On the Uniqueness of Real Closures and the Existence of Real Places

MANFRED KNEBUSCH (Saarbrücken)

σ is the *restriction* of τ to K, if $\sigma=\tau\circ\varphi_*$. If K and L are fields, this means that φ is compatible with the orderings corresponding to σ and τ. If $\mathfrak{p}$ is a prime ideal of $W(K)$ lying minimally over $\Lambda(\varphi)$ then there exists a (minimal) prime ideal $\mathfrak{q}$ of $W(L)$ with $\varphi_*^{-1}(\mathfrak{q})=\mathfrak{p}$ (cf. [5] Chap. II. §2 no 7 Prop. 16). This remark immediately implies

PROPOSITION 2.1. *A signature σ of K can be extended to a signature of L with respect to φ if and only if $\sigma(\Lambda(\varphi))=0$.*

§3. The Transfer Map

Now assume in addition that $\varphi: K\to L$ is finite and separable. In our case, this means that in the decomposition $L=L_1\times\cdots\times L_r$ of L as a K-module corresponding to the decomposition $K=K_1\times\cdots\times K_r$, any L_i is either zero or a product of finite separable field extensions of K_i. For later convenience I recall the definition of the *regular trace* $Tr=Tr_\varphi$ from L to K ([3], p. 397, [7]). We identify the ring $\mathrm{End}_K(L)$ of K-linear endomorphisms of L with $\mathrm{Hom}_K(L, K)\otimes_K L$. This is possible since L is a finitely generated projective K-module. For any x in L we denote by $\mathsf{L}(x)$ the K-linear endomorphism $y\mapsto xy$ of L. We further denote by e: $\mathrm{End}_K(L)\to K$ the evaluation map $f\otimes x\mapsto f(x)$ $\{f$ in $\mathrm{Hom}(L, K)$, x in $L\}$. The regular trace is defined by

$$Tr(x)=e(\mathsf{L}(x)).$$

As is well known (loc. cit.) the K-bilinear form $Tr(xy)$ on L is non singular. From this one sees immediately (cf [6]) that for any space (E, B) over L the K-module E equipped with the bilinear form $Tr\circ B$: $E\times E\to K$ is a space over K. We thus get an additive map ([18])

$$Tr^*: W(L)\to W(K).$$

sending $\{(E, B)\}$ to $\{(E, Tr\circ B)\}$. This "transfer-map" is related to $\varphi_*: W(K)\to W(L)$ by the following "Frobenius law"

PROPOSITION 3.1. ([18]).

$$x\cdot Tr^*(y)=Tr^*(\varphi_*(x)\,y)\ \textit{for } x \textit{ in } W(K) \textit{ and } y \textit{ in } W(L).$$

In fact, for spaces (E_1, B_1) over K and (E_2, B_2) over L the canonical map from $E_1\otimes_K E_2$ to $(L\otimes_K E_1)\otimes_L E_2$ is an isomorphism of spaces with respect to the forms $B_1\otimes_K(Tr\circ B_2)$ and $Tr\circ(B_1'\otimes_L B_2)$ where B_1' is obtained from B_1 by base extension.

We denote the image $Tr^*(W(L))$ by $M(\varphi)$. The Frobenius law implies

COROLLARY 3.2. *$M(\varphi)$ is an ideal of $W(K)$ and $\Lambda(\varphi)\ M(\varphi)=0$.*

The transfer map is compatible with base extensions. This is the content of the following "Mackey law", whose importance has been revealed by Dress in [6].

PROPOSITION 3.3. *Assume $\varphi: K \to L$ and $\psi: K \to M$ are homomorphisms between (semisimple) rings and that φ is finite and separable. Then the diagram*

$$\begin{array}{ccc} W(L) & \xrightarrow{(\psi\otimes 1)_*} & W(M\otimes_K L) \\ {\scriptstyle Tr^*_\varphi}\downarrow & & \downarrow{\scriptstyle Tr^*_{1\otimes\varphi}} \\ W(K) & \xrightarrow{\psi_*} & W(M) \end{array}$$

is commutative.

