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Hartmut Weiß and Frederik Witt

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Abstract

On the space of positive 3–forms on a seven–manifold, we study the negative gradient flow of a natural functional and prove short–time existence and uniqueness.

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1 Introduction

The group G_2 is – apart from the generic $SO(7)$ – the only possible holonomy group of an irreducible, non–symmetric simply–connected manifold in dimension 7. Together with $Spin(7)$ it forms the class of *exceptional holonomy groups* whose associated geometries have been widely studied in Riemannian geometry and theoretical physics. A G_2–metric is induced by a special 3–form Ω which is non–degenerate or *positive* in the sense that it gives rise to a complementary 4–form $\Theta(\Omega)$, so that $vol_\Omega = \Omega \wedge \Theta(\Omega)$ is a nowhere vanishing volume form. By a theorem of Fernández and Gray [7], the holonomy condition translates into the equations

$$d\Omega = 0, \quad d\Theta(\Omega) = 0. \tag{1}$$

For M compact, Hitchin [9] interpreted the second equation as the Euler–Lagrange equation for the functional on positive 3–forms

$$\Omega \mapsto \int_M vol_\Omega$$

restricted to the cohomology class of a closed positive 3–form Ω_0. Nevertheless, existence of critical points is a delicate issue. Since Joyce's seminal work [10] we know compact holonomy G_2–manifolds to exist, but a Yau–Aubin type theorem which guarantuees a priori existence is yet missing.

A natural idea in this context is to look for a geometric evolution equation on the space of positive 3–forms which evolves forms towards an Ω satisfying (1). In principle, this makes sense for any "special metric" induced by an underlying form Ω satisfying equations of type (1), such as $PSU(3)$– (cf. for instance [9], [13]) or $Spin(7)$–metrics (cf. for instance [10]). However, we shall focus on G_2 for two reasons. The set of positive 3–forms Ω is an open subset of $\Omega^3(M)$, and G_2 acts transitively on the sphere. Both these features greatly simplify technicalities.

A first candidate for a flow equation has been proposed in [3], where one uses the Laplacian Δ_Ω induced by the G_2–metric g_Ω, namely

$$\frac{\partial}{\partial t}\Omega = \Delta_\Omega \Omega.$$

Restricted to *closed* positive 3–forms, we can think of this flow as the gradient flow of Hitchin's functional. However, as we are going to show, the resulting flow equation is not even weakly parabolic so that standard techniques do not apply directly. This is reminiscent of the Einstein–Hilbert functional whose negative gradient flow fails to work on the same grounds, a fact which subsequently led to the definition of Ricci flow. We thus consider the Dirichlet energy functional

$$\mathcal{D} : \Omega \mapsto \frac{1}{2}\int_M \left(|d\Omega|^2_\Omega + |d\Theta(\Omega)|^2_\Omega\right) vol_\Omega$$

whose absolute minima also satisfy (1). By appealing to the standard theory of quasilinear parabolic equations and the so–called DeTurck trick as introduced in [5], we can prove for the associated negative gradient flow the following

Theorem 1.1 *Let $Q = -\mathrm{grad}\,\mathcal{D}$. Given a positive 3–form Ω_0, there exists $\epsilon > 0$ and a smooth family of positive 3–forms $\Omega(t)$ for $t \in [0, \epsilon]$ such that*

$$\begin{cases} \frac{\partial \Omega}{\partial t} = Q(\Omega), & t \in [0,\epsilon] \\ \Omega(0) = \Omega_0 \end{cases}.$$

Furthermore, if $\Omega(t)$ and $\Omega'(t)$ are solutions to (15), then $\Omega(t) = \Omega'(t)$ whenever defined.

Hence we can speak of *the* Dirichet energy flow for some initial value Ω_0 defined on a maximal time–interval $[0, T)$ for $0 < T \le \infty$. End of lifetime analysis will be dealt with in a forthcoming paper.

2 $\mathbf{G_2}$–structures

We recall some basic features of G_2–geometry to fix notations. Good references are [2], [3], [10] and [11].

There are two *open* $GL(7)$–orbits in $\Lambda^3\mathbb{R}^{7*}$, one of which is diffeomorphic to $GL(7)/G_2$. We denote this orbit by Λ^3_+ and refer to its elements as *positive* forms. Since G_2 is a subgroup of $SO(7)$, any $\Omega \in \Lambda^3_+$ induces an orientation and a Euclidean metric g_Ω.

Let $\Omega^3_+(M)$ denote the open set of sections of $\Lambda^3_+ M$, the fibre bundle associated with the $GL(7)$–representation Λ^3_+. Then $\Omega \in \Omega^3_+(M)$ induces a reduction of the frame bundle to a principal G_2–bundle. We also refer to the pair (M, Ω) as a *G_2–structure.* Such a structure (which exists if and only if the first and second Stiefel–Whitney class of M vanish) singles out a principal $SO(7)$–bundle. Hence, Ω induces a well–defined global orientation and a metric g_Ω giving rise to a Hodge star operator $\star_\Omega$. Locally, one can find an orthonormal frame $(e_1, \ldots, e_7)$ of TM such that

$$\Omega = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}.$$

Such a frame will be referred to as a *G_2–frame.*

The holonomy of the G_2–metric g_Ω is contained in G_2 if and only if the underlying G_2–form Ω is parallel with respect to the Levi–Civita connection induced by g_Ω, i.e. $\nabla^{g_\Omega}\Omega = 0$. In this case we shall say that the G_2–structure is *torsion–free* while we call (M, Ω) a *holonomy*

G_2–manifold if the holonomy of g_Ω is actually equal[1] to G_2. In case M is compact, a torsion–free G_2–structure has holonomy G_2 if the fundamental group $\pi_1(M)$ is finite. By a theorem of Fernández and Gray [7], torsion–freeness is equivalent to $d\Omega = 0$ and $d\star_\Omega\Omega = 0$. The latter equation can be viewed as the Euler–Lagrange equation of a non–linear variational problem set up by Hitchin [9]. Consider the smooth $GL(7)$–equivariant map

$$\phi : \Omega \in \Lambda^3_+ \mapsto vol_\Omega \stackrel{\text{Def}}{=} \Omega \wedge \star_\Omega\Omega \in \Lambda^7\mathbb{R}^{7*},$$

whose first derivative at Ω evaluated on $\dot\Omega \in \Lambda^3$ is

$$D_\Omega\phi(\dot\Omega) = \frac{7}{3}\star_\Omega \Omega \wedge \dot\Omega. \tag{2}$$

If M is compact, integrating ϕ gives the functional

$$\Phi : \Omega \in \Omega^3_+(M) \mapsto \int_M \phi(\Omega) = (\Omega, \Omega)_\Omega, \tag{3}$$

where $(\cdot\,,\cdot)_\Omega$ denotes the induced L^2–norm. In analogy with Hodge theory, we can ask for critical points of this functional restricted to a fixed cohomology class. From (2) it follows that a closed Ω is a critical point in its cohomology class if and only if $d\star_\Omega\Omega = 0$ holds, that is (M, Ω) defines a torsion–free G_2–structure.

3 Representation theory

Next we recall some elements of G_2–representation theory. The material is standard or follows from straightforward computations (cf. for instance [2], [4], [8] and [10]).

