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Existence of Equilibrium in Models of International Trade with Perfect or Imperfect Competition

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Abstract

Dixit and Norman (1980) showed that, under certain conditions, the world economy replicates the equilibrium of the hypothetical integrated economy without national borders in the traditional trade model with perfect competition. Helpman and Krugman (1985) extended their analysis to settings with imperfectly competitive sectors. This paper derives necessary and sufficient conditions for the existence of free trade equilibria with replication in models with perfect or imperfect competition.
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1 Introduction

Dixit and Norman (1980) gave a new impetus to the theory of international trade by showing how to deal with standard trade models from a general equilibrium theory perspective. The crucial idea is to first consider the world economy as a hypothetical integrated economy with no obstacles to the movement of factors of production and then check the conditions under which the world economy with free trade but immobile factors of production replicates the integrated equilibrium.[1] In the second central contribution to this strand of the literature, Helpman and Krugman (1985) showed how this approach can also be exploited when one leaves the traditional theory of international trade with perfect competition and turns to models with scale economies and imperfect competition.[2]

The present paper is concerned with the question of existence of free trade equilibria which replicate the hypothetical integrated equilibrium in models with perfect or imperfect competition. Following the Dixit-Norman approach, this question can be split into two parts: First, does an integrated equilibrium exist? Second, is replication possible? Dixit and Norman (1980, pp. 67, 77, 81, 108) note that the standard existence proofs apply in models with perfect competition. Helpman and Krugman (1985) push questions of existence aside in their analysis of models with imperfect competition. We provide necessary and sufficient conditions for the existence of an equilibrium with replication both in the traditional perfect competition model and in models with monopolistic competition.

To do so, we make several simplifying assumptions, which are somewhat stronger than the assumptions conventionally made in general equilibrium theory but which are quite standard in the theory of international trade: production and utility functions are strictly quasi-concave, returns to scale are constant, all inputs are essential, and preferences are homothetic.

To begin with, we consider a standard model with perfect competition. We show that a simple proof of the existence of an integrated equilibrium via Brouwer's fixed point theorem is possible, even though, given constant returns to scale, the supply correspondences are not single-valued. The proof follows standard trade theory by making use of input coefficients and unit cost functions and is no more difficult than the the usual proof for exchange economies.[3]

[1]Travis (1964) is a precursor of Dixit and Norman (1980). Dixit and Norman (1980, p. 125) remark that Samuelson (1953) already "saw through the whole problem, and we think that if he had filled out some of the asides and terse remarks he makes, he would have developed the argument much as we have done here."

[2]Harris (1984) initiated the investigation of computable general equilibrium models of this type, surveyed by Kehoe and Kehoe (1994). Grossman and Helpman (1991) went one step further and used the Dixit-Norman (1980) methodology to analyze endogenous growth models. Arnold (2007) generalizes their analysis to a broad class of endogenous growth models.

[3]It seems not to be widely noticed how easy it is to integrate production under constant returns to scale. For instance,

The second, more important, consequence of our simplifying assumptions is that the results derived for the traditional trade model can be adapted to models with imperfect competition, as pioneered by Helpman and Krugman (1985).[4] Following Helpman and Krugman (1985), we assume that consumers have Dixit-Stiglitz (1977) preferences for differentiated goods and that producers of the different varieties are monopolistic competitors. We start with a model with given numbers of monopolists. The proof of existence of an integrated equilibrium is straightforward: essentially, what has to be done is replace unit cost functions with markup-pricing rules and take account of pure profits in the income identities. Then we allow for free entry into the monopolistically competitive sectors. As the "numbers" of varieties produced are then endogenous, the construction of the mapping whose fixed point gives the integrated equilibrium has to be adapted.

Section 2 presents the perfectly competitive baseline model. Section 3 introduces imperfect competition. Free entry is introduced in Section 4. Section 5 concludes.

2 Perfect competition

2.1 Model

There are I consumers (distinguished by a superscript $i = 1, \ldots, I$), J firms (distinguished by a superscript $j = 1, \ldots, J$), K goods (indexed $k = 1, \ldots, K$), and L factors of production (indexed $l = 1, \ldots, L$).

Let x^j_{kl} denote the input of factor l in the production of good k in firm j and $\mathbf{x}^j_k = (x^j_{k1}, \ldots, x^j_{kL}) \in \mathbb{R}^L_+$. The production of each good k obeys a production function $f_k(\mathbf{x}^j_k) : \mathbb{R}^L_+ \to \mathbb{R}_+$ that is accessible to all firms j.[5]

Assumption 1: *For all k, f_k is continuous, strictly quasi-concave, linearly homogeneous (constant-returns-to-scale), and strictly increasing, and each factor l is essential.*[6]

Let $p_k \in \mathbb{R}_+$ denote the price of good k and $\mathbf{w} = (w_1, \ldots, w_L) \in \mathbb{R}^L_+$ the vector of factor prices. Firm j's profit is $\pi^j = \sum_{k=1}^K [p_k f_k(\mathbf{x}^j_k) - \mathbf{w}\mathbf{x}^j_k]$.

Consumers own the factors of production and shares in the firms. They use their (factor and profit) income to buy consumption goods. Let $\mathbf{x}^i = (x^i_1, \ldots, x^i_L) \in \mathbb{R}^L_+$ denote i's factor endowments. $\mathbf{x}^i$ is

Dixit and Norman (1980, p. 77) refer the reader to chapter 5 in Arrow and Hahn (1971), which allows for multi-valued supply and demand correspondences. For our purposes, the material in the "first approach" in chapter 2 of Arrow and Hahn (1971) is sufficient.

[4] Existence of equilibrium in models with monopolistic competition was pioneered by Negishi (1961).

[5] We write $\mathbb{R}^L_+ = \{\mathbf{x} | \, \mathbf{x} \in \mathbb{R}^L, x_l \geq 0 \text{ for all } l = 1, \ldots L\}$ and $\mathbb{R}^L_{++} = \{\mathbf{x} | \, \mathbf{x} \in \mathbb{R}^L, x_l > 0 \text{ for all } l = 1, \ldots L\}$.

[6] I.e., $f_k(\mathbf{x}^j_k) = 0$ if $\mathbf{x}^j_{kl} = 0$ for any l.

exogenous for all i ($\mathbf{x} = \sum_{i=1} \mathbf{x}^i \in \mathbb{R}^L_{++}$). For all i and j, let $\theta^{ij} \in \mathbb{R}_+$ denote consumer i's share in firm j ($\sum_{i=1}^I \theta^{ij} = 1$ for all j). i's income is $I^i = \mathbf{w}\mathbf{x}^i + \boldsymbol{\theta}^i \boldsymbol{\pi} \in \mathbb{R}_+$, where $\boldsymbol{\theta}^i = (\theta^{i1}, \ldots, \theta^{iJ}) \in \mathbb{R}^J_+$ and $\boldsymbol{\pi} = (\pi^1, \ldots, \pi^J) \in \mathbb{R}^J_+$. Let $\mathbf{y}^i = (y^i_1, \ldots, y^i_K) \in \mathbb{R}^K_+$ denote a consumption bundle for consumer i, where y^i_k is his consumption of good k. Given a vector of commodity prices, $\mathbf{p} = (p_1, \ldots, p_K) \in \mathbb{R}^K_+$, his budget set is $\mathcal{B}^i = \{\mathbf{y}^i | \, \mathbf{y}^i \in \mathbb{R}^K_+, \mathbf{p}\mathbf{y}^i \leq I^i\}$. All consumers have identical preferences represented by a common utility function $u : \mathbb{R}^K_+ \to \mathbb{R}$, which gives i's utility obtained from $\mathbf{y}^i_k \in \mathbb{R}^K_+$. u is normalized such that $u(\mathbf{0}) = 0$.

Assumption 2: *u is continuous, strictly quasi-concave, homothetic, and strictly increasing.*

An equilibrium prevails if firms maximize profit, consumers maximize utility, and the markets for all goods and factors of production clear:

Definition: *A price system* $(\mathbf{w}, \mathbf{p})$ *and an allocation* $(\{\{\mathbf{x}^j_k\}^K_{k=1}\}^J_{j=1}, \{\mathbf{y}^i\}^I_{i=1})$ *constitute an* integrated equilibrium *if*

(I) $\{\mathbf{x}^j_k\}^K_{k=1}$ *maximizes profit for all* j,

(II) $\mathbf{y}^i$ *maximizes utility in* $\mathcal{B}^i$ *for all* i,

(III) $\sum_{j=1}^J f_k(\mathbf{x}^j_k) = \sum_{i=1}^I y^i_k$ *for all* k,

(IV) $\sum_{k=1}^K \sum_{j=1}^J \mathbf{x}^j_k = \sum_{i=1}^I \mathbf{x}^i$.

This is a simplified Arrow-Debreu economy in which the quasi-concavity of production and utility functions is strict, returns to scale are constant, preferences are identical and homothetic, and factor supplies are exogenous. Later on, we will turn to the open economy version of this model by assuming the presence of national borders which inhibit factor movements but not flows of goods.[7] This model is a good starting point for our analysis because (it seems fair to say) it is a standard trade model, it is easy to analyze, and we can make frequent use of the results when it comes to models with imperfect competition.

2.2 Integrated equilibrium

Because of constant returns to scale, firms j minimize the unit cost of Production for any good k. That is, for all j and k, $\mathbf{x}^j_k$ solves $\min_{\mathbf{x}^j_k} : \mathbf{w}\mathbf{x}^j_k$ s.t.: $f_k(\mathbf{x}^j_k) = 1$. Denote the solution to this problem as $\mathbf{a}_k(\mathbf{w}) : \mathbb{R}^L_+ \to \mathbb{R}^L_+$ and the ensuing unit cost, $\mathbf{w}\mathbf{a}_k(\mathbf{w})$, as $c_k(\mathbf{w}) : \mathbb{R}^L_+ \to \mathbb{R}_+$. Because of strict quasi-concavity of f_k, if a minimum exists, it is unique, so that $\mathbf{a}_k(\mathbf{w})$ is single-valued. The assumption of

[7] This follows Dixit and Norman (1980, pp. 26-7). Their analysis is more general in two respects: they allow for diminishing returns and weak quasi-concavity in production (cf. Dixit and Norman, 1980, pp. 31, 66).

constant returns to scale has two further important implications. First, profit maximization implies $\mathbf{p} = \mathbf{c}(\mathbf{w})$, where $\mathbf{c}(\mathbf{w}) = (c_1(\mathbf{w}), \ldots, c_L(\mathbf{w})) : \mathbb{R}^L_+ \to \mathbb{R}^L_{++}$, and each firm j makes zero profit: $\boldsymbol{\pi} = \mathbf{0}$. Second, $\mathbf{a}_k(\mathbf{w})$ and $c_k(\mathbf{w})$ are homogeneous of degree zero and one, respectively.[8]

Consumer i solves $\max_{\mathbf{y}^i} u(\mathbf{y}^i)$ s.t.: $\mathbf{y}^i \in \mathcal{B}^i$. Assumption 2 ensures that the resulting demand curves are well-behaved: suppose a solution $\mathbf{y}(\mathbf{p}, I^i) : \mathbb{R}^{K+1}_+ \to \mathbb{R}^K_+$ to the utility maximization problem exists; then, strict quasi-concavity of u implies that $\mathbf{y}(\mathbf{p}, I^i)$ is single-valued; continuity of u implies continuity of $\mathbf{y}(\mathbf{p}, I^i)$; and the homotheticity of u implies a unitary income elasticity everywhere, so that we can write $\mathbf{y}(\mathbf{p}, I^i) = \mathbf{d}(\mathbf{p})I^i$, where $\mathbf{d} : \mathbb{R}^K_+ \to \mathbb{R}^K_+$ is single-valued, continuous, and homogeneous of degree minus one. The aggregate demand for goods is $\mathbf{y} = \mathbf{d}(\mathbf{p})I$, where $I = \sum_{i=1}^I I^i$ is aggregate income.

