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On the equivariant Tamagawa number conjecture in tame CM-extensions, II

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Abstract

We use the notion of non-commutative Fitting invariants to give a reformulation of the equivariant Iwasawa main conjecture (EIMC) attached to an extension F/K of totally real fields with Galois group $\mathcal{G}$, where K is a global number field and $\mathcal{G}$ is a p-adic Lie group of dimension 1 for an odd prime p. We attach to each finite Galois CM-extension L/K with Galois group G a module $SKu(L/K)$ over the center of the group ring $\mathbb{Z}G$ which coincides with the Sinnott-Kurihara ideal if G is abelian. We state a conjecture on the integrality of $SKu(L/K)$ which follows from the equivariant Tamagawa number conjecture (ETNC) in many cases, and is a theorem for abelian G. Assuming the validity of the EIMC and the vanishing of the Iwasawa μ-invariant, we compute Fitting invariants of certain Iwasawa modules, and we show that this implies the minus part of the ETNC at p for an infinite class of (non-abelian) Galois CM-extensions of number fields which are at most tamely ramified above p, provided that (an appropriate p-part of) the integrality conjecture holds.

Introduction

Let L/K be a finite Galois extension of number fields with Galois group G. D. Burns [Bu01] used complexes arising from étale cohomology of the constant sheaf $\mathbb{Z}$ to define a canonical element $T\Omega(L/K)$ of the relative K-group $K_0(\mathbb{Z}G, \mathbb{R})$. This element relates the leading terms at zero of Artin L-functions attached to L/K to natural arithmetic invariants. It was shown that the vanishing of $T\Omega(L/K)$ is equivalent to the ETNC for the pair $(h^0(\mathrm{Spec}(L))(0), \mathbb{Z}G)$ (cf. loc.cit., Th. 2.4.1).
The ETNC is known to be true if L is absolutely abelian as proved by D. Burns and C. Greither [BG03] with the exclusion of the 2-primary part; M. Flach [Fl02] extended the argument to cover the 2-primary part as well. If L is in addition totally real, the ETNC was independently proved in [RW02, RW03]. Some relatively abelian results are due to W. Bley [Bl06]; he showed that if L/K is a finite abelian extension, where K is an imaginary quadratic field which has class number one, then the ETNC holds for all intermediate extensions L/E such that $[L : E]$ is odd and divisible only by primes which split completely in $K/\mathbb{Q}$. Finally, if L/K is a CM-extension and p is odd, the ETNC at p naturally decomposes into a plus and a minus part; it was shown by the author [Nia] that the minus part of the ETNC at p holds if L/K is abelian and at most tamely ramified above p, and the Iwasawa μ-invariant vanishes if p divides $|G|$ (and some additional technical condition is fulfilled). Note that the vanishing of μ is a long standing conjecture of Iwasawa theory; the most general result is still due to B. Ferrero and L. Washington [FW79] and says that $\mu = 0$ for absolutely abelian extensions.
These results make heavily use of the validity of the EIMC attached to the extension L_∞^+/K, where L_∞^+ is the cyclotomic $\mathbb{Z}_p$-extension of L^+ which is the maximal real subfield of L. Note that the

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EIMC is known for abelian extensions of totally real number fields with Galois group $\mathcal{G}$ such that $\mathcal{G}$ is a p-adic Lie group of dimension 1 (cf. [Wi90a, RW02]).
In the abelian case, there is a natural formulation of the EIMC in terms of Fitting ideals. The theory of Fitting ideals also plays an important role within the descent methods used in [BG03, Bl06, Wi90b, Gr00, Ku03, Nia]. For not necessarily abelian $\mathcal{G}$, we will introduce a reformulation of the EIMC in terms of non-commutative Fitting invariants which have been introduced by the author [Nib]. This is the main purpose of section 2; we give some algebraic preparations on Iwasawa modules and their Fitting invariants in section 3.
Now let L/K be a Galois CM-extension with Galois group G. Assuming the validity of the EIMC attached to the extension L_∞^+/K and the vanishing of μ, we compute Fitting invariants of some natural Iwasawa modules in section 4; this generalizes results of C. Greither [Gr04]. In section 5, we introduce a module $SKu(L/K)$ over the center of the group $\mathbb{Z}G$ which is a non-commutative analogue of the Sinnot-Kurihara ideal (cf. [Si80], p. 193) and was already implicitly used in [Nic] and [BJ]. We formulate an integrality conjecture on $SKu(L/K)$ which is implied by the ETNC in many cases and follows from the results in [Ba77], [Ca79], [DR80] if G is abelian. Assuming the validity of this integrality conjecture, we generalize a descent method due to A. Wiles [Wi90b] in the equivariant version of C. Greither [Gr00] to the non-abelian situation; this shows that the EIMC implies the minus part of the ETNC at p provided that μ vanishes, the integrality conjecture holds and the ramification above p is at most tame (and, as in the abelian case, some technical extra assumption holds). For a special class of extensions, where no "trivial zeros" occur, the EIMC in fact implies the relevant part of the integrality conjecture. This generalizes [Nia], Th. 4 to the non-abelian situation. Moreover, it follows from the results in [Nic] that for the case at hand the EIMC implies the non-abelian analogues of Brumer's conjecture, of the Brumer-Stark conjecture and of the strong Brumer-Stark property as formulated in loc.cit., provided that $\mu = 0$ and the integrality conjecture holds.

1. Preliminaries

1.0.1 *K-theory* Let Λ be a left noetherian ring with 1 and $\mathrm{PMod}(\Lambda)$ the category of all finitely generated projective Λ-modules. We write $K_0(\Lambda)$ for the Grothendieck group of $\mathrm{PMod}(\Lambda)$, and $K_1(\Lambda)$ for the Whitehead group of Λ which is the abelianized infinite general linear group. If S is a multiplicatively closed subset of the center of Λ which contains no zero divisors, $1 \in S$, $0 \notin S$, we denote the Grothendieck group of the category of all finitely generated S-torsion Λ-modules of finite projective dimension by $K_0S(\Lambda)$. Writing Λ_S for the ring of quotients of Λ with denominators in S, we have the following Localization Sequence (cf. [CR87], p. 65)

$$K_1(\Lambda) \to K_1(\Lambda_S) \xrightarrow{\partial} K_0S(\Lambda) \xrightarrow{\rho} K_0(\Lambda) \to K_0(\Lambda_S). \tag{1}$$

In the special case where Λ is an $\mathfrak{o}$-order over a commutative ring $\mathfrak{o}$ and S is the set of all nonzero-divisors of $\mathfrak{o}$, we also write $K_0T(\Lambda)$ instead of $K_0S(\Lambda)$. Moreover, we denote the relative K-group corresponding to a ring homomorphism $\Lambda \to \Lambda'$ by $K_0(\Lambda, \Lambda')$ (cf. [Sw68]). Then we have a Localization Sequence (cf. [CR87], p. 72)

$$K_1(\Lambda) \to K_1(\Lambda') \xrightarrow{\partial_{\Lambda,\Lambda'}} K_0(\Lambda, \Lambda') \to K_0(\Lambda) \to K_0(\Lambda').$$

It is also shown in [Sw68] that there is an isomorphism $K_0(\Lambda, \Lambda_S) \simeq K_0S(\Lambda)$. For any ring Λ we write $\zeta(\Lambda)$ for the subring of all elements which are central in Λ. Let L be a subfield of either $\mathbb{C}$ or $\mathbb{C}_p$ for some prime p and let G be a finite group. In the case where Λ' is the group ring LG the reduced norm map $\mathrm{nr}_{LG} : K_1(LG) \to \zeta(LG)^\times$ is always injective. If in addition $L = \mathbb{R}$, there exists a canonical map $\hat{\partial}_G : \zeta(\mathbb{R}G)^\times \to K_0(\mathbb{Z}G, \mathbb{R}G)$ such that the restriction of $\hat{\partial}_G$ to the image of the reduced norm equals $\partial_{\mathbb{Z}G,\mathbb{R}G} \circ \mathrm{nr}_{\mathbb{R}G}^{-1}$. This map is called the extended boundary homomorphism and was introduced by Burns and Flach [BF01].

1.0.2 *Non-commutative Fitting invariants* For the following we refer the reader to [Nib]. We denote the set of all $m \times n$ matrices with entries in a ring R by $M_{m\times n}(R)$ and in the case $m = n$ the group of all invertible elements of $M_{n\times n}(R)$ by $\mathrm{Gl}_n(R)$. Let A be a separable K-algebra and Λ be an $\mathfrak{o}$-order in A, finitely generated as $\mathfrak{o}$-module, where $\mathfrak{o}$ is a complete commutative noetherian local ring with field of quotients K. Moreover, we will assume that the integral closure of $\mathfrak{o}$ in K is finitely generated as $\mathfrak{o}$-module. The group ring $\mathbb{Z}_pG$ of a finite group G will serve as a standard example. Let N and M be two $\zeta(\Lambda)$-submodules of an $\mathfrak{o}$-torsionfree $\zeta(\Lambda)$-module. Then N and M are called $\mathrm{nr}(\Lambda)$-*equivalent* if there exists an integer n and a matrix $U \in \mathrm{Gl}_n(\Lambda)$ such that $N = \mathrm{nr}(U) \cdot M$, where $\mathrm{nr} : A \to \zeta(A)$ denotes the reduced norm map which extends to matrix rings over A in the obvious way. We denote the corresponding equivalence class by $[N]_{\mathrm{nr}(\Lambda)}$. We say that N is $\mathrm{nr}(\Lambda)$-contained in M (and write $[N]_{\mathrm{nr}(\Lambda)} \subset [M]_{\mathrm{nr}(\Lambda)}$) if for all $N' \in [N]_{\mathrm{nr}(\Lambda)}$ there exists $M' \in [M]_{\mathrm{nr}(\Lambda)}$ such that $N' \subset M'$. Note that it suffices to check this property for one $N_0 \in [N]_{\mathrm{nr}(\Lambda)}$. We will say that x is contained in $[N]_{\mathrm{nr}(\Lambda)}$ (and write $x \in [N]_{\mathrm{nr}(\Lambda)}$) if there is $N_0 \in [N]_{\mathrm{nr}(\Lambda)}$ such that $x \in N_0$.

Now let M be a finitely presented (left) Λ-module and let

$$\Lambda^a \xrightarrow{h} \Lambda^b \twoheadrightarrow M \tag{2}$$

be a finite presentation of M. We identify the homomorphism h with the corresponding matrix in $M_{a\times b}(\Lambda)$ and define $S(h) = S_b(h)$ to be the set of all $b \times b$ submatrices of h if $a \geqslant b$. In the case $a = b$ we call (2) a quadratic presentation. The Fitting invariant of h over Λ is defined to be

$$\mathrm{Fitt}_\Lambda(h) = \begin{cases} [0]_{\mathrm{nr}(\Lambda)} & \text{if} \quad a < b \\ \left[\langle \mathrm{nr}(H) | H \in S(h)\rangle_{\zeta(\Lambda)}\right]_{\mathrm{nr}(\Lambda)} & \text{if} \quad a \geqslant b. \end{cases}$$

We call $\mathrm{Fitt}_\Lambda(h)$ a Fitting invariant of M over Λ. One defines $\mathrm{Fitt}_\Lambda^{\max}(M)$ to be the unique Fitting invariant of M over Λ which is maximal among all Fitting invariants of M with respect to the partial order "$\subset$". If M admits a quadratic presentation h, one also puts $\mathrm{Fitt}_\Lambda(M) := \mathrm{Fitt}_\Lambda(h)$ which is independent of the chosen quadratic presentation.
Now let C and C' be two finitely generated $\mathfrak{o}$-torsion Λ-modules of finite projective dimension and denote by $[C]$ and $[C']$ the corresponding classes in $K_0T(\Lambda)$, respectively. If $\rho([C] - [C']) = 0$, we choose $x \in K_1(A)$ such that $\partial(x) = [C] - [C']$ and define (cf. [Nib], Def. 3.6)

$$\mathrm{Fitt}_\Lambda(C : C') := \left[\langle \mathrm{nr}_A(x)\rangle_{\zeta(\Lambda)}\right]_{\mathrm{nr}(\Lambda)}.$$