The group G_2 acts irreducibly in its vector representation $\Lambda^1 \cong \mathbb{R}^7$ (in presence of a metric, we tacitly identify vectors with their duals). This action extends to the exterior algebra in the standard fashion, though Λ^p, the representation over p–forms, is no longer irreducible for $2 \leq p \leq 5$. More precisely, we have orthogonal decompositions

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14}, \quad \Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27},$$

where the subscript indicates the dimension of the irreducible module. We denote the corresponding components by $[\alpha^p]_q$. By equivariance, $\star$ induces isomorphisms $\Lambda^p_q \cong \Lambda^{n-p}_q$ from which an analogous decomposition of Λ^4 and Λ^5 follows. More precisely, we have

$$\begin{array}{ll} \Lambda^2_7 = \{\alpha \in \Lambda^2 \mid \star(\alpha \wedge \Omega) = 2\alpha\}, & \Lambda^2_{14} = \{\alpha \in \Lambda^2 \mid \star(\alpha \wedge \Omega) = -\alpha\} \cong \mathfrak{g}_2, \\ \Lambda^3_7 = \{\star(X \wedge \Omega) \mid X \in \Lambda^1\}, & \Lambda^3_{27} = \{\alpha \in \Lambda^3 \mid \star\Omega \wedge \alpha = 0,\, \Omega \wedge \alpha = 0\}. \end{array} \tag{4}$$

Note that Λ^2_{14} corresponds to the Lie algebra of $\mathfrak{g}_2$ sitting inside $\mathfrak{so}(7) \cong \Lambda^2$, while Λ^3_1 simply consists of multiples of Ω. These characterisations are obtained from a routine application of Schur's lemma. For illustration, we derive for $\eta \in \Lambda^2$ the identity[2]

$$(\eta \llcorner \Omega) \llcorner \Omega = [\eta]_7. \tag{5}$$

Indeed, $\eta \llcorner \Omega$ is a G_2–equivariant map taking values in the irreducible module $\Lambda^1 = \Lambda^1_7$ so that $\Lambda^2_{14} \subset \ker \llcorner\Omega$ by Schur, whence $\eta \llcorner \Omega = [\eta]_7 \llcorner \Omega$. Therefore, the identity (5) needs only to be checked for one element in Λ^2_7 (again by Schur). Fixing a G_2–frame as in the previous section, we find $e_1 \llcorner \Omega = e_{27} + e_{35} - e_{46} \in \Lambda^2_7$. Hence $(e_1 \llcorner \Omega) \llcorner \Omega = e_1$ and the assertion follows.

[1] This convention is by no means universal in the literature.

[2] Here, we adopt the convention of [4] for the metric contraction $\llcorner$: $\Lambda^k V^* \otimes \Lambda^l V^* \to \Lambda^{l-k} V^*$, e.g. $e^{12} \llcorner e^{12345} = e^{345}$ etc.

Remark: If M is endowed with a G_2–structure, these decompositions acquire global meaning. Hence we can speak of Ω^p_q–forms, where $\Omega^p_q(M) = C^\infty(\Lambda^p_q T^*M)$ are the smooth sections of the bundles with fibre Λ^p_q.

Next we pick a unit vector $\xi \in \Lambda^1$. Since the unit sphere S^6 is diffeomorphic with $G_2/SU(3)$, ξ gives rise to an $SU(3)$–representation over $\xi^\perp$, namely the real representation underlying the complex vector representation $\mathbb{C}^3$. In particular, $\xi^\perp$ carries a complex structure. In terms of forms, the group $SU(3)$ can be regarded as the stabiliser of a non–degenerate 2–form ω and a complex volume form $\Psi = \psi_+ + i\psi_- \in \Lambda^{3,0}\xi^\perp$. These forms relate to Ω and $\star_\Omega \Omega$ via

$$\begin{aligned} \Omega &= \omega \wedge \xi + \psi_+ \\ \star_\Omega \Omega &= \psi_- \wedge \xi + \frac{1}{2}\omega^2. \end{aligned}$$

Moreover, the decomposition of the exterior algebra over $\xi^\perp$ into irreducibles is given by

$$\lambda^1 = \xi^\perp, \quad \lambda^2 = \lambda^2_1 \oplus \lambda^2_6 \oplus \lambda^2_8, \quad \lambda^3 = \lambda^3_{1+} \oplus \lambda^3_{1-} \oplus \lambda^3_6 \oplus \lambda^3_{12}, \tag{6}$$

where as above the numerical subscript keeps track of the dimension. We also use these subscripts to denote the corresponding components of a form, e.g. $\gamma \in \lambda^3$ can be decomposed into the direct sum $\gamma = \gamma_{1+} \oplus \gamma_{1-} \oplus \gamma_6 \oplus \gamma_{12}$. The two trivial representations $\lambda^3_{1\pm}$ are spanned by ψ_+ and ψ_- respectively, while λ^2_8 corresponds to the Lie algebra of $\mathfrak{su}(3)$ sitting inside $\mathfrak{so}(6) \cong \lambda^2$. More importantly for our purposes, we can consider the decomposition of the exterior algebra over Λ^1 into $SU(3)$–irreducibles. Here, we shall denote by $(\mathbf{n})^p_q$ the n–dimensional irreducible $SU(3)$–representation inside Λ^p_q. Then

$$\begin{aligned} &\Lambda^1 \cong (\mathbf{1})^1_7 \oplus (\mathbf{6})^1_7, \quad \Lambda^2 \cong (\mathbf{1})^2_7 \oplus (\mathbf{6})^2_7 \oplus (\mathbf{6})^2_{14} \oplus (\mathbf{8})^2_{14}, \\ &\Lambda^3 \cong (\mathbf{1})^3_1 \oplus (\mathbf{1})^3_7 \oplus (\mathbf{6})^3_7 \oplus (\mathbf{1})^3_{27} \oplus (\mathbf{6})^3_{27} \oplus (\mathbf{8})^3_{27} \oplus (\mathbf{12})^3_{27}, \end{aligned}$$

so that no confusion shall occur. The decomposition of Λ^3 is of particular importance for the sequel. The occuring modules can be characterised as follows:

$$\begin{aligned} (\mathbf{1})^3_1 &= \{a(\omega \wedge \xi + \psi_+) \,|\, a \in \mathbb{R}\}, \\ (\mathbf{1})^3_7 &= \{b\psi_- \,|\, b \in \mathbb{R}\}, \\ (\mathbf{1})^3_{27} &= \{c(-4\omega \wedge \xi + 3\psi_+) \,|\, c \in \mathbb{R}\}, \\ (\mathbf{6})^3_7 &= \{(X \llcorner \psi_-) \wedge \xi - (X \llcorner \omega) \wedge \omega \,|\, X \in \xi^\perp\}, \\ (\mathbf{6})^3_{27} &= \{(Y \llcorner \psi_-) \wedge \xi + (Y \llcorner \omega) \wedge \omega \,|\, Y \in \xi^\perp\}, \\ (\mathbf{8})^3_{27} &= \{\beta_8 \wedge \xi \,|\, \beta_8 \in \lambda^2_8\}. \end{aligned}$$

Therefore, any $\dot{\Omega} \in \Lambda^3$ can be written as

$$\begin{aligned} \dot{\Omega} &= [\dot{\Omega}]_1 \oplus [\dot{\Omega}]_7 \oplus [\dot{\Omega}]_{27} \\ &= \big[a(\omega \wedge \xi + \psi_+)\big] \oplus \big[b\psi_- + (X_0 \llcorner \psi_-) \wedge \xi - (X_0 \llcorner \omega) \wedge \omega\big] \\ &\quad \oplus \big[c(-4\omega \wedge \xi + 3\psi_+) + (Y_0 \llcorner \psi_-) \wedge \xi + (Y_0 \llcorner \omega) \wedge \omega + \beta_8 \wedge \xi + \gamma_{12}\big] \end{aligned} \tag{7}$$

for constants a, b, $c \in \mathbb{R}$, vectors X_0, $Y_0 \in \xi^\perp$ and forms $\beta_8 \in \lambda^2_8$, $\gamma_{12} \in \lambda^3_{12}$. In particular, decomposing $\dot{\Omega} = \beta \wedge \xi + \gamma$, where β and γ are the uniquely determined 2– and 3–forms in $\Lambda^* \xi^\perp$ such that $\xi \llcorner \beta, \gamma = 0$, we obtain

$$\beta = (a - 4c)\omega \oplus (X_0 + Y_0) \llcorner \psi_- \oplus \beta_8 \tag{8}$$

$$\gamma = (a + 3c)\psi_+ \oplus b\psi_- \oplus \big((Y_0 - X_0) \llcorner \omega\big) \wedge \omega \oplus \gamma_{12}. \tag{9}$$

Thus $\beta_1 = (a-4c)\omega$ etc. For later applications, we need for $X \in \xi^\perp$ the identities

$$\star_\Omega\big((X\llcorner\psi_-)\wedge\Omega\big) = X\llcorner\psi_- + 2X\wedge\xi \tag{10}$$

and

$$|X\llcorner\psi_-|^2 = 2|X|^2, \tag{11}$$

where we use the expressions $\omega = e^{12}+e^{34}+e^{56}$, $\psi_+ = e^{135}-e^{146}-e^{236}-e^{245}$ and $\psi_- = e^{136}+e^{145}+e^{235}-e^{246}$ with respect to a G_2–frame (cf. [4]). Then (10) is proven along the lines of (5), while (11) uses the transitive and isometric action of $SU(3)$ on S^5 so that up to a rotation we may assume that $X = |X|e_1$.

