mathrm{H}^0(\mathrm{D}^{\cdot}) = \mathbb{Q}$, a contracting homotopy being given by

$$H^n : \mathrm{D}^n = \mathrm{A}_{\mathbb{Q}}^{\otimes(n+1)}\longrightarrow\mathrm{D}^{n-1} = \mathrm{A}_{\mathbb{Q}}^{\otimes n}\,;\, a_0\otimes\ldots\otimes a_n\mapsto\tau(a_0)a_1\otimes\ldots\otimes a_n. \tag{4}$$

Define a subcomplex $\Sigma^{\cdot}\subseteq\mathrm{D}^{\cdot}$ by

$$\Sigma^n := \sum_{i=1}^{n} \partial^i(\mathrm{D}^{n-1}) = \sum_{i=1}^{n} \mathrm{A}_{\mathbb{Q}}^{\otimes i} \otimes \mathbb{Q} \otimes \mathrm{A}_{\mathbb{Q}}^{\otimes(n-i)} \subseteq \mathrm{D}^n = \mathrm{A}_{\mathbb{Q}}^{\otimes(n+1)},$$

for $n \geqslant 1$ and $\Sigma^0 := 0$. One checks that the composition

$$\iota^n : \overline{\Gamma}^n \hookrightarrow \Gamma^n \hookrightarrow \Gamma^n \otimes \mathbb{Q} \xrightarrow{(\phi^n)^{-1}} \mathrm{D}^n \longrightarrow \mathrm{D}^n/\Sigma^n$$

is injective for all $n \geqslant 0$. This is obvious for $n = 0$, and for $n \geqslant 1$ it follows from $(\cap_{i=0}^{n-1} \ker(\sigma^i)) \cap (\sum_{i=1}^{n} \mathrm{im}\,(\partial^i)) = 0$, which in turn is an easy consequence of the cosimplicial identities: One shows by descending induction on $1 \leqslant j \leqslant n$ that $(\cap_{i=0}^{n-1} \ker(\sigma^i)) \cap (\sum_{i=j}^{n} \mathrm{im}\,(\partial^i)) = 0$. Observe that D^n/Σ^n is isomorphic to the group labelled $E_* \otimes G_*^n$ in [L1, p. 404] for $\mathrm{A} = \mathrm{BP}$.
We define a cochain complex $\mathrm{Q}^{\cdot}$ by the exactness of

$$0 \longrightarrow \overline{\Gamma}^{\cdot} \xrightarrow{\iota^{\cdot}} \mathrm{D}^{\cdot}/\Sigma^{\cdot} \xrightarrow{\pi^{\cdot}} \mathrm{Q}^{\cdot} \longrightarrow 0, \tag{5}$$

hence $\mathrm{Q}^n \simeq \mathrm{A}_{\mathbb{Q}}^{\otimes(n+1)}/\Sigma^n + \mathrm{im}\,(\overline{\Gamma}^{\otimes_{\mathrm{A}} n})$.
From the definitions of the differential of $\mathrm{D}^{\cdot}$ and Σ^n one obtains ($\mathrm{BQ}^n \subseteq \mathrm{Q}^n$ denoting the boundaries)

$$\mathrm{Q}^n/\mathrm{BQ}^n \simeq \mathrm{A}_{\mathbb{Q}}^{\otimes(n+1)}/\tilde{\Sigma}^n + \mathrm{im}\,(\overline{\Gamma}^{\otimes_{\mathrm{A}} n}),$$

where $\tilde{\Sigma}^n := \Sigma^n + \mathbb{Q} \otimes \mathrm{A}_{\mathbb{Q}}^{\otimes n} = \sum_{i=1}^{n+1} \mathrm{A}_{\mathbb{Q}}^{\otimes(i-1)} \otimes \mathbb{Q} \otimes \mathrm{A}_{\mathbb{Q}}^{\otimes(n+1-i)}$.

The alternative description of Ext we are aiming for is the following.

Proposition 3. *For any $n \geqslant 1$, the connecting homomorphism δ of (5) is an isomorphism*

$$\mathrm{H}^n(\mathrm{Q}^{\cdot}) \xrightarrow{\delta} \mathrm{H}^{n+1}(\overline{\Gamma}^{\cdot}) = \mathrm{Ext}^{n+1}.$$

Proof. One readily sees that the contracting homotopy (4) of $\mathrm{D}^{\cdot}$ respects the subcomplex $\Sigma^{\cdot}$ in positive dimensions, hence the middle term of (5) is acyclic in these dimensions. □

To explicitly compute δ, it is useful to note that the differential of $\mathrm{D}^{\cdot}/\Sigma^{\cdot}$ has the simple form

$$\mathrm{D}^n/\Sigma^n \xrightarrow{d} \mathrm{D}^{n+1}/\Sigma^{n+1},\ [a_0 \otimes \ldots \otimes a_n] \mapsto [1 \otimes a_0 \otimes \ldots \otimes a_n],$$

as is immediate from the definitions. To compute δ^{-1}, we consider the zig-zag

$$\begin{tikzcd} \mathrm{Ext}^{n+1} \arrow[drrrrr, dotted, "\delta^{-1}"] & \mathrm{Z}(\overline{\Gamma}^{n+1}) \arrow[l, two heads] \arrow[r, "\iota^{n+1}"] & \mathrm{D}^{n+1}/\Sigma^{n+1} \arrow[r, "H^{n+1}"] & \mathrm{D}^n/\Sigma^n \arrow[r, two heads, "\pi^n"] & \mathrm{Q}^n \arrow[r, two heads] & \mathrm{Q}^n/\mathrm{BQ}^n \\ & & & & & \mathrm{H}^n(\mathrm{Q}^{\cdot}). \arrow[u, hook] \end{tikzcd}$$

One checks that the dotted arrow exists and is the inverse of δ.
Now let p be a prime, $r \geqslant 1$ an integer, and let $\delta' : \mathrm{Ext}^n_\Gamma(\mathrm{A}, \mathrm{A}/p^r) \longrightarrow \mathrm{Ext}^{n+1}$ be the connecting homomorphism associated to the short exact sequence of Γ-comodules $0 \longrightarrow \mathrm{A} \xrightarrow{\cdot p^r} \mathrm{A} \longrightarrow \mathrm{A}/p^r \longrightarrow 0$. Consider the diagram

$$
\begin{tikzcd}
\delta' : \mathrm{Ext}^n_\Gamma(\mathrm{A}, \mathrm{A}/p^r) & \mathrm{Z}(\overline{\Gamma}^n \otimes_\mathrm{A} \mathrm{A}/p^r) \arrow[l, two heads] & \{z \in \overline{\Gamma}^n | dz \in p^r \overline{\Gamma}^{n+1}\} \arrow[l, two heads] \arrow[r, "\alpha"] \arrow[d, "\nu"'] \arrow[dr, dotted] & \mathrm{Ext}^{n+1} \\
 & & \mathrm{Q}^n/\mathrm{BQ}^n & \mathrm{H}^n(\mathrm{Q}^\cdot). \arrow[l, hook'] \arrow[u, "\delta", "\simeq"']
\end{tikzcd} \tag{6}
$$

Here, $\alpha(z) := [y]$ for any $y \in \overline{\Gamma}^{n+1}$ satisfying $dz = p^r y$ and $\nu(z) := \pi^n(p^{-r}\iota^n(z)) \bmod \mathrm{BQ}^n$. The upper horizontal line is δ' by definition and one checks that ν factors through the dotted arrow and makes the diagram commutative. Hence, when displaying an element of $\mathrm{im}\,(\delta')$ in $\mathrm{H}^\cdot(\mathrm{Q}^\cdot)$ rather than in the usual cobar complex, one does not have to compute the cobar differential implicit in α but only the (easier) map ν.
Finally, assume that everything in sight is graded where the grading on D^n,Γ^n, etc. is by total degree. For a fixed $k \in \mathbb{Z}$, consider the commutative diagram

$$
\begin{tikzcd}
\mathrm{Q}^n/\mathrm{BQ}^n \cong \mathrm{A}_\mathbb{Q}^{\otimes(n+1)}/\tilde{\Sigma}^n + \mathrm{im}\,(\overline{\Gamma}^{\otimes_\mathrm{A} n}) & \mathrm{H}^n(\mathrm{Q}^\cdot) \arrow[l, hook'] \arrow[r, "\delta", "\simeq"'] & \mathrm{Ext}^{n+1} \\
\mathrm{A}_\mathbb{Q}^{\otimes(n+1),k}/\tilde{\Sigma}^{n,(k)} + \mathrm{im}\,(\overline{\Gamma}^{\otimes_\mathrm{A} n})^k \arrow[u, hook, "j"'] & \mathrm{H}^{n,k}(\mathrm{Q}^\cdot) \arrow[l, hook'] \arrow[u, hook] \arrow[r, "\simeq"] & \mathrm{Ext}^{n+1,k}. \arrow[u, hook]
\end{tikzcd}
$$