Indeed, by erasing the letters "W" and the asterics, one obtains a commutative diagram, as follows easily from the description of the regular trace above. Starting from this fact the proof of Proposition 3.3 is straightforward.

We shall often write $Tr_{L/K}$ instead of Tr_φ, when there is no doubt as to which mapping $\varphi: K \to L$ is being considered.

Remark. Clearly the statements of this section remain true for the Witt rings of arbitrary commutative rings in the sense of [12], if $\varphi: K \to L$ is finite étale, in other words, if L is a finitely generated projective K-module and a projective $L\otimes_K L$-module ([8] Prop. 18.3.1, p. 114).

§ 4. Proof of the Uniqueness Theorem 1.3

Throughout this section L/K will be a finite extension of fields of characteristic zero. Denote by $Tr^*(1)$ the value of the unit element (1) of $W(L)$ under the transfer map $Tr^*: W(L) \to W(K)$ with respect to the inclusion $i: K \to L$.

LEMMA 4.1. *For any signature σ of K we have $\sigma(Tr^*(1)) \geqslant 0$. The signature σ can be extended to L if and only if $\sigma(Tr^*(1)) > 0$.*

Proof. We proceed by induction on $n=[L:K]$. For $n=1$ the assertion is trivial. Assume $n>1$. If σ can not be extended to L then by Prop. 2.1 we have $\sigma(\Lambda(i)) \neq 0$ and hence by Cor. 3.2 $\sigma(M(i))=0$, in particular $\sigma(Tr^*(1))=0$. Now we assume that σ has at least one extension τ in Sign L. Then

$$\sigma(Tr^*(1)) = \tau(i_* Tr^*(1)).$$

By the Mackey law 3.3 the diagram

$$\begin{array}{ccc} W(L) & \xrightarrow{(i\otimes 1)_*} & W(L\otimes_K L) \\ {\scriptstyle Tr^*_{L/K}}\downarrow & & \downarrow{\scriptstyle Tr^*_{L\otimes L/L}} \\ W(K) & \xrightarrow{i_*} & W(L) \end{array}$$

is commutative. Here $L \otimes_K L$ is considered as an extension of L by $1 \otimes i: L = L \otimes_K K \to \to L \otimes_K L$. This extension is L-isomorphic with a product $E_1 \times E_2 \times \cdots \times E_t$ where $E_1 = L$ and $E_2, \ldots, E_t$ are algebraic field extensions of L with degrees smaller than n.

Thus

$$\sigma(Tr^*_{L/K}(1)) = \sum_{i=1}^{t} \tau(Tr^*_{E_i/L}(1)).$$

The first summand equals 1 and the others are ≥ 0 by the induction hypothesis, which shows $\sigma(Tr^*_{L/K}(1)) > 0$. q.e.d.

For later convenience we recall the following

EXAMPLE 4.2. Assume $L = K(\sqrt{a})$ with some $a \neq 0$ in K. Then a signature σ of K is extendable to L if and only if $\sigma(a) = 1$. Indeed, without loss of generality assume $L \neq K$. $Tr^*(1)$ is represented by the space L over K with bilinear form $Tr(xy)$. This space is isomorphic to $(2) \perp (2a)$ (Consider the basis $1, \sqrt{a}$).

For the remainder of this section R denotes a real closed field and ϱ denotes the signature of R.

LEMMA 4.3 *Assume φ is a homomorphism from K into R such that*

$$\varrho(\varphi_* Tr^*(1)) > 0.$$

Then φ can be extended to homomorphism from L into R.