Because of zero profit, income equals factor income: $I^i = \mathbf{w}\mathbf{x}^i$ and $I = \mathbf{w}\mathbf{x}$. Using $\mathbf{p} = \mathbf{c}(\mathbf{w})$, the aggregate demand for goods is $\mathbf{y} = \mathbf{d}(\mathbf{c}(\mathbf{w}))\mathbf{w}\mathbf{x}$. The demand for factor l by producers of good k is $\sum_{j=1}^J x^j_{kl} = \sum_{j=1}^J a_{kl}(\mathbf{w})y^j_k = a_{kl}(\mathbf{w})y_k$, and the total demand for factor l is $\sum_{k=1}^K \sum_{j=1}^J x^j_{kl} = \sum_{k=1}^K a_{kl}(\mathbf{w})y_k$. Using the demands for goods and factors, define the excess factor demand function $\mathbf{z}(\mathbf{w}) : \mathbb{R}^L_+ \to \mathbb{R}^L$ by

$$z_l(\mathbf{w}) = \sum_{k=1}^K a_{kl}(\mathbf{w})d_k(\mathbf{c}(\mathbf{w}))\mathbf{w}\mathbf{x} - x_l, \quad l = 1, \ldots, L.$$

Theorem 2.1: *An integrated equilibrium exists.*

Proof: Consider the following price system $(\mathbf{w}^*, \mathbf{p}^*)$ and allocation $(\{\{\mathbf{x}^{j*}_k\}_{k=1}^K\}_{j=1}^J, \{\mathbf{y}^{i*}\}_{i=1}^I)$. The factor prices $\mathbf{w}^*$ are such that $\mathbf{z}(\mathbf{w}^*) = \mathbf{0}$; commodity prices satisfy $\mathbf{p}^* = \mathbf{c}(\mathbf{w}^*)$; the firms' factor inputs $\mathbf{x}^{j*}_k$ satisfy $\sum_{j=1}^J \mathbf{x}^{j*}_k = \mathbf{a}_k(\mathbf{w}^*)d_k(\mathbf{c}(\mathbf{w}^*))\mathbf{w}^*\mathbf{x}$ and $\mathbf{x}^{j*}_k \geq \mathbf{0}$ for all j and k;[9] and the consumption vectors are $\mathbf{y}^{i*} = \mathbf{d}(\mathbf{c}(\mathbf{w}^*))\mathbf{w}^*\mathbf{x}^i$ for all i. This price system and allocation satisfy the definition of an integrated equilibrium. The mappings $\mathbf{c}(\mathbf{w})$, $\mathbf{a}_k(\mathbf{w})$, and $\mathbf{d}(\mathbf{p})$ are well-defined, single-valued, and continuous provided that the domains of the cost minimization and utility maximization problems are compact.[10] So proving the existence of an integrated equilibrium boils down to (1) dealing with compactness problems and (2) proving the existence of $\mathbf{w}^*$ such that $\mathbf{z}(\mathbf{w}^*) = \mathbf{0}$.

(1) Let $\bar{\mathcal{X}} = \{\mathbf{x}^j_k | \mathbf{x}^j_k \in \mathbb{R}^L_+, \mathbf{x}^j_k \leq 2\mathbf{x}\}$ and $\bar{\mathcal{Y}} = \{\mathbf{y}^i | \mathbf{y}^i \in \mathbb{R}^K_+, y^i_k \leq 2f_k(\mathbf{x}), k = 1, \ldots, K\}$. $\bar{\mathcal{X}}$ and $\mathcal{B}_i \cap \bar{\mathcal{Y}}$ are compact. Input vectors $\mathbf{x}^j_k \notin \bar{\mathcal{X}}$ or consumption bundles $\mathbf{y}^i \notin \bar{\mathcal{Y}}$ are not feasible. A price system $(\mathbf{w}, \mathbf{p})$ and an allocation $(\{\{\mathbf{x}^j_k\}_{k=1}^K\}_{j=1}^J, \{\mathbf{y}^i\}_{i=1}^I)$ are an *integrated equilibrium on restricted domains* if they satisfy (I)-(IV) in the definition of an integrated equilibrium when $\mathbf{x}^j_k \in \bar{\mathcal{X}}$ is added in (I) and $\mathcal{B}^i$ is replaced with $\mathcal{B}_i \cap \bar{\mathcal{Y}}$ in (II). It is readily shown that a price system and an allocation are an

[8]These results hold true in particular when there is only one factor of production $l = 1$, in which case $a_{k1}(\mathbf{w}) = a_{k1}$ is exogenous and $c_k(\mathbf{w}) = w_1 a_{k1}$ for all k.

[9]Needless to say that, because of constant returns to scale, the division of $\sum_{j=1}^J \mathbf{x}^{j*}_k$ across firms j is indeterminate.

[10]Notice that the firms' supply correspondences are multi-valued, but we do not make use of them.

equilibrium if they are an equilibrium on restricted domains (see the Appendix). This allows us to focus on the latter, so that the domains in the optimization problems are compact and the functions $\mathbf{c}(\mathbf{w})$, $\mathbf{a}_k(\mathbf{w})$, and $\mathbf{d}(\mathbf{p})$ are well-defined.

(2) Consider the Gale-Nikaido mapping $\mathbf{g}(\mathbf{w})$ defined by

$$g_l(\mathbf{w}) = \frac{w_l + \max\{z_l(\mathbf{w}), 0\}}{\sum_{l'=1}^{L} [w_{l'} + \max\{z_{l'}(\mathbf{w}), 0\}]}, \quad l = 1, \ldots, L.$$

Since $\mathbf{c}(\mathbf{w})$, $\mathbf{a}_k(\mathbf{w})$, and $\mathbf{d}(\mathbf{p})$ are homogeneous of degree one, zero, and minus one, respectively, $\mathbf{z}(\mathbf{w})$ and $\mathbf{g}(\mathbf{w})$ are homogeneous of degree zero, and we can restrict the domain to the $L-1$-dimensional unit simplex $\Delta = \{\mathbf{w} | \, \mathbf{w} \in \mathbb{R}_+^L, \sum_{l=1}^{L} w_l = 1\}$. Clearly, $\mathbf{g}(\mathbf{w}) \in \mathbb{R}_+^L$ and $\sum_{l=1}^{L} g_l(\mathbf{w}) = 1$ for all $\mathbf{w}$, so that $\mathbf{g}(\mathbf{w}) : \Delta \to \Delta$ maps the unit simplex on itself. Since $\mathbf{g}(\mathbf{w})$ is continuous, Brouwer's theorem implies that a fixed point $\mathbf{w}^*$ exists.

The fixed point $\mathbf{w}^*$ of $\mathbf{g}(\mathbf{w})$ satisfies $\mathbf{z}(\mathbf{w}^*) = \mathbf{0}$. To see this, notice that since, by Assumption 2, $u(\mathbf{y}^i)$ is strictly increasing, the budget constraint $\mathbf{p}\mathbf{y}^i \leq I^i$ holds with equality for all i. Using $\mathbf{y}^i = \mathbf{d}(\mathbf{p})I^i$, it follows that $\mathbf{p}\mathbf{d}(\mathbf{p}) = 1$. Using $\sum_{l=1}^{L} w_l a_{kl}(\mathbf{w}) = c_k(\mathbf{w}) = p_k$, we can prove Walras' law: $\sum_{l=1}^{L} w_l z_l(\mathbf{w}) = 0$ for all $\mathbf{w}$ (see the Appendix). The validity of this equality for $\mathbf{w}^*$ implies $z_l(\mathbf{w}^*) = 0$ for $l = 1, \ldots, L$ (see the Appendix). ||

Remark 2.1.1: Some parts of Assumptions 1 and 2 are made solely for convenience. For example, the proof of Theorem 2.1 goes through without modification if only a subset of the factors l are used in the production each good k. Likewise, the homotheticity of u is inessential for our arguments. Generally, the individuals' demand functions $\mathbf{y}^i(\mathbf{p}, I^i)$ are continuous and homogeneous of degree zero, so the same holds true for the factor excess demand functions $z_l(\mathbf{w}) = \sum_{k=1}^{K} a_{kl}(\mathbf{w}) \sum_{i=1}^{I} \mathbf{y}^i(\mathbf{c}(\mathbf{w}), \mathbf{w}\mathbf{x}^i) - x_l$ ($l = 1, \ldots, L$), and the analysis goes through without substantial modifications. Finally, maintaining the assumption of constant returns to scale, one can dispense with the assumption that the quasi-concavity of production and utility functions is strict. The mappings $a_{kl}(\mathbf{w})$ and $d_k(\mathbf{p})$ are, then, possibly multi-valued but certainly convex-valued, and the proof that the Gale-Nikaido mapping has a fixed point goes via Kakutani's theorem.

2.3 Free trade equilibrium

Suppose the world economy described in Subsection 2.1 is divided into M countries, which are distinguished by a superscript m $(= 1, \ldots, M)$. Each consumer i lives in exactly one country m. The set of consumers living in country m is denoted $\mathcal{I}^m$ $(m = 1, \ldots, M)$. Similarly, $\mathcal{J}^m$ is the set of all firms j operating in country m $(m = 1, \ldots, M)$, and each firm j belongs to one of these sets. There is free international trade in goods, factors of production are immobile internationally.

Definition: *A price system* $(\mathbf{w}, \mathbf{p})$ *and an allocation* $(\{\{\mathbf{x}_k^j\}_{k=1}^K\}_{j=1}^J, \{\mathbf{y}^i\}_{i=1}^I)$ *constitute a* free trade equilibrium with replication *if (I)-(III) in the definition of an integrated equilibrium hold and*

(IV') $\sum_{k=1}^K \sum_{j\in\mathcal{J}^m} \mathbf{x}_k^j = \sum_{i\in\mathcal{I}^m} \mathbf{x}^i$ *for all* m.

That is, a free trade equilibrium prevails if firms maximize profit, consumers maximize utility, the world markets for goods clear, and each country's factor markets clear.

Theorem 2.2: *Let* $(\mathbf{w}, \mathbf{p})$ *and* $(\{\{\hat{\mathbf{x}}_k^j\}_{k=1}^K\}_{j=1}^J, \{\mathbf{y}^i\}_{i=1}^I)$ *be an integrated equilibrium. Then,* $(\mathbf{w}, \mathbf{p}, \{\{\mathbf{x}_k^j\}_{k=1}^K\}_{j=1}^J, \{\mathbf{y}^i\}_{i=1}^I)$ *is a free trade equilibrium with replication if, and only if,*

(V) $\mathbf{x}_k^j = \lambda_k^j \sum_{j'=1}^J \hat{\mathbf{x}}_k^{j'}$ *for some* $\lambda_k^j \in \mathbb{R}_+$ *and for all* j *and* k*,*

(VI) $\sum_{m=1}^M \sum_{j\in\mathcal{J}^m} \mathbf{x}_k^j = \sum_{j'=1}^J \hat{\mathbf{x}}_k^{j'}$ *for all* k*,*

and (IV') holds.