Here $\tilde{\Sigma}^{n,(k)} := \tilde{\Sigma}^n \cap \mathrm{A}_\mathbb{Q}^{\otimes(n+1),k}$. One checks that j is well defined and injective, and $\mathrm{H}^{n,k}(\mathrm{Q}^\cdot)$ is defined to be the pull back of $\mathrm{H}^n(\mathrm{Q}^\cdot)$ along j. The commutative diagram induces an isomorphism $\mathrm{H}^{n,k}(\mathrm{Q}^\cdot) \xrightarrow{\simeq} \mathrm{Ext}^{n+1,k}$ as indicated.
For example, for $(\mathrm{A}, \Gamma) = (\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$ and $n = 1$ we obtain an inclusion

$$
\mathrm{Ext}^{2,k} \subseteq \frac{(\mathrm{BP}_\mathbb{Q} \otimes \mathrm{BP}_\mathbb{Q})^{(k)}}{\mathrm{BP}_k\mathrm{BP} + (\mathrm{BP}_\mathbb{Q} \otimes \mathbb{Q} + \mathbb{Q} \otimes \mathrm{BP}_\mathbb{Q})^{(k)}} \tag{7}
$$

which is important for us since the f-invariant is defined in terms of the group on the right hand side.
To effectively compute representatives of β-elements in the complex $\mathrm{Q}^\cdot$ one proceeds as follows. Let $t, s, r \geqslant 1$ be integers as in section 2.1 and δ, δ' the coboundary maps introduced there. Fix $k \in \mathbb{Z}$ and $x \in \mathrm{H}^{0,k}(\mathrm{A}/(p^r, v_1^s))$, that is

$$
x \in \mathrm{C}_\Gamma^{0,k}(\mathrm{A}/(p^r, v_1^s)) = (\mathrm{A}/(p^r, v_1^s))^k
$$

is an invariant element (C_Γ indicates the reduced cobar complex). As δ is the connecting homomorphism determined by the short exact sequence of complexes obtained by applying C_Γ to (2), we compute $\delta(x)$ as follows: Lift x to $y \in (\mathrm{A}/(p^r))^k$ and compute the cobar differential

$$d = \eta_R - \eta_L : \mathrm{C}_\Gamma^{0,k}(\mathrm{A}/p^r) = (\mathrm{A}/p^r)^k \longrightarrow (\mathrm{A}/p^r \otimes_{\mathrm{A}} \overline{\Gamma})^k = (\overline{\Gamma}/p^r)^k = \mathrm{C}_\Gamma^{1,k}(\mathrm{A}/p^r)$$

obtaining $d(y) \in (\overline{\Gamma}/p^r)^k = \overline{\Gamma}^k/p^r$. Note that this computation requires knowledge of $\eta_R(y)$ mod p^r. Now $d(y)$ will be divisible by v_1^s, hence

$$d(y) = v_1^s z \text{ in } (\overline{\Gamma}/p^r)^k$$

with $z \in (\overline{\Gamma}/p^r)^{k-2s(p-1)} = \mathrm{C}_\Gamma^{1,k-2s(p-1)}(\mathrm{A}/p^r)$ representing $\delta(x)$. Lift z to some $w \in \overline{\Gamma}^{k-2s(p-1)}$. To proceed, we use diagram (6): w lies in the last but one group of the top row, hence we compute $\nu(w) \in \mathrm{H}^2(Q^{\cdot})$ which is our representative for $\delta'(\delta(x)) \in \mathrm{Ext}^2$. This requires to compute ϕ^{-1}. Observe for example that $\phi^{-1}(t_1) = \frac{1\otimes v_1 - v_1 \otimes 1}{p}$.

2.3 Elliptic homology theories and divided congruences: the f-invariant

We refer the reader to [L1] or [HBJ] for the notion of elliptic homology with respect to the congruence subgroup $\Gamma_1(N)$. In this section, E denotes the spectrum associated to the homology theory with coefficient ring $\mathrm{E}_* = \mathrm{M}_*(\mathbb{Z}_{(p)}, \Gamma_1(N))$, see section 3.2 for the notation. Finally, $\alpha : \mathrm{BP} \longrightarrow \mathrm{E}$ denotes the orientation.

By the naturality of the constructions in section 2.2 we have a commutative diagram for any $k \geqslant 0$

$$\begin{tikzcd}
\mathrm{Ext}^{2,k}[\mathrm{BP}] \arrow[r, hook, "\alpha"] \arrow[d, hook, "(7)"'] & \mathrm{Ext}^{2,k}[\mathrm{E}] \arrow[d, hook, "(7)"', "\iota"] \\
\frac{(\mathrm{BP}_{\mathbb{Q}} \otimes \mathrm{BP}_{\mathbb{Q}})^{(k)}}{\mathrm{BP}_k\mathrm{BP} + (\mathrm{BP}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \mathrm{BP}_{\mathbb{Q}})^{(k)}} \arrow[r, "\alpha\otimes\alpha"] & \frac{(\mathrm{E}_{\mathbb{Q}} \otimes \mathrm{E}_{\mathbb{Q}})^{(k)}}{\mathrm{E}_k\mathrm{E} + (\mathrm{E}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \mathrm{E}_{\mathbb{Q}})^{(k)}}\,.
\end{tikzcd}$$

The injectivity of α holds for any Landweber exact theory E of height at least two, [L2, Proof of 4.3.2]. To proceed, however, we will use a more subtle property of E, namely the topological q-expansion principle. We put

$$\underline{\underline{\mathrm{D}}}_k := \{f = \sum_{i=0}^{k} f_i \in \bigoplus_{i=0}^{k} \mathrm{E}_{\mathbb{Q},2i} | \text{there are } g_0 \in \mathbb{Q}, g_k \in \mathrm{E}_{\mathbb{Q},2k} \text{ such that } (f + g_0 + g_k)(q) \in \mathbb{Z}_{(p)}{}^{\Gamma}[[q]]\}$$

where $f(q)$ denotes the q-expansion of f at the cusp infinity, and for $\Gamma = \Gamma_1(N)$ we set $\mathbb{Z}^\Gamma := \mathbb{Z}[\frac{1}{N}, \zeta_N]$ if $N > 1$ and $\mathbb{Z}^{SL_2(\mathbb{Z})} := \mathbb{Z}[\frac{1}{6}]$ as in [L1]. We then define

$$(8) \qquad \iota^2 : \frac{(\mathrm{E}_{\mathbb{Q}} \otimes \mathrm{E}_{\mathbb{Q}})^{(2k)}}{\mathrm{E}_{2k}\mathrm{E} + (\mathrm{E}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \mathrm{E}_{\mathbb{Q}})^{(2k)}} \longrightarrow \underline{\underline{\mathrm{D}}}_k \otimes \mathbb{Q}/\mathbb{Z}\,, \sum_{i+j=k} f_i \otimes g_j \mapsto \sum_{i+j=k} -q^0(f_i) g_j\,,$$

where $q^0(f)$ is the constant term of the q-expansion of f at the cusp infinity. The composition $\iota^2 \circ \iota$ is injective [L1, Proposition 3.9] and hence so is the f-invariant

$$f : \mathrm{Ext}^{2,2k}[\mathrm{BP}] \hookrightarrow \underline{\underline{\mathrm{D}}}_k \otimes \mathbb{Q}/\mathbb{Z}\,.$$

We remark that our grading of E_* is the topological one, i.e. elements of dimension $2k$ correspond to modular forms of weight k.

G. Laures describes Ext^2 using the canonical Adams resolution [R, Definition 2.2.10] instead of the cobar resolution.

Proposition 4. *For $k > 0$ even, the above definition of the f-invariant $\mathrm{Ext}^{2,k} \longrightarrow \underline{\underline{\mathrm{D}}}_{k/2} \otimes \mathbb{Q}/\mathbb{Z}$ coincides with the one given in [L1].*

Proof. We have to show that the map $\mathrm{Ext}^{2,k} \hookrightarrow \frac{(\mathrm{BP}_{\mathbb{Q}} \otimes \mathrm{BP}_{\mathbb{Q}})^{(k)}}{\mathrm{BP}_k\mathrm{BP} + (\mathrm{BP}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \mathrm{BP}_{\mathbb{Q}})^{(k)}}$ we constructed in section 2.2,(7) coincides with the one of [L1]. We know [R, Lemma A.1.2.9 (b)] that, up to chain homotopy, there is a unique map from the unreduced cobar resolution $\Gamma \otimes_{\mathrm{A}} \Gamma^{\otimes_{\mathrm{A}} \cdot}$ to the canonical Adams resolution $\mathrm{BP}_*(\mathrm{BP} \wedge \Sigma\bar{\mathrm{BP}}^{\wedge \cdot})$ where $\bar{\mathrm{BP}} \to S^0 \xrightarrow{\eta} \mathrm{BP} \xrightarrow{d} \Sigma\bar{\mathrm{BP}}$ is an exact triangle in the stable homotopy category, see also [Br, Lemma 3.7] Now one checks that $\Gamma^{\otimes_{\mathrm{A}}(n+1)} \cong \pi_*(\mathrm{BP}^{\wedge n+2}) \xrightarrow{\pi_*(id_{\mathrm{BP}^{\wedge 2}} \wedge d^{\wedge n})} \pi_*(\mathrm{BP}^{\wedge 2} \wedge (\Sigma\bar{\mathrm{BP}})^{\wedge n})$ is a map of chain complexes where the isomorphism follows from [R, Lemma 2.2.7] and induction. So the claim reduces to the fact that the triangle

$$\begin{tikzcd}
\mathrm{Z}^{2,k}(\Gamma^{\otimes_{\mathrm{A}} \cdot}) \arrow[rr, "\pi_*(\mathrm{id}_{\mathrm{BP}^{\wedge 2}} \wedge d)"] \arrow[dr] & & \mathrm{Z}^{2,k}(\pi_*(\mathrm{BP}^{\wedge 2} \wedge (\Sigma\bar{\mathrm{BP}})^{\wedge \cdot})) \arrow[dl] \\
& \frac{(\mathrm{BP}_{\mathbb{Q}} \otimes \mathrm{BP}_{\mathbb{Q}})^{(k)}}{\mathrm{BP}_k\mathrm{BP} + (\mathrm{BP}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \mathrm{BP}_{\mathbb{Q}})^{(k)}} &
\end{tikzcd}$$

commutes. This follows using that our maps τ, H^2 and $\pi_*(d \wedge d)$ correspond to r, ρ and the isomorphism $\mathrm{D}^1/\tilde{\Sigma}^1 \cong G^2 = \pi_*((\Sigma\bar{\mathrm{BP}})^{\wedge 2}) \otimes \mathbb{Q}$ in [L1, section 3.1] (where we define τ by mapping all v_i to zero). $\square$

Note the degree shift and the factor 2 for the f-invariant $\pi^s_{2k} \longrightarrow \underline{\underline{\mathrm{D}}}_{k+1} \otimes \mathbb{Q}/\mathbb{Z}$ (both are missing in [L1, p. 411]).

3. Arithmetic computations

In section 3.1, we review results of N. Katz on divided congruences and establish a relation between BP-theory and the mod p Igusa tower (Theorem 5). In section 3.2 we give explicit computations for elliptic homology of level 3 and the corresponding divided congruences.