4 The Dirichlet energy functional

In the following, we shall always take the underlying manifold M to be closed.

The space $\Omega^3_+(M)$ of positive 3–forms is open in $\Omega^3(M)$ and we can equip the tangent space $T_\Omega\Omega^3_+(M)\cong\Omega^3(M)$ at Ω with the induced L^2–metric

$$G_\Omega(\dot\Omega_0,\dot\Omega_1) = \int_M g_\Omega(\dot\Omega_0,\dot\Omega_1)\, vol_\Omega = \int \dot\Omega_0\wedge\star_\Omega\dot\Omega_1,$$

$\dot\Omega_0$, $\dot\Omega_1\in\Omega^3(M)$. Gradients will always be taken with respect to G. We also define a natural non–linear differential operator of second order, namely

$$F : \Omega\in\Omega^3_+(M)\mapsto\Delta_\Omega\Omega = d\delta_\Omega\Omega+\delta_\Omega d\Omega\in\Omega^3(M). \tag{12}$$

Here, δ_Ω is the codifferential induced by g_Ω, i.e. for a form α^p of degree p, we have

$$\delta_\Omega = (-1)^p\star_\Omega d\star_\Omega\alpha^p.$$

Definition 4.1 *The* Dirichlet energy functional $\mathcal{D}:\Omega^3_+(M)\to\mathbb{R}$ *is defined by*

$$\mathcal{D}(\Omega) = \frac{1}{2}\big(\Delta_\Omega\Omega,\Omega\big)_\Omega = \frac{1}{2}(\|d\Omega\|^2_\Omega+\|\delta_\Omega\Omega\|^2_\Omega).$$

It follows that

$$\mathcal{D}(\Omega) = \frac{1}{2}\int_M\big(|d\Omega|^2_\Omega+|\delta_\Omega\Omega|^2_\Omega\big)\, vol_\Omega = \frac{1}{2}\int_M d\Omega\wedge\star_\Omega d\Omega+\delta_\Omega\Omega\wedge\star_\Omega\delta_\Omega\Omega.$$

Remark: Since $\star_{\varphi^*\Omega} = \varphi^*\star_\Omega\varphi^{-1*}$ for $\varphi\in \mathit{Diff}(M)$, $\mathcal{D}$ is diffeomorphism invariant, i.e. $\mathcal{D}(\varphi^*\Omega)=\mathcal{D}(\Omega)$ for all $\varphi\in \mathit{Diff}(M)$, $\Omega\in\Omega^3_+(M)$.

We compute the gradient of $\mathcal{D}$ next. To that end, we introduce the following piece of notation. If $A:\Omega^3_+(M)\to E$ is any vector bundle valued differential operator and $\Omega\in\Omega^3_+(M)$, we write $\dot A_\Omega$ for the linearisation of A at $\Omega\in\Omega^3_+(M)$ evaluated on some 3–form $\dot\Omega$ tangent to Ω, i.e.

$$\dot A_\Omega \overset{\text{Def}}{=} D_\Omega A(\dot\Omega).$$

We illustrate this convention by two examples which will be needed later.

Example: (i) Consider the map

$$\Theta : \Omega \in \Omega^3_+(M) \mapsto \star_\Omega \Omega \in \Omega^4(M).$$

By Prop. 10.3.5 in [10],

$$\dot{\Theta}_\Omega = \frac{4}{3} \star_{\Omega_0} [\dot{\Omega}]_1 + \star_\Omega [\dot{\Omega}]_7 - \star_\Omega [\dot{\Omega}]_{27} = \star_\Omega (\dot{\Omega} + \frac{1}{3}[\dot{\Omega}]_1 - 2[\dot{\Omega}]_{27}).$$

(ii) In continuation of the first example, we consider the map $F : \Omega^3_+(M) \to \Omega^3(M)$ defined in (12). Then

$$\begin{aligned}
\dot{F}_\Omega &= \dot{\star}_\Omega d \star_\Omega d\Omega + \star_\Omega d \dot{\star}_\Omega d\Omega + \star_\Omega d \star_\Omega d\dot{\Omega} - d\dot{\star}_\Omega d\Theta(\Omega) - d \star_\Omega d\dot{\Theta}(\Omega) \\
&\overset{(i)}{=} \star_\Omega d \star_\Omega d\dot{\Omega} - d \star_\Omega d \star_\Omega (\dot{\Omega} + \frac{1}{3}[\dot{\Omega}]_1 - 2[\dot{\Omega}]_{27}) + \text{ terms of lower order in } \dot{\Omega} \\
&= \Delta_\Omega \dot{\Omega} + d\delta_\Omega (\frac{1}{3}[\dot{\Omega}]_1 - 2[\dot{\Omega}]_{27}) + \text{ terms of lower order in } \dot{\Omega}. \qquad (13)
\end{aligned}$$

Lemma 4.2 *We have*

$$\dot{\mathcal{D}}_\Omega = \int_M \dot{\Omega} \wedge \star_\Omega \big(\Delta_\Omega \Omega + \frac{1}{3}[d\delta_\Omega \Omega]_1 - 2[d\delta_\Omega \Omega]_{27} + q(d\Omega)\big)$$

for some smooth quadratic function q. In particular, the L^2–gradient of $\mathcal{D}$ at Ω is

$$\operatorname{grad} \mathcal{D}(\Omega) = \Delta_\Omega \Omega + \frac{1}{3}[d\delta_\Omega \Omega]_1 - 2[d\delta_\Omega \Omega]_{27} + q(d\Omega).$$

Proof: As in the previous example,

$$\begin{aligned}
\dot{\mathcal{D}} &= \frac{1}{2}\int_M d\dot{\Omega} \wedge \star_\Omega d\Omega + d\Omega \wedge \big(\dot{\star}_\Omega d\Omega + \star_\Omega d\dot{\Omega}\big) \\
&\quad + \frac{1}{2}\int_M d\Theta(\Omega) \wedge \big(\dot{\star}_\Omega d\Theta(\Omega) + \star_\Omega d\dot{\Theta}_\Omega\big) + d\dot{\Theta}_\Omega \wedge \star_\Omega d\Theta(\Omega) \\
&= \int_M d\dot{\Omega} \wedge \star_\Omega d\Omega + d\dot{\Theta}_\Omega \wedge \star_\Omega d\Theta(\Omega) \\
&\quad + \frac{1}{2}\int_M d\Omega \wedge \dot{\star}_\Omega d\Omega + \dot{\star}_\Omega d\Theta(\Omega) \wedge d\Theta(\Omega). \qquad (14)
\end{aligned}$$

Now $l_{d\Omega} : \dot{\Omega} \mapsto \dot{\star}_\Omega d\Omega$ is a linear map from $\Omega^3(M) \to \Omega^3(M)$ depending (linearly) on $d\Omega$, so that we can consider the formal adjoint $l^*_{d\Omega}$. Thus

$$\int_M d\Omega \wedge \dot{\star}_\Omega d\Omega = G_\Omega\big(l_{d\Omega}(\dot{\Omega}), \star_\Omega d\Omega\big) = G_\Omega\big(\dot{\Omega}, l^*_{d\Omega}(\star_\Omega d\Omega)\big).$$

