ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 4, Fall 1984

AN INTRODUCTION TO LOCALLY SEMIALGEBRAIC SPACES

HANS DELFS AND MANFRED KNEBUSCH

Dedicated to the memory of Gus Efroymson

In our paper [4] we introduced the category of semialgebraic spaces over an arbitrary real closed field R. This category seems to be the natural framework in which to describe "topological" phenomena over R. Nevertheless it turns out to be too small for some purposes. For example, every semialgebraic covering $p: M \to N$ of a semialgebraic space N (defined in the obvious way) necessarily has finite degree since its fibres are zero-dimensional semialgebraic spaces and thus are finite sets. One also wants to consider "covering spaces of infinite degree over N. These spaces should, at least locally, look like semialgebraic spaces.

So we are forced to enlarge the category of semialgebraic spaces and to introduce locally semialgebraic spaces. In the first two sections of this paper we give a review of the basic definitions and properties of locally semialgebraic spaces and maps and illustrate this new concept by examples. In §3 we consider locally finite simplicial complexes. They are the most important examples of locally semialgebraic spaces since a large class of locally semialgebraic spaces, namely all paracompact and regular spaces, can be triangulated. In §5 we discuss the coverings of a space. These coverings were the initial reason for developing the whole theory. As in topology they are classified by the subgroups of the semialgebraic fundamental group defined in §4.

Since this paper is meant to be a survey, we have omitted most proofs. We are planning a more detailed treatment of the whole subject in the near future. Complete proofs of all statements in this paper will appear at this time.

Throughout the paper R denotes an arbitrary real closed field.

1. Definition of locally semialgebraic spaces and examples. Locally semialgebraic spaces are ringed spaces which locally look like semialgebraic spaces. In order to explain and interpret this in detail we start with the following ad hoc definition.

DEFINITION 1. A generalized topological space is a set M together with a set $\mathring{\mathscr{T}}(M)$ of subsets of M, called the "open subsets", and a set Cov_M of families $(U_\alpha \mid \alpha \in I)$ in $\mathring{\mathscr{T}}(M)$, called the "admissible coverings", such that the following eight properties hold.

i) $\emptyset \in \mathring{\mathscr{T}}(M)$, $M \in \mathring{\mathscr{T}}(M)$.

ii) If $U_1 \in \mathring{\mathscr{T}}(M)$ and $U_2 \in \mathring{\mathscr{T}}(M)$, then $U_1 \cap U_2 \in \mathring{\mathscr{T}}(M)$, $U_1 \cup U_2 \in \mathring{\mathscr{T}}(M)$.

iii) Every finite family $(U_\alpha \mid \alpha \in I)$ in $\mathring{\mathscr{T}}(M)$ is an element of Cov_M.

iv) If $(U_\alpha \mid \alpha \in I) \in \mathrm{Cov}_M$, then the union $U := \bigcup(U_\alpha \mid \alpha \in I)$ of this family is an element of $\mathring{\mathscr{T}}(M)$.

For any $U \in \mathring{\mathscr{T}}(M)$ we denote the subset of all $(U_\alpha \mid \alpha \in I) \in \mathrm{Cov}(M)$ with $\bigcup(U_\alpha \mid \alpha \in I) = U$ by $\mathrm{Cov}_M(U)$ and call these coverings the "admissible coverings of U".

v) If $(U_\alpha \mid \alpha \in I)$ is an admissible covering of $U \in \mathring{\mathscr{T}}(M)$ and if $V \in \mathring{\mathscr{T}}(M)$ is a subset of U, then $(U_\alpha \cap V \mid \alpha \in I)$ is an admissible covering of V.

vi) If $(U_\alpha \mid \alpha \in I) \in \mathrm{Cov}_M(U)$ and $(V_{\alpha\beta} \mid \beta \in J_\alpha) \in \mathrm{Cov}_M(U_\alpha)$ for every α, then $(V_{\alpha\beta} \mid \alpha \in I, \beta \in J_\alpha) \in \mathrm{Cov}_M(U)$.

vii) If $(V_\beta \mid \beta \in J) \in \mathrm{Cov}_M(U)$ and if $(U_\alpha \mid \alpha \in I)$ is a family in $\mathring{\mathscr{T}}(M)$ with $\bigcup(U_\alpha \mid \alpha \in I) = U$ which is refined by $(V_\beta \mid \beta \in J)$ {i.e., there is a map $\lambda: J \to I$ with $V_\beta \subset U_{\lambda(\beta)}$ for every $\beta \in J$}, then $(U_\alpha \mid \alpha \in I) \in \mathrm{Cov}_M(U)$.

viii) If $U \in \mathring{\mathscr{T}}(M)$, $(U_\alpha \mid \alpha \in I) \in \mathrm{Cov}_M(U)$ and if V is a subset of U with $V \cap U_\alpha \in \mathring{\mathscr{T}}(M)$ for every $\alpha \in I$, then $V \in \mathring{\mathscr{T}}(M)$.

REMARKS. a) We usually write M for the triple $(M, \mathring{\mathscr{T}}(M), \mathrm{Cov}_M)$.

b) A generalized, topological space is a (rather special) Grothendieck topology. Thus we have the theory of sheaves on such spaces at our disposal (cf. [**2**, **15**]). Notice that, in contrast to the topological case, the sheaf condition must be fulfilled only for admissible coverings.

DEFINITION 2. A ringed space over R is a pair $(M, \mathcal{O}_M)$ consisting of a generalized topological space M and a sheaf $\mathcal{O}_M$ of commutative R-algebras on M. A morphism (f, θ): $(M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ between ringed spaces is defined as usual: f is a continuous map from M to N, i.e., for every $V \in \mathring{\mathscr{T}}(N)$ the preimage $f^{-1}(V) \in \mathring{\mathscr{T}}(M)$ and for every covering $(V_\alpha \mid \alpha \in I) \in \mathrm{Cov}_N$ the family $(f^{-1}(V_\alpha) \mid \alpha \in I)$ is an element of Cov_M. $\mathscr{D}$ is a family $(\theta_V)_{V \in \mathring{\mathscr{T}}(N)}$ of R-algebra homomorphisms θ_V: $\mathcal{O}_N(V) \to \mathcal{O}_M(f^{-1}(V))$ compatible with restriction.

EXAMPLES. i) Let M be an affine semialgebraic space over R ([**4**, §7], [**5**, §1]). Choose for $\mathring{\mathscr{T}}(M)$ the set $\mathring{\mathfrak{S}}(M)$ of all open semialgebraic subsets of M. For $U \in \mathring{\mathfrak{S}}(M)$ define $\mathrm{Cov}_M(U)$ as the set of all families $(U_i \mid i \in I)$ in $\mathring{\mathfrak{S}}(M)$ such that $\bigcup(U_i \mid i \in I) = U$ and U is already covered by finitely many U_i, $i \in I$. Then M is a generalized topological space. (In this way every restricted topological space, as defined in [**4**, §7], [**5**, §1], may be regarded as a generalized topological space). Obviously the sheaves on this generalized topological space M are the same as the sheaves on M, considered as a restricted topological space. In particular, we have on M the sheaf $\mathcal{O}_M$: $\mathcal{O}_M(U)$ is the R-algebra of semialgebraic functions $f\colon U \to R$ for any $U \in \mathring{\mathfrak{S}}(M) = \mathring{\mathcal{T}}(M)$. This ringed space $(M, \mathcal{O}_M)$ over R is really the same as the semialgebraic space M and henceforth will be identified with it.

ii) Let $(M, \mathcal{O}_M)$ be a ringed space over R and $U \in \mathring{\mathcal{T}}(M)$. U bears the "induced" generalized topology. Restricting $\mathcal{O}_M$ to U we get the open subspace $(U, \mathcal{O}_M \mid U)$ of M. If $(U, \mathcal{O}_M \mid U)$ is a semialgebraic space over R, as defined in the preceding example, U is called an open semialgebraic subset of M.