1.0.3 *Equivariant L-values* Let us fix a finite Galois extension L/K of number fields with Galois group G. For any prime $\mathfrak{p}$ of K we fix a prime $\mathfrak{P}$ of L above $\mathfrak{p}$ and write $G_\mathfrak{P}$ resp. $I_\mathfrak{P}$ for the decomposition group resp. inertia subgroup of L/K at $\mathfrak{P}$. Moreover, we denote the residual group at $\mathfrak{P}$ by $\overline{G_\mathfrak{P}} = G_\mathfrak{P}/I_\mathfrak{P}$ and choose a lift $\phi_\mathfrak{P} \in G_\mathfrak{P}$ of the Frobenius automorphism at $\mathfrak{P}$.
If S is a finite set of places of K containing the set S_∞ of all infinite places of K, and χ is a (complex) character of G, we denote the S-truncated Artin L-function attached to χ and S by $L_S(s,\chi)$ and define $L_S^*(0,\chi)$ to be the leading coefficient of the Taylor expansion of $L_S(s,\chi)$ at $s = 0$. Recall that there is a canonical isomorphism $\zeta(\mathbb{C}G) = \prod_{\chi\in\mathrm{Irr}\,(G)} \mathbb{C}$, where $\mathrm{Irr}\,(G)$ denotes the set of irreducible characters of G. We define the equivariant Artin L-function to be the meromorphic $\zeta(\mathbb{C}G)$-valued function

$$L_S(s) := (L_S(s,\chi))_{\chi\in\mathrm{Irr}\,(G)}.$$

We put $L_S^*(0) = (L_S^*(0,\chi))_{\chi\in\mathrm{Irr}(G)}$ and abbreviate $L_{S_\infty}(s)$ by $L(s)$. If T is a second finite set of places of K sucht that $S\cap T=\emptyset$, we define $\delta_T(s) := (\delta_T(s,\chi))_{\chi\in\mathrm{Irr}(G)}$, where $\delta_T(s,\chi) = \prod_{\mathfrak{p}\in T}\det(1-N(\mathfrak{p})^{1-s}\phi_{\mathfrak{P}}^{-1}|V_\chi^{I_{\mathfrak{P}}})$ and V_χ is a G-module with character χ. We put

$$\Theta_{S,T}(s) := \delta_T(s)\cdot L_S(s)^\sharp,$$

where we denote by $^\sharp : \mathbb{C}G\to\mathbb{C}G$ the involution induced by $g\mapsto g^{-1}$. These functions are the so-called (S,T)-modified G-equivariant L-functions and we define Stickelberger elements

$$\theta_S^T := \Theta_{S,T}(0)\in\zeta(\mathbb{C}G).$$

If T is empty, we abbreviate θ_S^T by θ_S. Note that the χ-part of θ_S^T vanishes for a non-trivial character χ if there is an (infinite) prime $\mathfrak{p}\in S$ such that $V_\chi^{G_{\mathfrak{P}}}\neq 0$. Now let L/K be a Galois CM-extension, i.e. L is a CM-field, K is totally real and complex conjugation induces an unique automorphism j of L which lies in the center of G. If R is a subring of either $\mathbb{C}$ or $\mathbb{C}_p$ for a prime p such that 2 is invertible over R, we put $RG_- := RG/(1+j)$ which is a ring, since the idempotent $\frac{1-j}{2}$ lies in RG. For any RG-module M we define $M^- = RG_-\otimes_{RG} M$ which is an exact functor since $2\in R^\times$. Now Stark's conjecture (which is a theorem for odd characters, see [Ta84], Th. 1.2, p. 70) implies

$$\theta_S^T\in\zeta(\mathbb{Q}G_-). \tag{3}$$

Note that we actually have to exclude the special case $|S_\infty(L)|=1$ (cf. the proof of [Nia], Prop. 3, where (3) is shown in the relevant case $S=S_\infty$ and $T=\emptyset$), but in this situation the extension L/K is abelian. Here, we write $S(L)$ for the set of places in L which lie above those in S, and S is any (finite) set of places of K. Let us fix an embedding $\iota:\mathbb{C}\rightarrowtail\mathbb{C}_p$; then the image of θ_S^T in $\zeta(\mathbb{Q}_pG_-)$ via the canonical embedding

$$\zeta(\mathbb{Q}G_-)\rightarrowtail\zeta(\mathbb{Q}_pG_-) = \bigoplus_{\substack{\chi\in\mathrm{Irr}_p(G)/\sim\\ \chi\text{ odd}}}\mathbb{Q}_p(\chi),$$

is given by $\sum_\chi(\delta_T(0,\chi^{\iota^{-1}})\cdot L_S(0,\check{\chi}^{\iota^{-1}}))^\iota$, where we write $\check{\chi}$ for the character contragredient to χ. Here, the sum runs over all $\mathbb{C}_p$-valued irreducible odd characters of G modulo Galois action. Note that we will frequently drop ι and ι^{-1} from the notation.

1.0.4 *Ray class groups* Let T and S be as above. We write cl_L^T for the ray class group of L to the ray $\mathfrak{M}_T := \prod_{\mathfrak{P}\in T(L)}\mathfrak{P}$ and $\mathfrak{o}_S$ for the ring of $S(L)$-integers of L. Let S_f be the set of all finite primes in $S(L)$; then there is a natural map $\mathbb{Z}S_f\to\mathrm{cl}_L^T$ which sends each prime $\mathfrak{P}\in S_f$ to the corresponding class $[\mathfrak{P}]\in\mathrm{cl}_L^T$. We denote the cokernel of this map by $\mathrm{cl}_S^T(L) =: \mathrm{cl}_S^T$. Further, we denote the $S(L)$-units of L by E_S and define $E_S^T := \{x\in E_S : x\equiv 1 \bmod \mathfrak{M}_T\}$. All these modules are equipped with a natural G-action and we have the following exact sequences of G-modules

$$E_{S_\infty}^T\rightarrowtail E_S^T\xrightarrow{v}\mathbb{Z}S_f\to\mathrm{cl}_L^T\twoheadrightarrow\mathrm{cl}_S^T, \tag{4}$$

where $v(x)=\sum_{\mathfrak{P}\in S_f}v_{\mathfrak{P}}(x)\mathfrak{P}$ for $x\in E_S^T$, and

$$E_S^T\rightarrowtail E_S\to(\mathfrak{o}_S/\mathfrak{M}_T)^\times\xrightarrow{\nu}\mathrm{cl}_S^T\twoheadrightarrow\mathrm{cl}_S, \tag{5}$$

where the map ν lifts an element $\overline{x}\in(\mathfrak{o}_S/\mathfrak{M}_T)^\times$ to $x\in\mathfrak{o}_S$ and sends it to the ideal class $[(x)]\in\mathrm{cl}_S^T$ of the principal ideal (x). Note that the G-module $(\mathfrak{o}_S/\mathfrak{M}_T)^\times$ is c.t. (short for cohomologically trivial) if no prime in T ramifies in L/K. If L/K is a CM-extension, we define

$$A_S^T := (\mathbb{Z}[\tfrac{1}{2}]\otimes_{\mathbb{Z}}\mathrm{cl}_S^T)^-.$$

If $S=S_\infty$, we also write A_L^T and E_L^T instead of $A_{S_\infty}^T$ and $E_{S_\infty}^T$. Finally, we suppress the superscript T from the notation if T is empty. If M is a finitely generated $\mathbb{Z}$-module and p is a prime, we put

$M(p) := \mathbb{Z}_p \otimes_{\mathbb{Z}} M$. In particular, $A_L(p)$ is the p-part of the minus class group if p is odd.

2. A reformulation of the equivariant Iwasawa main conjecture

Let $p \neq 2$ be a prime and let F/K be a Galois extension of totally real fields with Galois group $\mathcal{G}$, where K is a global number field, F contains the cyclotomic $\mathbb{Z}_p$-extension K_∞ of K and $[F : K_\infty]$ is finite. Hence $\mathcal{G}$ is a p-adic Lie group of dimension 1 and there is a finite normal subgroup H of $\mathcal{G}$ such that $\mathcal{G}/H = \mathrm{Gal}(K_\infty/K) =: \Gamma_K$. Here, Γ_K is isomorphic to the p-adic integers $\mathbb{Z}_p$ and we fix a topological generator γ_K. We denote the completed group algebra $\mathbb{Z}_p[[\mathcal{G}]]$ by $\Lambda(\mathcal{G})$ and the total ring of fractions of $\Lambda(\mathcal{G})$ by $Q(\mathcal{G})$. If we pick a preimage γ of γ_K in $\mathcal{G}$, we can choose an integer m such that γ^{p^m} lies in the center of $\mathcal{G}$. Hence the ring $R := \mathbb{Z}_p[[\Gamma^{p^m}]]$ belongs to the center of $\Lambda(\mathcal{G})$, and $\Lambda(\mathcal{G})$ is an R-order in the separable $Quot(R)$-algebra $Q(\mathcal{G})$. Note that R is isomorphic to the power series ring $\mathbb{Z}_p[[T]]$. Let S be a finite set of places of K containing all the infinite places S_∞ and the set S_p of all places of K above p. Moreover, let M_S be the maximal abelian pro-p-extension of F unramified outside S, and denote the Iwasawa module $\mathrm{Gal}(M_S/F)$ by X_S. There is a canonical complex

$$C^{\cdot}(F/K) : \ldots \to 0 \to C^{-1} \to C^0 \to 0 \to \ldots \tag{6}$$

of R-torsion $\Lambda(\mathcal{G})$-modules of projective dimension at most 1 such that $H^{-1}(C^{\cdot}(F/K)) = X_S$ and $H^0(C^{\cdot}(F/K)) = \mathbb{Z}_p$. We put (cf. [RW04], §4)

$$\mho_S = \mho_S(F/K) := (C^{-1}) - (C^0) \in K_0T(\Lambda(\mathcal{G})).$$

Since $\rho(\mho_S) = 0$, there is a well defined Fitting invariant of $\mho_S$; more precisely,

$$\mathrm{Fitt}_{\Lambda(\mathcal{G})}(\mho_S) := \mathrm{Fitt}_{\Lambda(\mathcal{G})}(C^{-1} : C^0).$$

Moreover, if $\mathcal{F}$ is an exact functor from the category of R-torsion $\Lambda(\mathcal{G})$-modules of projective dimension at most 1 to itself, we also set

$$\mathrm{Fitt}_{\Lambda(\mathcal{G})}(\mathcal{F}(\mho_S)) := \begin{cases} \mathrm{Fitt}_{\Lambda(\mathcal{G})}(\mathcal{F}(C^{-1}) : \mathcal{F}(C^0)) & \text{if } \mathcal{F} \text{ is covariant} \\ \mathrm{Fitt}_{\Lambda(\mathcal{G})}(\mathcal{F}(C^0) : \mathcal{F}(C^{-1})) & \text{if } \mathcal{F} \text{ is contravariant.} \end{cases}$$

We recall some results concerning the algebra $Q(\mathcal{G})$ due to Ritter and Weiss [RW04]. Let $\mathbb{Q}_p^{\mathrm{c}}$ be an algebraic closure of $\mathbb{Q}_p$ and fix an irreducible ($\mathbb{Q}_p^{\mathrm{c}}$-valued) character χ of $\mathcal{G}$ with open kernel. Choose a finite field extension E of $\mathbb{Q}_p$ such that the character χ has a realization V_χ over E. Let η be an irreducible constituent of $\mathrm{res}^{\mathcal{G}}_H \chi$ and set

$$St(\eta) := \{g \in \mathcal{G} : \eta^g = \eta\},\ e_\eta = \frac{\eta(1)}{|H|} \sum_{h \in H} \eta(h^{-1})h,\ e_\chi = \sum_{\eta | \mathrm{res}^{\mathcal{G}}_H \chi} e_\eta.$$

For any finite field extension k of $\mathbb{Q}_p$ with ring of integers $\mathfrak{o}$, we set $Q^k(\mathcal{G}) := k \otimes_{\mathbb{Q}_p} Q(\mathcal{G})$ and $\Lambda^{\mathfrak{o}}(\mathcal{G}) = \mathfrak{o}[[\mathcal{G}]]$. By [RW04], corollary to Prop. 6, e_χ is a primitive central idempotent of $Q^E(\mathcal{G})$. By loc.cit., Prop. 5 there is a distinguished element $\gamma_\chi \in \zeta(Q^E(\mathcal{G})e_\chi)$ which generates a procyclic p-subgroup Γ_χ of $(Q^E(\mathcal{G})e_\chi)^\times$ and acts trivially on V_χ. Moreover, γ_χ induces an isomorphism $Q^E(\Gamma_\chi) \xrightarrow{\simeq} \zeta(Q^E(\mathcal{G})e_\chi)$ by loc.cit., Prop. 6. For $r \in \mathbb{N}_0$, we define the following maps

$$j_\chi^r : \zeta(Q^E(\mathcal{G})) \twoheadrightarrow \zeta(Q^E(\mathcal{G})e_\chi) \simeq Q^E(\Gamma_\chi) \to Q^E(\Gamma_K),$$

where the last arrow is induced by mapping γ_χ to $\kappa^r(\gamma_\chi)\gamma_K^{w_\chi}$, where $w_\chi = [\mathcal{G} : St(\eta)]$ and κ denotes the cyclotomic character of $\mathcal{G}$. Note that $j_\chi := j_\chi^0$ agrees with the corresponding map j_χ in loc.cit. It is shown that for any matrix $\Theta \in M_{n\times n}(Q(\mathcal{G}))$ we have

$$j_\chi(\mathrm{nr}(\Theta)) = \det{}_{Q^E(\Gamma_K)}(\Theta | \mathrm{Hom}_{EH}(V_\chi, Q^E(\mathcal{G})^n)). \tag{7}$$

Here, Θ acts on $f \in \mathrm{Hom}_{EH}(V_\chi, Q^E(\mathcal{G})^n)$ via right multiplication, and γ_K acts on the left via $(\gamma_K f)(v) = \gamma_K \cdot f(\gamma_K^{-1} v)$ for all $v \in V_\chi$. Hence the map

$$\begin{aligned} \mathrm{Det}\,(\ )(\chi) : K_1(Q(\mathcal{G})) &\to Q^E(\Gamma_K)^\times \\ [P, \alpha] &\mapsto \det{}_{Q^E(\Gamma_K)}(\alpha | \mathrm{Hom}_{EH}(V_\chi, E \otimes_{\mathbb{Q}_p} P)), \end{aligned}$$

where P is a projective $Q(\mathcal{G})$-module and α a $Q(\mathcal{G})$-automorphism of P, is just $j_\chi \circ \mathrm{nr}$. If ρ is a character of $\mathcal{G}$ of type W, i.e. $\mathrm{res}^{\mathcal{G}}_H \rho = 1$, then we denote by $\rho^\sharp$ the automorphism of the field $Q^c(\Gamma_K) := \mathbb{Q}_p^c \otimes_{\mathbb{Q}_p} Q(\Gamma_K)$ induced by $\rho^\sharp(\gamma_K) = \rho(\gamma_K)\gamma_K$. Moreover, we denote the additive group generated by all $\mathbb{Q}_p^c$-valued characters of $\mathcal{G}$ with open kernel by $R_p(\mathcal{G})$; finally, $\mathrm{Hom}^*(R_p(\mathcal{G}), Q^c(\Gamma_K)^\times)$ is the group of all homomorphisms $f : R_p(\mathcal{G}) \to Q^c(\Gamma_K)^\times$ satisfying

$$\begin{array}{ll} f(\chi \otimes \rho) = \rho^\sharp(f(\chi)) & \text{for all characters } \rho \text{ of type } W \text{ and} \\ f(\chi^\sigma) = f(\chi)^\sigma & \text{for all Galois automorphisms } \sigma \in \mathrm{Gal}(\mathbb{Q}_p^c/\mathbb{Q}_p). \end{array}$$

We have an isomorphism

$$\begin{aligned} \zeta(Q(\mathcal{G}))^\times &\simeq \mathrm{Hom}^*(R_p(\mathcal{G}), Q^c(\Gamma_K)^\times) \\ x &\mapsto [\chi \mapsto j_\chi(x)]. \end{aligned}$$