The second term in (14) is dealt with in a similar fashion. The last line is therefore of the form $\int_M \dot{\Omega} \wedge q(d\Omega)$ with q quadratic in the first derivatives of Ω, as asserted. On the other hand, Stokes implies

$$\begin{aligned}
\int_M d\dot{\Omega} \wedge \star_\Omega d\Omega + d\dot{\Theta}_\Omega \wedge \star_\Omega d\Theta(\Omega) &= \int_M \dot{\Omega} \wedge d \star_\Omega d\Omega - \dot{\Theta}_\Omega \wedge d \star_\Omega d\Theta(\Omega) \\
&= G_\Omega(\dot{\Omega}, \delta_\Omega d\Omega) + G_\Omega(\star_\Omega \dot{\Theta}_\Omega, d\delta_\Omega \Omega) \\
&= G_\Omega(\dot{\Omega}, \Delta_\Omega \Omega) + \frac{1}{3} G_\Omega([\dot{\Omega}]_1, d\delta_\Omega \Omega) \\
&= -2G_\Omega([\dot{\Omega}]_{27}, d\delta_\Omega \Omega),
\end{aligned}$$

whence the assertion for $[\cdot]_p$ is self–adjoint. ■

5 Short–time existence

>From now on, let

$$Q(\Omega) = -\mathrm{grad}\,\mathcal{D}(\Omega)$$

denote the negative gradient of $\mathcal{D}$.

Definition 5.1 *The* Dirichlet energy flow *with initial condition* $\Omega_0 \in \Omega^3_+(M)$ *is the negative gradient flow of* $\mathcal{D}$*, i.e. a smooth family of positive* 3*–forms* $\Omega(t) \in \Omega^3_+(M)$ *such that*

$$\frac{\partial}{\partial t}\Omega = Q(\Omega), \quad \Omega(0) = \Omega_0. \tag{15}$$

The goal of this section is to prove the existence part of the following

Theorem 5.2 *Given* $\Omega_0 \in \Omega^3_+(M)$*, there exists* $\epsilon > 0$ *and a smooth family of positive* 3*–forms* $\Omega(t)$ *for* $t \in [0,\epsilon]$ *such that*

$$\begin{cases} \frac{\partial \Omega}{\partial t} = Q(\Omega), & t \in [0,\epsilon] \\ \Omega(0) = \Omega_0 \end{cases}.$$

Furthermore, if $\Omega(t)$ *and* $\Omega'(t)$ *are solutions to* (15)*, then* $\Omega(t) = \Omega'(t)$ *whenever defined.*

Hence we can speak of *the* Dirichet energy flow for some initial value Ω_0 defined on a maximal time–interval $[0,T)$ with $0 < T \leq \infty$.

We will prove short–time existence and uniqueness by invoking the standard theory of quasilinear parabolic equations which we briefly recall (cf. for instance Chapter 4.4.2 [1] or Chapter 4 in [12]). Consider a Riemannian vector bundle $\big(E, (\cdot\,,\cdot)\big)$ and a nonlinear partial differential equation of the form

$$\frac{\partial u}{\partial t} = P_t(u), \tag{16}$$

where $P_t : \mathcal{U} \subset C^\infty(E) \to C^\infty(E)$ is a *quasilinear* second order differential operator defined on an open subset $\mathcal{U}$ of $C^\infty(E)$, possibly depending on time. More concretely, in terms of local coordinates $\{x^i\}$ with partial derivatives ∂_i and a local basis $\{s_\alpha\}$ of E, one has

$$P_t(u)(x) \overset{\mathrm{loc}}{=} \big(a^{ij\alpha}_\beta(t,x,u,\nabla u)\partial_i\partial_j u^\beta + b^\alpha(t,x,u,\nabla u)\big)s_\alpha$$

for smooth functions $a^{ij\alpha}_\beta$ and b^α. Fix $u_0 \in C^\infty(E)$ and compute $D_{u_0}P_0 : C^\infty(E) \to C^\infty(E)$, the linearisation of P_0 at u_0. We say that equation (16) is *strongly parabolic* at u_0 if the linearisation of P_0 at u_0 is strongly elliptic, i.e. there exists $\lambda > 0$ such that

$$\big(\sigma(D_{u_0}P_0)(x,\xi)v, v\big) \geq \lambda|\xi|^2|v|^2$$

for all $x \in M$, $\xi \in T^*_x M$ and $v \in E_x$. If the symbol is merely positive semi–definite, we call (16) *weakly parabolic.* Then:

Theorem 5.3 *If equation* (16) *is strongly parabolic at* u_0*, then there exists* $\epsilon > 0$ *and a unique smooth family* $u(t) \in C^\infty(E)$*,* $t \in [0,t]$ *such that*

$$\begin{cases} \frac{\partial u}{\partial t} = P_t(u), & t \in [0,\epsilon] \\ u(0) = u_0 \end{cases}.$$

Returning to the mainstream development, we note that Q in (15) is a second–order quasilinear partial differential operator, as follows from Lemma 4.2. In view of theorem 5.3, it remains to check parabolicity.

Lemma 5.4 *The principal symbol of the linearisation $D_\Omega Q$ of Q at $\Omega\in\Omega^3_+(M)$,*

$$\sigma(D_\Omega Q)(x,\cdot):T^*M\backslash\{0\}\to\mathrm{End}(\Lambda^3T^*_xM),$$

is given by

$$\begin{aligned}\sigma(D_\Omega Q)(x,\xi)\dot\Omega &= |\xi|^2_\Omega\dot\Omega+\xi\wedge\big(\xi\llcorner(\frac{1}{3}[\dot\Omega]_1-2[\dot\Omega]_{27})\big)\\ &\quad+\frac{1}{3}\big[\xi\wedge\big(\xi\llcorner(\dot\Omega+\frac{1}{3}[\dot\Omega]_1-2[\dot\Omega]_{27})\big)\big]_1\\ &\quad-2\big[\xi\wedge\big(\xi\llcorner(\dot\Omega+\frac{1}{3}[\dot\Omega]_1-2[\dot\Omega]_{27})\big)\big]_{27}\end{aligned}$$

where projections are taken with respect to Ω. In particular, the symbol is positive semi–definite.

Proof: As the principal symbol involves highest order terms only, we need to linearise the expression

$$Q(\Omega)=-\Delta_\Omega\Omega-\frac{1}{3}[d\delta_\Omega\Omega]_1+2[d\delta_\Omega\Omega]_{27}.$$

In our convention, $\sigma(d)(x,\xi)\dot\Omega=\xi\wedge\Omega$ and $\sigma(\delta_\Omega)(x,\xi)\dot\Omega=-\xi\llcorner\Omega$, where $\llcorner$ denotes metric contraction with respect to g_Ω. Hence, from (13) and the standard symbolic calculus we get the asserted symbol.