DEFINITION 3. a) A locally semialgebraic space over R is a ringed space $(M, \mathcal{O}_M)$ over R (we often simply write "M") which posesses an admissible covering $(M_\alpha | \alpha \in I) \in \mathrm{Cov}_M(M)$ such that all sets M_α are open semialgebraic subsets of M.

b) A morphism between locally semialgebraic spaces is a morphism in the category of ringed spaces.

The category of semialgebraic spaces is a full subcategory of the category of locally semialgebraic spaces.

Let $(M_\alpha \mid \alpha \in I)$ be an admissable covering of the locally semialgebraic space $(M, \mathcal{O}_M)$ by open semialgebraic subsets. An open subset $U \in \mathring{\mathcal{T}}(M)$ of M is an open semialgebraic subset of M if and only if U is contained in finitely many sets M_α. As in the semialgebraic case, the elements $f \in \mathcal{O}_M(U)$, where $U \in \mathring{\mathcal{T}}(M)$, can and will be considered as R-valued functions on U. They will be called the locally semialgebraic functions on U.

A morphism $(f, \theta)\colon (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ between locally semialgebraic spaces is determined by its first component f: For $V \in \mathring{\mathcal{T}}(N)$ and $h \in \mathcal{O}_N(V)$ $\theta_V(h) = h \circ f$. This is easily derived from [**4**, Th. 7.2]. Henceforth we will simply denote a morphism (f, θ) by f. Morphisms are also called locally semialgebraic maps.

A locally semialgebraic map $f\colon M \to N$ necessarily maps every open semialgebraic subset U of M into an open semialgebraic subset V of N, and f is a semialgebraic map from U to V.

The locally semialgebraic functions $f \in \mathcal{O}_M(M)$ on a locally semialgebraic space M are just the locally semialgebraic maps from M to the semialgebraic space R.

In general we have the following description of local semialgebraic maps in terms of semialgebraic maps. (Notice that coverings as described below exist for any continuous map f).

PROPOSITION 1.1. *Let $f\colon M \to N$ be a map between locally semialgebraic spaces. Assume $(M_\alpha | \alpha \in I)$ and $(N_\beta | \beta \in J)$ are admissible coverings of M and N by open semialgebraic subsets and that $\lambda\colon I \to J$ is a map such that $f(M_\alpha) \subset N_{\lambda(\alpha)}$ for every $\alpha \in I$. Then f is a locally semialgebraic map if and* *only if for every* $\alpha \in I$ *the restriction* $f | M_\alpha : M_\alpha \to N_{\lambda(\alpha)}$ *is a semialgebraic map between the semialgebraic spaces* M_α *and* $N_{\lambda(\alpha)}$.

Before giving examples of locally semialgebraic spaces, we will show that suitable inductive limits exist in this category.

LEMMA 1.2. *Let* M *be a set and let* $(M_\alpha | \alpha \in I)$ *be a directed system of subsets of* M *with* $\bigcup(M_\alpha | \alpha \in I) = M$. $(M_\alpha \subset M_\beta$ *if* $\alpha \leqq \beta)$. *Assume that every set* M_α *carries the structure of a locally semialgebraic space* $(M_\alpha, \mathcal{O}_\alpha)$ *over* R, *and that for* $\beta > \alpha$ *the space* $(M_\beta, \mathcal{O}_\beta)$ *is an open subspace of* $(M_\alpha, \mathcal{O}_\alpha)$. *Then* M *can be given (in an unique way) the structure* $(M, \mathcal{O}_M)$ *of a locally semialgebraic space such that* M *becomes the inductive limit of the system* $((M_\alpha, \mathcal{O}_\alpha) | \alpha \in I)$ *in the category of locally semialgebraic spaces over* R.

The proof of Lemma 1.2 is easy. Take as open subsets U of M those subsets U whose intersection $U \cap M_\alpha$ is an element of $\mathring{\mathscr{T}}(M_\alpha)$ for every $\alpha \in I$. A family $(U_\lambda | \lambda \in J)$ is an element of Cov_M if and only if $(U_\lambda \cap M_\alpha | \lambda \in J) \in \mathrm{Cov}_{M_\alpha}$ for every α. Finally an R-valued function $f\colon U \to R$ $(U \in \mathring{\mathscr{T}}(M))$ is an element of $\mathcal{O}_M(U)$ if and only if the restriction $f | U \cap M_\alpha$ is an element of $\mathcal{O}_{M_\alpha}(U \cap M_\alpha)$ for every α.

Observe that M_α is an open subspace of M and that $(M_\alpha | \alpha \in I)$ is an admissible covering of M.

EXAMPLE 1.3. Let M be a locally semialgebraic space and $(M_\alpha | \alpha \in I)$ an admissible covering of M by open semialgebraic subsets. Then M is the inductive limit of the directed system of semialgebraic spaces which consists of all unions of finitely many M_α, considered as open subspaces of M.

EXAMPLE 1.4. (Direct sums). Let $(M_\alpha | \alpha \in I)$ be a family of locally semialgebraic spaces over R and $M = \bigsqcup(M_\alpha | \alpha \in I)$ be the disjoint union of the sets M_α. Then the set $(N_\beta | \beta \in J)$ consisting of all finite unions of sets M_α is a directed system of locally semialgebraic spaces. Lemma 1.2 provides M with the structure of a locally semialgebraic space such that $M = \varinjlim N_\beta$. Obviously M is the direct sum of the spaces $(M_\alpha | \alpha \in I)$ in the category of locally semialgebraic spaces.

A locally semialgebraic space M is called discrete if it is the direct sum of a family $(M_\alpha | \alpha \in I)$ of one point spaces M_α.

EXAMPLE 1.5. (Direct products). Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be locally semialgebraic spaces. Then the cartesian product $M \times N$ of the sets M, N becomes a locally semialgebraic space as follows: Choose admissible coverings $(M_\alpha | \alpha \in I)$ and $(N_\beta | \beta \in J)$ of M and N by open semialgebraic subspaces which are directed systems of subsets. Thus $M = \varinjlim M_\alpha$ and $N = \varinjlim N_\beta$. For every $(\alpha, \beta) \in I \times J$ we introduce the direct product structure of the semialgebraic spaces M_α and N_β on the set $M_\alpha \times N_\beta$ ([**4**, §7]). By Lemma 1.2 we obtain the structure of a locally semialgebraic space on $M \times N$ with $M \times N = \varinjlim M_\alpha \times N_\beta$. $M \times N$ is the direct product of M and N in the category of locally semialgebraic spaces.