By loc.cit., Th. 5 the map $\Theta \mapsto [\chi \mapsto \mathrm{Det}\,(\Theta)(\chi)]$ defines a homomorphism

$$\mathrm{Det}\, : K_1(Q(\mathcal{G})) \to \mathrm{Hom}^*(R_p(\mathcal{G}), Q^c(\Gamma_K)^\times)$$

such that we obtain a commutative triangle

$$\begin{array}{ccc} & K_1(Q(\mathcal{G})) & \\ \swarrow^{\mathrm{nr}} & & \searrow^{\mathrm{Det}} \\ \zeta(Q(\mathcal{G}))^\times & \xrightarrow{\sim} & \mathrm{Hom}^*(R_p(\mathcal{G}), Q^c(\Gamma_K)^\times). \end{array} \tag{8}$$

We put $u := \kappa(\gamma_K)$. Each topological generator γ_K of Γ_K permits the definition of a power series $G_{\chi,S}(T) \in \mathbb{Q}_p^c \otimes_{\mathbb{Q}_p} Quot(\mathbb{Z}_p[[T]])$ by starting out from the Deligne-Ribet power series for abelian characters of open subgroups of $\mathcal{G}$ (cf. [DR80]). One then has an equality

$$L_{p,S}(1-s, \chi) = \frac{G_{\chi,S}(u^s - 1)}{H_\chi(u^s - 1)},$$

where $L_{p,S}(s, \chi)$ denotes the p-adic Artin L-function, and where, for irreducible χ, one has

$$H_\chi(T) = \begin{cases} \chi(\gamma_K)(1+T) - 1 & \text{if } H \subset \ker(\chi) \\ 1 & \text{otherwise.} \end{cases}$$

Now [RW04], Prop. 11 implies that

$$L_{K,S} : \chi \mapsto \frac{G_{\chi,S}(\gamma_K - 1)}{H_\chi(\gamma_K - 1)}$$

is independent of the topological generator γ_K and lies in $\mathrm{Hom}^*(R_p(\mathcal{G}), Q^c(\Gamma_K)^\times)$. Diagram (8) implies that there is a unique element $\Phi_S \in \zeta(Q(\mathcal{G}))^\times$ such that

$$j_\chi(\Phi_S) = L_{K,S}(\chi).$$

The EIMC as formulated in [RW04] now states that there is a unique $\Theta_S \in K_1(Q(\mathcal{G}))$ such that $\mathrm{Det}\,(\Theta_S) = L_{K,S}$ and $\partial(\Theta_S) = \mho_S$. By means of diagram (8), the following conjecture is equivalent to the EIMC without the uniqueness of Θ_S.

CONJECTURE 2.1. The element $\Phi_S \in \zeta(Q(\mathcal{G}))^\times$ is a generator of $\mathrm{Fitt}_{\Lambda(\mathcal{G})}(\mho_S)$.

We also discuss Conjecture 2.1 within the framework of the theory of [CFKSV05], §3. For this, let

$$\pi : \mathcal{G} \to \mathrm{Gl}_n(\mathfrak{o}_E)$$

be a continuous homomorphism, where $\mathfrak{o}_E$ denotes the ring of integers of E and n is some integer greater or equal to 1. There is a ring homomorphism

$$\Phi_\pi : \Lambda(\mathcal{G}) \to M_{n\times n}(\Lambda^{\mathfrak{o}_E}(\Gamma_K)) \tag{9}$$

induced by the continuous group homomorphism

$$\begin{aligned} &\mathcal{G} \to (M_{n\times n}(\mathfrak{o}_E) \otimes_{\mathbb{Z}_p} \Lambda(\Gamma_K))^\times = \mathrm{Gl}_n(\Lambda^{\mathfrak{o}_E}(\Gamma_K)) \\ &\sigma \mapsto \pi(\sigma) \otimes \overline{\sigma}, \end{aligned}$$

where $\overline{\sigma}$ denotes the image of σ in $\mathcal{G}/H = \Gamma_K$. By loc.cit., Lemma 3.3 the homomorphism (9) extends to a ring homomorphism

$$\Phi_\pi : Q(\mathcal{G}) \to M_{n\times n}(Q^E(\Gamma_K))$$

and this in turn induces a homomorphism

$$\Phi'_\pi : K_1(Q(\mathcal{G})) \to K_1(M_{n\times n}(Q^E(\Gamma_K))) = Q^E(\Gamma_K)^\times.$$

Let $\mathrm{aug} : \Lambda^{\mathfrak{o}_E}(\Gamma_K) \twoheadrightarrow \mathfrak{o}_E$ be the augmentation map and put $\mathfrak{p} = \ker(\mathrm{aug})$. Writing $\Lambda^{\mathfrak{o}_E}(\Gamma_K)_\mathfrak{p}$ for the localization of $\Lambda^{\mathfrak{o}_E}(\Gamma_K)$ at $\mathfrak{p}$, it is clear that aug naturally extends to a homomorphism $\mathrm{aug} : \Lambda^{\mathfrak{o}_E}(\Gamma_K)_\mathfrak{p} \to E$. One defines an evaluation map

$$\begin{aligned} \phi : Q^E(\Gamma_K) &\to E \cup \{\infty\} \\ x &\mapsto \begin{cases} \mathrm{aug}\,(x) & \text{if } x \in \Lambda^{\mathfrak{o}_E}(\Gamma_K)_\mathfrak{p} \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

If Θ is an element of $K_1(Q(\mathcal{G}))$, we define $\Theta(\pi)$ to be $\phi(\Phi'_\pi(\Theta))$. We need the following lemma.

Lemma 2.2. *If $\pi = \pi_\chi$ is a representation of $\mathcal{G}$ with character χ and $r \in \mathbb{N}_0$, then*

$$\begin{array}{ccc} K_1(Q(\mathcal{G})) & \xrightarrow{\Phi'_{\pi_\chi \kappa^r}} & K_1(M_{n\times n}(Q^E(\Gamma_K))) \\ \downarrow \mathrm{nr} & & \simeq \downarrow \mathrm{nr} \\ \zeta(Q(\mathcal{G}))^\times & \xrightarrow{j_\chi^r} & Q^E(\Gamma_K)^\times \end{array}$$

commutes. In particular, we have $\mathrm{nr} \circ \Phi'_{\pi_\chi} = \mathrm{Det}(\ )(\chi)$.

Proof. We recall that the map j_χ induces a field extension $Q^E(\Gamma_K)/Q^E(\Gamma_\chi)$, where $Q^E(\Gamma_\chi) = \zeta(Q^E(\mathcal{G})e_\chi)$. The results in [RW04] imply that in fact $Q^E(\Gamma_K)$ is a splitting field of $Q^E(\mathcal{G})e_\chi$ and we thus have an isomorphism

$$Q^E(\Gamma_K) \otimes_{Q^E(\Gamma_\chi)} Q^E(\mathcal{G})e_\chi \simeq M_{n\times n}(Q^E(\Gamma_K)). \tag{10}$$

Since $1 \otimes \gamma_\chi = \gamma_K^{w_\chi} \otimes 1$ in $Q^E(\Gamma_K) \otimes_{Q^E(\Gamma_\chi)} Q^E(\mathcal{G})e_\chi$ and $\pi_\chi(\gamma_\chi) \otimes \overline{\gamma}_\chi = 1 \otimes \gamma_K^{w_\chi}$ in $M_{n\times n}(Q^E(\Gamma_K))$, the homomorphism Φ_{π_χ} induces a realization of the above isomorphism (10). Hence $\mathrm{nr} \circ \Phi'_{\pi_\chi}$ is just the reduced norm on $Q^E(\mathcal{G})e_\chi$ which takes values in $Q^E(\Gamma_\chi) \overset{j_\chi}{\rightarrowtail} Q^E(\Gamma_K)$. This shows the lemma in the case $r = 0$. For arbitrary r, we similarly have $j_\chi^r(\mathrm{nr}(\Theta)) = \det_{Q^E(\Gamma_K)}(\Theta|\mathcal{V}_\chi(r)) = \mathrm{nr}(\Phi'_{\pi_\chi\kappa^r}(\Theta))$, where $\Theta \in K_1(Q(\mathcal{G}))$ and $\mathcal{V}_\chi(r)$ is the r-th Tate twist of the absolutely irreducible (right) module $\mathcal{V}_\chi := \mathrm{Hom}_{EH}(V_\chi, Q^E(\mathcal{G}))$ over $Q^E(\Gamma_K) \otimes_{Q^E(\Gamma_\chi)} Q^E(\mathcal{G})$. □

Conjecture 2.1 now implies that there is an element $\Theta_S \in K_1(Q(\mathcal{G}))$ such that $\partial(\Theta_S) = \mho_S$ and for any $r \geqslant 1$ divisible by $p-1$ we have

$$\Theta_S(\pi_\chi \kappa^r) = \phi(j_\chi^r(\Phi_S)) = L_S(1-r,\chi).$$

3. Algebraic preparations

Let $p \neq 2$ be a prime and let $\mathcal{G}$ be a p-adic Lie group of dimension 1, i.e. there is a finite normal subgroup H of $\mathcal{G}$ such that $\Gamma := \mathcal{G}/H$ is isomorphic to $\mathbb{Z}_p$. For any ring Λ and any Λ-module M, we write $\mathrm{pd}_\Lambda(M)$ for the projective dimension of M over Λ. For any finitely generated $\Lambda(\mathcal{G})$-module M, we write $\mu(M)$ for the Iwasawa μ-invariant of M. As before, let $\Gamma' \simeq \mathbb{Z}_p$ be a subgroup of $\mathcal{G}$ which is central in $\mathcal{G}$ and put $R = \mathbb{Z}_p[[\Gamma']]$.

Proposition 3.1. *Let M be a finitely generated R-torsion $\Lambda(\mathcal{G})$-module which has no non-trivial finite submodule, has $\mu(M) = 0$ and is cohomologically trivial as H-module. Then*

$$\mathrm{pd}_{\Lambda(\mathcal{G})}(M) \leqslant 1.$$

Proof. For any topological ring Λ, we denote the category of compact Λ-modules by $\mathcal{C}(\Lambda)$ and the category of discrete Λ-modules by $\mathcal{D}(\Lambda)$. We have a functor

$$\mathcal{H}om_{\Lambda(\mathcal{G})}(\ ,\ ) : \mathcal{C}(\Lambda(\mathcal{G})) \times \mathcal{D}(\Lambda(\mathcal{G})) \longrightarrow \mathcal{D}(\mathbb{Z}_p)$$

and we can use either projective resolutions in $\mathcal{C}(\Lambda(\mathcal{G}))$ or injective resolutions in $\mathcal{D}(\Lambda(\mathcal{G}))$ to define functors

$$\mathcal{E}xt^i_{\Lambda(\mathcal{G})}(\ ,\ ) : \mathcal{C}(\Lambda(\mathcal{G})) \times \mathcal{D}(\Lambda(\mathcal{G})) \longrightarrow \mathcal{D}(\mathbb{Z}_p),\ i \geqslant 0.$$

By [NSW00], Prop. (5.2.11), we have to show that $\mathcal{E}xt^2_{\Lambda(\mathcal{G})}(M,N) = 0$ for all simple N. We consider the spectral sequence (cf. loc.cit., Ch. V, §2, Ex. 4):

$$E_2^{i,j} = H^i(\Gamma_K, \mathcal{E}xt^j_{\mathbb{Z}_pH}(M,N)) \Longrightarrow E^{i+j} = \mathcal{E}xt^{i+j}_{\Lambda(\mathcal{G})}(M,N).$$

Since M has no non-trivial finite submodules and $\mu(M) = 0$, it is free and finitely generated as $\mathbb{Z}_p$-module. Moreover, it is c.t. as H-module by assumption and hence $\mathbb{Z}_pH$-projective. This implies $E_2^{i,j} = 0$ for $j > 0$. Since N and hence $\mathcal{H}om_{\mathbb{Z}_pH}(M,N)$ are p-torsion and the cohomological p-dimension of Γ_K is 1, we also have $E_2^{i,j} = 0$ if $i > 1$. This implies $\mathcal{E}xt^2_{\Lambda(\mathcal{G})}(M,N) = E^2 \simeq E_2^{2,0} = 0$. □

Proposition 3.2. *Let M be a finitely generated R-torsion $\Lambda(\mathcal{G})$-module such that $\mathrm{pd}_{\Lambda(\mathcal{G})}(M) \leqslant 1$ and $\mu(M) = 0$. Assume that $\mathrm{Fitt}_{\mathbb{Q}_p\Lambda(\mathcal{G})}(\mathbb{Q}_p \otimes M)$ is generated by $\mathrm{nr}(\phi)$ over $\mathbb{Q}_p\Lambda(\mathcal{G})$, where $\phi \in M_{n\times n}(\Lambda(\mathcal{G})) \cap \mathrm{Gl}_n(Q(\mathcal{G}))$ such that $\mu(\Lambda(\mathcal{G})^n/\phi(\Lambda(\mathcal{G})^n)) = 0$. Then also*

$$\mathrm{Fitt}_{\Lambda(\mathcal{G})}(M) = [\langle \mathrm{nr}(\phi)\rangle_{\zeta(\Lambda(\mathcal{G}))}]_{\mathrm{nr}(\Lambda(\mathcal{G}))}.$$

Proof. By [Nib], Lemma 6.2 the module M admits a quadratic presentation over $\Lambda(\mathcal{G})$ such that $\mathrm{Fitt}_{\Lambda(\mathcal{G})}(M)$ exists and is generated by $\mathrm{nr}(\psi)$ for some $\psi \in M_{n\times n}(\Lambda(\mathcal{G})) \cap \mathrm{Gl}_n(Q(\mathcal{G}))$. By assumption, we have

$$\langle \mathrm{nr}(\phi)\rangle_{\zeta(\mathbb{Q}_p\Lambda(\mathcal{G}))} = \langle \mathrm{nr}(\psi)\rangle_{\zeta(\mathbb{Q}_p\Lambda(\mathcal{G}))}$$

such that there is a unique $x \in \zeta(\mathbb{Q}_p\Lambda(\mathcal{G}))^\times$ with $\mathrm{nr}(\psi) = x \cdot \mathrm{nr}(\phi)$. Let us denote the integral closure of $\zeta(\Lambda(\mathcal{G}))$ in $\zeta(Q(\mathcal{G}))$ by $\mathfrak{Z}$. Then the reduced norm maps $K_1(\Lambda(\mathcal{G}))$ into $\mathfrak{Z}^\times$ and $K_1(\Lambda_{(p)}(\mathcal{G}))$ into $\mathfrak{Z}_{(p)}^\times$, where the subscript (p) means localization at the prime (p). We have shown that there is a natural number N such that $p^N \cdot x \in \mathfrak{Z}$. Since the μ-invariants vanish, the maps ϕ and ψ become isomorphisms after localization at (p) and we find $x \in \mathrm{nr}(K_1(\Lambda_{(p)}(\mathcal{G}))) \subset \mathfrak{Z}_{(p)}^\times$. Thus we can choose a Weierstraß polynomial f such that $f \cdot x \in \mathfrak{Z}$ and hence $x \in \mathfrak{Z}$ by Lemma 3.3 below. Now [RW05], Th. B in conjunction with diagram (8) implies

$$x \in \mathfrak{Z} \cap \mathrm{nr}(K_1(\Lambda_{(p)}(\mathcal{G}))) \subset \mathrm{nr}(K_1(\Lambda(\mathcal{G}))).$$

Hence the $\zeta(\Lambda(\mathcal{G}))$-modules generated by $\mathrm{nr}(\phi)$ and $\mathrm{nr}(\psi)$ are $\mathrm{nr}(\Lambda(\mathcal{G}))$-equivalent. □

We have used the following easy lemma.

