Canadian Mathematical Society
Conference Proceedings
Volume 4 (1984)

An invitation to real spectra

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Michel Coste and Marie-Françoise Coste-Roy have associated to every commutative ring A with unit a locally ringed space Sper A = $(X, \mathcal{O}_X)$, called the "real spectrum" of A ([CC_1], [CC_2], [C-R_1]). The real spectrum seems to reflect well the "semialgebraic" properties of A, in particular the phenomena depending on inequalities between elements of A for orderings of the residue class fields $A(\mathfrak{p}) = \mathrm{Quot}(A/\mathfrak{p})$ of the various prime ideals $\mathfrak{p}$ of A. (Remark: My notation Sper A for the real spectrum seems to be somewhat new. "Sper" means, read in French, "spectre réel". Read in English or German, it means "spears". We shall see in §3 that in fact the topological space X is a union of spears.)

The real spectra Sper A, more precisely their constructible subsets (cf. §3 below), seem to be the building blocks of an abstract semialgebraic geometry in much the same way as the usual spectra Spec A, invented by Grothendieck, are the building blocks of abstract algebraic geometry. I am sure that the Costes have found the "right" notion of real spectrum, and I conjecture that real spectra are the medium in which most interactions between quadratic form theory and semialgebraic geometry will take place in the years to come. In this talk I want to give some evidence for both these assertions.

My talk is much less than an introduction to real spectra. Although real spectra still are a rather unknown territory there already exist good introductions to this theory for non specialists. I mention the two articles [CC] and [C-R] of the Costes, §4 and §7 of Lam's article [L], and the first two sections of Bröcker's article [B]. Quadratic form theorists who have already gathered experience with orderings and real algebraic geometry should ignore most parts of my talk and proceed directly to those articles and the papers cited here.

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§1 - Some reminiscences

Before entering the subject of real spectra it might be appropriate to give an example of how semialgebraic geometry can be useful for quadratic form theory. I will choose an example from my own history and relate how I stumbled into semialgebraic geometry trying to solve a special problem on Witt rings of quadratic forms. I suspect that other Q.F. people have had similar experiences.

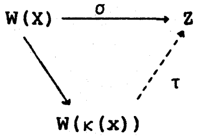
Let $W(X)$ denote the Wittring of symmetric bilinear forms on a quasiprojective scheme X (or more generally a divisorial scheme, cf.[K]). Around 1975 it was known that every prime ideal P of $W(X)$ is the inverse image of a prime ideal Q of the Wittring $W(\kappa(x))$ of the residue class field $\kappa(x)$ at some point $x \in X$ under the restriction homomorphism $W(X) \to W(\kappa(x))$. (This theory goes back to Dress [Dr] and Kanzaki-Kitamura [KK], cf. the exposition in [K]). Thus if P is not maximal,

$$P = \{\xi \in W(X) \mid \mathrm{sign}_\alpha(\xi) = 0\},$$

where sign_α denotes the Sylvester signature of ξ at some ordering α of $\kappa(x)$. In other words, the non-maximal prime ideals P of $W(X)$ correspond uniquely to the signatures of X, i.e. the ring homomorphisms $\sigma : W(X) \to \mathbb{Z}$, by the relation $P = \mathrm{Ker}(\sigma)$; and every signature σ of X factors through a signature of $\kappa(x)$ for some $x \in X$. Around 1975-80 I thought about a proof of the following more refined theorem.

<u>Theorem 1.1.</u> Suppose X is algebraic over some field k and let $\sigma : W(X) \to \mathbb{Z}$ be a signature of X. Then there exists some <u>closed</u>

point x of X such that σ factors through a signature of $\kappa(x)$,



I could prove this 1976 in the case $k = \mathbb{R}$ by use of Whitney's theorem that the set $X(\mathbb{R})$ of rational points of X has only finitely many connected components [K, Chap. V]. Already then it was apparent that the theorem could be proved for <u>any</u> field k by base extension to a suitable real closure of k, as soon as we had at our disposal an analogue of Whitney's theorem for an arbitrary real closed field R instead of $\mathbb{R}$. (N.B. If k is not formally real the theorem is trivial since then X has no signatures).

Now, whenever $R \neq \mathbb{R}$, the set $X(R)$ of rational points of an algebraic scheme X over R is totally disconnected in the strong topology. (This is the topology coming from the ordering of R). So Whitney's theorem cannot hold in a naive sense. Realizing this, I sorely felt the need to develop a meaningful geometry over an arbitrary real closed field R instead just the field $\mathbb{R}$ of real numbers. After a lot of experiments with curves over R, Hans Delfs and I established a theory of semialgebraic paths in $X(R)$, and we could prove that $X(R)$ has only finitely many path components. More generally, we could do this for any semialgebraic subset M of $X(R)$ instead of $X(R)$ itself [DK]. Then we could prove Theorem 1.1. in general by the same method as in the case $R = \mathbb{R}$ [DK, I §5].

In the meantime G.Brumfiel and the Costes independently had obtained similar results avoiding paths, namely that every semialgebraic set over R has only finitely many "semialgebraic connected components" ([Br, p. 260ff],[C-R_1, p. 46ff]). Their results also suffice to prove Theorem 1.1., but due to lack of communication we became aware of their work only much later.

These three approaches to connectedness have the same underlying philosophy - namely that in X(R) only semialgebraic subsets should be admitted as reasonable subsets. I recall that a subset M of X(R) is defined to be semialgebraic if for any Zariski-open subset U of X we have

$$M \cap U(R) = \bigcup_{i=1}^{r} \{x \in U(R) \mid f_{i0}(x) = 0,\ f_{i1}(x) > 0, \ldots\ldots, f_{i,s(i)}(x) > 0\}$$

with algebraic functions $f_{ij} \in R[U]$. (If X is affine, it suffices to check this for U = X). A semialgebraic subset M of X(R) has to be regarded as connected if M is not the disjoint union of two non empty <u>semialgebraic</u> subsets M_1, M_2 which are open in M (in the strong topology). Every semialgebraic subset M of X(R) is the disjoint union of finitely many open connected semialgebraic subsets. These connected components turn out to be the same as our path components.

Delfs and I developed a general framework for these and other semialgebraic considerations, the theory of "semialgebraic spaces" over R. These are suitable locally ringed spaces (not quite in the classical sense, cf. [DK, II §7]). An affine semialgebraic space is, up to isomorphism, just a semialgebraic subset M of some R^n, equipped with the sheaf $\mathcal{O}_M$ of continuous R-valued functions which have semialgebraic graphs, cf. [DK, II §7]. Every semialgebraic subset M of X(R), for X an algebraic scheme over R "is" an affine semialgebraic space [DK_2, §3]. Regarding M as a semialgebraic space means roughly to forget the embedding $M \hookrightarrow X(R)$.

Our theory of path components may be considered as the first step in semialgebraic homotopy theory, namely as the theory of $\pi_o(M)$. How about groups $\pi_q(M,x_o)$ for $q \geq 1$? Delfs and I later established a satisfactory theory of these groups (cf. [DK_3], [DK_4, Chap. III]), and also of semialgebraic homology and cohomology groups $H_q(M,G)$, $H^q(M,G)$, with G an arbitrary abelian group of coefficients ([D], [DK_1], [D_2]). Quite recently Delfs more generally developed (co)homology with arbitrary support, which allows him to do intersection theory on semialgebraic manifolds, and to describe the algebraic intersection theory on an algebraic scheme over $R(\sqrt{-1})$ in a purely semialgebraic way (cf. [D_1]). We also obtained a satisfactory orthogonal K-theory of affine semialgebraic spaces. Analogous to topological K-theory, $KO(M) = KO^o(M)$ is defined as the Grothendieck-ring of semialgebraic R-vector-bundles on M. (Such a vector bundle has to be trivial on a finite covering of M by open semialgebraic subsets.)

Lack of time prevents me from going into the details of any of these theories. I just mention that in every one of them we have as central result a "main theorem", which connects the theory to the corresponding classical topological theory. These main theorems all follow the same pattern (while the proofs are very different, mostly long and intricate). I want to state the main theorem for KO(M) here, since KO(M) is particularly interesting for quadratic form theory, as we shall see later. This needs some preparation.

If $\tilde{R}$ is a real closed overfield of R then we have a functor "base extension" $M \mapsto M(\tilde{R})$ from the category of semialgebraic spaces over R to the category of semialgebraic spaces over $\tilde{R}$. In the case that M is a semialgebraic subset of R^n the space $M(\tilde{R})$ is the semialgebraic subset of $\tilde{R}^n$ described by the same polynomial relations (equalities and strict inequalities) as M. (N.B. The polynomials have their coefficients in R and thus can be read as polynomials over $\tilde{R}$. By Tarski's principle the subset $M(\tilde{R})$ of $\tilde{R}^n$ is independent of the chosen description of M.)

For any semialgebraic vector bundle E over M we obtain by base extension a semialgebraic vector bundle $E(\tilde{R})$ over $M(\tilde{R})$. This yields a natural ring homomorphism $KO(M) \to KO(M(\tilde{R}))$. In the case $R = \mathbb{R}$ we may regard a semialgebraic real vector bundle as a topological vector bundle on the underlying topological space M_{top} of M. We thus obtain a natural ring homomorphism $KO(M) \to KO(M)_{top}$ from KO(M) to the classical KO of M_{top}.

<u>Theorem 1.2.</u> (Main theorem for KO). Let M be an affine semialgebraic space over R.

a) For any real closed overfield $\tilde{R}$ of R the natural homomorphism $KO(M) \to KO(M(\tilde{R}))$ is an isomorphism.

b) In the case $R = \mathbb{R}$ the natural homomorphism $KO(M) \to KO(M)_{top}$ is an isomorphism.

Let me indicate how this theorem in principle reduces the computation of KO(M) to the computation of a classical KO-ring. We assume for simplicity that M is also complete, i.e. isomorphic to a bounded closed semialgebraic subset of some R^n (cf. [DK, II §9]). Then M can be triangulated by a finite simplicial complex [DK_1, §2]. This means that M is isomorphic to the realization $|L|_R$ over R of a finite abstract simplicial complex (= simplicial complex in [Sp],= simplicial scheme in [Go]), where of course $|L|_R$ has to be regarded as a semialgebraic space. Now R and $\mathbb{R}$ both contain the field R_o of real algebraic numbers, and $|L|_R$, $|L|_{\mathbb{R}}$ are obtained from $|L|_{R_o}$ by base extension to R and $\mathbb{R}$ respectively. According to the main theorem $KO(M) \cong KO(|L|_R) \cong KO(|L|_{R_o}) \cong KO(|L|_{\mathbb{R}}) \cong KO(|L|_{\mathbb{R}})_{top}$.