Proof: Suppose (IV'), (V), and (VI) hold. Since the factor prices $\mathbf{w}$ are the same as in the integrated equilibrium, the input coefficients $\mathbf{a}_k(\mathbf{w})$ are identical, so (V) implies cost minimization ((I) is valid). All firms make zero profit. According to (VI), the world supply of each good j is equal to the respective integrated equilibrium quantity. Since factor and goods prices are the same as in the integrated equilibrium and consumers do not obtain any positive profit income, $\mathbf{y}^i$ maximizes utility ((II) is valid). So the world demands are also equal to the integrated equilibrium demands, and market clearing in the integrated equilibrium implies that the world commodities markets clear ((III) is valid). Given that (IV') holds by assumption, this proves the "if part" of the theorem.
As for the "only if part", if (V) is violated for some j and k, then firm j does not minimize the cost of producing k ((I) is violated). Given the aggregate demand for goods $\sum_{i=1}^I \mathbf{y}^i$, if (VI) does not hold for some k, then the world market for good k does not clear ((III) is violated). Finally, a free trade equilibrium cannot exist unless (IV') holds, as this is part of its definition. ||

Put briefly, Theorem 2.2 says that the replication of the integrated equilibrium as a free trade equilibrium is feasible if, and only if, it is possible to split up all world input vectors exhaustively into non-negative portions allocated to the individual countries in such a way that each country's factor markets clear. Importantly for our purposes, the "if part" of Theorem 2.2 establishes an existence result for M-country open economies: given that an integrated equilibrium exists (Theorem 2.1), the theorem gives a set of conditions sufficient for the existence of a free trade equilibrium. Figure 1 illustrates the standard example with $K = L = M = 2$. A free trade equilibrium with replication exists if the endowment point is located inside the parallelogram formed by the integrated equilibrium input vectors.

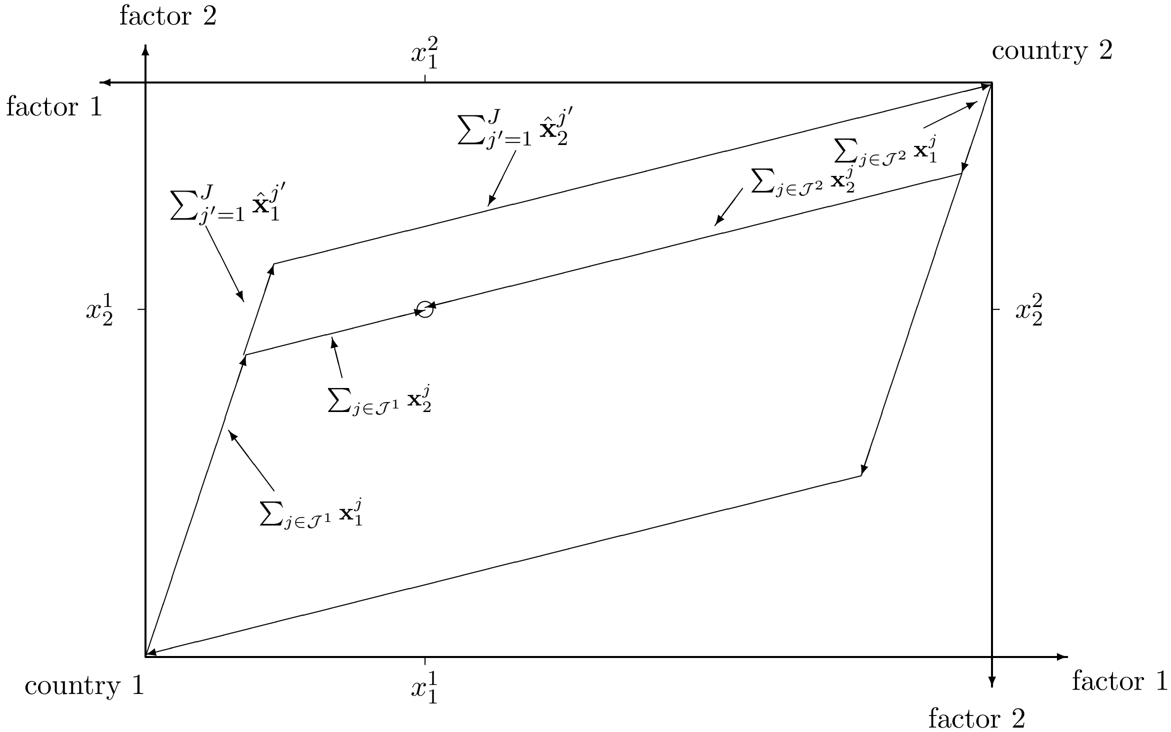

Figure 1: Perfect competition

Remark 2.2.1: The homotheticity assumption is inessential for the validity of Theorem 2.2 (cf. Remark 2.1.1).

Remark 2.2.2 (sufficient conditions for replication): A sufficient condition for the existence of a free trade equilibrium with replication is uniformity of the relative factor endowments across countries: if $\sum_{i\in\mathcal{I}^m} \mathbf{x}^i = \mu^m \sum_{i=1}^{I} \mathbf{x}^i$ for some $(\mu^1, \ldots, \mu^M) \in \mathbb{R}_+^M$, then $\lambda_k^m = \mu^m$ yields replication, where $\lambda_k^m = \sum_{j\in\mathcal{J}^m} \lambda_k^j$. If $K = L = M = 2$, this is the case in which the endowment point is located on the diagonal (cf. Dixit-Norman, 1980, p. 110). The condition is trivially satisfied if $L = 1$.

Remark 2.2.3 (number of goods and factors of production): Conditions (IV') and (V) for the feasibility of replication can be rewritten as

$$\sum_{k=1}^{K} \lambda_k^m \sum_{j'=1}^{J} \hat{\mathbf{x}}_k^{j'} = \sum_{i\in\mathcal{I}^m} \mathbf{x}^i.$$

This is a system of L equations in the K unknowns λ_k^m. So a free trade equilibrium with replication is "unlikely" to exist if the number of factors of production L exceeds the number of commodities K. For instance, if $K = 1$, $L = 2$, and $M = 2$, replication is feasible only if the conditions of Remark 2.2.2 is satisfied (i.e., if the endowment point is located on the diagonal, which coincides with the single integrated equilibrium input vector). If $K \geq L$, the more "dissimilar" the integrated equilibrium input vectors $\sum_{j'=1}^{J} \hat{\mathbf{x}}_k^{j'}$ for the different goods k and the more "similar" the countries' endowment vectors, the more "likely" is replication (cf. Dixit-Norman, 1980, pp. 111 ff.).

Remark 2.2.4: As the integrated equilibrium input vectors $\sum_{j'=1}^{J} \hat{\mathbf{x}}_k^{j'}$ are endogenous, Theorem 2.2 gives conditions for the feasibility of replication in terms of endogenous variables.

Remark 2.2.5 (non-traded goods): It is easy to add non-traded goods to the analysis. Letting $\mathcal{K}^n$ denote the subset of non-traded goods k, all one has to do is add the condition $\sum_{j\in\mathcal{J}^m} f_k(\mathbf{x}_k^j) = \sum_{i\in\mathcal{I}^m} y_k^i$ for all $k \in \mathcal{K}^n$ to the definition of a free trade equilibrium and to the conditions in Theorem 2.2. The case $L = M = 2$ with two traded goods can be illustrated with a box whose length and height are the endowments of factors 1 and 2, respectively, *net of the resources needed to produce a fraction* $(\mathbf{wx}^i)/(\mathbf{wx})$ *of the integrated equilibrium quantities of the non-traded goods.* A free trade equilibrium with replication exists if the net factor endowments point is located inside the parallelogram spanned by the integrated equilibrium input vectors for the two traded goods.

The standard results of traditional international trade theory are straightforward corollaries to the analysis conducted so far.

Remark 2.2.6 (balanced trade): That trade is balanced for each country follows from zero profit and the consumers' budget constraints: for each country, both the aggregate value of production and consumption expenditures are equal to aggregate factor incomes.

Remark 2.2.7 (gains from trade): Let $\tilde{\mathbf{y}}^m$ and $\tilde{\mathbf{w}}^m$ denote country m's autarky consumption vector (aggregated over all $i \in \mathcal{I}^m$) and factor price vector, respectively. From the fact that, at factor prices in a free trade equilibrium, production is no cheaper using the autarky input coefficients $\mathbf{a}_k(\tilde{\mathbf{w}})$ rather than the input coefficients $\mathbf{a}_k(\mathbf{w})$, it follows that the economy can afford its autarky consumption bundle $\tilde{\mathbf{y}}^m$ in a free trade equilibrium:

$$\mathbf{p}\tilde{\mathbf{y}}^m = \sum_{k=1}^{K} p_k \tilde{y}_k^m = \sum_{k=1}^{K} \mathbf{wa}_k(\mathbf{w})\tilde{y}_k^m \leq \sum_{k=1}^{K} \mathbf{wa}_k(\tilde{\mathbf{w}})\tilde{y}_k^m = \mathbf{w}\sum_{k=1}^{K} \mathbf{a}_k(\tilde{\mathbf{w}})\tilde{y}_k^m = \mathbf{wx}^m.$$

Consequently, each inhabitant of m gains from trade if income is redistributed such that the individual shares in aggregate income remain constant when trade is opened up (i.e., if $(\mathbf{wx}^i)/(\mathbf{wx}^m) = (\tilde{\mathbf{w}}\mathbf{x}^i)/(\tilde{\mathbf{w}}\mathbf{x}^m)$ for all $i \in \mathcal{I}^m$).

Remark 2.2.8 (comparative advantage): Let $M = 2$. Denote country m's autarky price vector as $\tilde{\mathbf{p}}^m$ and its net import vector as $\mathbf{z}^m \in \mathbb{R}^K$. Affordability implies $\tilde{\mathbf{p}}^m\mathbf{z}^m \geq 0$. Adding up and using $\mathbf{z}^1 + \mathbf{z}^2 = \mathbf{0}$ yields $(\tilde{\mathbf{p}}^1 + \tilde{\mathbf{p}}^2)\mathbf{z}^1 \geq 0$. That is, on average country 1 is a net importer $(z_k^1 > 0)$ of those goods which are relatively expensive in autarky $(\tilde{p}_k^1 > \tilde{p}_k^2)$ (see Dixit and Norman, 1980, pp. 94-5).

Remark 2.2.9 (trade pattern): The homotheticity assumption (cf. Remark 2.2.1) *is* essential if one wants to say more about the factor content of trade than Remark 2.2.8 does. The factor-l content of country m's production is x_l^m. Given homotheticity, the factor-l content of country m's consumption is equal to its share in world income $(\mathbf{wx}^m)/(\mathbf{wx})$ times x_l. So the factor-l content of country m's net exports is

$$x_l \left(\frac{x_l^m}{x_l} - \frac{\mathbf{wx}^m}{\mathbf{wx}} \right)$$

for all l. In a free trade equilibrium, this expression is positive for some l and negative for others. Evidently, it is positive for those factors l country m is relatively richly endowed with (i.e., for which x_l^m/x_l is high).