Lemma 3.3. *Let Λ be a ring, $x \in \Lambda$ and $y \in \zeta(\Lambda)$. Assume that y is a nonzerodivisor and x is a nonzerodivisor modulo y. Let S be a multiplicatively closed subset of $\zeta(\Lambda)$ which contains no zero divisors, $1 \in S$, $0 \notin S$ and let $\Psi \in \Lambda_S$ such that $x \cdot \Psi \in \Lambda$ and $y \cdot \Psi \in \Lambda$. Then also $\Psi \in \Lambda$.*

Proof. The equation $x \cdot \Psi \cdot y = x \cdot y \cdot \Psi$ implies that $y \cdot \Psi \equiv 0 \bmod y$, since x is a nonzerodivisor modulo y. Hence there is $\lambda \in \Lambda$ such that $y \cdot \Psi = y \cdot \lambda$. But y is a nonzerodivisor and thus $\lambda = \Psi$. □

If M is an Iwasawa torsion module, we write $\alpha(M)$ for the Iwasawa adjoint of M. If H is a finite group and M is a $\mathbb{Z}_p[H]$-module, we denote the Pontryagin dual $\mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ of M by $M^\vee$ which is equipped with the natural H-action $(hf)(m) = f(h^{-1}m)$ for $f \in M^\vee$, $h \in H$ and $m \in M$.

Lemma 3.4. *Let U be a subgroup of $\mathcal{G}$ of finite index.*

i) *For any $\Lambda(U)$-module N, we have an isomorphism $\mathrm{ind}_U^{\mathcal{G}}(N(1)) \simeq (\mathrm{ind}_U^{\mathcal{G}} N)(1)$.*
ii) *If $M = \mathrm{ind}_U^{\mathcal{G}} \mathbb{Z}_p$, then $\alpha(M) \simeq M$.*

Proof. Let us put $N' := N(1)$. Then $\mathrm{ind}_U^{\mathcal{G}} N' = \bigoplus_\sigma N'_\sigma$, where σ runs through a set of (left) coset representatives, and where $N'_\sigma = N'$ as sets and $gn' = u_\sigma n' \in N'_{\tilde{\sigma}}$ if $g\sigma = \tilde{\sigma} u_\sigma$ for $g \in \mathcal{G}$, $u_\sigma \in U$, $n' \in N'_\sigma$; similarly, $\mathrm{ind}_U^{\mathcal{G}} N = \bigoplus_\sigma N_\sigma$. An easy computation shows that

$$\bigoplus_\sigma N'_\sigma \longrightarrow (\bigoplus N_\sigma)(1), \ \sum_\sigma n'_\sigma \mapsto \sum_\sigma \kappa(\sigma) n'_\sigma$$

is an isomorphism of $\Lambda(\mathcal{G})$-modules. This shows (i). For (ii) we compute

$$\begin{aligned} \alpha(M) &= \varprojlim_n \mathrm{Hom}(M/p^n, \mathbb{Q}_p/\mathbb{Z}_p) \\ &= \varprojlim_n (\mathrm{ind}_U^{\mathcal{G}} \mathbb{Z}_p/p^n)^\vee \\ &\simeq \varprojlim_n \mathrm{ind}_U^{\mathcal{G}} \mathbb{Z}_p/p^n \\ &= M. \end{aligned}$$

□

We point out that Lemma 3.4 and Proposition 3.2 are non-abelian generalizations of [Gr04], Lemma 1 and Lemma 2, respectively.

4. Fitting invariants of Iwasawa modules

In this section we fix the following setting: let L/K be a Galois CM-extension of number fields with Galois group G, i.e. K is totally real and L is a totally imaginary quadratic extension of a totally real number field. This field is the maximal real subfield of L and will be denoted by L^+. Complex conjugation on $\mathbb{C}$ induces an automorphism j on L which is independent of the embedding into $\mathbb{C}$ and lies in the center of G. Let $p \neq 2$ be a prime and assume that j lies in the decomposition group $G_{\mathfrak{P}}$ for each prime $\mathfrak{P}$ of L above p which is wildly ramified in L/K (we will call this condition *almost tame* above p). In particular, we consider all Galois CM-extension wich are at most tamely ramified

above p.
We choose a prime $\mathfrak{p}_0 \nmid p$ of K which is unramified in L/K and define a set of places of K by

$$T = T_0 := \{\mathfrak{p}_0\} \cup S_{\mathrm{ram}} \setminus (S_{\mathrm{ram}} \cap S_p).$$

We may choose $\mathfrak{p}_0$ such that E_S^T is torsionfree. Then $A_L^T(p)$,the p-part of the minus ray class group $\mathrm{cl}_L^{T,-}$, is c.t. as G-module by [Nia], Th. 1.

Let L_∞ and K_∞ be the cyclotomic $\mathbb{Z}_p$-extensions of L and K, respectively. We denote the Galois group of K_∞/K by Γ_K. Hence Γ_K is isomorphic to $\mathbb{Z}_p$, and we fix a topological generator γ_K. Furthermore, we denote the n-th layer in the cyclotomic extension K_∞/K by K_n such that K_n/K is cyclic of order p^n. Accordingly, we set $\Gamma_L = \mathrm{Gal}(L_\infty/L)$ with a topological generator γ_L whose restriction to K_∞ is γ_K^a for an appropriate integer a. We enumerate the intermediate fields starting with $L = L_a$ such that L_n/L is cyclic of order p^{n-a}. This is because in this case L_n is the smallest intermediate field of L_∞/L which lies above K_n. It may also be convenient to define $L_n = L$ if $n \leqslant a$. We put

$$\mathcal{X}_T^- := \varprojlim A_{L_n}^T(p).$$

We denote the Galois group of L_∞/K by $\mathcal{G}$, hence $\mathcal{G} = H \rtimes \Gamma$, where H is a subgroup of G and Γ is topologically generated by a preimage γ of γ_K under the canonical epimorphism $\mathcal{G} \twoheadrightarrow \mathcal{G}/H = \Gamma_K$. Then $\mathcal{X}_T^-$ is a finitely generated R-torsion $\Lambda(\mathcal{G})_- := \Lambda(\mathcal{G})/(1+j)$-module, where as before $R = \mathbb{Z}_p[[\Gamma']]$ with $\Gamma' \simeq \mathbb{Z}_p$ central in $\mathcal{G}$. Let L' be the maximal subfield of L_∞ fixed by Γ. Since L' is contained in L_n if n is sufficiently large, the layers of the cyclotomic extensions of L and L' agree for $n >> 0$ and $A_{L_n}^T(p)$ is $\mathrm{Gal}(L_n/K_n)$-c.t., since each of the extensions L_n/K_n inherits the required properties from the extension L/K. Hence $\mathcal{X}_T^-$ is c.t. as H-module and has no nontrivial finite submodule (cf. [Gr04], first step of the proof of Prop. 6) such that Proposition 3.1 implies the following result.

Proposition 4.1. *If L/K is almost tame above p and the Iwasawa μ-invariant $\mu(\mathcal{X}_T^-)$ vanishes, then the projective dimension of $\mathcal{X}_T^-$ over $\Lambda(\mathcal{G})_-$ is at most* 1.

Now let S be a finite set of places of K containing S_∞ (but not necessarily S_p) and let M_S be the maximal abelian pro-p-extension of L_∞ unramified outside S. Moreover, let M_∞ be the maximal abelian unramified extension of L_∞ and define $\Lambda(\mathcal{G})$-modules

$$X_S := \mathrm{Gal}(M_S/L_\infty),\ X_{\mathrm{std}} := \mathrm{Gal}(M_\infty/L_\infty).$$

Hence X_{std} is the "standard" Iwasawa module which is the projective limit of the p-parts of the class groups in the cyclotomic tower of L. If $S = S_\infty \cup S_p$, we also write $X_{\{p\}}$ instead of $X_{S_\infty \cup S_p}$. Moreover, if $S = T \cup S_\infty$, there is an isomorphism $X_{T\cup S_\infty}^- \simeq \mathcal{X}_T^-$. Following Greither [Gr04], we will also define a "dual" Iwasawa module X_{du}: There is a minimal integer n_0 such that all the p-adic places ramify in L_∞/L_{n_0}. We denote the p-class field of L_{n_0} by M_{n_0} and put $X_{\mathrm{du}} := \mathrm{Gal}(M_\infty/M_{n_0}L_\infty)$. So X_{du} is a submodule of X_{std} of finite index and the subscript "du" is chosen because of the following description of X_{du}^- in the case $\zeta_p \in L$, where ζ_p denotes a primitive p-th root of unity (cf. [Gr04], beginning of §2 - note that G is assumed to be abelian in loc.cit., but in all cases, where we will cite [Gr04], this assumption is not necessary):

$$X_{\mathrm{du}}^- \simeq \alpha(X_{\{p\}}^+)(1).$$

If S contains all the p-adic places, we define an Iwasawa module $Z_S = Z_{L,S}$ by

$$Z_S = \alpha(X_S^+)(1) \text{ if } \zeta_p \in L,$$

$$Z_S = (Z_{L(\zeta_p),S})_\Delta \ \text{ otherwise},$$

where $\Delta = \mathrm{Gal}(L(\zeta_p)/L)$. Note that this definition slightly differs from the definition of the corresponding module in loc.cit. But since $p \nmid |\Delta|$, multiplication by $N_\Delta := \sum_{\delta\in\Delta}\delta$ induces an isomorphism $(Z_{L(\zeta_p),S})_\Delta \simeq (Z_{L(\zeta_p),S})^\Delta$. For any prime $\mathfrak{p}$ of K, we choose a prime $\wp$ in L_∞ above $\mathfrak{p}$ and put $\mathfrak{P} = \wp \cap L$. Setting $Z_\mathfrak{p} := \mathrm{ind}_{\mathcal{G}_\wp}^{\mathcal{G}}\mathbb{Z}_p$, class field theory gives an exact sequence (cf. loc.cit., sequence (1); for the proof replace loc.cit., Lemma 1 (i) by Lemma 3.4 (i)):

$$\bigoplus_{\mathfrak{p}\in S\setminus S_p} Z_\mathfrak{p}(1)^+ \rightarrowtail X_S^+ \twoheadrightarrow X_{\{p\}}^+ \tag{11}$$

We claim that this sequence induces an exact sequence

$$X_{\mathrm{du}}^- \rightarrowtail Z_S \twoheadrightarrow \bigoplus_{\mathfrak{p}\in S\setminus S_p} Z_\mathfrak{p}^- . \tag{12}$$

This is clear if $\zeta_p \in L$, since taking Iwasawa adjoints is exact on sequences of torsion Iwasawa modules without finite submodules and $\alpha(Z_\mathfrak{p}(1))(1) = \alpha(Z_\mathfrak{p}) = Z_\mathfrak{p}$ by Lemma 3.4 (ii). If $\zeta_p \notin L$, we put $L' = L(\zeta_p)$, $L'_\infty = L_\infty(\zeta_p)$ etc. Since $p \nmid |\Delta|$, the p-class groups of the layers in the cyclotomic tower are c.t. as Δ-modules and we have thus isomorphisms $A_{L'_n}(p)_\Delta \simeq A_{L_n}(p)$ which combine to induce an isomorphism $(X'^-_{\mathrm{std}})_\Delta \simeq X^-_{\mathrm{std}}$. We have a commutative diagram

$$\begin{array}{ccccc}
(X'^-_{\mathrm{du}})_\Delta & \hookrightarrow & (X'^-_{\mathrm{std}})_\Delta & \twoheadrightarrow & A_{L'_{n_0}}(p)_\Delta \\
\downarrow & & \downarrow \simeq & & \downarrow \simeq \\
X^-_{\mathrm{du}} & \hookrightarrow & X^-_{\mathrm{std}} & \twoheadrightarrow & A_{L_{n_0}}(p)
\end{array}$$

Hence also the leftmost vertical arrow is an isomorphism and we obtain (12) in general, as we may adjoin ζ_p first and then apply Δ-coinvariants to sequence (12) for L'.
Let $x \mapsto \dot{x}$ be the automorphism on $\Lambda(\mathcal{G})$ induced by $g \mapsto \kappa(g)g^{-1}$ for $g \in \mathcal{G}$. Let $\mathcal{G}^+ := \mathcal{G}/\langle j\rangle = \mathrm{Gal}(L_\infty^+/K)$ and let $\Phi_S \in \zeta(Q(\mathcal{G}^+))^\times$ be the unique element satisfying $j_\chi(\Phi_S) = L_{K,S}(\chi)$ for each even character of $\mathcal{G}$ with open kernel. We define idempotents

$$e^- = \frac{1-j}{2},\ e^+ = \frac{1+j}{2}.$$

The following is a non-abelian generalization of [Gr04], Th. 2.