Of course, this reduction to topological K-theory is most often of no practical value, since it is usually difficult to write down an explicit isomorphism $|L|_R \xrightarrow{\sim} M$, and anyway computing the topological KO of an explicitly given simplicial complex may be much too complicated.

If, for example, V is an algebraic variety defined over R_o in the classical sense then we have, for any real closed field R, by the main theorem $KO(V(R)) \cong KO(V(\mathbb{R}))_{top}$. (Here V(R) is the set of R-rational points on V regarded as a semialgebraic space.) So, in this case the main theorem gives a reduction to the classical theory which is useful in practice.

§2 - Definition of the real spectrum

We start with local considerations.

Definition 1. Let A be a commutative ring (always with 1). An *ordering* P *of* A is a subset $P \subset A$ with $P + P \subset P$, $P \cdot P \subset P$, $P \cup (-P) = A$, and $P \cap (-P) = \mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of A, called the *support* $\operatorname{supp} P$ of P.

Clearly, an ordering P of A with support $\mathfrak{p}$ is the preimage of an ordering $\overline{P}$ of the residue class field $A(\mathfrak{p}) = \operatorname{Quot}(A/\mathfrak{p})$, uniquely determined by P, under the natural homomorphism $A \to A(\mathfrak{p})$. In this way the orderings of A correspond bijectively to the pairs $(\mathfrak{p}, \overline{P})$, where $\mathfrak{p}$ runs through the points of Spec A and $\overline{P}$ runs through the orderings (in the usual sense) of $A(\mathfrak{p})$. For elements f,g of A we write $f >_P g$ (resp. $f \geq_P g$) if the images $\overline{f}, \overline{g}$ in $A(\mathfrak{p})$ fulfill the relation $\overline{f} > \overline{g}$ (resp. $\overline{f} \geq \overline{g}$) with respect to $\overline{P}$. Clearly $f \geq_P 0$ iff $f \in P$ and $f >_P 0$ iff $f \notin -P$.

If $\varphi : A \to B$ is a ring homomorphism (always with $\varphi(1) = 1$), then every ordering Q of B yields an ordering $P := \varphi^{-1}(Q)$ of A. For elements f,g in A we have $f \geq_P g$ iff $\varphi(f) \geq_Q \varphi(g)$, thus also $f >_P g$ iff $\varphi(f) >_Q \varphi(g)$. Notice that $\operatorname{supp} P = \varphi^{-1}(\operatorname{supp} Q)$.

Definition 2*). A local ring $(B, \mathfrak{m})$ is called *strictly real* if B is henselian and $B/\mathfrak{m}$ is a real closed field.

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*) This and other definitions should be regarded as ad hoc. The subject has not yet aged enough for a generally accepted terminology.

If $(B,\mathfrak{m})$ is strictly real then B has a unique ordering P(B) with support $\mathfrak{m}$. Relations $f>g$ or $f\geq g$ in B without further specification are meant with respect to P(B). If $\varphi(T)$ is any normed polynomial in one variable T with coefficients in B, and if

$$\bar{\alpha}_1 < \bar{\alpha}_2 < \ldots < \bar{\alpha}_r$$

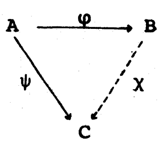
are the roots of the mod$\mathfrak{m}$ reduced polynomial $\bar{\varphi}(T) \in (B/\mathfrak{m})[T]$ in the real closed field $B/\mathfrak{m}$, and if all these roots are simple, then - by the henselian property of B - every $\bar{\alpha}_i$ has a unique lifting α_i to a root of $\varphi(T)$ in B,

$$\alpha_1 < \alpha_2 < \ldots < \alpha_r,$$

and these are all the roots of $\varphi(T)$ in B. In particular $(\varphi(T) = T^2-b)$, for any $b \in B$ we have $b>0$ iff $b \notin \mathfrak{m}$ and $b=c^2$ for some $c \in B$. In this case there exists a unique $c \in B$ with $c>0$ and $c^2 = b$.

Proposition 2.1. Given a ring A and an ordering P of A there exists a universal homomorphism $\varphi: A \to B$ of A into a strictly real local ring B with $\varphi^{-1}(P(B)) = P$.

Here the word "universal" means, as usual, the following. Given any homomorphism $\psi: A \to C$ of A into a strictly real local ring C with $\psi^{-1}(P(C)) = P$ there exists a unique local homomorphism $\chi: B \to C$ with $\chi \circ \varphi = \psi$,



Of course, since χ is local, we have $\chi^{-1}(P(C)) = P(B)$.

Proposition 2.1. is well known. Let $\mathfrak{y} = \text{supp}\, P$ and let $A(P)$ denote the real closure of $A(\mathfrak{y})$ with respect to $\bar{P}$. One takes for B the strict henselization of the local ring $A_{\mathfrak{y}}$ with respect to the extension $A(P)$ of its residue class field $A(\mathfrak{y})$ (cf. [R, Chap. VIII §2], there replacing the separable closure of $A(\mathfrak{y})$ by $A(P)$.) We write $B = A_P$ and call A_P the <u>strict real localization</u> of A at the ordering P.

We want to get something similar to Proposition 2.1. in a truly global setting. Our global objects are locally ringed spaces, as is common in many parts of geometry. Henceforth, in this section, a locally ringed space $X = (X, \mathcal{O}_X)$ will simply be called a "space" if no further specification is given. We often write $\mathcal{O}$ instead of $\mathcal{O}_X$ if no confusion can occur.

<u>Definition 3.</u> A space X is called <u>strictly real</u> if all local rings $\mathcal{O}_{X,x}$ are strictly real.

For example, any C^r-manifold $(1 \leq r \leq \infty)$ equipped with the sheaf of $\mathbb{R}$-valued C^r-functions is a strictly real space. Also every analytic space with its structure sheaf of real analytic functions is strictly real.

A strictly real space X is amenable to inequality considerations. Let me explain this by two examples. If $f \in \mathcal{O}(U)$ is a section of the structure sheaf of X over some open subset U of X, then for any $x \in U$ we denote by $f(x)$ the image of f in the residue class field $\kappa(x) = \mathcal{O}_x/m_x$ of $\mathcal{O}_x$. Since $\kappa(x)$ is real closed we can ask whether $f(x) > 0$, $f(x) = 0$, or $f(x) < 0$.

If $f(x) \neq 0$ then f is a unit in $\mathcal{O}_x$ and thus $f(y) \neq 0$ for all y in some neighbourhood U' of x in U. If $f(x) > 0$ then there exists a unique $g \in \mathcal{O}_x$ with $f = g^2$ and $g(x) > 0$, and we conclude that we have $f = g^2$ on some open neighbourhood $U'' \subset U'$ of x with a unique $g \in \mathcal{O}(U'')$ which is everywhere positive on U". This proves

<u>Proposition 2.2.</u> For a given $f \in \mathcal{O}(U)$ the set $V := \{x \in U \mid f(x) > 0\}$ is open in U. There exists a unique $g \in \mathcal{O}(V)$ with $f = g^2$ and g positive everywhere on V.

More generally one can extract a global result from the local consideration on roots of polynomials above. Let $\varphi(T)$ be a normed polynomial in one variable with coefficients in $\mathcal{O}(U)$ for some open $U \subset X$. For every $x \in U$ we have a polynomial $\varphi(x,T)$ over $\kappa(x)$ by inserting the point x in the coefficients of φ.

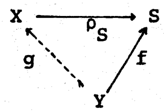
<u>Proposition 2.3.</u> The set V of all $x \in U$ such that the roots of $\varphi(x,T)$ in $\kappa(x)$ are all simple is open. If W is a connected open subset of V then, for every $x \in W$, $\varphi(x,T)$ has the same number r of roots in $\kappa(x)$. In case $r > 0$ there exist sections $\alpha_1,\ldots,\alpha_r \in \mathcal{O}(W)$ such that $\alpha_1(x) < \ldots < \alpha_r(x)$ for every $x \in W$ and $\varphi(\alpha_1) = \ldots = \varphi(\alpha_r) = 0$. The α_i are all the roots of $\varphi(T)$ in $\mathcal{O}(W)$.

Applying this proposition to the irreducible polynomials in $\mathbb{Q}[T]$ it is not difficult to prove that, for every open subset U of X, the field R_o of real algebraic numbers embeds in $\mathcal{O}(U)$ in a unique way. Thus all rings $\mathcal{O}(U)$ and all local rings $\mathcal{O}_x$ are R_o-algebras.

We now ask for a systematic way to associate a strictly real space to a scheme S (in the sense of Grothendieck). The Costes have given the best possible answer to this problem.

<u>Theorem 2.4.</u> For a given scheme S there exists a universal morphism $\rho_S : X \to S$ of spaces with X strictly real.

This means, there exists a morphism $\rho_S : X \to S$ with X strictly real, such that any morphism $f : Y \to S$ with Y strictly real factors through ρ_S in a unique way,



I call X the <u>strict real hull</u> of S and write $X = \mathcal{R}(S)$.

We can guess from the universal property of ρ_S what the points of the space $X = \mathcal{R}(S)$ are. Let $x \in X$ be given, and $s := \rho_S(x)$. The residue class field $\kappa(x)$ is a real closed field extension of $\kappa(s)$, hence induces an ordering $\overline{P}$ of $\kappa(s)$. The algebraic closure R of $\kappa(s)$ in $\kappa(x)$ is a real closure of $\kappa(s)$ with respect to $\overline{P}$. Corresponding to these field extensions we have a commutative diagram of morphisms between spaces (solid arrows)

$$\begin{array}{ccccc} X & & \xrightarrow{\quad \rho_S \quad} & & S \\ \uparrow & \nwarrow & & & \uparrow \\ \text{Spec } \kappa(x) & \longrightarrow & \text{Spec } R & \longrightarrow & \text{Spec } \kappa(s) \, . \end{array}$$

The space Spec R is strictly real. Thus we have a unique morphism from Spec R to X (dotted arrow) which divides the diagram into two commutative subdiagrams. This is only possible if $R = \kappa(x)$. Thus $\kappa(x)$ is the real closure of $\kappa(s)$ with respect to $\overline{P}$. Conversely, if $s \in S$ and an ordering $\overline{P}$ of $\kappa(s)$ are given, let R be "the" real closure of $\kappa(s)$ with respect to $\overline{P}$, and apply the universal property of ρ_S to the morphism Spec $R \to S$ corresponding to the field extension $\kappa(s) \to R$. We find a unique point $x \in X$ with $\rho_S(x) = s$ and R isomorphic to $\kappa(x)$ over $\kappa(s)$.

