Decay of Solutions of the Scalar Wave Equation in the Schwarzschild Geometry

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Contents

Introduction

At the beginning of the 20th century, Albert Einstein proposed a new physical theory: *Special Relativity.* The main difference to Newtonian physics lies in the fact that time and space are no longer regarded as separate objects, and furthermore the speed of light is a universal constant, i.e. not depending on the reference frame. Since the Newtonian law of gravitation as an action-at-distance law was not compatible with this theory, Einstein and other workers developed a relativistic theory of gravitation. The basic idea lies in the principle of equivalence, which states [cf. [25]]

Einstein equivalence principle. *In an arbitrary gravitational field no local non-gravitational experiment can distinguish a freely falling non-rotating system (local inertial system) from a uniformly moving system in the absence of a gravitational field.*

In short, gravity can be transformed away locally. As a consequence, gravity is no longer regarded as a force, but is related to curvature of spacetime itself, which is caused by its matter content. As a mathematical model of spacetime, one uses a four dimensional differentiable manifold M endowed with a metric tensor g of signature $(+,-,-,-)$, also called *Lorentzian manifold.* In such a manifold, it is a mathematical fact that one can choose local coordinates around any point $p \in M$, called *normal coordinates*, such that $g_{ij}(p) = \mathrm{diag}(1,-1,-1,-1)$, and in addition the components have vanishing first derivatives, $g_{ij,k} = 0$. Neglecting higher derivatives, relative to normal coordinates the neighborhood of p looks like Minkowski space. Because of the signature of the metric g, one distinguishes non-zero vectors $X \in T_pM$ into three classes, namely we call X

$$\begin{aligned} &\textit{timelike}, &&\text{when } g(X,X) > 0\ , \\ &\textit{null} \text{ or } \textit{lightlike}, &&\text{when } g(X,X) = 0\ , \\ &\textit{spacelike}, &&\text{when } g(X,X) < 0\ . \end{aligned}$$

At each point $p \in M$, the null vectors form a double cone in T_pM, the so called *light cone*, which separates the timelike from the spacelike vectors. It is now postulated that massive objects can only move on timelike curves, i.e. the velocity vectors of these curves are everywhere timelike, while photons follow lightlike paths. In short, this postulate is called *local causality.* So far, the metric has not been specified. As a consequence of the equivalence principle and the requirement that all equations should be independent of the coordinate choice, the world lines of free objects are described by geodesics. The correct generalization of Newton's equations to this framework are *Einstein's field equations*, which state that the spacetime metric g has to satisfy

$$R_{ij} - \frac{1}{2} R g_{ij} = T_{ij}\ ,$$

where R_{ij} is the Ricci tensor, R is scalar curvature and T_{ij} the *energy momentum tensor*, which is divergence-free and symmetric and describes the matter fields that are contained in spacetime. Using that both sides are symmetric, the Einstein field equations provide a set of ten independent coupled nonlinear partial differential equations in the metric and its first and second derivatives. Due to the high complexity of these equations, at present only a few exact solutions are known. An important family of solutions, which are a model for the stationary axisymmetric asymptotically flat field outside a rotating massive object, is the *Kerr solution.* In Boyer-Lindquist coordinates $(t, r, \vartheta, \varphi)$ with $r > 0$, $0 \leq \vartheta \leq \pi$, $0 \leq \varphi < 2\pi$, these solutions are given by the line element

$$ds^2 = \frac{\Delta}{U}\left(dt - a\sin^2\vartheta\, d\varphi\right)^2 - U\left(\frac{dr^2}{\Delta} + d\vartheta^2\right) - \frac{\sin^2\vartheta}{U}\left(adt - \left(r^2 + a^2\right) d\varphi\right)^2,$$

where $U(r, \vartheta) = r^2 + a^2\cos^2\vartheta$ and $\Delta(r) = r^2 - 2Mr + a^2$. Here, M and a are constants, M representing the mass and $J = aM$ the angular momentum as measured from infinity. $\Delta(r)$ vanishes at the values

$$r_\pm = M \pm \sqrt{M^2 - a^2}\ ,$$

where r_- characterizes the *Cauchy horizon* and r_+ the *event horizon.*

One can classify the Kerr solution by the parameters a and M:

$$\begin{array}{rl} M = |a| = 0: & \text{Minkowski metric} \\ M > |a| = 0: & \text{Schwarzschild metric} \\ M = |a| > 0: & \text{extreme Kerr metric} \\ M > |a| > 0: & \text{non-extreme Kerr metric}\,. \end{array}$$

Since this work mainly deals with the background of the Schwarzschild spacetime, we present a few characteristics of it. Due to the fact that $a = 0$, the Schwarzschild solution represents a spherical symmetric empty spacetime outside a spherical symmetric (non-rotating) massive body. It was the first non-trivial solution of the Einstein field equations found by K. Schwarzschild in 1916. In fact, most of the experiments to test the difference between General Relativity and Newtonian theory rely on this solution. We briefly mention a few examples, for more details see e.g. [25, 12]:

- *Perihelion advance.* General Relativity predicts the advance of the perihelion of planetary orbits.

- *Deflection of light.* It was observed, that light is deflected when passing by massive objects, as for example the sun.

- *Spectral shift.* Light falling towards a massive object should be blue-shifted, while it is red-shifted when departing therefrom.

All these phenomena were not explained by the standard Newtonian theory. The Schwarzschild solution as a special case of the Kerr solution is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2) \, .$$

We first consider the region $r > 2M$. In this situation, the vector $\frac{\partial}{\partial t}$ is a timelike Killing vector, which means that the flow of this vector field generates local isometries of the metric. Since the vector field $\frac{\partial}{\partial t}$ is orthogonal to the family of hypersurfaces $\{t = const\}$, the Schwarzschild solution is called static. Moreover, it is a spherical symmetric solution in the sense that it is invariant uner the group of isometries $SO(3)$ operating on the spacelike two spheres $\{t, r \; constant\}$. According to Birkhoff's Theorem this solution is unique in the sense that, if there exists another static, spherically symmetric solution it is already locally isometric to the Schwarzschild solution [cf. [14]]. One can also consider this solution for arbitrary r. Obviously, the metric has two singularities at $r = 0$ and $r = 2M$. In the region $0 < r < 2M$ there is a change in the dynamics in the sense that the vectors $\frac{\partial}{\partial t}$ become spacelike, while $\frac{\partial}{\partial r}$ is timelike. Note that any object which is once inside this region has inevitably go to $r = 0$. Hence, if the radius of a star becomes smaller than $2M$, collapse to a singularity cannot be avoided. Moreover, if any signals are emmited from inside, they will never reach a distant observer. Thus, we call the region $r = 2M$ the *event horizon* and such a singularity a *black hole*. Note that the singularity $r = 2M$ can be resolved by a simple coordinate transformation. To this end, we take the Regge Wheeler coordinate $r_* = r + 2M \log |r - 2M|$. Then, $v = t + r_*$ is an advanced null coordinate and the Schwarzschild metric with respect to the coordinates $(v, r, \vartheta, \varphi)$ takes the Eddington-Finkelstein form

$$ds^2 = \left(1 - \frac{2M}{r}\right) dv^2 - 2dvdr - r^2(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2) \, .$$

Obviously, in these coordinates, the surface $r = 2M$ is no longer a singularity. Moreover, one sees that the region $r = 2M$ acts as a one-way membrane in the sense that particles and photons can enter the region $0 < r < 2M$ but never leave it.

The main interest of this work lies in the investigation of the stability of the Schwarzschild spacetime. In general, the equations that describe the physical fields couple to the Einstein equations. On the one hand, the metric g affects the behavior of the physical fields; on the other hand, the behavior of the physical fields yields a non-zero contribution to the energy momentum tensor and thus influences the metric g as a solution of the Einstein field equations. Since the Einstein equations is a system of nonlinear partial differential equations, the full perturbation equations are in general too complicated to analyze. Hence, we here restrict attention to the linearized equations. The first major contribution in this topic was made in 1957, when Regge and Wheeler studied the linearized equations for perturbations of the Schwarzschild metric [23]. This work was continued in [30, 35], while more recently the decay of the perturbation and all of its derivatives was shown in [13] using a theorem by Wilcox. By heuristic arguments, in 1972 Price [19] got evidence for polynomial decay of solutions of the scalar wave equation in Schwarzschild, where the power depends explicitely on the angular mode. In 1973, Teukolsky [27] could derive by means of the Newman Penrose formalism one single master equation that describes in the Kerr background the evolution of a test scalar field ($s = 0$), a test neutrino field ($s = \pm 1/2$), a test electromagnetic field ($s = \pm 1$) and linearized gravitational waves ($s = \pm 2$). In the vacuum case, the Teukolsky equation has the form [note that we use a simplified formulation due to Whiting [32]]:

$$\Big[\partial_r \Delta \partial_r - \frac{1}{\Delta}\Big\{ \left(r^2 + a^2\right) \partial_t + a\partial_\varphi - (r - M)s\Big\}^2 - 4s(r + ia\cos\vartheta)\partial_t \\ + \partial_{\cos\vartheta} \sin^2\vartheta\, \partial_{\cos\vartheta} + \frac{1}{\sin^2\vartheta}\left(a\sin^2\vartheta\partial_t + \partial_\varphi + is\cos\vartheta\right)^2\Big]\Phi(t, r, \vartheta, \varphi) = 0.$$

Here, the parameter s is also called the spin weight of the field. Note that it is a quite complicated task in the case $s \neq 0$ to recover all the components of the corresponding field from a solution of this equation. For further details see [3, 33]. In two subsequent papers [28, 29], Teukolsky and Press discussed the physical consequences of these perturbations. From the mathematical point of view, the Teukolsky equation can be regarded as a hyperbolic PDE, hence one is generally interested in the Cauchy problem for general initial data of the field and its first time derivative on the hypersurface $t = 0$. Though any linearized perturbation is given by this equation, the rigorous analysis of the equation remains a quite subtle point. Note that in the case $s \neq 0$ *complex* coefficients are involved, which makes the analysis very complicated. Hence, until now there are just a few rigorous results in this case. In [9] local decay was proven for the Dirac equation ($s = \frac{1}{2}$) in the Kerr geometry (in the massless and massive case). Moreover, a precise decay rate has been specified in the massive case [10]. More recently, there has been a linear stability result for the Schwarzschild geometry under electromagnetic and gravitational perturbations [11]. This result relies on the mode analysis, which has been carried out in [32]. More work has been done on the case $s = 0$, where the Teukolsky equation reduces to the scalar wave equation. In the Schwarzschild case, Kay and Wald [17] proved a time independent L^∞-bound for solutions of the Klein-Gordon equation. In [5], a mathematical proof is given for the decay rate of solutions with spherical symmetric initial data, which has been predicted by Price [19]. For general initial data, the same authors derived another decay result [6], which is not sharp, however. Furthermore, Morawetz and Strichartz-type estimates for a massless scalar field without charge in a Reissner Nordstrøm background with naked singularity are developed in [24]. And in [2] a Morawetz-type inequality was proven for the semi-linear wave equation in Schwarzschild, which is also supposed to yield decay rates.

In this work, we consider the Cauchy problem for the scalar wave equation (i.e. Teukolsky's equation for $s = 0$) in the Schwarzschild background (i.e. $a = 0$)

$$\Box_g \Phi = 0 \;, \quad \Phi\big|_{t=0} = \Phi_0 \;, \quad \partial_t \Phi\big|_{t=0} = \Phi_1 \;,$$

with smooth initial data Φ_0, Φ_1, which for simplicity is compactly supported *outside* the event horizon. We first work out the program that has been used to obtain decay of the scalar wave equation in the Kerr geometry in [7, 8]. The main difference lies in the fact that in Kerr the metric is only axisymmetric (instead of static and spherical symmetric). This has the consequence, that the classical energy density may be negative inside the *ergosphere*, a region outside the event horizon in which the Killing vector corresponding to time translations becomes spacelike. This makes it necessary to apply special methods (spectral theory in Pontrjagin spaces, energy splitting estimates, causality arguments) which are technically demanding and not easily accessible. In Schwarzschild, however, due to the spherical symmetry, the classical energy density is positive everywhere outside the event horizon. This gives rise to a positive definite scalar product, making it possible to apply Hilbert space methods, which let us derive an integral representation for the solution of the Cauchy problem and out of this prove pointwise decay. Moreover, using this integral representation, we obtain a rigorous proof of Price's law for solutions with spherical symmetric initial data and also a decay rate for arbitrary initial data, which is not sharp, however. More precisely, our main results are:

Theorem 1. *Consider the Cauchy problem of the scalar wave equation in the Schwarzschild geometry*

$$\Box \phi = 0 \;, \quad (\phi, i\partial_t \phi)(0, r, x) = \Phi_0(r, x)$$

for smooth initial data $\Phi_0 \in C_0^\infty((2M, \infty) \times S^2)^2$ *which is compactly supported outside the event horizon. Then there exists a unique global solution* $\Phi(t) = (\phi(t), i\partial_t \phi(t)) \in C^\infty(\mathbb{R} \times (2M, \infty) \times S^2)^2$ *which is compactly supported for all times* t*. Moreover, for fixed* (r, x) *this solution decays as* $t \to \infty$.

Theorem 2. *Consider the Cauchy problem of the scalar wave equation in the Schwarzschild geometry*

$$\Box \phi = 0 \;, \quad (\phi_0, i\partial_t \phi_0)(0, r, x) = \Phi_0(r, x)$$

for smooth spherical symmetric initial data $\Phi_0 \in C_0^\infty((2M, \infty) \times S^2)^2$ *which is compactly supported outside the event horizon. Let* $\Phi(t) = (\phi(t), i\partial_t \phi(t)) \in$

$C^\infty(\mathbb{R} \times (2M, \infty) \times S^2)^2$ be the unique global solution which is compactly supported for all times t. Then for fixed r there is a constant $c = c(r, \Phi_0)$ such that for large t

$$|\phi(t)| \leq \frac{c}{t^3} \,.$$

Moreover, if we have initially momentarily static initial data, i.e. $\partial_t \phi_0 \equiv 0$, the solution $\phi(t)$ satisfies

$$|\phi(t)| \leq \frac{c}{t^4} \,.$$

These results are proven at the end of Section 5, and Section 7, respectively. In general, this work is organized as follows:

- In the first section we formulate the mathematical framework. In order to get a more convenient form of the wave equation, we introduce the Regge-Wheeler variable and rewrite the wave equation as a first-order Hamiltonian system. The resulting Hamiltonian is a symmetric operator with respect to the scalar product arising from the conserved energy. Exploiting the spherical symmetry of the problem, it turns out that it suffices to restrict in the following the problem for fixed angular modes l and m.

- In Section 2, we show, using standard results of the theory of symmetric hyperbolic systems together with Stone's theorem, that the corresponding Hamiltonian is essentially self-adjoint.

- It is our goal to apply Stone's formula, which relates the propagator to an integral over the resolvent. Thus, in Section 3 we give an explicit construction for the resolvent. This construction is based on special solutions $\acute{\phi}, \grave{\phi}$ with exponential decay at $\pm\infty$ of the radial equation. This is an ordinary differential equation of Schrödinger type

$$(-\partial_u^2 - \omega^2 + V_l(u))\phi(\omega, u) = 0 \,,$$

with potential

$$V_l(u) = \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{l(l+1)}{r^2}\right) \,,$$

where the parameter ω arises by separating the time t in the Hamiltonian formalism via $e^{-i\omega t}$.

- In Section 4 we prove the existence of $\acute{\phi}, \grave{\phi}$ by the formalism of the Jost equation. Moreover, we obtain regularity results in the parameter ω, showing that the main difficulty is to analyze the solutions $\grave{\phi}$ at $\omega = 0$, because there we find a loss of regularity which depends on the angular mode l.

- In Section 5 we show that the regularity results for $\acute{\phi}, \grave{\phi}$ lead to an integral representation for the solutions of the Cauchy problem for fixed l and m which holds pointwise. According to the theory of symmetric hyperbolic systems, the Cauchy problem has a unique smooth solution. Thus, summing over the angular modes yields the desired representation of this solution. Combining this representation with a Sobolev imbedding argument, we obtain pointwise decay in time.

- In order to prove Price's law, we have to determine the nature of the irregularity in ω at $\omega = 0$ of the Jost solutions $\grave{\phi}$. To this end, in Section 6 we develop a method to obtain an explicit expansion of the Jost solutions in ω.

- In Section 7 we improve the preceding expansion in the spherical symmetric case $l = 0$. Then we investigate the integral representation with this more detailed expansion, which yields some regularity results of the integrand of this representation with respect to ω. By a Fourier transform argument we get precisely the decay rate predicted by Price.

- In the last section we give a brief discussion on a possible strategy to prove Price's law in the case $l \neq 0$.

Finally, I would like to thank my supervisor Felix Finster for his suggestion to study this problem and his constant encouragement. Furthermore, I would like to thank Thierry Daudé for numerous discussions on the subject. Thanks go to the DFG for their financial support. At last, I want to thank my family, friends, especially my girlfriend Cornelia, who let me think about other things than purely mathematics.

1 Preliminaries

In this section we reformulate the wave equation as a first order Hamiltonian system. This will make it possible to analyze the dynamics of the waves with Hilbert space methods.

In the following we consider only the region $r > 2M$ outside the event horizon in the Schwarzschild background. Hence, writing down the scalar wave equation with respect to Schwarzschild coordinates it has the explicit form

$$\left[\frac{\partial^2}{\partial t^2} - \left(1 - \frac{2M}{r}\right) \frac{1}{r^2} \left(\frac{\partial}{\partial r}(r^2 - 2Mr)\frac{\partial}{\partial r} + \Delta_{S^2}\right)\right] \phi = 0 \,. \tag{1.1}$$

Here Δ_{S^2} denotes the standard Laplacian on the two sphere, which in the coordinates (ϑ, φ) is given by

$$\Delta_{S^2} = \frac{1}{\sin^2\vartheta} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial}{\partial(\cos\vartheta)} \sin^2\vartheta \frac{\partial}{\partial(\cos\vartheta)} \,. \tag{1.2}$$

In order to bring the equation (1.1) into a more convenient form, we first introduce the Regge-Wheeler coordinate u by

$$u(r) := r + 2M \log\left(\frac{r}{2M} - 1\right) \,. \tag{1.3}$$

The variable u takes values in the whole interval $(-\infty, \infty)$ as r ranges over $(2M, \infty)$. It satisfies the relations

$$\frac{du}{dr} = \frac{1}{1 - \frac{2M}{r}} \,, \qquad \frac{\partial}{\partial u} = \left(1 - \frac{2M}{r}\right) \frac{\partial}{\partial r} \,. \tag{1.4}$$

In what follows the variable r is always implicitly given by u. Using the Regge-Wheeler coordinate, the wave equation (1.1) transforms to

$$\left[\frac{\partial^2}{\partial t^2} - \frac{1}{r} \frac{\partial^2}{\partial u^2} r + \left(1 - \frac{2M}{r}\right) \left(\frac{2M}{r^3} - \frac{\Delta_{S^2}}{r^2}\right)\right] \phi = 0 \,. \tag{1.5}$$

To simplify this equation we multiply by r and substitute $r\phi = \psi$ This leads us to the Cauchy problem

$$\left.\begin{aligned} \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial u^2} + \left(1 - \frac{2M}{r}\right) \left(\frac{2M}{r^3} - \frac{\Delta_{S^2}}{r^2}\right)\right] \psi(t, u, \vartheta, \varphi) = 0 \\ (\psi, i\partial_t \psi)(0, u, \vartheta, \varphi) = \Psi_0(u, \vartheta, \varphi) \end{aligned}\right\} \tag{1.6}$$

where the initial data $\Psi_0 \in C_0^\infty(\mathbb{R} \times S^2)^2$ is smooth and compactly supported.

The equation in (1.6) can be reformulated as the Euler-Lagrange equation corresponding to the action

$$S = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} du \int_{-1}^{1} d(\cos\vartheta) \int_0^{2\pi} d\varphi\, \mathcal{L}(\psi, \nabla\psi) , \tag{1.7}$$

where the Lagrangian is given by

$$\begin{aligned} 2\,\mathcal{L} \;&=\; |\partial_t\psi|^2 - |\partial_u\psi|^2 - \left(1 - \frac{2M}{r}\right) \frac{2M}{r^3}|\psi|^2 - \\ &\left(1 - \frac{2M}{r}\right) \frac{1}{r^2} \left(\frac{1}{\sin^2\vartheta} |\partial_\varphi\psi|^2 + \sin^2\vartheta\, |\partial_{\cos\vartheta}\, \psi|^2 \right) . \end{aligned} \tag{1.8}$$

As one sees immediately, the Lagrangian is invariant under time translations, and thus Noether's theorem gives rise to a conserved quantity, the energy E,

$$E[\psi] = \int_{-\infty}^{\infty} du \int_{-1}^{1} d(\cos\vartheta) \int_0^{2\pi} \frac{d\varphi}{\pi}\, \mathcal{E}, \tag{1.9}$$

where $\mathcal{E}$ is the energy density

$$\begin{aligned} 2\mathcal{E} = 2\left(\frac{\partial\mathcal{L}}{\partial\psi_t}\, \psi_t - \mathcal{L} \right) = |\partial_t\psi|^2 + |\partial_u\psi|^2 + \\ + \left(1 - \frac{2M}{r}\right) \left\{ \frac{2M}{r^3}|\psi|^2 + \frac{1}{r^2} \left(\frac{1}{\sin^2\vartheta}|\partial_\varphi\psi|^2 + \sin^2\vartheta\, |\partial_{\cos\vartheta}\, \psi|^2 \right) \right\}. \end{aligned} \tag{1.10}$$

It is also easy to check directly that the above energy is conserved in time for all smooth solutions of the wave equation that are compactly supported for all times. Since we consider the wave equation outside the event horizon, i.e. $r > 2M$, it is clear that the energy density is positive everywhere.

Next we rewrite the Cauchy problem (1.6) in first-order Hamiltonian form.
Letting

$$\Psi = \begin{pmatrix} \psi \\ i\partial_t\psi \end{pmatrix} , \tag{1.11}$$

the Cauchy problem takes the form

$$i\partial_t\Psi = H\Psi\,, \quad \Psi\big|_{t=0} = \Psi_0 \tag{1.12}$$

where H is the Hamiltonian

$$\begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} . \tag{1.13}$$

Here A is the differential operator

$$A = -\partial_u^2 + \left(1 - \frac{2M}{r}\right) \left(\frac{2M}{r^3} - \frac{1}{r^2}\Delta_{S^2} \right) . \tag{1.14}$$

We use the energy E in order to introduce a scalar product such that the Hamiltonian H is symmetric with respect to it. More precisely, we endow the space $C_0^\infty(\mathbb{R}\times S^2)^2$ with the *energy scalar product* $\langle .,. \rangle$ by polarizing E, thus

$$\langle \Psi, \Phi \rangle := \int_{-\infty}^{\infty} du \int_{-1}^{1} d(\cos\vartheta) \int_0^{2\pi} \frac{d\varphi}{2\pi} \Bigg\{ \overline{\partial_t \psi} \partial_t \phi + \overline{\partial_u \psi} \partial_u \phi + \left(1 - \frac{2M}{r}\right) \times \left[\frac{2M}{r^3} \, \overline{\psi} \, \phi + \frac{1}{r^2} \left(\frac{1}{\sin^2\vartheta} \, \overline{\partial_\varphi \psi} \partial_\varphi \phi + \sin^2\vartheta \, \overline{\partial_{\cos\vartheta} \, \psi} \partial_{\cos\vartheta} \, \phi \right) \right] \Bigg\} , \tag{1.15}$$

where again $\Psi = (\psi, i\partial_t \psi)^T$ and $\Phi = (\phi, i\partial_t \phi)^T$. Energy conservation implies that for a solution Ψ of the Cauchy problem (1.12) which is compactly supported for all times,

$$\begin{aligned} 0 &= \frac{d}{dt} E[\Psi] = \frac{d}{dt} \langle \Psi, \Psi \rangle = \\ &= \langle \dot{\Psi}, \Psi \rangle + \langle \Psi, \dot{\Psi} \rangle = i \langle H\Psi, \Psi \rangle - i \langle \Psi, H\Psi \rangle \ . \end{aligned}$$

Since the initial data $\Psi_0 \in C_0^\infty(\mathbb{R} \times S^2)^2$ can be chosen arbitrarily, polarization yields

$$\langle H\Psi, \Phi \rangle = \langle \Psi, H\Phi \rangle \ , \quad \text{for all } \Psi, \Phi \in C_0^\infty(\mathbb{R} \times S^2)^2 \ . \tag{1.16}$$

Hence the operator H is symmetric on $C_0^\infty(\mathbb{R} \times S^2)^2$ with respect to $\langle .,. \rangle$.

We will now use the spherical symmetry to simplify the problem. More precisely, we make use of the fact that the angular dependence of the wave equation in the Schwarzschild geometry involves only the Laplacian on the two sphere. It is well-known that the spherical harmonics $Y_{lm}(\vartheta, \varphi)$, where $l \in \mathbb{N}_0, |m| \leq l$, are smooth eigenfunctions of Δ_{S^2} with the eigenvalues $-l(l+1)$. Moreover, they form an orthonormal basis of the space $L^2(S^2)$. Thus we can decompose an arbitrary $\Psi \equiv (\psi_1, \psi_2)^T \in C_0^\infty(\mathbb{R} \times S^2)^2$ in the following way,

$$\Psi(u, \vartheta, \varphi) = \sum_{l=0}^{\infty} \sum_{|m| \leq l} \Psi^{lm}(u) Y_{lm}(\vartheta, \varphi) \ , \tag{1.17}$$

where for each component the sum converges for fixed u in $L^2(S^2)$. Since the $\Psi^{lm} \equiv (\psi_1^{lm}, \psi_2^{lm})^T$ are uniquely determined by $\psi_i^{lm}(u) = \langle Y_{lm}, \psi_i(u) \rangle_{L^2(S^2)}$ it is clear that $\Psi^{lm}(u) \in C_0^\infty(\mathbb{R})^2$ for all l, m. Using this decomposition, we rewrite the norm of Ψ corresponding to the energy scalar product as

$$\langle \Psi, \Psi \rangle = \int_{-\infty}^{\infty} du \int_{-1}^{1} d(\cos\vartheta) \int_0^{2\pi} \frac{d\varphi}{2\pi} \Big\{ |\psi_2|^2 + |\partial_u \psi_1|^2 +$$

$$
+\overline{\psi_1}\left(1-\frac{2M}{r}\right)\left(\frac{2M}{r^3}-\frac{1}{r^2}\Delta_{S^2}\right)\psi_1\Big\}
$$

$$
=\sum_{l=0}^{\infty}\sum_{|m|\leq l}\int_{-\infty}^{\infty}du\Big\{|\psi_2^{lm}(u)|^2+|\partial_u\psi_1^{lm}(u)|^2+
$$

$$
\left(1-\frac{2M}{r}\right)\left(\frac{2M}{r^3}+\frac{l(l+1)}{r^2}\right)|\psi_1^{lm}(u)|^2\Big\}\ , \quad (1.18)
$$

where in the first equation we have integrated by parts with respect to (ϑ, φ). The second equation follows from the properties of the Y_{lm}. As one can immediately see, the integrand for every summand in (1.18) is positive. Hence again by polarizing we obtain for any angular mode l an scalar product $\langle .,. \rangle_l$ on $C_0^\infty(\mathbb{R})^2$ given by

$$
\langle \Psi, \Phi \rangle_l = \int_{-\infty}^{\infty}\Big\{\overline{\psi_2}\phi_2+\overline{\psi_1'}\phi_1'+V_l\,\overline{\psi_1}\phi_1\Big\}\,du\ , \quad (1.19)
$$

with the potential $V_l(u)$ defined as

$$
V_l(u)=\left(1-\frac{2M}{r}\right)\left(\frac{2M}{r^3}+\frac{l(l+1)}{r^2}\right)\ . \quad (1.20)
$$

This definition leads to an isometry

$$
\begin{array}{ccc}
\left(C_0^\infty(\mathbb{R}\times S^2)^2, \langle .,. \rangle\right) & \longrightarrow & \bigoplus_{l=0}^{\infty}\bigoplus_{|m|\leq l}\left(C_0^\infty(\mathbb{R})^2, \langle .,. \rangle_l\right) \\
\Psi & \mapsto & \Psi^{lm}\ .
\end{array} \quad (1.21)
$$

Using (1.17), the Hamiltonian H also decomposes in the following way,

$$
H\Psi(u,\vartheta,\varphi)=\sum_{l=0}^{\infty}\sum_{|m|\leq l}H_l\Psi^{lm}(u)Y_{lm}(\vartheta,\varphi)\ .
$$

Here the H_l act on $C_0^\infty(\mathbb{R})^2$ and are given by

$$
H_l=\begin{pmatrix} 0 & 1 \\ -\partial_u^2+V_l(u) & 0 \end{pmatrix}\ . \quad (1.22)
$$

Thus for fixed angular modes l and m the Cauchy problem (1.12) simplifies to

$$
i\partial_t\Psi^{lm}=H_l\Psi^{lm}\ , \quad \Psi^{lm}\big|_{t=0}=\Psi_0^{lm}\ , \quad (1.23)
$$

where the initial data is in $C_0^\infty(\mathbb{R})^2$. Moreover, the H_l are symmetric on $C_0^\infty(\mathbb{R})^2$ with respect to $\langle .,. \rangle_l$, because for any $\Psi, \Phi \in C_0^\infty(\mathbb{R})^2$ the functions $\Psi(u)Y_{lm}$ and $\Phi(u)Y_{lm}$ are in $C_0^\infty(\mathbb{R}\times S^2)^2$. Thus

$$
\langle H_l\Psi, \Phi\rangle_l=\langle H(\Psi\,Y_{lm}), \Phi Y_{lm}\rangle=\langle \Psi Y_{lm}, H(\Phi\,Y_{lm})\rangle=\langle \Psi, H_l\Phi\rangle_l\ .
$$

In particular, the norm with respect to $\langle .,. \rangle_l$ is constant for solutions of (1.23) with compact support in u for all times. Therefore we again refer to $\langle .,. \rangle_l$ as the energy scalar product.