Theorem 4.2. *Assume that the EIMC holds for L_∞^+/K.*

i) *If $\zeta_p \in L$, then*

$$\mathrm{Fitt}^{\max}_{\Lambda(\mathcal{G})}(Z_S) = \mathrm{Fitt}^{\max}_{\Lambda(\mathcal{G})}(\mathbb{Z}_p(1))^\sharp[\langle\dot{\Phi}_S e^- + e^+\rangle]_{\mathrm{nr}(\Lambda(\mathcal{G}))}.$$

ii) *If $\zeta_p \notin L$, then $\mathrm{pd}_{\Lambda(\mathcal{G})}(Z_S) \leqslant 1$ and*

$$\mathrm{Fitt}_{\Lambda(\mathcal{G})}(Z_S) = [\langle\dot{\Phi}_S e^- + e^+\rangle]_{\mathrm{nr}(\Lambda(\mathcal{G}))}.$$

Proof. Assume that $\zeta_p \in L$. The canonical complex (6) for the extension L_∞^+/K gives an exact sequence

$$X_S^+ \rightarrowtail C^{-1} \to C^0 \twoheadrightarrow \mathbb{Z}_p.$$

Applying the functor $\alpha(\ )(1)$ to this sequence yields

$$\mathbb{Z}_p(1) \rightarrowtail \alpha(C^0)(1) \to \alpha(C^{-1})(1) \twoheadrightarrow Z_S. \tag{13}$$

Now it follows from [Nib], Prop. 6.3 (ii) that

$$\begin{aligned}\mathrm{Fitt}^{\max}_{\Lambda(\mathcal{G})}(Z_S) &= \mathrm{Fitt}^{\max}_{\Lambda(\mathcal{G})}(\mathbb{Z}_p(1))^\sharp \cdot \mathrm{Fitt}_{\Lambda(\mathcal{G})}(\alpha(\mho_S)(1)) \\ &= \mathrm{Fitt}^{\max}_{\Lambda(\mathcal{G})}(\mathbb{Z}_p(1))^\sharp \cdot [\langle\dot{\Phi}_S e^- + e^+\rangle]_{\mathrm{nr}(\Lambda(\mathcal{G}))},\end{aligned}$$

where the second equality holds by the EIMC and loc.cit., Prop. 6.3 (i).
If $\zeta_p \notin L$, we again put $L' = L(\zeta_p)$. We apply $\Delta = \mathrm{Gal}(L'/L)$-coinvariants to sequence (13) (for L') and obtain an exact sequence

$$\alpha(C^0)(1)_\Delta \rightarrowtail \alpha(C^{-1})(1)_\Delta \twoheadrightarrow Z_S.$$

Hence Z_S has projective dimension at most 1 and

$$\begin{aligned}\mathrm{Fitt}_{\Lambda(\mathcal{G})}(Z_S) &= \mathrm{Fitt}_{\Lambda(\mathcal{G})}(\alpha(\mho_S((L'_\infty)^+/K))(1)_\Delta) \\ &= \mathrm{Fitt}_{\Lambda(\mathcal{G})}(\alpha(\mho_S(L^+_\infty/K))(1)) \\ &= [\langle \dot{\Phi}_S e^- + e^+\rangle]_{\mathrm{nr}(\Lambda(\mathcal{G}))},\end{aligned}$$

where the second equality follows from [RW04], Prop. 12, whereas the last equality is the EIMC. □

As in [Gr04], Prop. 6 we have an exact sequence

$$\mathbb{Z}_p(1) \rightarrowtail \bigoplus_{\mathfrak{p}\in T} Z_{\mathfrak{p}}(1)^- \to \mathcal{X}_T^- \twoheadrightarrow X_{\mathrm{std}}^- \tag{14}$$

if $\zeta_p \in L$, and without the leftmost term if $\zeta_p \notin L$. For $\mathfrak{p} \notin S_p$ we put

$$\Xi_{\mathfrak{p}} := \varepsilon_{\mathfrak{p}} \frac{\kappa(\phi_\wp) - \phi_\wp}{1-\phi_\wp} + 1 - \varepsilon_{\mathfrak{p}} \in \mathbb{Q}_p\Lambda(\mathcal{G}_\wp), \text{ where } \varepsilon_{\mathfrak{p}} = |I_{\mathfrak{P}}|^{-1} N_{I_{\mathfrak{P}}} \in \mathbb{Q}_p H,$$

$$\xi_{\mathfrak{p}} := \mathrm{nr}(1 \otimes \Xi_{\mathfrak{p}}).$$

Here, $\phi_\wp \in \mathcal{G}$ and $I_{\mathfrak{P}}$ are the Frobenius and the inertia subgroup at a chosen prime $\wp$ in L_∞ above $\mathfrak{p}$, respectively; note that the inertia subgroup depends only on the prime $\mathfrak{P}$ in L above p, since $\mathfrak{p}$ lies not above p and is thus unramified in the cyclotomic extension. The element $1 \otimes \Xi_{\mathfrak{p}}$ belongs to $\mathbb{Q}_p\Lambda(\mathcal{G}) = \mathrm{ind}_{\mathcal{G}_\wp}^{\mathcal{G}} \mathbb{Q}_p\Lambda(\mathcal{G}_\wp)$. Note that $\phi_\wp$ and $I_{\mathfrak{P}}$ depend on the choice of $\wp$, but $\xi_{\mathfrak{p}}$ does not. If S is a finite set of places of K containing $S_p \cup S_\infty$, we put

$$\Psi_S := \prod_{\mathfrak{p}\in S\setminus S_p} \xi_{\mathfrak{p}} \cdot \dot{\Phi}_S e^- \in \zeta(Q(\mathcal{G})_-).$$

Proposition 4.3. *The Fitting invariant* $\mathrm{Fitt}_{\mathbb{Q}_p\Lambda(\mathcal{G})_-}(\mathbb{Q}_p\mathcal{X}_T^-)$ *is generated by* $\Psi_{T\cup S_p}$. *In particular,* $\Psi_{T\cup S_p} \in \zeta(\mathbb{Q}_p\Lambda(\mathcal{G})_-)$.

Proof. We first observe that $\mathbb{Q}_p\Lambda(\mathcal{G})$ is a maximal $\mathbb{Q}_p \otimes R$-order in $Q(\mathcal{G})$. In this case every finitely generated $\mathbb{Q}_p\Lambda(\mathcal{G})$-module has a quadratic presentation, and taking Fitting invariants is multiplicative on short exact sequences of $\mathbb{Q}_p \otimes R$-torsion $\mathbb{Q}_p\Lambda(\mathcal{G})$-modules. It suffices to assume the EIMC in the "maximal order case" which is a theorem ([RW04], Th. 16; cf. also loc.cit., remark H), and we may use Theorem 4.2 over $\mathbb{Q}_p\Lambda(\mathcal{G})$ without assuming the EIMC. We put $i = -1$ if $\zeta_p \in L$ and $i = 0$ otherwise. Since $\mathbb{Q}_pX_{\mathrm{du}} = \mathbb{Q}_pX_{\mathrm{std}}$, the exact sequences (12) and (14) imply that

$$\begin{aligned}\mathrm{Fitt}(\mathbb{Q}_p\mathcal{X}_T^-) &= \mathrm{Fitt}(\mathbb{Q}_pZ_{T\cup S_p}^-) \cdot \mathrm{Fitt}(\mathbb{Q}_p(1))^i \cdot \prod_{\mathfrak{p}\in T} \mathrm{Fitt}(\mathbb{Q}_pZ_{\mathfrak{p}}^-)^{-1} \cdot \mathrm{Fitt}(\mathbb{Q}_pZ_{\mathfrak{p}}(1)^-) \\ &= \langle \dot{\Phi}_{T\cup S_p} e^-\rangle \cdot \prod_{\mathfrak{p}\in T} \mathrm{Fitt}(\mathbb{Q}_pZ_{\mathfrak{p}}^-)^{-1} \cdot \mathrm{Fitt}(\mathbb{Q}_pZ_{\mathfrak{p}}(1)^-),\end{aligned}$$

where all Fitting invariants are taken over $\mathbb{Q}_p\Lambda(\mathcal{G})_-$ and the second equality holds by Theorem 4.2. The Fitting invariant of $\mathbb{Q}_pZ_{\mathfrak{p}}^-$ is generated by $\mathrm{nr}(1 \otimes x_{\mathfrak{p}})$ with $x_{\mathfrak{p}} = 1 - \varepsilon_{\mathfrak{p}} + (1-\phi_\wp)\varepsilon_{\mathfrak{p}}$, since $\mathbb{Q}_pZ_{\mathfrak{p}} = \mathrm{ind}_{\mathcal{G}_\wp}^{\mathcal{G}} \mathbb{Q}_p$ and $\mathbb{Q}_p$ is isomorphic to $\mathbb{Q}_p\Lambda(\mathcal{G}_\wp)/x_{\mathfrak{p}}$ as $\mathbb{Q}_p\Lambda(\mathcal{G}_\wp)$-module. Likewise, the Fitting invariant of $\mathbb{Q}_pZ_{\mathfrak{p}}(1)^-$ is generated by $\mathrm{nr}(1 \otimes \dot{x}_{\mathfrak{p}})$. We obtain

$$\mathrm{Fitt}(\mathbb{Q}_pZ_{\mathfrak{p}}^-)^{-1} \cdot \mathrm{Fitt}(\mathbb{Q}_pZ_{\mathfrak{p}}(1)^-) = \langle \mathrm{nr}(1 \otimes (x_{\mathfrak{p}}\dot{x}_{\mathfrak{p}}^{-1}))\rangle = \langle \xi_{\mathfrak{p}}\rangle.$$

□

We now prove the non-abelian analogue of [Gr04], Th. 6.

Theorem 4.4. *Let L/K be almost tame above p. Assume that the EIMC holds for the extension L_∞^+/K and $\mu(\mathcal{X}_T^-)=0$. Then $\Psi_{T\cup S_p}$ generates the Fitting invariant* $\mathrm{Fitt}_{\Lambda(\mathcal{G})_-}(\mathcal{X}_T^-)$.

Proof. By Proposition 4.3, we know that $\Psi_{T\cup S_p}$ is a generator of $\mathrm{Fitt}_{\mathbb{Q}_p\Lambda(\mathcal{G})_-}(\mathbb{Q}_p\mathcal{X}_T^-)$. Let us write $\Psi_{T\cup S_p}=\mathrm{nr}(B)$, where $B=\Theta\cdot\prod_{\mathfrak{p}\in T}(1\otimes\Xi_\mathfrak{p})$ and $\mathrm{nr}(\Theta)=\dot{\Phi}_{T\cup S_p}$. Since $\mathcal{X}_T^-$ is of projective dimension at most 1 by Proposition 4.1, it suffices to show that $B\in M_{n\times n}(\Lambda(\mathcal{G})_-)$ for some n by Proposition 3.2.
The validity of the EIMC implies that Θ is in the image of $K_1(\Lambda_{(p)}(\mathcal{G}))$; in fact, this is equivalent to the EIMC by [RW05], Th. A, but it is clearly necessary, since $\mho_S$ vanishes if we localize at (p). Hence we can choose a Weierstraß polynomial f such that $f\cdot\Theta$ lies in $M_{n\times n}(\Lambda(\mathcal{G})_-)$. We claim that $(1-\phi_\wp)\Xi_\mathfrak{p}$ belongs to $\Lambda(\mathcal{G}_\wp)$. Taking this for granted for the moment, we see that $f\cdot\prod_{\mathfrak{p}\in T}1\otimes(1-\phi_\wp^N)\cdot B$ belongs to $M_{n\times n}(\Lambda(\mathcal{G})_-)$, where N is a large integer such that $\phi_\wp^N$ is central in $\Lambda(\mathcal{G})$ for all $\mathfrak{p}\in T$. Since we already know that $p^b\cdot B\in M_{n\times n}(\Lambda(\mathcal{G})_-)$ for sufficiently large b, Lemma 3.3 implies that also B belongs to $M_{n\times n}(\Lambda(\mathcal{G})_-)$.
We are left with the claim. Setting $q_\mathfrak{p}=\kappa(\phi_\wp)$ and $|I_\mathfrak{P}|=p^r\cdot e'$ with $p\nmid e'$, we find that

$$(1-\phi_\wp)\Xi_\mathfrak{p}=1-\phi_\wp-\varepsilon_\mathfrak{p}(q_\mathfrak{p}-1)$$

and we have to show that p^r divides $(q_\mathfrak{p}-1)$. We may assume $r>0$. Let $q\neq p$ be the rational prime below $\mathfrak{p}$ and denote the q-Sylow subgroup of $I_\mathfrak{P}$ by $R_\mathfrak{P}$. There is a natural inclusion $I_\mathfrak{P}/R_\mathfrak{P}\rightarrowtail(\mathfrak{o}_L/\mathfrak{P})^\times$ which preserves the group action of $G_\mathfrak{P}/I_\mathfrak{P}$, where $\mathfrak{P}$ is a place in L above $\mathfrak{p}$. Let $\mathcal{I}$ be the subgroup of $G_\mathfrak{P}/I_\mathfrak{P}$-invariants of $I_\mathfrak{P}/R_\mathfrak{P}$. Then $\mathcal{I}$ maps into $(\mathfrak{o}_K/\mathfrak{p})^\times$ and hence $|\mathcal{I}|$ divides $(q_\mathfrak{p}-1)$. We have to show that the p-parts of $|\mathcal{I}|$ and $|I_\mathfrak{P}|$ coincide. Let a be a generator of $I_\mathfrak{P}/R_\mathfrak{P}$ and choose a lift $b\in G_\mathfrak{P}/R_\mathfrak{P}$ of $\phi_\mathfrak{P}^{-1}$ which is of maximal order of all such elements; here, $\phi_\mathfrak{P}=\phi_\wp \bmod \Gamma_L$. By [Ch85], Lemma p. 369 we have $b^{-1}ab=a^{q_\mathfrak{p}}$ and hence a^i belongs to $\mathcal{I}$ if and only if $|I_\mathfrak{P}/R_\mathfrak{P}|$ divides $(q_\mathfrak{p}^i-i)$. Since $r>0$, this in particular implies that $q_\mathfrak{p}^i-i\equiv 0 \bmod p$ if $a^i\in\mathcal{I}$. But $p\neq q$ such that $p\nmid i$. It follows that $\mathcal{I}$ is generated by a^m for some divisor m of $|I_\mathfrak{P}/R_\mathfrak{P}|$ which is not divisible by p. Hence p^r is the exact power of p which divides $|\mathcal{I}|=|I_\mathfrak{P}|\cdot(|R_\mathfrak{P}|\cdot m)^{-1}$. □