3.1 Divided congruences

We review parts of [K1]. Some technical remarks are in order: In *loc. cit.* N. Katz works with level-N structures of fixed determinant for $N \geqslant 3$. To confirm with general policy in algebraic topology we wish to consider $\Gamma_1(N)$-structures instead, which are representable only for $N \geqslant 5$. More seriously, one has to check that the relevant part of [K1] works for this different moduli problem. This we did, but we will not explain the details here and only remark that both the geometric irreducibility of the moduli spaces and the irreducibility of the Igusa tower are valid for $\Gamma_1(N)$. Furthermore, we will use these results for $p \geqslant 5$ and $N = 1$ and for $p = 2$ and $N = 3$. These cases can be handled by using auxiliary rigid level structures and taking invariants under suitable finite groups as in [K1].
Fix a level $N \geqslant 5$, a prime p not dividing N and a primitive N-th root of unity $\zeta \in \overline{\mathbb{F}_p}$. We

put $k := \mathbb{F}_p(\zeta)$, $W := W(k)$ (Witt vectors) and also denote by $\zeta \in W$ the Teichmüller lift of ζ. Finally, K denotes the field of fractions of W and for any $\mathbb{Z}_{(p)}$-algebra R we denote by $\mathrm{M}_k(R, \Gamma_1(N))$ the R-module of holomorphic modular forms for $\Gamma_1(N)$ of weight k and defined over R , see e.g. [K3] or [L1]. The $\Gamma_1(N)$ is omitted from the notation if it is clear from the context. We fix a lift $E_{p-1} \in \mathrm{M}_{p-1}(W)$ of the Hasse invariant. The existence of such a lift puts further restrictions on p and N which are satisfied in our applications.
We define the ring of divided congruences D by

$$\mathrm{M}_*(W) \subseteq \mathrm{D} := \{f \in \mathrm{M}_*(K) | f(q) \in W[[q]]\} \subseteq \mathrm{M}_*(K),$$

where $f(q)$ is the q-expansion of f at the cusp infinity. For $n \geqslant 0$ we also define

$$\mathrm{D}_n := \mathrm{D} \cap \left(\bigoplus_{i=0}^{n} \mathrm{M}_i(K) \right) \subseteq \mathrm{D} \text{ and}$$

$$\underline{\underline{\mathrm{D}}}_n := \mathrm{D}_n + K + \mathrm{M}_n(K) \subseteq \bigoplus_{i=0}^{n} \mathrm{M}_i(K)$$

which is consistent with the definition of the previous section. The group $\underline{\underline{D}}_k$ considered in [L1] differs from the $\underline{\underline{\mathrm{D}}}_k$ above because the ring of holomorphic modular forms has been localised in [L1]. This difference is not serious because the f-invariant factors through holomorphic modular forms [L1, Proposition 3.13].

The ring D carries a uniformly continuous $\mathbb{Z}_p^*$-action (the diamond operators) defined by

$$[\alpha](\sum_i f_i) := \sum_i \alpha^i f_i,$$

where $\alpha \in \mathbb{Z}_p^*$ and $f_i \in \mathrm{M}_i(K)$. We put $\Gamma_0 := \mathbb{Z}_p^*$ and $\Gamma_n := 1 + p^n\mathbb{Z}_p$ for $n \geqslant 1$ and $V_{1,n} := (\mathrm{D}/p\mathrm{D})^{\Gamma_n}$ for $n \geqslant 0$. Then

$$V_{1,0} \subseteq V_{1,1} \subseteq \ldots \subseteq \mathrm{D}/p\mathrm{D}$$

is an ind-étale $\mathbb{Z}_p^*$-Galois extension, the mod p Igusa tower. So, $V_{1,0} \subseteq V_{1,1}$ is a $(\mathbb{Z}/p)^*$-Galois extension and for all $n \geqslant 2$ $V_{1,n-1} \subseteq V_{1,n}$ is an étale $\mathbb{Z}/p$-extension and hence an Artin-Schreier extension. An immediate computation with diamond operators shows that the composition $\mathrm{M}_*(W) \to \mathrm{D} \to \mathrm{D}/p\mathrm{D}$ factors through $V_{1,1} \subseteq \mathrm{D}/p\mathrm{D}$. It is a result of P. Swinnerton-Dyer that this induces an isomorphism

$$\mathrm{M}_*(W)/(p, E_{p-1} - 1) \xrightarrow{\simeq} V_{1,1}, \tag{9}$$

see [K1, Corollary 2.2.8].
By Artin-Schreier theory, given $n \geqslant 2$ and $x \in \mathrm{D}/p\mathrm{D}$ satisfying $[\alpha](x) = x$ for all $\alpha \in \Gamma_n$ and $[1 + p^{n-1}](x) = x + 1$ one has $V_{1,n} = V_{1,n-1}[x]$ and the minimal polynomial of x over $V_{1,n-1}$ is $T^p - T - a$ for some $a \in V_{1,n-1}$.

At this point we can establish a first relation between BP-theory and divided congruences. Consider the ring extensions

$$\mathrm{M}_*(W) \subseteq \mathrm{D} \subseteq W[[q]].$$

We have a formal group $\mathcal{F}$ over $\mathrm{M}_*(W)$ induced by the universal elliptic curve. The base change of $\mathcal{F}$ to $W[[q]]$ is the formal completion of a Tate elliptic curve and is thus isomorphic to $\widehat{\mathbb{G}_m}$. Implicit in [K1] is the fact that D is the *minimal* extension of $\mathrm{M}_*(W)$ over which $\mathcal{F}$ becomes isomorphic to $\widehat{\mathbb{G}_m}$, i.e. D is obtained from $\mathrm{M}_*(W)$ by adjoining the coefficients of an isomorphism $\mathcal{F} \simeq \widehat{\mathbb{G}_m}$ defined over $W[[q]]$. This is what underlies N. Katz' construction [K1, Section 5] of a sequence of elements $d_n \in \mathrm{D}$ which modulo p constitute a sequence of Artin-Schreier generators for the mod p Igusa tower.
Since the elements $t_n \in \mathrm{BP}_*\mathrm{BP}$ are the coefficients of the universal isomorphism of a p-typical formal group law, one may expect a relation between the t_n and the d_n. To formulate this, denote by $\alpha : \mathrm{BP}_* \longrightarrow \mathrm{M}_*(W)$ the classifying map of $\mathcal{F}$ and consider the composition

$$\phi : \mathrm{BP}_*\mathrm{BP} \subseteq \mathrm{BP}_*\mathrm{BP} \otimes \mathbb{Q} \overset{(3)}{\simeq} \mathrm{BP}_{\mathbb{Q}}^{\otimes 2} \xrightarrow{\alpha\otimes\alpha} \mathrm{M}_*(K)^{\otimes 2} \xrightarrow{-q^0\otimes \mathrm{id}} \mathrm{M}_*(K).$$

Note that the map ι_2 in (8) composed with the orientation α is a quotient of ϕ.
The topological q-expansion principle guarantees that $T_n := \phi(t_n) \in \mathrm{D}$ for $n \geqslant 1$ and we can thus define $\overline{T}_n := (T_n \bmod p\mathrm{D}) \in \mathrm{D}/p\mathrm{D}$.

Theorem 5. *For any $n \geqslant 1$ we have $[1+p^k](\overline{T}_n) = \overline{T}_n$ for $k > n$ and $[1+p^n](\overline{T}_n) = \overline{T}_n + 1$. Hence $\overline{T}_n$ is an Artin-Schreier generator for the extension $V_{1,n} \subseteq V_{1,n+1}$.*

Proof. Let $\omega = (\sum_{n\geqslant 1} a_n t^{n-1})dt$ be the expansion along infinity of a normalised (i.e. $a_1 = 1$) invariant differential on the universal elliptic curve. Then $a_n \in \mathrm{M}_{n-1}(W)$ and the logarithm of the p-typification is $\sum_{n\geqslant 0} \frac{a_{p^n}}{p^n} t^{p^n} \in \mathrm{M}_*(K)[[t]]$, i.e. the classifying map $\alpha : \mathrm{BP}_* \longrightarrow \mathrm{M}_*(W)$, when tensored with $\mathbb{Q}$, sends $l_n \in \mathrm{BP}_{\mathbb{Q},2(p^n-1)}$ to $\frac{a_{p^n}}{p^n} \in \mathrm{M}_{p^n-1}(K)$, see [R, Theorem A.2.1.27] for the definition of the l_n (=λ_n in the notation of *loc. cit.*).
Defining $d_0 := 1$ and d_n ($n \geqslant 1$) recursively by

$$\sum_{i=0}^{n} \frac{d_{n-i}^{p^i}}{p^i} = \frac{a_{p^n}}{p^n}, \tag{10}$$

N. Katz shows in [K1, Corollary 5.7] that the $\overline{d}_n := (d_n \bmod p) \in \mathrm{D}/p\mathrm{D}$ behave under the diamond operators as claimed for the $\overline{T}_n$. In $\mathrm{BP}_*\mathrm{BP} \otimes \mathbb{Q}$ we have $\eta_R(l_n) = \sum_{i=0}^{n} l_i t_{n-i}^{p^i}$ and we apply ϕ to this relation to obtain

$$\frac{a_{p^n}}{p^n} = \sum_{i=0}^{n} \frac{q^0(a_{p^i})}{p^i} T_{n-i}^{p^i}$$

which, using (10), implies

$$\sum_{i=0}^{n} \frac{d_{n-i}^{p^i}}{p^i} = \sum_{i=0}^{n} \frac{q^0(a_{p^i})}{p^i} T_{n-i}^{p^i}. \tag{11}$$

We now proceed by induction on $n \geqslant 1$. For $n = 1$ we have $d_1 + 1/p = q^0(a_1)T_1 + q^0(a_p)/p$. Also, $q^0(a_1) = 1$ since $a_1 = 1$ and $q^0(a_p) \in 1 + pW$ because a_p reduces mod p to the Hasse invariant which has q-expansion equal to 1. We obtain $\overline{T}_1 = \overline{d}_1 + \alpha$ for some $\alpha \in k$. As α is invariant under all diamond operators, our claim for $n = 1$ is obvious.
Assume that $n \geqslant 2$. From (11) and $a_1 = 1$ we obtain

$$T_n = \sum_{i=0}^{n} \frac{d_{n-i}^{p^i}}{p^i} - \sum_{i=1}^{n} \frac{q^0(a_{p^i})}{p^i} T_{n-i}^{p^i}.$$

For $k > n$ we know that the terms involving d_i are invariant mod p under $[1 + p^k]$ whereas the remaining terms are likewise invariant by the induction hypothesis.
Finally, we have

$$[1+p^n]T_n = [1+p^n]d^n + [1+p^n](\sum_{i=1}^{n} \frac{d_{n-i}^{p^i}}{p^i}) - [1+p^n](\sum_{i=1}^{n} \frac{q^0(a_{p^i})}{p^i} T_{n-i}^{p^i}).$$