Our strategy is to solve for a given inital data $\Psi_0 \in C_0^\infty(\mathbb{R} \times S^2)^2$ the Cauchy problem (1.23) for fixed angular modes l and m, and to sum up the solutions afterwards. Therefore, in what follows we will fix the angular modes l, m and consider the problem (1.23). In order to avoid too many indices, we usually omit the subscript l in the Hamiltonian and energy scalar product.

2 Spectral Properties of the Hamiltonian

In the previous section we introduced the energy scalar product $\langle \, , \rangle$ on the space $C_0^\infty(\mathbb{R})^2$. Since we cannot expect $C_0^\infty(\mathbb{R})^2$ to be complete with respect to this inner product (and indeed it is not, because the energy scalar product in the second component is just the usual L^2-scalar product), we define the Hilbert space $H^1_{V_l 0}(\mathbb{R})$ as the completion of $C_0^\infty(\mathbb{R})$ within the Hilbert space

$$H^1_{V_l}(\mathbb{R}) = \left\{ u \text{ with } u' \in L^2(\mathbb{R}) \text{ and } V_l^{1/2} u \in L^2(\mathbb{R}) \right\}$$

endowed with the scalar product

$$\langle u, v \rangle_1 := (u', v')_{L^2} + (V_l \, u, v)_{L^2} \, .$$

Note that this coincides with the energy scalar product on the first component. Therefore, we choose $\mathscr{H} \equiv H^1_{V_l 0}(\mathbb{R}) \oplus L^2(\mathbb{R})$ endowed with the energy scalar product as the underlying Hilbert space for our Hamiltonian H.

In the previous section we have seen that the Hamiltonian H is symmetric on $C_0^\infty(\mathbb{R})^2$. Before we can use functional analytic methods, we need to construct a self-adjoint extension of H. In fact, we are able to prove the following lemma:

Lemma 2.1. *The operator H with domain $\mathscr{D}(H) = C_0^\infty(\mathbb{R})^2$ is essentially self-adjoint in the Hilbert space $\mathscr{H}$.*

In order to prove this lemma, we use the following version of Stone's theorem about strongly continuous one-parameter unitary groups. A proof of this theorem can be found in [20, Section VIII.4].

Theorem 2.2. *Let $U(t)$ be a strongly continuous one-parameter unitary group on a Hilbert space $\mathscr{H}$. Then there is a self-adjoint operator A on $\mathscr{H}$ such that $U(t) = e^{itA}$.*

Furthermore, let D be a dense domain which is invariant under $U(t)$ and on which $U(t)$ is strongly differentiable. Then i^{-1} times the strong derivative of $U(t)$ is essentially self-adjoint on D, and its closure is A.

Now we apply this theorem:

Proof of Lemma 2.1. According to the theory of symmetric hyperbolic systems (cf. [15, Section 5.3]), the Cauchy problem

$$\left.\begin{array}{c}\left(\partial_t^2 - \partial_u^2 + V_l(u)\right) \psi(t,u) = 0 \\ \psi|_{t=0} = f \ , \ i\partial_t \psi|_{t=0} = g\end{array}\right\}$$

with smooth, compactly supported initial data $f, g \in C_0^\infty(\mathbb{R})$ has a unique solution $\psi(t,u) \in C^\infty(\mathbb{R} \times \mathbb{R})$ which is also compactly supported in u for all times. Using this solution, we define for arbitrary $t \in \mathbb{R}$ the operators

$$\begin{array}{rcl} U(t) : C_0^\infty(\mathbb{R})^2 & \to & C_0^\infty(\mathbb{R})^2 \\ \begin{pmatrix} f \\ g \end{pmatrix} & \mapsto & \begin{pmatrix} \psi(t,.) \\ i\partial_t \psi(t,.) \end{pmatrix} , \end{array}$$

which leave the dense subspace $C_0^\infty(\mathbb{R})^2 \subseteq \mathscr{H}$ invariant for all times t. Due to the energy conservation, the $U(t)$ are unitary with respect to the energy scalar product and hence extend to unitary operators on the entire Hilbert space $\mathscr{H}$. Furthermore, since the solution is uniquely determined by the initial data, the $U(t)$ have the following properties,

$$U(0) = Id, \quad U(t+s) = U(t)U(s) \quad \text{for all } t, s \in \mathbb{R} \ ,$$

and thus they form a one-parameter unitary group. Due to the fact that smooth initial data yields smooth solutions in t and u, this group is strongly continuous on $\mathscr{H}$ and strongly differentiable on the domain $C_0^\infty(\mathbb{R})^2$. Calculating i^{-1} times the strong derivative one gets

$$i^{-1} \lim_{h \searrow 0} \frac{1}{h} \left(U(h) \begin{pmatrix} f \\ g \end{pmatrix} - \begin{pmatrix} f \\ g \end{pmatrix} \right) = i^{-1} \begin{pmatrix} -ig \\ i(\partial_u^2 - V_l) f \end{pmatrix} = -H \begin{pmatrix} f \\ g \end{pmatrix}$$

for all $f, g \in C_0^\infty(\mathbb{R})$, and the lemma follows from Theorem 2.2. □

For the further investigations of the Hamiltonian H, we consider its self-adjoint closure which, for the sake of simplicity, we again denote by H. For our purposes, it is not important to know the exact domain of definition $\mathscr{D}(H)$ of the self-adjoint extension.

3 Construction of the Resolvent

Stone's formula for the spectral projections of a self-adjoint operator A (cf. [20] Theorem VII.13),

$$\frac{1}{2}\left[P_{[a,b]}+P_{(a,b)}\right] = \text{s-}\lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\int_a^b \left[(A-\omega-i\epsilon)^{-1}-(A-\omega+i\epsilon)^{-1}\right]\,d\omega\;, \tag{3.1}$$

relates the spectral projections to the resolvent (here s-lim denotes the strong limit of operators). In view of this relation, it is of interest to derive an explicit representation of the resolvent.

In the preceding section we have seen that there is a domain $\mathscr{D}(H)$ such that our Hamiltonian H is self-adjoint in the Hilbert space $(\mathscr{H}, \langle\,,\rangle)$. From this it immediately follows that the spectrum $\sigma(H)\subseteq\mathbb{R}$ is on the real line and therefore the resolvent $(H-\omega)^{-1}:\mathscr{H}\to\mathscr{H}$ exists for every $\omega\in\mathbb{C}\setminus\mathbb{R}$.

Let us now fix $\omega\in\mathbb{C}\setminus\mathbb{R}$. We often denote the ω-dependence by a subscript $_\omega$ .We begin by reducing the eigenvalue equation $H\Psi=\omega\Psi$ by substituting the equation for the first component in the second equation. We thus obtain the Schrödinger-type equation

$$\left(-\partial_u^2+V_\omega(u)\right)\phi(u)=0 \tag{3.2}$$

with the potential

$$V_\omega(u) = -\omega^2+V_l(u) = -\omega^2+\left(1-\frac{2M}{r}\right)\left(\frac{2M}{r^3}+\frac{l(l+1)}{r^2}\right)\,. \tag{3.3}$$

In what follows we refer to this equation simply as the Schrödinger equation. It can be regarded as the radial equation associated to the wave equation in (1.6). Our goal is to construct the resolvent $(H-\omega)^{-1}$ out of special solutions of this equation. We introduce fundamental solutions $\acute{\phi}_\omega$ and $\grave{\phi}_\omega$ of the Schrödinger equation (3.2) which satisfy asymptotic boundary conditions at $u=\pm\infty$ (the existence of these solutions will be proved in Section 4). More precisely, in the case $\mathrm{Im}\,(\omega)>0$ we impose that

$$\lim_{u\to-\infty} e^{i\omega u}\acute{\phi}_\omega(u)=1\;,\qquad \lim_{u\to-\infty}\left(e^{i\omega u}\acute{\phi}_\omega(u)\right)'=0 \tag{3.4}$$

$$\lim_{u\to+\infty} e^{-i\omega u}\grave{\phi}_\omega(u)=1\;,\qquad \lim_{u\to+\infty}\left(e^{-i\omega u}\grave{\phi}_\omega(u)\right)'=0\;, \tag{3.5}$$

whereas in the case $\mathrm{Im}\,(\omega)<0$,

$$\lim_{u\to-\infty} e^{-i\omega u}\acute{\phi}_\omega(u)=1\;,\qquad \lim_{u\to-\infty}\left(e^{-i\omega u}\acute{\phi}_\omega(u)\right)'=0 \tag{3.6}$$

$$\lim_{u\to+\infty} e^{i\omega u}\grave{\phi}_\omega(u)=1\;,\qquad \lim_{u\to+\infty}\left(e^{i\omega u}\grave{\phi}_\omega(u)\right)'=0\,. \tag{3.7}$$

Since the resolvent exists, the map $(H - \omega) : \mathscr{D}(H) \to \mathscr{H}$ is bijective and in particular the kernel is trivial. Hence the solutions $\acute{\phi}_\omega, \grave{\phi}_\omega$ are linearly independent (otherwise they would give rise to a vector in the kernel due to the exponential decay). Thus $\acute{\phi}_\omega$ and $\grave{\phi}_\omega$ are indeed a system of fundamental solutions with non-vanishing Wronskian

$$w(\acute{\phi}_\omega, \grave{\phi}_\omega) := \acute{\phi}_\omega(u)\grave{\phi}'_\omega(u) - \acute{\phi}'_\omega(u)\grave{\phi}_\omega(u) \,. \tag{3.8}$$

Note that the Wronskian is independent of the variable u, as is easily verified by differentiating with respect to u and substituting the Schrödinger equation.

In the next lemma, we use this fundamental system to derive the Green's function corresponding to (3.2).

Lemma 3.1. *The function*

$$s_\omega(u,v) := -\frac{1}{w(\acute{\phi}_\omega, \grave{\phi}_\omega)} \times \begin{cases} \acute{\phi}_\omega(u)\grave{\phi}_\omega(v)\,, & \textit{if } u \le v \\ \acute{\phi}_\omega(v)\grave{\phi}_\omega(u)\,, & \textit{if } u > v \end{cases} \tag{3.9}$$

satisfies the distributional equations

$$\left(-\frac{\partial^2}{\partial u^2} + V_\omega(u)\right) s_\omega(u,v) = \delta(u-v) = \left(-\frac{\partial^2}{\partial v^2} + V_\omega(v)\right) s_\omega(u,v) \,.$$

Proof. By definition of the distributional derivative we have for every test function $\eta \in C_0^\infty(\mathbb{R})$,

$$\int_{-\infty}^{\infty} \eta(u) \left[-\partial_u^2 + V_\omega(u)\right] s_\omega(u,v) du = \int_{-\infty}^{\infty} \left[(-\partial_u^2 + V_\omega(u))\eta(u)\right] s_\omega(u,v) du \,.$$

It is obvious from its definition that the function $s(.,v)$ is smooth except at the point $u = v$, where its first derivative has a discontinuity. Thus, after splitting up the integral, we can integrate by parts twice to obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left[(-\partial_u^2 + V_\omega(u))\eta(u)\right] s_\omega(u,v)\, du = \\
&= \int_{-\infty}^{v} \eta(u)(-\partial_u^2 + V_\omega(u)) s_\omega(u,v)\, du + \lim_{u \nearrow v} \left[\eta(u)\partial_u s_\omega(u,v)\right] + \\
&\quad + \int_{v}^{\infty} \eta(u)(-\partial_u^2 + V_\omega(u)) s_\omega(u,v)\, du - \lim_{u \searrow v} \left[\eta(u)\partial_u s_\omega(u,v)\right] \,.
\end{aligned}$$

Since for $u \neq v$, s_ω is a solution of (3.2), the obtained integrals vanish. Computing the limits with the definition (3.9), we get

$$\int_{-\infty}^{\infty} \left[(-\partial_u^2 + V_\omega(u))\eta(u)\right] s_\omega(u,v)\, du = \left(\lim_{u \nearrow v} - \lim_{u \searrow v}\right) \eta(u)\partial_u s_\omega(u,v) =$$

$$= -\frac{1}{w(\acute{\phi}_\omega, \grave{\phi}_\omega)}\, \eta(v) \left[\acute{\phi}'_\omega(v) \grave{\phi}_\omega(v) - \grave{\phi}'_\omega(v) \acute{\phi}_\omega(v) \right] = \eta(v) \ ,$$

where in the last step we used the definition of the Wronskian (3.8). This yields the first equation. The second equation is proven exactly in the same way. □

With this function s_ω we are now able to construct the resolvent. More precisely,

Proposition 3.2. *For every $\omega \in \mathbb{C} \setminus \mathbb{R}$, the resolvent $(H-\omega)^{-1} : \mathscr{H} \to \mathscr{H}$ can be represented as an integral operator with the integral kernel*

$$k_\omega(u,v) = \delta(u-v) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + s_\omega(u,v) \begin{pmatrix} \omega & 1 \\ \omega^2 & \omega \end{pmatrix} . \tag{3.10}$$

Proof. We introduce the integral operator S_ω with the integral kernel $k_\omega(u,v)$ on the domain

$$\mathscr{D}(S_\omega) := \left\{ (H-\omega)\Psi \,\middle|\, \Psi \in C_0^\infty(\mathbb{R})^2 \right\} .$$

Let us verify that $\mathscr{D}(S_\omega)$ is a dense subset of $\mathscr{H}$. Let $\phi \in \mathscr{H}$ be an arbitrary vector. Because of the existence of the resolvent, the operator $H - \omega : \mathscr{D}(H) \to \mathscr{H}$ is onto, and thus there is a vector $\psi \in \mathscr{D}(H)$ with $(H-\omega)\psi = \phi$. Then due to the definition of the closure of H, there is a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subseteq C_0^\infty(\mathbb{R})^2$ with $\psi_n \to \psi$ and $H\psi_n \to H\psi$ as $n \to \infty$. This shows that $\{(H-\omega)\psi_n\}_{n\in\mathbb{N}} \subseteq \mathscr{D}(S_\omega)$ converges to $(H-\omega)\psi = \phi$. We conclude that $\mathscr{D}(S_\omega)$ is dense. We now calculate the operator product $S_\omega\,(H-\omega)$ on $C_0^\infty(\mathbb{R})^2$. For an arbitrary $\Psi = (\psi_1, \psi_2)^T \in C_0^\infty(\mathbb{R})^2$ we have

$$\begin{aligned} (S_\omega(H-\omega)\psi)\,(u) &= \int_{-\infty}^{\infty} k_\omega(u,v)(H-\omega)\psi(v)\, dv = \\ &= \begin{pmatrix} 0 \\ -\omega\psi_1 + \psi_2 \end{pmatrix}(u) + \\ &\quad + \int_{-\infty}^{\infty} s_\omega(u,v) \begin{pmatrix} -\partial_v^2 + V_\omega(v) & 0 \\ \omega\left(-\partial_v^2 + V_\omega(v)\right) & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}(v)\, dv . \end{aligned}$$

Hence, according to Lemma 3.1,

$$S_\omega(H-\omega) = Id \quad \text{on } C_0^\infty(\mathbb{R})^2 .$$

This yields that $S_\omega = (H-\omega)^{-1}$ on the dense set $\mathscr{D}(S_\omega)$. Since $(H-\omega)^{-1}$ is a bounded operator, the claim follows. □

As mentioned at the beginning of this section, we can now apply Stone's formula for the spectral projections of H and get for every $\Psi \in \mathscr{H}$

$$\frac{1}{2}\left[P_{[a,b]} + P_{(a,b)}\right]\Psi =$$

$$= \lim_{\epsilon \searrow 0} \frac{1}{2\pi i} \int_a^b \left[(H - (\omega + i\epsilon))^{-1} - (H - (\omega - i\epsilon))^{-1} \right] \Psi \, d\omega \, ,$$

and this yields together with Proposition 3.2

$$= \lim_{\epsilon \searrow 0} \frac{1}{2\pi i} \int_a^b \left(\int_{\mathbb{R}} \left(k_{\omega+i\epsilon}(.,v) - k_{\omega-i\epsilon}(.,v) \right) \Psi(v) dv \right) d\omega \, , \qquad (3.11)$$

where the limit is with respect to the norm in $\mathscr{H}$. It is therefore of special interest how the kernels $k_{\omega+i\epsilon}(u,v)$ and $k_{\omega-i\epsilon}(u,v)$ behave as $\epsilon \searrow 0$. Since these kernels are given explicitly in terms of the fundamental solutions $\acute{\phi}_{\omega\pm i\epsilon}$ and $\grave{\phi}_{\omega\pm i\epsilon}$, we will discuss their behavior in the next section.

4 The Jost Solutions of the Radial Equation

In this section we want to discuss the existence and the behavior of the solutions $\acute{\phi}_\omega, \grave{\phi}_\omega$ of the Schrödinger equation (3.2), which in Section 3 we used for the construction of the resolvent. We will prove the following theorem.

Theorem 4.1.

(i) For every $\omega \in D = \left\{ \omega \in \mathbb{C} \mid \mathrm{Im}\, \omega \le \frac{1}{4M} \right\}$, there exists a unique solution $\phi_1(\omega, u)$ of the Schrödinger equation (3.2) satisfying the boundary conditions (3.6) such that for every fixed $u \in \mathbb{R}$ the function $\phi_1(\omega, u)$ is holomorphic in $\omega \in \mathring{D}$ and continuous in D.

(ii) For every angular momentum number l, the solutions $\grave{\phi}_\omega$ of the Schrödinger equation (3.2) with boundary conditions (3.7) are well-defined and uniquely determined on the set

$$E = \left\{ \omega \in \mathbb{C} \,\middle|\, \mathrm{Im}\, \omega \le 0, \ \omega \neq 0 \right\} \, .$$

For each fixed $u \in \mathbb{R}$, the function $\grave{\phi}_\omega(u)$ is holomorphic in $\omega \in \mathring{E}$ and continuous in E.

Furthermore, in the case $l = 0$, $\grave{\phi}_\omega(u)$ may be continuously extended to $\omega = 0$.

Once having proven this theorem, we simply set

$$\acute{\phi}_\omega(u) := \begin{cases} \overline{\phi_1(\overline{\omega}, u)} & , \text{ if } \mathrm{Im}\, \omega > 0 \\ \phi_1(\omega, u) & , \text{ if } \mathrm{Im}\, \omega \le 0 \end{cases} \, , \qquad (4.1)$$

as well as

$$\grave{\phi}_\omega(u) := \overline{\grave{\phi}_{\overline{\omega}}(u)} \quad \text{if } \mathrm{Im}\, \omega > 0 \, , \qquad (4.2)$$

to obtain the solutions of Section 3. For $\mathrm{Im}\, \omega < 0$ this is clear by definition. But in the case of $\mathrm{Im}\, \omega > 0$ the above defined $\acute{\phi}_\omega(u), \grave{\phi}_\omega(u)$ are indeed the unique solutions of the Schrödinger equation (3.2) with the desired boundary conditions (3.4) and (3.5), respectively. This follows immediately by complex conjugation of the Schrödinger equation due to the fact that our potential V_l is real.

For the proof of Theorem 4.1 we will formally manipulate the Schrödinger equation with boundary conditions (3.6, 3.7) in order to get an appropriate integral equation (which in different contexts is called the Jost or Lipman-Schwinger equation). Then we will perform a perturbation expansion and get estimates for all the terms of the expansion. A reference for this method can be found e.g. in [22, Section XI.8]. Since this reference contains only an outline of the proof, it seems worth working out the details.

To introduce the method, we begin with the solutions $\phi_1(\omega, u)$. First we write the Schrödinger equation (3.2) in the form

$$\left(-\frac{d^2}{du^2} - \omega^2\right) \phi_\omega(u) = -W(u)\phi_\omega(u) \ , \tag{4.3}$$

where W is a potential in $L^1(\mathbb{R})$ (later on, W will be replaced by V_l). Next we define for $\omega \in \mathbb{C}$ the function $G_\omega(u)$ by

$$G_\omega(u) := \begin{cases} 0 & \text{, if } u \leq 0 \\ -\frac{1}{\omega}\sin(\omega u) & \text{, if } u > 0 \text{ and } \omega \neq 0 \\ -u & \text{, if } u > 0 \text{ and } \omega = 0 \end{cases} \ . \tag{4.4}$$

A simple computation shows that $G_\omega(u)$ defines a *Green's function* for the operator on the left hand side of the equation (4.3) in the sense that the distributional equation

$$\left(-\frac{d^2}{du^2} - \omega^2\right) G_\omega(u) = \delta(u)$$

holds. In order to build in the boundary condition (3.6), we make in equation (4.3) the substitution $\phi_\omega(u) = e^{i\omega u} + \tilde{\phi}_\omega(u)$ to obtain

$$\left(-\frac{d^2}{du^2} - \omega^2\right) \tilde{\phi}_\omega(u) = -W(u)\phi_\omega(u) \ .$$

Solving this equation formally by convoluting the right hand side with G_ω, we get the formal solution

$$\tilde{\phi}_\omega(u) = \big((-W\phi_\omega) * G_\omega\big)(u) \equiv -\int_{-\infty}^{\infty} G_\omega(u-v)W(v)\phi_\omega(v)\, dv \ .$$

Hence $\phi_\omega(u)$ satisfies the equation

$$\phi_\omega(u) = e^{i\omega u} - \int_{-\infty}^{u} G_\omega(u-v)W(v)\phi_\omega(v)\, dv \ , \tag{4.5}$$

which is referred to as the *Jost equation with boundary conditions at* $-\infty$. Its significance lies in the fact that we can now easily perform a perturbation expansion in the potential W. Namely, making for ϕ_ω the ansatz as the perturbation series

$$\phi_\omega = \sum_{k=0}^{\infty} \phi_\omega^{(k)} \;, \tag{4.6}$$

we are led to the iteration scheme

$$\left.\begin{array}{rcl} \phi_\omega^{(0)}(u) & = & e^{i\omega u} \\ & \vdots & \\ \phi_\omega^{(k+1)}(u) & = & -\displaystyle\int_{-\infty}^{u} G_\omega(u-v) W(v) \phi_\omega^{(k)}(v)\, dv \end{array}\right\} . \tag{4.7}$$

This iteration scheme can be used to construct solutions of the Jost equation.

We remark that under certain assumptions on W like continuity, the Jost equation is equivalent to the corresponding Schrödinger equation with appropriate boundary conditions. We will show this for our special case $W \equiv V_l$. A systematic method to rewrite second-order differential equations with boundary conditions as integral equations can be found e.g. in [22, Section XI.8 Appendix 2].

We now state a theorem about solutions of the Jost equation. We consider more general potentials W than we have in our case, because it might be of interest by itself.

Theorem 4.2. *Suppose that W is a measurable function obeying for a given $u_0 < 0$ the condition $\int_{-\infty}^{u_0} |W(v)| dv < \infty$. Define for $u \leq u_0$ the function $P_\omega(u)$ by*

$$P_\omega(u) = \int_{-\infty}^{u} \frac{4|v|}{1+|\omega v|}\, |W(v)| e^{-(\operatorname{Im}\omega + |\operatorname{Im}\omega|)v}\, dv \;. \tag{4.8}$$

Then:

(i) For each $\omega \in E = \{\omega \in \mathbb{C} \mid \operatorname{Im}\omega \leq 0, \omega \neq 0\}$ the Jost equation (4.5) has a unique solution $\phi_\omega(u)$ obeying $\lim_{u\to-\infty} \left|e^{-i\omega u}\phi_\omega(u)\right| < \infty$. Moreover, $\phi_\omega(u)$ is continuously differentiable in u on $(-\infty, u_0)$ with $\lim_{u\to-\infty} e^{-i\omega u}\, \phi_\omega(u) = 1$ and $\lim_{u\to-\infty} e^{-i\omega u}\, \phi'_\omega(u) = i\omega$. For each fixed u, the functions $\phi_\omega(u)$ and $\phi'_\omega(u)$ are holomorphic in $\mathring{E}$ and continuous in E. They satisfy the bounds

$$\left|\phi_\omega(u) - e^{i\omega u}\right| \;\leq\; e^{-u\operatorname{Im}\omega} \left|e^{P_\omega(u)} - 1\right| \tag{4.9}$$

$$\left|\phi'_\omega(u) - i\omega e^{i\omega u}\right| \;\leq\; e^{-u\operatorname{Im}\omega} e^{P_\omega(u)} \int_{-\infty}^{u} |W(v)|\, dv \;. \tag{4.10}$$

(ii) If $\int_{-\infty}^{u_0} |v||W(v)|\,dv < \infty$, for every $u \le u_0$ the function $\phi_\omega(u)$ may be continuously extended to $\omega = 0$. Moreover, (4.9), (4.10) hold also at $\omega = 0$.

(iii) If $\int_{-\infty}^{u_0} e^{-mv}|W(v)|dv < \infty$, for every $u \le u_0$ the function $\phi_\omega(u)$ can be extended to a holomorphic function in $\{\omega \mid \operatorname{Im}\omega < \frac{1}{2}m\}$, continuous in $\{\omega \mid \operatorname{Im}\omega \le \frac{1}{2}m\}$. Moreover, in the interval $0 < \operatorname{Im}\omega < \frac{1}{2}m$ the inequalities (4.9),(4.10) are replaced by

$$\left|\phi_\omega(u) - e^{i\omega u}\right| \le \frac{1}{|\omega|} e^{u \operatorname{Im}\omega} e^{P_\omega(u)} \int_{-\infty}^{u} e^{-2v \operatorname{Im}\omega} |W(v)|\,dv \quad (4.11)$$

$$\left|\phi_\omega'(u) - i\omega e^{i\omega u}\right| \le e^{u \operatorname{Im}\omega} e^{P_\omega(u)} \int_{-\infty}^{u} e^{-2v \operatorname{Im}\omega} |W(v)|\,dv\,. \quad (4.12)$$

In each case, ϕ obeys $\overline{\phi_\omega(u)} \equiv \overline{\phi(\omega, u)} = \phi(-\overline{\omega}, u)$.

We call this solution ϕ the **Jost solution**. For the proof of this theorem we need a good estimate for the Green's function G_ω.

Lemma 4.3. *For all $u \in \mathbb{C}$,*

$$|\sin u| \le \frac{2|u|}{1+|u|}\, e^{|\operatorname{Im} u|}\,. \quad (4.13)$$

Moreover, if $\omega \neq 0$ and $v \le u \le 0$,

$$\left|\frac{1}{\omega}\sin(\omega(u-v))\right| \le \frac{4|v|}{1+|\omega v|}\, e^{-v|\operatorname{Im}\omega| - u\,\operatorname{Im}\omega}\,. \quad (4.14)$$

Proof. In the case $|u| \ge 1$, the inequality (4.13) follows directly from the Euler formula $\sin u = \frac{1}{2i}\left(e^{iu} - e^{-iu}\right)$ and the estimate

$$(1+|u|)|\sin u| \le \frac{1}{2}(1+|u|)2e^{|\operatorname{Im} u|} \le 2|u|e^{|\operatorname{Im} u|}\,.$$

In the remaining case $|u| < 1$, we again use the Euler formula to obtain

$$(1+|u|)|\sin u| = \frac{1}{2}(1+|u|)\left|e^{iu} - e^{-iu}\right| = \frac{1}{2}(1+|u|)\left|\int_{-1}^{1} iue^{iu\tau}\,d\tau\right|,$$

and hence

$$(1+|u|)|\sin u| \le \frac{1}{2}(|u|+|u|^2)\int_{-1}^{1}\left|e^{iu\tau}\right|d\tau \le \frac{1}{2}(|u|+|u|^2)2\,e^{|\operatorname{Im} u|}\,.$$

Now (4.13) follows by the assumption $|u| < 1$.

In order to show (4.14) we use the identity

$$\frac{1}{\omega}\sin(\omega(u-v)) = \frac{1}{\omega}\left(\sin(\omega u)e^{i\omega v} - \sin(\omega v)e^{i\omega u}\right)$$

and apply (4.13),

$$\begin{aligned}\left|\frac{1}{\omega}\sin(\omega(u-v))\right| &\leq \frac{1}{|\omega|}\left(\left|\sin(\omega u)e^{i\omega v}\right| + \left|\sin(\omega v)e^{i\omega u}\right|\right) \\ &\overset{(4.13)}{\leq} \frac{2|u|}{1+|\omega u|}\, e^{|u\,\mathrm{Im}\,\omega|}e^{-v\,\mathrm{Im}\,\omega} + \frac{2|v|}{1+|\omega v|}\, e^{|v\,\mathrm{Im}\,\omega|}e^{-u\,\mathrm{Im}\,\omega}\,. \end{aligned} \tag{4.15}$$

Due to the assumption $0 \geq u \geq v$, we know that $|v| \geq |u|$ and thus

$$\frac{2|u|}{1+|\omega u|} \leq \frac{2|v|}{1+|\omega v|}\,, \qquad -u|\,\mathrm{Im}\,\omega| - v\,\mathrm{Im}\,\omega \leq -v|\,\mathrm{Im}\,\omega| - u\,\mathrm{Im}\,\omega\,.$$

Using these inequalities in (4.15) the claim follows. □

Note that the estimate (4.14) remains valid in the limit $0 \neq \omega \to 0$, if one replaces $\frac{1}{\omega}\sin(\omega(u-v))$ by the function $u-v$.