3 Imperfect competition

3.1 Model

We now introduce imperfectly competitive markets into the baseline model of Section 2, following the Dixit-Stiglitz (1977) approach popularized in trade theory by Krugman (1979a, 1980) and Helpman and Krugman (1985, Chapters 6 and 7). In consumer i's consumption vector $\mathbf{y}^i \in \mathbb{R}_+^K$, the first N components $(y_1^i, \ldots, y_N^i)$ are now composites of variants of N differentiated goods $k \in \{1, \ldots, N\} = \mathcal{K}^d$. For $k \in \{N+1, \ldots, K\} = \mathcal{K}^h$, as before, y_k^i denotes the consumption of a homogeneous good. y_k^i is given by $y_k^i = [\int_0^{h_k} (z_{kh}^i)^{\alpha_k} dh]^{1/\alpha_k}$ for $k \in \mathcal{K}^d$, where $z_{kh}^i \in \mathbb{R}_+$ is consumer i's consumption of variant h of good k, $h_k \in \mathbb{R}_{++}$ is the exogenously given "number" of variants of good k, and $0 < \alpha_k < 1$ for all $k \in \mathcal{K}^d$. In the following section, h_k is endogenized using free entry into the differentiated goods sector. We use the continuum-of-variants formulation in order to avoid problems of integer numbers then. Each variant h of a good k is produced from inputs $\mathbf{x}_{kh} \in \mathbb{R}_+^L$ using the same production function $f_k : \mathbb{R}_+^L \to \mathbb{R}_+$, which satisfies the requirements of Assumption 1.[11] There is a single, monopolistic, producer of each variant h of any good k, who chooses the price p_{kh} of his variant so as to maximize profit π_{kh} (Chamberlinian monopolistic competition). Letting $\theta_{kh}^i \in \mathbb{R}_+$ denote i's share in the firm producing variant h of good k (where $\sum_{i=1}^I \theta_{kh}^i = 1$ for all $k \in \mathcal{K}^d$ and all h), i's income is $I^i = \mathbf{wx}^i + \sum_{j=1}^J \theta^{ij}\pi^j + \sum_{k\in\mathcal{K}^d} \int_0^{h_k} \theta_{kh}^i \pi_{kh} dh$, and his budget set is $\mathcal{B}^i = \{[z_{10}^i, z_{1h_1}^i], \ldots, [z_{N0}^i, z_{Nh_N}^i], (y_{N+1}^i, \ldots, y_K^i), |\, z_{hk}^i \in \mathbb{R}_+, h \in [0, h_k], k = 1, \ldots, N, y_k^i \in \mathbb{R}_+, k = N+1, \ldots, K, \sum_{k\in\mathcal{K}^d} \int_0^{h_k} p_{kh} z_{kh}^i dh + \sum_{k\in\mathcal{K}^h} p_k y_k^i \le I^i\}$.

Definition: *A price system* $(\mathbf{w}, \{(p_{kh})_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{p_k\}_{k\in\mathcal{K}^h})$ *and an allocation* $(\{(\mathbf{x}_{kh})_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{\{\mathbf{x}_k^j\}_{k\in\mathcal{K}^h}\}_{j=1}^J, \{\{(z_{kh}^i)_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{y_k^i\}_{k\in\mathcal{K}^h}\}_{i=1}^I)$ *constitute an* integrated equilibrium *if*

[11] As the Dixit-Stiglitz index is strictly quasi-concave, the analysis in Section 2 applies if one sticks to the assumption of perfect competition.

(I) p_{kh} *and* $\mathbf{x}_{kh}$ *maximize profit for all* $k \in \mathcal{K}^d$ *and* h*, and* $\{\mathbf{x}_k^j\}_{k\in\mathcal{K}^h}$ *maximizes profit for all* j*,*

(II) $\{\{(z_{kh}^i)_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{y_k^i\}_{k\in\mathcal{K}^h}\}$ *maximizes utility in* $\mathcal{B}^i$ *for all* i*,*

(III) $f_k(\mathbf{x}_{kh}) = \sum_{i=1}^I z_{kh}^i$ *for all* $k \in \mathcal{K}^d$ *and* h*, and* $\sum_{j=1}^J f_k(\mathbf{x}_k^j) = \sum_{i=1}^I y_k^i$ *for all* $k \in \mathcal{K}^h$*,*

(IV) $\sum_{k\in\mathcal{K}^d} \int_0^{h_k} \mathbf{x}_{kh} dh + \sum_{k\in\mathcal{K}^h} \sum_{j=1}^J \mathbf{x}_k^j = \sum_{i=1}^I \mathbf{x}^i$.

3.2 Integrated equilibrium

In this subsection, we characterize the integrated equilibrium of the model described in Subsection 3.1. To begin with, we characterize the consumers' demands for the variants of a differentiated good for given expenditure on this goods and the resulting pricing behavior of the firms which supply the variants of this good.

Let I_k^i denote income spent on all the variants h of good $k \in \mathcal{K}^d$ by consumer i. A necessary condition for utility maximization is that $(z_{kh}^i)_{h\in[0,h_k]}$ solves $\max_{(z_{kh}^i)_{h\in[0,h_k]}}$: $[\int_h^{h_k} (z_{kh}^i)^{\alpha_k} dh]^{1/\alpha_k}$ s.t.: $\int_0^{h_k} p_{kh} z_{kh}^i dh = I_k^i$ given I_k^i. Solving this maximization problem yields the demand functions

$$z_{kh}^i = \left[\frac{p_{kh}}{\left(\int_0^{h_k} p_{kh'}^{-\frac{\alpha_k}{1-\alpha_k}} dh' \right)^{-\frac{1-\alpha_k}{\alpha_k}}} \right]^{-\frac{1}{1-\alpha_k}} \frac{I_k^i}{\left(\int_0^{h_k} p_{kh'}^{-\frac{\alpha_k}{1-\alpha_k}} dh' \right)^{-\frac{1-\alpha_k}{\alpha_k}}}.$$

The demands are homogeneous of degree one, so that the aggregate demand $z_{kh} = \sum_{i=1}^I z_{kh}^i$ is obtained by replacing I_k^i with $I_k = \sum_{i=1}^I I_k^i$. Profit maximization yields the constant-markup price $p_{kh} = c_k(\mathbf{w})/\alpha_k$ for all h, where $c_k(\mathbf{w})$ is the minimum unit cost of production. This follows from standard concave programming arguments. Because of uniform prices of the variants h of good k and diminishing marginal utility of each variant, the cheapest way to achieve $y_k^i = 1$ is to choose z_{kh}^i uniform for all variants $h \in [0, h_k]$, i.e., $z_{kh}^i = h_k^{-1/\alpha_k}$. The ensuing cost of $y_k^i = 1$ is $p_k = h_k p_{kh} z_{kh}^i = h_k^{-(1-\alpha_k)/\alpha_k} p_{kh}$. Letting $\mathbf{p} = (p_1, \ldots, p_K) \in \mathbb{R}_+^K$, consumer i's utility maximization problem, given the optimal allocation of I_k^i across the variants h of k, becomes $\max_{\mathbf{y}^i} u(\mathbf{y}^i)$ s.t.: $\mathbf{y}^i \in \mathcal{B}^i$, where $\mathcal{B}^i = \{\mathbf{y}^i | \, \mathbf{y}^i \in \mathbb{R}_+^K, \mathbf{p}\mathbf{y}^i \leq I^i\}$. This is the same problem as in Section 2, and the solution can be expressed as $\mathbf{y}^i = \mathbf{d}(\mathbf{p}) I^i$, where $\mathbf{d} : \mathbb{R}_+^K \to \mathbb{R}_+^K$ is single-valued, continuous, and homogeneous of degree minus one. Aggregating over all consumers i yields the demand function $\mathbf{y} = \mathbf{d}(\mathbf{p}) I$. The aggregate demand for each variant h of good k is $z_{kh} = \sum_{i=1}^I z_{kh}^i = \sum_{i=1}^I h_k^{-1/\alpha_k} y_k^i = h_k^{-1/\alpha_k} d_k(\mathbf{p}) I$.

By the same reasoning as in Section 2, profit maximization entails $p_k = c_k(\mathbf{w})$ for all $k \in \mathcal{K}^h$ and $\sum_{k\in\mathcal{K}^h} \sum_{j=1}^J x_{kl}^j = \sum_{k\in\mathcal{K}^h} a_{kl}(\mathbf{w}) y_k = \sum_{k\in\mathcal{K}^h} a_{kl}(\mathbf{w}) d_k(\mathbf{p}) I$. The producer of variant h of a differentiated good $k \in \mathcal{K}^d$ demands $x_{khl} = a_{kl}(\mathbf{w})z_{kh} = a_{kl}(\mathbf{w})h_k^{-1/\alpha_k}d_k(\mathbf{p})I$ units of factor l, where $a_{kl}(\mathbf{w})$ is the input coefficient resulting from minimizing the unit cost of producing a variant of differentiated product k. Aggregating over all variants h of k yields $x_{kl} = a_{kl}(\mathbf{w})h_k^{-(1-\alpha_k)/\alpha_k}d_k(\mathbf{p})I$. Finally, aggregating over all k yields the total demand for factor l:

$$\sum_{k\in\mathcal{K}^d} a_{kl}(\mathbf{w})h_k^{-\frac{1-\alpha_k}{\alpha_k}}d_k(\mathbf{p})I + \sum_{k\in\mathcal{K}^h} a_{kl}(\mathbf{w})d_k(\mathbf{p})I.$$

Producers of homogeneous commodities make zero profit: $\pi^j = 0$ for all j. The producer of variant h of differentiated product k makes profit $\pi_{kh} = [p_{kh} - c_k(\mathbf{w})]z_{kh} = [(1-\alpha_k)/\alpha_k]c_k(\mathbf{w})h_k^{-1/\alpha_k}d_k(\mathbf{p})I$. Aggregate profit is $\sum_{k\in\mathcal{K}^d}\int_0^{h_k}\pi_{kh}dh = I\sum_{k\in\mathcal{K}^d}[(1-\alpha_k)/\alpha_k]c_k(\mathbf{w})h_k^{-(1-\alpha_k)/\alpha_k}d_k(\mathbf{p})$. From $I = \mathbf{wx} + \sum_{k\in\mathcal{K}^d}\int_0^{h_k}\pi_{kh}dh$, we get aggregate income:

$$I = \frac{\mathbf{wx}}{1 - \sum_{k\in\mathcal{K}^d}\frac{1-\alpha_k}{\alpha_k}c_k(\mathbf{w})h_k^{-\frac{1-\alpha_k}{\alpha_k}}d_k(\mathbf{p})}.$$