For the second assertion we may assume $|\xi|_\Omega=1$. To keep notation tight we put $\mu(\dot\Omega)=\frac{1}{3}[\dot\Omega]_1-2[\dot\Omega]_{27}$ and $\eta(\dot\Omega)=\star_\Omega\dot\Theta_\Omega=\dot\Omega+\mu(\dot\Omega)$. Then

$$\begin{aligned}g_\Omega\big(\sigma(D_\Omega Q)(x,\xi)\dot\Omega,\dot\Omega\big) &= g_\Omega\big(\dot\Omega+\xi\wedge(\xi\llcorner\mu)+\frac{1}{3}[\xi\wedge(\xi\llcorner\eta)]_1-2[\xi\wedge(\xi\llcorner\eta)]_{27},\dot\Omega\big)\\ &= |\dot\Omega|^2_\Omega+g_\Omega(\xi\llcorner\mu,\xi\llcorner\dot\Omega)+\frac{1}{3}g_\Omega(\xi\llcorner\eta,\xi\llcorner[\dot\Omega]_1)-2g_\Omega(\xi\llcorner\eta,\xi\llcorner[\dot\Omega]_{27})\\ &= |\dot\Omega|^2_\Omega+g_\Omega(\xi\llcorner\mu,\xi\llcorner\dot\Omega)+g_\Omega(\xi\llcorner\eta,\xi\llcorner\mu)\\ &= |\dot\Omega|^2_\Omega+2g_\Omega(\xi\llcorner\mu,\xi\llcorner\dot\Omega)+|\xi\llcorner\mu|^2_\Omega\\ &= |\dot\Omega|^2_\Omega+|\xi\llcorner\eta|^2_\Omega-|\xi\llcorner\dot\Omega|^2_\Omega\\ &= |\gamma|^2_\Omega+|\xi\llcorner\eta|^2_\Omega\\ &\geq 0,\end{aligned}$$

where we used the decomposition $\dot\Omega=\beta\wedge\xi+\gamma$ as in Section 3. ■

Remark: By diffeomorphism invariance, we cannot expect the principal symbol to be strictly positive–definite. Indeed, $g_\Omega\big(\sigma(D_\Omega Q)(x,\xi)\dot\Omega,\dot\Omega\big)=0$ if and only if $|\gamma|^2_\Omega$, $|\xi\llcorner\eta|^2_\Omega=0$. From (9) we deduce $a=-3c$, $b=0$,$X_0=Y_0$ and $\gamma_{12}=0$. Using this, (7) gives $\xi\llcorner\eta=(X_0-Y_0)\llcorner\psi_--\beta_8$, hence $X_0=Y_0$ and $\beta_8=0$. Finally, (8) implies

$$\ker\sigma(D_\Omega Q)(x,\xi)=\{(v\omega+V\llcorner\psi_-)\wedge\xi\,|\,v\in\mathbb{R},\,V\in\xi^\perp\}.$$

This problem is reminiscent of what happens with Ricci flow where it is circumvented by so–called DeTurck's trick [5]. First we remark that for $\varphi \in \mathit{Diff}(M)$

$$\varphi^* Q(\Omega) = Q(\varphi^*\Omega) \tag{17}$$

as $\mathcal{D} \circ \varphi^* = \mathcal{D}$ and $G_{\varphi^*\Omega}(\dot\Omega_0, \dot\Omega_1) = G_\Omega(\varphi^{-1*}\dot\Omega_0, \varphi^{-1*}\dot\Omega_1)$. Thus, given a family of diffeomorphisms $\partial_t \varphi_t = X_t \circ \varphi_t$ induced by a (time–dependent) vector field X_t on M, differentiating (17) yields the intertwining formula

$$\mathcal{L}_X\big(Q(\Omega)\big) = D_\Omega Q(\mathcal{L}_X\Omega), \tag{18}$$

where $\mathcal{L}_X$ denotes Lie derivative with respect to X. While the left hand side of (18) is first order in X, the right hand side, as the composition of a second with a first order operator, is of third order in X. Hence, passing to symbol level, we find

$$\sigma(D_\Omega Q)(x,\xi) \circ \sigma(X \mapsto \mathcal{L}_X\Omega)(x,\xi) = 0 \tag{19}$$

so that $\sigma(X \mapsto \mathcal{L}_X\Omega)(x,\xi)$ takes values in the kernel of $\sigma(D_\Omega Q)(x,\xi)$. Put differently, we can think of the symbol of the map

$$\Omega \in \Omega^3_+(M) \mapsto X(\Omega) \in C^\infty(TM) \mapsto \Lambda(\Omega) = \mathcal{L}_{X(\Omega)}\Omega \in \Omega^3(M) \tag{20}$$

where $X(\Omega)$ is a vector field depending non–trivially on the 1–jet of Ω, as a kind of projector to the kernel of $\sigma(D_\Omega Q)$. Summarising, one expects the symbol of the modified operator

$$\widetilde{Q}(\Omega) = Q(\Omega) + \Lambda(\Omega) \tag{21}$$

to have trivial kernel for a suitably chosen vector field. Given a fixed initial condition $\Omega_0 \in \Omega^3_+(M)$ we shall employ

$$X_{\Omega_0} : \Omega^3(M) \to \Omega^1(M), \quad X_{\Omega_0}(\Omega) = \big((\delta_{\Omega_0}\Omega) \llcorner \Omega_0\big), \tag{22}$$

where we contract and dualise with respect to the metric g_{Ω_0}. Again, any distinction between forms and (multi–) vectors will be dropped in presence of a fixed metric. We think of X_{Ω_0} as a first order, linear differential operator.

Lemma 5.5 *The operator $\widetilde{Q}$ in* (21) *is strongly elliptic at $\Omega_0 \in \Omega^3_+(M)$ for X_{Ω_0} as in* (22).

Proof: We need to compute the principal symbol of the linearisation of Λ at Ω_0. Again we take $|\xi|_\Omega = 1$. Firstly, since X_{Ω_0} is linear, we find for the linearisation in virtue of Cartan's formula

$$\dot\Lambda_{\Omega_0} = d\big(X_{\Omega_0} \llcorner \Omega_0\big) + \text{ lower order terms in } \dot\Omega,$$

whence

$$\begin{aligned} \sigma(D_{\Omega_0}\Lambda)(x,\xi)\dot\Omega &= \xi \wedge \big(\sigma(X_{\Omega_0})(x,\xi)\dot\Omega \llcorner \Omega_0\big) \\ &= \xi \wedge \big((\xi \llcorner \dot\Omega) \llcorner \Omega_0 \llcorner \Omega_0\big). \end{aligned}$$

Decomposing $\dot\Omega = \beta \wedge \xi + \gamma$ as above we therefore find using (5)

$$\xi \wedge \big((\beta \llcorner \Omega_0) \llcorner \Omega_0\big) = \xi \wedge \big(([\beta]_7 \llcorner \Omega_0) \llcorner \Omega_0\big) = \xi \wedge [\beta]_7.$$

The projection of β onto Λ^2_7 is given by $\big($bearing in mind $\star([\beta]_{14}\wedge\Omega)=-[\beta]_{14}$ by (4)$\big)$

$$\begin{aligned}[\beta]_7 &= \frac{1}{3}\big(\beta+\star_{\Omega_0}(\beta\wedge\Omega_0)\big)\\ &= \beta_1\oplus\frac{1}{3}\big(\beta_6+\star_{\Omega_0}(\beta_6\wedge\Omega_0)\big)\\ &\overset{(8),(10)}{=} \beta_1\oplus\frac{2}{3}\big((X_0+Y_0)\llcorner\psi_-+(X_0+Y_0)\wedge\xi\big)\end{aligned}$$

whence

$$\xi\wedge[\beta]_7=\xi\wedge\big(\beta_1+\frac{2}{3}(X_0+Y_0)\llcorner\psi_-\big).$$

Consequently

$$g_{\Omega_0}(\dot\Omega,\xi\wedge[\beta]_7)=|\beta_1|^2+\frac{2}{3}|(X_0+Y_0)\llcorner\psi_-|^2$$

so that the computation from Lemma 5.4 implies

$$g_\Omega\big(\sigma(D_\Omega Q)(x,\xi)\dot\Omega,\dot\Omega\big)=|\gamma|^2+|\xi\llcorner\eta|^2+|\beta_1|^2+\frac{2}{3}|(X_0+Y_0)\llcorner\psi_-|^2. \tag{23}$$

Now $\xi\llcorner\eta=\sigma\oplus\beta_8$ with $g_\Omega(\sigma,\beta_8)=0$, while $|\beta|^2=|\beta_1|^2+|\beta_6|^2+|\beta_8|^2$ by (6). But (8) gives $|\beta_6|^2=|(X_0+Y_0)\llcorner\psi_-|^2$, whence

$$g_\Omega\big(\sigma(D_\Omega Q)(x,\xi)\dot\Omega,\dot\Omega\big)\geq\frac{2}{3}(|\beta|^2+|\gamma|^2)=\frac{2}{3}|\dot\Omega|^2$$

by (23). ■

Now by Theorem 5.3 we obtain uniqueness and short–time existence of the modified flow

$$\frac{\partial}{\partial t}\widetilde\Omega=\widetilde Q(\widetilde\Omega),\quad \widetilde\Omega(0)=\Omega_0. \tag{24}$$