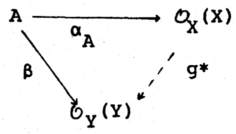
In a similar way we can guess what the local ring $\mathcal{O}_{X,x}$ of a given point $x \in X$ looks like. Let $s = \rho_S(x)$ and $\overline{P}$ be again the ordering of $\kappa(s)$ corresponding to x. Consider the one point space $\{x\}$ equipped with the strict real localisation $(\mathcal{O}_S)_P$ of the local ring $\mathcal{O}_{S,s}$ at the ordering P induced by $\overline{P}$. This space admits an evident morphism into X. Applying the universal property of ρ_S to this morphism we see that $\mathcal{O}_x = (\mathcal{O}_S)_P$. Let us summarise these observations.

<u>Corollary 2.5.</u> The points x of $X = \mathcal{R}(S)$ correspond uniquely with the pairs $(s,\overline{P})$ consisting of all $s \in S$ and all orderings $\overline{P}$ of $\kappa(s)$ as follows: $s = \rho_S(x)$, $\overline{P}$:= ordering of $\kappa(s)$ induced by the ordering of the real closed field extension $\kappa(x)$ of $\kappa(s)$.

The field $\kappa(x)$ is the real closure of $\kappa(s)$ with respect to $\overline{P}$ and $\mathcal{O}_{X,x}$ is the strict real localization of $\mathcal{O}_{S,s}$ at the ordering P of $\mathcal{O}_{S,s}$ corresponding to $\overline{P}$.

I want to reformulate Theorem 2.4. in the case $S = \operatorname{Spec} A$ for A a commutative ring. Recall that a morphism $f : Y \to \operatorname{Spec} A$ is uniquely determined by its comorphism $f^* : A = \mathcal{O}_S(S) \to \mathcal{O}_Y(Y)$, and that in this way we have a bijection of the set of morphisms from Y to $\operatorname{Spec} A$ to the set of ring homomorphisms from A to $\mathcal{O}_Y(Y)$ [EGA I, Cor. 1.6.4.]. Thus we can restate Theorem 2.4. for S affine as follows.

__Theorem 2.4.a.__ Let A be a commutative ring. There exists a pair (X,α_A) consisting of a strictly real space X and a ring homomorphism $\alpha_A : A \to \mathcal{O}_X(X)$ such that for any pair (Y,β) with Y a strictly real space and $\beta : A \to \mathcal{O}_Y(Y)$ a ring homomorphism there is a unique morphism $g : Y \to X$ such that $\beta = \alpha_A \circ g^*$,



__Definition 4.__ This space X is called the __real spectrum__ $\operatorname{Sper} A$ of A, i.e. $\operatorname{Sper} A = \mathcal{R}(\operatorname{Spec} A)$. The universal morphism from $\operatorname{Sper} A$ to $\operatorname{Spec} A$, which has α_A as comorphism, will be denoted by ρ_A, i.e. $\rho_A = \rho_{\operatorname{Spec} A}$. The structure sheaf $\mathcal{O}_X$ will often be denoted by $\mathcal{N}_A$ and its ring $\mathcal{O}_X(X)$ of global sections by $\mathcal{N}(A)$. The sections of $\mathcal{N}_A$ on some open $U \subset X$ are called the __Nash functions__ on U.

It will become apparent later that the name "Nash functions" is appropriate here, since these "functions" generalize the classical Nash functions on open subsets of $\mathbb{R}^n$.

Corollary 2.5. can be reformulated as follows (cf. beginning of this section).

Corollary 2.5.a. The points x of Sper A correspond uniquely with the orderings P of A by the relation

$$P = \{f \in A \mid \alpha_A(f)(x) \geq 0\}.$$

We have $\rho_A(x) = \operatorname{supp} P$, $\kappa(x) = A(P)$, and $(\mathcal{N}_A)_x = A_P$.

From now on we identify the points of Sper A with the orderings of A. The universal property of the real spectrum implies that for every ring homomorphism $\varphi : A \to B$ there exists a unique morphism $\operatorname{Sper} \varphi : \operatorname{Sper} B \to \operatorname{Sper} A$ such that the diagram

$$\begin{array}{ccc} \operatorname{Sper} B & \xrightarrow{\operatorname{Sper} \varphi} & \operatorname{Sper} A \\ \rho_B \downarrow & & \downarrow \rho_A \\ \operatorname{Spec} B & \xrightarrow[\operatorname{Spec} \varphi]{} & \operatorname{Spec} A \end{array}$$

commutes. The image of a point $Q \in \operatorname{Sper} B$ under $\operatorname{Sper} \varphi$ is the ordering $\varphi^{-1}(Q)$ of A.

Remark 2.6. By a well known result from real commutative algebra ([L, Th. 3.9], [BDS], [K,III, Prop. 4]) the space Sper A is not empty, i.e. A has orderings, if and only if -1 is not a sum of squares in A. We call such a ring A here "semireal" following Lam [L] (instead of "formally real" in other papers).

I did not say anything about a proof of Theorem 2.4. Of course, it suffices to prove Theorem 2.4.a. Then we get the strict real hull of an arbitrary scheme by gluing the strict real hulls of its open affine subschemes. The papers of the Costes deal exclusively with the affine case. In [CR] M.F. Coste-Roy proves Theorem 2.4.a by explicit construction of the space $(\mathrm{Sper}\, A, \mathcal{H}_A)$ and direct verification of the universal property. In this way she obtains a lot of extra information on Sper A. We do not repeat her very interesting arguments here, refering the reader to [CR], but we shall quote in the next two sections some of the results obtained by the Costes in this and other papers.

Right now I cannot resist to mention another beautiful theorem of the Costes, although the relevance of this theorem for quadratic form theory is not clear.

Theorem 2.7. ([CR, Th. 5.1.], the theorem there is more general.) For any semireal ring A the morphism $\mathrm{Sper}(\alpha_A)$ from $\mathrm{Sper}\,\mathcal{H}(A)$ to Sper(A) is an isomorphism. Consequently the ring homomorphism $\alpha_{\mathcal{H}(A)} : \mathcal{H}(A) \to \mathcal{H}(\mathcal{H}(A))$ is an isomorphism.

§3 - The topology of Sper A and the spears (cf. [CC], [L])

Let A be a semireal ring and X = Sper A. We know from §2 that for any element f of A the set

$$\{P \in X \mid \alpha_A(f)(P) > 0\} = \{P \in X \mid f \notin -P\} = \{P \in X \mid f >_P 0\}$$

is open in X (cf. Prop. 2.2). It turns out that these sets form a subbasis of the topological space X, in other words

Theorem 3.1. The sets

$$H(f_1,\dots,f_r) = \{P \in X \mid f_1 >_P 0,\dots,f_r >_P 0\},$$

with $(f_1,\dots,f_r)$ running through the finite families in A, are a basis of X.

This basis is called the Harrison basis of A. Indeed, in the case that A is a field, this basis had been introduced on the set of orderings of A long ago by Harrison [H], so Sper A is then - as a topological space - identical with the space of orderings usually considered in quadratic form theory.

Definition 1. A subset D of X = Sper A is called constructible if D can be obtained from finitely many basic open sets $H(f_1,\dots,f_r)$ by finitely many boolean operations. This means that D is the union of finitely many sets of the form

$$\{P \in X \mid f_0 \in P \cap (-P),\ f_1 >_P 0,\dots,f_r >_P 0\}$$

with elements $f_0, f_1,\dots,f_r$ of A. The set of all constructible subsets of X is denoted by $\mathfrak{r}(X)$.

The constructible subsets of X - in particular X itself - are compact*). Indeed, even the following stronger statement holds.

Proposition 3.2. If a constructible set D is covered by a family in $\mathscr{V}(X)$, then D is already covered by a finite subfamily.

Corollary 3.3. Let U be a subset of a constructible set D, and assume that U is open in D. Then U is constructible iff U is compact. In this case U is the union of finitely many sets $D \cap H(f_1, \ldots f_r)$.

We denote the set of constructible open subsets of a constructible set D by $\mathring{\mathscr{V}}(D)$, and the set of all constructible subsets of D by $\mathscr{V}(D)$.

We now consider the closure $\overline{\{P\}}$ of a one point set in Sper A. In general, this set is not constructible. As in the theory of usual spectra we call the points in $\overline{\{P\}}$ the specializations of P, and we write $P \succ Q$ if Q is a specialization of P.

Let $\mathfrak{q} :=$ supp P. In order to describe the specializations of P we need the following standard definition from real commutative algebra [Br_1, p. 57].

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*) "Compact" here always means "quasicompact" and does not include "Hausdorff".

Definition 2. An ideal $\mathfrak{a}$ of A is called P-convex, if for any elements x and y of P with $x+y \in \mathfrak{a}$ we have $x \in \mathfrak{a}$ and $y \in \mathfrak{a}$.

Clearly $\mathfrak{p}$ is P-convex, and $\mathfrak{p}$ is contained in every other P-convex ideal.

Theorem 3.4. Let P and Q be orderings of A with supports $\mathfrak{p}$ and $\mathfrak{q}$.

i) $P \succ Q$ iff $P \subset Q$. In this case $Q = P \cup \mathfrak{q} = P + \mathfrak{q}$, and the prime ideal $\mathfrak{q}$ is P-convex.

ii) The specializations Q of P correspond bijectively with the P-convex prime-ideals $\mathfrak{q}$ under the relation $\mathfrak{q} = \text{supp}\, Q$.

iii) For any two specializations Q_1 and Q_2 of P either $Q_1 \succ Q_2$ or $Q_2 \succ Q_1$. Thus the set $\overline{\{P\}}$ is totally ordered. It contains a maximal element, i.e. a closed point, P^*.

iv) If $P \not\succ Q$ and $Q \not\succ P$ then there exists some $f \in A$ with $P \in H(f)$ and $Q \in H(-f)$.

v) The sets $\overline{\{P\}}$ are precisely all closed irreducible subsets of Sper A. (Recall that a topological space is called irreducible if it is not the union of two proper closed subsets.)