Let $\tilde{c}_k(\mathbf{w}) = h_k^{-(1-\alpha_k)/\alpha_k}c_k(\mathbf{w})/\alpha_k$ for $k \in \mathcal{K}^d$, $\tilde{c}_k(\mathbf{w}) = c_k(\mathbf{w})$ for $k \in \mathcal{K}^h$, and $\tilde{\mathbf{c}}(\mathbf{w}) = (\tilde{c}_1(\mathbf{w}),\ldots,\tilde{c}_K(\mathbf{w}))$, so that $\mathbf{p} = \tilde{\mathbf{c}}(\mathbf{w})$. $\tilde{\mathbf{c}}(\mathbf{w})$ is continuous and homogeneous of degree one. Substituting this and the expression for aggregate income into the factor demands, we obtain the excess factor demand function $\mathbf{z}(\mathbf{w})$ defined by

$$z_l(\mathbf{w}) = \frac{\sum_{k\in\mathcal{K}^d} a_{kl}(\mathbf{w})h_k^{-\frac{1-\alpha_k}{\alpha_k}}d_k(\tilde{\mathbf{c}}(\mathbf{w})) + \sum_{k\in\mathcal{K}^h} a_{kl}(\mathbf{w})d_k(\tilde{\mathbf{c}}(\mathbf{w}))}{1 - \sum_{k\in\mathcal{K}^d}(1-\alpha_k)\tilde{c}_k(\mathbf{w})d_k(\tilde{\mathbf{c}}(\mathbf{w}))}\mathbf{wx} - x_l.$$

Theorem 3.1: *An integrated equilibrium exists.*

Proof: Suppose there is a $\mathbf{w}^*$ such that $\mathbf{z}(\mathbf{w}^*) = \mathbf{0}$. Let $p_{kh}^* = c_k(\mathbf{w}^*)/\alpha_k$ for all $h \in [0, h_k]$ and all $k \in \mathcal{K}^d$ and $p_k^* = c_k(\mathbf{w}^*)$ for $k \in \mathcal{K}^h$. Let $\mathbf{x}_{kh}^* = \mathbf{a}_k(\mathbf{w}^*)z_{kh}^*$ for all $h \in [0, h_k]$ and all $k \in \mathcal{K}^d$, where $z_{kh}^* = \sum_{i=1}^I h_k^{-1/\alpha_k}d_k(\tilde{\mathbf{c}}(\mathbf{w}^*))I^*$ and I^* is obtained by evaluating the equation for I at $\mathbf{w} = \mathbf{w}^*$ and $\mathbf{p} = \mathbf{d}(\tilde{\mathbf{c}}(\mathbf{w}^*))$. Let $\mathbf{x}_k^{j*}$ satisfy $\sum_{j=1}^J \mathbf{x}_k^{j*} = \mathbf{a}_k(\mathbf{w}^*)d_k(\tilde{c}(\mathbf{w}^*))I^*$ and $\mathbf{x}_k^{j*} \geq \mathbf{0}$ for all j and $k \in \mathcal{K}^h$. Finally, for all i, let $z_{kh}^{i*} = h_k^{-1/\alpha_k}d_k(\tilde{\mathbf{c}}(\mathbf{w}^*))I^{i*}$ for all $h \in [0, h_k]$ and all $k \in \mathcal{K}^d$, where $I^{i*} = \mathbf{w}^*\mathbf{x}^i + \sum_{k\in\mathcal{K}^d}\int_0^{h_k}\theta_{kh}^i\pi_{kh}^*dh$ and $\pi_{kh}^* = [p_{kh}^* - c_k(\mathbf{w}^*)]z_{kh}^*$, and let $y_k^{i*} = d_k(\mathbf{p})I^{i*}$ for $k \in \mathcal{K}^h$. The price system $(\mathbf{w}^*, \{(p_{kh}^*)_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{p_k^*\}_{k\in\mathcal{K}^h})$ and the allocation $(\{(\mathbf{x}_{kh}^*)_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{\{\mathbf{x}_k^{j*}\}_{k\in\mathcal{K}^h}\}_{j=1}^J, \{\{(z_{kh}^{i*})_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{y_k^{i*}\}_{k\in\mathcal{K}^h}\}_{i=1}^I)$ are an integrated equilibrium. So, as in the proof of Theorem 2.1, all we have to do is (1) deal with compactness problems and (2) prove the existence of $\mathbf{w}^*$ such that $\mathbf{z}(\mathbf{w}^*) = \mathbf{0}$.

(1) The maximization of $[\int_0^{h_k}(z_{kh}^i)^{\alpha_k}dh]^{1/\alpha_k}$ given I_k^i and of monopoly profit given the resulting demand functions and $c_k(\mathbf{w})$ is conducted via concave programming, which does not presuppose compactness

of the domain. Let $\bar{\mathcal{X}}$ be defined as in Subsection 2.2. Let $\bar{\mathcal{X}}_k = \{\mathbf{x}_{kh} | \, \mathbf{x}_{kh} \in \mathbb{R}^L_+, \mathbf{x}_{kh} \le 2\mathbf{x}/h_k\}$ for all $k \in \mathcal{K}^d$, and redefine $\bar{\mathcal{Y}} = \{\mathbf{y}^i | \, \mathbf{y}^i \in \mathbb{R}^K_+, y^i_k \le 2h_k^{1/\alpha_k} f_k(\mathbf{x}), k \in \mathcal{K}^d, y^i_k \le 2f_k(\mathbf{x}), k \in \mathcal{K}^h\}$. $\bar{\mathcal{X}}$, $\bar{\mathcal{X}}_k$, and $\mathcal{B}_i \cap \bar{\mathcal{Y}}$ are compact. Input vectors $\mathbf{x}^j_k \notin \bar{\mathcal{X}}$ or $\mathbf{x}_{kh} \notin \bar{\mathcal{X}}_k$ are not feasible in an integrated equilibrium with symmetry with respect to the variants h of given differentiated goods k (i.e., with $h_k \mathbf{x}_{kh} \le \mathbf{x}$ for all $k \in \mathcal{K}^d$). Similarly, consumption bundles $\mathbf{y}^i \notin \bar{\mathcal{Y}}$ are not feasible in a symmetric equilibrium (since $y^i_k = h_k^{1/\alpha_k} z^i_{kh} \le h_k^{1/\alpha_k} f_k(\mathbf{x})$ for $k \in \mathcal{K}^d$). A price system and an allocation are an *integrated equilibrium on restricted domains* if they satisfy (I)-(IV) in the definition of an integrated equilibrium when $\mathbf{x}_{kh} \in \bar{\mathcal{X}}_k$ and $\mathbf{x}^j_k \in \bar{\mathcal{X}}$ are added in (I) and $\mathcal{B}^i$ is replaced with $\mathcal{B}_i \cap \bar{\mathcal{Y}}$ in (II). The same arguments as used in the proof of Theorem 2.1 establish that a price system and an allocation are an equilibrium if they are an equilibrium on restricted domains.

(2) Define the Gale-Nikaido mapping $\mathbf{g}(\mathbf{w})$ as in the Proof of Theorem 2.1. Since $\tilde{\mathbf{c}}(\mathbf{w})$, $\mathbf{a}_k(\mathbf{w})$, and $\mathbf{d}(\mathbf{p})$ are homogeneous of degree one, zero, and minus one, respectively, $\mathbf{z}(\mathbf{w})$ and $\mathbf{g}(\mathbf{w})$ are homogeneous of degree zero, and we can let $\mathbf{g} : \Delta \to \Delta$. By Brouwer's theorem, there exists a fixed point $\mathbf{w}^*$. From $\mathbf{p}\mathbf{y}^i = I^i$ and $\mathbf{y}^i = \mathbf{d}(\mathbf{p})I^i$, we have $\mathbf{p}\mathbf{d}(\mathbf{p}) = 1$. As above, it follows that $\sum_{l=1}^L w_l z_l(\mathbf{w}) = 0$ for all $\mathbf{w}$ (see the Appendix). The validity of this equality for the fixed point $\mathbf{w}^*$ of the Gale-Nikaido mapping establishes $z_l(\mathbf{w}^*) = \mathbf{0}$. ||

3.3 Free trade equilibrium

Suppose the world economy is divided, in the same way as in Subsection 2.3, into M countries engaged in free trade with each other. For each differentiated good k, let $\mathcal{H}^m_k$ denote the set of all variants h with a monopolist located in country m, where $\cup_{m=1}^M \mathcal{H}^m_k = [0, h_k]$ for all $k \in \mathcal{K}^d$. Assume that the monopolist for variant $h \in \mathcal{H}^m_k$ of k has to produce in his home country m.[12]

Definition: *A price system* $(\mathbf{w}, \{(p_{kh})_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{p_k\}_{k\in\mathcal{K}^h})$ *and an allocation* $(\{(\mathbf{x}_{kh})_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{\{\mathbf{x}^j_k\}_{k\in\mathcal{K}^h}\}_{j=1}^J, \{\{(z^i_{kh})_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{y^i_k\}_{k\in\mathcal{K}^h}\}_{i=1}^I)$ *constitute a* free trade equilibrium with replication *if (I)-(III) in the definition of an integrated equilibrium hold and*

(IV') $\sum_{k\in\mathcal{K}^d} \int_{h\in\mathcal{H}^m_k} \mathbf{x}_{kh} dh + \sum_{k\in\mathcal{K}^h} \sum_{j\in\mathcal{J}^m} \mathbf{x}^j_k = \sum_{i\in\mathcal{I}^m} \mathbf{x}^i$ *for all* m.

Theorem 3.2: *Let* $(\mathbf{w}, \{(p_{kh})_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{p_k\}_{k\in\mathcal{K}^h})$ *and* $(\{(\mathbf{x}_{kh})_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{\{\hat{\mathbf{x}}^j_k\}_{k\in\mathcal{K}^h}\}_{j=1}^J,$ $\{\{(z^i_{kh})_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{y^i_k\}_{k\in\mathcal{K}^h}\}_{i=1}^I)$ *be an integrated equilibrium. Then,* $(\mathbf{w}, \{(p_{kh})_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d},$ $\{p_k\}_{k\in\mathcal{K}^h})$ *and* $(\{(\mathbf{x}_{kh})_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{\{\mathbf{x}^j_k\}_{k\in\mathcal{K}^h}\}_{j=1}^J, \{\{(z^i_{kh})_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{y^i_k\}_{k\in\mathcal{K}^h}\}_{i=1}^I)$ *is a free trade equilibrium with replication if, and only if,*

(V) $\mathbf{x}^j_k = \lambda^j_k \sum_{j'=1}^J \hat{\mathbf{x}}^{j'}_k$ *for some* $\lambda^j_k \in \mathbb{R}_+$ *and for all* j *and* k*,*

[12] Giving up this assumption enlarges the set of endowments compatible with replication of the integrated equilibrium.

(VI) $\sum_{m=1}^{M} \sum_{j \in \mathcal{J}^m} \mathbf{x}_k^j = \sum_{j'=1}^{J} \hat{\mathbf{x}}_k^{j'}$ *for all* k,

and (IV') holds.

Proof: Suppose (IV'), (V), and (VI) hold. Since the factor prices $\mathbf{w}$ are the same as in the integrated equilibrium, the input coefficients $\mathbf{a}_k(\mathbf{w})$ are identical, so (V) implies cost minimization ((I) is valid). According to (VI), the world supply of each good j is equal to the respective integrated equilibrium quantity. Since income is the same as in the integrated equilibrium, $\mathbf{y}^i$ maximizes utility ((II) is valid). So the world demands are also equal to the integrated equilibrium demands, and market clearing in the integrated equilibrium implies that the world commodities markets clear ((III) is valid). Given that (IV') holds by assumption, this proves the "if part" of the theorem.