Here we have $[1 + p^n]d^n \equiv d_n + 1\,(pD)$ and the remaining terms are invariant. Thus, indeed, $[1 + p^n](\overline{T}_n) = \overline{T}_n + 1$. $\square$

3.2 Modular forms

For a prime $p \geqslant 5$, the following is well known [L1, Appendix]:

$$\mathrm{M}_*(\mathbb{Z}_{(p)}, \Gamma_1(1)) = \mathbb{Z}_{(p)}[E_4, E_6],$$

where E_4 and E_6 are the Eisenstein series of level one of the indicated weight. For the discriminant Δ, the ring of meromorphic modular forms is given by $\mathbb{Z}_{(p)}[E_4, E_6, \Delta^{-1}]$ and the usual orientation

$$\mathrm{BP}_* \longrightarrow \mathbb{Z}_{(p)}[E_4, E_6, \Delta^{-1}]$$

is Landweber exact of height 2 and factors through $\mathbb{Z}_{(p)}[E_4, E_6]$. A similar result holds for $p \geqslant 3$ and $\mathrm{M}_*(\mathbb{Z}_{(p)}, \Gamma_1(2)) = \mathbb{Z}_{(p)}[\delta, \epsilon]$.
The purpose of this section is to give analogous results for $\Gamma_1(3)$ and $p = 2$, c.f. [St] for related results.
Consider the elliptic curve

$$E : y^2 + a_1xy + a_3y = x^3$$

defined over $R := \mathbb{Z}[1/3][a_1, a_3, \Delta^{-1}]$ where $\Delta = a_3^3(a_1^3 - 27a_3)$ is the discriminant of the given Weierstrass equation. Note that, unlike in level one, Δ is not irreducible as a polynomial in a_1 and

a_3 and we put $f := a_3$, $g := a_1^3 - 27a_3$, hence $\Delta = f^3 g$.
The section $P := (0,0) \in E(R)$ is of exact order 3 in every geometric fibre as follows from [Si1, III,2.3] and $\omega := dx/(2y + a_1 x + a_3)$ is an invariant differential on E.

The following may be compared with [St, Lemma 11]:

Proposition 6. *The above tuple $(E/R, \omega, P)$ is the universal example of an elliptic curve over a $\mathbb{Z}[1/3]$-scheme together with a point of order 3 and a non-zero invariant differential.*

Proof. We have to show that whenever T is a $\mathbb{Z}[1/3]$-scheme and E'/T is an elliptic curve with non-zero invariant differential ω' and $P' \in E'(T)$ of exact order 3, there is a unique map $\phi : T \longrightarrow \mathrm{Spec}\,(R)$ such that $\phi^*(E, P, \omega) = (E', P', \omega')$. We show the uniqueness of ϕ first. This amounts to seeing that the only change of coordinates

$$x = u^2 x' + r\,,\ y = u^3 y' + u^2 s x' + t$$

with $r, s, t \in R$ and $u \in R^*$ (see [Si1, III Table 1.2]) preserving (E, P, ω) is the identity, i.e. $r = s = t = 0$ and $u = 1$.
From $x'(P) = y'(P) = 0$ we obtain $r = t = 0$. Next, $a_4 = a_4'$ implies $-sa_3 = 0$, hence $s = 0$ because $\Delta = f^3 g$ and thus also $f = a_3$ is a unit in R. Finally, $\omega' = u\omega$ forces $u = 1$.
Given the uniqueness of ϕ in general, its existence is a local problem on T and we can assume that $T = \mathrm{Spec}\,(S)$ is affine and E'/T is given by a Weierstrass equation with coefficients $a_i' \in S$. Moving P' to $(0,0)$ gives $a_6' = 0$. We claim that $a_3' \in S^*$: This can be checked on geometric fibres where it follows from [Si1, III,2.3] and the fact that $(0,0)$ has order 3 (if a_3' vanished on some geometric fibre the point $(0,0)$ would have order 2 in that fibre). Using this, one finds a transformation such that $(dy)_{P'} = 0$ in $\Omega_{E'/T,P'}$, hence $a_2' = a_4' = 0$. We thus have some $\psi : T \longrightarrow \mathrm{Spec}\,(R)$ such that $\psi^*(E, P) = (E', P')$ and $\psi^*(\omega) = u\omega'$ for some $u \in S^*$. Adjusting ψ using u, i.e. multiplying the a_i' by u^{-i}, we obtain the desired ϕ. □

We conclude that the ring of meromorphic modular forms is given as

$$\mathrm{M}_*^{mer}(\mathbb{Z}_{(2)}, \Gamma_1(3)) = \mathbb{Z}_{(2)}[a_1, a_3, \Delta^{-1}]$$

with a_i of weight i, and likewise for any other prime different from 3 in place of 2.
As usual, $t = -x/y$ is a local parameter at infinity for E/R which is normalised for ω and hence determines a 2-typical formal group law over $\mathrm{M}_*^{mer}(\mathbb{Z}_{(2)}, \Gamma_1(3))$. Using [Si1, p. 113] one checks that the corresponding classifying map

$$\alpha : \mathrm{BP}_* \longrightarrow \mathrm{M}_*^{mer}(\mathbb{Z}_{(2)}, \Gamma_1(3))$$

satisfies $\alpha(v_1) = a_1$ and $\alpha(v_2) = a_3$ for the Hazewinkel generators v_i. Thus α makes $\mathrm{M}_*^{mer}(\mathbb{Z}_{(2)}, \Gamma_1(3))$ a Landweber exact BP algebra of height 2.
Using the orders of f and g at the two cusps 0 and ∞ of $X_1(3)$, one can check that the ring of holomorphic modular forms is given by

$$\mathrm{M}_*(\mathbb{Z}_{(2)}, \Gamma_1(3)) = \mathbb{Z}_{(2)}[a_1, a_3]. \tag{12}$$

We stick to the notations of section 3.1 for $p = 2$ and $N = 3$. For example, ζ denotes a primitive cube-root of unity and $W = W(\mathbb{F}_2(\zeta)) = W(\mathbb{F}_4) = \mathbb{Z}_2[\zeta]$ is the unique unramified quadratic extension of $\mathbb{Z}_2$.
To study divided congruences we will need to know the q-expansions of a_1 and a_3. Given a Dirichlet character χ, we consider it as a function on $\mathbb{Z}$ as usual and define for $k \geqslant 0$ and $n \geqslant 1$

$$\sigma_k^{\chi}(n) := \sum_{1 \leqslant d | n} \chi(d) d^k.$$

In the following, χ will always denote the unique non-trivial character mod 3

$$\chi : (\mathbb{Z}/3\mathbb{Z})^* \longrightarrow \mathbb{C}^*.$$

Proposition 7. *The q-expansions of a_1 and a_3 at the cusp infinity are given as follows.*

$$a_1(q) = (1 + 2\zeta)(1 + 6 \sum_{n \geqslant 1} \sigma_0^{\chi}(n) q^n) \text{ and}$$

$$a_3(q) = (1 + 2\zeta)(-\frac{1}{9} + \sum_{n \geqslant 1} \sigma_2^{\chi}(n) q^n) \text{ in } W[[q]].$$

Proof. From (12) we know that $\mathrm{rk}\, \mathrm{M}_1(\mathbb{Z}_{(2)}) = 1$ and $\mathrm{rk}\, \mathrm{M}_3(\mathbb{Z}_{(2)}) = 2$. Using [K2, section 2.1.1] we see that

$$6G_{1,\chi}(q) = 1 + 6 \sum_{n \geqslant 1} \sigma_0^{\chi}(n) q^n \in \mathrm{M}_1(\mathbb{Z}_{(2)}) \text{ and} \tag{13}$$

$$G_{3,\chi}(q) = -\frac{1}{9} + \sum_{n \geqslant 1} \sigma_2^{\chi}(n) q^n \in \mathrm{M}_3(\mathbb{Z}_{(2)}).$$

We have evaluated $\mathrm{L}(0, \chi) = 1/3$ and $\mathrm{L}(-2, \chi) = -2/9$ using [Ne, Theorem VII.2.9] and [Wa, formula following Proposition 4.1 and Exercise 4.2(b)].
It is easy to see that $G_{3,\chi}(0) = 0$, i.e. $G_{3,\chi}$ vanishes at the cusp 0. Below, we explain how to compute the following values of a_1 and a_3 at the cusps zero and infinity.

$$a_1(\infty) = 1 + 2\zeta \tag{14}$$

$$a_3(\infty) = -\frac{1}{9}(1 + 2\zeta) \tag{15}$$

$$a_3(0) = 0. \tag{16}$$

Using these values and the dimensions of the spaces of modular forms of weight 1 and 3, we conclude that $a_1 = 6(1 + 2\zeta)G_{1,\chi}$ and $a_3 = (1 + 2\zeta)G_{3,\chi}$, hence that a_1 and a_3 have desired q-expansions by (13). We are using the fact that the map $\mathrm{M}_3(\mathbb{Z}_{(2)}) \otimes \mathbb{C} = \mathrm{M}_3(\mathbb{C}) \longrightarrow \mathbb{C}^2$, $f \mapsto (f(\infty), f(0))$ is an isomorphism, as follows from the theory of Eisenstein series.
To establish (14) and (15) one has to evaluate a_1 and a_3 at the tuple $(T(q), \omega_{can}, P)$ consisting of the Tate curve $T(q)/\mathbb{Z}((q))$, its canonical invariant differential ω_{can} and a specific section $P \in T(q)(\mathbb{Z}[\zeta]((q)))[3]$. To do so, one may use J. Tate's uniformisation [Si2, p. 426] to

write $T(q)/(\mathbb{Z}[[q]]/(q^3))$ in Weierstrass form, the point P having coordinates $(X(q,\zeta), Y(q,\zeta))$. One then uses Weierstrass transformations to bring $(T(q), \omega_{can}, P)/\mathbb{Z}[[q]]/(q^3)$ to the standard form of Proposition 6. The coefficients a_1 and a_3 of the Weierstrass equation thus obtained are by definition $a_1(\infty)$ and $a_3(\infty)$. The computation for (16) is similar, the point P has to be replaced by $Q = (X(q, q^{1/3}), Y(q, q^{1/3}))$. □

Remark 8. *In E. Hecke's notation [He], we have* $a_1 = \frac{9i}{\pi} G_1(\tau, 0, 1, 3)$ *and* $a_3 = \frac{27i}{4\pi^3} G_3(\tau, 0, 1, 3)$.