EXAMPLE 1.6. Let $(M, \mathcal{O}_M)$ be an affine locally complete (i.e., every point of M has a complete semialgebraic neighborhood) semialgebraic space. Choose an embedding $M \subset R^n$ with M bounded in R^n. Let d: $\bar{M} \to R$ be the function giving the distance to the "boundary" $\bar{M} - M$ of M: $d(x) = \min(\|x - y\| \mid y \in \bar{M} - M)$. d is a semialgebraic function on $\bar{M}$ and $d^{-1}(0) = \bar{M}\backslash M$. Hence $(M(\varepsilon) \mid \varepsilon > 0)$ with $M(\varepsilon) := \{x \in M \mid d(x) > \varepsilon\}$ is a directed system of open semialgebraic subspaces of M with $\bigcup(M(\varepsilon) \mid \varepsilon > 0) = M$. Applying Lemma 1.2, we get a locally semialgebraic structure on M, denoted by M_{loc}, such that $M_{\text{loc}} = \varinjlim_{\varepsilon>0} M(\varepsilon)$. This structure does not depend on the chosen embedding $M \subset R^n$, since M_{loc} is also the inductive limit of the system of all complete semialgebraic subspaces of M. M_{loc} is a semialgebraic space if and only if M is complete. In this case $M = M_{\text{loc}}$.

EXAMPLE 1.7. (Base extension). Let $\tilde{R}$ be a real closed field extension of R. There is a canonical functor "base extension" from the category of locally semialgebraic spaces over R to the category of locally semialgebraic spaces over $\tilde{R}$. To a locally semialgebraic space M over R we associate a space $M(\tilde{R})$ in the following way: Choose an admissible covering $(M_\alpha \mid \alpha \in I)$ of M by a directed system of open semialgebraic subspaces. Then $M = \varinjlim M_\alpha$ (Ex. 1.3). Take as $M(\tilde{R})$ the inductive limit of the base extensions $M_\alpha(\tilde{R})$ which were defined in [**3**, §9], [**5**, §6], [**6**, §4].

Using the description of locally semialgebraic maps in terms of semialgebraic maps (Proposition 1.1) and the base extension of semialgebraic maps [loc. cit.], one can in a similar way associate to a locally semialgebraic map f: $M \to N$ a locally semialgebraic map $f_{\tilde{R}}$: $M(\tilde{R}) \to N(\tilde{R})$.

Let M be a locally semialgebraic space over R and let $(M_\alpha \mid \alpha \in I)$ be a fixed admissible covering of M by open semialgebraic subsets. We denote the set of open semialgebraic subsets of M by $\mathring{\mathfrak{S}}(M)$.

DEFINITION 4. A subset X of M is called locally semialgebraic if for every $W \in \mathring{\mathfrak{S}}(M)$ the intersection $X \cap W$ is a semialgebraic subset of the semialgebraic space W. It is enough to check that $X \cap M_\alpha$ is semialgebraic in M_α for every $\alpha \in I$. The set of all locally semialgebraic subsets of M is denoted by $\mathring{\mathscr{T}}(M)$.

DEFINITION 5. A family $(X_\lambda \mid \lambda \in \Lambda)$ of subsets of M is called locally finite if for every $W \in \mathring{\gamma}(M)$ the intersection $W \cap X_\lambda$ is empty for all but finitely many $\lambda \in \Lambda$. This is the case, if the set M_α meets only finitely many X_λ for every $\alpha \in I$.

Clearly the union and intersection of any locally finite family in $\mathscr{T}(M)$

is again an element of $\mathscr{T}(M)$. Also, the preimage of a locally semialgebraic subset of M under a locally semialgebraic map $f: M \to N$ is a locally semialgebraic subset of N. But in contrast to the semialgebraic theory, the image of a locally semialgebraic set under a locally semialgebraic map is not necessarily locally semialgebraic. For example, one can define in an obvious way a locally semialgebraic "spiral map" h from $\mathbf{R}_{loc}$ to $\mathbf{R}^2$ (infinite spiral with center 0). The "spiral" $h(\mathbf{R}_{loc})$ is not a semialgebraic subset of $\mathbf{R}^2$.

EXAMPLE 1.8. (Subspaces). Let X be a locally semialgebraic subset of M. We can provide X with the structure of a locally semialgebraic space in a natural way. Adding all finite unions we assume that our covering $(M_\alpha | \alpha \in I)$ is a directed system. For every $\alpha \in I$, $X \cap M_\alpha$ is a semialgebraic subset of M_α and hence is a semialgebraic subspace of the semialgebraic space M_α ([**4**, §7], [**5**, §1]). Applying Lemma 1.2, we endow X with the structure $(X, \mathcal{O}_X)$ of a locally semialgebraic space such that X is the inductive limit of $(X \cap M_\alpha | \alpha \in I)$.

These spaces $(X, \mathcal{O}_X)$ are called the (locally semialgebraic) subspaces of $(M, \mathcal{O}_M)$. Of course the structure on X does not depend on the choice of the covering $(M_\alpha | \alpha \in I)$ of M.

EXAMPLE 1.9. (Existence of fibre products). Let $f: M \to S$, $g: N \to S$ be locally semialgebraic maps over R. The subset $M \times_S N$ of $M \times N$ (cf. Ex. 1.5) consisting of all pairs (x, y) in $M \times N$ with $f(x) = g(x)$ is a locally semialgebraic subset of $M \times N$ and hence a locally semialgebraic space over R. $M \times_S N$ is the fibred product of M and N over S in the category of locally semialgebraic spaces.

DEFINITION 6. Let M be a locally semialgebraic space over R. A subset X of M is called semialgebraic, if X is locally semialgebraic and if the subspace $(X, \mathcal{O}_X)$ of M is a semialgebraic space. The set of all semialgebraic subsets of M is denoted by $\mathfrak{S}(M)$.

It is easily seen that a locally semialgebraic subset $X \in \mathscr{T}(M)$ of M is semialgebraic if and only if it is contained in some set $W \in \mathfrak{S}(M)$. The image of a semialgebraic subset under a locally semialgebraic map is again semialgebraic.

The sets in $\mathring{\mathscr{T}}(M)$, M a given locally semialgebraic space, are a basis of open sets for a topology in the classical sense on M. This topology is called the strong topology. The open sets in the strong topology are unions of arbitrary families in $\mathring{\mathscr{T}}(M)$ (or in $\mathfrak{S}(M)$). A set $X \in \mathscr{T}(M)$ is an element of $\mathring{\mathscr{T}}(M)$ if and only if X is open in the strong topology. Hence it is justified to use from now on the following terminology: The words open, closed, dense etc. refer always to the strong topology on M. The sets $U \in \mathring{\mathscr{T}}(M)$ are now called "open locally semialgebraic" subsets of M.

The closure $\bar{X}$ and the interior X of a locally semialgebraic subset X of M are again locally semialgebraic. A locally semialgebraic map $f: M \to N$ is continuous in the strong topology and its graph is a locally semialgebraic subset of $M \times N$. But in contrast to the semialgebraic theory, the converse is not true in general (consider, e.g., the map $R \to R_{\text{loc}}$, $x \mapsto x$).