The third part of this theorem reveals a fundamental difference between the real spectrum Sper A and the usual spectrum Spec A. In Spec A a set $\overline{\{\mathfrak{p}\}}$ is only in rare cases totally ordered under specialization.

If A has finite Krull dimension n then, by parts i) - iii) of the theorem, every set $\overline{\{P\}}$ is a finite chain

$$P = P_0 \succ P_1 \succ \ldots \succ P_t$$

with $t \leq n$. We shall see in the next section that in important cases there exist many sets $\overline{\{P\}}$ with $t = n$. Let me call, for any A, a set $\overline{\{P\}}$ a <u>spear</u> of Sper A and its closed point P^* the <u>tip</u> of the spear.

We denote the set of closed points (= "maximal" points) of Sper A by $(\text{Sper } A)^{\max}$. The topological subspace $(\text{Sper } A)^{\max}$ of Sper A is compact and Hausdorff. {Compactness follows from the compactness of Sper A; Hausdorff is clear from Th. 3.4.iv}. The specialization map $\lambda : P \mapsto P^*$ from Sper A to $(\text{Sper } A)^{\max}$ is continuous and, moreover, identifying, i.e. $(\text{Sper } A)^{\max}$ - as a subspace of Sper A - carries the quotient topology of Sper A with respect to λ ([S], cf. [L] §4). From this it is evident that λ yields a bijection between the connected components of Sper A and $(\text{Sper } A)^{\max}$.

Although $(\text{Sper } A)^{\max}$ is such an honest space, I have the impression that this space only plays a minor role in semialgebraic geometry (but see §4 for the case that A is a finitely generated algebra over $\mathbb{R}$, and §5 for the case that A is semilocal). Usually the full real spectrum is more useful.

Our considerations about spears and their tips generalize to a constructible subset D of Sper A instead of Sper A itself. A <u>spear</u> <u>in</u> D, is the closure $\overline{\{P\}} \cap D$ of some point $P \in D$ in D. It again is totally ordered under specialization (clear!) and has a maximal element P'. But notice that the point P', while closed in D, is perhaps not closed in Sper A. Again the subspace D^{max} of closed points of D is compact and Hausdorff and is a retract and quotient of D under the specialization map $P \mapsto P'$.

<u>Proposition 3.5.</u> A constructible subset D of Sper A is closed iff, for every $P \in D$, the full set of specializations $\overline{\{P\}}$ is contained in D.

For any point $\mathfrak{y}$ of Spec A the fibre $\rho^{-1}(\mathfrak{y})$ of the canonical map $\rho = \rho_A$ from Sper A to Spec A is the set of orderings of the field $A(\mathfrak{y})$, hence can be identified with the underlying set of Sper $A(\mathfrak{y})$. One can check directly that, under this identification, the Harrison topology of Sper $A(\mathfrak{y})$ coincides with the subspace topology in Sper A. A more conceptual proof of this fact runs as follows. Let Y be the topological subspace $\rho^{-1}(\mathfrak{y})$ of Sper A and $i : Y \hookrightarrow \text{Sper}\, A$ the inclusion map. We equip Y with the sheaf of rings $(i^*\mathcal{K}_A) \otimes_A A(\mathfrak{y})$. (N.B. There is a natural map from A into $\mathcal{K}(A)$, hence into every stalk of $i^*\mathcal{K}_A$.) The ringed space Y, with this structure sheaf, is strictly real. We have a commutative diagram of ringed spaces

$$\begin{array}{ccc} Y & \hookrightarrow & \text{Sper}\, A \\ \rho' \downarrow & & \downarrow \rho \\ \text{Spec}\, A(\mathfrak{y}) & \hookrightarrow & \text{Spec}\, A \end{array}$$

Starting from this diagram it is easily verified that ρ' is a universal morphism from a strictly real space to Spec $A(\mathfrak{y})$. Thus $Y = \text{Sper}\, A(\mathfrak{y})$.

Clearly $\rho^{-1}(\mathfrak{p})$ is not empty iff the prime ideal $\mathfrak{p}$ is <u>real</u>, i.e., for any $x_1,\ldots,x_n \in A$, $x_1^2 + \ldots + x_n^2 \in \mathfrak{p} \Rightarrow x_1,\ldots,x_n \in \mathfrak{p}$.

The topological space Sper A is the disjoint union of the sets $\rho^{-1}(\mathfrak{p})$ with $\mathfrak{p}$ running through the real prime ideals of A. The subspaces $\rho^{-1}(\mathfrak{p})$ = Sper $A(\mathfrak{p})$ of Sper A are totally disconnected, compact, and Hausdorff. (They should be regarded as "transverse" to the spears of Sper A, which causes their disconnectedness.)

Here a major advantage of working with real specta may be seen: Quadratic form theorists have long been interested in the geometry of the space of orderings Sper F of a field F (cf. e.g. [ELW]). It turned out to be difficult to observe geometric phenomena in these spaces since they are totally disconnected. Now, if you embed Sper F into the real spectrum Sper A of a sufficiently global ring A by an isomorphism of F with a suitable residue class field $A(\mathfrak{p})$, then you have a better chance to do geometry, since in important cases Sper A has only finitely many connected components (cf. next section).

§4 - The geometric case

We look at the real spectrum Sper A of a finitely generated commutative algebra A over some real closed field R. Without serious loss of generality we assume that A has no nilpotent elements. Now A is the ring R[V] of algebraic functions on some affine algebraic variety defined over R. We call this situation "the geometric case". (More generally we could look at the strict real hull of the scheme associated with a quasiprojective variety over R without much extra effort.) Every rational point $x \in V(R)$ corresponds to a maximal ideal $\mathfrak{y}$ of A with residue class field $A(\mathfrak{y}) = A/\mathfrak{y} = R$. Thus $A(\mathfrak{y})$ has a unique ordering, and we have a unique point P in Sper A with $\rho_A(P) = \mathfrak{y}$. We identify x with this point P and regard V(R) as a subset of Sper A. Clearly $V(R) \subset (\text{Sper } A)^{max}$.

For any set $H(f_1,\ldots,f_r)$ of the Harrison basis of Sper A we have

$$H(f_1,\ldots,f_r) \cap V(R) = \{x \in V(R) \mid f_1(x) > 0,\ldots,f_r(x) > 0\}$$

and we see that the subspace topology of V(R) in Sper A is the usual strong topology on V(R). More generally, the intersection of a constructible set D of Sper A with V(R) is a semialgebraic subset of V(R), determined by the same polynomial relations as D. {N.B. To be in harmony with the terminology in §1, observe that V(R) = X(R) with X = Spec A.}

The following remarkable fact can be proved by use of Tarski's principle ([CC, §3], in fact some version of the Positivstellensatz ([L, §7], [C-T]) suffices.)

<u>Theorem 4.1.</u> The correspondance $D \mapsto D \cap V(R)$ is a bijection from the set $\mathfrak{r}(\text{Sper } A)$ of constructible subsets of Sper A to the set $\mathfrak{r}(V(R))$ of semialgebraic subsets of V(R).

This theorem implies that, for any constructible subset D of Sper A, the set $D \cap V(R)$ is dense in D. A fortiori, $D \cap V(R)$ is dense in D^{max}. One might look for cases in which $D \cap V(R) = D^{max}$. Now D^{max} is always a compact Hausdorff space. Thus equality holds if and only if the semialgebraic set $D \cap V(R)$ is compact. If $R \neq \mathbb{R}$ then, as is well known, every compact semialgebraic set must be finite. But in the case $R = \mathbb{R}$ we have many non trivial compact semialgebraic sets. For example, if V is the affine standard variety $\mathbb{C}^n$, then a semialgebraic set N in $V(\mathbb{R}) = \mathbb{R}^n$ is compact if and only if N is bounded and closed in $\mathbb{R}^n$.

For any semialgebraic subset S of V(R) we denote by $\tilde{S}$ the unique constructible subset D of Sper A with $D \cap V(R) = S$. If T is a second semialgebraic subset of V(R), then clearly

$$(S \cap T)^{\sim} = \tilde{S} \cap \tilde{T}, \quad (S \cup T)^{\sim} = \tilde{S} \cup \tilde{T},$$
$$(V(R) \setminus S)^{\sim} = \text{Sper } A \setminus \tilde{S}.$$

Theorem 4.2. Let M be a semialgebraic subset of V(R) and S a semialgebraic subset of M. Then S is open in M if and only if $\tilde{S}$ is open in $\tilde{M}$.

This theorem is a little deeper than Theorem 4.1. To get a grasp at its content, we state a purely semialgebraic consequence of Theorem 4.2.

Theorem 4.2a. Again, let $S \subset M$ be semialgebraic subsets of V(R) and assume that S is open in M. Then there exist finitely many algebraic functions $f_{ij} \in A$ $(1 \leq i \leq r,\ 1 \leq j \leq s(i))$ such that

$$S = \bigcup_{i=1}^{r} \{x \in M \mid f_{i1}(x) > 0, \dots, f_{i,s(i)}(x) > 0\}.$$

Indeed, since $\widetilde{S}$ is compact and open in $\widetilde{M}$, we have

$$\widetilde{S} = \bigcup_{i=1}^{r} \widetilde{M} \cap H(f_{i1}, \dots, f_{i,s(i)})$$

for suitable $f_{ij} \in A$. Intersecting with $V(R)$ we obtain the desired conclusion.

Conversely, Theorem 4.2 follows easily from Theorem 4.2a. Now Theorem 4.2a is a well known fact in semialgebraic geometry (Brumfiel's "unproved proposition 8.1.2" in [Br]), but all existing proofs need refined methods from semialgebraic geometry. Proofs can be found in [CC, §3], [De, §1], [D, §3], and, in the case $R = \mathbb{R}$, already in [BE, §5].