Proof. Let σ denote the signature $\varrho \circ \varphi_*$ of K. The tensor product $R \otimes_K L$ constructed from $\varphi: K \to R$ and the inclusion $i: K \to L$ has a decomposition

$$R \otimes_K L = E_1 \times \cdots \times E_t$$

into fields. We regard $R \otimes_K L$ as an extension of R by the map $1 \otimes i$ from $R = R \otimes_K K$ to $R \otimes_K L$. From the Mackey law 3.3 we obtain as in the proof of lemma 4.1:

$$\sigma(Tr^*_{L/K}(1)) = \varrho \circ Tr^*_{R \otimes L/R}(1) = \sum_{j=1}^{t} \varrho \circ Tr^*_{E_j/R}(1).$$

Now according to (1.1) any E_j is R-isomorphic either to R or to the algebraic closure $\bar{R}$ of R. Since $W(\bar{R}) \cong \mathbf{Z}/2\mathbf{Z}$ is a torsion group, $Tr^*_{\bar{R}/R}(1) = 0$. Thus we see that $\sigma(Tr^*_{L/K}(1))$ equals the number of components E_j which are isomorphic to R. Recall that $E_1, \ldots, E_t$ form a full system of inequivalent field composites of L and R over K ([11] Chap. I, §16). Thus it follows that $\sigma(Tr^*_{L/K}(1))$ is the number of different homomorphisms from L to R which extend φ. Since by assumption $\sigma(Tr^*_{L/K}(1)) > 0$, the proof is complete.

PROPOSITION 4.4. *Assume L is ordered and $\varphi: K \to R$ is an order preserving homomorphism. Then there exists an order preserving homomorphism $\psi: L \to R$ which extends φ.*

Proof. i) Let σ denote the signature of K corresponding to the restriction of the ordering of L to K. The assumption that φ is order preserving means $\sigma = \varrho \circ \varphi_*$. Since σ can be extended to L we know from Lemma 4.1 that $\sigma(Tr^*(1)) > o$ and thus by Lemma 3.4 that there exists at least one homomorphism from L to R which extends φ.

ii) Let $\psi_1, \ldots, \psi_r$ denote the different homomorphisms from L to R which extend φ. We must show that at least one of the ψ_i is order preserving. Assume the contrary is true. Then for any ψ_i, $1 \leqslant i \leqslant r$, we have an element $a_i > 0$ in L such that $\psi_i(a_i) < 0$. By example 4.2 the ordering of L can be extended to the field $M = L(\sqrt{a_1}, \ldots, \sqrt{a_r})$. By part i) of our proof there exists a homomorphism $\chi: M \to R$ which extends φ. Certainly $\chi(a) = \chi(\sqrt{a_i})^2 > 0$ for $1 \leqslant i \leqslant r$. Thus $\chi \mid L$ can not coincide with any of the ψ_i, which yields the desired contradiction. q.d.e.

After these preparations the proof of Theorem 1.3 is easy. From Proposition 4.4 and an application of Zorn's lemma it follows that the given order preserving homomorphism $\varphi: K \to R$ can be extended to a homomorphism ψ from the real closure F to R. We still must show that ψ is the only extension of φ from F to R. For any $\chi: F \to R$ extending φ the images $\chi(F)$ and $\psi(F)$ both coincide with the algebraic closure of K in R. Thus there exists an automorphism λ of F/K with $\chi = \psi \circ \lambda$. Of course λ is order preserving. Assume λ is not the identity. Then we can find some x in F with $x < \lambda(x)$. Applying λ repeatedly to this inequality we obtain

$$x < \lambda(x) < \lambda^2(x) < \cdots < \lambda^n(x)$$

for all n, which is impossible since $\lambda^m(x) = x$ for some m (depending on x). This completes the proof of Theorem 1.3.