Finally:

Lemma 5.6 *Let $\widetilde\Omega(t)$ be a solution to the modified flow equation* (24) *with initial condition Ω_0. Let φ_t be the family of diffeomorphisms determined by $\partial_t\varphi_t=-X_{\Omega_0}\big(\widetilde\Omega(t)\big)\circ\varphi_t$ and $\varphi_0=\mathrm{Id}_M$. Then $\Omega(t)=\varphi_t^*\widetilde\Omega(t)$ is a solution to the Dirichlet energy flow* (15) *with same initial condition Ω_0.*

Proof: By definition,

$$\begin{aligned}\frac{\partial}{\partial t}\Omega &= \varphi_t^*\big(\frac{\partial}{\partial t}\widetilde\Omega+\mathcal{L}_{-X_{\Omega_0}(\widetilde\Omega)}\widetilde\Omega\big)\\ &\overset{(21)}{=} \varphi_t^*Q(\widetilde\Omega)\\ &\overset{(17)}{=} Q(\Omega).\end{aligned}$$

Moreover, the initial condition is satisfied, for $\Omega(0)=\mathrm{Id}_M^*\Omega_0=\Omega_0$. ■

Remark: A further natural flow to consider is the gradient flow attached to the Hitchin functional Φ from Section 2. First, (2) and (3) show the L^2–gradient of Φ to be

$$\mathrm{grad}\,\Phi(\Omega)=\frac{3}{7}\Omega.$$

For closed $\Omega_0 \in \Omega^3_+(M)$ we study the restriction of Φ to the cohomology class of Ω_0 and write $\Omega = \Omega_0 + d\alpha$ for $\Omega \in [\Omega_0]$. Let $\Phi_0(\alpha) = \Phi(\Omega)$. Its L^2–gradient is $\operatorname{grad}\Phi_0(\alpha) = \frac{7}{3}\delta_{\Omega_0+d\alpha}(\Omega_0 + d\alpha)$, which gives rise to the gradient flow equation

$$\frac{\partial}{\partial t}\alpha = \delta_{\Omega_0+d\alpha}(\Omega_0 + d\alpha) \tag{25}$$

subject to the initial condition $\alpha(0) = 0$. On the other hand, we can study the non–linear heat equation

$$\frac{\partial}{\partial t}\Omega = F(\Omega) = \Delta_\Omega\Omega \tag{26}$$

subject to $\Omega(0) = \Omega_0$. Note that this flow preserves closedness of the initial condition, i.e. if $d\Omega_0 = 0$, then $d\Omega(t) = 0$ if defined. Furthermore, a closed initial condition induces a 1–1 correspondence between solutions of (25) and (26): If α is a solution to (25), then $\Omega = \Omega_0 + d\alpha$ solves (26). Conversely, if Ω solves (26), then $\alpha(\,\cdot\,, t) := -\int_0^t (\delta_\Omega\Omega)(\,\cdot\,,\tau)d\tau$ is a solution to (25) with $\Omega = \Omega_0 + d\alpha$.

Now by the example in Section 4 we have

$$\sigma(D_\Omega F)(x,\xi)\dot\Omega = |\xi|^2_\Omega\dot\Omega + \xi \wedge \big(\xi \llcorner (\frac{1}{3}[\dot\Omega]_1 - 2[\dot\Omega]_{27})\big). \tag{27}$$

As before, the kernel is given by

$$\ker\sigma(D_\Omega F)(x,\xi) = \{(v\omega + V\llcorner\psi_-) \wedge \xi \mid v \in \mathbb{R},\, V \in \xi^\perp\}.$$

However, $\dot\Omega = \beta_8 \wedge \xi$ implies (taking $|\xi|_\Omega = 1$ for simplicity)

$$g_\Omega\big(\dot\Omega, \dot\Omega + \xi \wedge \big(\xi\llcorner(\frac{1}{3}[\dot\Omega]_1 - 2[\dot\Omega]_{27})\big) = -|\beta_8|^2_\Omega < 0$$

while for instance $\dot\Omega = 7c\psi_+$ gives

$$g_\Omega\big(\dot\Omega, \dot\Omega + \xi \wedge \big(\xi\llcorner(\frac{1}{3}[\dot\Omega]_1 - 2[\dot\Omega]_{27})\big) = 49c^2|\psi_+|^2_\Omega > 0.$$

Since by (19), the symbol of $X \mapsto \mathcal{L}_X\Omega$ takes values in the kernel $K = \ker\sigma(D_\Omega F)$, DeTurck's trick cannot modify the component $A = \mathrm{pr}_{K^\perp} \circ \sigma(D_\Omega F)_{|K^\perp} : K^\perp \to K^\perp$ of $\sigma(D_\Omega F) : K \oplus K^\perp \to K \oplus K^\perp$. Hence, the linearisation of $\widetilde{Q} = Q + \Lambda$ will be indefinite no matter how the vector field X in (20) is chosen (though $\widetilde{Q}$ might have trivial kernel). We therefore deal with a heat equation of mixed forwards/backwards type for which short–time existence is in general not expected.

6 Uniqueness

In the final section we settle the uniqueness part of Theorem 5.2.

We pursue a strategy along the lines of the uniqueness proof for Ricci flow. As shown by Lemma 5.6, a solution of the modified flow $\widetilde\Omega(t)$ with initial condition $\widetilde\Omega(t) = \Omega_0$ yields a solution $\Omega(t) = \varphi_t^*\widetilde\Omega(t)$ to the unmodified flow (15) by integrating the time–dependent vector field

$$-X_{\Omega_0}\big(\widetilde\Omega(t)\big) \circ \varphi_t = \frac{\partial}{\partial t}\varphi_t, \quad \varphi_0 = \mathrm{Id}_M. \tag{28}$$

Conversely, how can we build a solution of the modified flow (necessarily unique in virtue of strong parabolicity)?

In the situation above we substitute $\widetilde{\Omega}$ by $\varphi^{-1*}\Omega$ and turn the ordinary differential equation (28) into the partial differential equation

$$\frac{\partial}{\partial t}\varphi_t = -X_{\Omega_0}\big(\varphi_t^{-1*}\Omega(t)\big) \circ \varphi_t, \quad \varphi_0 = \mathrm{Id}_M. \tag{29}$$

Remark: Equation (29) should be considered as an analogue of the harmonic map heat flow

$$\frac{\partial}{\partial t}\varphi_t = \tau_{g(t),g_0}(\varphi_t)$$

introduced by Eells and Sampson [6], albeit with a time–dependent *tension field* $\tau_{g(t),g_0}(\varphi_t)$. We can think of $\tau_{g(t),g_0}(\varphi_t)$ as a differential operator defined by Riemannian metrics $g(t)$ and g_0 on M and taking a smooth map $\varphi : \big(M, g(t)\big) \to (M, g_0)$ to a section $\tau_{g(t),g_0}(\varphi) \in C^\infty(\varphi^*TM)$.