As for the "only if part", if (V) is violated for some j and k, then firm j does not minimize the cost of producing k ((I) is violated). Given the aggregate demand for goods $\sum_{i=1}^{I} \mathbf{y}^i$, if (VI) is invalid for some k, then the world market for good k does not clear ((III) is violated). Finally, a free trade equilibrium cannot exist unless (IV') holds, as this is part of its definition. ||

Remark 3.2.1 (balanced trade): That trade is balanced for each country follows from zero profit and the consumers' budget constraints: both the aggregate value of production and consumption expenditures are equal to aggregate factor incomes.

Remark 3.2.2: Let $N = L = M = 2$ and $K = 3$; that is, two countries trade three goods, one of which is a differentiated Dixit-Stiglitz commodity. Given the assumption that the suppliers of the differentiated variants of the Dixit-Stiglitz good have to produce in their respective home countries, the necessary and sufficient condition for the existence of a free trade equilibrium with replication is that, in the box whose length and height are the endowments of factors 1 and 2, respectively, *net of the resources needed to produce the integrated equilibrium quantities of the variants with a domestic monopolist,* the net factor endowment point is located inside the parallelogram spanned by the integrated equilibrium input vectors for the two homogeneous commodities (cf. Remark 2.2.5).

Remark 3.2.3 (failure of replication): Let $N = K = L = 1$; that is, there are no homogeneous commodities and one differentiated good which is produced using a single factor of production. Then, replication is not generally feasible. Factor market clearing requires that the "number" of variants produced h_1^m if proportional to the supply of the single factor of production x_1^m in each country m: $h_1^m a_{11} z_1 = x_1^m$. Hence, a necessary condition for replication is that country m's share in the total "number" of variants is equal to its share in the world factor supply: $h_1^m / h_1 = x_1^m / x_1$ for all m.

Remark 3.2.3 (North-South trade): Consider the model with $N = K = L = 1$ when replication is not feasible (cf. the previous remark).[13] From the demand curves for the variants of a differentiated

[13] This model is similar to Krugman (1979b, Section II). The differences are that in Krugman (1979b), markets are

good, $z_{1h}/z_{1h'} = (p_{1h}/p_{1h'})^{-1/(1-\alpha_1)}$ for any h, h'. Let country m produce h^m different variants h, and let country m' produce $h^{m'}$ different variants h'. Then, from the factor market clearing conditions $h^m a_{11} z_{1h} = x_1^m$ and $h^{m'} a_{11} z_{1h'} = x_1^{m'}$,

$$\frac{w_1^m}{w_1^{m'}} = \left(\frac{\frac{h^m}{x_1^m}}{\frac{h^{m'}}{x_1^{m'}}} \right)^{1-\alpha_1} .$$

That is, the single factor of production gets a higher reward in country m than in m' if the "number" of variants per unit of factor supply is higher in m than in m'.

4 Free entry

4.1 Model

In the previous section, we considered the h_k's ($k \in \mathcal{K}^d$) as exogenous, now we endogenize them.[14] Assume that to run a firm producing variant h of k a producer has to incur an upfront fixed cost. Specifically, he has to produce $\bar{f}_k(\bar{\mathbf{x}}_{kh}) = 1$ using inputs $\bar{\mathbf{x}}_{kh} \in \mathbb{R}_+^L$, where $\bar{f}_k : \mathbb{R}_+^L \to \mathbb{R}_+$ satisfies Assumption 1. There is free entry into the firm sector. Let $\mathbf{h} = (h_1, \ldots, h_N) \in \mathbb{R}_+^N$ denote the vector of the numbers of firms in the differentiated goods sectors $k \in \mathcal{K}^d$.

Definition: *A price system* $(\mathbf{w}, \{(p_{kh})_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{p_k\}_{k\in\mathcal{K}^h})$, *an allocation* $(\{(\mathbf{x}_{kh})_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{\{\mathbf{x}_k^j\}_{k\in\mathcal{K}^h}\}_{j=1}^J, \{\{(z_{kh}^i)_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{y_k^i\}_{k\in\mathcal{K}^h}\}_{i=1}^I)$, *and a vector of firm numbers* $\mathbf{h}$ *constitute an* integrated equilibrium *if*

(I) p_{kh} *and* $\mathbf{x}_{kh}$ *maximize profit for all* $k \in \mathcal{K}^d$ *and all* h*, the maximum profit is equal to the cost of entry, and* $\{\mathbf{x}_k^j\}_{k\in\mathcal{K}^h}$ *maximizes profit for all* j*,*

(II) $\{\{(z_{kh}^i)_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{y_k^i\}_{k\in\mathcal{K}^h}\}$ *maximizes utility in* $\mathcal{B}^i$ *for all* i*,*

(III) $f_k(\mathbf{x}_{kh}) = \sum_{i=1}^I z_{kh}^i$ *for all* $k \in \mathcal{K}^d$ *and* h*, and* $\sum_{j=1}^J f_k(\mathbf{x}_k^j) = \sum_{i=1}^I y_k^i$ *for all* $k \in \mathcal{K}^h$*,*

(IV) $\sum_{k\in\mathcal{K}^d} \int_0^{h_k} (\mathbf{x}_{kh} + \bar{\mathbf{x}}_{kh}) dh + \sum_{k\in\mathcal{K}^h} \sum_{j=1}^J \mathbf{x}_k^j = \sum_{i=1}^I \mathbf{x}^i$.

4.2 Integrated equilibrium

The demand functions for the variants of the differentiated goods derived in the previous section are unchanged, the monopolistic competitors charge the constant-markup price $p_{kh} = c_k(\mathbf{w})/\alpha_k$ for

competitive and the North can produce all known goods, while the South can produce only a subset of the goods.

[14] Cf. Dixit and Norman (1980, Section 9.3) and Helpman and Krugman (1985, Part III).

all $k \in \mathcal{K}^d$ and all h, the minimum expenditure needed to get $y_k^i = 1$ is $p_k = h_k^{-(1-\alpha_k)/\alpha_k} p_{kh} = h_k^{-(1-\alpha_k)/\alpha_k} c_k(\mathbf{w})/\alpha_k$, utility maximization yields $\mathbf{y} = \mathbf{d}(\mathbf{p})I$, and profit gross of the fixed cost is $\pi_{kh} = [(1-\alpha_k)/\alpha_k] c_k(\mathbf{w}) h_k^{-1/\alpha_k} d_k(\mathbf{p})I$.

Let $\bar{c}_k(\mathbf{w}) : \mathbb{R}_+^L \to \mathbb{R}_+$ denote the minimum cost of producing $\bar{f}_k(\bar{\mathbf{x}}_{kh}) = 1$ and $\bar{\mathbf{a}}_k(\mathbf{w}) : \mathbb{R}_+^L \to \mathbb{R}_+^L$ the associated input vector. The demand for factor l is

$$\sum_{k \in \mathcal{K}^d} \left[a_{kl}(\mathbf{w}) h_k^{-\frac{1-\alpha_k}{\alpha_k}} d_k(\mathbf{p})I + h_k \bar{a}_{kl}(\mathbf{w}) \right] + \sum_{k \in \mathcal{K}^h} a_{kl}(\mathbf{w}) d_k(\mathbf{p})I.$$

Free entry implies that there are no pure profits: $I = \mathbf{wx}$. Define $\tilde{c}_k(\mathbf{w}, \mathbf{h}) = h_k^{-(1-\alpha_k)/\alpha_k} c_k(\mathbf{w})/\alpha_k$ for $k \in \mathcal{K}^d$, $\tilde{c}_k(\mathbf{w}, \mathbf{h}) = c_k(\mathbf{w})$ for $k \in \mathcal{K}^h$, and $\tilde{\mathbf{c}}(\mathbf{w}, \mathbf{h}) = (\tilde{c}_1(\mathbf{w}, \mathbf{h}), \ldots, \tilde{c}_K(\mathbf{w}, \mathbf{h}))$, so that $\mathbf{p} = \tilde{\mathbf{c}}(\mathbf{w}, \mathbf{h})$. The excess factor demand function $\mathbf{z}(\mathbf{w}, \mathbf{h})$ is given by

$$z_l(\mathbf{w}, \mathbf{h}) = \sum_{k \in \mathcal{K}^d} \left[a_{kl}(\mathbf{w}) h_k^{-\frac{1-\alpha_k}{\alpha_k}} d_k(\tilde{\mathbf{c}}(\mathbf{w}, \mathbf{h}))\mathbf{wx} + h_k \bar{a}_{kl}(\mathbf{w}) \right] + \sum_{k \in \mathcal{K}^h} a_{kl}(\mathbf{w}) d_k(\tilde{\mathbf{c}}(\mathbf{w}, \mathbf{h}))\mathbf{wx} - x_l, \quad l = 1, \ldots, L.$$

Free entry into the firm sector implies $\pi_{kh} = \bar{c}_k(\mathbf{w})$ for all h and for all $k \in \mathcal{K}^d$. Using the equation for π_{kh}, the definition of $\tilde{c}_k(\mathbf{w}, \mathbf{h})$ for $k \in \mathcal{K}^d$, and $\mathbf{p} = \tilde{\mathbf{c}}(\mathbf{w}, \mathbf{h})$, the free entry condition can be rewritten as $h_k = v_k(\mathbf{w}, \mathbf{h})$, where

$$v_k(\mathbf{w}, \mathbf{h}) = (1-\alpha_k) \frac{\tilde{c}_k(\mathbf{w}, \mathbf{h}) d_k(\tilde{\mathbf{c}}(\mathbf{w}, \mathbf{h}))\mathbf{wx}}{\bar{c}_k(\mathbf{w})}, \quad k = 1, \ldots, N.$$

Theorem 4.1: *An integrated equilibrium exists.*

Proof: A price system $(\mathbf{w}^*, \{(p_{kh})_{h \in [0,h_k]}\}_{k \in \mathcal{K}^d}, \{p_k^*\}_{k \in \mathcal{K}^h})$, an allocation $(\{(\mathbf{x}_{kh}^*)_{h \in [0,h_k]}\}_{k \in \mathcal{K}^d}, \{\{\mathbf{x}_k^{j*}\}_{k \in \mathcal{K}^h}\}_{j=1}^J, \{\{(z_{kh}^{i*})_{h \in [0,h_k]}\}_{k \in \mathcal{K}^d}, \{y_k^{i*}\}_{k \in \mathcal{K}^h}\}_{i=1}^I)$, and a vector of firm numbers $\mathbf{h}^*$ are an integrated equilibrium if $\mathbf{z}(\mathbf{w}^*) = \mathbf{0}$, $p_{kh}^* = c_k(\mathbf{w}^*)/\alpha_k$ for all $h \in [0, h_k]$ and all $k \in \mathcal{K}^d$, $p_k^* = c_k(\mathbf{w}^*)$ for $k \in \mathcal{K}^h$, $I^* = \mathbf{w}^*\mathbf{x}^*$, $z_{kh}^* = \sum_{i=1}^I h_k^{-1/\alpha_k} d_k(\tilde{\mathbf{c}}(\mathbf{w}^*))I^*$, $\mathbf{x}_{kh}^* = \mathbf{a}_k(\mathbf{w}^*) z_{kh}^*$ for all $h \in [0, h_k]$ and all $k \in \mathcal{K}^d$, $\sum_{j=1}^J \mathbf{x}_k^{j*} = \mathbf{a}_k(\mathbf{w}^*) d_k(\tilde{c}(\mathbf{w}^*))I^*$, $\mathbf{x}_k^{j*} \geq \mathbf{0}$ for all j and $k \in \mathcal{K}^h$, $I^{i*} = \mathbf{w}^*\mathbf{x}^i$, $z_{kh}^{i*} = h_k^{-1/\alpha_k} d_k(\tilde{\mathbf{c}}(\mathbf{w}^*))I^{i*}$ for all $h \in [0, h_k]$ and all $k \in \mathcal{K}^d$, $y_k^{i*} = d_k(\mathbf{p}^*)I^{i*}$ for $k \in \mathcal{K}^h$, and $h_k^* = v_k(\mathbf{w}^*, \mathbf{h}^*)$ for all $k \in \mathcal{K}^d$.