§5. A Trace Formula for Signatures

I want to throw some more light on Lemma 4.1. We first return to Proposition 4.4. We denote by F a fixed real closure of L with respect to the given ordering. By Theorem 1.3 any order preserving extension $\psi: L \to R$ of φ can be further extended to F. Thus the uniqueness statement in Theorem 1.3, applied to F/K, shows that there is a *unique* order preserving $\psi: L \to R$ which extends φ. We obtain the well known

COROLLARY 5.1. Let L/K be a finite algebraic field extension and $\varphi: K \to R$ a homomorphism from K into a real closed field R. Denote by ϱ the signature of R and by σ the induced signature $\varrho \circ \varphi_*$ of K. Then the signatures τ of L extending σ correspond bijectively with the homomorphisms $\psi: L \to R$ extending φ via $\tau = \varrho \circ \psi_*$.

We stay in the situation of this corollary. Denote by S the set of all $\psi: L \to R$ extending φ. Repeating the proof of Lemma 4.3 with an arbitrary element ξ of $W(L)$ instead of the unit element we obtain

$$\sigma(Tr^*_{L/K}(\xi)) = \varrho \circ Tr^*_{R\otimes L/R}((\varphi \otimes 1)_* \xi) = \sum_{\psi \in S} \varrho \circ \psi_*(\xi),$$

with the convention that if S is empty this sum is zero. Corollary 5.1 now implies

PROPOSITION 5.2 *Let L/K be a finite algebraic field extension and σ a signature of K. For any ξ in $W(L)$*

$$\sigma(Tr^*_{L/K}(\xi)) = \sum_{\tau|\sigma} \tau(\xi),$$

where τ runs through all signatures of L extending σ.

This proposition generalizes Lemma 4.1.

§6. Existence of Real Places on Function Fields

In this section R denotes a real closed field, K a real finitely generated field extension of R of transcendency degree $r \geqslant 1$, and S a fixed real closed field extension of R (e. g. $S = R$). We are interested in places $\varphi: K \to S \cup \infty$ over R, i. e. with φ the identity on R. The aim of this section is to prove the following theorem by the methods of §4.

THEOREM 6.1. *Assume that $(t_1, \dots, t_r)$ is a transcendency basis of K over R. Then there exist elements $a_1, \dots a_r, b_1, \dots, b_r$ in R with $a_i < b_i$, such that for every r-tuple $(c_1, \dots, c_r)$ of elements in S with $a_i < c_i < b_i$ there exists a place $\varphi: K \to S \cup \infty$ over R with $\varphi(t_i) = c_i$ for $1 \leqslant i \leqslant r$.*

Remark 6.2. Assume that S has transcendency degree $\geqslant r$ over R. Then for given elements $a_1, \dots, a_r, b_1, \dots, b_r$ in R with $a_i < b_i$ there exists an r-tuple $(c_1, \dots, c_r)$ of elements in S which are algebraically independent over R and such that $a_i < c_i < b_i$ for $1 \leqslant i \leqslant r$. Indeed, take any system $(u_1, \dots, u_r)$ of algebraically independent elements in S. Replacing u_i by $\pm u_i$ or $\pm u_i^{-1}$ we may assume that all u_i are positive and not infinitely large over R, i.e. that there exist elements d_i in R such that $0 \leqslant u_i \leqslant d_i$. Then the elements

$$c_i = a_i + d_i^{-1} u_i (b_i - a_i)$$

have the required property. Any place $\varphi: K \to S \cup \infty$ over R with $\varphi(t_i) = c_i$ must be an injection of K into S. Thus Theorem 6.1 contains Lang's theorem about embeddings of real function fields ([15], Th. 10). Our proof will be very close to the proof in [15], the main difference being that we replace the use of Sturm's theorem by results of section 4.

Remark 6.3. It is evident from the arguments in [1] that one may use theorem 6.1 to obtain generalizations of Artin's results on definite functions in [1] §2 and §4, see e.g. Lang's theorem 8 in [15].