Now we are ready to prove Theorem 4.2:

Proof of Theorem 4.2.
Using the perturbation expansion (4.6) together with the iteration scheme (4.7), one easily sees that we have already found a *formal* solution. So our goal is to show that this series is well-defined and has the desired properties. To this end, we shall prove inductively that

$$\left|\phi_\omega^{(k)}(u)\right| \leq e^{-u\ \mathrm{Im}\,\omega}\,\frac{1}{k!}P_\omega(u)^k \quad \text{for all } k \in \mathbb{N}_0\,, \text{ for all } \omega, u \tag{4.16}$$

such that $P_\omega(u)$ is well-defined by (4.8). Due to the integrability conditions on the potential W in the statement of the theorem this is the case for $u \leq u_0$ and for all $\omega \in E$ (cf. (i)), $\omega \in \overline{E}$ (cf. (ii)), $\omega \in \{\mathrm{Im}\,\omega \leq \frac{1}{2}m\}$ (cf. (iii)), respectively. Furthermore, $P_\omega(u)$ is continuous in u as well as in ω in these domains. The first statement is obvious while the latter is due to the fact that the integrand in the definition (4.8) is continuous in ω and one directly finds an integrable dominating function such that one can apply Lebegue's dominated convergence theorem.

We start the induction with the case $k=0$ for which (4.16) certainly is satisfied. Thus assume that (4.16) holds for a given k. Then, estimating the integral equation in (4.7) using (4.14) and (4.8), we obtain

$$\left|\phi_\omega^{(k+1)}(u)\right| \leq \int_{-\infty}^{u} |G_\omega(u-v)||W(v)|\left|\phi_\omega^{(k)}(v)\right| dv$$

$$
\begin{aligned}
&\leq \int_{-\infty}^{u} \frac{4|v|}{1+|\omega v|}\, e^{-v|\operatorname{Im}\omega|-u \operatorname{Im}\omega} |W(v)| e^{-v \operatorname{Im}\omega} \frac{1}{k!}\, P_\omega(v)^k\, dv \\
&= e^{-u \operatorname{Im}\omega} \frac{1}{k!} \int_{-\infty}^{u} \frac{dP_\omega}{dv}(v)\, P_\omega(v)^k\, dv \\
&= e^{-u \operatorname{Im}\omega} \frac{1}{(k+1)!} P_\omega(u)^{k+1} ,
\end{aligned}
$$

where in the last step we used that $P_\omega(u)$ vanishes when u goes to $-\infty$. This concludes the proof of (4.16).

Summing over k, (4.16) yields the inequality

$$
\sum_{k=0}^{\infty} \left|\phi_\omega^{(k)}(u)\right| \leq e^{-u \operatorname{Im}\omega} e^{P_\omega(u)}. \tag{4.17}
$$

Because of the continuity of $P_\omega(u)$, the series (4.6) converges uniformly for u and ω in compact sets. Using the iteration scheme (4.7), this series can be written as

$$
\sum_{k=0}^{\infty} \phi_\omega^{(k)}(u) = e^{i\omega u} - \sum_{k=1}^{\infty} \int_{-\infty}^{u} G_\omega(u-v) W(v) \phi_\omega^{(k-1)}(v)\, dv ,
$$

and the bound (4.17) allows us to apply Lebesgue's dominated convergence theorem and to interchange the sum and the integral. Hence the series is indeed a solution of the Jost equation (4.5).

Next we want to show that a solution of the Jost equation is continuously differentiable with respect to u. To this end, we first compute for an arbitrary $u < u_0$ the difference quotient,

$$
\begin{aligned}
&\frac{1}{h}\left(\phi_\omega(u+h) - \phi_\omega(u) - e^{i\omega(u+h)} + e^{i\omega u}\right) \overset{(4.5)}{=} \\
&\int_{-\infty}^{u+h} \frac{1}{h\omega} \left[\sin(\omega(u+h-v)) - \sin(\omega(u-v))\right] W(v) \phi_\omega(v)\, dv + \qquad (4.18) \\
&+\frac{1}{h} \int_{u}^{u+h} \frac{1}{\omega} \sin(\omega(u-v)) W(v) \phi_\omega(v)\, dv , \qquad (4.19)
\end{aligned}
$$

where $h \neq 0$. We may restrict attention to the case $\operatorname{Im}\omega \leq 0$ and $h > 0$ (the other cases are analogous). Using the estimate

$$
\begin{aligned}
\left|\partial_u \left(\frac{1}{\omega} \sin(\omega(u-v))\right)\right| &= \left|\cos(\omega(u-v))\right| \leq \\
&\leq \frac{1}{2}\left(e^{-u \operatorname{Im}\omega + v \operatorname{Im}\omega} + e^{u \operatorname{Im}\omega - v \operatorname{Im}\omega}\right)
\end{aligned}
$$

together with (4.17), we can apply the mean value theorem to the first integrand to obtain the dominating function

$$
\frac{1}{2}\left(e^{-\xi(v) \operatorname{Im}\omega} e^{v \operatorname{Im}\omega} + e^{\xi(v) \operatorname{Im}\omega} e^{-v \operatorname{Im}\omega}\right) |W(v)|\, e^{-v \operatorname{Im}\omega} e^{P_\omega(v)} ,
$$

where $\xi(v) \in [u, u+h]$. Due to the integrability conditions on W, it is clear that this function is integrable. Hence Lebesgue's dominated convergence theorem allows us to take the limit $h \to 0$ in (4.18). This gives

$$\int_{-\infty}^{u} \cos(\omega(u-v)) W(v) \phi_\omega(v)\, dv \ .$$

In order to treat the second integral, we choose $h < h_0$, where h_0 is so small that

$$\max_{v \in [u,u+h]} \left| \frac{1}{\omega h} \sin(\omega(u-v)) \right| \leq 2 \quad \text{for all } h < h_0.$$

This is possible because $\lim_{h \to 0} \frac{1}{\omega h} \sin(\omega h) = 1$. Thus we can estimate (4.19) by

$$\begin{aligned}
&\left| \frac{1}{h} \int_u^{u+h} \frac{1}{\omega} \sin(\omega(u-v)) W(v) \phi_\omega(v)\, dv \right| \leq \\
&\quad \leq 2\, e^{-(u+h)\, \mathrm{Im}\, \omega} e^{P_\omega(u+h)} \int_{(-\infty, u_0)} \left| 1_{[u,u+h]}(v) W(v) \right| dv \ ,
\end{aligned}$$

and the last integral goes to 0 as $h \to 0$ by Lebesgue's monotone convergence theorem using the fact that $W \in L^1(-\infty, u_0)$. Hence (4.19) vanishes.

Alltogether we conclude that $\phi_\omega(u)$ is differentiable with derivative

$$\phi'_\omega(u) = i\omega e^{i\omega u} + \int_{-\infty}^{u} \cos(\omega(u-v)) W(v) \phi_\omega(v) dv \ , \tag{4.20}$$

which is continuous on $(-\infty, u_0)$ because of the estimate (4.17).

The estimate (4.9) is a simple consequence of (4.16) together with the perturbation expansion (4.6). For the proof of (4.10) we use the representation of the derivative (4.20) together with the inequality (4.17):

$$\begin{aligned}
\left| \phi'_\omega(u) - i\omega e^{i\omega u} \right| &\overset{(4.20)}{\leq} \int_{-\infty}^{u} |\cos(\omega(u-v))|\, |W(v)|\, |\phi_\omega(v)| dv \\
&\overset{(4.17)}{\leq} \int_{-\infty}^{u} \frac{1}{2} \left(e^{-u\, \mathrm{Im}\, \omega} e^{v\, \mathrm{Im}\, \omega} + e^{u\, \mathrm{Im}\, \omega} e^{-v\, \mathrm{Im}\, \omega} \right) |W(v)|\, e^{-v\, \mathrm{Im}\, \omega} e^{P_\omega(v)}\, dv \\
&\leq\ e^{-u\, \mathrm{Im}\, \omega} e^{P_\omega(u)} \int_{-\infty}^{u} |W(v)|\, dv \ ,
\end{aligned}$$

where in the last step we used the fact that $P_\omega(v)$ and $e^{-v\, \mathrm{Im}\, \omega}$ (with $\mathrm{Im}\, \omega \leq 0$) are monotone increasing. The estimates (4.11) as well as (4.12) are shown in the same way.

Let us now verify that for any fixed u, the function $\phi_\omega(u)$ is holomorphic in ω, and continuous on the domains as specified in (i), (ii) and (iii). Due to the locally uniform convergence of the perturbation series, it suffices to show that every $\phi_\omega^{(k)}(u)$ has the desired properties. We do this inductively, where the case $k = 0$ is trivial. Let us now assume that $\phi_\omega^{(k)}(u)$ is holomorphic in $\mathring{E}$ ($\{\operatorname{Im}\omega < \frac{1}{2}m\}$, respectively). In order to prove that $\phi_\omega^{(k+1)}$ is holomorphic, we want to apply Morera's theorem. Thus we must show that $\phi_\omega^{(k+1)}(u)$ is continuous in ω and that the integral

$$\oint_\gamma \phi_\omega^{(k+1)}(u)\, d\omega \stackrel{(4.7)}{=} \oint_\gamma \int_{-\infty}^u \frac{1}{\omega} \sin(\omega(u-v)) W(v) \phi_\omega^{(k)}(v)\, dv\, d\omega \tag{4.21}$$

vanishes for every closed contour γ in $\mathring{E}$ (or in case (iii), for every contour in $\{\operatorname{Im}\omega < \frac{1}{2}m\}$, respectively). Using the above estimates (4.14),(4.16) together with the monotonicity of $P_\omega(u)$ in u we get the following bound for the integrand

$$\begin{aligned} &\Big|\frac{1}{\omega} \sin(\omega(u-v)) W(v) \phi_\omega^{(k)}(v)\Big| \leq \\ &\qquad \leq |W(v)| \frac{4|v|}{1+|\omega v|} e^{-u \operatorname{Im}\omega - v|\operatorname{Im}\omega| - v \operatorname{Im}\omega} \frac{1}{k!} P_\omega(u)^k\ . \end{aligned} \tag{4.22}$$

Due to the induction hypothesis, the integrand is continuous in ω. Moreover, for a compact neighborhood $K(\omega_0)$ of a fixed ω_0 contained in the specified domains, (4.22) yields for the family $\frac{1}{\omega}\sin(\omega(u-v))W(v)\phi_\omega^{(k)}(v)$, $\omega \in K(\omega_0)$ the uniformly dominating function

$$|W(v)| \frac{4|v|}{1+|v| \min|\omega|} e^{-u \operatorname{Im}\omega - v\ \max(|\operatorname{Im}\omega| + \operatorname{Im}\omega)} \frac{1}{k!} P_\omega(u)^k\ ,$$

where the minimum and the maximum are taken in $K(\omega_0)$. This function is integrable for $K(\omega_0)$ chosen sufficiently small due to the integrability conditions on W. This lets us apply Lebesgue's dominated convergence theorem to show the continuity in ω for $\phi_\omega^{(k+1)}(u)$, which is given by the integral (4.7). Moreover, (4.22) together with the continuity in ω of $P_\omega(u)$ yield that the integral

$$\oint_\gamma \int_{-\infty}^u \Big|\frac{1}{\omega} \sin(\omega(u-v)) W(v) \phi_\omega^{(k)}(v)\Big|\, dv\, d\omega < \infty$$

exists for an arbitrary closed contour γ in $\mathring{E}$ (or $\{\operatorname{Im}\omega < \frac{1}{2}m\}$, respectively). By the theorem of Fubini, we may interchange the orders of integration in (4.21). Because of the induction hypothesis, the integrand of (4.21) on the right hand side is holomorphic. Thus the integral vanishes due to the Cauchy integral theorem. We conclude that $\phi_\omega^{(k)}$ is holomorphic for every k. Since $\phi_\omega(u)$ is holomorphic, the same argument together with equation (4.20) yields that ϕ'_ω is also holomorphic.

It remains to prove uniqueness. Let $\psi_\omega(u)$ be another solution of the Jost equation obeying $\lim_{u\to-\infty}\left|e^{-i\omega u}\psi_\omega(u)\right| < \infty$. Then we can find a $c > 0$ with $\left|\psi_\omega(u)\right| \le ce^{-u\,\mathrm{Im}\,\omega}$ for all $u \le u_0$. Then as above one shows inductively that

$$\left|\psi_\omega(u) - \sum_{l=0}^{N} \phi_\omega^{(k)}(u)\right| \le c\, e^{-u\,\mathrm{Im}\,\omega}\frac{1}{(N+1)!}P_\omega(u)^{N+1} ,$$

and taking $N \to \infty$ we obtain $\psi_\omega = \phi_\omega$.
The uniqueness also implies that $\overline{\phi(\omega, u)} = \phi(-\overline{\omega}, u)$, concluding the proof. □

Remark 4.4. In order to treat the Schrödinger equation (3.2) with boundary conditions at infinity (3.7), we derive the corresponding Jost equation with boundary conditions at $+\infty$ using the same procedure as on page 21:

$$\phi_\omega(u) = e^{-i\omega u} - \int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))W(v)\phi_\omega(v)\,dv \; . \tag{4.23}$$

It is obvious that the solution $\tilde{\phi}_\omega(u)$ of the Jost equation with boundary conditions at $-\infty$ with potential $W(-v)$ constructed in Theorem 4.2 gives rise to a solution ϕ_ω of (4.23) by defining $\phi_\omega(u) := \tilde{\phi}_\omega(-u)$.

With the results of Theorem 4.2 it is now easy to prove Theorem 4.1:

Proof of Theorem 4.1. Let us apply Theorem 4.2 to the potential $V_l(u)$ given by (1.20), which is obviously a smooth function in u. Furthermore, it vanishes on the event horizon $2M$ with the asymptotics $V_l = \mathcal{O}(r-2M)$. Using the definition (1.3) of the Regge-Wheeler coordinate u, this means that $V_l(u)$ decays exponentially as $u \to -\infty$. More precisely, there is a constant $c > 0$ such that

$$|V_l(u)| \le c\, e^{\frac{u}{2M}} \quad \text{for small } u \; .$$

Theorem 4.2 (*iii*) yields for $u \le u_0 < 0$ a solution $\phi_1(\omega, u)$ of the Jost equation (4.5) with the desired properties. It remains to show that ϕ_1 is also a solution of the Schrödinger equation (3.2) for $u \le u_0$. (Due to the Picard-Lindelöf theorem, this solution of the linear equation can be uniquely extended to $u \in \mathbb{R}$; the resulting function is analytic in ω due to the analytical dependence in ω from the coefficients and initial conditions.) But this follows immediately by differentiating equation (4.20) and using that $V_l \equiv W$ is smooth, so that the whole integrand is at least differentiable with respect to v. We have then proven the existence of $\acute{\phi}_\omega$. For the uniqueness, we show that in our special case every solution of (3.2) with boundary conditions (3.6) is a solution of (4.5). This can be done by integration by parts: For let $\psi_\omega(u)$ be such a solution. Then

$$\int_{-\infty}^{u} \frac{1}{\omega}\sin(\omega(u-v))V_l(v)\psi_\omega(v)\,dv =$$

$$= \int_{-\infty}^{u} \frac{1}{\omega} \sin(\omega(u-v))(\partial_v^2 + \omega^2)\psi_\omega(v)\, dv = \psi_\omega(u) - e^{i\omega u} \,,$$

where the remaining terms vanish due to the boundary conditions. Since we know that the solution of the Jost equation is uniquely determined, this must be also the case for the solution of the Schrödinger equation. Thus we have proven part (i).

For the proof of (ii) we refer to Remark 4.4. In contrast to the exponential decay at $-\infty$, the potential $V_l(u)$ has only polynomial decay at $+\infty$. More precisely, according to the definition of u, $V_l(u) = \mathcal{O}(\frac{l(l+1)}{u^2})$ for $l \geq 1$, $V_0(u) = \mathcal{O}(\frac{2M}{u^3})$, respectively, as $u \to \infty$. Thus we can apply the analogs of Theorem 4.2 (i), (ii), respectively. This gives the existence and uniqueness of the solution $\grave{\phi}_\omega$ for the Schrödinger equation with the stated properties. $\square$

When taking the limit $\epsilon \searrow 0$ in Stone's formula (3.11), the behavior of $\grave{\phi}_\omega(u)$ at $\omega = 0$ still causes problems. While in the case $l = 0$ we know from Theorem 4.1 that $\grave{\phi}_\omega$ can be continuously extended there, we do not yet know what happens for $l \neq 0$. The following theorem settles this problem by showing that, after suitable rescaling, the solutions $\grave{\phi}_\omega$ have a well-defined limit at $\omega = 0$:

Theorem 4.5. *For every angular momentum number l, there is a solution ϕ_0 of the Schrödinger equation (3.2) for $\omega = 0$ with the asymptotics*

$$\lim_{u\to\infty} u^l \phi_0(u) = i^l \frac{2^l \sqrt{\pi}}{\Gamma(\frac{1}{2} - l)} = (-i)^l (2l-1)!! \,, \tag{4.24}$$

where

$$(2l-1)!! := \begin{cases} (2l-1)\cdot(2l-3)\cdot \ldots \cdot 3 \cdot 1 & , \ \text{if } l \neq 0 \\ 1 & , \ \text{if } l = 0 \,. \end{cases}$$

This solution can be obtained as a limit of the solutions from Theorem 4.1, in the sense that for all $u \in \mathbb{R}$,

$$\phi_0(u) = \lim_{E \ni \omega \to 0} \omega^l \grave{\phi}_\omega(u) \quad \text{and} \quad \phi_0'(u) = \lim_{E\ni\omega\to 0} \omega^l \grave{\phi}_\omega'(u) \,. \tag{4.25}$$

Note that the above properties of the solution ϕ_0 really coincide in the case $l = 0$ with that of the solution $\grave{\phi}_0$ already constructed in Theorem 4.1 (ii).

For the proof of this theorem we use the same method as in the proof for Theorem 4.1. However, the iteration scheme (4.7) does not work for $l \neq 0$ in the limit $\omega \to 0$, because the integral

$$\phi_0^{(1)}(u) = -\int_u^\infty (u-v) V_l(v)\, dv$$

diverges ($V_l(u)$ decays only quadratically at infinity for $l \neq 0$). We avoid this problem by adding the leading asymptotic term of the potential V_l to the unperturbed equation,

$$\left(-\frac{d^2}{du^2} - \omega^2 + \frac{(l+\frac{1}{2})^2 - \frac{1}{4}}{u^2} \right) \phi_\omega(u) = -W_l(u)\phi_\omega(u) \ . \tag{4.26}$$

Now the perturbation term $W_l(u) = V_l(u) - \frac{l(l+1)}{u^2}$ has the asymptotics $W_l(u) = \mathcal{O}(\frac{\log u}{u^3})$.

Fortunately, the unperturbed differential equation corresponding to (4.26) can still be solved exactly. The solutions can be expressed in terms of Bessel functions. For our further consideration, the two functions

$$h_1(l, \omega u) = \sqrt{\frac{\pi \omega u}{2}} J_{l+\frac{1}{2}}(\omega u) \ , \quad h_2(l, \omega u) = \sqrt{\frac{\pi \omega u}{2}} J_{-l-\frac{1}{2}}(\omega u) \tag{4.27}$$

play an important role. Here the function $J_\nu(u)$ is the Bessel function of the first kind (a good reference for the theory of the Bessel functions is [31]). It solves Bessel's differential equation

$$u^2 y''(u) + u y'(u) + (u^2 - \nu^2) y(u) = 0 \ .$$

In addition, it is an analytic function in ν and u for all values of ν and $u \neq 0$ (if $\mathrm{Re}\, \nu \geq 0$, it can be analytically extended even to $u = 0$). It has the series expansion

$$J_\nu(u) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(\nu + m + 1)} \left(\frac{u}{2}\right)^{\nu + 2m} \tag{4.28}$$

and the following asymptotics for $|u| \gg 1$ (cf. [31] 7.21):

$$\begin{aligned} J_\nu(u) \sim \sqrt{\frac{2}{\pi u}} \Bigg[&\cos\left(u - \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right) \cdot \sum_{m=0}^{\infty} \frac{(-1)^m (\nu, 2m)}{(2u)^{2m}} \\ &- \sin\left(u - \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right) \cdot \sum_{m=0}^{\infty} \frac{(-1)^m (\nu, 2m+1)}{(2u)^{2m+1}} \Bigg] \ , \end{aligned} \tag{4.29}$$

where we have used the notation

$$(\nu, m) := \frac{\Gamma\left(\nu + m + \frac{1}{2}\right)}{m!\Gamma\left(\nu - m + \frac{1}{2}\right)} \ .$$

Moreover, the derivatives satisfy the recurrence formulas

$$\begin{aligned} u J_\nu'(u) &= u J_{\nu-1}(u) - \nu J_\nu(u) \quad \text{and} \\ u J_\nu'(u) &= \nu J_\nu(u) - u J_{\nu+1}(u) \ . \end{aligned}$$

The Wronskian of the functions $J_\nu, J_{-\nu}$ (which both solve the same differential equation, since Bessel's differential equation is symmetric in ν) is given by the formula

$$w\left(J_\nu(u), J_{-\nu}(u)\right) = -\frac{2\sin(\nu\pi)}{\pi u} \,. \tag{4.30}$$

This yields that these functions form a fundamental system for Bessel's differential equation provided that ν is not an integer.

In our applications we choose $\nu = l + \frac{1}{2}$. Thus the functions $h_1(l, \omega u)$ and $h_2(l, \omega u)$ have the following asymptotics,

$$h_1(l,\omega u) \quad \sim \quad \left\{ \begin{array}{lcl} \cos\left(\omega u - (l+1)\frac{\pi}{2}\right) & , & \text{if } |\omega u| \gg 1 \\ \frac{\sqrt{\pi}}{\Gamma(\frac{3}{2}+l)} \left(\frac{\omega u}{2}\right)^{l+1} & , & \text{if } |\omega u| \ll 1 \end{array} \right\} \tag{4.31}$$

$$h_2(l,\omega u) \quad \sim \quad \left\{ \begin{array}{lcl} \cos\left(\omega u + l\frac{\pi}{2}\right) & , & \text{if } |\omega u| \gg 1 \\ \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-l)} \left(\frac{\omega u}{2}\right)^{-l} & , & \text{if } |\omega u| \ll 1 \end{array} \right\} . \tag{4.32}$$

Furthermore, the formula (4.30) for the Wronskian simplifies to

$$w\big(h_1(l,\omega u), h_2(l,\omega u)\big) = (-1)^{l+1}\omega \,, \qquad \text{if } l \text{ is an integer} \,, \tag{4.33}$$

and this yields that in the case $\omega \neq 0$ the solutions h_1, h_2 form a fundamental system.

Thus for $\omega \neq 0$ we take as the Green's function for the operator on the left hand side of (4.26) the standard formula

$$S_\omega(u,v) = \Theta(v-u)\frac{1}{w(h_1,h_2)}\big(h_1(\omega v)h_2(\omega u) - h_1(\omega u)h_2(\omega v)\big) \,, \tag{4.34}$$

where $h_{1/2}(\omega u) \equiv h_{1/2}(l,\omega u)$ and Θ denotes the Heaviside function defined by $\Theta(x) = 1$ if $x \geq 0$ and $\Theta(x) = 0$ otherwise. Note that S_ω is also well-defined in the limit $\omega \to 0$. For this we use the asymptotics and the value of the Wronskian and get for very small ω,

$$\begin{aligned} \lim_{\omega\to 0} S_\omega(u,v) &= \lim_{\omega\to 0} \frac{(-1)^{l+1}}{\omega} \cdot \frac{\pi\omega}{2\Gamma\left(\frac{3}{2}+l\right)\Gamma\left(\frac{1}{2}-l\right)} \left(v^{l+1}u^{-l} - u^{l+1}v^{-l}\right) \\ &= \frac{(-1)^{l+1}\pi}{2\left(\frac{1}{2}+l\right)\Gamma\left(\frac{1}{2}+l\right)\Gamma\left(\frac{1}{2}-l\right)} \left(v^{l+1}u^{-l} - u^{l+1}v^{-l}\right) \\ &= \frac{(-1)^{l+1}\pi\cos(\pi l)}{(2l+1)\pi} \left(v^{l+1}u^{-l} - u^{l+1}v^{-l}\right) \\ &= -\frac{1}{2l+1} \left(v^{l+1}u^{-l} - u^{l+1}v^{-l}\right) \,, \end{aligned}$$

where we have used some elementary properties of the Gamma function. This also shows that the Green's function converges to the Green's function $S_0(u,v)$ given by the above formula for the solutions u^{l+1}, u^{-l} of the unperturbed differential operator on the left hand side of (4.26) for $\omega = 0$.
We now proceed with the perturbation series ansatz

$$\phi_\omega(u) = \sum_{m=0}^{\infty} \phi_\omega^{(m)}(u) \;, \tag{4.35}$$

which, as at the beginning of this section, leads to the iteration scheme

$$\phi_\omega^{(m+1)}(u) = -\int_u^\infty S_\omega(u,v) W_l(v) \phi_\omega^{(m)}(v)\, dv \;. \tag{4.36}$$

As initial function we take

$$\phi_\omega^{(0)}(u) = \omega^l e^{-i(l+1)\frac{\pi}{2}} \sqrt{\frac{\pi \omega u}{2}} H^{(2)}_{l+\frac{1}{2}}(\omega u) \;,$$

where $H^{(2)}_\nu$ is another solution of Bessels equation (called Bessel function of the third kind or second Hankel function). It is related to J_ν by

$$H^{(2)}_\nu(u) = \frac{J_{-\nu}(u) - e^{\nu \pi i} J_\nu(u)}{-i \sin(\nu\pi)} \;,$$

and has for large $|u|$ the asymptotics

$$H^{(2)}_\nu(u) \sim \sqrt{\frac{2}{\pi u}} e^{-i\left(u - \frac{1}{2}\pi\left(\nu + \frac{1}{2}\right)\right)} \sum_{m=0}^{\infty} \frac{(\nu, m)}{(2iu)^m} \;. \tag{4.37}$$

Thus, our initial function $\phi_\omega^{(0)}(u)$ solves the unperturbed equation, and we have the relation

$$\phi_\omega^{(0)}(u) = \omega^l \left((-i)^{l+1} h_1(l, \omega u) + i^l h_2(l, \omega u) \right) \tag{4.38}$$

together with the asymptotics

$$\phi_\omega^{(0)}(u) = \omega^l e^{-i\omega u} \left(1 + \mathcal{O}\left(\frac{1}{\omega u} \right) \right) \quad , \text{ if } |\omega u| \gg 1 \;. \tag{4.39}$$

Moreover, the function $\phi_\omega^{(0)}$ converges in the limit $\omega \to 0$ pointwise for all $u \geq u_0 > 0$:

$$\lim_{\omega \to 0} \phi_\omega^{(0)}(u) = i^l \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2} - l\right)} \left(\frac{u}{2} \right)^{-l} \;. \tag{4.40}$$

Since we are interested in statements for $\omega = 0$, it is convenient in what follows to restrict ω to the domain

$$F := \{ \omega \in \mathbb{C} \mid \operatorname{Im} \omega \leq 0, |\omega| \leq 1 \} \;.$$

The following lemma yields that our perturbation series (4.35) is well-defined.