We will close this section with some remarks on the components and the dimension of a locally semialgebraic space M.

DEFINITION 7. A path in M is a locally semialgebraic map α: $[0, 1] \to M$ from the semialgebraic space $[0, 1]$ ($=$ unit interval in R) to M. (Notice that the image $\alpha([0, 1])$ is a semialgebraic subset of M). The path component of a point $x \in M$ in M is the set of all points $y \in M$ such that there is a path α in M with $\alpha(0) = x$ and $\alpha(1) = y$.

It is an immediate consequence of the corresponding results for semialgebraic spaces that every path component of M is a closed and open locally semialgebraic subset of M. The family $(M_\lambda | \lambda \in \Lambda)$ of all path components is locally finite, in particular it is an admissible covering of M. Thus M is the direct sum of the spaces M_λ.

Every path component M' is connected, i.e., there does not exist a partition of M' into two non-empty open locally semialgebraic subsets. Thus the path components of M are also the "connected components". They are often simply called components.

The dimension dim M of a locally semialgebraic space M is defined as the supremum of the dimensions of all semialgebraic subsets of M. dim $M = 0$ if and only if M is a discrete space (Ex. 1.4).

2. More basic definitions and results. From now on we will tacitly assume that all locally semialgebraic spaces are "separated", i.e., Hausdorff in the strong topology, and we will often call them simply "spaces".

DEFINITION 1. A space M is called regular if for every closed locally semialgebraic subset A of M and every point $x \in M \backslash A$ there exist sets $U, V \in \mathring{\mathscr{T}}(M)$ with $x \in U$, $A \subset V$ and $U \cap V = \emptyset$.

A semialgebraic space X is regular if and only if X is affine (**[14]**). Hence in a regular space every semialgebraic subspace is affine. Since most results in semialgebraic geometry are known for affine spaces, it seems to be reasonable to restrict to regular locally semialgebraic spaces. But to obtain deeper results on the topology of locally semialgebraic spaces from the semialgebraic theory we are forced, at least at the moment, to consider only spaces which are in addition paracompact. Fortunately this suffices for many applications.

DEFINITION 2. A locally semialgebraic space M is called paracompact, if there exists a locally finite (cf. §1, Def. 5) covering $(M_\alpha \mid \alpha \in I)$ of M by open semialgebraic subsets. (Notice: Such a covering is admissible).

EXAMPLE 2.1. Assume that our real closed base field R contains a sequence $\varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \cdots$ with $\lim_{n\to\infty} \varepsilon_n = 0$. In practice, this is nearly always the case. For example, every real closure of a finitely generated field is even "microbial", i.e., R contains an element $\theta \neq 0$ with $\lim_{n\to\infty}\theta^n = 0$ ([9]). Also every real closed field is the inductive limit of its microbial real closed subfields. Then for every locally complete semialgebraic space M over R the space M_{loc} introduced in Example 1.6 is paracompact. Indeed, the sets

$$M_0 =: \{x \in M \mid d(x) > \varepsilon_1\},$$
$$M_n := \{x \in M \mid \varepsilon_{n-1} > d(x) > \varepsilon_{n+1}\} \ (n \geqq 1)$$

form a locally finite covering of M by open semialgebraic subsets. Conversely it can be shown that R contains a nontrivial zero sequence if M_{loc} is paracompact.

EXAMPLES 2.2. Every locally finite simplicial complex (see §3) is paracompact as is every covering space of a semialgebraic space (§5). Every subspace of a paracompact space is again paracompact. (Notice the difference from the topological theory.)

Paracompact spaces inherit many of the nice properties of semialgebraic spaces. All the following statements can be proved using a locally finite covering and the semialgebraic theory.

REMARKS 2.3. Let M be a paracompact locally semialgebraic space.

i) M is regular if and only if every semialgebraic subspace is affine.

ii) The closure $\bar{X}$ of any semialgebraic subset X of M is semialgebraic.

Now assume additionally that M is regular.

iii) (Shrinking Lemma). For every locally finite covering $(U_\lambda \mid \lambda \in \Lambda)$ of M by open semialgebraic subsets, there is a covering $(V_\lambda \mid \lambda \in \Lambda)$ of M by open semialgebraic subsets with $\bar{V}_\lambda \subset U_\lambda$ for every $\lambda \in \Lambda$.

iv) For every locally finite covering $(U_\lambda \mid \lambda \in \Lambda)$ of M by open semialgebraic subsets, there is a subordinate partition of unity — i.e., there exists a family $(\varphi_\lambda \mid \lambda \in \Lambda)$ of locally semialgebraic functions $\varphi_\lambda\colon M \to [0, 1]$ such that $\varphi_\lambda | M\backslash U_\lambda \equiv 0$ for every $\lambda \in \Lambda$ and $\sum_{\lambda\in\Lambda}\varphi_\lambda(x) = 1$ for every $x \in M$.

v) ("Tietze's extension theorem"). Let A be a closed subspace of M and $f\colon A \to K$ be a locally semialgebraic function on A with values in a convex semialgebraic subset K of R. Then there is a locally semialgebraic function $g\colon M \to K$ with $g|A = f$.

vi) If A, B are disjoint closed locally semialgebraic subsets of M, then there is a locally semialgebraic function $f\colon M \to [0, 1]$ with $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

DEFINITION 3. A map $f\colon M \to N$ between locally semialgebraic spaces is called semialgebraic if f is locally semialgebraic and if $f^{-1}(Y)$ is a semialgebraic subset of M for every semialgebraic subset Y of N.

Every locally semialgebraic map with domain a semialgebraic space M is semialgebraic. The image $f(X)$ of every locally semialgebraic subset X of M under a semialgebraic map $f\colon M \to N$ is a locally semialgebraic subset of N.

NOTATION. We denote by $\bar{\mathcal{T}}(M)$ (resp. $\bar{\mathfrak{S}}(M)$) the set of all closed locally semialgebraic (resp. closed semialgebraic) subsets of a locally semialgebraic space M.

DEFINITION 4. a) A locally semialgebraic map $f\colon M \to N$ is called proper if, given any locally semialgebraic map $g\colon N' \to N$, $f'\colon M \times_N N' \to N'$ maps every $X \in \bar{\mathcal{T}}(M \times_N N')$ onto a set $f'(X) \in \bar{\mathcal{T}}(N')$. Here f' is the map obtained from f by base extension with g.

b) A locally semialgebraic space M is called complete if the map from M to the one point space is proper.

For a semialgebraic map $f\colon M \to N$ between semialgebraic spaces the notion "proper" as defined here coincides with the previous one (introduced in [**4**, §9], [**5** §2]).

PROPOSITION 2.4. *Let* $f\colon M \to N$ *be a locally semialgebraic map and* $(N_\beta | \beta \in J)$ *be an admissible covering of* N. *Then* f *is proper if and only if the restriction* $f | f^{-1}(N_\beta)\colon f^{-1}(N_\beta) \to N_\beta$ *is proper for every* $\beta \in J$.

This is easy to prove. More difficult is the next theorem.