(1) Restricting the domains of the cost minimization and utility maximization problems as in the proof of Theorem 3.1 gives rise to well-defined solutions.

(2) Let $\bar{h}_k$ denote the number of variants of k developed that obtains if factors are used for no other purpose: $\bar{h}_k = \bar{f}_k(\mathbf{x}/\bar{h}_k)$. Evidently, $\bar{h}_k$ is uniquely determined. Let $\mathbf{g}$ be defined by

$$\begin{aligned} g_l(\mathbf{w}, \mathbf{h}) &= \frac{w_l + \max\{z_l(\mathbf{w}, \mathbf{h}), 0\}}{\sum_{l'=1}^L [w_{l'} + \max\{z_{l'}(\mathbf{w}, \mathbf{h}), 0\}]}, \quad l = 1, \ldots, L \\ g_{L+k}(\mathbf{w}, \mathbf{h}) &= \min\left\{\bar{h}_k, v_k(\mathbf{w}, \mathbf{h})\right\}, \quad k = 1, \ldots, N. \end{aligned}$$

As $\mathbf{g}$ is homogeneous of degree zero in $\mathbf{w}$, we can assume $\mathbf{w} \in \Delta$. Let $\Theta = \times_{k \in \mathcal{K}^d}[0, \bar{h}_k]$, and consider the mapping $\mathbf{g} : \Delta \times \Theta \to \Delta \times \Theta$. The presence of the term $h_k^{-(1-\alpha_k)/\alpha_k}$ in $z_l(\mathbf{w}, \mathbf{h})$ and of $\bar{c}(\mathbf{w})$ in the denominator of the expression for $v_k(\mathbf{w}, \mathbf{h})$ suggests two possible discontinuities of $\mathbf{g}$. However, substituting for $h_k^{-(1-\alpha_k)/\alpha_k}$ from $p_k = h_k^{-(1-\alpha_k)/\alpha_k} c_k(\mathbf{w})/\alpha_k$ and $\mathbf{p} = \tilde{\mathbf{c}}(\mathbf{w}, \mathbf{h})$ into $z_l(\mathbf{w}, \mathbf{h})$ shows that the demand for factor l for the production of the differentiated variants of a good $k \in \mathcal{K}^d$ is $\alpha_k a_{kl}(\mathbf{w}) p_k d_k(\mathbf{p})/c_k(\mathbf{w}) < \alpha_k a_{kl}(\mathbf{w}) I/c_k(\mathbf{w})$. Moreover, $c_k(\mathbf{w}) > 0$ and $\bar{c}_k(\mathbf{w}) > 0$, because all factors are essential in any productive activity and not all factor prices can simultaneously be equal to zero (since $\mathbf{w} \in \Delta$). So $\mathbf{g}$ is continuous, Brouwer's theorem applies, and a fixed point $(\mathbf{w}^*, \mathbf{h}^*)$ exists.

By Walras' law ($\mathbf{wz}(\mathbf{w}, \mathbf{h}) = 0$), we have $z_l(\mathbf{w}^*, \mathbf{h}^*) = 0$ for all $l = 1, \ldots, L$. Furthermore, $h_k^* = \min\left\{\bar{h}_k, v_k(\mathbf{w}^*, \mathbf{h}^*)\right\}$ for $k \in \mathcal{K}^d$. Suppose $h_k^* = \bar{h}_k$ for some $k \in \mathcal{K}^d$. Since $\bar{f}_k(\bar{\mathbf{x}}_{kh})$ is strictly increasing in each factor of production l, this requires that the entire factor endowments $\mathbf{x}$ are used to develop variants of k, which contradicts factor market clearing. So $h_k^* < \bar{h}_k$ for all k, and $h_k^* = v_k(\mathbf{w}^*, \mathbf{h}^*)$, which implies that the free entry conditions are satisfied. This proves that the allocation described at the beginning of this proof is an integrated equilibrium. ||

Remark 4.1.1 (Cobb-Douglas example): If the utility function is Cobb-Douglas, then $\tilde{c}_k(\mathbf{w}, \mathbf{h}) d_k(\tilde{\mathbf{c}}(\mathbf{w}, \mathbf{h}))$ $(= p_k d_k(\mathbf{p}))$ is a constant, and the free entry condition gives h_k in reduced form as a function of $\mathbf{w}$ alone. h_k can be substituted into the factor excess demand functions, and existence of an integrated equilibrium can be proved by considering the factor excess demands alone.

Remark 4.1.2: As in Section 2, the assumption that all factors l are essential in the production of each good k can easily be weakened substantially. Suppose that at least one factor of production l which is essential in the development of the variants of any good k is also essential in the production of k or in the development or production of some other good $k' \neq k$. Then, $h_k^* = \bar{h}_k$ implies excess demand for l. This contradiction implies that the free entry condition is satisfied.

4.3 Free trade equilibrium

Definition: *A price system* $(\mathbf{w}, \{(p_{kh})_{h \in [0,h_k]}\}_{k \in \mathcal{K}^d}, \{p_k\}_{k \in \mathcal{K}^h})$, *an allocation* $(\{\{(\mathbf{x}_{kh})_{h \in [0,h_k]}\}_{k \in \mathcal{K}^d}, \{\mathbf{x}_k^j\}_{k \in \mathcal{K}^h}\}_{j=1}^J, \{\{(z_{kh}^i)_{h \in [0,h_k]}, \{y_k^i\}_{k \in \mathcal{K}^h}\}_{k \in \mathcal{K}^d}\}_{i=1}^I)$, *and a collection of sets* $\{\{\mathcal{H}_k^m\}_{m=1}^M\}_{k \in \mathcal{K}^d}$ *constitute a* free trade equilibrium with replication *if (I)-(III) in the definition of an integrated equilibrium hold and*

(IV') $\sum_{k \in \mathcal{K}^d} \int_{h \in \mathcal{H}_k^m} (\mathbf{x}_{kh} + \bar{\mathbf{x}}_{kh}) dh + \sum_{k \in \mathcal{K}^h} \sum_{j \in \mathcal{J}^m} \mathbf{x}_k^j = \sum_{i \in \mathcal{I}^m} \mathbf{x}^i$ *for all* m, *and*

(V) $\cup_{m=1}^M \mathcal{H}_k^m = [0, h_k]$ *for all* $k \in \mathcal{K}^d$.

Theorem 4.2: *Let* $(\mathbf{w}, \{(p_{kh})_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{p_k\}_{k\in\mathcal{K}^h})$, $(\{(\mathbf{x}_{kh})_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{\{\hat{\mathbf{x}}_k^j\}_{k\in\mathcal{K}^h}\}_{j=1}^J$, $\{\{(z_{kh}^i)_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{y_k^i\}_{k\in\mathcal{K}^h}\}_{i=1}^I)$ *and* $\mathbf{h}$ *be an integrated equilibrium. Then,* $(\mathbf{w}, \{(p_{kh})_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{p_k\}_{k\in\mathcal{K}^h})$, $(\{(\mathbf{x}_{kh})_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}, \{\{\mathbf{x}_k^j\}_{k\in\mathcal{K}^h}\}_{j=1}^J, \{\{(z_{kh}^i)_{h\in[0,h_k]}$, $\{y_k^i\}_{k\in\mathcal{K}^h}\}_{k\in\mathcal{K}^d}\}_{i=1}^I$, *and* $\{\{\mathcal{H}_k^m\}_{m=1}^M\}_{k\in\mathcal{K}^d}$ *are a free trade equilibrium with replication if, and only if,*

(VI) $\mathbf{x}_k^j = \lambda_k^j \sum_{j'=1}^J \hat{\mathbf{x}}_k^{j'}$ *for some* $\lambda_k^j \in \mathbb{R}_+$ *and for all* j *and* k,

(VII) $\sum_{m=1}^M \sum_{j\in\mathcal{J}^m} \mathbf{x}_k^j = \sum_{j'=1}^J \hat{\mathbf{x}}_k^{j'}$ *for all* k,

and (IV') and (V) hold.

Proof: Suppose (IV'), (V), (VI), and (VII) hold. Since the factor prices $\mathbf{w}$ are the same as in the integrated equilibrium, the input coefficients $\mathbf{a}_k(\mathbf{w})$ are identical, so (VI) implies cost minimization. Since the prices are the same, producers of the homogeneous goods make zero profit. Since, moreover, the quantities produced of the variants of the differentiated goods $(\{(\mathbf{x}_{kh})_{h\in[0,h_k]}\}_{k\in\mathcal{K}^d}$ are also the same, producers in the differentiated goods sectors maximize profit, and the free entry conditions are satisfied ((I) is valid). According to (VII), the world supply of each homogeneous good j is equal to the respective integrated equilibrium quantity. Since factor and goods prices are the same as in the integrated equilibrium, $\mathbf{y}^i$ maximizes utility ((II) is valid). So the world demands are also equal to the integrated equilibrium demands, and market clearing in the integrated equilibrium implies that the world commodities markets clear ((III) is valid). Given that (IV') and (V) hold by assumption, this proves the "if part" of the theorem.

If (VI) is violated for some j and k, then firm j does not minimize the cost of producing k ((I) is violated). Given the aggregate demand for goods $\sum_{i=1}^I \mathbf{y}^i$, if (VII) is invalid for some k, then the world market for good k does not clear ((III) is violated). Finally, (IV') and (V) must not be violated, since they are part of the definition of a free trade equilibrium with replication. ||

The 2x2x2 example with one differentiated good ($N = 1$) and one homogeneous good can be illustrated by a figure similar to Figure 1. Think of the input vector for good 1 as giving the factor inputs in both the development and the production of the variants of the differentiated good. Let a measure h_1^m of the variants be produced in m. Then the demand for factor l by producers of variants of the differentiated good located in country m is $h_1^m a_{1l}(\mathbf{w}) h_1^{-1/\alpha_1} d_1(\mathbf{p}) + h_1^m \bar{a}_{1l}(\mathbf{w})$. Linearity in h_1^m implies that the national input vectors are linearly dependent, so that the feasibility of replication can be checked graphically be the same reasoning as in Subsection 2.3.

Remark 4.2.1 (trade pattern): As emphasized by Helpman and Krugman (1985, pp. 2 ff. and Section 7.3), in a free trade equilibrium with replication, there is both interindustry trade (by the same arguments as in Remark 2.2.9, the factor-l content of country m's net exports is positive for those factors l country m is relatively richly endowed with, and vice versa) and intraindustry trade (countries export some variants of the differentiated goods and import others). A similar remark applies to the model of Section 3.

Remark 4.2.2 (multinational firms): Helpman and Krugman (1985, Chapter 12) point out that the set of endowments consistent with replication grows if it is possible to develop and produce variants of the differentiated products in different countries. In the example with $K = L = M = 2$ and $N = 1$, the input vector for good 1 is split up into two separate input vectors for the development and production of variants of good 1, respectively, and the parallelogram in Figure 1 is replaced by a hexagon that includes the parallelogram.