Lemma 4.6. *For every $u_0 > 0$, the iteration scheme (4.35),(4.36),(4.38) converges locally uniformly for all $u \geq u_0$ and $\omega \in F$. In particular, the functions $\phi_\omega(u)$ are for fixed u a continuous family in $\omega \in F$. They satisfy the integral equation*

$$\phi_\omega(u) = \phi_\omega^{(0)}(u) - \int_u^\infty S_\omega(u,v) W_l(v) \phi_\omega(v)\, dv \,. \tag{4.41}$$

Proof. In order to prove the lemma, we need to derive good bounds for the initial function $\phi_\omega^{(0)}(u)$ as well as for the Green's function $S_\omega(u,v)$. To this end, we exploit the asymptotics of h_1, h_2. We thus obtain the bound

$$\frac{1}{C_1} \leq |\phi_\omega^{(0)}(u)| e^{-u \operatorname{Im} \omega} \left(\frac{u}{1+|\omega| u} \right)^l \leq C_1 \,. \tag{4.42}$$

Likewise, for the Green's function we have (note that $v \geq u > 0$),

$$|S_\omega(u,v)| \;\leq\; C_2 \left(\frac{u}{1+|\omega| u} \right)^{-l} \left(\frac{v}{1+|\omega| v} \right)^{l+1} , \quad \text{if } |\omega v| \ll 1$$

and

$$|S_\omega(u,v)| \;\leq\; C_3 \frac{v}{1+|\omega| v} e^{v |\operatorname{Im} \omega| + u \operatorname{Im} \omega} , \qquad \text{if } |\omega u| \gg 1.$$

The last inequality follows from the asymptotics

$$|S_\omega(u,v)| \;\sim\; \left| \frac{1}{\omega} \sin(\omega(u-v)) \right|, \quad \text{if } |\omega u| \gg 1 \,,$$

in the same way as the second inequality of Lemma 4.3. Combining these cases we find a constant such that

$$|S_\omega(u,v)| \leq C_4 \left(\frac{u}{1+|\omega| u} \right)^{-l} \left(\frac{v}{1+|\omega| v} \right)^{l+1} e^{v |\operatorname{Im} \omega| + u \operatorname{Im} \omega} \,. \tag{4.43}$$

Hence defining the function Q_ω by

$$Q_\omega(u) := C_4 \int_u^\infty \frac{v}{1+|\omega| v} \, |W_l(v)| \, dv \,, \tag{4.44}$$

which is well-defined for all $\omega \in F$ and $u \geq u_0 > 0$ due to the asymptotic of W_l, it is straightforward to show inductively (cf. proof of Theorem 4.2) that for all $m \in \mathbb{N}$

$$\left| \phi_\omega^{(m)}(u) \right| \leq C_1 \left(\frac{u}{1+|\omega| u} \right)^{-l} e^{u \operatorname{Im} \omega} \frac{Q_\omega(u)^m}{m!} \,. \tag{4.45}$$

Now we proceed exactly as in the proof of Theorem 4.2, where the inequality (4.45) can be considered as the analogue of (4.16). It follows that the series (4.35) converges locally uniformly in ω and u and satisfies the integral equation (4.41). Furthermore, one shows inductively applying Lebesgue's dominated convergence theorem, that for fixed u each $\phi_\omega^{(m)}(u)$ depends continuously of $\omega \in F$. It follows that the same is true for the series due to local uniform convergence. ☐

We are now ready to prove Theorem 4.5:

Proof of Theorem 4.5. According to Lemma 4.6, our perturbation series (4.35) satisfies the integral equation (4.41). Using the recurrence formulas for the derivatives of $J_\nu(u)$, one obtains

$$\begin{aligned} \partial_u h_1(l, \omega u) &= -\frac{l}{u} h_1(l, \omega u) + \omega h_1(l-1, \omega u), \\ \partial_u h_2(l, \omega u) &= -\frac{l}{u} h_2(l, \omega u) - \omega h_2(l-1, \omega u) \;, \text{ respectively.} \end{aligned}$$

This allows us to estimate the behavior of $\partial_u S_\omega(u, v)$. Exactly as for $S_\omega(u, v)$, we obtain the following asymptotic formulas,

$$|\partial_u S_\omega(u, v)| \le C_5 \left(\frac{u}{1 + |\omega| u} \right)^{-l-1} \left(\frac{v}{1 + |\omega| v} \right)^{l+1} e^{v |\operatorname{Im} \omega| + u \operatorname{Im} \omega} \;.$$

Following the same arguments of the proofs of Theorems 4.1 and 4.2, and combining them with the above estimates and asymptotic formulas we now have the following results:

1) One can differentiate $\phi_\omega(u)$ with respect to u, and $\phi'_\omega(u)$ is given by

$$\phi'_\omega(u) = \left(\phi^{(0)}_\omega \right)'(u) - \int_u^\infty \partial_u S_\omega(u, v) W_l(v) \phi_\omega(v) \, dv \;.$$

In particular, Lebesgue's dominated convergence theorem yields that for fixed u, $\phi'_\omega(u)$ is continuous in $\omega \in F$.

2) $\phi_\omega(u)$ and $\phi'_\omega(u)$ obey the following estimates,

$$\begin{aligned} |\phi_\omega(u) - \phi^{(0)}_\omega(u)| \;&\le\; C_1 \left(\frac{u}{1 + |\omega| u} \right)^{-l} e^{u \operatorname{Im} \omega} \left(e^{Q_\omega(u)} - 1 \right) \\ \left| \phi'_\omega(u) - \left(\phi^{(0)}_\omega \right)'(u) \right| \;&\le\; C_5 \left(\frac{u}{1 + |\omega| u} \right)^{-l-1} e^{Q_\omega(u_0)} e^{u \operatorname{Im} \omega} \int_u^\infty v |W_l(v)| dv \;. \end{aligned}$$

Thus $\phi_\omega(u) \sim \omega^l e^{-i\omega u}$ and $\phi'_\omega(u) \sim -i\omega^{l+1} e^{-i\omega u}$ as $u \to \infty$.

3) Differentiating $\phi_\omega(u)$ twice with respect to u shows that $\phi_\omega(u)$ is a solution of the Schrödinger equation (3.2) for all $u \ge u_0$. Furthermore, from the asymptotics at infinity combined with the uniqueness statement in Theorem 4.1, we know that

$$\phi_\omega(u) = \omega^l \grave{\phi}_\omega(u) \quad , \text{ if } \omega \neq 0, u \ge u_0 \;. \tag{4.46}$$

Obviously, this extends to all $u \in \mathbb{R}$.

Thus we have proven the continuity statement (4.25) for all $u \ge u_0$. On the other hand, we know from the Picard-Lindelöf theorem that for u on compact intervals, the solutions depend continuously on ω. This yields (4.25) for all $u \in \mathbb{R}$.

Finally, the asymptotics (4.24) is a simple consequence of (4.40). □

5 An Integral Spectral Representation

In the previous section we derived some regularity results for the solutions $\acute{\phi}_\omega$ and $\grave{\phi}_\omega$. We already know (cf. Section 3) that these solutions are a system of fundamental solutions of the Schrödinger equation (3.2) in the cases $\mathrm{Im}\,\omega < 0$ and $\mathrm{Im}\,\omega > 0$, respectively. Thus the Wronskian $w(\acute{\phi}_\omega, \grave{\phi}_\omega)$ is non-vanishing in these regions, which implies that the integral kernel $k_\omega(u, v)$ of the resolvent is well defined. Since our next goal is to get the limit in (3.11), we prove in the next lemma that the continuous extension of the solutions $\acute{\phi}_\omega, \grave{\phi}_\omega$ to the real axis again yields a system of fundamental solutions. More precisely,

Lemma 5.1. *The Wronskian $w(\acute{\phi}_\omega, \grave{\phi}_\omega)$ does not vanish for $\omega \in \mathbb{R} \setminus \{0\}$. In particular, $\acute{\phi}_\omega, \grave{\phi}_\omega$ are fundamental solutions for the Schrödinger equation (3.2). In addition, this remains true for the solutions $\acute{\phi}_0$ and ϕ_0 in the case $\omega = 0$.*

Proof. Let us begin with the statement for $\acute{\phi}_0, \phi_0$:
For $\omega = 0$, the solutions $\acute{\phi}_0(u), \phi_0(u)$ have the asymptotics

$$\lim_{u \to -\infty} \acute{\phi}_0(u) = 1 \quad \text{and} \quad \lim_{u \to \infty} u^l \phi_0(u) = (-i)^l (2l-1)!!\ .$$

Looking at the construction of these solutions, one sees that $\acute{\phi}_0$ is a real solution, while ϕ_0 is either purely real or imaginary (depending on the value of l). The Schrödinger equation for $\omega = 0$ reduces to $\phi''(u) = V_l(u)\phi(u)$ with a everywhere positive potential V_l. Hence, exploiting the special asymptotics, the solution $\acute{\phi}_0$ is convex and $\mathrm{Re}\,\phi_0$ ($\mathrm{Im}\,\phi_0$, respectively) is either convex or concave depending on l. In any case, we see that $\acute{\phi}_0$ and ϕ_0 are linearly independent, and thus $w(\acute{\phi}_0, \phi_0) \neq 0$.

In order to prove the main part of the Lemma, we consider a complex solution $z = z_1 + i z_2$ of the Schrödinger equation, where $\{z_1, z_2\}$ is a fundamental system of real solutions, especially $w(z_1, z_2) \equiv c \neq 0$. Setting $y = \frac{z'}{z}$, a simple computation shows that

$$\mathrm{Im}\, y = \frac{w(z_1, z_2)}{|z|^2}\ ,$$

where the right hand side is well defined because $w(z_1, z_2) \neq 0$ implies that $|z| \neq 0$ everywhere. As a consequence, we have $\mathrm{Im}\, y \neq 0$ everywhere. Thus it follows that for all u either $\mathrm{Im}\, y(u) > 0$ or < 0, due to the continuity of the solution z in u.

Applying this result to the solutions $\acute{\phi}_\omega$ and $\grave{\phi}_\omega$, respectively, and exploiting their asymptotics, one sees that $\mathrm{Im}\, \acute{y}_\omega(u)$ and $\mathrm{Im}\, \grave{y}_\omega(u)$ have different signs for all u. Therefore,

$$w(\acute{\phi}_\omega, \grave{\phi}_\omega) = \acute{\phi}_\omega(u)\grave{\phi}'_\omega(u) - \acute{\phi}'_\omega(u)\grave{\phi}_\omega(u) = \acute{\phi}_\omega(u)\grave{\phi}_\omega(u)\big(\grave{y}_\omega(u) - \acute{y}_\omega(u)\big) \neq 0\ .$$

□

As a consequence we have the following

Corollary 5.2. *The function* $s_\omega(u,v)$ *given by (3.9) is continuous in* (ω,u,v) *for* $\omega \in \{\operatorname{Im}\omega \le 0\}$, $(u,v)\in\mathbb{R}^2$.

Proof. We already know that for fixed $u_0 < 0$, $\acute{\phi}_\omega(u_0)$ is continuous in ω on $\{\operatorname{Im}\omega \le 0\}$. Thus as solutions of the linear differential equation (3.2), which depends analytically on ω and smooth on u, the family $\acute{\phi}_\omega(u)$ is (at least) continuous in (ω,u) in the region $\{\operatorname{Im}\omega\le 0\}\times\mathbb{R}$. Analogously this holds for $\omega^l\grave{\phi}_\omega(u)$ according to Theorems 4.1 and 4.5. Since $s_\omega(u,v)$ is invariant if we substitute $\omega^l\grave{\phi}_\omega(u)$ for $\grave{\phi}_\omega(u)$, the preceding lemma yields the claim. □

Note that the corollary is also true if ω is in the upper half plane. The essential statement in this corollary is that one can extend $s_\omega(u,v)$ continuously in ω up to the real axis.

From the definitions (4.1) and (4.2), we have for $\omega\in\{\operatorname{Im}\omega\neq 0\}$ the relations

$$\overline{s_\omega(u,v)} = s_{\overline{\omega}}(u,v)\ , \quad \text{hence} \quad \overline{k_\omega(u,v)} = k_{\overline{\omega}}(u,v)\ .$$

This allows us to simplify the expression (3.11). Evaluating for fixed u the right hand side of (3.11) we obtain for any $\Psi\in\mathscr{H}$ as well as for any bounded interval $[a,b]\subseteq\mathbb{R}$

$$\lim_{\epsilon\searrow 0} -\frac{1}{\pi}\int_a^b\left(\int_{\mathbb{R}}\operatorname{Im}\left(k_{\omega-i\epsilon}(u,v)\right)\Psi(v)dv\right)d\omega\ .$$

According to the above corollary, we know that $\operatorname{Im}k_\omega(u,v)$ is continuous in (ω,u,v) for $\omega\in\{\operatorname{Im}\omega\le 0\}$, $(u,v)\in\mathbb{R}^2$. Thus, if we restrict Ψ to the dense set $C_0^\infty(\mathbb{R})^2$, we integrate a continuous integrand over a compact interval. Hence, considering the limit as a pointwise limit for any u, we may interchange the limit and integration. Thus for any $\Psi\in C_0^\infty(\mathbb{R})^2$, $[a,b]\subset\mathbb{R}$ bounded and u the right hand side of (3.11) converges pointwise to

$$-\frac{1}{\pi}\int_a^b\left(\int_{\operatorname{supp}\Psi}\operatorname{Im}\big(k_\omega(u,v)\big)\Psi(v)dv\right)d\omega\ .$$

Hence, together with the norm convergence in (3.11), the spectral projections of H are for every u described by the formula

$$\frac{1}{2}\big(P_{[a,b]}+P_{(a,b)}\big)\Psi(u) = -\frac{1}{\pi}\int_a^b\left(\int_{\operatorname{supp}\Psi}\operatorname{Im}\big(k_\omega(u,v)\big)\Psi(v)dv\right)d\omega\ . \quad (5.1)$$

In particular, this representation yields that $P_{[a,b]}\equiv P_{(a,b)}$.

As an immediate consequence we have the following

Corollary 5.3. *The spectrum $\sigma(H)$ of the operator H is absolutely continuous, i.e. $\sigma(H) \equiv \sigma_{ac}(H)$.*

Proof. The corollary is equivalent to the statement that the spectral measure $\langle \Psi, dP_\omega \Psi \rangle$ of any $\Psi \in \mathscr{H}$ is absolutely continuous. This is clear by (5.1) for any $\Psi \in C_0^\infty(\mathbb{R})^2$. But since this subset is dense, this also holds on the whole Hilbert space $\mathscr{H}$. □

Next we want to write the integrand in (5.1), i.e. $\int_{\mathrm{supp}\Psi} \dots dv$, in a more compact way. We first note that for real ω the complex conjugates of $\acute{\phi}_\omega$ and $\grave{\phi}_\omega$ are again solutions of (3.2). Hence, for any $\omega \in \mathbb{R} \setminus \{0\}$ the pair $\left\{ \acute{\phi}_\omega, \overline{\acute{\phi}_\omega} \right\}$ forms a fundamental system for this equation due to the boundary conditions. Thus we can express $\grave{\phi}_\omega$ as a linear combination of $\acute{\phi}_\omega$ and $\overline{\acute{\phi}_\omega}$,

$$\grave{\phi}_\omega(u) = \lambda(\omega) \acute{\phi}_\omega(u) + \mu(\omega) \overline{\acute{\phi}_\omega(u)} \quad (\omega \in \mathbb{R} \setminus \{0\}) ,$$

where λ and μ are referred to as *transmission coefficients.* The Wronskian of $\acute{\phi}_\omega$ and $\grave{\phi}_\omega$ can be expressed by

$$w\left(\acute{\phi}_\omega, \grave{\phi}_\omega\right) = \mu(\omega) w\left(\acute{\phi}_\omega, \overline{\acute{\phi}_\omega}\right) = -2i\omega\mu(\omega) ,$$

where in the last step we used the asymptotics (3.6). Moreover, we introduce the real fundamental solutions

$$\phi_\omega^1(u) = \mathrm{Re}\, \acute{\phi}_\omega(u) , \quad \phi_\omega^2(u) = \mathrm{Im}\, \acute{\phi}_\omega(u)$$

and denote the corresponding eigenvectors of the Hamiltonian H by $\Phi_\omega^a(u) = (\phi_\omega^a(u), \omega\phi_\omega^a(u))^T$.

Using the above definitions, a short calculation shows that for $\omega \neq 0$ we can express the imaginary part of the Green's function $s_\omega(u,v)$ by

$$\mathrm{Im}\, s_\omega(u,v) = -\frac{1}{2\omega} \sum_{a,b=1}^{2} t_{ab}(\omega) \phi_\omega^a(u) \phi_\omega^b(v) , \tag{5.2}$$

where the coefficients $t_{ab}(\omega)$ are given by

$$\begin{aligned} t_{11}(\omega) &= 1 + \mathrm{Re}\left(\frac{\lambda}{\mu}(\omega)\right), \quad t_{12}(\omega) = t_{21}(\omega) = -\,\mathrm{Im}\left(\frac{\lambda}{\mu}(\omega)\right) , \\ t_{22}(\omega) &= 1 - \mathrm{Re}\left(\frac{\lambda}{\mu}(\omega)\right) . \end{aligned} \tag{5.3}$$

Since we know that $\mathrm{Im}\, s_\omega(u,v)$ is continuous for $\omega \in \mathbb{R}$ and the expression (5.2) holds for all $\omega \in \mathbb{R} \setminus \{0\}$, it extends to $\omega = 0$. With (5.2), the integrand in (5.1) can be written as

$$-\frac{1}{2\omega} \Bigg(\int_{\mathrm{supp}\Psi} \sum_{a,b=1}^{2} t_{ab}(\omega) \phi_\omega^a(u) \phi_\omega^b(v) \begin{pmatrix} \omega & 1 \\ \omega^2 & \omega \end{pmatrix} \Psi(v) dv \Bigg) =$$

$$= -\frac{1}{2\omega^2} \sum_{a,b=1}^{2} t_{ab}(\omega)\Phi^a_\omega(u)\Big(\int_{\text{supp}\Psi} \omega^2 \phi^b_\omega(v)\psi_1(v) + \omega\phi^b_\omega(v)\psi_2(v)\, dv\Big) ,$$

where the ψ_i denote the two components of Ψ.

Since $\phi^b_\omega(u)$ solves the Schrödinger equation (3.2), it satisfies the relation $\big(-\partial_v^2 + V_l(v)\big)\phi^b_\omega(v) = \omega^2\phi^b_\omega(v)$. Using this and integrating by parts, this simplifies to

$$-\frac{1}{2\omega^2} \sum_{a,b=1}^{2} t_{ab}(\omega)\Phi^a_\omega(u)\langle\Phi^b_\omega, \Psi\rangle . \tag{5.4}$$

(Note that in this case the energy scalar product of Φ^b_ω and Ψ is well defined, because Ψ has compact support. Whereas in general this does not exist for arbitrary $\Psi \in \mathscr{H}$, due to the fact that $\Phi^b_\omega \notin \mathscr{H}$.)

With (5.4), we now obtain a more compact representation for the spectral projections. Moreover, we can use (5.4) to express the solution operators e^{-itH}.

Proposition 5.4. *Consider the Cauchy Problem (1.23) for compactly supported smooth initial data $\Psi_0 \in C_0^\infty(\mathbb{R})^2$. Then the solution has the integral representation*

$$\begin{aligned} \Psi(t) &= e^{-itH}\Psi_0 = \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega t}\frac{1}{\omega^2}\sum_{a,b=1}^{2} t_{ab}(\omega)\Phi^a_\omega \left\langle\Phi^b_\omega, \Psi_0\right\rangle d\omega . \end{aligned} \tag{5.5}$$

Here the integral converges in norm in the Hilbert space $\mathscr{H}$.

Proof. We use the following variation of Stone's formula to obtain for any bounded interval $(c,d) \subseteq \mathbb{R}$

$$\begin{aligned} &\frac{1}{2}e^{-itH}\left(P_{[c,d]} + P_{(c,d)}\right)\Psi \\ &\quad = \lim_{\epsilon\searrow 0}\int_c^d e^{-i\omega t}\left[(H-\omega-i\epsilon)^{-1} - (H-\omega+i\epsilon)^{-1}\right]\Psi\, d\omega , \end{aligned}$$

where the limit is with respect to the norm of $\mathscr{H}$. Since we know that $P_{[c,d]} \equiv P_{(c,d)}$, it follows that this expression is equal to $e^{-itH}P_{(c,d)}\Psi$. Using the explicit formula for the resolvent, for every $u \in \mathbb{R}$ the right hand side is equal to

$$\lim_{\epsilon\searrow 0} -\frac{1}{\pi}\int_c^d e^{-i\omega t}\left(\int_{\mathbb{R}} \text{Im}\big(k_{\omega-i\epsilon}(u,v)\big)\Psi(v)dv\right) d\omega . \tag{5.6}$$

Due to the continuity of the imaginary part of the kernel $k_\omega(u,v)$, we may take for $\Psi_0 \in C_0^\infty(\mathbb{R})^2$ and (c,d) bounded the pointwise limit for any $u \in \mathbb{R}$.

Hence, using (5.4) we can simplify (5.6) to

$$\frac{1}{2\pi}\int_c^d e^{-i\omega t}\frac{1}{\omega^2}\sum_{a,b=1}^{2} t_{ab}(\omega)\Phi^a_\omega(u)\langle\Phi^b_\omega,\Psi_0\rangle\,d\omega\;,$$

and together with the norm convergence it follows that this term is equal to $e^{-itH}P_{(c,d)}\Psi_0(u)$. Using the abstract spectral theorem and that e^{-itH} is a unitary operator, it is clear that $e^{-itH}P_{(-n,n)}\Psi_0 \to e^{-itH}\Psi_0$ in norm as $n\to\infty$. □

This proposition extends to the following theorem.

Theorem 5.5. *For any fixed $u\in\mathbb{R}$ the integrand in the representation (5.5) is in $L^1(\mathbb{R},\mathbb{C}^2)$ as a function of ω. In particular, the representation (5.5) of the solutions holds pointwise for every $u\in\mathbb{R}$, i.e.*

$$\Psi(t,u)=\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega t}\frac{1}{\omega^2}\sum_{a,b=1}^{2} t_{ab}(\omega)\Phi^a_\omega(u)\,\langle\Phi^b_\omega,\Psi_0\rangle\,d\omega\;. \tag{5.7}$$

Moreover, for u fixed, the function $\Psi(t,u)$ vanishes as $t\to\infty$.

Proof. Since we know that the integrand is continuous in ω, it is in $L^1([a,b],\mathbb{C}^2)$ for any bounded interval $[a,b]$. Thus it remains to analyze the integrand for large $|\omega|$.

To this end, we must investigate the asymptotic behavior of the fundamental solutions $\acute{\phi}_\omega$ and $\grave{\phi}_\omega$ in ω. We constructed these solutions with the iteration scheme (4.7) as solutions of the Jost equation. For the proof of this, the estimate (4.14) played an essential role. Since in this case we consider real ω with $|\omega|\gg 1$, we can use the simple estimate $\left|\frac{1}{\omega}\sin(\omega(u-v))\right|\le\frac{1}{|\omega|}$ instead of (4.14), which now holds for every $u,v\in\mathbb{R}$. Thus, proceeding exactly in the same way as in the proof of Theorem 4.2, we now obtain the following estimates for the several terms in the series expansion (4.6)

$$\left|\phi^{(k)}_\omega(u)\right|\le\frac{1}{k!}\hat{P}_\omega(u)^k\;,\quad\text{where }\hat{P}_\omega(u):=\int_{-\infty}^{u}\frac{1}{|\omega|}V_l(v)dv\;,$$

for any $k\in\mathbb{N}$ and $u\in\mathbb{R}$. Thus the solution $\acute{\phi}_\omega(u)$ ($\omega\neq 0$) is given *for all* $u\in\mathbb{R}$ by this series expansion and obeys the uniform bound

$$\left|\acute{\phi}_\omega(u)-e^{i\omega u}\right|\;\le\;e^{\hat{P}_\omega(u)}-1\;\le\;e^{\frac{1}{|\omega|}\|V_l\|_{L^1}}-1\;,$$

since $V_l\in L^1(\mathbb{R})$. In particular,

$$\left|\acute{\phi}_\omega(u)\right|\le 1+\mathcal{O}\left(\frac{1}{|\omega|}\right)\quad\text{for }|\omega|\gg 1\;. \tag{5.8}$$

Next we investigate the ω-dependence of $\langle \Phi^b_\omega, \Psi_0 \rangle$. We integrate by parts to obtain

$$\langle \Phi^b_\omega, \Psi_0 \rangle = \int\limits_{\mathrm{supp}\Psi_0} \phi^b_\omega(v)\big(\omega\psi_2(v) - \psi_1(v)'' + V_l(v)\psi_1(v)\big)dv\ ,$$

where $\Psi_0 = (\psi_1, \psi_2)^T$ (note that the boundary terms drop out, because $\Psi_0 \in C_0^\infty(\mathbb{R})^2$). Since $\phi^b_\omega(u)$ is a solution of the Schrödinger equation (3.2), we substitute $\frac{1}{\omega^2}\left(-\phi^b_\omega(u)'' + V_l(u)\phi^b_\omega(u)\right)$ for $\phi^b_\omega(u)$ and integrate by parts twice,

$$= \frac{1}{\omega^2} \int\limits_{\mathrm{supp}\Psi_0} \phi^b_\omega\big(- (\omega\psi_2 - \psi_1'' + V_l\psi_1)'' + V_l(\omega\psi_2 - \psi_1'' + V_l\psi_1)\big)dv\ .$$

We can now iterate this procedure as often as we like due to the fact that $\Psi_0 \in C_0^\infty(\mathbb{R})^2$ and $V_l \in C^\infty(\mathbb{R})$. Thus using the bound (5.8), we obtain arbitrary polynomial decay in ω for $\langle \Phi^b_\omega, \Psi_0 \rangle$.

Thus it remains to control the coefficients $t_{ab}(\omega)$ for large $|\omega|$. According to the definition of the transmission coefficients $\lambda(\omega)$ and $\mu(\omega)$, they satisfy the following relations,

$$w(\grave{\phi}_\omega, \acute{\phi}_\omega) = 2i\omega\mu(\omega) \quad \text{and} \quad w(\grave{\phi}_\omega, \overline{\acute{\phi}_\omega}) = -2i\omega\lambda(\omega)\ .$$

In order to calculate the Wronskians, we substitute the Jost integral equations (4.5),(4.23) for $\acute{\phi}_\omega$ and $\grave{\phi}_\omega$, respectively, as well as the corresponding integral equations for the derivatives (for instance (4.20) in the case $\acute{\phi}_\omega$) and obtain immediately

$$\mu(\omega) = 1 + \mathcal{O}\left(\frac{1}{\omega}\right)\ , \quad \lambda(\omega) = \mathcal{O}\left(\frac{1}{\omega}\right).$$

Hence the coefficients $t_{ab}(\omega)$ remain (at least) bounded, according to their definition (5.3).

We conclude that the integrand in (5.7) is in $L^1(\mathbb{R}, \mathbb{C}^2)$ as a function of ω. Thus the right hand side in the integral representation (5.5) converges also pointwise and, together with the norm convergence, (5.7) follows.

Since for u fixed, $\Psi(t, u)$ is a Fourier transform of a L^1 -function, the Riemann-Lebesgue lemma applies. Hence $\Psi(t, u)$ vanishes as $t \to \infty$. □

In the next step we extend this proposition to the Cauchy problem (1.6).

Theorem 5.6. *Consider the Cauchy problem (1.6) for smooth and compactly supported initial data. Then there exists a unique smooth solution, which is compactly supported for all times.*

Moreover, decomposing the initial data Ψ_0 into spherical harmonics, the solution has the representation

$$\Psi(t,u,\vartheta,\varphi) = \sum_{l=0}^{\infty} \sum_{|m|\leq l} e^{-itH_l} \Psi_0^{lm}(u) Y_{lm}(\vartheta,\varphi) \; . \tag{5.9}$$

Proof. For the existence and uniqueness of such a solution we apply the theory of linear symmetric hyperbolic systems (cf. [15]). Since the equation in (1.6) is defined on $\mathbb{R} \times \mathbb{R} \times S^2$ we have to work in local coordinates for S^2. We demonstrate the idea in the chart $\big(U, (\vartheta,\varphi)\big)$, where U is an open, relative compact subset of S^2 such that (ϑ,φ) are well defined on $\overline{U}$.

Letting $\Phi = (\partial_t\psi, \partial_u\psi, \partial_{\cos\vartheta}\psi, \partial_\varphi\psi, \psi)^T$ we can write the equation as the first order system

$$A^0 \partial_t \Phi + A^1 \partial_u \Phi + A^2 \partial_{\cos\vartheta} \Phi + A^3 \partial_\varphi \Phi + B\Phi = 0 \; ,$$

where the matrices A^i, B are given by

$$A^0 = \mathrm{diag}\left(1, 1, \left(1 - \frac{2M}{r}\right) \frac{1}{r^2} \sin^2\vartheta, \left(1 - \frac{2M}{r}\right) \frac{1}{r^2} \frac{1}{\sin^2\vartheta}, 1\right) \; ,$$

$$\begin{aligned}
A^1 &= (a^1_{ij}) \; , &\quad \text{with} \quad & a^1_{12} = a^1_{21} = -1 \; , \\
A^2 &= (a^2_{ij}) \; , &\quad \text{with} \quad & a^2_{13} = a^2_{31} = -\left(1 - \frac{2M}{r}\right) \frac{1}{r^2} \sin^2\vartheta \; , \\
A^3 &= (a^3_{ij}) \; , &\quad \text{with} \quad & a^3_{14} = a^3_{41} = -\left(1 - \frac{2M}{r}\right) \frac{1}{r^2} \frac{1}{\sin^2\vartheta} \; , \\
B &= (b_{ij}) \; , &\quad \text{with} \quad & b_{13} = \left(1 - \frac{2M}{r}\right) \frac{1}{r^2} 2\cos\vartheta \; , \\
& & & b_{15} = \left(1 - \frac{2M}{r}\right) \frac{2M}{r^3}, \; b_{51} = -1 \; ,
\end{aligned}$$

and all other coefficients vanish. By multiplying this system with the term $\left(1 - \frac{2M}{r}\right)^{-1} r^2$, we obtain a linear symmetric hyperbolic system on $\mathbb{R}\times\mathbb{R}\times U$ in the sense that the A^i are symmetric and A^0 is uniformly positive definite on $\mathbb{R}\times\mathbb{R}\times U$. Since the initial data Ψ_0 has compact support, we can restrict the system to $\mathbb{R}\times V \times U$, where $V \subseteq \mathbb{R}$ is an open, relative compact set with $\mathrm{supp}\Psi_0 \subseteq V \times S^2$. Considering the system on this domain, the matrices A^i, B remain uniformly bounded. Since we can cover S^2 by a finite number of such charts, the theory of symmetric hyperbolic systems yields an $\varepsilon_1 > 0$ such that there is unique and smooth solution $\psi(t,u,x)$ for all $t < \varepsilon_1$ on $\mathbb{R} \times V \times S^2$ with initial data Ψ_0.