THEOREM 2.5. *Let* $f\colon M \to N$ *be a proper map between locally semialgebraic spaces. Assume that* M *is paracompact. Then* f *is semialgebraic. In particular, every paracompact complete locally semialgebraic space is semialgebraic.*

We see that considering proper locally semialgebraic maps essentially means considering proper maps between semialgebraic spaces. So this is nothing really new. But there is a more general class of morphisms behaving nearly as well as proper maps.

DEFINITION 5. a) A locally semialgebraic map $f\colon M \to N$ is called partially proper, if the restriction $f|A\colon A \to N$ is proper for every $A \in \bar{\mathfrak{S}}(M)$.

b) A locally semialgebraic space M is called partially complete if the map from M to the one point space is partially proper. This means that every closed semialgebraic subset of M is a complete semialgebraic space.

Of course a proper map is partially proper.

EXAMPLE 2.5. For every locally complete semialgebraic space M the space M_{loc} (Example 1.6) is partially complete.

The importance of partially complete spaces is stressed by the following.

THEOREM 2.6. *Every regular paracompact locally semialgebraic space M is isomorphic to a locally semialgebraic subspace of a partially complete regular space $\bar{M}$.*

In [4, 12.5] and [8, 2.3] we gave two useful characterizations of proper maps between semialgebraic spaces. These characterizations generalize to partially proper maps in the category of locally semialgebraic spaces.

PROPOSITION 2.7. *A locally semialgebraic map $f: M \to N$ is partially proper if and only if the following two conditions are fulfilled*:

i) *$f(A) \in \bar{\mathfrak{S}}(N)$ for every $A \in \bar{\mathfrak{S}}(M)$.*

ii) *All fibres $f^{-1}(y)$, $y \in N$, are partially complete spaces.*

THEOREM 2.8. *A locally semialgebraic map $f: M \to N$ is partially proper if and only if for any path $\alpha: [0, 1] \to N$, and any semialgebraic map $\beta: [0, 1[\to M$ with $f \circ \beta = \alpha \mid [0, 1[$, there exists a (unique) path $\bar{\beta}: [0, 1] \to M$ such that $f \circ \bar{\beta} = \alpha$ and $\bar{\beta} \mid [0, 1[= \beta$.*

COROLLARY 2.9. *A locally semialgebraic space M is partially complete if and only if every semialgebraic map $\alpha: [0, 1[\to M$ can be completed to a path $\bar{\alpha}: [0, 1] \to M$.*

One can derive from Theorem 2.8 that any pull-back of a partially proper map is again partially proper.

Important examples of partially proper maps are given by Proposition 2.10 below.

DEFINITION 6. a) A locally semialgebraic map $f: M \to N$ is called trivial if for one point $y \in N$ (and thus for every $y \in N$) there is a locally semialgebraic isomorphism $\varphi: M \to N \times f^{-1}(y)$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow[\sim]{\varphi} & N \times f^{-1}(y) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & N & \end{array}$$

commutes, where pr_1 denotes the natural projection.

b) f is called weakly locally trivial if every $y \in N$ has an open locally semialgebraic neighborhood $L \in \mathring{\mathcal{T}}(N)$ such that the restriction $f|f^{-1}(L)$: $f^{-1}(L) \to L$ is trivial.

c) f is called locally trivial if N has an admissible covering $(N_\beta|\beta \in J)$ such that for every $\beta \in J$ the restriction $f|f^{-1}(N_\beta): f^{-1}(N_\beta) \to N_\beta$ is trivial.

PROPOSITION 2.10. *Every weakly locally trivial map $f: M \to N$ with partially complete fibres is partially proper.*

This can be easily seen using Proposition 2.7.

3. Simplicial complexes and triangulations. We need some definitions of a combinatorial nature, cf. [6, §2]. Recall that an open n-simplex in some vector space V over R is a set

$$\sigma = \{\sum_{i=0}^{n} t_i e_i | t_i \in R,\ t_i > 0,\ \sum_{i=0}^{n} t_i = 1\}$$

with affinely independent points $e_0, \ldots, e_n$ of V which are called the vertices of σ. The closure $\bar{\sigma}$ of σ is defined to be the convex hull of $e_0, \ldots, e_n$ in V.

DEFINITION 1. a) A (geometric) simplicial complex over R is a pair $(X, \Sigma(X))$ consisting of a subset X of some vector space V over R and a family of pairwise disjoint open simplices σ in V such that the following two properties hold:

i) X is the union of the family $\Sigma(X)$,

ii) the intersection $\bar{\sigma} \cap \bar{\tau}$ of the closure of any two simplices $\sigma, \tau \in \Sigma(X)$ is either empty or a face (defined as usual) of both $\bar{\sigma}$ and $\bar{\tau}$.

b) The closure of the complex $(X, \Sigma(X))$ is the pair $(\bar{X}, \Sigma(\bar{X}))$, where $\Sigma(\bar{X})$ is the set of all open faces of all $\sigma \in \Sigma(X)$ and $\bar{X}$ is the union of all $\tau \in \Sigma(\bar{X})$. This is again a simplicial complex.

c) The complex $(X, \Sigma(X))$ is called closed if $X = \bar{X}$, or, equivalently, if $\Sigma(X) = \Sigma(\bar{X})$.

REMARK. Our notion of simplicial complex differs slightly from the classical one (Classical simplicial complexes are the closed complexes in our terminology). The reason is that the combinatorial semialgebraic topology used by us is based at least as much upon open simplices as upon closed simplices. Classically, an open simplex σ is usually regarded as a simplicial complex consisting of infinitely many closed simplices. In our theory, this would erringly replace the semialgebraic space σ by the locally semialgebraic space σ_{loc} introduced in Example 1.6.

We often denote a simplicial complex $(X, \Sigma(X))$ simply by X. Let X be a simplicial complex in a vector space V over R. A subcomplex Y of X is called closed in X if Y if the intersection of the subcomplexes X and $\bar{Y}$ of $\bar{X}$. It is called open in X if the complex $X \backslash Y$ is closed in X.

DEFINITION 2. a) X is called finite if $\Sigma(X)$ is finite.

b) X is called locally finite if every $\sigma \in \Sigma(X)$ is contained in a finite open subcomplex of X.

c) X is called strictly locally finite if $\bar{X}$ is locally finite. This means that every vertex of X is a vertex of only finitely many $\sigma \in \Sigma(X)$.

There are many examples of locally finite complexes which are not strictly locally finite (e.g., infinitely many open 1-simplices with one common vertex).

Locally finite simplicial complexes may be equipped with the structure of a locally semialgebraic space in a natural way. First observe that a finite simplicial complex X is an affine semialgebraic space because it is a semialgebraic subset of the finite dimensional vector space spanned by the vertices of X.

Let now X be a locally finite simplicial complex over R. The set $(X_\lambda | \lambda \in \Lambda)$ of all open and finite subcomplexes of X is a directed system ($\lambda \leq \mu$ iff $X_\lambda \subset X_\mu$) with union X. If $X_\lambda \subset X_\mu$, the semialgebraic space X_λ is an open subspace of the semialgebraic space X_μ. According to Lemma 1.2 X has the structure of a locally semialgebraic space over R such that $X = \varinjlim X_\lambda$. In this way we consider every locally finite simplicial complex to be a locally semialgebraic space.