We close this section with a few preparations for the Galois descent. If χ is a character of $\mathcal{G}$ with open kernel, we define

$$S_\chi:=\{\mathfrak{p}\subset K | I_\mathfrak{P}\not\subset\ker(\chi)\}.$$

Lemma 4.5. *Let S bet a finite set of primes of K containing S_∞. Let χ be an even character of $\mathcal{G}$ with open kernel and put $\Sigma:=S\cup S_p$ and $\Sigma_\chi:=(S\cap S_\chi)\cup S_p$.*

i) *If χ is of type S (i.e. $\Gamma\subset\ker(\chi)$), we have an equality*

$$L_{p,\Sigma}(s,\chi)=L_{p,\Sigma_\chi}(s,\chi)\prod_{\mathfrak{p}\in\Sigma\setminus\Sigma_\chi}\det(1-\sigma_\wp u^{-sc_\mathfrak{p}}|V_{\chi\omega^{-1}}^{I_\mathfrak{P}}), \tag{15}$$

where we write $\phi_\wp=\sigma_\wp\cdot\gamma^{c_\mathfrak{p}}$ with $\sigma_\wp\in H$, $c_\mathfrak{p}\in\mathbb{Z}_p$, and where ω denotes the Teichmüller character.

ii) *We have an equality*

$$G_{\chi,\Sigma}(T)=G_{\chi,\Sigma_\chi}(T)\prod_{\mathfrak{p}\in\Sigma\setminus\Sigma_\chi}g_{\mathfrak{p},\chi\omega^{-1}}(T),$$

where $g_{\mathfrak{p},\chi}(T):=\det_{Q^c(\Gamma_K)}(1-\phi_\wp^{-1}\varepsilon_\mathfrak{p}|\mathcal{V}_{\chi^{-1}})$.

Proof. For (i), we have to evaluate both sides at $s=1-r$, where $r\geqslant 1$ is divisible by $(p-1)$. We observe that

$$u^{(r-1)c_\mathfrak{p}}=\kappa(\phi_\wp)^{r-1}\kappa(\sigma_\wp)^{1-r}=N(\mathfrak{p})^{r-1}\omega(\sigma_\wp).$$

Now we compute that the right hand side of equation (15) at $s = 1 - r$ equals

$$\begin{aligned} L_{p,\Sigma_\chi}(1-r,\chi) \prod_{\mathfrak{p}\in\Sigma\setminus\Sigma_\chi} \det(1-\sigma_\wp\omega(\sigma_\wp)N(\mathfrak{p})^{r-1}|V_{\chi\omega^{-1}}^{I_\mathfrak{P}}) &= L_{\Sigma_\chi}(1-r,\chi) \prod_{\mathfrak{p}\in\Sigma\setminus\Sigma_\chi} \det(1-\sigma_\wp N(\mathfrak{p})^{r-1}|V_\chi^{I_\mathfrak{P}}) \\ &= L_\Sigma(1-r,\chi) \\ &= L_{p,\Sigma}(1-r,\chi). \end{aligned}$$

This proves (i). For (ii) we observe that $g_{\mathfrak{p},\chi}(u^s-1) = \det(1-\sigma_\wp u^{-sc_\mathfrak{p}}\varepsilon_\mathfrak{p}|V_\chi)$ if χ is of type S. Hence (i) implies (ii) in this case. If $\chi = \psi \otimes \rho$, where ψ is of type S and ρ is of type W, then we have an equality

$$g_{\mathfrak{p},\psi\otimes\rho} = g_{\mathfrak{p},\chi}(\rho(\gamma_K)(1+T)-1).$$

Since similar equalities hold for $G_{\chi,\Sigma}$ and G_{χ,Σ_χ}, we get (ii) in general. □

Corollary 4.6. *Keep the notation of Lemma 4.5, but assume that χ is an odd character and Σ contains S_{ram}. Then*

$$j_\chi(\dot{\Phi}_\Sigma) = L_{K,\Sigma_\chi}(\chi^{-1}\omega) \prod_{\mathfrak{p}\in\Sigma\setminus\Sigma_\chi} g_{\mathfrak{p},\chi^{-1}}(\gamma_K - 1).$$

The following proposition is still contained in the author's dissertation [Ni08], Prop. 3.2.7, but it was not yet published in a peer reviewed journal:

Proposition 4.7. *Let L/K be a Galois CM-extension with Galois group G, $p \neq 2$ a rational prime and T a finite G-invariant set of places of L such that $T \cap S_p = \emptyset$. If $\mathcal{X}_T^-$ denotes the projective limit of the minus p-ray class groups $A_{L_n}^T(p)$, there is an exact sequence of $\mathbb{Z}_pG_-$-modules*

$$\bigoplus_{\mathfrak{p}\in S_p} (\mathrm{ind}\,_{G_\mathfrak{P}}^G \mathbb{Z}_p)^- \to (\mathcal{X}_T^-)_{\Gamma_L} \twoheadrightarrow A_L^T(p).$$

Proof. The canonical restriction map $X_T \to \mathrm{cl}_L^T(p)$ is surjective on minus parts, since the cokernel is a quotient of Γ_L on which j acts trivially. It clearly factors through $(\mathcal{X}_T^-)_{\Gamma_L}$.
Recall that M_T is the maximal abelian pro-p-extension of L_∞ unramified outside T. We put $\mathcal{Y}_T = \mathrm{Gal}(M_T/L)$. Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_s$ be the primes in L above p. Exactly these primes ramify in L_∞/L, and we denote the finitely many primes in L_∞, which lie above $\mathfrak{P}_1, \ldots, \mathfrak{P}_s$, by $\mathfrak{P}_{ik}^\infty$, $1 \leqslant i \leqslant s$. Moreover, we choose above each $\mathfrak{P}_{ik}^\infty$ a prime $\tilde{\mathfrak{P}}_{ik}$ in M_T, and denote its inertia group in $\mathcal{Y}_T$ by I_{ik}.
We obviously have an isomorphism $\mathcal{Y}_T/X_T \simeq \Gamma_L$. So we can pick a preimage $\gamma \in \mathcal{Y}_T$ of γ_L, and thus

$$\mathcal{Y}_T = X_T \cdot \overline{\langle\gamma\rangle}. \tag{16}$$

Let $\mathcal{Y}_T'$ be the closure of the commutator subgroup of $\mathcal{Y}_T$. Then G acts on $\mathcal{Y}_T/\mathcal{Y}_T'$ via conjugation, and we may assume that $\gamma^j \equiv \gamma \bmod \mathcal{Y}_T'$, as we may choose a lift $\tilde{j} \in \mathrm{Gal}(M_T/K)$ of j and replace γ by $\gamma^{(1+\tilde{j})/2}$. The condition on the set T forces that the extension M_T/L_∞ does not ramify above p. Therefore $I_{ik} \cap X_T = 1$, and we get inclusions

$$I_{ik} \rightarrowtail \mathcal{Y}_T/X_T = \Gamma_L.$$

Hence, each I_{ik} is isomorphic to $\Gamma_L^{p^{n_{ik}}}$ for an appropriate integer n_{ik}. We fix a topological generator σ_{ik} of I_{ik} which maps to $\gamma_L^{p^{n_{ik}}}$ via the above inclusion. But for fixed i, each two of these inertia groups are conjugate, and hence $n_i := n_{ik}$ does not depend on k. Corresponding to (16) we write $\sigma_{ik} = a_{ik}\gamma^{p^{n_i}}$ with $a_{ik} \in X_T$.
Let M_0 be the p-ray class field of L to the ray $\mathfrak{M}_T$ such that $\mathrm{Gal}(M_0/L) \simeq \mathrm{cl}_L^T(p)$. Because of the obvious exact sequence

$$\mathrm{Gal}(M_T/M_0) \rightarrowtail \mathcal{Y}_T \twoheadrightarrow \mathrm{cl}_L^T(p)$$

we are interested in the Galois group $\mathrm{Gal}(M_T/M_0)$. We claim that it equals the subgroup $\mathcal{N}$ of $\mathcal{Y}_T$ generated by $\mathcal{Y}'_T$ and the inertia groups I_{ik}. For this, let N be the intermediate field of the extension M_T/L fixed by $\mathcal{N}$. Then N is the largest subfield of M_T which is abelian over L and unramified above p. Thus $M_0 \subset N$. If we assume that $M_0 \neq N$, we find an intermediate field N_0 of finite degree over L such that $M_0 \subsetneq N_0 \subset N$. Let $\mathfrak{N}$ be the conductor of N_0/L. Then the primes which divide $\mathfrak{N}$ are exactly the primes in T. The commutative diagram

$$\begin{array}{ccccccc} \mathfrak{o}_L & \longrightarrow & (\mathfrak{o}_L/\mathfrak{N})^\times & \longrightarrow & \mathrm{cl}_L^{\mathfrak{N}} & \twoheadrightarrow & \mathrm{cl}_L \\ \| & & \downarrow & & \downarrow & & \| \\ \mathfrak{o}_L & \longrightarrow & (\mathfrak{o}_L/\mathfrak{M}_T)^\times & \longrightarrow & \mathrm{cl}_L^T & \twoheadrightarrow & \mathrm{cl}_L \end{array}$$

now implies that the order m of the kernel of the surjection $\mathrm{cl}_L^{\mathfrak{N}} \twoheadrightarrow \mathrm{cl}_L^T$ is prime to p, since the primes dividing m are below the primes in T. What we have shown is $N_0 = M_0$, in contradiction to our assumption.

Lemma 4.8. *Let $\mathcal{Y}'_T$ be the closure of the commutator subgroup of $\mathcal{Y}_T$. Then*

$$\mathcal{Y}'_T = X_T^{\gamma_L - 1}.$$

Proof. The proof of [Wa82], Lemma 13.14 nearly remains unchanged. We only have to replace the inertia subgroup I_1 in loc.cit. by $\overline{\langle\gamma\rangle}$. ☐

Since $\gamma^j = \gamma \bmod \mathcal{Y}'_T$, the above Lemma implies that we obtain an isomorphism

$$A_L^T(p) \simeq \mathcal{X}_T^- / \langle (\mathcal{X}_T^-)^{\gamma_L - 1}, a_{ik}^{e_-} \rangle.$$

As already mentioned, the inertia groups I_{ik} are conjugate for fixed i, hence $\sigma_{ik} \equiv \sigma_{i1} \bmod \mathcal{Y}'_T$ and likewise $a_{ik} \equiv a_{i1} \bmod \mathcal{Y}'_T$ for all k. Hence

$$A_L^T(p) \simeq \mathcal{X}_T^- / \langle (\mathcal{X}_T^-)^{\gamma_L - 1}, a_1, \ldots, a_s \rangle,$$

where we have defined $a_i := a_{i1}^{e_-}$. Since $\mathcal{X}_T^- / (\mathcal{X}_T^-)^{\gamma_L - 1} = (\mathcal{X}_T^-)_{\Gamma_L}$, Proposition 4.7 follows from the following lemma. ☐

Lemma 4.9. *If $\mathfrak{P}_j = \mathfrak{P}_i^g$ for an element $g \in G$, then $a_j \equiv a_i^g \bmod (\mathcal{X}_T^-)^{\gamma_L - 1}$.*

Proof. Let $\tau \in \mathrm{Gal}(M_T/K)$ be a lift of g. Then g acts on $(\mathcal{X}_T^-)_{\Gamma_L}$ via conjugation by τ. $\tilde{\mathfrak{P}}_{i1}^\tau$ is a prime in M_T above $\mathfrak{P}_j$, hence there exists an $x \in \mathcal{Y}_T$ such that $\tilde{\mathfrak{P}}_{i1}^\tau = \tilde{\mathfrak{P}}_{j1}^x$. Replacing τ by $x^{-1}\tau$ we may assume that $x = 1$. Hence

$$\overline{\langle \sigma_{j1} \rangle} = I_{j1} = I_{i1}^\tau = \overline{\langle \sigma_{i1}^\tau \rangle}.$$

Since the restriction to L_∞ induces an isomorphism $I_{j1} \simeq \Gamma_L^{p^{n_j}}$ and

$$\sigma_{i1}^\tau|_{L_\infty} = (\gamma_L^{p^{n_i}})^\tau = (\gamma_L^{p^{n_i}})^g = \gamma_L^{p^{n_i}},$$

we have $n_i = n_j$ and $\sigma_{j1} = \sigma_{i1}^\tau$, i.e.

$$a_{j1} = (a_{i1}\gamma^{p^{n_j}})^\tau \cdot \gamma^{-p^{n_j}}.$$

But $\gamma^\tau|_{L_\infty} = \gamma_L$ implies that $\gamma^\tau = x_\tau \cdot \gamma$ for an element $x_\tau \in X_T$. Hence, the assertion follows from the above equation, since $x_\tau^{e_-}$ vanishes in $(\mathcal{X}_T^-)_{\Gamma_L}$, as j trivially acts on $\gamma \bmod \mathcal{Y}'_T$ and commutes with τ. ☐

5. An integrality conjecture

Let L/K be a Galois CM-extension with Galois group G. Let S and T be two finite sets of places of K such that

- S contains all the infinite places of K and all the places which ramify in L/K, i.e. $S \supset S_{\mathrm{ram}} \cup S_\infty$.
- $S \cap T = \emptyset$.
- E_S^T is torsionfree.