Moreover, this solution has finite propagation speed, which is independent of u (this is physically clear from causality; it follows mathematically by considering lens-shaped regions for our symmetric hyperbolic systems). Thus there exists an $\varepsilon > 0$ (possibly smaller than ε_1) such that the solution $\psi(t,u,x)$ has compact support in $V \times S^2$ for all times $t \leq \varepsilon$. Iterating this argument for the Cauchy problem with initial data $(\psi(\varepsilon,u,x), i\partial_t \psi(\varepsilon,u,x))$ (and choosing $V \subseteq \mathbb{R}$ sufficiently large), we get a unique and smooth solution $\psi(t,u,x)$ with compact support for all $t \leq 2\varepsilon$ and so forth. Thus we have proven the existence of a global solution $\psi(t,u,x) \in C^\infty(\mathbb{R} \times \mathbb{R} \times S^2)$ which is unique and compactly supported for all times t.

In order to prove the representation (5.9), we consider the restriction of the solution $\Psi(t,u,x) = (\psi(t,u,x), i\partial_t \psi(t,u,x))^T$ of the Cauchy problem (1.12) in Hamiltonian form to fixed modes l, m

$$\Psi^{lm}(t,u) Y_{lm}(\vartheta,\varphi) = \langle \Psi(t,u), Y_{lm} \rangle_{L^2(S^2)} Y_{lm}(\vartheta,\varphi) \ .$$

Then $\Psi^{lm}(t,u) Y_{lm}(\vartheta,\varphi)$ is a solution of (1.12) with the smooth and compactly supported initial data $\Psi_0^{lm}(u) Y_{lm}(\vartheta,\varphi)$. Thus $\Psi^{lm}(t,u)$ is a solution of the Cauchy problem (1.23), and due to the uniqueness of such solutions

$$\Psi^{lm}(t,u) = e^{-itH_l} \Psi_0^{lm}(u) \ .$$

The uniqueness of the decomposition into spherical harmonics yields (5.9). □

We are now ready to prove pointwise decay.

Theorem 5.7. *Consider the Cauchy problem of the scalar wave equation in the Schwarzschild geometry*

$$\Box \phi = 0 \ , \quad (\phi, i\partial_t \phi)(0,r,x) = \Phi_0(r,x)$$

for smooth initial data $\Phi_0 \in C_0^\infty((2M,\infty) \times S^2)^2$ which is compactly supported outside the event horizon. Then there exists a unique global solution $\Phi(t) = (\phi(t), i\partial_t \phi(t)) \in C^\infty(\mathbb{R} \times (2M,\infty) \times S^2)^2$ which is compactly supported for all times t. Moreover, for fixed (r,x) this solution decays as $t \to \infty$.

Proof. The existence and uniqueness of solutions of the Cauchy problem follow directly from Theorem 5.6 after the substitution $\psi = r\phi$. Thus it remains to show the pointwise decay.

The conserved energy for solutions which are compactly supported for all times t implies that for every t

$$\|\Psi(t,u,\vartheta,\varphi)\|^2 = \|\Psi_0(u,\vartheta,\varphi)\|^2 = \sum_{l=0}^{\infty} \sum_{|m| \leq l} \|\Psi_0^{lm}(u)\|_l^2 \ ,$$

where for the second equation we used the isometry (1.21). Hence, defining

$$\Psi^L(t,u,\vartheta,\varphi) := \sum_{l=L}^{\infty} \sum_{|m| \leq l} \Psi^{lm}(t,u) Y_{lm}(\vartheta,\varphi) \ ,$$

we can find for every $\varepsilon > 0$ a number L_0 such that

$$\|\Psi^{L_0}(t,u,\vartheta,\varphi)\|^2 = \sum_{l=L_0}^{\infty} \sum_{|m|\le l} \|\Psi_0^{lm}(u)\|_l^2 < \varepsilon \,.$$

Let us now consider the Cauchy problem (1.6) with initial data

$$H\Psi_0 = \sum_{l=0}^{\infty} \sum_{|m|\le l} (H_l \Psi_0^{lm}) Y_{lm} \,.$$

Obviously, this data is also smooth and compactly supported and thus gives rise to the solution

$$\sum_{l=0}^{\infty} \sum_{|m|\le l} \left(e^{-itH_l} H_l \Psi_0^{lm}\right) Y_{lm} = \sum_{l=0}^{\infty} \sum_{|m|\le l} \left(H_l e^{-itH_l} \Psi_0^{lm}\right) Y_{lm} = H\Psi \,,$$

where in the second equation we again used the uniqueness of the decomposition into spherical harmonics. Thus for every $\varepsilon > 0$ there is a L_1 (without restriction $\ge L_0$) such that

$$\|H\Psi^{L_1}(t)\| < \varepsilon \,, \quad \text{for all times } t \,.$$

Proceeding inductively, we find for every number N and for every $\varepsilon > 0$ a number L_N such that

$$\|H^n \Psi^{L_N}(t)\| < \varepsilon \,, \quad \text{for all } t \text{ and } n \le N \,.$$

Let $K \subseteq \mathbb{R} \times S^2$ be an arbitrary compact subset with smooth boundary. Then, due to the definition of the energy, there exists a constant $C_0(K) > 0$ such that for $\Psi^{L_N} = (\psi_1^{L_N}, \psi_2^{L_N})^T$,

$$\|\psi_1^{L_N}\|_{H^1(K)} + \|\psi_2^{L_N}\|_{L^2(K)} \;\le\; C_0(K) \|\Psi^{L_N}\| \,.$$

Applying the same argument to $H\Psi^{L_N} = (\psi_2^{L_N}, A\psi_1^{L_N})^T$, where A is the differential operator given by (1.14), there is a $C_1(K) > 0$ such that

$$\|A\psi_1^{L_N}\|_{L^2(K)} + \|\psi_2^{L_N}\|_{H^1(K)} \;\le\; C_1(K) \|H\Psi^{L_N}\| \,.$$

Since the differential operator A is of the form $A = -\Delta + X$, where X is a first order differential operator, it is in particular a second order elliptic partial differential operator. Thus, for $u \in C^\infty(\mathbb{R} \times S^2)$ and for each $U \subset\subset V \subset\subset \mathbb{R} \times S^2$ ($\subset\subset$ denotes relative compact) there is an estimate (cf. [26, p.379 (11.3)])

$$\|u\|_{H^{k+2}(U)} \le C\|Au\|_{H^k(V)} + C\|u\|_{H^{k+1}(V)} \quad \text{for all } k \ge 0.$$

It follows that there exist new constants $C_0(K), C_1(K)$ such that

$$\|\psi_1^{L_N}\|_{H^2(K)} + \|\psi_2^{L_N}\|_{H^1(K)} \;\leq\; C_0(K)\|\Psi^{L_N}\| + C_1(K)\|H\Psi^{L_N}\| \;.$$

Iterating this inequality, we obtain constants $C_0(K), ..., C_k(K)$ such that

$$\|\psi_1^{L_N}\|_{H^{k+1}(K)} + \|\psi_2^{L_N}\|_{H^k(K)} \;\leq\; \sum_{n=0}^{k} C_n(K)\|H^n\Psi^{L_N}\| \;.$$

In particular, for every $\varepsilon > 0$ there is a number L such that

$$\|\Psi^L(t)\|_{H^2(K)} < \varepsilon \quad \text{for all } t \;.$$

Thus the Sobolev embedding theorem yields (possibly after enlarging L)

$$\|\Psi^L(t)\|_{L^\infty(K)} < \varepsilon \quad \text{for all } t \;.$$

Furthermore, due to the pointwise decay for fixed modes l, m which was shown in Theorem 5.5, we can find for any $\varepsilon > 0$ and $(u, x) \in \mathbb{R} \times S^2$ a time t_0 and a number L such that for the solution $\Psi(t, u, x)$ of the Cauchy problem (1.6),

$$|\Psi(t,u,x)| \leq \sum_{l=0}^{L-1} \sum_{|m|\leq l} |\Psi^{lm}(t,u) Y_{lm}(x)| + |\Psi^L(t,u,x)| < \varepsilon \quad \text{for all } t \geq t_0 \;.$$

Since $\psi = r\phi$, this concludes the proof. ☐

6 Expansion of the Jost solutions $\grave{\phi}_\omega$

Since the ω-dependence of the Jost solutions $\grave{\phi}_\omega$ plays an essential role in the analysis of the integral representation, we show in this section a method to expand these solutions at the critical point $\omega = 0$. We start with an explicit calculation:

Lemma 6.1. *For all $u > 0$, $\omega \in \mathbb{R} \setminus \{0\}$, $\varepsilon > 0$, $q \in \mathbb{N}_0$ and $p \in \mathbb{N}$,*

$$\begin{aligned}\int_u^\infty e^{-2i\omega x - \varepsilon x} \frac{\log^q(x)}{x^p}\, dx &= \sum_{m=0}^{q} \binom{q}{m} \log^{q-m}(u) \Bigg\{ (2i\omega+\varepsilon)^{p-1} \\ \times \Bigg[\frac{(-1)^{p-1}}{(p-1)!} \frac{(-1)^{m+1}}{m+1} \log^{m+1}[(2i\omega+\varepsilon)u] &+ \sum_{k=0}^{m} c_k(m) \log^k [(2i\omega+\varepsilon)u] \Bigg] \\ -u^{-p+1} \sum_{k=0, k\neq p-1}^{\infty} \frac{(-1)^k (-1)^m\, m!}{(k-p+1)^{m+1} k!} &[(2i\omega+\varepsilon)u]^k \Bigg\} \;, \end{aligned} \tag{6.1}$$

where the coefficients c_k involve the coefficients $a_0, ..., a_q$ of the series expansion of the Γ-function at $1-p$.

Proof. In order to prove this, we write the integral as λ-derivatives,

$$\int_u^\infty e^{-2i\omega x-\varepsilon x}\frac{\log^q(x)}{x^p}\,dx = \frac{d^q}{d\lambda^q}F_p(\lambda)\bigg|_{\lambda=0}\;, \tag{6.2}$$

with the generating functional,

$$F_p(\lambda) = \int_u^\infty e^{(-2i\omega-\varepsilon)x}\frac{1}{x^{p-\lambda}}\,dx = u^{-p+\lambda+1}\int_1^\infty e^{(-2i\omega-\varepsilon)uv}\frac{1}{v^{p-\lambda}}\,dv\;,$$

where in the last step we introduced the new integration variable $v=\frac{x}{u}$. In the following we will write $z=(2i\omega+\varepsilon)u$ for reasons of convenience. The integral on the right hand side is also known as the Exponential Integral $E_{p-\lambda}(z)$ with the series expansion

$$E_{p-\lambda}(z) = \Gamma(1-p+\lambda)\,z^{p-\lambda-1} - \sum_{k=0}^{\infty}\frac{(-1)^k}{(k-p+\lambda+1)k!}\,z^k\;,$$

for small $\lambda\neq 0$ [as a reference cf. [34]]. Using the series expansion of the Γ-function at $1-p\in\mathbb{Z}\setminus\mathbb{N}$, where the Γ-function has a pole of first-order, we obtain

$$\begin{aligned} F_p(\lambda) &= u^{-p+\lambda+1}\left[\left(\frac{(-1)^{p-1}}{(p-1)!\,\lambda}+\sum_{n=0}^{\infty}a_n\lambda^n\right)z^{p-\lambda-1}\right.\\ &\qquad\qquad\left.-\sum_{k=0}^{\infty}\frac{(-1)^k}{(k-p+\lambda+1)k!}\,z^k\right]\\ &= u^{-p+\lambda+1}\left[z^{p-1}\left(\frac{(-1)^{p-1}}{(p-1)!}\left(\frac{z^{-\lambda}-1}{\lambda}\right)+z^{-\lambda}\sum_{n=0}^{\infty}a_n\lambda^n\right)\right.\\ &\qquad\qquad\left.-\sum_{k=0,k\neq p-1}^{\infty}\frac{(-1)^k}{(k-p+\lambda+1)k!}\,z^k\right]. \end{aligned} \tag{6.3}$$

Using $z^{-\lambda}=e^{-\lambda\log z}$, we immediately get the formulas

$$\begin{aligned} \frac{d^n}{d\lambda^n}\left(\frac{z^{-\lambda}-1}{\lambda}\right)\bigg|_{\lambda=0} &= \frac{(-1)^{n+1}\log^{n+1}(z)}{n+1}\\ \frac{d^m}{d\lambda^m}\left(u^{\lambda}\right)\bigg|_{\lambda=0} &= \log^m(u)\\ \frac{d^m}{d\lambda^m}\left(z^{-\lambda}\right)\bigg|_{\lambda=0} &= (-1)^m\log^m(z) \end{aligned}$$

one directly verifies the claim setting (6.3) in (6.2). □

Directly in the same way, one proves an analogue lemma for the case $p\in\mathbb{Z}\setminus\mathbb{N}$:

Lemma 6.2. *For all $u > 0$, $\omega \in \mathbb{R} \setminus \{0\}$, $\varepsilon > 0$, $q \in \mathbb{N}_0$ and $p \in \mathbb{Z} \setminus \mathbb{N}$,*

$$\int_u^\infty e^{-2i\omega x - \varepsilon x} \frac{\log^q(x)}{x^p}\, dx =$$

$$\sum_{m=0}^{q} \binom{q}{m} \log^{q-m}(u) \Bigg\{ (2i\omega + \varepsilon)^{p-1} \sum_{k=0}^{m} c_k(m) \log^k \left[(2i\omega + \varepsilon)u\right]$$

$$- u^{-p+1} \sum_{k=0}^{\infty} \frac{(-1)^k (-1)^m\, m!}{(k-p+1)^{m+1} k!} \left[(2i\omega + \varepsilon)u\right]^k \Bigg\} \quad (6.4)$$

where the coefficients c_k involve the coefficients $a_0, ..., a_q$ of the series expansion of the Γ-function at $1 - p$.

Compared to Lemma 6.1, here the logarithmic term is of lower order due to the fact that the Gamma-function has no singularity for positive integers.

In order to apply this lemma to our integral representation, we have to derive an asymptotic expansion for the potential $V_l(u)$ at $+\infty$. Therefore, we have the following

Lemma 6.3. *For the potential $V_l(u) = \left(1 - \frac{2M}{r(u)}\right)\left(\frac{2M}{r(u)^3} + \frac{l(l+1)}{r(u)^2}\right)$ we have the asymptotic expansion*

$$V_l(u) = \sum_{p=2}^{k} \sum_{q=0}^{p-2} c_{pq} \frac{\log^q(u)}{u^p} + c_{k+1,k-1} \frac{\log^{k-1}(u)}{u^{k+1}} + \mathcal{O}\left(\frac{\log^{k-2}(u)}{u^{k+1}}\right), \quad (6.5)$$

as $u \to \infty$, with $k \geq 2$ and real coefficients c_{pq}, where e.g. the first coefficients are given by

$$c_{20} = l(l+1)\,, \quad c_{31} = 4l(l+1)M\,,$$

$$c_{30} = 2M - 2Ml(l+1)(1 + 2\log(2)) - 4Ml(l+1)\log(M)$$

$$c_{42} = 12l(l+1)M^2\,,$$

$$c_{41} = -4M^2(-3 + l(l+1)(5 + 8\log(8)) + 6l(l+1)\log(M))\,, \ \dots\ .$$

Furthermore, in the case $l = 0$ the coefficients $c_{n,n-2}$ vanish.

Proof. First we have to find an expression for r in terms of the Regge-Wheeler coordinate u. Remember that $u = r + 2M\log(\frac{r}{2M} - 1)$, which is equivalent to

$$e^{\frac{u}{2M} - 1} = \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M} - 1}\,.$$

In order to resolve this equation with respect to r, we use the principal branch of the Lambert W function denoted by $W(z)$. This is just the inverse function of $f(x) = xe^x$ on the positive real axis. [As a reference cf. [4].] Hence, we obtain

$$r = 2M + 2M\, W(e^{\frac{u}{2M}-1}) \ . \tag{6.6}$$

Moreover, for W we have the asymptotic expansion

$$W(z) = \log z - \log(\log z) + \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} c_{km}\,(\log(\log z))^{m+1}(\log z)^{-k-m-1} \ , \tag{6.7}$$

as $z \to \infty$. Here, the coefficients c_{km} are given by $c_{km} = \frac{1}{m!}(-1)^k \begin{bmatrix} k+m \\ k+1 \end{bmatrix}$, where $\begin{bmatrix} k+m \\ k+1 \end{bmatrix}$ is a Stirling cycle number. In particular, applying this expansion to (6.6), we get the series representation

$$\begin{aligned} r(u) = 2M + 2M\Big[&\frac{u}{2M} - 1 - \log\left(\frac{u}{2M} - 1\right) \\ &+ \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} c_{km}\left(\log\left(\frac{u}{2M}-1\right)\right)^{m+1}\left(\frac{u}{2M}-1\right)^{-k-m-1}\Big] \ . \end{aligned}$$

This allows us to expand the powers $\frac{1}{r^2}$, $\frac{1}{r^3}$ and $\frac{1}{r^4}$ to any order in $u/2M - 1$ using the method of the geometric series. Together with the expansion

$$\log\left(\frac{u}{2M} - 1\right) = \log\left[\frac{u}{2M}\left(1 - \frac{2M}{u}\right)\right] = \log u - \log(2M) - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{2M}{u}\right)^n ,$$

which holds for $u > 2M$, the result follows. □

These two lemmas let us expand the solution $\grave{\phi}_\omega(u)$ in the following way.

Lemma 6.4. *For $l = 0$, $\omega \in \mathbb{R}\setminus\{0\}$ and fixed $u > 0$, the fundamental solution $\grave{\phi}_\omega(u)$ can be represented as*

$$\grave{\phi}_\omega(u) = e^{-i\omega u} + g_0(\omega, u) + 2i\omega\log(2i\omega)g_1(\omega, u) + 2i\omega g_2(\omega, u) \ , \tag{6.8}$$

where the functions g_0, g_1 and g_2 are $C^1(\mathbb{R})$ with respect to ω.

In order to prove this, we need the following version of Lemma 4.3.

Lemma 6.5. *For all $u \in \mathbb{C}$ and $n \in \mathbb{N}_0$,*

$$\left|\partial_u^n\left(\frac{1}{u}\sin u\right)\right| \le \frac{2^{n+1}}{1+|u|}e^{|\operatorname{Im} u|} \ . \tag{6.9}$$

Moreover, if $\omega \neq 0$ and $v \ge u > 0$,

$$\left|\partial_\omega^n\Big[\frac{1}{\omega}\sin(\omega(u-v))\Big]\right| \le \frac{C(n)\,v^{n+1}}{1+|\omega v|}e^{v|\operatorname{Im}\omega| + u\operatorname{Im}\omega} \ , \tag{6.10}$$

for some constant $C(n)$, which is just depending on n.

Proof. The proof follows essentially the arguments of the proof of Lemma 4.3, where we have shown the case $n = 0$, when one substitutes $v \to -v$, $u \to -u$. Namely, in the case $|u| \geq 1$, Eulers formula for the sin-function immediately yields

$$(1+|u|)\Big|\partial_u\left(\frac{1}{u}\sin u\right)\Big| = (1+|u|)\Big| -\frac{1}{u^2}\sin u + \frac{1}{u}\cos u\Big| \leq 4e^{|\operatorname{Im} u|} \;.$$

And inductively we get

$$(1+|u|)\Big|\partial_u^n\left(\frac{1}{u}\sin u\right)\Big| \leq 2^{n+1}e^{|\operatorname{Im} u|} \;.$$

For $|u| < 1$ we rewrite $(1/u)\sin u$ as an integral, in order to obtain the estimate

$$(1+|u|)\Big|\partial_u^n\left(\frac{1}{u}\sin u\right)\Big| = (1+|u|)\Big|\frac{1}{2}\int_{-1}^{1}(i\tau)^n e^{iu\tau}\,d\tau\Big| \leq 2e^{|\operatorname{Im} u|} \;,$$

which shows the first claim. As a consequence, we get for $\omega \neq 0$ and all $n \in \mathbb{N}$ the estimate

$$\begin{aligned}\Big|\partial_\omega^n\left(\frac{1}{\omega}\sin(\omega u)\right)\Big| &= \Big|u^{n+1}\partial_{\omega u}^n\left(\frac{1}{\omega u}\sin(\omega u)\right)\Big| \\ &\leq \frac{2^{n+1}|u|^{n+1}}{1+|\omega u|}e^{|\operatorname{Im}(\omega u)|} \;. \end{aligned} \tag{6.11}$$

Using the identity

$$\frac{1}{\omega}\sin(\omega(u-v)) = \frac{1}{\omega}\sin(\omega u)e^{-i\omega v} - \frac{1}{\omega}\sin(\omega v)e^{-i\omega u} \;,$$

we get

$$\begin{aligned}\Big|\partial_\omega\left(\frac{1}{\omega}\sin(\omega(u-v))\right)\Big| &\leq \Big|\frac{1}{\omega}\left(\sin(\omega u)(-iv)e^{-i\omega v} - \sin(\omega v)(-iu)e^{-i\omega u}\right)\Big| \\ &\quad + \Big|\partial_\omega\left(\frac{1}{\omega}\sin(\omega u)\right)e^{-i\omega v} - \partial_\omega\left(\frac{1}{\omega}\sin(\omega v)\right)e^{-i\omega u}\Big| \;.\end{aligned}$$

Using the estimates (6.11) and (6.9) for $n = 0$ together with the assumption $v \geq u > 0$, we see as in the proof of Lemma 4.3 that the first term is bounded by

$$\Big|\frac{1}{\omega}\left(\sin(\omega u)(-iv)e^{-i\omega v} - \sin(\omega v)(-iu)e^{-i\omega u}\right)\Big| \leq \frac{4v^2}{1+|\omega v|}e^{v|\operatorname{Im}\omega| + u\operatorname{Im}\omega} \;.$$

For the second term we use (6.11)

$$\Big|\partial_\omega\left(\frac{1}{\omega}\sin(\omega u)\right)e^{-i\omega v} - \partial_\omega\left(\frac{1}{\omega}\sin(\omega v)\right)e^{-i\omega u}\Big|$$

$$\leq \frac{4u^2}{1+|\omega u|} e^{u|\operatorname{Im}\omega| + v \operatorname{Im}\omega} + \frac{4v^2}{1+|\omega v|} e^{v|\operatorname{Im}\omega| + u \operatorname{Im}\omega} ,$$

and obtain due to the assumption $v \geq u > 0$

$$\leq \frac{8v^2}{1+|\omega v|} e^{v|\operatorname{Im}\omega| + u \operatorname{Im}\omega} .$$

Thus, we have shown (6.10) for $n = 1$. We proceed inductively to conclude the proof. □

Note that the estimate (6.10) remains valid in the limit $0 \neq \omega \to 0$ for all n, because

$$\lim_{\omega \to 0} \partial_\omega^n \left(\frac{1}{\omega} \sin(\omega(u-v)) \right) = \begin{cases} (-1)^{n/2} \dfrac{1}{n+1} (u-v)^{n+1} & , \text{ if } n \text{ even,} \\ 0 & , \text{ if } n \text{ odd.} \end{cases}$$

Proof of Lemma 6.4: First, remember that the solution $\grave{\phi}_\omega(u)$ is given by the perturbation series

$$\grave{\phi}_\omega(u) = \sum_{k=0}^{\infty} \phi_\omega^{(k)}(u) ,$$

where the summands follow the iteration scheme

$$\phi_\omega^{(0)}(u) = e^{-i\omega u} , \; \phi_\omega^{(k+1)}(u) = -\int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) V_0(v) \phi_\omega^{(k)}(v) \, dv , \quad (6.12)$$

with potential $V_0(u) = \left(1 - \frac{2M}{r(u)}\right) \frac{2M}{r(u)^3}$. According to Lemma 6.3, this potential can be represented for large u as $V_0(u) = \frac{c_{30}}{u^3} + h(u)$, with $h(u) = \mathcal{O}\left(\frac{\log u}{u^4}\right)$. Next, we split this iteration scheme up. To this end, we define

$$\tilde{\phi}_\omega^{(1)}(u) := -\int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) h(v) e^{-i\omega v} \, dv , \quad (6.13)$$

and analogously,

$$\hat{\phi}_\omega^{(1)}(u) := -\int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) \frac{c_{30}}{v^3} e^{-i\omega v} \, dv . \quad (6.14)$$

Thus, obviously $\phi_\omega^{(1)}(u) = \hat{\phi}_\omega^{(1)}(u) + \tilde{\phi}_\omega^{(1)}(u)$. Now we iterate these two functions

$$\tilde{\phi}_\omega^{(k+1)}(u) := -\int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) V_0(v) \tilde{\phi}_\omega^{(k)}(v) \, dv , \; k \geq 1 ,$$

analogously for $\hat{\phi}_\omega^{(k+1)}(u)$. Hence, we have the formal decomposition

$$\grave{\phi}_\omega(u) = e^{-i\omega u} + \sum_{k=1}^{\infty} \hat{\phi}_\omega^{(k)}(u) + \sum_{k=1}^{\infty} \tilde{\phi}_\omega^{(k)}(u) \ . \tag{6.15}$$

Both series are well-defined. In order to show this, we use the bound

$$\left| \frac{1}{\omega} \sin(\omega(u-v)) \right| \ \leq \ \frac{4|v|}{1+|\omega v|} \ , \tag{6.16}$$

from Lemma 6.5 for real ω [Note that this estimate is also valid for the case $v \geq u > 0$]. Hence, we get inductively the estimates (cf. Section 4)

$$\begin{aligned} \left|\hat{\phi}_\omega^{(k+1)}(u)\right| &\leq \hat{R}_\omega(u) \frac{P_\omega(u)^k}{k!} \ , \\ \left|\tilde{\phi}_\omega^{(k+1)}(u)\right| &\leq \tilde{R}_\omega(u) \frac{P_\omega(u)^k}{k!} \ , \end{aligned}$$

for all $k \geq 0$, where the functions $\hat{R}, \tilde{R}$ and P are given by

$$\begin{aligned} \hat{R}_\omega(u) &:= \int_u^\infty \frac{4v}{1+|\omega|v} \left| \frac{c_{30}}{v^3} \right| dv \ , \\ \tilde{R}_\omega(u) &:= \int_u^\infty \frac{4v}{1+|\omega|v} |h(v)| \, dv \ , \\ P_\omega(u) &:= \int_u^\infty \frac{4v}{1+|\omega|v} |V_0(v)| \, dv \ . \end{aligned}$$

Thus, the series $\sum \hat{\phi}_\omega^{(k)}(u)$ as well as $\sum \tilde{\phi}_\omega^{(k)}(u)$ converge locally uniformly with respect to u and ω. In the next step we show that, for fixed $u > 0$, $\sum \tilde{\phi}_\omega^{(k)}(u)$ is $C^1(\mathbb{R})$ with respect to ω. To this end, it suffices to prove that each summand $\tilde{\phi}_\omega^{(k)}$, $k \geq 1$, is C^1 and that the series $\sum \partial_\omega \tilde{\phi}_\omega^{(k)}$ converges locally uniformly in ω. Due to the estimates (6.10),(6.16), we have the inequality

$$\begin{aligned} &\left| \partial_\omega \left[\frac{1}{\omega} \sin(\omega(u-v)) h(v) e^{-i\omega v} \right] \right| \leq \\ &\qquad \leq \left| \frac{12\, v^2}{1+|\omega|v} h(v) \right| + \left| \frac{4v^2}{1+|\omega|v} h(v) \right| = \frac{16 v^2}{1+|\omega|v} |h(v)| \ . \end{aligned} \tag{6.17}$$

Hence, the second term is an integrable bound, uniformly in ω, for the first derivative of the integrand. It follows that $\tilde{\phi}_\omega^{(1)}(u)$ is C^1 with respect to ω, bounded by

$$\left| \partial_\omega \tilde{\phi}_\omega^{(1)}(u) \right| \leq \int_u^\infty \frac{16 v^2}{1+|\omega|v} |h(v)| \, dv =: \tilde{R}_\omega^{(1)}(u) \ .$$

Together with the estimate

$$\tilde{R}_\omega(u) \leq \frac{1}{4u}\int_u^\infty \frac{16v^2}{1+|\omega|v}|h(v)|\,dv \leq \frac{1}{u}\tilde{R}^{(1)}_\omega(u)\;,$$

one shows inductively that $\tilde{\phi}^{(k+1)}_\omega(u)$ is C^1 with respect to ω, bounded by

$$\left|\partial_\omega \tilde{\phi}^{(k+1)}_\omega(u)\right| \leq \tilde{R}^{(1)}_\omega(u)\frac{(4P_\omega(u))^k}{k!}\;.$$