We claim that a curve $\varphi_t \subset \mathit{Diff}(M)$ which solves (29) for some solution $\Omega(t)$ to (15) yields a solution $\widetilde{\Omega}(t) = \varphi^{-1*}\Omega(t)$ to the modified flow (24). Indeed, let $Y_t \circ \varphi_t = \partial_t\varphi_t$ be the induced time–dependent vector field. Then differentiating the identity $\varphi_t^{-1} \circ \varphi_t = \mathrm{Id}_M$ with respect to t gives

$$Y_t(x) = -d_{\varphi_t(x)}\varphi_t^{-1}\big(-X_{\Omega_0}\big(\varphi_t^{-1*}\Omega(t)\big) \circ \varphi_t(x)\big) = \varphi_{t*}^{-1}X_{\Omega_0}\big(\varphi_t^{-1*}\Omega(t)\big)(x), \tag{30}$$

where for $\varphi \in \mathit{Diff}(M)$ and $X \in C^\infty(TM)$,

$$(\varphi_*X)(x) \overset{\mathrm{Def}}{=} d_{\varphi^{-1}(x)}\varphi\big(X(\varphi^{-1}(x)\big).$$

As a consequence, we get

$$\begin{aligned}\frac{\partial}{\partial t}\varphi^{-1*}\Omega(t) &= \varphi^{-1*}\big(\frac{\partial}{\partial t}\Omega(t) + \mathcal{L}_{Y_t}\Omega(t)\big)\\ &= Q\big(\varphi^{-1*}\Omega(t)\big) + \mathcal{L}_{\varphi_{t*}Y_t}\varphi^{-1*}\Omega(t)\\ &= Q\big(\varphi^{-1*}\Omega(t)\big) + \mathcal{L}_{X_{\Omega_0}(\varphi_t^{-1*}\Omega(t))}\varphi^{-1*}\Omega(t)\end{aligned}$$

by (30).

We establish the existence of a solution to (29) next. Let

$$P_t = P_{\Omega(t),\Omega_0} : \varphi \in \mathit{Diff}(M) \subset C^\infty(M,M) \mapsto -d\varphi\big(X_{\varphi^*\Omega_0}\big(\Omega(t)\big)\big) \in C^\infty(\varphi^*TM). \tag{31}$$

In view of the functoriality of the definition of X_{Ω_0}

$$\varphi_*X_{\Omega_0}\big(\Omega(t)\big) = X_{\varphi^{-1*}\Omega_0}\big(\varphi^{-1*}\Omega(t)\big),$$

whence

$$P_t(\varphi) = -\varphi_*X_{\varphi^*\Omega_0}\big(\Omega(t)\big) \circ \varphi = -X_{\Omega_0}\big(\varphi^{-1*}\Omega(t)\big) \circ \varphi,$$

or alternatively,

$$P_t(\varphi)(x) = P_{\Omega(t),\Omega_0}(\varphi)(x) = P_{\varphi^{-1*}\Omega(t),\Omega_0}(\mathrm{Id}_M)\big(\varphi(x)\big). \tag{32}$$

Since $\mathit{Diff}(M)$ is open in $C^\infty(M,M)$, a solution to the flow equation

$$\partial_t \varphi_t = P_t(\varphi_t), \quad \varphi_0 = \mathrm{Id}_M$$

gives us the desired solution to (29).

To get formally in a situation to apply Theorem 5.3, we choose an embedding $f_0 : M \to \mathbb{R}^n$. We keep on denoting by the same letter all tensors on M pushed forward to $f_0(M)$, i.e. we write g for $f_{0*}g$ etc. Let $\mathcal{N}$ be a tubular neighbourhood of $f_0(M)$ in $\mathbb{R}^n$ which we think of as an open neighbourhood inside the normal bundle

$$\pi : \nu f_0(M) \to f_0(M).$$

By choosing a Riemannian metric h on the fibres, we obtain first the induced metric $\pi^* g + h$ on $\mathcal{N}$ and thus, by using a partition of unity, a metric on $\mathbb{R}^n$ which restricts to g on $f_0(M)$ so that f_0 is an isometry. Similarly, we extend Ω_0 by $\pi^*\Omega_0$ to $\mathcal{N}$ and subsequently to $\mathbb{R}^n$. Then $f^*\Omega_0$ is still a positive 3–form on M for any f in a suitably small open neighbourhood $\mathcal{U}$ of $f_0 \in C^\infty(M,\mathbb{R}^n)$, and we can consider P_t as an operator

$$P_t : f \in \mathcal{U} \subset C^\infty(M,\mathbb{R}^n) \mapsto -df\big(X_{f^*\Omega_0}\big(\Omega(t)\big)\big) \in C^\infty(M,\mathbb{R}^n).$$

We start by showing that P_t is a quasilinear, second order differential operator. Let $y^1,\ldots,y^n$ be the standard coordinates on $\mathbb{R}^n$. Further, we fix a chart $U \subset M$ with coordinates $x^1,\ldots,x^7$ and partial derivatives $\partial_1,\ldots,\partial_7$. Denote by $\star^{ijk}_{opqr}$ the components of $\star_{f^*\Omega_0} : \Omega^3(U) \mapsto \Omega^4(U)$ with respect to these coordinates. These depend on the components of $f^*\Omega_0$ given by $\Omega_{0,\alpha\beta\gamma}\partial_l f^\alpha \partial_m f^\beta \partial_n f^\gamma$. Schematically,

$$\star_{f^*\Omega_0}\Omega(t) \overset{\mathrm{loc}}{=} \star^{ijk}_{opqr}\big(\Omega_{0,\alpha\beta\gamma}\partial_l f^\alpha \partial_m f^\beta \partial_n f^\gamma\big)\Omega(t)_{ijk}dx^{opqr}$$

so that by the chain rule

$$d\star_{f^*\Omega_0}\Omega(t) \overset{\mathrm{loc}}{=} \big(a^{ij}_{opqrs,\alpha}(t,x,\nabla f)\partial_i\partial_j f^\alpha + b_{opqrs}(t,x,\nabla f)\big)dx^{opqrs}$$

for smooth coefficients $a^{\cdots}_{\cdots}$ and $b_{\cdots}$. Finally, computing $\llcorner f^*\Omega_0$ after applying once more $\star_{f^*\Omega_0}$ leads to

$$\big(\star_{f^*\Omega_0} d\star_{f^*\Omega_0}\Omega(t)\big)\llcorner f^*\Omega_0 \overset{\mathrm{loc}}{=} (\Omega_0)^{kl}_m\big(\widetilde{a}^{ij}_{kl,\alpha}(t,x,\nabla f)\partial_i\partial_j f^\alpha + \widetilde{b}_{kl}(t,x,\nabla f)\big)dx^m$$

which after dualising and contracting with $df = \partial_i f^\gamma dx^i \otimes \partial_{y^\gamma}$ shows that P_t is a quasilinear, second order differential operator.

Lemma 6.1 *There exists $\epsilon > 0$ and a smooth family of embeddings $f(t) \in C^\infty(M,\mathbb{R}^n)$, $t \in [0,\epsilon]$ such that*

$$\frac{\partial}{\partial t} f(t) = P_t\big(f(t)\big), \quad f(0) = f_0. \tag{33}$$

Furthermore, $f(t)(M) \subset f_0(M)$ for all t.

Proof: First we identify M with its image $f_0(M)$ in $\mathbb{R}^n$ in order to keep notation tight. Shrinking $\mathcal{U}$ if necessary we may assume that for $f \in \mathcal{U}$, $f(x)$ lies in a unique fibre νM_y of the normal bundle. To turn f into a section of νM we fix a connection ∇^h compatible with the fibre metric h on νM whose induced parallel transport along the path γ we denote by $\mathcal{P}^{\nabla^h}_\gamma$. Then

$$\sigma(f)(x) \stackrel{\mathrm{Def}}{=} \mathcal{P}^{\nabla^h}_{\gamma_{\pi(f(x))\to x}} f(x),$$

where $\gamma_{\pi(f(x))\to x}$ is the unique geodesic joining $\pi\big(f(x)\big)$ and x (shrink $\mathcal{U}$ possibly further to guarantuee existence and uniqueness of such a geodesic). In particular, $\sigma(f_0)$ identifies f_0 with the zero section of νM.