Note that $a_1(q) \equiv 1 \bmod 2$, hence $a_1 \in \mathrm{M}_1(\mathbb{Z}_{(2)}, \Gamma_1(3))$ is a lift of the Hasse invariant for $p = 2$. From section 3.1 we know that

$$V_{1,0} = V_{1,1} \overset{(9)}{\simeq} \mathrm{M}_*(W, \Gamma_1(3))/(2, a_1 - 1) = k[a_3]$$

$(k := W/2W = \mathbb{F}_4)$ and that, for $T := \frac{q^0(a_1) - a_1}{2} \in \mathrm{D}$, $\overline{T} := (T \bmod 2) \in \mathrm{D}/2\mathrm{D}$ is an Artin-Schreier generator for $V_{1,1} \subseteq V_{1,2}$, in particular $\overline{T}^2 + \overline{T} \in V_{1,1} \simeq k[a_3]$ and for later use we will need the following more precise result.

Proposition 9. $\overline{T}^2 + \overline{T} = 1 + a_3$.

Proof. Recall that the q-expansion map $V_{1,1} \subseteq \mathrm{D}/p\mathrm{D} \hookrightarrow k[[q]]$ is injective [K1, (1.4.6) for $m = 1$]. In $k[[q]]$ we have $T = \sum_{n \geqslant 1} \sigma_0^\chi(n) q^n$ and $T^2 = \sum_{n \geqslant 1} \sigma_0^\chi(n) q^{2n}$, hence

$$T^2 + T = \sum_{n \geqslant 1} (\sigma_0^\chi(n/2) + \sigma_0^\chi(n)) q^n,$$

where we understand that $\sigma_0^\chi(n/2) = 0$ for n odd. To complete the proof, one needs to check that for all $n \geqslant 1$ one has

$$\sigma_0^\chi(n/2) + \sigma_0^\chi(n) \equiv \sigma_2^\chi(n) \bmod 2,$$

and we leave this exercise in elementary number theory to the reader. □

4. f-invariants and Kervaire invariant one

In this section, we compute the f-invariants of two infinite families of β-elements including the Kervaire elements $\beta_{2^n, 2^n}$ and explain the relation of our results with the Kervaire invariant one problem.

4.1 $f(\beta_t)$ for t not divisible by p

Fix a prime p and the level N as $N = 1$ for $p \geqslant 5$, $N = 5$ for $p = 3$ and $N = 3$ for $p = 2$. We keep the notations of section 3.1 for this choice of p and N. Given an integer $t \geqslant 1$ not divisible by p, recall that $\beta_t = \delta' \delta(v_2^t) \in \mathrm{Ext}^{2, 2t(p^2-1)-2(p-1)}[\mathrm{BP}]$ has its f-invariant in $\underline{\underline{\mathrm{D}}}_n \otimes \mathbb{Q}/\mathbb{Z}$, $n := t(p^2 - 1) - (p - 1)$. When trying to express $f(\beta_t)$ in terms of divided congruences, we encounter what is in fact the major obstacle at the moment for using the arithmetic of divided congruences in homotopy theory: The group $\underline{\underline{\mathrm{D}}}_n \otimes \mathbb{Q}/\mathbb{Z}$ is not *directly* related to D. Instead, we have $\underline{\underline{\mathrm{D}}}_n = \mathrm{D} + K + \mathrm{M}_n(K)$ by definition and there is a canonical surjection

$$\pi : \mathrm{D}_n \otimes \mathbb{Q}/\mathbb{Z} \simeq \left(\bigoplus_{i=0}^{n} \mathrm{M}_i(K) \right) / \mathrm{D}_n \longrightarrow \underline{\underline{\mathrm{D}}}_n \otimes \mathbb{Q}/\mathbb{Z} \simeq \left(\bigoplus_{i=0}^{n} \mathrm{M}_i(K) \right) / \underline{\underline{\mathrm{D}}}_n$$

which is split because its kernel is divisible, hence W-injective. In particular, π remains surjective when restricted to p-torsion

$$\pi : \mathrm{D}_n \otimes \mathbb{Q}/\mathbb{Z}[p] \longrightarrow \underline{\underline{\mathrm{D}}}_n \otimes \mathbb{Q}/\mathbb{Z}[p], \tag{17}$$

note that $f(\beta_t) \in \underline{\underline{\mathrm{D}}}_n \otimes \mathbb{Q}/\mathbb{Z}[p]$. The group $\mathrm{D}_n \otimes \mathbb{Q}/\mathbb{Z}[p]$ is related to the ring of divided congruences as follows:

$$\psi : \mathrm{D}_n \otimes \mathbb{Q}/\mathbb{Z}[p] \xrightarrow{\simeq} \mathrm{D}_n/p\mathrm{D}_n \hookrightarrow \mathrm{D}/p\mathrm{D}, \tag{18}$$

where the first arrow is multiplication by p and the injectivity of the last map is immediate. What we will do is to compute some element in $\mathrm{D}/p\mathrm{D}$ in the image of ψ which under π projects to $f(\beta_t)$. At the low risk of confusion we will continue to label such an element, which is in general not unique, as $f(\beta_t)$. Recall that we have fixed an elliptic orientation $\alpha : \mathrm{BP}_* \longrightarrow \mathrm{M}_*(W)$ and denote $T := \frac{\alpha(v_1) - q^0(\alpha(v_1))}{p} \in \mathrm{D}/p\mathrm{D}$. We also put $b := ((q^0(\alpha(v_2)) \bmod p) \in k$.

Theorem 10. *For an integer $t \geqslant 1$ not divisible by the fixed prime p, we have*

$$f(\beta_t) = b^t - (T^p - T + b)^t \in V_{1,0} \subseteq \mathrm{D}/p\mathrm{D}.$$

Proof. Note first that from section 3.1 we know that T is an Artin-Schreier generator for $V_{1,1} \subseteq V_{1,2}$, hence $T^p - T \in V_{1,1}$. A short computation with diamond operators, which we leave to the reader, shows that in fact $T^p - T \in V_{1,0}$, hence also $b^t - (T^p - T + b)^t \in V_{1,0}$.
We introduce $a := q^0(\phi(v_1))$ and compute as explained at the end of section 2.2 using the notations introduced there. From $\eta_R v_2 \equiv v_2 + v_1 t_1^p - v_1^p t_1 \bmod p$ we obtain

$$\eta_R v_2^t \equiv v_2^t + \sum_{i=1}^{t} \binom{t}{i} v_2^{t-i} v_1^i t_1^i (t_1^{p-1} - v_1^{p-1})^i \bmod p,$$

hence

$$w = \sum_{i=1}^{t} \binom{t}{i} v_2^{t-i} v_1^{i-1} t_1^i (t_1^{p-1} - v_1^{p-1})^i$$

and

$$\nu(w) = \frac{1}{p} \sum_{i=1}^{t} \binom{t}{i} (v_2^{t-i} v_1^{i-1} \otimes 1) \left(\frac{1 \otimes v_1 - v_1 \otimes 1}{p} \right)^i \cdot$$

$$\left(\left(\frac{1 \otimes v_1 - v_1 \otimes 1}{p} \right)^{p-1} - v_1^{p-1} \otimes 1 \right)^i \in \frac{(\mathrm{BP}_{\mathbb{Q}} \otimes \mathrm{BP}_{\mathbb{Q}})^{(2n)}}{\mathrm{BP}_{2n}\mathrm{BP} + (\mathrm{BP}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \mathrm{BP}_{\mathbb{Q}})^{(2n)}}.$$

As in section 2.3 we apply $\iota^2 \circ (\alpha \otimes \alpha)$ to this expression to obtain, denoting $\alpha(v_1) \in \mathrm{M}_{p-1}(W)$ as v_1 for simplicity,

$$-\frac{1}{p}\sum_{i=1}^{t}\binom{t}{i}b^{t-i}a^{i-1}\left(\frac{v_1-a}{p}\right)^i\left(\left(\frac{v_1-a}{p}\right)^{p-1}-a^{p-1}\right)^i=$$

$$-\frac{1}{pa}\left[-b^t+\left(a\left(\frac{v_1-a}{p}\right)\left(\left(\frac{v_1-a}{p}\right)^{p-1}-a^{p-1}\right)+b\right)^t\right]=$$

$$\frac{-1}{pa}\left(\left(a\left(\frac{v_1-a}{p}\right)\left(\left(\frac{v_1-a}{p}\right)^{p-1}-a^{p-1}\right)+b\right)^t-b^t\right)\in\bigoplus_{i=0}^{n}\mathrm{M}_i(K).$$

This is a representative for $f(\beta_t)$ in $\mathrm{D}_n\otimes\mathbb{Q}/\mathbb{Z}[p]$ to which we have to apply the map ψ from (18) to obtain an element in $\mathrm{D}/p\mathrm{D}$. For this, note that $a\equiv 1$ mod p because v_1 reduces to the Hasse invariant mod p. This allows us to put $a=1$ in the above expression (but not to replace $\frac{v_1-a}{p}$ by $\frac{v_1-1}{p}$; this would require the congruence $a\equiv 1$ mod p^2, which does not hold in general). We then obtain indeed

$$-((T(T^{p-1}-1)+b)^t-b^t)=b^t-(T^p-T+b)^t.$$

$\square$

Remark 11. *Assume that $p\geqslant 5$ in the situation of Theorem 10. In general, the elliptic orientation will not map v_1 to the Eisenstein series E_{p-1} of weight $p-1$ and level one. But $\alpha(v_1)$ and E_{p-1} can only differ by a modular form divisible by p and we may thus change the orientation to force $\alpha(v_1)=E_{p-1}$. Assuming this, we see that $f(\beta_{1,1,1})=\frac{E_{p-1}-1}{p^2}-\frac{1}{p}(\frac{E_{p-1}-1}{p})^p$, as first computed by G. Laures [L1, p. 414] (where the second summand is missing).*

Remark 12. *The injectivity of the f-invariant together with the known structure of $\mathrm{Ext}^2[\mathrm{BP}]$ provides some non-trivial information about the arithmetic of divided congruences as follows. Fix some $x\in\mathrm{Ext}^{2,k}[\mathrm{BP}]$ of order p^r. Then $f(x)\in\underline{\underline{\mathrm{D}}}_k\otimes\mathbb{Q}/\mathbb{Z}$ will be of order p^r, hence a representative of $f(x)$ in $\mathrm{D}_k\otimes\mathbb{Q}/\mathbb{Z}$ will be of order p^s for some $s\geqslant r$. Thus the f-invariant relates the order of a β-element to the (non-)existence of a certain divided congruence.*
Let us assume that $r=1$ as is the case for all $\beta_{t,s,r}$ considered in this article. Then our results show that our representatives in $\mathrm{D}\otimes\mathbb{Q}/\mathbb{Z}$ have order p and the non-trivial additional information on divided congruences is then that they do not lie in the kernel of $\mathrm{D}\otimes\mathbb{Q}/\mathbb{Z}\longrightarrow\underline{\underline{\mathrm{D}}}\otimes\mathbb{Q}/\mathbb{Z}$.
To give an example, assume that we are in the situation of Remark 11. The arithmetic of divided congruences shows that $F:=\frac{E_{p-1}-1}{p^2}-\frac{1}{p}(\frac{E_{p-1}-1}{p})^p\in\mathrm{D}_{p(p-1)}\otimes\mathbb{Q}/\mathbb{Z}$ is of order p, i.e. $F\in\bigoplus_{i=0}^{p(p-1)}\mathrm{M}_i(K)$ has a q-expansion with denominator exactly p. The additional information is then that for any $\alpha\in K$ and $f\in\mathrm{M}_{p(p-1)}(K)$ the q-expansion of $F+\alpha+f$ will still have exact denominator p.