In the proof of Theorem 6.1 we may always assume in addition that the r-tuple $(c_1, \ldots, c_r)$ is algebraically free over R. In fact, for a given S one easily constructs a real closed field extension T of S such that S is maximally archimedian in T and T contains a system $u_1, \ldots, u_r$ of elements which are algebraically independent and infinitely small over S (e. g. [15], p. 389). If $(c_1, \ldots, c_r)$ is an r-tuple in S such that there exists a homomorphism $\psi: K \to T$ over R with $\psi(t_i) = c_i + u_i$, then the composition $\varphi = \lambda \circ \psi$ with the canonical place $\lambda: T \to S \cup \infty$ of T/S ([15], p. 380) has the values $\varphi(t_i) = c_i$.

We make the following arrangements for the proof of Theorem 6.1: $Tr^*(1)$ denotes the image of the unit element of $W(K)$ under the transfer map Tr^* from $W(K)$ to $W(R(t_1, \ldots, t_r))$. We chose elements $g_1(t_1, \ldots, t_r), \ldots, g_n(t_1, \ldots, t_r)$ in $R[t_1, \ldots, t_r]$ such that $Tr^*(1)$ is represented by the space

$$(g_1(t_1, \ldots, t_r)) \perp \cdots \perp (g_n(t_1, \ldots, t_r))$$

over $R(t_1, \ldots, t_r)$. We further chose a fixed ordering of K and denote by σ the signature of $R(t_1, \ldots, t_r)$ corresponding with the restriction of this ordering to $R(t_1, \ldots, t_r)$. Finally we denote by ϱ the signature of S.

We first prove Theorem 6.1 in the case $r = 1$. Write t instead of t_1. Let

$$d_1 < d_2 < \cdots < d_N$$

be the elements of R which occur as roots of (at least one of) the polynomials $g_i(t)$ if these have any roots in R. We chose $a < b$ in R in the following way: If there are no roots let a and b be arbitrary with $a < b$. If $t < d_1$ let $b = d_1$ and a arbitrary $< d_1$. If $d_i < t < d_{i+1}$ let $a = d_i$, $b = d_{i+1}$. If $d_N < t$ let $a = d_N$ and b be arbitrary $> a$. Assume c is an element of S with $a < c < b$ and c not in R. By (1.1) every $g_i(t)$ decomposes in $R[t]$ in a product of a constant, some factors $t - d_i$ and some factors of type $(t-e)^2 + f^2$ with $f \neq 0$. Therefore clearly $g_i(c) \neq 0$ and $\varrho(g_i(c)) = \sigma(g_i(t))$ for $1 \leqslant i \leqslant n$ (cf [15] p. 386.) We denote by χ the homomorphism from $R(t)$ to S over R with $\chi(t) = c$. Then

$$\varrho \circ \chi_*(Tr^*(1)) = \sum_{i=1}^{n} \varrho(g_i(c)) = \sum_{i=1}^{n} \sigma(g_i(t)) = \sigma(Tr^*(1)).$$

Now σ can be extended to K. Thus by Lemma 4.1 we have $\varrho \circ \chi_*(Tr^*(1)) > 0$. By Lemma 4.3 χ can be extended to a homomorphism φ from K to S.

From the thus proved special case $r = 1$ of Theorem 6.1 one easily obtains (see [15], p. 387) a proof for all $r \geqslant 1$ of the following

PROPOSITION 6.4. ([15], Th. 8.). *Assume K is ordered and M is a finite set of non zero elements in K. Then there exists a place $\varphi: K \to R \cup \infty$ over R such that for all x in M the value $\varphi(x)$ is $\neq \infty$, $\neq 0$ and of the same sign as x.*

We now prove Theorem 6.1 for arbitrary $r \geqslant 1$. By Proposition 6.4 there exists a place $\varphi: K \to R \cup \infty$ over R such that the values $\varphi(t_i) = h_i$ are finite and such that the values $\varphi(g_j(t_1, \ldots, t_r)) = g_j(h_1, \ldots, h_r)$ are non zero and have the same signs as the corresponding $g_j(t_1, \ldots, t_r)$. {Recall that we have fixed an ordering on K.} Then there exists an element $\varepsilon > 0$ in R such that for all r-tuples $(c_1, \ldots, c_r)$ in S with $h_i - \varepsilon < c_i < h_i +$ $+ \varepsilon$ the values $g_j(c_1, \ldots, c_r)$ are non zero and of the same signs as the corresponding $g_j(c_1, \ldots, c_r)$ are non zero and of the same signs as the corresponding $g_j(t_1, \ldots, t_r)$. Assume that $(c_1, \ldots, c_r)$ is such an r-tuple which is in addition algebraically free over R. Denote by χ the homomorphism from $R(t_1, \ldots, t_r)$ to S over R which maps t_i to c_i for $1 \leqslant i \leqslant r$. As in the case $r = 1$ we obtain