This yields that the sum $\sum \partial_\omega \tilde{\phi}^{(k)}_\omega$ converges locally uniformly in ω. Hence, the sum $\sum \tilde{\phi}^{(k)}_\omega(u)$ is $C^1(\mathbb{R})$ with respect to ω. According to the decomposition (6.15), it remains to analyze the $\omega-$dependence of $\sum \hat{\phi}^{(k)}_\omega(u)$. To this end, we compute the first summand:

$$\begin{aligned}
\hat{\phi}^{(1)}_\omega(u) &= \frac{1}{2i\omega}\int_u^\infty \left(e^{-i\omega(u-v)} - e^{i\omega(u-v)}\right) e^{-i\omega v}\frac{c_{30}}{v^3}\,dv \\
&= \frac{1}{2i\omega}e^{-i\omega u}\int_u^\infty \frac{c_{30}}{v^3}\,dv - \frac{1}{2i\omega}e^{i\omega u}\int_u^\infty \frac{c_{30}}{v^3}e^{-2i\omega v}\,dv\;.
\end{aligned}$$

Integrating the second term by parts, we obtain

$$\begin{aligned}
&= \frac{1}{2i\omega}\left(e^{-i\omega u}\frac{c_{30}}{2u^2} - e^{i\omega u}\frac{c_{30}}{2u^2}e^{-2i\omega u} + e^{i\omega u}\int_u^\infty \frac{c_{30}}{-2v^2}(-2i\omega)e^{-2i\omega v}\,dv\right) \\
&= e^{i\omega u}\int_u^\infty \frac{c_{30}}{2v^2}\,e^{-2i\omega v}\,dv\;.
\end{aligned}$$

The series expansion of Lemma 6.1 in the limit $\varepsilon \to 0$ yields

$$\begin{aligned}
\hat{\phi}^{(1)}_\omega(u) = \frac{c_{30}}{2}e^{i\omega u}\Big\{ & 2i\omega\big(\log(2i\omega u) + c_0\big) \\
& -u^{-1}\sum_{k=0,k\neq 1}^\infty \frac{(-1)^k}{(k-1)k!}\,(2i\omega u)^k\Big\}\;. \qquad (6.18)
\end{aligned}$$

Intuitively, the only term which is not C^1 is the term involving $2i\omega\log(2i\omega u)$. More precisely, defining

$$\hat{\psi}^{(1)}_\omega(u) := \hat{\phi}^{(1)}_\omega(u) - c_{30}e^{i\omega u}i\omega\log(2i\omega u)\;, \qquad (6.19)$$

and iterating this by

$$\hat{\psi}^{(k+1)}_\omega(u) := -\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))V_0(v)\hat{\psi}^{(k)}_\omega(v)\,dv\;,\; k\geq 1,$$

we show next that the sum $\sum \hat{\psi}^{(k)}_\omega(u)$ is C^1 with respect to ω. By definition this holds for the initial function $\hat{\psi}^{(1)}_\omega(u)$. In order to prove this for the sum, we apply the same method as above. To this end, we need good estimates for the initial functions $\hat{\psi}^{(1)}_\omega(u)$ and $\partial_\omega \hat{\psi}^{(1)}_\omega(u)$. Estimating the integral representation of $\hat{\phi}^{(1)}_\omega(u)$, we obtain for arbitrary $u > 0$ and $\omega \in \mathbb{R}$,

$$\left| \hat{\psi}^{(1)}_\omega(u) + c_{30} e^{i\omega u} i\omega \log(2i\omega u) \right| = \left| \hat{\phi}^{(1)}_\omega(u) \right| \leq \int_u^\infty \left| \frac{c_{30}}{2v^2} \right| dv = \frac{c_{30}}{2u} \; .$$

On the other hand, looking at the series in (6.18), we obtain for all $u \leq \frac{1}{|\omega|}$ the estimate

$$\left| \hat{\psi}^{(1)}_\omega(u) \right| = \left| \frac{c_{30}}{2} e^{i\omega u} \left\{ 2i\omega c_0 - \frac{1}{u} \sum_{k=0, k\neq 1}^{\infty} \frac{(-1)^k}{(k-1)k!} (2i\omega u)^k \right\} \right| \leq \frac{\tilde{c}}{u} \; , \quad (6.20)$$

with a suitable constant $\tilde{c}$. Thus, we get for all $u > 0$ and $\omega \in \mathbb{R}$ the estimate

$$\left| \hat{\psi}^{(1)}_\omega(u) \right| \leq \frac{c}{u} + c|\omega| |\log(2i\omega u)| \, 1_{[\frac{1}{|\omega|}, \infty)}(u) \; , \quad (6.21)$$

where c is chosen suitably and $1_.(.)$ denotes the characteristic function. In order to estimate the derivative $\partial_\omega \hat{\psi}^{(1)}_\omega(u)$, we use in the domain $u \geq \frac{1}{|\omega|}$, $|\omega| \neq 0$, the following bound for $\partial_\omega \hat{\phi}^{(1)}_\omega(u)$ [see also (6.17)],

$$\begin{aligned} \left| \partial_\omega \left(\hat{\phi}^{(1)}_\omega(u) \right) \right| &\leq \int_u^\infty \frac{16 v^2}{1 + |\omega| v} \left| \frac{c_{30}}{v^3} \right| dv \\ &\leq \frac{16}{|\omega|} \int_u^\infty \frac{|c_{30}|}{v^2} \, dv \leq \frac{16 c_{30}}{|\omega| u} \leq 16 \, c_{30} \; . \end{aligned}$$

Together with the analogon to estimate (6.20) in the region $u \leq \frac{1}{|\omega|}$, we obtain the bound

$$\left| \partial_\omega \hat{\psi}^{(1)}_\omega(u) \right| \leq \tilde{c} + \tilde{c}(1 + u|\omega|) |\log(2i\omega u)| \, 1_{[\frac{1}{|\omega|}, \infty)}(u) \; , \quad (6.22)$$

where $u > 0, \omega \in \mathbb{R}$ and $\tilde{c}$ is an appropriate constant. For reasons of simplicity, we choose $c = \tilde{c}$ such that both inequalities (6.21),(6.22) hold. Using these inequalities, we show by induction, in the same way as above, that $\hat{\psi}^{(k)}_\omega(u)$ is C^1 with respect to ω and obeys the estimates

$$\begin{aligned} \left| \hat{\psi}^{(k)}_\omega(u) \right| &\leq \frac{c}{u} \frac{P_\omega(u)^{k-1}}{(k-1)!} + \frac{c}{u} r(|\omega|) \frac{P_\omega(u)^{k-2}}{(k-2)!} \; , & (6.23) \\ \left| \partial_\omega \left(\hat{\psi}^{(k)}_\omega(u) \right) \right| &\leq c \frac{\left(4P_\omega(u) \right)^{k-1}}{(k-1)!} + 5c \, r(|\omega|) \frac{\left(4P_\omega(u) \right)^{k-2}}{(k-2)!} \; , & (6.24) \end{aligned}$$

for all $k \geq 2, u > 0$ and $\omega \in \mathbb{R}$, where r is given by

$$r(|\omega|) := \int_{\frac{1}{|\omega|}}^\infty \frac{4|\omega| v^2}{1 + |\omega| v} |V_0(v)| |\log(2i\omega v)| \, dv \; .$$

Due to (6.23),(6.24), the sums $\sum \hat{\psi}_\omega^{(k)}(u)$ and $\sum \partial_\omega \hat{\psi}_\omega^{(k)}(u)$ converge locally uniformly in ω. Hence, we conclude that $\sum \hat{\psi}_\omega^{(k)}(u)$ is well defined and continuously differentiable with respect to ω.

Thus, it remains to look at the term we get by the iteration of

$$\vartheta_\omega^{(1)}(u) := c_{30} e^{i\omega u} i\omega \log(2i\omega u) = ic_{30}\, \omega \log(2i\omega) e^{i\omega u} + ic_{30}\, \omega \log(u) e^{i\omega u} .$$

To this end, we split up the iteration, exactly as we did for the iteration of $\phi_\omega^{(k)}(u)$, i.e. we define

$$\begin{aligned} \tilde{\vartheta}_\omega^{(2)}(u) &:= -\int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) h(v) \vartheta_\omega^{(1)}(v)\, dv \;, \\ \hat{\vartheta}_\omega^{(2)}(u) &:= -\int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) \frac{c_{30}}{v^3} \vartheta_\omega^{(1)}(v)\, dv \;, \end{aligned}$$

and iterate these functions,

$$\tilde{\vartheta}_\omega^{(k+1)}(u) := -\int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) V_0(v) \tilde{\vartheta}_\omega^{(k)}(v)\, dv \;,\; k \geq 2 \;,$$

analogously for $\hat{\vartheta}_\omega^{(k+1)}(u)$. Next, in exactly the same way as for $\tilde{\phi}^{(k)}$, one sees that

$$\sum_{k=2}^\infty \tilde{\vartheta}_\omega^{(k)}(u) = 2i\omega \log(2i\omega)\, f_1(\omega, u) + 2i\omega f_2(\omega, u) \;,$$

where $f_1(., u)$ and $f_2(., u)$ are C^1 with respect to ω. Finally, by an exact calculation

$$\begin{aligned} \hat{\vartheta}_\omega^{(2)}(u) = ic_{30}^2 \omega \log(2i\omega) e^{-i\omega u} \int_u^\infty e^{2i\omega v} \frac{1}{2v^2}\, dv \\ + ic_{30}^2 \omega e^{-i\omega u} \int_u^\infty e^{2i\omega v} \left(\frac{1}{4v^2} + \frac{\log v}{2v^2} \right) dv \;, \end{aligned}$$

together with the series expansion of Lemma 6.1 in the limit $\varepsilon \to 0$ we obtain

$$\begin{aligned} \hat{\vartheta}_\omega^{(2)}(u) &= \frac{1}{4} ic_{30}^2\, \omega (1 + 2\log(2i\omega)) e^{-i\omega u} \\ &\times \left[(-2i\omega) \big(\log(-2i\omega u) + c_0 \big) - u^{-1} \sum_{k=0, k\neq 1}^\infty \frac{(-1)^k}{(k-1)k!} (-2i\omega u)^k \right] \\ &+ \frac{1}{2} ic_{30}^2\, \omega e^{-i\omega u} \left[\sum_{m=0}^1 \binom{1}{m} \log^{1-m}(u) \Big\{ (-2i\omega) \times \right. \\ &\left(\frac{(-1)^{m+2}}{m+1} \log^{m+1}(-2i\omega u) + \sum_{k=0}^m c_k \log^k(-2i\omega u) \right) \end{aligned}$$

$$-u^{-1}\sum_{k=0,k\neq p-1}^{\infty}\frac{(-1)^k(-1)^m\, m!}{(k-1)^{m+1}k!}(-2i\omega u)^k\Big\}\Big]\ .$$

Proceeding in the same way as for $\sum\hat{\psi}_\omega^{(k)}(u)$ [i.e. we omit the $\log\omega$-terms in the square brackets, and iterate these functions], we again get terms of the form

$$2i\omega\log(2i\omega)\ f_3(\omega,u)+2i\omega f_4(\omega,u)$$

with continuously differentiable functions $f_3(.,u), f_4(.,u)$. So after simplifications there remain terms of the form

$$(2i\omega)^2\log^s(2i\omega)\log^r(-2i\omega)\log^t(u)e^{-i\omega u}\ .$$

These are obviously C^1 with respect to ω and so is their iteration, due to the fact that the additional ω-order yields directly integrable bounds for all ω. This completes the proof. □

Next, we try to apply the idea of this proof to the case $l\geq 1$. Therefore, we make some remarks about the fundamental solutions $\omega^l\grave{\phi}_{\omega l}(u)$ (see also Section 4). The fundamental solutions were constructed as the series

$$\omega^l\grave{\phi}_\omega(u)=\sum_{m=0}^{\infty}\phi_\omega^{(m)}(u)\ , \tag{6.25}$$

where the $\phi^{(m)}$ are given by the iteration scheme

$$\phi_\omega^{(m+1)}(u)=-\int_u^\infty S_\omega(u,v)W_l(v)\phi_\omega^{(m)}(v)\,dv\ , \tag{6.26}$$

with potential, cf. also Lemma 6.3,

$$\begin{aligned} W_l(u)=V_l(u)-\frac{l(l+1)}{u^2} &= c_{31}\frac{\log u}{u^3}+\frac{c_{30}}{u^3}+h(u)\ , \\ \text{where}\ \ h(u) &= \mathcal{O}\left(\frac{\log^2 u}{u^4}\right) \end{aligned} \tag{6.27}$$

for large u, and Green's function

$$S_\omega(u,v)=\frac{(-1)^{l+1}}{\omega}\big(h_1(l,\omega v)h_2(l,\omega u)-h_1(l,\omega u)h_2(l,\omega v)\big)\ , \tag{6.28}$$

where

$$h_1(l,\omega u)=\sqrt{\frac{\pi\omega u}{2}}J_{l+1/2}(\omega u)\ ,\quad h_2(l,\omega u)=\sqrt{\frac{\pi\omega u}{2}}J_{-l-1/2}(\omega u)\ , \tag{6.29}$$

and J_ν denotes the Bessel function of the first kind. As initial function $\phi_\omega^{(0)}(u)$ we have chosen

$$\phi_\omega^{(0)}(u)=\omega^l e^{-i(l+1)\frac{\pi}{2}}\sqrt{\frac{\pi\omega u}{2}}H_{l+1/2}^{(2)}(\omega u)\ ,$$

where $H_\nu^{(2)}$ denotes the second Hankel function. Since l is an integer, these functions are directly connected to the spherical Bessel functions and simplify significantly. Namely, h_1, h_2 have the following representations [cf. [1, Chapter 10]]

$$h_1(l,\omega u) = P(l+\frac{1}{2},\omega u)\sin(\omega u-\frac{1}{2}l\pi)+Q(l+\frac{1}{2},\omega u)\cos(\omega u-\frac{1}{2}l\pi) \quad (6.30)$$

$$h_2(l,\omega u) = P(l+\frac{1}{2},\omega u)\cos(\omega u+\frac{1}{2}l\pi)-Q(l+\frac{1}{2},\omega u)\sin(\omega u+\frac{1}{2}l\pi) \quad (6.31)$$

where P, Q are finite polynomials given by

$$P(l+\frac{1}{2},\omega u) = \sum_{k=0}^{[\frac{1}{2}l]}(-1)^k\frac{(l+\frac{1}{2},2k)}{(2\omega u)^{2k}} ,$$

$$Q(l+\frac{1}{2},\omega u) = \sum_{k=0}^{[\frac{1}{2}(l-1)]}(-1)^k\frac{(l+\frac{1}{2},2k+1)}{(2\omega u)^{2k+1}} ,$$

with

$$(l+\frac{1}{2},k) = \frac{(l+k)!}{k!\,\Gamma(l-k+1)} .$$

And the initial function can be expressed by

$$\phi_\omega^{(0)}(u) = \omega^l e^{-i\omega u}\sum_{k=0}^{l}\frac{(l+\frac{1}{2},k)}{(2i\omega u)^k} . \quad (6.32)$$

Due to the recurrence formulas for the derivatives of the Bessel functions, we have the identities

$$\partial_\omega h_1(l,\omega u) = uh_1(l-1,\omega u)-\frac{l}{\omega}h_1(l,\omega u) \quad ,$$

$$\partial_\omega h_2(l,\omega u) = -uh_2(l-1,\omega u)-\frac{l}{\omega}h_2(l,\omega u) .$$

As a consequence,

$$\begin{aligned}\partial_\omega S_\omega(u,v) = & -\frac{2l+1}{\omega}S_\omega \\ & + v\frac{(-1)^{l+1}}{\omega}(h_1(l-1,\omega v)h_2(l,\omega u)+h_1(l,\omega u)h_2(l-1,\omega v)) \\ & + u\frac{(-1)^l}{\omega}(h_1(l,\omega v)h_2(l-1,\omega u)+h_1(l-1,\omega u)h_2(l,\omega v)) .\end{aligned}$$

This allows us to derive the necessary estimates for the Green's function $S_\omega(u,v)$. In the proof of Lemma 4.6 we have already seen (exploiting the asymptotics)

$$|S_\omega(u,v)| \le C_1\left(\frac{u}{1+|\omega|u}\right)^{-l}\left(\frac{v}{1+|\omega|v}\right)^{l+1}e^{v|\operatorname{Im}\omega|+u\operatorname{Im}\omega} ,$$

for $v \geq u > 0$ and an appropriate constant C_1. In order to derive an estimate for $\partial_\omega S_\omega$ and small $|\omega|$, we make use of

$$\begin{aligned} h_1(l, \omega u) &\sim k_1(\omega u)^{l+1} + k_2(\omega u)^{l+3} \\ h_2(l, \omega u) &\sim k_3(\omega u)^{-l} + k_4(\omega u)^{-l+2} , \quad \text{if } |\omega| u \ll 1 , \end{aligned}$$

and certain constants $k_1, ..., k_4$ [refer to the series expansion of the Bessel functions [1, 9.1.10]] to obtain (note that $v \geq u > 0$),

$$|\partial_\omega S_\omega(u,v)| \leq C_2 \left(\frac{u}{1+|\omega|u} \right)^{-l} \left(\frac{v}{1+|\omega|v} \right)^{l+2} , \quad \text{if } |\omega| v \ll 1 .$$

For large arguments $|\omega|u \gg 1$ we use (6.30),(6.31) and get by a straightforward calculation

$$\partial_\omega S_\omega(u,v) \sim \frac{-2l}{\omega^2} \sin(\omega(u-v)) + \partial_\omega \left[\frac{1}{\omega} \sin(\omega(u-v)) \right] , \quad \text{if } |\omega| u \gg 1.$$

Together with (6.10), we obtain

$$|S_\omega(u,v)| \leq C_3 \frac{v^2}{1+|\omega|v} e^{v|\operatorname{Im}\omega| + u \operatorname{Im}\omega} , \quad \text{if } |\omega| u \gg 1.$$

Combining these estimates, we find a constant C such that

$$|\partial_\omega S_\omega(u,v)| \leq C \left(\frac{u}{1+|\omega|u} \right)^{-l} \left(\frac{v}{1+|\omega|v} \right)^{l+1} v \, e^{v|\operatorname{Im}\omega| + u \operatorname{Im}\omega} , \tag{6.33}$$

for $v \geq u > 0$. Moreover, looking at (6.32) we get the following bounds for the initial function,

$$|\phi_\omega^{(0)}(u)| \leq C_4 \left(\frac{u}{1+|\omega|u} \right)^{-l} e^{u \operatorname{Im}\omega} , \tag{6.34}$$

$$|\partial_\omega \phi_\omega^{(0)}(u)| \leq C_5 \left(\frac{u}{1+|\omega|u} \right)^{-l} u \, e^{u \operatorname{Im}\omega} . \tag{6.35}$$

These estimates allow us to proceed in exactly the same way as in the proof of Lemma 6.4. As analogon to $\hat{\phi}_\omega^{(1)}(u)$ we obtain the term

$$-\int_u^\infty S_\omega(u,v) \left(c_{31} \frac{\log v}{v^3} + \frac{c_{30}}{v^3} \right) \phi_\omega^{(0)}(v) \, dv,$$

which we calculate using (6.30),(6.31) and (6.32). Essentially, we get integrals of the shape

$$\frac{\omega^l}{(\omega u)^n \omega^{m+k+1}} \left(C_6 e^{i\omega u} \int_u^\infty e^{-2i\omega v} \frac{\log^q v}{v^{3+k+m}} \, dv + C_7 e^{-i\omega u} \int_u^\infty \frac{\log^q v}{v^{3+k+m}} \, dv \right) ,$$

where $q \in \{0,1\}, 0 \leq n, m, k \leq l$. Note that the terms involving ω singularities resolve, due to the fact that $\omega^l \grave{\phi}_\omega$ is continuous with respect to ω. Computing these integrals via Lemma 6.1 (in the limit $\varepsilon \to 0$), we see (as before) that the only terms not being C^1 with respect to ω are of the form

$$e^{i\omega u} \frac{1}{\omega^{m+k+1}} (2i\omega)^{k+m+2} \left(\log^2(2i\omega u) + \log u \log(2i\omega u) + \log(2i\omega u)\right) , \tag{6.36}$$

modulo coefficients. Now, we apply the same iteration with analog estimates and all in all we have shown:

Lemma 6.6. *For $l \geq 1$, $\omega \in \mathbb{R} \setminus \{0\}$ and fixed $u > 0$ the fundamental solutions $\omega^l \grave{\phi}_\omega(u)$ have the representation*

$$\begin{aligned} \omega^l \grave{\phi}_\omega(u) = \phi_\omega^{(0)}(u) + g_3(\omega, u) + 2i\omega \log^2(2i\omega) g_4(\omega, u) \\ +2i\omega \log(2i\omega) g_5(\omega, u) + 2i\omega g_6(\omega, u) , \end{aligned} \tag{6.37}$$

where the functions g_3, g_4, g_5 and g_6 are $C^1(\mathbb{R})$ with respect to ω.

7 The decay rate for spherical symmetric initial data

In this section we consider the case $l = 0$. According to Theorem 5.5, the solution of the Cauchy problem for compactly supported smooth initial data $\Psi_0 \in C_0^\infty(\mathbb{R})^2$ has the pointwise representation

$$\Psi(t, u) = -\frac{1}{\pi} \int_{\mathbb{R}} e^{-i\omega t} \left(\int_{\text{supp}\, \Psi_0} \text{Im}(s_\omega(u, v)) \begin{pmatrix} \omega & 1 \\ \omega^2 & \omega \end{pmatrix} \Psi_0(v) dv \right) d\omega , \tag{7.1}$$

where $s_\omega(u, v)$ is piecewise defined by (3.9) and represents the Green's function for the Schrödinger equation (3.2). According to the equations (5.2),(5.3), the imaginary part of $s_\omega(u, v)$ is symmetric with respect to the arguments u, v. Thus, for all $u, v \in \mathbb{R}$

$$\text{Im}(s_\omega(u, v)) = \text{Im}\left(\frac{\acute{\phi}_\omega(u) \grave{\phi}_\omega(v)}{w(\acute{\phi}_\omega, \grave{\phi}_\omega)} \right) , \tag{7.2}$$

where $\acute{\phi}_\omega, \grave{\phi}_\omega$ are the Jost solutions in the case $l = 0$. Our goal is now to use the Fourier transform (7.1), in order to get detailed decay rates. To this end, we have to analyze the integral kernel, hence essentially (7.2).

Since we already know that $\acute{\phi}_\omega$ is analytic on a neighborhood of the real line, it remains to understand $\grave{\phi}_\omega$ at the point $\omega = 0$. To this end, we want to use an expansion as in Lemma 6.4. The problem is that this expansion is not sufficient for this purpose. Thus, we apply a similar method in order to gain

Lemma 7.1. *For $l = 0$, $\omega \in \mathbb{R} \setminus \{0\}$, $n \geq 3$ and fixed $u > 0$, we get for the fundamental solution $\grave{\phi}_\omega(u)$ the representation*

$$\grave{\phi}_\omega(u) = e^{-i\omega u} + g_0(\omega, u) + \sum_{i \geq j+k=1}^{n} (2i\omega)^i \log^j(2i\omega) \log^k(-2i\omega) g_{ijk}(\omega, u) \ , \tag{7.3}$$

where the functions $g_0, g_{ijk} \in C^n(\mathbb{R})$ with respect to ω.

In order to prove this, we need the following lemma.

Lemma 7.2. *Let $u > 0$, $n \in \mathbb{N}$ and $h \in C^\infty(\mathbb{R}_+)$ be a smooth function satisfying $\int_u^\infty v^{n+1}|h(v)|\, dv < \infty$.*
Then:

(i)

$$f_\omega^{(1)}(u) := -\int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) h(v) e^{-i\omega v}\, dv$$

is $C^n(\mathbb{R})$ with respect to ω.

(ii) For all $k \geq 1$

$$f_\omega^{(k+1)}(u) := -\int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) V_0(v) f_\omega^{(k)}(v)\, dv \ ,$$

are $C^n(\mathbb{R})$ with respect to ω and the series $\sum_{k \geq 1} \partial_\omega^m f_\omega^{(k)}(u)$, $m \leq n$, converge locally uniformly.

In particular, $\sum f_\omega^{(k)}(u)$ is $C^n(\mathbb{R})$ with respect to ω.

Proof. This is shown in exactly the same way as the statement that the functions $\tilde{\phi}_\omega^{(k)}$ in the proof of Lemma 6.4 as well as the series are C^1 with respect to ω. In order to show the differentiability up to the n-th order, we use the estimates of Lemma 6.5. □

Proof of Lemma 7.1. Because of complex calculations we show this at first in the case $n = 3$. To this end, we split up the iteration scheme (6.12) of the fundamental solutions in the following way. According to Lemma 6.3, we can write the potential V_0 as

$$V_0(v) = \sum_{p=3}^{5} \sum_{q=0}^{p-3} c_{pq} \frac{\log^q(v)}{v^p} + r_6(v) \ ,$$

where r_6 is a smooth function for $v \geq u$ behaving asymptotically at infinity as $\mathcal{O}\left(\frac{\log^3(v)}{v^6}\right)$. Thus, defining

$$\tilde{\phi}_\omega^{(1)}(u) := -\int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) r_6(v) e^{-i\omega v}\, dv$$

and for all $k \geq 1$

$$\tilde{\phi}_\omega^{(k+1)}(u) := -\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))V_0(v)\tilde{\phi}_\omega^{(k)}(v)\,dv\ ,$$

Lemma 7.2 yields that $\sum \tilde{\phi}_\omega^{(k)} \in C^3(\mathbb{R})$ with respect to ω and is a contribution to $g_0(\omega,u)$ in the statement of the lemma. Thus, we have to compute the remaining term

$$\hat{\phi}_\omega^{(1)}(u) := -\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))\sum_{p=3}^{5}\sum_{q=0}^{p-3} c_{pq}\frac{\log^q(v)}{v^p}e^{-i\omega v}\,dv.$$

We do this essentially in the same way as we computed the terms $\hat{\phi}^{(1)}, \hat{\vartheta}^{(2)}$ in the proof of Lemma 6.4. We split up the $\sin(\omega(u-v))$ with Euler's formula and integrate by parts and obtain

$$= -e^{i\omega u}\int_u^\infty \left(\frac{c_{30}}{-2v^2} + \sum_{p=3}^{4}\sum_{q=0}^{p-2}\tilde{c}_{pq}\frac{\log^q v}{v^p}\right)e^{-2i\omega v}\,dv\ , \tag{7.4}$$

where the coefficients $\tilde{c}_{pq}$ depend on the integral functions of the terms $\log^r v/v^s$. Now, we apply Lemma 6.1 in the limit $\varepsilon \searrow 0$ and get

$$\begin{aligned} = \quad & e^{i\omega u}\frac{c_{30}}{2}\left\{2i\omega\log(2i\omega u) - \frac{1}{u}\sum_{k=0}^{\infty} d_k(2i\omega u)^k\right\} \\ + \quad & e^{i\omega u}\frac{c_{41}}{3}\left\{(2i\omega)^2\left(\frac{1}{4}\log^2(2i\omega u) + \log(2i\omega u)(c - \frac{1}{2}\log u)\right)\right. \\ & \left. \qquad + (c + c\log u)\frac{1}{u^2}\sum_{k=0}^{\infty} d_k(2i\omega u)^k\right\} \\ + \quad & e^{i\omega u}\left\{(2i\omega)^3\sum_{s+t=1}^{3} c\log^s(2i\omega u)\log^t u\right. \\ & \left. \qquad + \sum_{m=0}^{2} c\log^m u\frac{1}{u^3}\sum_{k=0}^{\infty} d_k(2i\omega u)^k\right\} \end{aligned} \tag{7.5}$$

with appropriate constants c and d_k, which are of the form

$$d_k = \frac{(-1)^k(-1)^m\, m!}{(k-p+1)^{m+1}k!}\ . \qquad \text{[cf. Lemma 6.1]}$$

Since the series-terms are obviously $C^3(\mathbb{R})$ with respect to ω, this expression of $\hat{\phi}_\omega^{(1)}(u)$ fits into the desired expansion (7.3). In the next step we have to iterate (7.5). To this end, we treat each term in the curly brackets separately. We show this exemplarily for the first term which we denote by

$$\begin{aligned} \alpha^{(1)}(u) &:= e^{i\omega u}\frac{c_{30}}{2}\left\{2i\omega\log(2i\omega u) - \frac{1}{u}\sum_{k=0}^{\infty} d_k(2i\omega u)^k\right\} \\ &= e^{i\omega u}\int_u^{\infty}\frac{c_{30}}{2v^2}e^{-2i\omega v}\,dv\ . \end{aligned} \tag{7.6}$$

In order to derive sufficient bounds for all $u > 0$, we use different methods for the regions $|\omega|u \geq 1$ and $|\omega|u < 1$. First, let u be such that $|\omega|u \geq 1$, and by integrating by parts we get:

$$\begin{aligned} \alpha^{(1)}(u) &= e^{i\omega u}\int_u^{\infty}\frac{c_{30}}{2v^2}\frac{1}{(-2i\omega)^3}\partial_v^3 e^{-2i\omega v}\,dv \\ &= \frac{c_{30}}{4i\omega}e^{-i\omega u}\frac{1}{u^2} - \frac{c_{30}}{(2i\omega)^2}e^{-i\omega u}\frac{1}{u^3} + \frac{3c_{30}}{(2i\omega)^3}e^{-i\omega u}\frac{1}{u^4} \\ &\quad -e^{i\omega u}\int_u^{\infty}\frac{12}{v^5}\frac{1}{(2i\omega)^3}e^{-2i\omega v}\,dv\ . \end{aligned} \tag{7.7}$$