Example 13. *Fix $p=5$ and set $g_2:=\frac{1}{12}E_4$ and $g_3:=\frac{-1}{216}E_6$ as in [K1]. The comparison of the logarithm of the universal p-typical formal group law [R] and the corresponding coefficients of the logarithm of the elliptic curve (E,ω) [K1, (5.0.3)] (p-typification does not change these coefficients) shows that the orientation a maps v_1 to a_p and v_2 to $\frac{a_{p^2}-a_p^{p+1}}{p}$, the a_i denoting the normalised (multiplied with $-1/2$) a_i of [K1, p. 351]. One deduces that v_1 maps to $-8g_2$ and v_2 maps to $\frac{a_{25}-a_5^6}{5}$. A computation with Maple shows that $a_{25}=129761280g_2^3g_3^2+32440320g_3^4+3784704g_2^6$ (and also that the correct value for the unnormalised a_{11} is $-2520g_2g_3$ and not $-512g_2g_3$). It follows that $q^0(v_1)=-\frac{2}{3}$ and $q^0(v_2)=-\frac{4900}{3^{10}}$, so Theorem 10 may be rephrased in terms of the Eisenstein series g_2 and g_3.*

4.2 Projecting to the Kervaire element

In this section, we compute $f(\beta_{s2^n,2^n})$ for $n \geqslant 0$ and $s \geqslant 1$ odd at the prime $p = 2$. Using this, we are able to determine a single coefficient in the f-invariant of a $(U, fr)^2$- manifold of dimension 2^n the non-vanishing of which is necessary and sufficient for the corner of X to be a Kervaire manifold, that is having Kervaire invariant one. See [L2] for the notion of cobordism of manifolds with corners. We begin by recalling the well-known relation of the Kervaire invariant to certain β-elements, due to W. Browder [B]. Fix some $n \geqslant 3$. We have a homomorphism

$$K : \pi^s_{2^n-2} \longrightarrow \mathbb{Z}/2$$

which sends the class of a stably framed manifold to its Kervaire invariant. Consider on the other hand the composition

$$K' : \pi^s_{2^n-2} \longrightarrow \pi^s_{2^n-2}[2] \longrightarrow \mathrm{E}_\infty^{2,2^n}[\mathrm{H}\mathbb{Z}/2] \hookrightarrow \mathrm{E}_2^{2,2^n}[\mathrm{H}\mathbb{Z}/2] = \mathbb{Z}/2 \cdot h_{n-1}^2.$$

Here, the first map is the projection to the 2-primary part, the second is the projection onto F^2/F^3 in the (classical) Adams spectral sequence at $p = 2$, the third is an edge homomorphism and the final equality is due to J. Adams, [R, 3.4.1, c)].

Proposition 14. $K = K'$.

Proof. For any $y \in \pi^s_{2^n-2}[2]$ we have $K(y) = 1$ if and only if y has Adams filtration 2. This is implicit in [B], c.f. [BJM2, p. 144]. □

We can easily obtain a similar homotopy theoretic description of K using BP instead of $\mathrm{H}\mathbb{Z}/2$.

Proposition 15. *Let $n \geqslant 2$. Then $\mathrm{Ext}^{2,2^n}$ is a direct sum of cyclic groups of order 2. It is generated by the element $\alpha_1 \cdot \alpha_{2^{n-1}-1}$ and the elements $\beta_{s2^i,2^i}$ with s odd and $i \geqslant 0$ such that $(3s-1)2^{i+1} = 2^n$ and the case $(s, i) = (1, 0)$ has to be omitted.*

Proof. This follows from [R, Corollary 5.4.5]. Observe that the $\overline{\alpha}_t$ in loc. cit equals α_t as t is odd, see [Sh, Theorem 1.5] or [R, Theorem 5.2.6]. □

Remark 16. *The Lemma shows that the number of generators of $\mathrm{Ext}^{2,2^n}$ is $[n/2] + 1$ for $n \geqslant 3$. The low dimensional cases are as follows.*

$$Ext^{2,4} \; : \; \alpha_1^2$$

$$Ext^{2,8} \; : \; \alpha_1\alpha_3, \beta_{2,2}$$

$$Ext^{2,16} \; : \; \alpha_1\alpha_7, \beta_{4,4}, \beta_{3,1}$$

$$Ext^{2,32} \; : \; \alpha_1\alpha_{15}, \beta_{8,8}, \beta_{6,2}$$

$$Ext^{2,64} \; : \; \alpha_1\alpha_{31}, \beta_{16,16}, \beta_{12,4}, \beta_{11,1}$$

$$Ext^{2,128} \; : \; \alpha_1\alpha_{63}, \beta_{32,32}, \beta_{24,8}, \beta_{22,2}$$

$$Ext^{2,256} \; : \; \alpha_1\alpha_{127}, \beta_{64,64}, \beta_{48,16}, \beta_{44,4}, \beta_{43,1}.$$

Now we consider the composition

$$K'' : \pi^s_{2^n-2} \longrightarrow \pi^s_{2^n-2}[2] \longrightarrow \mathrm{E}_\infty^{2,2^n}[\mathrm{BP}] \hookrightarrow \mathrm{E}_2^{2,2^n}[\mathrm{BP}] \longrightarrow \mathbb{Z}/2 \cdot \beta_{2^{n-2},2^{n-2}}$$

which is defined in analogy with K', the final map being the projection to the $\mathbb{Z}/2$-summand generated by $\beta_{2^{n-2},2^{n-2}}$.

Proposition 17. $K'' = K$.

Proof. We have the Thom reduction $\Phi : \mathrm{Ext}^*[\mathrm{BP}] \longrightarrow \mathrm{Ext}^*[\mathrm{H}\mathbb{Z}/2]$ which satisfies $\Phi(\beta_{2^{n-2},2^{n-2}}) = h_{n-1}^2$ and is zero on all other generators of $\mathrm{Ext}^{2,2^n}[\mathrm{BP}]$, see [R, 5.4.6, a)] and Proposition 15. The result then follows from Proposition 14. □

Let X be a $(U, fr)^2$-manifold of dimension 2^n. From the above, we see that the corner of X is a Kervaire manifold if and only if the f-invariant of X contains $\beta_{2^n,2^n}$ as a summand. Thus, one certainly wants a more geometric description of the f-invariant (or just its projection to $\beta_{2^{n-2},2^{n-2}}$). In principle, it is possible to obtain such a description in terms of Chern numbers of X, simply because they determine the $(U, fr)^2$-bordism class of X [L2], but the necessary computations become quite complicated already in low dimensions. At the end of this section, we will explain how divided congruences might simplify such computations. We then would like to generalise Theorem 18 below to higher dimensions.

Recall [L2, section 4.1] that if X is a $(U, fr)^2$-manifold then there is a decomposition of its stable tangent bundle $TX = TX^{(0)} \oplus TX^{(1)}$ and we have Chern classes $c_i^{(j)} \in H^{2i}(X, \mathbb{Z})$ accordingly $(i \geqslant 0, j = 0, 1)$.

Theorem 18. *a) Let X be a $(U, fr)^2$-manifold of dimension 4 and put $q := < c_1^{(0)} c_1^{(1)}, [X] > \in \mathbb{Z}$. Then q is odd if and only if the corner of X has Kervaire invariant 1. If q is even, then the corner of X is the boundary of a framed manifold.*
b) Let X be a $(U, fr)^2$-manifold of dimension 8 and put $q := < c_1^{(0)}(c_1^{(1)3} + c_1^{(1)} c_2^{(1)} + c_3^{(1)}) + (c_2^{(0)} + c_1^{(0)2})(c_2^{(1)} + c_1^{(1)2}), [X] > \in \mathbb{Z}$. Then q is odd if and only if the corner of X has Kervaire invariant 1. If q is even, then the corner of X is the boundary of a framed manifold.