Next consider minus the rough Laplacian $-\Delta^h = -\nabla^{h*}\nabla^h : C^\infty(\nu M) \to C^\infty(\nu M)$. This is a strongly elliptic, linear second differential operator. Thus $-\Delta^h \circ \sigma : C^\infty(M, \mathbb{R}^n) \to C^\infty(\nu M) \subset C^\infty(M, \mathbb{R}^n)$ is a quasilinear, second order differential operator, and so is

$$\widetilde{P}_t : f \in \mathcal{U} \subset C^\infty(M, \mathbb{R}^n) \mapsto P_t(f) - \Delta^h\sigma(f) \in C^\infty(M, \mathbb{R}^n).$$

We wish to establish short–time existence and uniqueness of the associated flow equation

$$\frac{\partial}{\partial t} f(t) = \widetilde{P}_t\big(f(t)\big), \quad f(0) = f_0. \tag{34}$$

To compute the linearisation $D_{f_0}\widetilde{P}_0(Y)$ we consider a curve $f_s \subset \mathcal{U}$ through f_0 with

$$Y(x) = \frac{d}{ds} f_s|_{s=0}(x) \in T_x\mathbb{R}^n \cong \mathbb{R}^n.$$

We write $Y^\parallel(x)$ and $Y^\perp(x)$ for the projections of $Y(x)$ to T_xM and νM_x. By design of the extension of Ω_0 over $\mathbb{R}^n$ (cf. our convention above),

$$\varphi_s \stackrel{\mathrm{Def}}{=} \pi \circ f_s \in \mathit{Diff}(M)$$

satisfies $f_s^*\Omega_0 = \varphi_s^*\Omega_0$. Further,

$$\frac{d}{ds}\varphi_s(x)|_{s=0} = d_x\pi\big(Y(x)\big) = Y^\parallel(x),$$

so that

$$\begin{aligned}
\frac{d}{ds}P_0(f_s)|_{s=0} &= -\frac{d}{ds} df_s\big(X_{\varphi_s^*\Omega_0}(\Omega_0)\big)|_{s=0} \\
&= -\frac{d}{ds} df_s\big(\varphi_{s*}^{-1}X_{\Omega_0}(\varphi_{s*}^{-1}\Omega_0)\big)|_{s=0} \\
&= X\big(\mathcal{L}_{Y^\parallel}\Omega_0\big) + \text{ terms of lower order in } Y \\
&= \delta_{\Omega_0}(dY^\parallel \llcorner \Omega_0)\llcorner\Omega_0 + \text{ terms of lower order in } Y.
\end{aligned}$$

For the linearisation of $-\Delta^h \circ \sigma$, we first assume Y to be tangent to M, i.e. $Y(x) \in T_xM \subset T_x\mathbb{R}^n$. Integrating Y yields a family f_s such that $\sigma(f_s)$ is the zero section. On the other hand, for Y normal to M, we take f_s such that the curves $f_s(x)$ are contained in the fibres νM_x. Thus $\pi\big(f_s(x)\big) = x$, whence $\sigma(f_s) = f_s$ and $\Delta^h\sigma(f_s) = \Delta^h f_s$. Consequently, for general $Y = Y^\parallel + Y^\perp$ we find

$$D_{f_0}\Delta^h \circ \sigma(Y) = \Delta^h Y^\perp,$$

for Δ^h is linear. Finally, it follows that the symbol of the linearised operator $D_{f_0}\widetilde{P}_0$ is

$$\sigma(D_{f_0}\widetilde{P}_0)(x,\xi)Y = -\big(\xi\llcorner\big(\xi \wedge \big(Y^\parallel(x)\llcorner\Omega_0(x)\big)\big)\big)\llcorner\Omega_0(x) + |\xi|^2_{\Omega_0}Y^\perp(x).$$

To check strong ellipticity we assume $|\xi|_{\Omega_0} = 1$ and write $Y^{\|}(x) = a\xi + Y_0$ for $a \in \mathbb{R}$, $Y_0 \in \xi^{\perp}$ and $\Omega_0(x) = \omega \wedge \xi + \psi_+$. Then $Y^{\|} \llcorner \Omega_0(x) = a\omega + Y_0 \llcorner \psi_+ + (Y_0 \llcorner \omega) \wedge \xi$. To keep notation tight we omit x, so that

$$\begin{aligned}
g_{\Omega_0}\big(\sigma(D_{f_0}\widetilde{P}_0)(x,\xi)Y, Y\big) &= -g_{\Omega_0}\big(\big(\xi \llcorner \big(\xi \wedge \big(Y^{\|} \llcorner \Omega_0\big)\big)\big) \llcorner \Omega_0, Y^{\|}\big) + |Y^{\perp}|^2_{\Omega_0} \\
&= g_{\Omega_0}\big(\Omega_0, \big(\xi \llcorner \big(\xi \wedge (Y^{\|} \llcorner \Omega_0)\big)\big) \wedge Y^{\|}\big) + |Y^{\perp}|^2_{\Omega_0} \\
&= g_{\Omega_0}\big(Y^{\|} \llcorner \Omega_0, \xi \llcorner \big(\xi \wedge (Y^{\|} \llcorner \Omega_0)\big)\big) + |Y^{\perp}|^2_{\Omega_0} \\
&= g_{\Omega_0}\big(a\omega + Y_0 \llcorner \psi_+ + (Y_0 \llcorner \omega) \wedge \xi, a\omega + Y_0 \llcorner \psi_+\big) + |Y^{\perp}|^2_{\Omega_0} \\
&= 3|a|^2 + |Y_0 \llcorner \psi_+|^2 + |Y^{\perp}|^2_{\Omega_0} \\
&\overset{(11)}{=} 3|a|^2 + 2|Y_0|^2 + |Y^{\perp}|^2_{\Omega_0} \\
&\geq |Y|^2_{\Omega_0}.
\end{aligned}$$

Theorem 5.3 applies once again to yield short–time existence and uniqueness of (34).

Last we show that a solution $f(t)$ to (34) satisfies $f(t)(M) \subset M$. Since in this case, $\sigma\big(f(t)\big)$ is just the zero section of νM so that $\Delta^h \sigma\big(f(t)\big) = 0$, we also obtain the desired solution to (33). First we remark that $r \in \mathrm{End}\,\nu M$ which is multiplication by -1 along the fibres commutes with $\widetilde{P}_t$ in the sense that $\widetilde{P}_t \circ r = dr \circ \widetilde{P}_t$. This is clear for P_t as $(r \circ f)^*\Omega_0 = f^*\Omega_0$. Furthermore, r commutes with $\Delta^h \circ \sigma$ by definition of σ and the linearity of Δ^h. Since the action of r and dr coincide on the fibres of νM, we can replace r by dr. Now if $f(t)(M)$ were not contained in M, $r \circ f$ would yield a second solution, contradicting uniqueness. ■

In particular, for a given Dirichlet energy flow $\Omega(t)$, a solution $f(t) \in \mathcal{U}$ to (31) yields a solution $\varphi(t) = f_0^{-1} \circ f(t) \in \mathit{Diff}(M)$ to (29). From there, uniqueness easily follows:

Corollary 6.2 *Suppose that* $\Omega(t)$ *and* $\Omega'(t)$ *are two solutions to* (15) *for* $t \in [0, \epsilon]$, $\epsilon > 0$. *If* $\Omega(0) = \Omega_0 = \Omega'(0)$, *then* $\Omega(t) = \Omega'(t)$ *for all* $t \in [0, \epsilon]$.

Proof: Solving for (29) with $P_{\Omega(t),\Omega_0}$, $P_{\Omega'(t),\Omega_0}$ gives two flows φ_t and φ'_t so that $\widetilde{\Omega}(t) = \varphi_t^*\Omega(t)$ and $\widetilde{\Omega}'(t) = \varphi_t'^*\Omega'(t)$ define a solution to (24) with initial value Ω_0. By uniqueness of the modified flow we find $\widetilde{\Omega}(t) = \widetilde{\Omega}'(t)$. But then (32) implies both φ_t and φ'_t to be solutions of the ordinary differential equation

$$\frac{\partial}{\partial t}\psi_t = P_{\widetilde{\Omega}(t),\Omega_0}(\mathrm{Id}_M) \circ \psi_t. \tag{35}$$

By uniqueness of the solution to (35), we conclude $\varphi_t = \varphi'_t$, whence $\Omega(t) = \Omega'(t)$. ■

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H. Weiss, F. Witt: Mathematisches Institut der Universität München, Theresienstr. 39, D–80333 München, F.R.G.
e-mail: `weiss,witt@math.lmu.de`