Using this expression and elementary integral estimates, we get for all $u > 0$ satisfying $|\omega|u \geq 1$ the bounds

$$\begin{aligned} \left|\alpha^{(1)}(u)\right| &\leq c\frac{1}{|\omega|u^2}\ , \\ \left|\partial_\omega\alpha^{(1)}(u)\right| &\leq c\frac{1}{|\omega|u}\ , \\ \left|\partial_\omega^2\alpha^{(1)}(u)\right| &\leq c\frac{1}{|\omega|}\ , \\ \left|\partial_\omega^3\alpha^{(1)}(u)\right| &\leq c\frac{u}{|\omega|}\ , \end{aligned} \tag{7.8}$$

with suitable constants c. Moreover, comparing the infinite sum of (7.6) with the exponential function, one directly sees that it is C^3 with respect to ω. It satisfies for all $u > 0$ with $|\omega|u < 1$ the bounds

$$\begin{aligned} \left|\frac{1}{u}\sum_{k=0}^{\infty} d_k(2i\omega u)^k\right| &\leq \frac{c}{u} \\ \left|\partial_\omega\left(\frac{1}{u}\sum_{k=0}^{\infty} d_k(2i\omega u)^k\right)\right| &\leq c \\ \left|\partial_\omega^2\left(\frac{1}{u}\sum_{k=0}^{\infty} d_k(2i\omega u)^k\right)\right| &\leq cu \\ \left|\partial_\omega^3\left(\frac{1}{u}\sum_{k=0}^{\infty} d_k(2i\omega u)^k\right)\right| &\leq cu^2\ . \end{aligned} \tag{7.9}$$

Using (7.8),(7.9), we verify that iterating the sum first with r_6 followed by the full iteration with the potential V_0, we obtain a C^3-function. For the remaining term $c_{30}/2e^{i\omega u}2i\omega\log(2i\omega u)$ we use the identity $\log(2i\omega u) = \log(2i\omega)+\log u$ together with Lemma 7.2 to show that the first iteration with r_6 followed by the full iteration with the potential V_0 yields a term of the form $2i\omega\log(2i\omega)f_{110}(\omega,u)+\omega f_{100}(\omega,u)$, $f_{110}, f_{100} \in C^3(\mathbb{R})$ with respect to ω, fitting into the expansion (7.3). Thus, it remains to compute the integral

$$-\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))\sum_{p=3}^{5}\sum_{q=0}^{p-3} c_{pq}\frac{\log^q(v)}{v^p}\alpha^{(1)}(v)\,dv\,.$$

We do this exemplarily for the term

$$\beta^{(2)}(u) := -\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))\frac{c_{30}}{v^3}\alpha^{(1)}(v)\,dv\,. \tag{7.10}$$

A complex calculation, using Lemma 6.1 and Lemma 7.3, which will be stated and proven afterwards, yields

$$\begin{aligned}\beta^{(2)}(u) = \quad & 2i\omega\log(2i\omega)e^{-i\omega u}c\Big\{(-2i\omega)\log(-2i\omega u) - \frac{1}{u}\sum_{k=0}^{\infty} d_k(-2i\omega u)^k\Big\} \\ &+ 2i\omega e^{-i\omega u}\frac{c_{30}^2}{4}\Big\{(-2i\omega)\big(-\frac{1}{2}\log^2(-2i\omega u) \\ &\quad + \log(-2i\omega u)(c+\log u)\big) - (c + c\log u)\frac{1}{u}\sum_{k=0}^{\infty} d_k(-2i\omega u)^k\Big\} \\ &+ e^{-i\omega u}\Big\{c(2i\omega)^2\log(-2i\omega u) + \frac{1}{u^2}\sum_{k=0}^{\infty} d_k(-2i\omega u)^k\Big\}\,, \end{aligned}\tag{7.11}$$

with suitable constants c, d_k. Hence $\beta^{(2)}(u)$ goes with (7.3). So far, we cannot finish this scheme, but if one has a close look, one sees that the most irregular term at $\omega = 0$, namely $2i\omega\log(2i\omega)$, now appears with a $1/u$ decay, while the other irregularities appear with an additional ω-power. Furthermore, due to the bounds (7.8) together with direct integral estimates, we obtain for all u with $|\omega|u \geq 1$ the bounds

$$\begin{aligned}\left|\beta^{(2)}(u)\right| &\leq c\int_u^\infty \frac{v}{1+|\omega|v}\frac{1}{v^3}\frac{1}{v^2|\omega|}\,dv \leq c\frac{1}{u^4|\omega|} \\ \left|\partial_\omega\beta^{(2)}(u)\right| &\leq c\frac{1}{u^3|\omega|} \\ \left|\partial_\omega^2\beta^{(2)}(u)\right| &\leq c\frac{1}{u^2|\omega|} \\ \left|\partial_\omega^3\beta^{(2)}(u)\right| &\leq c\frac{1}{u|\omega|}\,. \end{aligned}\tag{7.12}$$

Using in the region $|\omega|u < 1$ for the sum-terms in (7.11) estimates analog to (7.9), we conclude in the same way as before, that iterating these first with r_6 followed by the full iteration with the potential V_0 and summing up, we obtain C^3-terms. We split up the remaining log-terms by $\log(-2i\omega u) = \log(-2i\omega) + \log(u)$ and use Lemma 7.2 to show that applying the same procedure yields terms that go with (7.3). Hence, we have to analyze the integral

$$-\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))\sum_{p=3}^{5}\sum_{q=0}^{p-3} c_{pq}\frac{\log^q(v)}{v^p}\beta^{(2)}(v)\,dv\ ,$$

exemplarily we treat the term

$$\gamma^{(3)}(u) := -\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))\frac{c_{30}}{v^3}\beta^{(2)}(v)\,dv\ . \tag{7.13}$$

Computing this expression with the same methods one sees that the term with $2i\omega\log(2i\omega)$ decays as $1/u^2$ and the $\omega^2\log^s(\pm 2i\omega)$-terms decay as $\log^t(u)/u$. With bounds analog to (7.12),(7.9) the same procedure applies and yields terms that match with (7.3). Once again it remains to analyze

$$-\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))\sum_{p=3}^{5}\sum_{q=0}^{p-3} c_{pq}\frac{\log^q(v)}{v^p}\gamma^{(3)}(v)\,dv\ ,$$

and exemplarily

$$-\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))\frac{c_{30}}{v^3}\gamma^{(3)}(v)\,dv\ .$$

Calculating this, one checks that the $2i\omega\log(2i\omega)$-term decays as $1/u^3$, the $\omega^2\log^s(\pm 2i\omega)$-terms decay as $\log^t(u)/u^2$ and the $\omega^3\log^m(\pm 2i\omega)$-terms decay as $\log^n(u)/u$. Applying this scheme two times more, all terms which are not C^3 with respect to ω decay at least as $\log^s(u)/u^3$. Subtracting these terms from the full term, we obtain a C^3-term which is decaying at least as $\log^s(u)/u^3$, according to estimates analog to (7.12),(7.9) and estimating $|\omega|$ by $1/u$ in the region $|\omega|u < 1$. So Lemma 7.2 applies for the full iteration with the potential V_0 and we get a C^3-term. Due to their decay, we are able to iterate the subtracted $\log\omega$-terms also with the full potential V_0 and get terms that match with (7.3). Thus, the scheme can be stopped after finitely many calculations and the lemma is proven for $n = 3$. For $n \geq 4$ we split the potential in the way

$$V_0(v) = \sum_{p=3}^{n+2}\sum_{q=0}^{p-3} c_{pq}\frac{\log^q(v)}{v^p} + r_{n+3}(v)\ ,$$

and proceed with the same calculations. In (7.7) we have to integrate by parts up to the n-th order, in order to obtain as analogon to estimate (7.8)

$$\left|\partial_\omega^m \alpha^{(1)}(u)\right| \le \frac{c}{|\omega|} u^{2-l}\ , \quad m \le n\ .$$

The next difference appears in the estimates (7.12). These cannot be done for $n \ge 4$ by simple integral estimates as a matter of convergence. Thus, we have to subtract from the result of the analog calculation to (7.7) for $\alpha^{(1)}(u)$ the first $n-3$ exact terms of the form

$$\frac{c}{\omega u^2} e^{-i\omega u} + \dots + \frac{c}{\omega^{n-3} u^{n-2}} e^{-i\omega u} =: \rho^{(1)}(u)\ ,$$

and get for $m \le n$

$$\begin{aligned}
\left|\partial_\omega^m \beta^{(2)}(u)\right| &\le \left|\partial_\omega^m \int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))\frac{c_{30}}{v^3}\left(\alpha^{(1)}(u) - \rho^{(1)}(u)\right)\right| \\
&\quad + \left|\partial_\omega^m \int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))\frac{c_{30}}{v^3}\rho^{(1)}(u)\right| \\
&\le \frac{c}{|\omega|} u^{m-4}\ ,
\end{aligned}$$

where for the first integral this can be done by elementary integral estimates, and for the second integral we have to integrate the subtracted terms by parts, as we did to obtain the estimates for $\alpha^{(1)}(u)$. Keeping these differences in mind, we can conclude exactly in the same way as for $n = 3$, which yields the claim for arbitrary n.

□

We now state the missing lemma.

Lemma 7.3. *Let $u > 0$ and $\omega \in \mathbb{R}\backslash\{0\}$. For the calculation of the iteration of the infinite sums that appear in the integration in Lemma 6.1 with an arbitrary part of the potential, $\log^q u/u^p$, cf. Lemma 6.3, we obtain the identity*

$$\begin{aligned}
&-\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))\frac{\log^q v}{v^p}\frac{\log^s v}{v^t} e^{\pm i\omega v}\sum_{k=0}^\infty d_k(\pm 2i\omega v)^k\, dv \\
&= \frac{1}{u^{p+t-2}} e^{\mp i\omega u}\sum_{l=0}^{s+q} c\log^l(u)\sum_{k=0}^\infty d_{kl}(\mp 2i\omega u)^k \qquad (7.14) \\
&\quad + (2i\omega)^{p+t-2} e^{\mp i\omega u}\sum_{m=0}^{s+q}\sum_{r=1}^{m+1} c\log^r(\mp 2i\omega u)\log^{s+q-m}(u)\ , \qquad (7.15)
\end{aligned}$$

for suitable constants d_{kl}, c.

Proof. Let us denote $m = q + s \geq 0$ and $n = p + t \geq 4$. In order to compute the integral on the left hand side in the lemma, we insert a convergence generating factor

$$\begin{aligned}
&- \int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) \frac{\log^m v}{v^n} e^{\pm i\omega v} \sum_{k=0}^\infty d_k (\pm 2i\omega v)^k \, dv \\
&= \lim_{\varepsilon \searrow 0} \int_u^\infty e^{-\varepsilon v} \frac{1}{\omega} \sin(\omega(v-u)) \frac{\log^m v}{v^n} e^{\pm i\omega v} \sum_{k=0}^\infty d_k (\pm 2i\omega v)^k \, dv. \quad (7.16)
\end{aligned}$$

In the next step we interchange the integral and the infinite sum. This can be done for any $\varepsilon > 2|\omega|$ by a dominating convergence argument, if one estimates the modulus of the sum very roughly by $\exp(2|\omega|v)$. Thus, the two expressions

$$\int_u^\infty \frac{1}{\omega} \sin(\omega(v-u)) e^{-\varepsilon v} \frac{\log^m v}{v^n} e^{\pm i\omega v} \sum_{k=0}^\infty d_k (\pm 2i\omega v)^k \, dv$$

and

$$\sum_{k=0}^\infty d_k (\pm 2i\omega)^k \int_u^\infty \frac{1}{\omega} \sin(\omega(v-u)) e^{-\varepsilon v} \frac{\log^m v}{v^n} e^{\pm i\omega v} v^k \, dv$$

coincide for any $\varepsilon > 2|\omega|$. Moreover, both expressions are analytic in ε for $\operatorname{Re}\varepsilon > 0$. So by the identity theorem for analytic functions both expressions coincide for any $\varepsilon > 0$. So (7.16) is equal to

$$\lim_{\varepsilon \searrow 0} \sum_{k=0}^\infty d_k (\pm 2i\omega)^k \int_u^\infty \frac{1}{\omega} \sin(\omega(v-u)) e^{-\varepsilon v} \frac{\log^m v}{v^{n-k}} e^{\pm i\omega v} \, dv \, .$$

Once again we rewrite $\sin(\omega(v-u))$ with Eulers formula and integrate by parts [note that one has to be careful with the ε-terms that are generated by this integration by parts, but in the limit $\varepsilon \searrow 0$ they vanish] to obtain

$$= \lim_{\varepsilon \searrow 0} \sum_{k=0}^\infty d_k (\pm 2i\omega)^k e^{\mp i\omega u} \int_u^\infty e^{\pm 2i\omega v - \varepsilon v} \sum_{l=0}^m c \frac{\log^l v}{v^{n-k-1}} \, dv \, ,$$

with suitable constants c arising from the integral function of $\log^m v / v^{n-k}$. Now we apply Lemma 6.1, Lemma 6.2, take the limit $\varepsilon \searrow 0$ and get

$$\begin{aligned}
&= \sum_{l=0}^m \sum_{k=0}^\infty d_k (\pm 2i\omega)^k e^{\mp i\omega u} \sum_{i=0}^l c \binom{l}{i} \log^{l-i}(u) \, \times \\
&\left\{ (\mp 2i\omega)^{n-k-2} \sum_{j=0}^{i+1} c \log^j [\mp 2i\omega u] - \frac{1}{u^{n-k-2}} \sum_{r=0, r \neq n-k-2}^\infty d_r (\mp 2i\omega u)^r \right\}.
\end{aligned}$$

We reorder the two infinite sums to one infinite sum, which can be done because of the structure of the coefficients d_k, d_r of the exponential integral that lets us compare the new coefficients to the coefficients of the exponential series, and get the expression (7.15). $\square$

Next, we need a similar expansion for the derivative $\grave{\phi}'_\omega(u)$.

Lemma 7.4. *For $l = 0$, $\omega \in \mathbb{R} \setminus \{0\}$, $n \geq 3$ and fixed $u > 0$, the first u-derivative of $\grave{\phi}_\omega(u)$ satisfies the expansion*

$$\grave{\phi}'_\omega(u) = -i\omega e^{-i\omega u} + h_0(\omega, u) + \sum_{i \geq j+k=1}^{n} (2i\omega)^i \log^j(2i\omega) \log^k(-2i\omega) h_{ijk}(\omega, u) \,, \tag{7.17}$$

where the functions $h_0, h_{ijk} \in C^n(\mathbb{R})$ with respect to ω.

Proof. In order to prove this, we use the fact that $\grave{\phi}'_\omega(u)$ satisfies for $u > 0$ an integral equation analog to (4.20)

$$\grave{\phi}'_\omega(u) = -i\omega e^{-i\omega u} - \int_u^\infty \cos(\omega(u-v)) V_0(v) \grave{\phi}_\omega(v)\, dv \,.$$

We estimate $\cos(\omega(u-v))$ and its $\omega-$derivatives for real ω and $v \geq u > 0$ by

$$\left|\partial_\omega^n \cos(\omega(u-v))\right| \leq (v-u)^n \leq (2v)^n \,, \quad n \in \mathbb{N}_0 \,. \tag{7.18}$$

Thus, using this estimate for $n = 0$ together with the iteration scheme (6.12) for $\grave{\phi}_\omega(u)$, we obtain a well defined iteration scheme for the u-derivative, cf. proof of Theorem 4.2:

$$\begin{aligned} \grave{\phi}'_\omega(u) &= \sum_{k=0}^{\infty} \psi_\omega^{(k)}(u) \,, \quad \text{where} \\ \psi_\omega^{(0)}(u) &= -i\omega e^{-i\omega u} = \left(\phi_\omega^{(0)}\right)'(u) \,, \\ \psi_\omega^{(k+1)}(u) &= -\int_u^\infty \cos(\omega(u-v)) V_0(v) \phi_\omega^{(k)}(v)\, dv = \left(\phi_\omega^{(k+1)}\right)'(u) \,, \end{aligned} \tag{7.19}$$

with $k \geq 0$. Due to this iteration scheme together with the estimates (7.18) that replace the bounds (6.10) and the identity $\cos(\omega(u-v)) = 1/2(e^{i\omega(u-v)} + e^{i\omega(v-u)})$, we can use the decompositions of the $\phi_\omega^{(k)}$, which we have made in the proof of Lemma 7.1. In particular, we apply the procedure of this proof, in order to show the claim. $\square$

Now, we use the expansions (7.3),(7.17), in order to analyze the ω-dependence of the essential part of the integral kernel

$$\mathrm{Im}\left(\frac{\acute{\phi}_\omega(u)\grave{\phi}_\omega(v)}{w(\acute{\phi}_\omega, \grave{\phi}_\omega)}\right) \,.$$

At this stage, it is enough to set $n = 4$ in (7.3),(7.17) for our purposes. Looking at the integral representation (7.1) of the solution, we see that $u \in \mathbb{R}$ is fixed while $v \in \mathbb{R}$ varies in a compact set, the support of our initial data Ψ_0. Due to the Picard-Lindelöf theorem and the analytical dependence in ω of the Schrödinger equation from the coefficients, the expansions (7.3),(7.17) extend to any u, and v, respectively, on compact sets. Moreover, the following properties follow directly by the construction of the expansions.

Corollary 7.5. *For $4 \geq i = j + k \geq 1$ the function g_{ijk}, h_{ijk} can be constructed such that they obey the equalities*

$$\begin{array}{rcll} g_{ijk}(\omega,u) + o(\omega^\kappa) & = & c_{ijk}\left(e^{-i\omega u} + g_0(\omega,u)\right) & \text{and} \\ h_{ijk}(\omega,u) + o(\omega^\kappa) & = & c_{ijk}\, h_0(\omega,u)\,, & \textit{for } i \text{ even} \\ g_{ijk}(\omega,u) + o(\omega^\kappa) & = & c_{ijk}\overline{(e^{i\omega u} + g_0(\omega,u))} & \text{and} \\ h_{ijk}(\omega,u) + o(\omega^\kappa) & = & c_{ijk}\,\overline{h_0(\omega,u)}\,, & \textit{for } i \text{ odd.} \end{array} \tag{7.20}$$

where κ is an arbitrary integer and the c_{ijk} are real constants, in particular not depending on u.

Proof. We show this exemplarily for the first terms g_{110}, h_{110}. In this situation, (7.20) holds because the first term, where $(2i\omega)\log(2i\omega)$ appears, appears with $c_{30}/2\; e^{i\omega u}$ and there are no other terms with this ω-dependence except the terms that are generated by this [cf. the calculations (7.5),(7.11)]. Thus, $g_{110}(\omega,u)$ is generated by $e^{i\omega u}$, which is just the complex conjugate of $e^{-i\omega u}$, and this behavior is kept by the iteration scheme. So any C^4-term that is generated is the complex conjugate of a corresponding term of g_0. This is valid, until one finishes the iteration scheme with the arguments at the end of the proof of Lemma 7.1, by what the $o(\omega^\kappa)$-term arises. Since one can do arbitrary many calculations and in each iteration at least a $\pm 2i\omega\log(\pm 2i\omega)$ is generated, the κ can be chosen arbitrary. Moreover, looking at the iteration scheme (7.19), the equalities for $h_{110}(\omega,u)$ are a consequence of the arguments for $g_{110}(\omega,u)$, because of the fact that by the calculations concerning this scheme no additional highest order log-terms, i.e. $i = j + k$, are generated. $\square$

In the following assume that $\kappa = 5$. We expand the functions $g_{ijk}(\omega,u)$ and $h_{ijk}(\omega,u)$ in their Taylor polynom with respect to ω at $\omega = 0$ up to the fourth order:

$$g_{ijk}(\omega,u) = \sum_{m=0}^{4} \frac{1}{m!}\partial_\omega^m g_{ijk}\big|_{(0,u)}\omega^m + r_{ijk}(\omega,u)\;,$$

$$h_{ijk}(\omega,u) = \sum_{m=0}^{4} \frac{1}{m!}\partial_\omega^m h_{ijk}\big|_{(0,u)}\omega^m + q_{ijk}(\omega,u)\;,$$

where the remaining terms $r_{ijk}(\omega,u), q_{ijk}(\omega,u) \in C^4(\mathbb{R})$ behave for small ω as $o(|\omega|^4)$. Note that, due to this fact, any logarithmic irregularity multiplied with r_{ijk}, q_{ijk} yields a C^4-term with respect to ω. Moreover, we expand for fixed u the fundamental solution $\acute{\phi}_\omega(u)$ and its $u-$derivative $\acute{\phi}'_\omega(u)$

$$\acute{\phi}_\omega(u) = \sum_{k=0}^{\infty} c_k(u)\omega^k \ , \quad \acute{\phi}'_\omega(u) = \sum_{k=0}^{\infty} d_k(u)\omega^k \ ,$$

which exist, because these are analytic in ω for fixed u. Since the fundamental solutions $\acute{\phi}, \grave{\phi}$ are real for $\omega = 0$, the coefficients $g_0(0,u), h_0(0,u), c_0(u)$ and $d_0(u)$ are real for all $u \in \mathbb{R}$. Using all these properties, we expand

$$\frac{\acute{\phi}_\omega(u)\grave{\phi}_\omega(v)}{w(\acute{\phi}, \grave{\phi})} \ , \tag{7.21}$$

with the ansatz of a geometrical series with respect to ω. Note that, according to Lemma 5.1, the Wronskian does not vanish for $\omega = 0$. By a straightforward calculation it is shown that, essentially using (7.20), the terms with the highest logarithmic order, i.e. $(2i\omega)^i \log(2i\omega)^j \log(-2i\omega)^k, i = j+k$, vanish. Thus, we have to pick out the terms $(2i\omega)^2 \log(2i\omega)^j \log(-2i\omega)^k$ with $j + k = 1$, in order to get the lowest regularity. Looking at the calculations (7.5) and (7.11) [Note that these are the only possible terms, where a term with this irregularity appears the first time, according to our construction. The others are just a consequence out of these and hence a contribution to functions g_{2jk}], the desired terms appear in $\grave{\phi}$ the first time as

$$e^{-i\omega u}\left((2i\omega)^2 \log(2i\omega)(c + c\log u) - c(2i\omega)^2 \log(-2i\omega)\right) \ , \tag{7.22}$$

where a $(2i\omega)^2 \log(2i\omega) \log u$ shows up in the first line of (7.11), if one separates $\log(-2i\omega u) = \log(-2i\omega) + \log u$. All other such terms appearing in the second line of (7.11) as well as in the second line of (7.5) vanish because of their coefficients. Applying the same arguments as before, it follows that $g_{201}(\omega,u) + o(\omega^\kappa) = c(e^{-i\omega u} + g_0(\omega,u))$ and $h_{201}(\omega,u) + o(\omega^\kappa) = ch_0(\omega,u)$. Hence, the terms with $(2i\omega)^2 \log(-2i\omega)$ cancel in the ω-expansion of (7.21). Because of the additional $\log u$-term, we get

$$g_{210}(\omega,u) + o(\omega^\kappa) = c_1(e^{-i\omega u} + g_0(\omega,u)) + c_2(e^{-i\omega u}\log u + g(\omega,u)) \ ,$$

and

$$h_{210}(\omega,u) + o(\omega^\kappa) = c_1 h_0(\omega,u) + c_2 h(\omega,u) + \frac{1}{4}c_{30}e^{i\omega u} \ ,$$

with appropriate real constants c_1, c_2, where the last term appears by a direct calculation of $\psi^{(1)}(u)$ with the part c_{30}/v^3 of the potential $V_0(v)$. Furthermore, $g(\omega,u), h(\omega,u)$ are C^4-functions with respect to ω, where $g(\omega,u)$ is generated by the iteration of $e^{-i\omega u}\log u$ and $h(\omega,u)$ the consequence out of this in (7.19). One directly verifies that $g(0,u), h(0,u)$ are real, in general non-vanishing. Putting all these informations together, one sees that there appears a term with $(2i\omega)^2 \log(2i\omega)$ in the ω-expansion of (7.21), which is generated on the one hand by the $g(0,u), h(0,u)$, and on the other hand by the $2i\omega \log(2i\omega)$-part multiplied with the ω-contribution of first order of $\acute{\phi}, \acute{\phi}'$. This represents the part with the highest irregularity with respect to ω. Moreover, the related coefficients are purely *real*, depending on u, v and in general non-vanishing. Using the identity

$$\log(2i\omega) = i\frac{\pi}{2}\,\text{sign}(\omega) + \log(2|\omega|) ,$$

and taking the imaginary part of (7.21), which is just the essential part of our integral kernel, we obtain as the lowest regular ω-term in the expansion of (7.2) at $\omega = 0$

$$c_0(u) g_{20}(v)\, \omega^2\, \text{sign}(\omega) , \tag{7.23}$$

where the function $g_{20}(v)$ arises out of the foregoing calculation. The symmetry of (7.2) with respect to u, v yields immediately $g_{20}(v) = k c_0(v)$ with an appropriate constant $k \neq 0$.

In the next step we want to use (7.23), in order to derive the decay of the solution $\Psi(t,u)$ given by (7.1). To this end, first we have to analyze the behavior of the ω-derivatives of the integrand up to the fourth order for large $|\omega|$.

Lemma 7.6. *For $u \in \mathbb{R}$ and compactly supported smooth initial data $\Psi_0 \in C_0^\infty(\mathbb{R})^2$ of the Cauchy problem, the ω-derivatives of the integrand in the integral representation (7.1)*

$$\partial_\omega^m \left(\int_{\text{supp}\,\Psi_0} \text{Im}\left(\frac{\acute{\phi}_\omega(u) \grave{\phi}_\omega(v)}{w(\acute{\phi}_\omega, \grave{\phi}_\omega)} \right) \begin{pmatrix} \omega & 1 \\ \omega^2 & \omega \end{pmatrix} \Psi_0(v) dv \right) , \quad m \in \{0, ..., 4\} , \tag{7.24}$$

have arbitrary polynomial decay in ω for $|\omega| \to \infty$.