The closure of a locally finite complex X is a locally semialgebraic space only if X is strictly locally finite.

The following statements are easily checked.

PROPOSITION 3.1. *Let X be a locally finite simplicial complex and Y be a subcomplex of X.*

i) *X is a regular and paracompact locally semialgebraic space.*

ii) *X is partially complete if and only if X is closed.*

iii) *Y is a locally semialgebraic subspace of X.*

iv) *Y is an open (resp. closed) locally semialgebraic subset of X if and only if Y is an open (resp. closed) subcomplex of X.*

Conversely, we can prove that every paracompact regular locally semialgebraic space M is isomorphic to a strictly locally finite simplicial complex X.

THEOREM 3.2. *Let M be a regular paracompact locally semialgebraic space over R and let $(A_\lambda | \lambda \in \Lambda)$ be a locally finite family of locally semialgebraic subsets of M. Then there exist a strictly locally finite simplicial complex X over R, a family $(Y_\lambda | \lambda \in \Lambda)$ of subcomplexes of X, and a locally semialgebraic isomorphism $\varphi: X \simeq M$ which maps Y_λ onto A_λ for every $\lambda \in \Lambda$.*

For affine semialgebraic spaces this fact is well known ([7, §2], [13], [11] for $R = \mathbf{R}$). Hence, if $(M_\alpha | \alpha \in I)$ is a locally finite covering of M by open semialgebraic subsets, we can find, for every $\alpha \in I$, a simultaneous triangulation φ_α of M_α and the finitely many subsets $M_\alpha \cap A_\lambda$ which are not empty. The main problem in the proof of Theorem 3.2 is glueing all these "local" triangulations φ_α together into a global triangulation φ. It first appears that Theorem 2.6 is a trivial consequence of Theorem 3.5 (simply take the closure $\bar{X}$ of X for $\bar{M}$). But our glueing procedure only works for partially complete spaces. Theorem 2.6 guarantees that it suffices to consider this case.

4. The fundamental group. Below we will define the (semialgebraic) fundamental group of a locally semialgebraic space. This will enable us to discuss locally semialgebraic coverings in the next section.

NOTATION. Let M and N be locally semialgebraic spaces and A and B be locally semialgebraic subsets of M and N. Then, as usual, a locally semialgebraic map f: $(M, A) \to (N, B)$ is a locally semialgebraic map f: $M \to N$ with $f(A) \subset B$. The letter I will always denote the unit interval $[0, 1]$ in our real closed base field R.

DEFINITION 1. Two locally semialgebraic maps f, g: $(M, A) \rightrightarrows (N, B)$ are called homotopic if there is a locally semialgebraic map F: $(M \times I, A \times I) \to (N, B)$ (called a homotopy from f to g) with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for every $x \in M$.

DEFINITION 2. Let M be a locally semialgebraic space and $x_0 \in M$. Then $\pi_1(M, x_0)$ is the set of homotopy classes of semialgebraic maps α: $(I, \{0, 1\}) \to (M, \{x_0\})$. $\pi_1(M, x_0)$ with the composition defined as usual is a group called the (semialgebraic) fundamental group of M with base point x_0.

As in classical homotopy theory, every path β: $[0, 1] \to M$ with $\beta(0) = x_0$ and $\beta(1) = x_1$ induces an isomorphism from $\pi_1(M, x_0)$ onto $\pi_1(M, x_1)$. Thus, up to isomorphism, the fundamental group does not depend on the choice of the base point, provided the space is connected. Of course, every locally semialgebraic map f: $(M, x_0) \to (N, y_0)$ induces a group homomorphism f_* from $\pi_1(M, x_0)$ to $\pi_1(N, y_0)$ which only depends on the homotopy class of f.

Connected spaces with trivial fundamental group are called simply connected. Since a semialgebraic space can be covered by finitely many contractible open semialgebraic subsets (recall that it can be triangulated), every locally semialgebraic space has an admissible covering by open simply connected semialgebraic subsets.

If M is a connected locally semialgebraic space over R, then for any real closed field $\tilde{R} \supset R$ the space $M(\tilde{R})$ over $\tilde{R}$ obtained from M by base extension (cf. Example 1.7) is again connected. Indeed, for any space M over R and any point x of $M(\tilde{R})$, there exists a path over $\tilde{R}$ which connects x with a point of the subset M of $M(\tilde{R})$.

We now state a theorem which enables us to compute the fundamental group in many cases by transfer from known topological fundamental groups. The analogue of this theorem for the higher homotopy groups, to be defined in the obvious way, is also true.

THEOREM 4.1. *Let M be a regular space over R.*

i) *For any real closed field $\tilde{R} \supset R$ the natural homomorphism $\pi_1(M, x_0) \to \pi_1(M(\tilde{R}), x_0)$ which maps the class $[\alpha]$ of a loop α: $[0, 1] \to$*

M with $\alpha(0) = \alpha(1) = x_0$ *to the class* $[\alpha_{\tilde{R}}]$, *is an isomorphism. Here* $\alpha_{\tilde{R}}$ *denotes the loop obtained from* α *by base extension* (cf. Ex. 1.7).

ii) *If* $R = \mathbf{R}$ *then the obvious homomorphism from* $\pi_1(M, x_0)$ *to the topological fundamental group* $\pi_1(M_{\text{top}}, x_0)$ *of the associated topological space* M_{top} *(strong topology) is an isomorphism.*

We illustrate the possibility of transferring results on topological fundamental groups to semialgebraic fundamental groups by an example (See [**6**, §5] where a similar transfer method was used in homology).

EXAMPLE 4.2. Let X be a smooth and irreducible algebraic curve of genus g over an algebraically closed field C of characteristic zero. Denote by $X(C)$ the set of C-rational points. We choose a base point $x_0 \in X(C)$ and a real closed subfield R of C such that $C = R(\sqrt{-1})$. Identifying C with R^2, the set $X(C)$ becomes an affine semialgebraic space over R. Let $\bar{X}$ be the smooth completion of X, i.e., the smooth complete algebraic curve of genus g over C containing X as a Zariski open subset. We have $X(C) = \bar{X}(C)\setminus\{p_1, \ldots, p_r\}$ with points $p_1, \ldots, p_r \in \bar{X}(C)$ $(r \geqq 0)$. Then, as in the case $C = \mathbf{C}$ (cf. [**1**, chap. I]), $\pi_1(X(C), x_0)$ is the free group with $2g + r - 1$ generators, provided $r \geqq 1$. If X is complete, i.e., $r = 0$, $\pi_1(X(C), x_0)$ is generated by $2g$ elements $a_1, b_1, \ldots, a_g, b_g$ with the single relation $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1$.

It is a remarkable feature of Example 4.2 that the group $\pi_1(X(C), x_0)$ does not depend on the choice of the real closed subfield R of C. For X a smooth algebraic variety over C of dimension $\geqq 2$ this is no longer true in general, cf. [**12**, end of §1].

An important tool for the study of locally semialgebraic coverings is the following homotopy lifting property for locally trivial maps (recall Definition 6 in §2).