We refer to the above hypotheses as $Hyp(S,T)$. For a fixed set S we define $\mathfrak{A}_S$ to be the $\zeta(\mathbb{Z}G)$-submodule of $\zeta(\mathbb{Q}G)$ generated by the elements $\delta_T(0)$, where T runs through the finite sets of places of K such that $Hyp(S,T)$ is satisfied. Note that $\mathfrak{A}_S$ equals the $\mathbb{Z}G$-annihilator of the roots of unity of L if G is abelian by [Ta84], Lemma 1.1, p. 82.
For each finite prime $\mathfrak{p}$ of K, we define a $\mathbb{Z}G_\mathfrak{P}$-module $U_\mathfrak{p}$ by

$$U_\mathfrak{p} := \langle N_{I_\mathfrak{P}}, 1 - \varepsilon_\mathfrak{p} \phi_\mathfrak{P}^{-1} \rangle_{\mathbb{Z}G_\mathfrak{P}} \subset \mathbb{Q}G_\mathfrak{P},$$

where we recall that $\varepsilon_\mathfrak{p} = |I_\mathfrak{P}|^{-1} N_{I_\mathfrak{P}}$. Note that $U_\mathfrak{p} = \mathbb{Z}G_\mathfrak{P}$ if $\mathfrak{p}$ is unramified in L/K such that the definition of the following $\zeta(\mathbb{Z}G)$-module is indeed independent of the set S as long as S contains the ramified primes:

$$U := \langle \prod_{\mathfrak{p} \in S \setminus S_\infty} \mathrm{nr}(u_\mathfrak{p}) | u_\mathfrak{p} \in U_\mathfrak{p} \rangle_{\zeta(\mathbb{Z}G)} \subset \zeta(\mathbb{Q}G).$$

Definition 5.1. Let S be a finite set of primes which contains $S_{\mathrm{ram}} \cup S_\infty$. We define a $\zeta(\mathbb{Z}G)$-module by

$$SKu(L/K,S) := \mathfrak{A}_S \cdot U \cdot L(0)^\sharp \subset \zeta(\mathbb{Q}G).$$

We call $SKu(L/K) := SKu(L/K, S_{\mathrm{ram}} \cup S_\infty)$ the (fractional) *Sinnott-Kurihara ideal.*

For abelian G, this definition coincides with the Sinnott-Kurihara ideal $SKu(L/K)$ in [Gr07] (see also [Si80], p. 193).
Let $\mathcal{I}(G)$ be the $\zeta(\mathbb{Z}G)$-module generated by the elements $\mathrm{nr}(H)$, $H \in M_{n \times n}(\mathbb{Z}G)$, $n \in \mathbb{N}$. Actually, $\mathcal{I}(G)$ is a commutative ring and we have inclusions

$$\zeta(\mathbb{Z}G) \subset \mathcal{I}(G) \subset \zeta(\mathfrak{M}(G)),$$

where $\mathfrak{M}(G)$ is a maximal order in $\mathbb{Q}G$. We now state the following integrality conjecture:

Conjecture 5.2. The Sinnott-Kurihara ideal $SKu(L/K)$ is contained in $\mathcal{I}(G)$.

Remark 1. i) Since clearly $SKu(L/K,S) \subset SKu(L/K,S')$ if $S' \subset S$, Conjecture 5.2 implies $SKu(L/K,S) \subset \mathcal{I}(G)$ for all admissible sets S.

ii) If the sets S and T satisfy $Hyp(S,T)$, the Stickelberger element θ_S^T is contained in $SKu(L/K,S)$. Hence Conjecture 5.2 predicts that $\theta_S^T \in \mathcal{I}(G)$ which is part of [Nic], Conjecture 2.1.

iii) In the above definitions, we may replace $\mathbb{Z}$ and $\mathbb{Q}$ by $\mathbb{Z}_p$ and $\mathbb{Q}_p$, respectively. We obtain a local Sinnott-Kurihara ideal $SKu_p(L/K)$ contained in $\zeta(\mathbb{Q}_pG)$ and a $\zeta(\mathbb{Z}_pG)$-module $\mathcal{I}_p(G)$. Since we have an equality

$$\mathcal{I}(G) = \bigcap_p \mathcal{I}_p(G) \cap \zeta(\mathbb{Q}G),$$

we have an equivalence

$$SKu(L/K) \subset \mathcal{I}(G) \iff SKu_p(L/K) \subset \mathcal{I}_p(G) \ \forall p.$$

If G is abelian, we obviously have $\mathcal{I}(G) = \zeta(\mathbb{Z}G) = \mathbb{Z}G$ and the results in [Ba77], [Ca79], [DR80] each imply the following theorem (cf. [Gr07], §2).

Theorem 5.3. *Conjecture 5.2 holds if L/K is an abelian CM-extension.*

6. The ETNC in almost tame extensions

Let us fix a finite Galois extension L/K of number fields with Galois group G and a finite set S of places of K which contains $S_{\mathrm{ram}} \cup S_\infty$. In [Bu01] the author defines the following element of $K_0(\mathbb{Z}G, \mathbb{R})$:

$$T\Omega(L/K, 0) := \psi_G^*(\chi_{G,\mathbb{R}}(\tau_S, \lambda_S^{-1}) + \hat{\partial}_G(L_S^*(0)^\sharp)).$$

Here, ψ_G^* is a certain involution on $K_0(\mathbb{Z}G, \mathbb{R})$ which is not important for our purposes, since we will be only interested in the nullity of $T\Omega(L/K, 0)$. Furthermore, $\tau_S \in \mathrm{Ext}_G^2(E_S, \Delta S)$ is Tate's canonical class (cf. [Ta66]), where ΔS is the kernel of the augmentation map $\mathbb{Z}S(L) \twoheadrightarrow \mathbb{Z}$ which maps each $\mathfrak{P} \in S(L)$ to 1. Finally, λ_S denotes the negative of the usual Dirichlet map, so $\lambda_S : \mathbb{R} \otimes E_S \to \mathbb{R} \otimes \Delta S$, $u \mapsto -\sum_{\mathfrak{P} \in S(L)} \log |u|_{\mathfrak{P}} \mathfrak{P}$, and $\chi_{G,\mathbb{R}}(\tau_S, \lambda_S^{-1})$ is the refined Euler characteristic associated to the perfect 2-extension whose extension class is τ_S, metrised by λ_S^{-1}. For more precise definitions we refer the reader to [Bu01]. The ETNC for the motive $h^0(L)$ with coefficients in $\mathbb{Z}G$ in this context asserts that the element $T\Omega(L/K, 0)$ is zero. Note that this statement is also equivalent to the Lifted Root Number Conjecture formulated by Gruenberg, Ritter and Weiss [GRW99] (cf. [Bu01], Th. 2.3.3).
It is also proven in loc.cit. that $T\Omega(L/K, 0)$ lies in $K_0(\mathbb{Z}G, \mathbb{Q})$ if and only if Stark's conjecture holds. In this case the ETNC decomposes into local conjectures at each prime p by means of the isomorphism

$$K_0(\mathbb{Z}G, \mathbb{Q}) \simeq \bigoplus_{p \nmid \infty} K_0(\mathbb{Z}_p G, \mathbb{Q}_p).$$

Now let L/K be a Galois CM-extension. Since Stark's conjecture is known for odd characters (cf. [Ta84], Th. 1.2, p. 70), $T\Omega(L/K, 0)$ has a well defined image $T\Omega(L/K, 0)_p^-$ in $K_0(\mathbb{Z}_p G_-, \mathbb{Q}_p)$. Recall that T consists of a prime $\mathfrak{p}_0 \nmid p$ and all finite places of K which ramify in L/K and do not lie above p, and we have chosen $\mathfrak{p}_0$ such that E_S^T is torsionfree. We have the following reformulation of [Nia], Th. 2.

Theorem 6.1. *Let p be an odd prime and L/K a Galois CM-extension which is almost tame above p. Then*

$$T\Omega(L/K, 0)_p^- = 0 \iff \mathrm{Fitt}_{\mathbb{Z}_p G_-}(A_L^T(p)) = [\langle \theta_{S_1}^T \rangle]_{\mathrm{nr}(\mathbb{Z}_p G_-)},$$

where S_1 denotes the set of all wildly ramified primes above p.

We have the following connection to the integrality conjecture 5.2 (cf. [Nic], proof of Th. 5.1 and Cor. 5.6):

Theorem 6.2. *Let p be an odd prime and L/K a Galois CM-extension and assume that $T\Omega(L/K, 0)_p^-$ vanishes. If the p-part of the roots of unity of L is a c.t. G-module or if L/K is almost tame above p, then the p-part of Conjecture 5.2 holds, i.e. $SKu_p(L/K) \subset \mathcal{I}_p(G)$.*

The aim of this section is to prove a partial reverse of this theorem for almost tame extensions.

Lemma 6.3. *Let p be an odd prime and L/K a Galois CM-extension which is almost tame above p. Assume that the Iwasawa μ-invariant vanishes and that the EIMC holds for the extension L_∞^+/K. Then*

$$\mathrm{Fitt}_{\mathbb{Z}_p G_-}(\mathcal{X}_T^- / (\gamma_L - 1)) = [\langle \theta_{S_p}^T \rangle]_{\mathrm{nr}(\mathbb{Z}_p G_-)}.$$

Proof. By Theorem 4.4, the Fitting invariant of $\mathcal{X}_T^-$ over $\Lambda(\mathcal{G})_-$ is generated by Ψ_Σ, where we put $\Sigma = T \cup S_p$. Now [Nib], Th. 6.4 implies that $\mathrm{Fitt}_{\mathbb{Z}_p G_-}(\mathcal{X}_T^- / (\gamma_L - 1))$ is generated by

$$\sum_{\chi \in \mathrm{Irr}\,(G)} \mathrm{aug}_{\Gamma_L}(j_\chi(\Psi_\Sigma)) e_\chi. \tag{17}$$

But using Corollary 4.6 we compute

$$\begin{aligned} j_\chi(\Psi_\Sigma) &= \left(\prod_{\mathfrak{p}\in T} j_\chi(\xi_\mathfrak{p})\right) \cdot j_\chi(\dot{\Phi}_\Sigma) \\ &= \left(\prod_{\mathfrak{p}\in T} j_\chi(\mathrm{nr}(\Xi_\mathfrak{p} \cdot (1-\phi_\mathfrak{P}^{-1}\varepsilon_\mathfrak{p})))\right) L_{K,\Sigma_\chi}(\chi^{-1}\omega) \\ &= \left(\prod_{\mathfrak{p}\in T} j_\chi(\mathrm{nr}(\varepsilon_\mathfrak{p}(1-N(\mathfrak{p})\phi_\mathfrak{P}^{-1}) + 1 - \varepsilon_\mathfrak{p}))\right) L_{K,\Sigma_\chi}(\chi^{-1}\omega). \end{aligned}$$

Hence (17) equals

$$\sum_{\chi\in\mathrm{Irr}\,(G)} \left(\prod_{\mathfrak{p}\in T} \det(1-N(\mathfrak{p})\phi_\mathfrak{P}^{-1}|V_\chi^{I_\mathfrak{P}})\right) L_{\Sigma_\chi}(0,\chi^{-1}) = \theta_{S_p}^T.$$

□

We define an element $\alpha_p \in \zeta(\mathbb{Q}_p G_-)$ by

$$\alpha_p = \prod_{\mathfrak{p}\in S_p\setminus S_1} \mathrm{nr}(1-\varepsilon_\mathfrak{p}\phi_\mathfrak{P}^{-1})$$

such that we have an equality $\theta_{S_1}^T \cdot \alpha_p = \theta_{S_p}^T$. We start with the following special case, where we get Conjecture 5.2 for free.

Proposition 6.4. *Let p be an odd prime and L/K a Galois CM-extension such that $j \in G_\mathfrak{P}$ for all $\mathfrak{P}$ above p. Assume that the Iwasawa μ-invariant vanishes and that the EIMC holds for the extension L_∞^+/K. Then $T\Omega(L/K,0)_p^- = 0$ and the p-part of Conjecture 5.2 holds.*

Proof. As before, the canonical restriction map $\mathcal{X}_T^- \to A_L^T(p)$ is surjective. By [Nib], Prop. 3.5 (i) this implies

$$\mathrm{Fitt}_{\mathbb{Z}_pG_-}(\mathcal{X}_T^-/(\gamma_L-1)) \subset \mathrm{Fitt}_{\mathbb{Z}_pG_-}(A_L^T(p)).$$

Since we have $j \in G_\mathfrak{P}$ for all $\mathfrak{P}$ above p by assumption, the element α_p lies in $\mathrm{nr}(K_1(\mathbb{Z}_pG_-))$ and thus Lemma 6.3 implies $\theta_{S_1}^T \in \mathrm{Fitt}_{\mathbb{Z}_pG_-}(A_L^T(p))$. In particular, we have $\theta_{S_1}^T \in \mathcal{I}_p(G)$. Let E be a splitting field of $\mathbb{Q}_pG$. Since $\theta_{S_1}^T = (\delta_T(0,\chi)L_{S_1}(0,\chi^{-1}))_\chi$ and

$$|A_L^T(p)| = x \cdot \prod_{\substack{\chi\in\mathrm{Irr}\,(G)\\ \chi \text{ odd}}} (\delta_T(0,\chi)L_{S_1}(0,\chi^{-1}))^{\chi(1)}$$

with an appropriate unit $x \in \mathfrak{o}_E^\times$ by [Nia], Prop. 4, the Stickelberger element $\theta_{S_1}^T$ is actually a generator of $\mathrm{Fitt}_{\mathbb{Z}_pG_-}(A_L^T(p))$ by [Nib], Prop. 5.4. Now Theorem 6.1 implies the vanishing of $T\Omega(L/K,0)_p^-$ which also implies Conjecture 5.2 by Theorem 6.2. □

Let us denote the normal closure of L over $\mathbb{Q}$ by L^{cl} which is again a CM-field. We will henceforth make the following additional assumption:

$$L^{\mathrm{cl}} \not\subset (L^{\mathrm{cl}})^+(\zeta_p).$$

Note that this assumption fails only for finitely many primes p, since such a p has to ramify in $L^{\mathrm{cl}}/\mathbb{Q}$.