Proof. If $\sum_{i \geqslant 0} l_i x^{2^i}$ is the logarithm of the universal 2-typical formal group law, then (see [L2, Example 4.2.4])

$$exp(x) = x - l_1 x^2 + 2 l_1^2 x^3 - (5 l_1^3 + l_2) x^4 \ \ (mod\ x^5)$$

and thus

$$Q(x) := \frac{x}{exp(x)} = 1 + l_1 x - l_1^2 x^2 + (2 l_1^3 + l_2) x^3 \ \ (mod\ x^4).$$

For indeterminates x_i of dimension 2 we set $\Pi := \prod_i Q(x_i)$. Denoting by c_i the i-th elementary symmetric function in the x_i one gets (using the definition of the Hazewinkel generators)

$$\Pi^{(2)} = \frac{v_1}{2} c_1$$

$$\Pi^{(4)} = \frac{v_1^2}{4}(3c_2 - c_1^2) \text{ and}$$

$$\Pi^{(6)} = \frac{v_1^3}{8}(4c_1^3 - 13 c_1 c_2 + 16 c_3) + \frac{v_2}{2}(c_1^3 - 3 c_1 c_2 + 3 c_3).$$

To prove part a) one has that $\mathrm{BP}_{\mathbb{Q}}^{\otimes 2,4}/(\mathrm{BP}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \mathrm{BP}_{\mathbb{Q}})$ is a one-dimensional $\mathbb{Q}$-vector space generated by $v_1 \otimes v_1$. Moreover, one checks that the image of $\mathrm{BP}_4\mathrm{BP}$ is generated (over $\mathbb{Z}_{(2)}$) by $\frac{v_1 \otimes v_1}{2}$ and that $\frac{v_1 \otimes v_1}{4}$ is a representative of α_1^2. To see the latter, observe that α_1 is represented by t_1, hence [R, A.1.2.15] α_1^2 is represented in the cobar complex by $t_1 \otimes t_1 = (1 \otimes t_1)(t_1 \otimes 1)$, and then one applies the description of δ^{-1} given in section 2.2. Using the notations introduced in [L2], one computes that

$$K_{\mathrm{BP}<2>}(TX)^{(4)} = c_1^{(0)} c_1^{(1)} \frac{v_1 \otimes v_1}{4} \text{ in } \mathrm{BP}_{\mathbb{Q}}^{\otimes 2,4}/(\mathrm{BP}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \mathrm{BP}_{\mathbb{Q}})^{(4)},$$

hence the image of the corner of X in $\mathrm{Ext}^{2,4}$ is represented by $\frac{q}{2} \cdot \frac{v_1 \otimes v_1}{2} = q \cdot \alpha_1^2$. The final assertion follows because the only non-trivial element of π_2^s has Adams-Novikov filtration precisely 2.
For part b), we know that $\mathrm{Ext}^{2,8}$ is generated by $\alpha_1 \overline{\alpha}_3$ and $\beta_{2,2}$. As $\beta_{2,2}$ is a permanent cycle in the ANSS whereas $\alpha_1 \overline{\alpha}_3$ is not we know that the image of X in $\mathrm{Ext}^{2,8}$ is a multiple of $\beta_{2,2}$. One computes that in the notation of section 2.2 $\delta(v_2^2)$ is represented by $z = t_1^4 + v_1^2 t_1^2$ in $C^1(A/2)$. Hence $\beta_{2,2} = \delta' \delta(v_2^2)$ is represented in $\mathrm{BP}_{\mathbb{Q}}^{\otimes 2,8}/(\mathrm{BP}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \mathrm{BP}_{\mathbb{Q}})^{(8)}$ by $-\frac{1}{8}(v_1 \otimes v_1^3) + \frac{5}{16}(v_1^2 \otimes v_1^2) - \frac{3}{8}(v_1^3 \otimes v_1)$. Computing enough of the image of $v_1^3 t_1, v_1^2 t_1^2, v_1^3 t_1$ and t_1^4 under $\mathrm{BP}_8\mathrm{BP} \hookrightarrow \mathrm{BP}_8\mathrm{BP} \otimes \mathbb{Q} \simeq \mathrm{BP}_{\mathbb{Q}}^{\otimes 2,8} \longrightarrow \mathrm{BP}_{\mathbb{Q}}^{\otimes 2,8}/(\mathrm{BP}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \mathrm{BP}_{\mathbb{Q}})^{(8)}$, one sees that $\frac{v_1^2 \otimes v_1^2}{8} \in \mathrm{im}\,(\mathrm{BP}_8\mathrm{BP})$ and $\beta_{2,2}$ is represented by $\frac{v_1^2 \otimes v_1^2}{16}$. We provide the following argument for general n, for the proof here we need the case $n = 3$. Observe that by the computations of the previous sections, the image of $\mathrm{Ext}^{2,2^n}$ in $\mathrm{BP}_{\mathbb{Q}}^{\otimes 2,2^n}/(\mathrm{BP}_{\mathbb{Q},2^n} \otimes \mathbb{Q} + \mathbb{Q} \otimes \mathrm{BP}_{\mathbb{Q},2^n} + \mathrm{BP}_{2^n}\mathrm{BP})$ is given by representatives consisting of summands of the form $v_1^i v_2^k \otimes v_1^j$ with rational coefficients. Moreover, these elements map under the isomorphism ϕ of section 2.2 to polynomials in v_1, v_2 and t_1. In other words, the f-invariant in bidegree $(2, 2^n)$ factors through the subgroup generated by elements $v_1^i v_2^k \otimes v_1^j$ modulo elements of the form $\phi^{-1}(v_1^i v_2^j t_1^k)$, $1 \otimes v_1^j$ and $v_1^i v_2^j \otimes 1$. Denote this quotient by B_{2^n}. A computation using the results of the previous section shows that $\frac{c}{2^{2^{n-1}}} v_1^{2^{n-2}} \otimes v_1^{2^{n-2}}$ is not zero in B_{2^n} for c an odd integer. More precisely, no element in the relations defining the quotient B_{2^n} contains such a summand. Among the elements in $\mathrm{Ext}^{2,2^n}$, the image of $\beta_{2^{n-2},2^{n-2}}$ in B_{2^n} contains such a summand and the other generators exhibited in Proposition 15 do not. Thus we have a well-defined map $\mathrm{Ext}^{2,2^n} \to \mathbb{Z}/2$ given by mapping an element to 1 if it admits a representative in B_{2^n} having a summand $\frac{c}{2^{2^{n-1}}} v_1^{2^{n-2}} \otimes v_1^{2^{n-2}}$ with c odd. This map is a projection to $\beta_{2^{n-2},2^{n-2}}$. We would like to consider the summands of

$$K_{\mathrm{BP}<2>}(TX)^{(8)} \equiv \Pi^{(2)} \otimes \Pi^{(6)} + \Pi^{(4)} \otimes \Pi^{(4)} + \Pi^{(6)} \otimes \Pi^{(2)} \bmod (\mathrm{BP}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \mathrm{BP}_{\mathbb{Q}})^{(8)}$$

as elements in B_{2^3}. Once this is achieved, we have to consider only the summands involving $v_1^2 \otimes v_1^2$. The only summand which is not already given by a representative in B_{2^3} is $c_1^{(0)}(c_1^{(1)3} - 3c_1^{(1)} c_2^{(1)} + 3c_3^{(1)}) \frac{v_1 \otimes v_2}{4}$ in $\Pi^{(2)} \otimes \Pi^{(6)}$. One computes (for $p = 2$ and the Hazewinkel generators as before) that $\phi^{-1}(t_2) = \frac{1 \otimes v_2}{2} - \frac{v_2 \otimes 1}{1} + \frac{1 \otimes v_1^3}{4} - \frac{v_1 \otimes v_1^2}{8} + \frac{v_1^2 \otimes v_1}{4} - 3\frac{v_1^3 \otimes 1}{8}$. Hence we have $\phi^{-1}(t_1 t_2) = -\frac{v_1 \otimes v_2}{4} - \frac{v_1^2 \otimes v_1^2}{16} + \ldots$, so we have to look at the coefficients of $\frac{v_1 \otimes v_2}{4}$ and $\frac{v_1^2 \otimes v_1^2}{16}$ which by the above equal $c_1^{(0)}(c_1^{(1)3} - 3c_1^{(1)} c_2^{(1)} + 3c_3^{(1)})$ and $(3c_2^{(0)} - c_1^{(0)2})(3c_2^{(1)} - c_1^{(1)2})$. We also have that $c_1^{(0)}(c_1^{(1)3} - 3c_1^{(1)} c_2^{(1)} + 3c_3^{(1)})$ equals $c_1^{(0)}(c_1^{(1)3} + c_1^{(1)} c_2^{(1)} + c_3^{(1)})$ and $(3c_2^{(0)} - c_1^{(0)2})(3c_2^{(1)} - c_1^{(1)2})$ equals $(c_2^{(0)} + c_1^{(0)2})(c_2^{(1)} + c_1^{(1)2})$ modulo 2 when evaluated on $[X]$. Now the assertion follows as in part a). □

Of course, it is possible to do similar but more complicated computations for 2^n-dimensional $(U, fr)^2$-manifolds in case $n \geqslant 4$. We always have a projection to $\mathbb{Z}/2$ looking at the power of 2 in the denominator of the coefficient of the summand $v_1^{2^{n-2}} \otimes v_1^{2^{n-2}}$. The computation then reduces to compute those $\Pi^{(2i)}$ which contribute to this summand. The diligent reader may thus prove statements of the following form: The element h_{n-1}^2 survives (equivalently: there is a framed manifold in dimension $2^n - 2$ having Kervaire invariant 1) if and only if $< F_n(c_i^{(0)}, c_j^{(1)}), [X] >$ is odd for a certain explicit polynomial F_n. The main problem in the computation of F_n is to find representatives in B_{2^n} (that is sums of $v_1^i v_2^k \otimes v_1^j$) for elements arising in the $\Pi^{(2^n-i)} \otimes \Pi^{(i)}$. For $n = 3$ this was done using $t_1 t_2$. In the case $n > 3$, it will be necessary to find suitable elements in $\mathrm{BP}_{2^n}\mathrm{BP}$ which will involve t_i for larger i and the computation of ϕ^{-1} of these elements.

The rest of this section is devoted to the computation of the f-invariant in dimension 2^n at the prime 2. More generally, we compute the f-invariant of $\beta_{s2^n,2^n} \in \mathrm{Ext}^{2,(3s-1)2^{n+1}}$ for all $n \geqslant 0$ and $s \geqslant 1$ odd.
We use the notations of section 3.1 for $p = 2$ and $N = 3$ and those of section 3.2 and write $T := \frac{a_1-1}{2} \in \mathrm{D}/2\mathrm{D}$ which is an Artin-Schreier generator for the extension $V_{1,0} = V_{1,1} \simeq k[a_3] \subseteq V_{1,2}$.