Again, let M be a semialgebraic subset of $V(R)$. Since $\widetilde{M}$ is compact, every covering of $\widetilde{M}$ by open subsets can be refined to a covering by finitely many constructible open subsets. These coverings correspond, according to Theorem 4.2, uniquely to the coverings of M by finitely many open semialgebraic subsets. Thus the sheaves on the topological space $\widetilde{M}$ (with values, say, in abelian groups) correspond uniquely to the sheaves on M, as soon as we agree that in M only open semialgebraic subsets and finite coverings of such sets by open semialgebraic subsets are admitted. This is precisely what is done in the theory of semialgebraic spaces, cf. [DK, §7].

In other words, the "semialgebraic site" on M is equivalent to the usual topological site on $\tilde{M}$.

Taking M = V(R) we may ask: What is the semialgebraic sheaf $\mathcal{N}_{V(R)}$ on V(R) corresponding to the sheaf $\mathcal{N}_A$ on $\tilde{M}$ = Sper A? Assume for simplicity that V is regular at all rational points. Then it turns out that the sections of $\mathcal{N}_{V(R)}$ can be regarded as R-valued functions. In the case R = $\mathbb{R}$ these are the classical Nash functions, i.e. real analytic functions satisfying algebraic equations, as considered by John Nash [N] and Artin-Mazur [AM, §2]. Thus let us call $\mathcal{N}_{V(R)}$ the <u>sheaf of Nash functions</u> on V(R), by abuse of language even if the regularity condition is not satisfied. I do not have space here to go into the details of the theory of Nash functions (cf. [BE_1], [C-R], [C_1], [D, §12] for that.) Just let me say that, if again V is regular on V(R) and if R contains an element $\delta > 0$ with $\lim_{n\to\infty} \delta^n = 0$, such that convergence of power series makes sense in R, then a function $f : U \to R$ on some open semialgebraic $U \subset V(R)$ is Nash if and only if its graph is semialgebraic and f can be developed in a power series in a neighbourhood of any point $p \in U$, with respect to some regular system of parameters. It is a marvellous fact that the power series of a Nash function, which is of course the Taylor series of f, converges to f in a small neighbourhood, although usually R is not complete in any reasonable sense. {The fields R considered here are the "Cantor fields" of Dubois-Bukowski [DB]. For example, every real closure of a finitely generated field extension of $\mathbb{Q}$ or $\mathbb{R}$ is Cantor.}

We state another remarkable consequence of Theorem 4.2.

Corollary 4.3. Let M be a semialgebraic subset of V(R) and let $M_1, \dots M_r$ be the finitely many semialgebraic connected components of M (cf. §1). Then the sets $\tilde{M}_1, \dots, \tilde{M}_r$ are the connected components - in the topological sense - of $\tilde{M}$. In particular (M = V(R)), Sper A has only finitely many connected components.

More generally, one expects that the whole geometry of Sper A can be mirrored by the semialgebraic geometry of V(R). We now explain how the points of Sper A can be "seen" in the semialgebraic space M := V(R).

We consider the set Y(M) of ultrafilters of the Boolean lattice $\mathcal{Y}(M)$ of semialgebraic subsets of M. For any $S \in \mathcal{Y}(M)$ we introduce the set Y(S) consisting of all $F \in Y(M)$ with $S \in F$. If T is a second semialgebraic subset of M then clearly

$$Y(S \cap T) = Y(S) \cap Y(T), \; Y(S \cup T) = Y(S) \cup Y(T),$$
$$Y(M \setminus S) = Y(M) \setminus Y(S).$$

We provide Y(M) with the topology which has as a basis of open sets the sets Y(U) with U running through all open semialgebraic subsets of M.

In view of Theorem 4.1, every point P of Sper A yields an ultrafilter $\alpha(P) \in Y(M)$ in a natural way, namely

$$\alpha(P) = \{S \in \mathcal{Y}(M) \mid P \in \tilde{S}\}.$$

We thus have a map α from Sper A to Y(M) which obviously is continuous (N.B. M = V(R)).

<u>Theorem 4.4.</u> α is a homeomorphism from Sper A onto Y(M).

A good way to prove this theorem is to exhibit an inverse map $\beta : Y(M) \to \text{Sper } A$ explicitly. This can be done as follows (cf. [B, p. 260]).

Let an ultrafilter $F \in Y(M)$ be given. We want to construct an ordering P of A with $\alpha(P) = F$. Let d denote the minimum of the dimensions dim S of all $S \in F$. {The dimension dim S of a semialgebraic subset S of V(R) is defined as the (algebraic) dimension of the Zariski closure Z of S in V, cf. [DK, §8].} We choose some $T \in F$ with $\dim T = d$. Let $Z_1, \ldots, Z_r$ denote the irreducible components of the Zariski closure of T in V. Then T is the union of the sets $T_i := T \cap Z_i(R)$, $1 \leq i \leq r$. Since F is an ultrafilter, some T_i, say T_1, is an element of F. Thus we have found an irreducible subvariety Z of V, namely $Z = Z_1$, with $\dim Z = d$ and $Z(R) \in F$. {"subvariety" means here Zariski closed subset.}

Let $\mathfrak{p}$ be the prime ideal of A corresponding to Z. The ultrafilter F is generated by $F_o := F \cap \gamma(Z(R))$, and F_o is an ultrafilter of $\gamma(Z(R))$. Every $S \in F_o$ is Zariski dense in Z, since $\dim S = d$. Conversely, every $S \in \gamma(Z(R))$, whose complement $Z(R) \setminus S$ is <u>not</u> Zariski dense in Z, is an element of F_o. Indeed, otherwise $Z(R) \setminus S$ would be an element of F_o, but

$\dim(Z(R) \setminus S) < d$. We now define an ordering $\overline{P}$ on the function field $R(Z) = A(\mathfrak{p})$ as follows. Let a rational function $f \neq 0$ on Z be given. The intersection G of the domain of definition of f with Z(R) is an element of F_o, since $Z(R) \setminus G$ is not Zariski dense in Z. Now G is the disjoint union of the three sets $\{x \in G | f(x) > 0\}$, $\{x \in G | f(x) < 0\}$, $\{x \in G | f(x) = 0\}$. The last set has smaller dimensions than d, hence is not an element of F_o. Thus precisely one of the first two sets is an element of F_o. We decree that $f \in \overline{P}$ iff $\{x \in G | f(x) > 0\} \in F_o$. It is easily verified that we obtain in this way an ordering $\overline{P}$ of $A(\mathfrak{p})$. We denote the ordering P of A with support $\mathfrak{p}$, which corresponds to $\overline{P}$, by $\beta(F)$.

It is easily verified that the map $\beta : Y(M) \to \mathrm{Sper}\, A$ obtained in this way is continuous and inverse to α.

Our description of the ordering $\alpha^{-1}(F)$ reveals some geometric facts about the orderings of functions fields over R which are worth to be mentioned. We say that the ultrafilter F "lives" on the irreducible subvariety Z of V occuring above.

<u>Remark 4.5.</u> Let Z be the irreducible subvariety of V corresponding to a given prime ideal $\mathfrak{p}$ of A. Then $\mathfrak{p}$ is real, i.e. the function field $R(Z) = A(\mathfrak{p})$ is formally real, if and only if Z(R) is Zariski dense in Z.

Indeed, precisely in this case there exist ultrafilters $F \in Y(V(R))$ which live on Z. Remark 4.5 is a well known fact from the early times of real algebraic geometry, due to E. Artin [A, §4]. Moreover, it is known that Z(R) is Zariski dense in Z iff Z(R) contains regular points of Z (cf. e.g. [C-T, p. 103f]).

Remark 4.6. Assume that A is an integral domain, i.e. that V is irreducible. Assume further that V(R) is Zariski dense in V. Then the orderings of the quotient field R(V) of A can be identified with the ultrafilters $F \in \gamma(V(R))$ which do not contain semialgebraic sets of smaller dimension than $\dim V = d$.

Indeed, precisely these ultrafilters live on V. Remark 4.6 had already been observed by Brumfiel [Br, p. 232ff], presumably quite a while before the invention of real spectra (cf. also [B, p. 260]). The ultrafilters of $\gamma(V(R))$ living on V can be regarded as the ultrafilters of the Boolean lattice $\tilde{\gamma}(V(R))$ obtained from $\gamma(V(R))$ by factoring out the equivalence relation

$$S_1 \sim S_2 \iff \dim(S_1 \setminus S_2) < d \text{ and } \dim(S_2 \setminus S_1) < d$$

i.e. the Boolean lattice obtained from $\gamma(V(R))$ by neglecting lower dimensional sets. Every element of $\tilde{\gamma}(V(R))$ can be represented by an open semialgebraic set. This gives a geometric explanation "why" the space of orderings of Quot(A) is totally disconnected.

Remark 4.7. Assume again that A = R[V] is integral. Let P be an ordering of the quotient field K = R(V) of A, and let F be the corresponding ultrafilter of $\gamma(V(R))$. Further, let Φ denote the set of all open connected semialgebraic sets $U \in F$ which are contained in the regular part $V(R)_{reg}$ of V. Since the singular part of V(R) has dimension less than d, the set $V(R)_{reg}$ is an element of F. Thus Φ generates the ultrafilter F. We look at the stalk A_P of the sheaf $\mathcal{N}_A$ at the point $P \in \operatorname{Sper} A$. The open constructible sets $\tilde{U}$ with $U \in \Phi$ are a fundamental system of neighbourhoods of P in Sper A. Thus

$$A_P = \varinjlim_{U\in\Phi} \mathcal{N}_A(\tilde{U}) = \varinjlim_{U\in\Phi} \mathcal{N}_{V(R)}(U).$$

If $f: U \to R$ is a Nash function on some $U \in \Phi$, and f is not identically zero, then the set of zeros of f on U has dimension less than d, since Nash functions obey an identity principle, similar to real analytic functions. Thus f is invertible in the inductive limit A_P. We see that the stalk A_P coincides with its residue class field A(P), which is the real closure K(P) of K with respect to P. (By abuse of language we regarded the ordering P of K also as an ordering of A.)

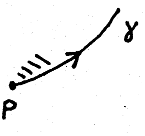
In this way the elements of any real closure of a function field K over R can be interpreted as the Nash functions living on the regular open sets of some ultrafilter in a given model of K.