Proof. We proceed essentially as in the proof of Theorem 5.5, where the case $m = 0$ was shown. To this end, we have to investigate the behavior of $\acute{\phi}_\omega(u), \grave{\phi}_\omega(v)$ in ω for $u \in \mathbb{R}$ fixed and v in the compact set $\text{supp}\Psi_0$. We start with $\grave{\phi}_\omega$. We assume that $|\omega| \geq 1$ and $u_0 \in \mathbb{R}$ is arbitrary. Obviously, we find for any $v \geq u \geq u_0$ and $m \in \{0, ..., 4\}$ a constant $C_1(u_0)$ such that

$$\left| \partial_\omega^m \left[\frac{1}{\omega} \sin(\omega(u - v)) \right] \right| \leq \frac{1}{|\omega|} C_1(u_0)(1 + |v|)^m . \tag{7.25}$$

Furthermore, splitting the potential as

$$V_0(u) = \sum_{p=3}^{5} \sum_{q=0}^{p-3} c_{pq} \frac{\log^q(v)}{v^p} + r_6(v)$$

and following an analog calculation as in (7.7), we obtain for the ω-derivatives of the first iteration $\phi_\omega^{(1)}(u)$ for all $u \geq 1$ and $m \in \{0, ..., 4\}$ the estimate

$$\left|\partial_\omega^m \phi_\omega^{(1)}(u)\right| \leq \frac{1}{|\omega|} C_2 u^{m-2} , \tag{7.26}$$

with an appropriate constant C_2. [Note that this is just an analogue to the estimate (7.8).] For all $u < 1$ and $m \in \{0, ..., 4\}$ we get

$$\begin{aligned} \left|\partial_\omega^m \phi_\omega^{(1)}(u)\right| &\leq \left|\partial_\omega^m \int_u^1 \frac{1}{\omega} \sin(\omega(u-v)) V_0(v) e^{-i\omega v}\, dv\right| \\ &\quad + \left|\partial_\omega^m \int_1^\infty \frac{1}{\omega} \sin(\omega(u-v)) V_0(v) e^{-i\omega v}\, dv\right| \\ &\leq \frac{1}{|\omega|} f(m,u) + C_3 \frac{1}{|\omega|} \sum_{k=0}^m |u|^k , \end{aligned}$$

where f is a continuous function with respect to u and the second term arises by the same method as we used for the estimate (7.26). Defining C_4 by

$$C_4 := \max_{m \in \{0,...,4\}} \max_{u \in [u_0, 1]} \left\{ \left(f(m,u) + C_3 \frac{1}{|\omega|} \sum_{k=0}^m |u|^k \right) (1+|u|)^{2-m} \right\} ,$$

and $C_5 := \max(C_2, C_4)$, we obtain for all $u \geq u_0$ and $m \in \{0, ..., 4\}$ the bound

$$\left|\partial_\omega^m \phi_\omega^{(1)}(u)\right| \leq \frac{1}{|\omega|} C_5 (1+|u|)^{m-2} . \tag{7.27}$$

In order to estimate the derivatives of the second iteration $\phi_\omega^{(2)}(u)$ up to the fourth order, we subtract the first exact term out of the integration by parts in (7.26), $\frac{c_{30}}{4i\omega u^2} e^{-i\omega u}$, from the first iteration $\phi_\omega^{(1)}(u)$ and obtain for $u \geq 1$ and $m \leq 4$ the bounds

$$\left|\partial_\omega^m \left(\phi_\omega^{(1)}(u) - \frac{c_{30}}{4i\omega u^2} e^{-i\omega u}\right)\right| \leq \frac{1}{|\omega|} C u^{m-3}. \tag{7.28}$$

Thus, in order to estimate the ω-derivatives of the second iteration:

$$\begin{aligned} \left|\partial_\omega^m \phi_\omega^{(2)}(u))\right| &\leq \left|\partial_\omega^m \int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) V_0(v) \left(\phi_\omega^{(1)}(v) - \frac{c_{30}}{4i\omega u^2} e^{-i\omega v}\right) dv\right| \\ &\quad + \left|\partial_\omega^m \int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) V_0(v) \frac{c_{30}}{4i\omega u^2} e^{-i\omega v}\, dv\right| . \end{aligned}$$

Using the estimates (7.28),(7.25), and once again the method of splitting up the potential and integrating by parts for the second integral, we get for $u \geq 1$ and $m \leq 4$ the bounds

$$\left|\partial_\omega^m \phi_\omega^{(2)}(u))\right| \leq \frac{1}{|\omega|} C u^{m-4},$$

and thus, following the foregoing arguments for all $u \geq u_0$ (after possibly enlarging C_5) the estimates

$$\left|\partial_\omega^m \phi_\omega^{(2)}(u))\right| \leq \frac{1}{|\omega|} C_5 (1+|u|)^{m-4} \ . \tag{7.29}$$

Using (7.25) and (7.29), we obtain for the ω-derivatives of the third iteration for all $u \geq u_0$

$$\begin{aligned} \left|\partial_\omega^m \phi_\omega^{(3)}(u)\right| &\leq \left|\sum_{k=0}^m \binom{m}{k} \int_u^\infty \partial_\omega^{m-k}\left(\frac{1}{\omega}\sin(\omega(v-u))\right) V_0(v) \partial_\omega^k \phi_\omega^{(2)}(v)\, dv\right| \\ &\leq 16 C_1(u_0) C_5 \frac{1}{|\omega|} \int_u^\infty (1+|v|)^{m-4} \frac{1}{|\omega|} V_0(v)\, dv \ . \end{aligned} \tag{7.30}$$

Note that interchanging the integral and the ω-derviatives is permitted, because the ω-derivatives of the integrand are integrable due to the estimates (7.25),(7.29) and the $1/v^3$-decay of the potential $V_0(v)$. We show by induction in n for all $u \geq u_0$ the inequality

$$\left|\partial_\omega^m \phi_\omega^{(n)}(u)\right| \leq 16 C_1(u_0) C_5 \frac{1}{|\omega|} Q_\omega(m,u) \frac{1}{(n-3)!} P_\omega(u)^{n-3}, \quad \forall n \geq 3 \ , \tag{7.31}$$

where the functions $Q_\omega(m,u)$ and $P_\omega(u)$ are given by the integrals

$$\begin{aligned} Q_\omega(m,u) &:= \int_u^\infty (1+|v|)^{m-4} \frac{1}{|\omega|} V_0(v)\, dv \\ P_\omega(u) &:= 16 C_1(u_0) C_6 \int_u^\infty \frac{1}{|\omega|} V_0(v)\, dv \ , \end{aligned}$$

where C_6 is a constant chosen such that for all $x \geq v \geq u_0$

$$(1+|x|)^{k-m} \leq C_6 (1+|v|)^{k-m} \ , \quad 0 \leq k \leq m \leq 4 \ .$$

The initial step is now given by (7.30). So assume that (7.31) holds for n. Then, according to the iteration scheme,

$$\begin{aligned} \left|\partial_\omega^m \phi_\omega^{(n+1)}(u)\right| \ \leq \ & \left|\sum_{k=0}^m \binom{m}{k} \int_u^\infty C_1(u_0)(1+|v|)^{m-k} \frac{1}{|\omega|} V_0(v)\right. \\ & \left. \times 16 C_1(u_0) C_5 \frac{1}{|\omega|} Q_\omega(k,v) \frac{1}{(n-3)!} P_\omega(v)^{n-3}\, dv\right| \ . \end{aligned}$$

Using the inequality

$$Q_\omega(k,v) \leq C_6 (1+|v|)^{k-m} Q_\omega(m,v)$$

and the monotonicity of Q_ω, we obtain

$$\begin{aligned}\left|\partial_\omega^m \phi_\omega^{(n+1)}(u)\right| &\le 16C_1(u_0)C_5\frac{1}{|\omega|}\int_u^\infty 16C_1(u_0)(1+|v|)^{m-k}\frac{1}{|\omega|}V_0(v) \\ &\qquad \times C_6(1+|v|)^{k-m}Q_\omega(m,v)\frac{1}{(n-3)!}P_\omega(v)^{n-3}\,dv \\ &\le 16C_1(u_0)C_5\frac{1}{|\omega|}Q_\omega(m,u)\int_u^\infty \frac{dP_\omega}{dv}(v)\frac{1}{(n-3)!}P_\omega(v)^{n-3}\,dv \\ &= 16C_1(u_0)C_5\frac{1}{|\omega|}Q_\omega(m,u)\frac{1}{(n-2)!}P_\omega(u)^{n-2}\,,\end{aligned}$$

and (7.31) follows. In particular, we get for all $u \ge u_0$ and $m \le 4$ the estimate

$$\begin{aligned}\left|\partial_\omega^m \grave{\phi}_\omega(u) - \partial_\omega^m e^{-i\omega u}\right| &\le C_5\frac{1}{|\omega|}(1+|u|)^{m-2} + \frac{1}{|\omega|}C_5(1+|u|)^{m-4} \\ &\quad +16C_1(u_0)C_5\frac{1}{|\omega|}Q_\omega(m,u)e^{P_\omega(u)}\,, \qquad (7.32)\end{aligned}$$

and the right hand side obviously tends to zero as $|\omega| \to \infty$.

In an analog way using the iteration scheme (4.7) for $\acute{\phi}$, one shows for all $u \le u_0$ and $m \in \{0,...,4\}$

$$\left|\partial_\omega^m \acute{\phi}_\omega(u) - \partial_\omega^m e^{i\omega u}\right| \le \sum_{n=1}^\infty \frac{1}{n!}M_\omega(m,u)^n = e^{M_\omega(m,u)} - 1\,, \qquad (7.33)$$

where $M_\omega(m,u)$ is given by

$$M_\omega(m,u) := \frac{C_7}{|\omega|}\int_{-\infty}^u (1+|v|)^m V_0(v)\,dv\,,$$

with a sufficiently large constant C_7. Note that this integral is well defined, and in particular the estimate is obtained easier, due to the fact that $V_0(v)$ decays exponentially as $v \to -\infty$. Moreover, the right hand side in (7.33) also goes to zero as $|\omega| \to \infty$. Thus, due to (7.32) and (7.33), the ω-derivatives of the fundamental solutions up to the fourth order $\partial_\omega^m \acute{\phi}_\omega(u)$,$\partial_\omega^m \grave{\phi}_\omega(v)$ are controlled for large $|\omega|$ by constants, which depend on u and the support of the initial data Ψ_0. One also shows with these results and applying the same arguments to $\acute{\phi}'_\omega, \grave{\phi}'_\omega$ that the Wronskian $w(\acute{\phi},\grave{\phi})$ behaves as $\mathcal{O}(|\omega|)$ and $\partial_\omega^m w(\acute{\phi},\grave{\phi}), m \le 4$ is bounded by constants as $|\omega| \to \infty$. Hence, interchanging in the representation (7.24) the differentiation with respect to ω and the integral, which is no problem because of the compact support of Ψ_0, making the substitutions

$$\grave{\phi}_\omega(v) = \frac{1}{\omega^2}\left(-\grave{\phi}''_\omega(v) + V_0(v)\grave{\phi}_\omega(v)\right)\,,$$

$$\partial_\omega \grave{\phi}_\omega(v) = \frac{-2}{\omega^3}\left(-\grave{\phi}''_\omega(v) + V_0(v)\grave{\phi}_\omega(v)\right) + \frac{1}{\omega^2}\left(-\partial_\omega \grave{\phi}''_\omega(v) + V_0(v)\partial_\omega \grave{\phi}_\omega(v)\right)$$

as well as the analog substitutions for the second, third and fourth ω-derivative [Note that in the region $|\omega| \geq 1$ $\grave{\phi}_\omega(v)$ is C^4 with respect to ω, cf. Lemma 7.1, $n = 4$] and integrating by parts with respect to v, one immediately has decay at least of $1/\omega^2$. Thus iterating this procedure, which can be done because V_0 and Ψ_0 are smooth, yields arbitrary decay in ω and the lemma is proven. □

Remark 7.7. Since the method of the proof does not depend on the highest order ω-derivative, the statement of Lemma 7.6 can be extended to arbitrary m. The only point where one has to be careful is the derivation of (7.29), since for ω-derivatives of higher order one has to calculate and subtract more exact terms than in (7.28), due to convergence problems. If (7.29) is not sufficient, in order to start the induction, one has to iterate this procedure appropriately many times.

We are now ready to state and prove our main theorem:

Theorem 7.8. *Consider the Cauchy problem of the scalar wave equation in the Schwarzschild geometry*

$$\Box\phi = 0\,, \quad (\phi_0, i\partial_t\phi_0)(0,r,x) = \Phi_0(r,x)$$

for smooth spherical symmetric initial data $\Phi_0 \in C_0^\infty((2M,\infty)\times S^2)^2$ *which is compactly supported outside the event horizon. Let* $\Phi(t) = (\phi(t), i\partial_t\phi(t)) \in C^\infty(\mathbb{R}\times(2M,\infty)\times S^2)^2$ *be the unique global solution which is compactly supported for all times* t. *Then for fixed* r *there is a constant* $c = c(r,\Phi_0)$ *such that for large* t

$$|\phi(t)| \leq \frac{c}{t^3}\,. \tag{7.34}$$

Moreover, if we have initially momentarily static initial data, i.e. $\partial_t\phi_0 \equiv 0$, *the solution* $\phi(t)$ *satisfies*

$$|\phi(t)| \leq \frac{c}{t^4}\,. \tag{7.35}$$

Proof. First, we decompose our initial data Φ_0 into spherical harmonics. Due to the spherical symmetry we obtain $\Phi_0(r,\vartheta,\varphi) = \tilde{\Phi}_0(r)Y_{00}(\vartheta,\varphi)$, where $\tilde{\Phi}_0(r) \in C_0^\infty((2M,\infty))^2$. Introducing the Regge-Wheeler coordinate $u(r)$ and making the substitution $\Psi(t,u) = r(u)\tilde{\Phi}(t,r(u))$, we apply Theorem 5.5. Thus, our solution has the representation

$$\Phi(t,r,\vartheta,\varphi) = \frac{1}{r}\Psi(t,u(r))Y_{00}(\vartheta,\varphi)\,,$$

where $\Psi(t,u)$ satisfies

$\Psi(t,u) =$

$$-\frac{1}{\pi}\int_{\mathbb{R}} e^{-i\omega t}\left(\int_{\text{supp}\,\Psi_0} \text{Im}\left(\frac{\acute{\phi}_\omega(u)\grave{\phi}_\omega(v)}{w(\acute{\phi}_\omega,\grave{\phi}_\omega)}\right)\begin{pmatrix}\omega & 1\\ \omega^2 & \omega\end{pmatrix}\Psi_0(v)dv\right)d\omega, \quad (7.36)$$

with initial data $\Psi_0(u) := r(u)\tilde{\Phi}_0(u)$ and the Jost solutions $\acute{\phi},\grave{\phi}$ in the case $l = 0$. According to the detailed analysis of (7.21) with respect to ω, the term

$$\text{Im}\left(\frac{\acute{\phi}_\omega(u)\grave{\phi}_\omega(v)}{w(\acute{\phi}_\omega,\grave{\phi}_\omega)}\right) - c_0(u)g_{20}(v)\omega^2\,\text{sign}(\omega) - c_{32}(u)g_{32}(v)\omega^3\log^2|\omega|$$
$$-c_{31}(u)g_{31}(v)\omega^3\log|\omega| - c_{30}(u)g_{30}(v)\omega^3\,\text{sign}(\omega)$$

is $C^3(\mathbb{R})$ with respect to ω for fixed $u \in \mathbb{R}$, $v \in \text{supp}\Psi_0$, where the $c_{ij}(u)$, $g_{ij}(v)$ denote the appropriate coefficient functions. [Note that these are linearly dependent due to the symmetry of (7.2) with respect to u, v.] Thus, defining

$$f(\omega,u) := \left(\int_{\text{supp}\,\Psi_0} \text{Im}\left(\frac{\acute{\phi}_\omega(u)\grave{\phi}_\omega(v)}{w(\acute{\phi}_\omega,\grave{\phi}_\omega)}\right)\begin{pmatrix}\omega & 1\\ \omega^2 & \omega\end{pmatrix}\Psi_0(v)dv\right)_1,$$

where the subscript denotes the first vector component, the term

$$\tilde{f}(\omega,u) := f(\omega,u) - \Big(c_0(u)d_{20}(\psi_0^2)\,\omega^2\,\text{sign}(\omega) + c_{32}(u)d_{32}(\psi_0^2)\,\omega^3\log^2|\omega|$$
$$+c_{31}(u)d_{31}(\psi_0^2)\,\omega^3\log|\omega| + c_{30}(u)d_{30}(\psi_0^2)\,\omega^3\,\text{sign}(\omega)\Big)\eta(\omega)$$
$$=: f(\omega,u) - r(\omega,u)\,,$$

is also $C^3(\mathbb{R})$ with respect to ω. Here, ψ_0^2 denotes the second component of the initial data Ψ_0,

$$d_{ij}(\psi_0^2) := \int_{\text{supp}\,\Psi_0} g_{ij}(v)\psi_0^2(v)\,dv\,,$$

and $\eta(\omega) \in C_0^\infty(\mathbb{R})$ is a smooth cutoff-function which is identically to 1 on a neighborhood of $\omega = 0$ and 0 outside a compact set. Moreover, because of Lemma 7.6 the $\partial_\omega^m\tilde{f}(\omega,u), m \in \{0,1,2,3\}$ have rapid decay for large $|\omega|$ and are in particular $L^1(\mathbb{R})$ with respect to ω. Thus, due to (7.36), the first component of Ψ satisfies

$$\begin{aligned}\psi^1(t,u) &= -\frac{1}{\pi}\int_{\mathbb{R}} e^{-i\omega t}\tilde{f}(\omega,u)\,d\omega - \frac{1}{\pi}\int_{\mathbb{R}} e^{-i\omega t}r(\omega,u)\,d\omega\\ &= -\frac{1}{(it)^3\pi}\left(\int_{\mathbb{R}}\tilde{f}(\omega,u)\partial_\omega^3 e^{-i\omega t}\,d\omega + \int_{\mathbb{R}} r(\omega,u)\partial_\omega^3 e^{-i\omega t}\,d\omega\right).\end{aligned}$$

We write the second integral as $\int_{-\infty}^0 + \int_0^\infty$, integrate every integral three times by parts and obtain

$$\psi^1(t,u) = \frac{1}{(it)^3\pi}\Bigg(4c_0(u)d_{20}(\psi_0^2) + \int_{\mathbb{R}} e^{-i\omega t}\partial_\omega^3\tilde{f}(\omega,u)\,d\omega$$

$$+\int_{-\infty}^{0} e^{-i\omega t}\partial_\omega^3 r(\omega,u)\,d\omega + \int_0^\infty e^{-i\omega t}\partial_\omega^3 r(\omega,u)\,d\omega\Bigg) \ .$$

Note that the other boundary terms vanish, because the $\partial_\omega^m \tilde{f}(\omega), m \leq 3$ have rapid decay and $\eta(\omega) \equiv 0$ outside of a compact set. Obviously, all integrals are well defined, and the Riemann-Lebesgue lemma shows the claim in the first case. If the initial data is initially momentarily static, all the $d_{ij}(\psi_0^2)$ vanish and the entries in the matrix in (7.36) yield an additional ω. Hence, the highest irregular term is $c_0(u)d_{20}(\psi_0^1)\omega^3\,\mathrm{sign}(\omega)$, and the same arguments as before conclude the proof. ☐

Remark 7.9. The decay rates $1/t^3$, and $1/t^4$, respectively, are optimal in the sense that there exists initial data such that these cannot be improved. This is obvious due to the fact that $c_0(u) > 0$.

8 Discussion and outlook on the case $l \neq 0$

According to Price's Law [19], the lm-component $\Phi^{lm}(t,u) = \frac{1}{r}\Psi^{lm}(t,u)$ of a solution for the Cauchy problem (1.6) in Schwarzschild spacetime with compactly supported smooth initital data generally falls off at late times t as t^{-2l-3} and t^{-2l-4} for initially momentarily static initial data, respectively. This has been confirmed in the previous section for spherical symmetric initial data, i.e. in the case $l = 0$ [cf. Theorem 7.8]. Moreover, there is numerical evidence which lets us conjecture this to be correct [16]. In this section, we briefly discuss whether the methods of the preceding section still apply to the case when the angular mode l is non-zero, and if not, about different ansatzes that might be worth trying.

To this end, let us reconsider the construction of the fundamental solutions $\grave{\phi}_{\omega l}$ of the Schrödinger equation (3.2), which are given by the equations (6.25) to (6.32). Hence, we still have finite expressions for the Green's function $S_\omega(u,v)$ as well as for the initial function $\phi_\omega^{(0)}(v)$, which involve essentially the plane waves $e^{\pm i\omega u}, e^{\pm i\omega v}$. Expanding all these expressions and deriving estimates analog to (6.33) and (6.35) for higher order ω-derivatives, we can improve Lemma 6.6 in the same way as Lemma 6.4 following the arguments of the proof of Lemma 7.1. Also, a similar result to Corollary 7.5 seems straightforward. The problem now arises, when we have to derive an $\omega-$expansion of the essential part of the integral kernel

$$\mathrm{Im}\left(\frac{\acute{\phi}_{\omega l}(u)\grave{\phi}_{\omega l}(v)}{w(\acute{\phi}_l,\grave{\phi}_l)}\right) \ . \tag{8.1}$$

The main difficulty can be seen as follows. If we proceeded in the same way as in the case $l = 0$, the lowest regular term with respect to ω should appear with the power ω^{2l+2} [cf. proof of Theorem 7.8] in order to satisfy Price's law. But due to the fact that the first irregularity in ω looks as follows,

$$e^{i\omega u}u^{-l}2i\omega\big(c\log^2(2i\omega u)+c\log u\log(2i\omega u)+c\log(2i\omega u)\big)\ ,$$

[cf. equation (6.36)], we would have to find a systematic way in order to check that the coefficients in front of terms with lower regularity vanish. Because of the complexity of the calculations we did not succeed in this point. Thus, following the same arguments as for $l=0$ together with the analog result to Corollary 7.5, which would involve $2i\omega\log^2(2i\omega)$ as highest irregularity, we would have to assume $\omega\log|\omega|$ as the lowest regular term in the expansion of (8.1). Except for this problem, we do not expect any further difficulties in extending Lemma 7.6 to $l\neq 0$, apart from the complexity of the calculations and the estimates. Thus, for arbitrary l it follows a similar statement to Theorem 7.8, but with the decay $|\phi(t)|\leq c/t^2$, and in the case of momentarily static initial data $|\phi(t)|\leq c/t^3$, respectively. The proof uses essentially the arguments of the proof of Theorem 7.8, with the difference that one basically has to check the inequality

$$\Big|\int_{-1}^{1}\log|\omega|e^{-i\omega t}\,d\omega\Big|\leq\frac{c}{t}\,.$$

To this end, one makes the substitution $z=\omega t$ and splits up the integrals to obtain

$$\begin{aligned}\int_{-1}^{1}\log|\omega|e^{-i\omega t}\,d\omega &= \frac{1}{t}\Big(\int_{-1}^{1}\log|z|e^{-iz}\,dz-\log t\int_{-t}^{t}e^{-iz}\,dz\\ &\quad+\int_{-t}^{-1}\log(-z)e^{-iz}\,dz+\int_{1}^{t}\log z e^{-iz}\,dz\Big)\,.\end{aligned}$$

Computing the second integral and integrating the last two integrals by parts yields

$$=\frac{1}{t}\Big(\int_{-1}^{1}\log|z|e^{-iz}\,dz+\frac{1}{i}\int_{-t}^{-1}\frac{1}{z}e^{-iz}\,dz+\frac{1}{i}\int_{1}^{t}\frac{1}{z}e^{-iz}\,dz\Big)\ ,$$

and the inequality follows, after having integrated the last two integrals once again by parts followed by standard integral estimates. However, in view of Price's law, this result is not satisfying. This led us to try another ansatz that seemed to be promising. This must be considered at first as a vague idea. But it might encourage the reader to go on with this problem.

The main idea is to rewrite the second fundamental solution $\grave{\phi}_\omega$ in terms of the first $\acute{\phi}_\omega$. To this end, we use the fact that for an ODE of Schrödinger type, which is in our case $(-\partial_u^2-\omega^2+V_l(u))\phi(u)=0$, and an associated solution $\phi_1(u)$, a linearly independent second solution is given by

$$\phi_2(u)=A\phi_1(u)\int^u\frac{1}{\phi_1(x)^2}\,dx\ ,$$

with an arbitrary constant A. Thus, for our given data $\acute{\phi}_\omega, \grave{\phi}_\omega$ and $w(\omega) := w(\acute{\phi}_\omega, \grave{\phi}_\omega)$, we formally define

$$\psi_\omega(u) := -\acute{\phi}_\omega(u) \int_u^\infty \frac{w(\omega)}{\acute{\phi}_\omega(x)^2}\, dx \ ,$$

in order to obtain a second linearly independent solution. We expect the integral for $\operatorname{Im}\omega < 0$ to be well defined, because of the asymptotic behavior of our fundamental solutions at $\pm\infty$. This suggests exponential increase of $|\acute{\phi}_\omega(u)|$ as $e^{|\operatorname{Im}\omega| u}$ when $u \to \infty$. Hence, suppose that everything is well defined for $\operatorname{Im}\omega < 0$. Then, due to the construction, $w(\acute{\phi}_\omega, \psi_\omega) = w(\omega)$, and thus the solutions $\acute{\phi}_\omega, \psi_\omega - \grave{\phi}_\omega$ are linearly dependent. For $\operatorname{Im}\omega < 0$ the boundary conditions are defined such that $\psi_\omega(u) - \grave{\phi}_\omega(u) \to 0$ as $u \to \infty$ and this would yield together with the linear dependency of $\acute{\phi}_\omega, \psi_\omega - \grave{\phi}_\omega$, that $\psi_\omega(u) - \grave{\phi}_\omega(u) \equiv 0$. Hence

$$\grave{\phi}_\omega(u) = -\acute{\phi}_\omega(u) \int_u^\infty \frac{w(\omega)}{\acute{\phi}_\omega(x)^2}\, dx \ , \quad \text{where } \operatorname{Im}\omega < 0 \, .$$

Since we are interested in real ω, we would like to obtain a similar formula for that case. A difficulty is that the asymptotic behaviors at $\pm\infty$ of $\acute{\phi}_\omega$ are expected to be plane waves. Hence, in order to ensure the convergence of the integral we must insert an additional convergence generating factor $e^{-\varepsilon x}$. Moreover, it is not clear that the boundary conditions of ψ_ω and $\grave{\phi}_\omega$ coincide. Provided that these technical difficulties can be handled, one gets the formula

$$\grave{\phi}_\omega(u) = -\acute{\phi}_\omega(u) \lim_{\varepsilon \searrow 0} \int_u^\infty e^{-\varepsilon x} \frac{w(\omega)}{\acute{\phi}_\omega(x)^2}\, dx \ , \tag{8.2}$$

valid for all $\omega \in \mathbb{R} \setminus \{0\}$. We now use this formula in the definition (3.9) of the Green's function $s_\omega(u,v)$ and obtain for all $u, v \in \mathbb{R}$ and $\omega \in \mathbb{R} \setminus \{0\}$ for the essential part of the integral kernel

$$\operatorname{Im}\big(s_\omega(u,v)\big) = \operatorname{Im}\left(\acute{\phi}_\omega(u) \acute{\phi}_\omega(v) \lim_{\varepsilon \searrow 0} \int_v^\infty e^{-\varepsilon x} \frac{1}{\acute{\phi}_\omega(x)^2}\, dx \right) . \tag{8.3}$$

Since $\acute{\phi}_\omega$ is analytic with respect to ω, it remains to analyze the integral in order to obtain the highest irregularity with respect to ω. At this stage, all the foregoing calculations indicate to consider the regions $|\omega| v < C$ and $|\omega| v \geq C$ separately. Hence, we split up the integral:

$$\int_v^\infty e^{-\varepsilon x} \frac{1}{\acute{\phi}_\omega(x)^2}\, dx = \int_v^{C/|\omega|} e^{-\varepsilon x} \frac{1}{\acute{\phi}_\omega(x)^2}\, dx + \int_{C/|\omega|}^\infty e^{-\varepsilon x} \frac{1}{\acute{\phi}_\omega(x)^2}\, dx \, .$$

The intention is to choose $C = C(\omega)$ such that the first integral of this splitting is analytic with respect to ω, whereas the second integral should yield the required irregularity. In order to treat the first integral it seems convenient to work in the "natural" variable ωr, where one might check that the fundamental solution $\acute{\phi}_\omega$ is analytic with respect to ωr. Hence, there might exist such a $C_1(\omega)$ to ensure analyticity of the integral. On the other hand, for the second integral we want to use a refined version of the expansion of Lemma 6.6. To this end, one could express the fundamental solution $\acute{\phi}_\omega$ in the denominator for all $\omega \in \mathbb{R} \setminus \{0\}$ by

$$\acute{\phi}_\omega(u) = \alpha(\omega)\grave{\phi}_\omega(u) + \beta(\omega)\overline{\grave{\phi}_\omega(u)} \ . \tag{8.4}$$

This is motivated by the fact that

$$w(\acute{\phi}_\omega, \omega^l \grave{\phi}_\omega) = -2i\omega^{l+1}\beta(\omega) \ , \quad w(\acute{\phi}_\omega, \omega^l \overline{\grave{\phi}_\omega}) = 2i\omega^{l+1}\alpha(\omega) \ ,$$

which follows from a short calculation using the asymptotics. Since the left hand sides in both equations are continuous in ω according to Lemma 5.1 and using the improved expansion of Lemma 6.6, we expect the coefficients $\alpha(\omega)$, $\beta(\omega)$ to behave like

$$\begin{aligned} \alpha(\omega) &= \frac{1}{\omega^{l+1}}(c + c\omega \log^2(2i\omega) + c\omega\log(2i\omega) + r_1(\omega)) \ , \\ \beta(\omega) &= \frac{1}{\omega^{l+1}}(c + c\omega \log^2(2i\omega) + c\omega\log(2i\omega) + r_2(\omega)) \ , \end{aligned}$$

where the c are appropriate constants and $r_1, r_2 \in C^{2l+4}(\mathbb{R})$ with respect to ω, $r_{1/2}(\omega) = \mathcal{O}(|\omega|)$ for small $|\omega|$. Moreover, a simplified calculation assuming that the fundamental solution $\grave{\phi}_\omega(x) \sim e^{-i\omega x}$ for $x > C/\omega$, $\omega > 0$, C arbitrary, yields

$$\lim_{\varepsilon \searrow 0} \int_{C/\omega}^{\infty} \frac{e^{-\varepsilon x}}{(\alpha(\omega)e^{-i\omega x} + \beta(\omega)e^{i\omega x})^2}\, dx = \frac{1}{2i\omega\Big(\alpha(\omega)e^{-2iC} + \beta(\omega)\Big)\alpha(\omega)} \ ,$$

where one uses the method of periodic integrals, i.e. one substitutes appropriately such that one can reduce the integral to a contour integral over the unit circle and applies the residue theorem. Hence, if one chose $C_2(\omega)$ appropriately, it should be possible that the right hand side is $\sim \omega^{2l+1}$. At this point, one has to take into account that $|\beta(\omega)|^2 - |\alpha(\omega)|^2 \equiv 1$, which is a consequence of a straightforward computation of $w(\acute{\phi}_\omega, \overline{\acute{\phi}_\omega})$ using (8.4) and the asymptotics of $\grave{\phi}_\omega$, so one has to choose $C_2(\omega)$ carefully. Furthermore, one wants to substitute in this integral a refined version of the expansion of $\grave{\phi}_\omega$ of Lemma 6.6. One has to be careful with the contour, because due to the perturbation there might appear an additional pole on or inside the unit circle. All in all, together with improved estimates it might be possible to obtain as leading irregularity $\omega^{2l+2}\log^2(2i\omega)$. At this point, there still remains the problem of matching these two expansions of the integral, i.e.

to choose one $C(\omega)$ such that both results hold or to consider the region, where $C_1(\omega)/|\omega|$ and $C_2(\omega)/\omega$ overlap. In particular, one has to be careful with the transformation $u \to r(u)$, which involves logarithms. But these logarithms should just change the phase of $\grave{\phi}$, which can be chosen arbitrary, which might also resolve this problem. As already mentioned, this should be regarded as an ansatz, where a lot of details had to be checked rigorously. This might be not easy but seems to be manageable at first view, and we would like to encourage the reader to go on with these ideas.

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