Theorem 19. *The image of the f-invariant in $V_{1,2}$ is given by*

$$f(\alpha_1\alpha_{2^{n+1}-1}) = T \text{ for } n \geqslant 0,$$
$$f(\beta_s) = 1 + a_3^s \text{ for } s \geqslant 3 \text{ odd },$$
$$f(\beta_{s2,2}) = 1 + a_3^{2s} \text{ for } s \geqslant 1 \text{ odd and}$$
$$f(\beta_{s2^n,2^n}) = (a_3^{2^n} + a_3^{3\cdot 2^{n-2}})^s \text{ for } s \geqslant 1 \text{ odd and } n \geqslant 2.$$

Proof. For the first line, recall that mod 2 we have $\alpha_t := \alpha_{t,1} := \delta(v_1^t)$. One computes that in the cobar complex $\alpha_1\alpha_t$ is represented by $t_1 \otimes \frac{1}{2}[(2t_1+v_1)^t - v_1^t]$, use the description of the product in the cobar complex of [R, A.1.2.15]. Using that ϕ is a ring isomorphism and the description of δ^{-1}, see section 2.2, one further computes that $\alpha_1\alpha_t$ is represented by $-\frac{1}{4}v_1 \otimes v_1^t$ in the usual quotient of $\mathrm{BP}_{\mathbb{Q}}^{\otimes 2}$, c.f. (7).
The second line is a special case of Theorem 10 (use Proposition 9) and implies the third line because $x_1 \equiv x_0^2 = v_2^2 \bmod v_1^2$, recall the invariant sequences $(2, v_1^{2^n}, x_n)$ from section 2.1.
The only case requiring a longer computation is $f(\beta_{4,4})$:
In the notation of section 2.1 we have $r = 1, s = t = 4$ and

$$x_2 = v_2^4 - v_1^3 v_2^3 \in \mathrm{H}^{0,24}(\mathrm{BP}/(2, v_1^4)).$$

This value of x_2 follows from the definition in [MRW, p. 476] or [R, Theorem 5.2.13]) after cancelling all possible multiples of v_1^4. One computes that, in the notation of section 2.2, $z \in (\overline{\Gamma}/2)^{16}$ is given as

$$\begin{aligned} z &= t_1^8 + v_1^4 t_1^4 + v_2^2 t_1(t_1+v_1) + v_1 v_2 t_1^2(t_1^2+v_1^2) + v_1^2 t_1^3 (t_1+v_1)^3 \\ &= v_1 v_2^2 t_1 + v_2^2 t_1^2 + v_1^3 v_2 t_1^2 + +v_1^5 t_1^3 + v_1 v_2 t_1^4 + v_1^3 t_1^5 + v_1^2 t_1^6 + t_1^8. \end{aligned}$$

One then computes that $f(\beta_{4,4}) = \nu(w)$, where $w \in \overline{\Gamma}$ is a lift of z as in section 2.2, is as claimed, using the relation in Proposition 9 and that $q^0(a_1) = q^0(a_3) = 1 \bmod 2$ (see Proposition 7).
Now the value for $f(\beta_{s4,4})$ for any s follows immediately from the derivation property of the connecting homomorphism δ, namely $\delta(x^n) = \delta(x)(\sum_{i=0}^{n-1} \eta_R(x)^i \eta_L(x)^{n-1-i})$. Alternatively, $f(\beta_{s4,4})$ may be computed directly for any odd s using that $(q^0 \otimes \mathrm{id})\eta_L(x_2) = 0 \bmod 2$, η_R and ϕ^{-1} are ring homomorphisms and Proposition 9. We obtain $f(\beta_{s2^n,2^n}) = f(\beta_{s4,4})^{n-2}$ for all s and $n \geqslant 3$ because $x_n = x_{n-1}^2$ for all $n \geqslant 3$. □

Fix some $n \geqslant 3$. To explain the relevance of the above computation for the problem of projecting the f-invariant to $\beta_{2^{n-2},2^{n-2}}$ we contemplate the following diagram.

(19)
$$\begin{tikzcd}
\mathrm{Ext}^{2,2^n}[\mathrm{BP}] \arrow[r, hook, "f"] \arrow[rrr, bend left, "\pi'"] \arrow[ddr, "\iota"'] & \underline{\underline{\mathrm{D}}}_{2^{n-1}} \otimes \mathbb{Q}/\mathbb{Z}[2] & & \mathbb{Z}/2 \arrow[dd, hook] \\
& \mathrm{D}_{2^{n-1}} \otimes \mathbb{Q}/\mathbb{Z}[2] \arrow[u, two heads, "(17)"] \arrow[r, hook, "(18)"] & \mathrm{D}/2\mathrm{D} & \\
& \tilde{V_{1,2}} \arrow[u, hook] \arrow[r, hook] & V_{1,2} \arrow[u, hook, "\text{(section 3.1)}"'] \arrow[r, "\pi"] & k = \mathbb{F}_4.
\end{tikzcd}$$

By the results in section 3.1, $V_{1,2}$ is k-free on the set $\{a_3^i T^j | i \geqslant 0, j = 0, 1\}$ and π is defined to be the projection to the coefficient of $a_3^{2^{n-2}}$. The map π' is defined to be the projection to the generator $\beta_{2^{n-2},2^{n-2}}$, c.f. Proposition 15. Theorem 19 determines representatives in $\tilde{V_{1,2}} := \mathrm{D}_{2^{n-1}} \otimes \mathbb{Q}/\mathbb{Z}[2] \cap V_{1,2}$ for all generators of $\mathrm{Ext}^{2,2^n}[\mathrm{BP}]$ and thus defines the map ι. We know that (19) commutes when π and π' are omitted.

THEOREM 20. *The diagram (19) is commutative.*

Proof. By inspection of Proposition 15 and Theorem 19 , the only generator of $\mathrm{Ext}^{2,2^n}[\mathrm{BP}]$ whose f-invariant contains $a_3^{2^{n-2}}$ is the Kervaire element $\beta_{2^{n-2},2^{n-2}}$. □

COROLLARY 21. *Let $n \geqslant 3$ and X a $(U, fr)^2$-manifold of dimension 2^n. Then the corner of X has Kervaire invariant one if and only if the f-invariant of X admits a representative in $\tilde{V_{1,2}}$ which contains the summand $a_3^{2^{n-2}}$.*

Note that the coefficient of $a_3^{2^{n-2}}$ in the f-invariant of X can rather easily be expressed in terms of Chern numbers of X. The very reason that this does not give us the Chern numbers determining the Kervaire elements is the indeterminacy in the above constructions caused by the projection $\mathrm{D}_{2^{n-1}} \otimes \mathbb{Q}/\mathbb{Z}[2] \longrightarrow \underline{\underline{\mathrm{D}}}_{2^{n-1}} \otimes \mathbb{Q}/\mathbb{Z}[2]$.

REFERENCES

APS M. Atiyah, V. Patodi, I. Singer, Spectral asymmetry and Riemannian geometry II, Math. Proc. Cambridge Philos. Soc. **78** (1975), no. 3, 405–432.

BJM M. Barratt, J. Jones, M. Mahowald, Relations amongst Toda brackets and the Kervaire invariant in dimension 62, J. London Math. Soc. (2) **30** (1984), no. 3, 533–550.

BJM2 M. Barratt, J. Jones, M. Mahowald, The Kervaire invariant and the Hopf invariant, in: Algebraic topology (Seattle, Wash., 1985), 135–173, Lecture Notes in Math., **1286**, Springer, Berlin, 1987.

B W. Browder, The Kervaire invariant of framed manifolds and its generalization, Ann. Math. **90** (1969), 157-186.

Br R. Bruner, The homotopy theory of H_∞ ring spectra, in: H_∞ Ring Spectra and their Applications, Lecture Notes in Math., **1176**, Springer, Berlin, 1986.

He E. Hecke, Theorie der Eisensteinschen Reihe höherer Stufe und ihre Anwendung auf Funktionentheorie und Arithmetik, Abh. Math. Seminar der Hamb. Univ. **5** (1927), 199-224 oder Kapitel 24 aus E. Hecke, Mathematische Werke, Vandenhoeck & Ruprecht, Gttingen 1959.

HBJ F. Hirzebruch, T. Berger, R. Jung, Manifolds and modular forms, Aspects of Mathematics, E20, Friedr. Vieweg & Sohn, Braunschweig, 1992.

HS M. Hovey, N. Strickland, Comodules and Landweber exact homology theories, Adv. Math. **192** (2005), 427–456.

K1 N. Katz, Higher congruences between modular forms, Ann. Math. **101** (1975), 332–367.

K2 N. Katz, The Eisenstein measure and p-adic interpolation, Amer. J. Math. **99** (1977), no. 2, 238–311.

K3 N. Katz, p-adic properties of modular schemes and modular forms, in: Modular functions of one variable III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 69–190. Lecture Notes in Mathematics, **350**, Springer, Berlin, 1973.

KM S. Kochman, M. Mahowald, On the computation of stable stems, Comtemp. Math. **181** (1995), 299–316.

L1 G. Laures, The topological q-expansion principle, Topology **38** (1999), 387–425.

L2 G. Laures, On cobordism of manifolds with corners, Trans. AMS **352** (2000), 5667-5688.

MRW H. Miller, D. Ravenel and W. Wilson, Periodic phenomena in the Adams-Novikov spectral sequence, Ann. Math. **106** (1977), 469–516.

Na N. Naumann, Comodule categories and the geometry of the stack of formal groups, math.AT/0503308.

Ne J. Neukirch, Algebraic number theory, Grundlehren der Mathematischen Wissenschaften, **322**, Springer-Verlag, Berlin, 1999.

R D. Ravenel, Complex cobordism and stable homotopy groups of spheres, Pure and Applied Mathematics, **121**, Academic Press, Inc., Orlando, FL, 1986.

Sh K. Shimomura, Novikov's Ext^2 at the prime 2, Hiroshima Math. J. **11** (1981), 499–513.

Si1 J. Silverman, The arithmetic of elliptic curves, Graduate Texts in Mathematics, **106**, Springer-Verlag, New York, 1986.

Si2 J. Silverman, Advanced topics in the arithmetic of elliptic curves, Graduate Texts in Mathematics, **151**, Springer-Verlag, New York, 1994.

St N. Strickland, Notes on level three structures on elliptic curves, preprint (2000), available at http://www.shef.ac.uk/personal/n/nps/papers/.

Sw R. Switzer, Algebraic topology—homotopy and homology, Reprint of the 1975 original, Classics in Mathematics, Springer-Verlag, Berlin, 2002.

Wa L. Washington, Introduction to cyclotomic fields, Second edition, Graduate Texts in Mathematics, **83**, Springer-Verlag, New York, 1997.

Jens Hornbostel and Niko Naumann jens.hornbostel@mathematik.uni-regensburg.de
niko.naumann@mathematik.uni-regensburg.de
NWF I- Mathematik, Universität Regensburg, 93040 Regensburg