We return to an arbitrary algebraic variety V over R and its affine ring A = R[V]. We now describe the specializations of a given point $P \in \operatorname{Sper} A$ in terms of ultrafilters, omitting proofs. Let $F = \alpha(P)$ and let Z denote the irreducible subvariety of V on which F lives, i.e. the subvariety corresponding to $\mathfrak{p}$ = supp P.

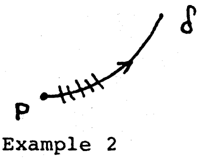
Definition. Let W be an irreducible subvariety of V. We say that the ultrafilter F touches W if the closure $\overline{S}$ of every $S \in F$ meets W in a set $\overline{S} \cap W(R)$ which is Zariski dense in W. Clearly then $W \subset Z$.

Theorem 4.8 [B, p. 262]. If F touches W then there exists a unique ultrafilter $F_W \in Y(V(R))$ which lives on W and contains the sets $\overline{S} \cap W(R)$ with $S \in F$. The orderings $\alpha^{-1}(F_W)$ corresponding to these ultrafilters are precisely all specializations of $\alpha^{-1}(F) = P$.

We finally give four examples of spears in Sper A for $A = \mathbb{R}[x,y]$, the ring of polynomials in two variables over $\mathbb{R}$. We have $V = \mathbb{C}^2$, $V(\mathbb{R}) = \mathbb{R}^2$.



Example 1



Example 2

The geometric ingredients of the first example are a point $p \in \mathbb{R}^2$, a half branch γ of a real algebraic curve W emanating at p, and one of the two " banks " of γ. The corresponding spear consists of three orderings $P_0 > P_1 > P_2$. Here P_0 consists of all $f \in A$ which are non negative near p at the chosen bank of γ. P_1 consists of all $f \in A$ which are non negative on γ near p, and P_2 consists of all $f \in A$ with $f(p) \geq 0$. P_0 lives on the whole variety $\mathbb{C}^2$, while P_1 lives on W and P_2 lives on $\{p\}$.

The geometric ingredients of the second example are a point $p \in \mathbb{R}^2$ and a transcendental curve δ starting at p, say $p = (0,1)$ and $\delta = \{(t,e^t) \mid t \geq 0\}$. The corresponding spear consists of two orderings $P_0 \succ P_1$. Here P_0 consists of all $f \in A$ which are non negative on δ near p, hence non negative in a neighbourhood of $\delta \setminus \{p\}$ near p (N.B. No polynomial changes sign at δ!), and P_1 consists of all $f \in A$ with $f(p) \geq 0$. P_0 lives on $\mathbb{C}^2$, while P_1 lives on $\{p\}$.

Example 3. Embedding $\mathbb{R}^2$ in the projective plane $\mathbb{P}^2(\mathbb{R})$, as usual, we modify Example 1 as follows. We choose p as a real point of the infinite hyperplane and again choose a half-branch γ of a real algebraic curve W emanating at p. Then we obtain a spear $P_0 \succ P_1$ with P_0 living on $\mathbb{C}^2$ and P_1 living on W. The point P_1 is closed in Sper A since the point p is "missing" on Sper A .

Example 4. Similarly we modify Example 2 by choosing p at the infinite hyperplane. We obtain a spear consisting of a single ordering P living on $\mathbb{C}^2$.

The last two examples show that the Hausdorff space $(\mathrm{Sper}\, A)^{max}$ is a very large compactification of the topological space $\mathbb{R}^2$.

§5 - Signatures

In this and the next section I finally come to those aspects of real spectra which, according to our present knowledge, are particularly relevant for quadratic form theory.

A signature of a commutative ring A is a ring homomorphism σ from the Witt ring W(A) to the ring of integers $\mathbb{Z}$. As already mentioned in §1, the kernels of the signatures are the non maximal prime ideals of W(A), and different signatures yield different prime ideals of W(A).

We equip the set Sign A of signatures of A with the coarsest topology such that for every $\varphi \in W(A)$ the $\mathbb{Z}$-valued function $\hat{\varphi} : \sigma \mapsto \sigma(\varphi)$ on Sign A is continuous ($\mathbb{Z}$ has the discrete topology). As is well known, this topology is the same as the subspace topology of Sign A in Spec W(A) under the natural injection Sign A $\hookrightarrow$ Spec W(A) (cf. [K_1, §1]). The space Sign A is compact, Hausdorff, and totally disconnected.

We have a natural map $\gamma :$ Sper A $\to$ Sign A from the real spectrum of A, more precisely its underlying topological space, to the space Sign A, which sends every ordering P of A to the signature

$$\gamma(P) : W(A) \longrightarrow W(A(\mathfrak{p})) \xrightarrow[\operatorname{sign}_P]{} \mathbb{Z}$$

Here the first arrow is the natural map from W(A) to the Witt ring of the residue class field $A(\mathfrak{p})$ at $\mathfrak{p} :=$ supp P, and $\text{sign}_{\overline{P}}$ is the Sylvester signature at the ordering $\overline{P}$ of $A(\mathfrak{y})$ corresponding to P. More explicitly the signature $\gamma(P)$ can be described as follows. Given a non singular symmetric bilinear form B over A (living on a finitely generated projective A-module), we choose an open neighbourhood U of $\mathfrak{y}$ in $X := \text{Spec}\, A$ such that B can be diagonalized over U,

$$B|U = \langle f_1, \ldots, f_r \rangle$$

with elements $f_i \in A$ which are units in $\mathcal{O}_X(U)$. The Witt class φ of B has under $\gamma(P)$ the value

$$\gamma(P)(\varphi) = \varepsilon_1 + \ldots + \varepsilon_r,$$

with $\varepsilon_i = +1$ if $f_i \in P$, $\varepsilon_i = -1$ if $f_i \in -P$. (N.B. $f_i \notin \mathfrak{y}$, since f_i is a unit on U.) From this description of $\gamma(P)$ it is clear that the map γ is continuous. Indeed, the function $\hat{\varphi} \circ \gamma$ is constant on the open neighbourhood $\rho_A^{-1}(U) \cap H(\varepsilon_1 f_1, \ldots, \varepsilon_r f_r)$ of P in Sper A.

As already said in §1, every signature σ of A factors through a signature of some residue class field $A(\mathfrak{y})$ of A. This means that the map $\gamma : \text{Sper}\, A \to \text{Sign}\, A$ is surjective (Recall that the signatures of $A(\mathfrak{y})$ correspond uniquely to the orderings of $A(\mathfrak{y})$.)

Since Sign A is totally disconnected, every connected component of Sper A maps under γ to a single point. A fortiori, every spear $\overline{\{P\}}$ maps to a single point $\gamma(P)$. (This is also evident from the explicit description of $\gamma(P)$ above.) Thus we obtain from γ by restriction a continuous map

$$\gamma^{max} : \mathrm{Sper}\ (A)^{max} \to \mathrm{Sign}\ A$$

between compact Hausdorff spaces which is still surjective and hence identifying. We also have a commutative triangle of identifying maps

$$\begin{array}{ccc} \mathrm{Sper}\ A & \overset{\lambda}{\twoheadrightarrow} & (\mathrm{Sper}\ A)^{max} \\ & {\scriptstyle\gamma}\searrow \quad \swarrow{\scriptstyle\gamma^{max}} & \\ & \mathrm{Sign}\ A & \end{array} \qquad (*)$$

with λ the specialization map considered in §3.

We now focus attention on the case where A is semilocal, i.e. has only finitely many maximal ideals. Then the connection between the spaces Sper A and Sign A is particularly narrow.

<u>Theorem 5.1</u> [S, §1]. If A is semilocal then γ^{max} is a homeomorphism from $(\mathrm{Sper}\ A)^{max}$ onto Sign A.

To prove this theorem it suffices to show that γ^{max} is bijective. We do this by explicit description of an inverse map. For this purpose we recall some well known facts about signatures of semilocal rings.

We assume without loss of generality that Spec A is connected. Then W(A) is generated by the free bilinear spaces <f> of rank 1 with f running through the group of units A^* of A. Denoting the value of a signature σ on <f> simply by $\sigma(f)$ we regard σ as a character on A^* with values ± 1. A character

$\sigma : A^* \to \{\pm 1\}$ turns out to be a signature precisely if $\sigma(-1) = -1$ and $\sigma(a_1^2 f_1 + \ldots + a_r^2 f_r) = +1$ whenever $f_1, \ldots, f_r$ are units of A with $\sigma(f_1) = \ldots = \sigma(f_r) = +1$ and $a_1, \ldots, a_r$ are elements of A with $a_1^2 f_1 + \ldots + a_r^2 f_r$ again a unit [KRW, §2].

For any signature σ of A we introduce the set $Q(\sigma)$ consisting of all elements $a_1^2 f_1 + \ldots + a_r^2 f_r$, for finitely many units f_i of A such that $\sigma(f_1) = \cdots = \sigma(f_r) = +1$ and elements $a_1, \ldots, a_r$ of A such that $Aa_1 + \ldots + Aa_r = A$. Then A is the disjoint union of the sets $Q(\sigma)$, $(-1)Q(\sigma)$, and a prime ideal $\mathfrak{p}(\sigma)$. Moreover $Q(\sigma) + \mathfrak{p}(\sigma) = Q(\sigma)$. ([KK], [$K_3$, p. 86ff]). It is now evident that $P(\sigma) := Q(\sigma) \cup \mathfrak{p}(\sigma)$ is an ordering of A with support $\mathfrak{p}(\sigma)$.

We call $P(\sigma)$ the <u>ordering of A associated with the signature</u> σ. From the definition of $Q(\sigma)$ and Theorem 3.4.i above it is clear, that $P(\sigma)$ has no proper specializations in Sper A, i.e. $P(\sigma) \in (\text{Sper } A)^{\max}$. Clearly $\gamma(P(\sigma)) = \sigma$. On the other hand, if we start with an ordering P of A, it is easily verified that the ordering associated with $\gamma(P)$ is a specialization of P. If $P \in (\text{Sper } A)^{\max}$, this ordering is again P. Thus we have found an inverse map $\sigma \mapsto P(\sigma)$ to $\gamma^{\max}$, and Theorem 5.1 is proved.