Lemma 6.5. *Let $N > 0$ be a natural number. Then there are infinitely many primes $r \in \mathbb{Z}$ such that*

– $r \equiv 1 \bmod p^N$.

– *$j \in G_{\mathfrak{R}}$ for all primes $\mathfrak{R}$ in L above r.*
– *The Frobenius automorphism* Frob_p *at p in* $\mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q})$ *generates* $\mathrm{Gal}(k_r/\mathbb{Q})$, *where k_r denotes the unique subfield of $\mathbb{Q}(\zeta_r)$ of degree p^N over $\mathbb{Q}$.*

Proof. The proof of [Gr00], Prop. 4.1 carries over unchanged to the present situation. □

Let $N \in \mathbb{N}$ be large and choose a prime r as in Lemma 6.5 which does not ramify in $L^{\mathrm{cl}}/\mathbb{Q}$. We put $L' := Lk_r$, $K' = Kk_r$ and $G' = \mathrm{Gal}(L'/K) = G \times C_N$, where $C_N \simeq \mathrm{Gal}(k_r/\mathbb{Q})$ is cyclic of order p^N, generated by Frob_p. Note that L'/K is again almost tame above p. Moreover, we define $T' := T \cup S_r$, where S_r denotes the set of places in K above r. Using the same arguments as in [Nia] following Prop. 9 we have an isomorphism

$$A_{L'}^{T'}(p) \simeq A_{L'}^{T}(p)$$

and hence $A_{L'}^{T}(p)$ is G'-c.t. by loc.cit., Th. 1. As in loc.cit. the restriction map induces an isomorphism

$$(A_{L'}^{T}(p))_{C_N} \simeq A_L^T(p). \tag{18}$$

We will need the following lemma.

Lemma 6.6. *Assume that G' is a direct product of a group G and an abelian group C. Then we have $|G| \cdot \mathcal{I}_p(G') \subset \zeta(\mathbb{Z}_p G')$ for all primes p.*

Proof. Choose a maximal order $\mathfrak{M}(G)$ containing $\mathbb{Z}_p G$. Then $\mathfrak{M}(G)$ is a direct sum of matrix rings of type $M_{n\times n}(\mathfrak{o}_D)$, where $\mathfrak{o}_D$ denotes the valuation ring of a skew field D. We have

$$\zeta(M_{n\times n}(\mathfrak{o}_D)) = \zeta(\mathfrak{o}_D) = \mathfrak{o}_F,$$

where $\mathfrak{o}_F$ is the ring of integers of the field $F = \zeta(D)$ which is finite over $\mathbb{Q}_p$. Since the reduced norm maps $\mathfrak{M}(G)$ into its center and $|G| \cdot \zeta(\mathfrak{M}(G)) \subset \zeta(\mathbb{Z}_p G)$, it suffices to show that the reduced norm maps $M_{m\times m}(M_{n\times n}(\mathfrak{o}_D)[C])$ into $\mathfrak{o}_F[C]$. Let us at first assume that $D = F$. Then the map

$$\begin{aligned} \sigma : M_{n\times n}(F)[C] &\longrightarrow M_{n\times n}(F[C]) \\ \sum_{c\in C} M_c c &\mapsto (\sum_{c\in C} \alpha_{ij}(c)c)_{i,j} \end{aligned}$$

is an isomorphism of rings, where $M_c = (\alpha_{ij}(c))_{i,j}$ lies in $M_{n\times n}(F)$. Likewise, σ induces an isomorphism

$$\sigma : M_{n\times n}(\mathfrak{o}_F)[C] \simeq M_{n\times n}(\mathfrak{o}_F[C]).$$

Therefore, we have

$$\mathrm{nr}(M_{m\times m}(M_{n\times n}(\mathfrak{o}_F)[C])) = \mathrm{nr}(M_{nm\times nm}(\mathfrak{o}_F[C])) = \mathrm{nr}(\mathfrak{o}_F[C]) = \mathfrak{o}_F[C].$$

For arbitrary D, there is a field E, Galois over F such that $E \otimes_F D \simeq M_{s\times s}(E)$ for some integer s. We have just proven that the reduced norm maps $M_{m\times m}(M_{n\times n}(\mathfrak{o}_D)[C])$ into $\mathfrak{o}_E[C]$. But the image is invariant under the action of $Gal(E/F)$ and is therefore contained in $\mathfrak{o}_F[C]$. □

Let $\alpha'_p \in \zeta(\mathbb{Q}_p G')$ be defined analogously to α_p such that $\theta_{S_1}^{T'} \cdot \alpha'_p = \theta_{S_p}^{T'}$. Now choose a second natural number $M \leqslant N$ and put

$$\nu := \sum_{i=0}^{p^M-1} \mathrm{Frob}_p^{ip^{N-M}} \in \mathbb{Z}_p C_N \subset \zeta(\mathbb{Z}_p G').$$

Lemma 6.7. *Let f be the least common multiple of the residual degrees $f_{\mathfrak{p}}(K/\mathbb{Q})$ of all $\mathfrak{p} \in S_p$. If $N - M \geqslant v_p(|G| \cdot f)$, then $|G| \cdot \alpha'_p$ is a nonzerodivisor in $\zeta(\mathbb{Z}_p G')/\nu$.*

Proof. We first observe that Lemma 6.6 implies that $|G| \cdot \alpha_p'$ lies in $\zeta(\mathbb{Z}_p G')$. Since $\mathbb{Z}_p C_N/\nu$ and likewise $\zeta(\mathbb{Z}_p G')/\nu$ are reduced rings, we have to show that no minimal prime of $\zeta(\mathbb{Z}_p G')$ contains both, $|G| \cdot \alpha_p'$ and ν. The minimal primes are given by

$$\mathfrak{p}_{\chi'} := \{x \in \zeta(\mathbb{Z}_p G') | \chi'(x) = 0\}, \ \chi' \in \mathrm{Irr}\,(G').$$

We may write χ' as a product $\chi \cdot \chi_N$ of irreducible characters χ of G and χ_N of C_N; then $\chi'(\mathrm{Frob}_p) = \chi(1) \cdot \zeta_{p^s}$ for some $s \leqslant N$. Assume that $\nu \in \mathfrak{p}_{\chi'}$; hence $0 = \chi'(\nu) = \chi(1) \sum_{i=0}^{p^M-1} \zeta_{p^s}^{ip^{N-M}}$. But since $\chi(1) \neq 0$, this implies $s > N - M$. If also $|G| \cdot \alpha_p' \in \mathfrak{p}_{\chi'}$, there is a prime $\mathfrak{p} \in S_p$ and a prime $\mathfrak{P}'$ in L' above $\mathfrak{p}$ such that the inertia group at $\mathfrak{P}'$ acts trivially on $V_{\chi'}$ and $\det(1 - \phi_{\mathfrak{P}'}^{-1} | V_\chi^{I_{\mathfrak{P}'}})$ vanishes. But this determinant is a product of some $1 - \zeta \cdot \zeta_{p^s}^{-f_\mathfrak{p}}$, where ζ is a root of unity of order dividing $|G|$ and, by assumption, we have $v_p(\mathrm{ord}\,(\zeta_{p^s}^{f_\mathfrak{p}})) = \frac{s}{v_p(f_\mathfrak{p})} > \frac{N-M}{v_p(f_\mathfrak{p})} \geqslant v_p(|G|)$. This is a contradiction. □

We are ready to prove the main result of this section which generalizes [Nia], Th. 4.

THEOREM 6.8. *Let p be an odd prime and L/K a Galois CM-extension which is almost tame above p. Assume that the Iwasawa μ-invariant vanishes and that $L^{\mathrm{cl}} \not\subset (L^{\mathrm{cl}})^+(\zeta_p)$. Moreover assume that for each integer M there is an integer $N \geqslant M$ such that there is a prime $r = r(N)$ as in Lemma 6.5, unramified in $L^{\mathrm{cl}}/\mathbb{Q}$ such that the p-part of Conjecture 5.2 is true for L'/K and the EIMC holds for the extension $(L'_\infty)^+/K$. Then $T\Omega(L/K,0)_p^- = 0$. In particular, the p-parts of the following conjectures hold:*

- i) *the strong Stark conjecture for odd characters as formulated by T. Chinburg [Ch83], Conj. 2.2.*
- ii) *the (weak) non-abelian Brumer conjecture of [Nic], Conj. 2.1 and 2.3.*
- iii) *the (weak) non-abelian Brumer-Stark conjecture of [Nic], Conj. 2.6 and 2.7.*
- iv) *the weak non-abelian strong Brumer-Stark conjecture of [Nic], Conj. 3.6.*

Moreover, L/K fulfills the non-abelian strong Brumer-Stark property at p (cf. [Nic], Def. 3.5).

Remark 2. i) Since the EIMC as well as Conjecture 5.2 are known for abelian Galois groups, it seems to be likely that we can prove these conjectures attached to the extensions L'/K if so for L/K. For example, the EIMC is known to be true for Galois groups $\mathcal{G}$ which are pro-p and have an abelian subgroup of index p (cf. [RW08]). But if this is true for $Gal(L_\infty^+/K)$, then also for $Gal((L'_\infty)^+/K)$.

ii) Since the strong Stark conjecture at p is a theorem for odd characters in the case at hand (cf. [Nia], Cor. 2), it follows from the results in [Nic] that the weak variants of the above conjectures are true unconditionally (cf. loc.cit., Cor. 4.2).

Proof of Theorem 6.8. We first observe that the Iwasawa μ-invariant does not change if we enlarge L to L' by [NSW00], Th. 11.3.8. Now choose natural numbers $M \leqslant N$ such that $r = r(N)$ fulfills the above conditions and $N - M \geqslant v_p(|G| \cdot f)$, where f was defined in Lemma 6.7. Let $\mathcal{G}' = \mathrm{Gal}(L'_\infty/K)$ and let $\mathcal{X}_{T'}^-$ be the projective limit of the minus p-ray class groups $A_{L'_n}^{T'}(p)$. Then $\mathcal{X}_{T'}^-$ has projective dimension at most one and the EIMC for the extension $(L'_\infty)^+/K$ implies

$$\mathrm{Fitt}_{\Lambda(\mathcal{G}')_-}(\mathcal{X}_{T'}^-) = \left[\langle \Psi_{T' \cup S_p} \rangle\right]_{\mathrm{nr}(\Lambda(\mathcal{G}')_-)}.$$

For each prime $\mathfrak{p}$ of K let $\mathfrak{P}' \subset L'$ be a prime above $\mathfrak{p}$. By Proposition 4.7, we have a right exact sequence

$$\bigoplus_{\mathfrak{p} \in S_p} \mathrm{ind}_{G_{\mathfrak{P}'}}^{G} \mathbb{Z}_p \to (\mathcal{X}_{T'}^-)_{\Gamma_{L'}} \twoheadrightarrow A_{L'}^{T'}(p).$$

The Fitting invariant of the leftmost term is generated by α_p', whereas $\theta_{S_p}^{T'} = \theta_{S_p}^{T'}(L'/K)$ is a generator of $\mathrm{Fitt}_{\mathbb{Z}_p G'_-}((\mathcal{X}_{T'}^-)_{\Gamma_{L'}})$ by Lemma 6.3. Since $j \in G_{\mathfrak{R}}$ for all primes above r, we may replace $\theta_{S_p}^{T'}$ by

$\theta_{S_p}^T$. The above sequence gives rise to the following inclusion of Fitting invariants (cf. [Nib], Prop. 3.5 (iii)):

$$\mathrm{Fitt}_{\mathbb{Z}_pG'_-}\left(\bigoplus_{\mathfrak{p}\in S_p}\mathrm{ind}_{G_{\mathfrak{P}'}}^{G}\mathbb{Z}_p\right)\cdot\mathrm{Fitt}_{\mathbb{Z}_pG'_-}(A_{L'}^{T'}(p))\subset\mathrm{Fitt}_{\mathbb{Z}_pG'_-}((\mathcal{X}_{T'}^-)_{\Gamma_{L'}}).$$

If we choose a generator ξ' of $\mathrm{Fitt}_{\mathbb{Z}_pG'_-}(A_{L'}^{T'}(p))$, there exists $x\in\zeta(\mathbb{Z}_pG')$ such that

$$\alpha_p'\xi' = x\cdot\theta_{S_p}^T = x\cdot\alpha_p'\theta_{S_1}^T.$$

It follows from Lemma 6.6 that multiplication by $|G|^2$ yields an equality in $\zeta(\mathbb{Z}_pG')$ (since Conjecture 5.2 holds by assumption) such that Lemma 6.7 gives

$$|G|\cdot\xi'\equiv|G|\cdot x\cdot\theta_{S_1}^T \bmod \nu. \tag{19}$$

Let $\mathrm{aug}:\mathbb{Z}_pG'\to\mathbb{Z}_pG$ be the natural augmentation map. Since Fitting invariants behave well under base change (cf. [Nib], Lemma 5.5), the element $\xi:=\mathrm{aug}(\xi')$ generates the Fitting invariant of $A_L^T(p)$ by (18). But since $\mathrm{aug}(\theta_{S_p}^T(L'/K))=\theta_{S_p}^T(L/K)$ and $\mathrm{aug}(\nu)=p^M$, equation (19) implies

$$\xi\equiv\mathrm{aug}(x)\cdot\theta_{S_1}^T(L/K) \bmod p^{M-m}\mathcal{I}_p(G),$$

where p^m is the exact power dividing $|G|$. This gives an inclusion

$$\mathrm{Fitt}_{\mathbb{Z}_pG}(A_L^T(p))\subset[\langle\theta_{S_1}^T\rangle]_{\mathrm{nr}(\mathbb{Z}_pG)},$$

as we may choose M arbitrarily large. Now we can conclude as in Proposition 6.4 that $\theta_{S_1}^T$ is in fact a generator of $\mathrm{Fitt}_{\mathbb{Z}_pG}(A_L^T(p))$ and we are done via Theorem 6.1. □

Remark 3. Note that we have not used the whole statement of Conjecture 5.2. It suffices to assume that the denominators of the elements $\theta_{S_1}^T(L'/K)$ for varying $r=r(N)$ are bounded, independently of N.

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