THEOREM 4.3. *Let* $p\colon M \to N$ *be a locally trivial locally semialgebraic map between locally semialgebraic spaces* M *and* N. *Let* L *be a regular paracompact locally semialgebraic space and* $H\colon L \times [0, 1] \to N$ *be a locally semialgebraic homotopy. Assume that a locally semialgebraic map* $f\colon L \to M$ *is given with* $p(f(x)) = H(x, 0)$ *for every* $x \in L$. *Then there exists a locally semialgebraic homotopy* $G\colon L \times [0, 1] \to M$ *such that* $G(x, 0) = f(x)$ *for every* $x \in L$ *and* $p \circ G = H$.

Since no compactness arguments are available, the proof is rather different from the topological one. After choosing a suitable triangulation of L, the lifted homotopy G is constructed successively on the k-skeleta of L. We strongly use the fact that an arbitrary pair (M, A) of affine semialgebraic spaces with A closed in M has the homotopy extension property ([**7**, §4, §5]).

5. Coverings. Coverings are important examples of the following class of locally semialgebraic morphisms.

DEFINITION 1. A locally semialgebraic map $f\colon M \to N$ over R is called a local isomorphism if every point $x \in M$ has an open locally semialgebraic neighborhood U such that the restriction $f|U\colon U \to f(U)$ is a locally semialgebraic isomorphism from U onto an open locally semialgebraic subset of N.

EXAMPLE 5.1. Denote by C the algebraic closure $R(\sqrt{-1})$ of R. Let $f\colon V \to W$ be an etale algebraic morphism between algebraic varieties over R (resp. over C). Then the induced semialgebraic map $f_R\colon V(R) \to W(R)$ (resp. $f\colon V(C) \to W(C)$) is a local isomorphism. This is an easy conclusion from the implicit function theorem ([**4**, 6.9]).

The following important result says that every local isomorphism is even a local isomorphism "in the strong sense".

THEOREM 5.2. *Let $f\colon M \to N$ be a local isomorphism between locally semialgebraic spaces M, N over R. Then M has an admissible covering $(M_\alpha | \alpha \in I)$ such that M_α is mapped isomorphically onto an open locally semialgebraic subset of N by f for every $\alpha \in I$.*

DEFINITION 2. A locally semialgebraic map $p\colon M \to N$ is called a locally semialgebraic covering of N (or "covering" in short), if p is surjective, locally trivial (cf. §2, Definition 6), and has discrete fibres (cf. Example 1.4). p is called connected if M, and hence also N, is connected.

DEFINITION 3. Let $f\colon M \to N$ be a locally semialgebraic map.

a) f is called finite if f is proper and all fibres $f^{-1}(y)$, $y \in N$, are finite sets.

b) f is called partially finite if $f|A\colon A \to N$ is finite for every closed semialgebraic subset A of M (or, equivalently, if f is partially proper and has discrete fibres).

PROPOSITION 5.3. *Every covering $p\colon M \to N$ is a partially finite map and a local isomorphism.*

The first statement follows from Proposition 2.10. The second is fairly obvious since p is locally trivial. For finite maps we have a converse to proposition 5.3.

PROPOSITION 5.4. *Let $p\colon M \to N$ be a finite locally semialgebraic map which is a local isomorphism. Assume that M is paracompact and N is connected. Then p is a covering.*

EXAMPLE 5.5. Let $f\colon V \to W$ be a finite algebraic morphism between varieties over R (resp. over $C = R(\sqrt{-1})$), and let N be a semialgebraic subset of $W(R)$ (resp. $W(C)$). Assume that f is etale at all points of $M :=$

$V(R) \cap f^{-1}(N)$ (resp. $M := f^{-1}(N)$). Then the map $p: M \to N$ obtained from f by restriction is finite. We conclude from Ex. 5.1 and Prop. 5.4 that p is a covering.

PROPOSITION 5.6. *Let $p: M \to N$ be a covering of a paracompact locally semialgebraic space N. Then M is paracompact.*

PROOF. We choose a locally finite covering $(N_\beta | \beta \in J)$ of N by open semialgebraic subsets such that p is trivial over N_β for every $\beta \in J$. Then $p^{-1}(N_\beta) = \bigsqcup (M_{\beta\alpha} | \alpha \in I(\beta))$ is the disjoint union of open semialgebraic subsets $M_{\beta\alpha}$ of M. $(M_{\beta\alpha} | \beta \in J, \alpha \in I(\beta))$ is a locally finite covering of M.

As in classical topology coverings have the "unique path lifting property".

PROPOSITION 5.7. *Let $p: M \to N$ be a locally semialgebraic covering and $\alpha: [0, 1] \to N$ be a path. Then for every $x_0 \in M$ with $p(x_0) = \alpha(0)$ there is a unique path $\beta: [0, 1] \to M$ with $\beta(0) = x_0$ and $p \circ \beta = \alpha$.*

The proof is very easy.

COROLLARY 5.8. *Let $p: M \to N$ be a covering. Assume that N is connected. Then all fibres of p have the same cardinality (if this cardinality is finite, we call it the degree of p).*

Homotopies can be lifted to a covering space (Theorem 4.3). This implies the following theorem.

THEOREM 5.9. *Let $p: M \to N$ be a locally semialgebraic covering, $x_0 \in M$, $p(x_0) = y_0$. Let $\alpha_0, \alpha_1: [0, 1] \rightrightarrows N$ be paths with $\alpha_0(0) = \alpha_1(0) = y_0$ and $\alpha_0(1) = \alpha_1(1)$ which are homotopic with fixed endpoints. Then the uniquely determined liftings $\beta_0, \beta_1: [0, 1] \rightrightarrows M$ of α_0, α_1 with $\beta_0(0) = \beta_1(0) = x_0$ have the common endpoint $\beta_0(1) = \beta_1(1)$ and are homotopic with fixed endpoints.*

Starting with this theorem, we are able to classify the coverings, essentially using the same arguments as in classical topology.

THEOREM 5.10. *Let $p: M \to N$ be a connected locally semialgebraic covering, $x_0 \in M$, $y_0 = p(x_0)$.*

i) *The induced homomorphism $p_*: \pi_1(M, x_0) \to \pi_1(N, y_0)$ is injective. If p is finite, the index of $\pi_1(M, x_0)$ in $\pi_1(N, y_0)$ is equal to the degree of p.*

ii) *If $f: (L, z_0) \to (N, y_0)$ is a locally semialgebraic map with L connected then f can be lifted to a locally semialgebraic map $g: (L, z_0) \to (M, x_0)$ with $p \circ g = f$ if and only if $f_*(\pi_1(L, z_0)) \subset p_*(\pi_1(M, x_0))$. g is uniquely determined provided it exists. If f is a covering, then g is also a covering.*

In particular, Theorem 5.10 says that every connected covering $p: M \to$

$\tilde{N}$ of N is determined up to isomorphy by the subgroup $p_*(\pi_1(M, x_0))$ of $\pi_1(N, y_0)$. Conversely we can prove the next theorem.