In the commutative triangle (*) above the fibres of the specialization map λ are obviously connected. On the other hand, we know that the fibres of γ are unions of connected components of Sper A. Thus Theorem 5.1 has the following consequence.

Corollary 5.2. If A is semilocal, then the fibres of $\gamma : \operatorname{Sper} A \twoheadrightarrow \operatorname{Sign} A$ are the connected components of Sper A. Every connected component C of Sper A contains a unique closed point P, and C is the set of generalisations of P in Sper A.

Henceforth we identify $(\operatorname{Sper} A)^{\max}$ with Sign A via $\gamma^{\max}$. Thus $\operatorname{Sign} A \subset \operatorname{Sper} A$. But there is a subtle point to be kept in mind. On Sper A we have a distinguished basis of open sets, the Harrison basis. Intersecting these sets with Sign A we obtain a distinguished basis on Sign A. On the other hand, in quadratic form theory Sign A is regarded as an abstract "space of orderings" in the sense of M. Marshall (cf. [KR, §6], [K_2, §2], [Ms, Chap. 10], and Marshall's papers cited there). As an abstract space of orderings Sign A carries also a distinguished basis (called again "Harrison basis"). This second basis consists of the sets $H(f_1,\ldots,f_r) \cap \operatorname{Sper}(A)^{\max}$ with $(f_1,\ldots,f_r)$ running through the finite systems of units of A. Thus the second basis is usually smaller than the first one, although both bases generate the same topology on $(\operatorname{Sper} A)^{\max}$.

Returning to an arbitrary commutative ring A we consider the semilocalization A_S of A at a given finite set $S \subset \operatorname{Spec} A$, $S = \{\mathfrak{p}_1,\ldots,\mathfrak{p}_r\}$. This is the semilocal ring obtained by localizing A with respect to the multiplicative set $A \setminus (\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_r)$. As usual, we regard the topological space $\operatorname{Spec} A_S$ as a subspace of the topological space Spec A. Let $j : \operatorname{Spec} A_S \to \operatorname{Spec} A$

be the inclusion map. The structure sheaf of Spec A_S is the inverse image of the structure sheaf of Spec A under j. In this strong sense, Spec A_S is a "ringed subspace" of Spec A.

I claim that analogously Sper A_S is a ringed subspace of Sper A. Indeed, let Y denote the inverse image $\rho_A^{-1}(\text{Spec } A_S)$ of Spec A_S in Sper A, considered as a topological subspace, and let $i: Y \to \text{Sper } A$ be the inclusion map. We equip Y with the structure sheaf $i^*\mathcal{O}_A$. Then Y is a strictly real ringed space. We have an obvious commutative diagram of ringed spaces

$$\begin{array}{ccc} Y & \overset{i}{\hookrightarrow} & \text{Sper } A \\ \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho_A} \\ \text{Spec } A_S & \underset{j}{\hookrightarrow} & \text{Spec } A \end{array}$$

Using this diagram it is easily checked that ρ' is a universal morphism from a strictly real ringed space to Spec A_S. Thus $Y = \text{Sper } A_S$.

We may omit from the set S every prime ideal $\mathfrak{p}_i$ which is contained in some $\mathfrak{p}_j$, $j \neq i$, without affecting A_S. We have - say as subsets of Spec A -

$$\text{Spec } A_S = \bigcup_{i=1}^{r} \text{Spec } A_{\mathfrak{p}_i}.$$

Taking inverse images with respect to ρ_A we obtain

$$\text{Sper } A_S = \bigcup_{i=1}^{r} \text{Sper } A_{\mathfrak{p}_i}.$$

Thus the real spectra of the semilocalizations of A all sit in Sper A, and finitely many of them glue together nicely. As a set, every real spectrum Sper $A_{\mathfrak{p}_i}$ is the disjoint union of the real spectra Sper $A(\mathfrak{p})$ with $\mathfrak{p}$ running through the prime ideals contained in $\mathfrak{p}_i$.

Notice that the spaces $(\text{Sper}\, A_S)^{max} = \text{Sign}\,(A_S)$ do not fit together so nicely, since a point $P \in \text{Sper}\, A_S \cap \text{Sper}\, A_T$ may be closed in Sper A_S but not closed in Sper A_T.

Having reached this point one may try to exploit the powerful theory of reduced quadratic forms on the subspaces Sign A_S = $= (\text{Sper}\, A_S)^{max}$ of Sper A, to obtain information on the geometry of Sper A. This has been done recently by Bröcker in the geometric case with great success. Given an open semialgebraic subset S of V(R), with V an arbitrary affine variety over a real closed field R, he obtains a criterion whether S is principal, i.e.

$$S = \{x \in V(R) \mid f_1(x) > 0, \ldots, f_r(x) > 0\}$$

with functions $f_i \in R[V]$. In case that S is principal, he further obtains a bound on the number r of inequalities needed to describe S, which depends only on dim V, cf. [B_1]. Finally - and most surprising - he gives a bound on the number of principal sets needed to write any open semialgebraic $S \subset V(R)$ as a union of principal sets, i.e. a bound on the number r in Theorem 4.2a, which again depends only on $\dim V$, cf. his talk at this conference.

Bröcker's proofs provide striking examples how abstract quadratic form theory can be used to obtain purely geometric results. It is important that one considers the spaces of signatures of suitable semilocalizations of $R[V]$. Just localizations do not suffice.

We now look again at the map $\gamma : \mathrm{Sper}\, A \twoheadrightarrow \mathrm{Sign}\, A$ for A an arbitrary commutative ring. Louis Mahé has proved the following deep theorem.

Theorem 5.3. [M, Prop. 3.1]. Let F_1 and F_2 be two disjoint closed constructible subsets of Sper A. Then there exists some integer $n \geq 0$ and some element $\varphi \in W(A)$ such that $\gamma(P)(\varphi) = 2^n$ for every $P \in F_1$ and $\gamma(P)(\varphi) = -2^n$ for every $P \in F_2$. In other words, the function $\Phi \circ \gamma : \mathrm{Sper}\, A \to \mathbb{Z}$ has constant values 2^n on F_1 and -2^n on F_2.

I cannot indicate even the idea of Mahé's ingenious proof of the theorem. The reason is, that Mahé makes essential use of the theory of abstract Nash functions on Sper A, whereas we have discussed Nash functions only casually. (Mahé also uses very interesting methods from global quadratic form theory.)

Corollary 5.4. The fibres of $\gamma : \mathrm{Sper}\, A \twoheadrightarrow \mathrm{Sign}\, A$ are the connected components of Sper A.

Mahé draws this consequence of his theorem only in the geometric case [M, Cor. 3.4]. Thus I indicate how the corollary follows from the theorem in general. (One has to be a little careful, since it is doubtful whether the closure $\overline{D}$ of a constructible subset D of Sper A is again constructible. We know this only in the geometric case.) I need the following

<u>Shrinking Lemma</u> (cf. [Br_2, Prop. 2.4] and, for a full proof, [D_2, §1]). Let $(U_i \mid 1 \le i \le r)$ be a finite covering of Sper A by open constructible sets. Then there exists a covering $(F_i \mid 1 \le i \le r)$ of Sper A by closed constructible sets with $F_i \subset \dot{U}_i$ for $1 \le i \le r$.

cf. Errata at the end

Let C_1 and C_2 be two different connected components of Sper A, and let $\gamma(C_i) = \{\sigma_i\}$. We have to prove that $\sigma_1 \neq \sigma_2$. Since the sets C_1 and C_2 are (quasi)compact we obtain, by topological standard arguments, open constructible sets U_1 and U_2 with $U_i \supset C_i$ and $U_1 \cap U_2 = \emptyset$. (Start with Theorem 3.4.iv.) Since Sper A is compact and $C_1 \cup C_2$ is closed we can find open constructible sets $U_3, \ldots, U_r$ such that Sper A $= U_1 \cup \ldots \cup U_r$ and $U_3, \ldots, U_r$ are disjoint from $C_1 \cup C_2$. Replacing the system $U_3, \ldots, U_r$ by the one set $U_3 \cup \ldots \cup U_r$, we assume that $r = 3$. Applying the shrinking lemma we obtain closed constructible sets F_1, F_2, F_3 such that Sper A $= F_1 \cup F_2 \cup F_3$ and $F_i \subset U_i$. We have $F_1 \supset C_1$, $F_2 \supset C_2$, and $F_1 \cap F_2 = \emptyset$. Applying Mahé's theorem to F_1 and F_2 we obtain some $\varphi \in W(A)$ with $\sigma_1(\varphi) = 2^n$ and $\sigma_2(\varphi) = -2^n$. Thus indeed $\sigma_1 \neq \sigma_2$, and the corollary is proved.

From now on we denote, for any $P \in \mathrm{Sper}\,A$ and $\varphi \in W(A)$, the value of the signature $\gamma(P)$ on φ simply by $\mathrm{sign}_P(\varphi)$. (It really is the "Sylvester signature of φ at the ordering P".) The _total_ _signature_ $\mathrm{sign}(\varphi)$ of φ is the continuous (= locally constant) function $P \mapsto \mathrm{sign}_P(\varphi)$ of $\mathrm{Sper}\,A$, i.e. in previous notation, $\mathrm{sign}(\varphi) = \hat{\varphi} \circ \gamma$. We then have a ring homomorphism

$$\mathrm{sign} : W(A) \to C(\mathrm{Sper}\,A, \mathbb{Z})$$

from $W(A)$ to the ring of continuous $\mathbb{Z}$-valued functions on $\mathrm{Sper}\,A$ which we call the _total_ _signature_ _map_. Notice that, according to Corollary 5.4, the rings $C(\mathrm{Sper}\,A, \mathbb{Z})$ and $C(\mathrm{Sign}\,A, \mathbb{Z})$ really are the same object, so sign is the total signature map considered usually in quadratic form theory.

As is well known, the kernel of the total signature map is the nil radical of $W(A)$ (cf. [Dr], [K, Chap, III]). As a second consequence of Theorem 5.3 we have

Corollary 5.5. [M, Th. 3.2]. The cokernel of the total signature map is a 2-primary torsion group.

In order to deduce this corollary from the theorem, just observe that, as an abelian group, $C(\mathrm{Sper}\,A, \mathbb{Z})$ is generated by the characteristic functions of the constructible subsets of $\mathrm{Sper}\,A$ which are closed and open in $\mathrm{Sper}\,A$.