$$\sigma(Tr^*(1)) = \varrho \circ \chi_*(Tr^*(1))$$

and we see again by Lemma 4.1 and 4.3 that χ can be extended to a homomorphism φ from K to S. Theorem 6.1 is proved.

Certainly there are other relevant questions about real places on function fields which can be treated by the methods used here. I hope to come back to this subject in the near future.

REFERENCES

[1] ARTIN, E., *Über die Zerlegung definiter Funktionen in Quadrate*, Hamb. Abh. *5* (1927), 100–115.

[2] ARTIN, E. and SCHREIER, O., *Algebraische Konstruktion reeller Körper*, Hamburger Abh. *5* (1926), 85–99.

[3] AUSLANDER, M. and GOLDMAN, O., *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. *97* (1960), 367–409.

[4] BOURBAKI, N., *Éléments de mathématique, Algébre*, (Hermann, Paris).

[5] ——, *Éléments se mathématique, Algèbre commutative*, (Hermann, Paris).

[6] DRESS, A., *The Witt ring as Mackey-functor*, Notes on the theory of representations of finite groups I, Chap. 2, Appendix A, Univ. Bielefeld 1971.

[7] DEMEYER, FR., *The trace map and separable algebras*, Osaka J. Math. *3* (1966), 7–11.

[8] GROTHENDIECK, A., *Éléments de géométrie algébrique*, Chap. IV, 4, Publ. Math. IHES No 32, Bures Sur Yvette 1967.

[9] GROSS, H. and HAFNER, P., *Über die Eindeutigkeit des reellen Abschlusses eines angeordneten Körpers*, Comm. Math. Helv. *44* (1969), 491–494.

[10] HARRISON, D. K., *Witt rings*, Lecture Notes, Department of Mathematics, University of Kentucky, Lexington, Kentucky 1970.

[11] JACOBSON, N., *Lectures in abstract algebra*, vol. III (van Nostrand 1964).

[12] KNEBUSCH, M., *Grothendieck- und Wittringe von nichtausgearteten symmetrischen Bilinearformen*, Sitzber. Heidelberg. Akad. Wiss. (1969/70), 93–157.

[13] KNEBUSCH, M., ROSENBERG, A. and WARE, R., *Structure of Witt rings and Quotients of abelian group rings*, to appear Amer. J. Math.

[14] ——, *Signatures on semi-local rings*, to appear Bull. Amer. Math. Soc.

[15] LANG, S., *The theory of real places*, Annals Math. *57* (1953), 378–391.
[16] LEICHT, J. and LORENZ, F., *Die Primideale des Wittschen Ringes*, Invent. math. *10* (1970), 82–88.
[17] MILNOR, J., *Symmetric bilinear forms*, Lecture Notes, Institute for Advanced Study Princeton N. J. 1971.
[18] SCHARLAU, W., *Zur Pfisterschen Theorie der quadratischen Formen*, Invent. Math. *6* (1969), 327–328.
[19] ——, *Quadratic Forms*, Queen's Papers in pure appl. math. No 22, Queen's University, Kingston, Ontario 1969.
[20] WITT, E., *Theorie der quadratischen Formen in beliebigen Körpern*, J. reine angew. Math. *176*, (1937), 31–44.

Mathematisches Institut der
Universität des Saarlandes
Deutschland

Received March 4, 1972