THEOREM 5.11. *Let N be a connected locally semialgebraic space, $y_0 \in N$ and H be a subgroup of $\pi_1(N, y_0)$. Then there exists a connected locally semialgebraic covering $p: M \to N$ with $p_*(\pi_1(M, x_0)) = H$ for some $x_0 \in M$ with $p(x_0) = y_0$.*

Again the proof essentially runs along the same lines as the corresponding proof in topology. The set M is defined as follows:

$$M := \{\alpha: [0, 1] \to N | \alpha \text{ a semialgebraic path}, \alpha(0) = y_0\}/\sim.$$

Here two paths $\alpha, \beta \in M$ are defined to be equivalent if and only if $\alpha(1) = \beta(1)$ and the homotopy class of $\alpha * \beta^{-1}$ (where $*$ denotes the composition of paths) is an element of H. Mapping an equivalence class $[\alpha] \in M$ to $\alpha(1)$ we get a map $p: M \to N$. For $U \in \mathring{\mathscr{T}}(N)$ and a path $\alpha: [0, 1] \to N$ with $\alpha(0) = y_0$ and $\alpha(1) \in U$ we define

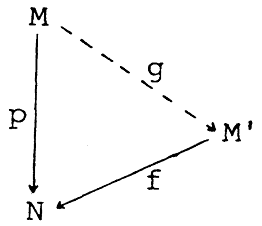
$$M(\alpha, U) := \{[\alpha * \gamma] \in M \mid \gamma: [0, 1] \to U, \gamma(0) = \alpha(1)\}.$$

Then we choose an admissible covering $(N_i | i \in I)$ of N by open simply connected locally semialgebraic subsets. The preimage $p^{-1}(N_i)$ of N_i is the disjoint union $\bigsqcup(M(\alpha_j, N_i) | j \in J(i))$ of sets $M(\alpha_j, N_i)$, α_j running through a suitably chosen system of representants. If U is simply connected, then $M(\alpha, U)$ is mapped bijectively to U by p. Thus p allows us to transfer the locally semialgebraic structure of N_i to $M(\alpha_j, N_i)$ for every $j \in J(i)$. But then $p^{-1}(N_i)$ also becomes a locally semialgebraic space in a canonical way (Ex. 1.4). One may now check that the structures on $p^{-1}(N_i)$ and $p^{-1}(N_j)$ fit together for $N_i \cap N_j \neq \emptyset$. We equip M with the "inductive limit structure" (cf. Lemma 1.2) and in this way obtain the desired locally semialgebraic covering $p: M \to N$ with $p_*(\pi_1(M, x_0)) = H$. Here x_0 is the point given by the constant path $[0, 1] \to \{y_0\}$.

Applying Theorem 5.11 to the trivial subgroup of $\pi_1(N, y_0)$ we derive the following corollary.

COROLLARY 5.12. *Let N be a connected locally semialgebraic space and $y_0 \in N$.*

i) *Up to isomorphy there exists a uniquely determined connected locally semialgebraic covering $p: M \to N$, called the universal covering of N, with the following universal property. For every $x_0 \in M$ with $p(x_0) = y_0$ and every connected covering $f: (M', z_0) \to (N, y_0)$ there exists a uniquely determined locally semialgebraic map $g: (M, x_0) \to (M', z_0)$ such that the diagram*



communes. The map g is again a covering.

ii) *A connected covering $p'\colon M' \to N$ is universal and hence isomorphic to $p\colon M \to N$, if and only if M' is simply connected.*

EXAMPLE 5.13. Let V be an algebraic variety over the algebraic closure $C = R(\sqrt{-1})$ of R. (N.B. Every algebraically closed field of characteristic zero is of this type). Then V in general has no algebraic universal covering. But the universal covering exists in the category of locally semialgebraic spaces over R.

EXAMPLE 5.14. Let A be the union of all intervals $]-n, n[$ in R with $n \in \mathbf{N}$. A is the smallest valuation ring of R compatible with the ordering. We equip A with the "inductive limit structure" of the open subspaces $]-n, n[$ of R. Then we choose a surjective semialgebraic map e from $[0, 1]$ to the unit circle $S^1 \subset R^2$ which maps 0 and 1 to the end point $(0, 1)$ and is injective on $]0, 1[$ (cf. [**8**, §6]). e extends to a locally semialgebraic map $p\colon A \to S^1$ by $p(m + x) = e(x)$ $(m \in \mathbf{Z},\ x \in [0, 1[)$. p is the universal covering of S^1.

Let N be a locally semialgebraic space over $\mathbf{R}$. In both theories, the topological and the semialgebraic, the coverings are classified by the subgroups of the fundamental group. Hence we obtain from Theorem 4.1.

COROLLARY 5.15. *Every topological covering of N is topologically isomorphic to a locally semialgebraic covering. Two locally semialgebraic coverings of N are topologically isomorphic if and only if they are locally semialgebraically isomorphic.*

We thank Roland Huber at Regensburg who first proved Theorem 5.2 in full generality.

REFERENCES

1. L .V. Ahlfors and L. Sario, *Riemann Surfaces*, Princeton University Press 1960.

2. M. Artin, *Grothendieck topologies*, Seminar Harvard University 1962.

3. H. Delfs, *Kohomologie affiner semialgebraischer Räume*, Diss. Univ. Regensburg, 1980.

4. H. Delfs and M. Knebusch, *Semialgebraic topology over real closed fields* II; *Basic theory of semialgebraic spaces*, Math. Z. **178** (1981), 175–213.

5. ———, *Semialgebraic topology over a real closed field*, in: *Ordered Fields and Real Algebraic Geometry*, Ed. D. W. Dubois and T. Recio, Contemporary Math. 8, Amer. Math. Soc. 1981, 61–78.

6. ———, *On the homology of algebraic varieties over real closed fields*, J. reine angew. Math. **335** (1982), 122–163.

7. ———, *Separation, retractions and homotopy extension in semialgebraic spaces*, to appear Pacific J. Math. 114 No. 1 (1984).

8. ———, *Zur Theorie der semialgebraischen Wege und Intervalle über einem reell abgeschlossenen Körper*, *in*: *Geometrie algébrique réelle et formes quadratiques*, Ed. J. L. Colliot-Thélène et al., Lecture Notes Math. 959, 299–323, Springer Verlag 1982.

9. D. W. Dubois, *Real algebraic curves*, Technical Report N . 227, Univ. New Mexico. Albuquerque 1971.

10. A. Grothendieck and J. Dieudonné, *Elements de géométrie algébrique II*", Publ. Math. IHES No. 8 (1961).

11. H. Hironaka, *Triangulations of algebraic sets*, in: *Algebraic geometry Arcata 1974*, Ed. R. Hartshorne, Proc. Sympos. Pure Math. 29, 165–185, Amer. Math. Soc. 1975.

12. M. Knebusch, *Isoalgebraic geometry*: *First steps*, *Seminaire de Théorie des nombres*, *Delange-Pisot-Poitou* 1981, to appear.

13. S. Lojasiewicz, *Triangulation of semi-analytic sets*, Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 449–474.

14. R. Robson, *Embedding semi-algebraic spaces*, Math. Z. **183** (1983), 365–370.

15. M. Artin, A. Grothendieck and J. L. Verdier, *Théorie des topos et cohomologie étale de schemas*, Lecture Notes Math. 269, Springer Verlag 1972.

FAKULTÄT FUR MATHEMATIK DER UNIVERSITÄT D – 8400 REGENSBURG, F.R.G.