§6 - Brumfiel's big theorem

We have seen in the geometric case A = R[V] (cf. §4) that the category of sheaves on the semialgebraic space V(R) is equivalent to the category of sheaves on the topological space Sper A. The sheaf of Nash functions $\mathcal{N}_{V(R)}$ on V(R) corresponds to the structure sheaf $\mathcal{N}_A$ of Sper A. Now, the sheaf $\mathcal{C}_{V(R)}$ of semialgebraic functions (= continuous R-valued functions with semialgebraic graph) also lives on V(R), and we have a natural sheaf homomorphism $\mathcal{N}_{V(R)} \to \mathcal{C}_{V(R)}$ which, in good cases (say, V is regular everywhere on V(R)), is an injection. Thus we have a sheaf of rings $\mathcal{C}_A$ on Sper A corresponding to $\mathcal{C}_{V(R)}$ with a homomorphism $\mathcal{N}_A \to \mathcal{C}_A$, and, in good cases, $\mathcal{N}_A$ is even a subsheaf of $\mathcal{C}_A$. It turns out that the stalks of $\mathcal{C}_A$ are local rings and that, at every point P of Sper A, the sheaves $\mathcal{C}_A$ and $\mathcal{N}_A$ have the same residue class field, namely A(P).

In his recent paper [Br_2], G.W. Brumfiel has constructed a sheaf $\mathcal{C}_A$ of "abstract semialgebraic functions" on the real spectrum of any commutative ring A, with analogous properties. He obtains $\mathcal{C}_A$ as the sheaf of "constructible continuous sections" of the natural morphism Sper A[T] → Sper A, T being an indeterminate. Of course, in the geometric case his sheaf is identical with the sheaf $\mathcal{C}_A$ above. N. Schwartz gives another construction of $\mathcal{C}_A$ in [S_1], which perhaps allows an easier deduction of the basic properties of $\mathcal{C}_A$, in particular that $\mathcal{C}_A$ is a sheaf of rings. (It is not so easy to work with continuous constructible sections. Actually Brumfiel's notion of "continuity" has to be slightly modified.) More generally, both authors construct a sheaf $\mathcal{C}_D$ of abstract semialgebraic functions on every constructible subset D of Sper A.

Denoting the ring of global sections of $\mathcal{C}_A$ by $\mathcal{C}(A)$, we have a natural ring homomorphism $\mathcal{R}(A) \to \mathcal{C}(A)$, which in good cases is an injection. Brumfiel states in [Br_2] the following

Theorem 6.1. The homomorphism of Witt rings $W(A) \to W(\mathcal{C}(A))$ induced by the natural map $A \to \mathcal{R}(A) \to \mathcal{C}(A)$ is an isomorphism up to 2-torsion, i.e. its kernel and cokernel are 2-primary torsion groups.

The question arises whether $W(\mathcal{C}(A))$ can be interpreted in some more geometric way. Let us consider a somewhat analogous but easier case, namely the Witt ring of the ring $\mathcal{C}(X)$ of continuous $\mathbb{R}$-valued functions on a compact Hausdorff space X. By a well known result, due to Serre and Swan [Sw], the Grothendieck ring $K(\mathcal{C}(X))$ of finitely generated projective modules over $\mathcal{C}(X)$ can be identified with the Grothendieck ring KO(X) of real (continuous) vectorbundles on X. Similarly, the arguments in [Sw] yield that $W(\mathcal{C}(X))$ can be identified with Witt ring W(X) of bilinear spaces (E,B) on X, i.e. real vector bundles E on X equipped with a non-degenerated $\mathbb{R}$-valued bilinear form B. Now every bilinear space (E,B) admits an orthogonal decomposition $E = E_+ \perp E_-$, with B positive definite on E_+ and negative definite on E_- (cf. e.g.

[MH, p. 106]). Sending the Witt class [E,B] to the element $[E_+] - [E_-]$ of KO(X) we obtain an isomorphism of rings $W(X) \xrightarrow{\sim} KO(X)$ (loc.cit., Milnor and Husemoller attribute this result to G. Lusztig.)

In the geometric case one can prove in a similar way that for any semialgebraic subset M of V(R)

$$W(\mathcal{C}(M)) \cong KO(M) \cong K(\mathcal{C}(M)).$$

Here $\mathcal{C}(M)$ is the ring of semialgebraic functions on M, and KO(M) is the orthogonal K-ring of M considered in §1.

We return to an arbitrary commutative ring A. Brumfiel defines in [Br_2] the Grothendieck ring KO(D) of "constructible vector bundles" on every constructible subset D of Sper A, and proves (or, at least, makes plausible) that $KO(D) = K(\mathcal{C}(D))$, with $\mathcal{C}(D)$ the ring of abstract semialgebraic functions on D (= global sections of $\mathcal{C}_D$). Of course, in the geometric case $KO(D) = KO(M)$ and $\mathcal{C}(D) = \mathcal{C}(M)$, with M the semialgebrais set $D \cap V(R)$ corresponding to D. Brumfiel then states

<u>Theorem 6.2.</u> $W(\mathcal{C}(D)) \cong KO(D)$ for any constructible subset D of Sper A.

According to Theorems 6.1 and 6.2 we have a natural ring homomorphism

$$\tau : W(A) \to KO(\mathrm{Sper}\ A)$$

which is an isomorphism up to 2-torsion. This is what I call "Brumfiel's big theorem". It is a vast improvement of Mahé's result, Corollary 5.5 above. Indeed, there is the rank map

$$KO(\mathrm{Sper}\,A) \rightarrow C(\mathrm{Sper}\,A, \mathbb{Z})$$

which sends every virtual vector bundle to its rank, a locally constant $\mathbb{Z}$-valued function on $\mathrm{Sper}\,A$. The rank map is a surjective ring homomorphism. Composing it with τ we obtain the total signature map sign described in §5. Mahé's theorem follows from Brumfiel's big theorem.

To be fair, it must be said that the proofs of Theorem 6.1 and 6.2 are merely sketched in Brumfiel's paper [Br_2]. Brumfiel uses an extension of Mahé's method together with a vast amount of new foundational techniques in abstract semialgebraic geometry (for example abstract semialgebraic homotopy). Once these foundations are firmly established, Brumfiel's ingenious sketch should be a rigorous proof.

Brumfiel's big theorem is an eye-opening message to quadratic form theorists: Up to 2-torsion, the Witt ring W(A) is a "topological" object, where the word "topological" has to be interpreted within the theory of real spectra. You are invited to become friends with real spectra and to study Mahé's and Brumfiel's proofs. They contain very fine examples how semialgebraic geometry can be useful for quadratic form theory.

Addendum (June 1984). Lou van den Dries has kindly pointed out to me that Theorem 4.2a is proved in his paper "Some applications of a model theoretic fact to (semi-)algebraic geometry" (Indag. math. 44 (1982), 397-401) using an elementary model theoretic principle and valuation theory. In fact, in this special case model theory is not really necessary.

A Corrigendum to this article appears at the end of the book.

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We use the following abbreviations.

"Proc. San Francisco 1981" means:
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"Proc. Rennes 1981" means:
J.-L. Colliot-Thélène, M. Coste, L. Mahé, M.-F. Roy, ed., "Géométrie algébrique réelle et formes quadratiques", Lecture Notes Math. 959, Springer, Berlin-Heidelberg-New York 1982

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CORRIGENDUM TO "An invitation to real spectra"

by Manfred Knebusch

In §5 of that article the hypothesis of Theorem 5.3 should read: "Let F_1 and F_2 be two disjoint <u>clopen</u> (= closed and open) constructible subsets of Sper A".

In order to deduce Corollary 5.4 from Theorem 5.3, in its corrected form, we do not need the shrinking lemma but the following general fact.

<u>Lemma.</u> Let A be a commutative ring and x a point of Sper A. The connected component of x in Sper A is the intersection of all clopen subsets of Sper A which contain x.

If C_1 and C_2 are different connected components of Sper A then, by this lemma, there exists a clopen subset F of Sper A with $C_1 \subset F$ and $C_2 \subset \mathrm{Sper}\, A \setminus F$. Theorem 5.3 provides us with an element $\varphi \in W(A)$ such that $\hat{\varphi} \circ \gamma | C_1 = 2^n$ and $\hat{\varphi} \circ \gamma | C_2 = -2^n$. This means that the signatures $\sigma_1 := \gamma(C_1)$ and $\sigma_2 := \gamma(C_2)$ have the values 2^n and -2^n on φ. Thus $\sigma_1 \neq \sigma_2$, and Corollary 5.4 is proved.

<u>Proof of the lemma.</u> Let $(F_\alpha | \alpha \in I)$ be the family of all clopen subsets of Sper A which contain x, indexed in some way, and let Y denote the intersection of all F_α. Let Z be the connected component of x in Sper A. Then, for every $\alpha \in I$, we have $Z \subset F_\alpha$, hence $Z \subset Y$. We verify that Y is connected, which will imply that Y = Z, as desired.

Suppose on the contrary, that Y is the disjoint union of two non empty closed sets M and N and, without loss of generality, that $x \in M$. Since M and N are closed and (quasi)compact there exist, as a consequence of Theorem 3.4.iv, two disjoint open constructible subsets U and V of Sper A with $M \subset U$ and $N \subset V$. The intersection Y of the F_α is contained in $U \cup V$. The family $(F_\alpha | \alpha \in I)$ is closed under finite intersections, and Sper A $\setminus (U \cup V)$ is compact. Thus there exists some $\beta \in I$ with $F_\beta \subset U \cup V$. The clopen set F_β is the disjoint union of the open, hence clopen, subsets $F_\beta \cap U$ and $F_\beta \cap V$. Moreover, $x \in F_\beta \cap U$ since $x \in M$. Thus $F_\beta \cap U = F_\gamma$ for some $\gamma \in I$. We have

$$N \subset Y \subset F_\gamma \subset U,$$

which contradicts the disjointness of U and V. Thus Y is indeed connected. q.e.d.

Remark. As the proof shows the statement of the lemma holds more generally for any topological space which is (quasi)compact and normal.

I thank Roland Huber for detecting the mistake in Theorem 5.3 and for providing me with the lemma above.