Remark 4.2.3 (Krugman model): Krugman (1979a) considers the model with $N = K = L = 1$. In an integrated equilibrium, prices and profits are $p_{1h} = w_1 a_{11}/\alpha$ and $\pi_{1h} = [(1-\alpha_1)/\alpha_1] w_1 a_{11} z_1$. The free entry condition $\pi_{1h} = w_1 \bar{a}_{11}$ yields $z_1 = [\alpha_1/(1-\alpha_1)]\bar{a}_{11}/a_{11}$. Together with the factor market clearing condition $h_1(a_{11}z_1 + \bar{a}_{11})$, it follows that

$$h_1 = (1-\alpha_1)\frac{x_1}{\bar{a}_{11}}.$$

Other than in the case of h_1^m exogenous (cf. Remark 3.2.3), there is a free trade equilibrium that replicates this integrated equilibrium. The number of variants produced in country m is $h_1 = (1-\alpha_1)x_1^m/\bar{a}_{11}$. The value of exports is $(1 - I^m/I)p_{1h}z_{1h}h_1^m$, the value of imports is $(I^m/I)\sum_{m'=1}^{M} h_1^{m'} p_{1h} z_{1h}$. Trade is balanced if $h_1^m / \sum_{\substack{m'=1 \\ m' \neq m}} h_1^{m'} = I^m/I$. The validity of this condition follows from the definition of income: $I^m = w_1 x_1^m$ and $I = w_1 x_1 = w_1 \sum_{m'=1}^{M} x_1^{m'}$. There are gains from trade. Inserting $z_{1h}^i = I^i/(h_1 p_{1h})$, $p_{1h} = w_1/\alpha_1$, and $I^i = w_1 x_1^i$ into the utility function $u = [\int_0^{h_1} z_{1h}^{\alpha_1} dh]^{1/\alpha_1}$ gives $u = h_1^{(1-\alpha_1)/\alpha_1}\alpha_1 x_1^i$. u rises due to the increase in the "number" of available varieties.

5 Conclusion

Dixit and Norman (1980) showed that, under certain conditions, the world economy replicates the equilibrium of the hypothetical integrated economy without national borders in the traditional trade model with perfect competition. Helpman and Krugman (1985) extended their analysis to settings with imperfectly competitive sectors. This paper derives necessary and sufficient conditions for the existence of free trade equilibria with replication in models with perfect or imperfect competition.

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Appendix

A.1 Theorem 2.1

Proof that an allocation and a price system are an integrated equilibrium if they are an integrated equilibrium on restricted domains: Let $(\mathbf{w}^*, \mathbf{p}^*, \{\{\mathbf{x}_k^{j*}\}_{k=1}^K\}_{j=1}^J, \{\mathbf{y}^{i*}\}_{i=1}^I)$ be an integrated equilibrium on restricted domains. To prove the assertion that $(\hat{\mathbf{w}}, \hat{\mathbf{p}}, \{\{\hat{\mathbf{x}}_k^j\}_{k=1}^K\}_{j=1}^J, \{\hat{\mathbf{y}}^i\}_{i=1}^I) = (\mathbf{w}^*, \mathbf{p}^*, \{\{\mathbf{x}_k^{j*}\}_{k=1}^K\}_{J=1}^J, \{\mathbf{y}^{i*}\}_{i=1}^I)$ is an integrated equilibrium, we have to show (1) that there is no $\mathbf{x}_k^j \in \mathbb{R}_+^L$ such that $\mathbf{x}_k^j \notin \bar{\mathcal{X}}$, $f_k(\mathbf{x}_k^j) \geq 1$, and $\mathbf{w}^*\mathbf{x}_k^j < \mathbf{w}^*\mathbf{x}_k^{j*}$ and (2) that for all i there is no $\mathbf{y}^i \in \mathbb{R}_+^K$ such that $\mathbf{y}^i \notin \bar{\mathcal{Y}}$, $\mathbf{y}^i \in \mathcal{B}^i$, and $u(\mathbf{y}^i) > u(\mathbf{y}^{i*})$.

(1) Suppose to the contrary $\mathbf{w}^*\mathbf{x}_k^j < \mathbf{w}^*\mathbf{x}_k^{j*}$ for some $\mathbf{x}_k^j \in \mathbb{R}_+^L$ not contained in $\bar{\mathcal{X}}$ such that $f_k(\mathbf{x}_k^j) \geq 1$. Then $\mathbf{w}^*\tilde{\mathbf{x}}_k^j < \mathbf{w}^*\mathbf{x}_k^{j*}$ and, because of quasi-concavity, $f_k(\tilde{\mathbf{x}}_k^j) \geq 1$ for all $\tilde{\mathbf{x}}_k^j$ on the line segment joining $\mathbf{x}_k^{j*}$ and $\mathbf{x}_k^j$. This holds true in particular for points on the segment close to $\mathbf{x}_k^{j*}$, i.e., in $\bar{\mathcal{X}}$. This contradicts the fact that $\mathbf{x}_k^{j*}$ minimizes unit cost on the domain $\bar{\mathcal{X}}$.

(2) Suppose there exists a $\mathbf{y}^i \in \mathbb{R}_+^K$ such that $\mathbf{y}^i \notin \bar{\mathcal{Y}}$, $\mathbf{y}^i \in \mathcal{B}^i$, and $u(\mathbf{y}^i) > u(\mathbf{y}^{i*})$. Since both $\mathbf{y}^i$ and $\mathbf{y}^{i*}$ are in $\mathcal{B}^i$, $\tilde{\mathbf{y}}^i \in \mathcal{B}^i$ for all $\tilde{\mathbf{y}}^i$ on the line segment joining $\mathbf{y}^i$ and $\mathbf{y}^{i*}$. $u(\tilde{\mathbf{y}}^i) > u(\mathbf{y}^{i*})$ because of strict quasi-concavity of u. Finally, points $\tilde{\mathbf{y}}^i$ sufficiently close to $\mathbf{y}^{i*}$ satisfy $\tilde{\mathbf{y}}^i \in \bar{\mathcal{Y}}$. Hence, $u(\tilde{\mathbf{y}}^i) > u(\mathbf{y}^{i*})$ for some $\tilde{\mathbf{y}}^i \in \mathcal{B}^i \cap \bar{\mathcal{Y}}$, a contradiction.[15] ||

[15] The converse, "only if", proposition is trivial. If $(\hat{\mathbf{w}}, \hat{\mathbf{p}}, \{\{\hat{\mathbf{x}}_k^j\}_{k=1}^K\}_{j=1}^J, \{\hat{\mathbf{y}}^i\}_{i=1}^I)$ is an integrated equilibrium, then $(\mathbf{w}^*, \mathbf{p}^*, \{\{\mathbf{x}_k^{j*}\}_{k=1}^K\}_{j=1}^J, \{\mathbf{y}^{i*}\}_{i=1}^I) = (\hat{\mathbf{w}}, \hat{\mathbf{p}}, \{\{\hat{\mathbf{x}}_k^j\}_{k=1}^K\}_{j=1}^J, \{\hat{\mathbf{y}}^i\}_{i=1}^I)$ is an integrated equilibrium on restricted domains. From the factor market clearing conditions (III), $\hat{\mathbf{x}}_k^j \leq \mathbf{x} < 2\mathbf{x}$ and, hence, $\hat{\mathbf{x}}_k^j \in \bar{\mathcal{X}}$ for all j and k. Similarly, $\hat{\mathbf{y}}^i \in \bar{\mathcal{Y}}$ for all i. That is, the solutions to the producers' cost minimization problem and to the consumers' utility maximization problem on the unrestricted domains lie in the restricted domains. Obviously, they are also optimal on the restricted domains.

Proof that $\sum_{l=1}^{L} w_l z_l(\mathbf{w}) = 0$ *for all* $\mathbf{w}$*:*

$$\begin{aligned}
\sum_{l=1}^{L} w_l z_l(\mathbf{w}) &= \sum_{l=1}^{L} w_l \sum_{k=1}^{K} a_{kl}(\mathbf{w}) d_k(\mathbf{c}(\mathbf{w}))\mathbf{wx} - \mathbf{wx} \\
&= \mathbf{wx}\left[\sum_{k=1}^{K} d_k(\mathbf{c}(\mathbf{w})) \sum_{l=1}^{L} w_l a_{kl}(\mathbf{w}) - 1\right] \\
&= \mathbf{wx}\left[\sum_{k=1}^{K} p_k d_k(\mathbf{p}) - 1\right] \\
&= \mathbf{wx}\,[\mathbf{pd}(\mathbf{p}) - 1] \\
&= 0. \quad ||
\end{aligned}$$

Proof that the fixed point of $\mathbf{g}(\mathbf{w})$ *is an integrated equilibrium:*

$$\begin{aligned}
\sum_{l=1}^{L} w_l^* z_l(\mathbf{w}^*) &= \sum_{l=1}^{L} g_l(\mathbf{w}^*) z_l(\mathbf{w}^*) \\
&= \frac{\sum_{l=1}^{L} w_l^* z_l(\mathbf{w}^*) + \sum_{l=1}^{L} \max\{z_l(\mathbf{w}^*), 0\} z_l(\mathbf{w}^*)}{\sum_{l'=1}^{L} [w_{l'}^* + z_{l'}^+(\mathbf{w}^*)]} \\
&= \frac{\max\{z_l(\mathbf{w}^*)^2, 0\}}{\sum_{l'=1}^{L} [w_{l'}^* + z_{l'}^+(\mathbf{w}^*)]} \\
&= 0.
\end{aligned}$$

This implies $z_l(\mathbf{w}^*) = 0$ for $l = 1, \ldots, L$. ||

A.2 Theorem 3.1

Proof that $\sum_{l=1}^{L} w_l z_l(\mathbf{w}) = 0$ *for all* $\mathbf{w}$*:* Using $\sum_{l=1}^{L} w_l a_{kl}(\mathbf{w}) = c_k(\mathbf{w})$ for all k, $h_k^{-(1-\alpha_k)/\alpha_k} c_k(\mathbf{w})/\alpha_k = \tilde{c}_k(\mathbf{w}) = p_k$ for $k \in \mathcal{K}^d$, $c_k(\mathbf{w}) = p_k$ for $k \in \mathcal{K}^h$, $\tilde{\mathbf{c}}(\mathbf{w}) = \mathbf{p}$, and $\mathbf{pd}(\mathbf{p}) = 1$, we get

$$\begin{aligned}
\sum_{l=1}^{L} w_l z_l(\mathbf{w}) &= \sum_{l=1}^{L} w_l \frac{\sum_{k\in\mathcal{K}^d} a_{kl}(\mathbf{w}) h_k^{-\frac{1-\alpha_k}{\alpha_k}} d_k(\tilde{\mathbf{c}}(\mathbf{w})) + \sum_{k\in\mathcal{K}^h} a_{kl}(\mathbf{w}) d_k(\tilde{\mathbf{c}}(\mathbf{w}))}{1 - \sum_{k\in\mathcal{K}^d}(1-\alpha_k)\tilde{c}_k(\mathbf{w}) d_k(\tilde{\mathbf{c}}(\mathbf{w}))} \mathbf{wx} - \mathbf{wx} \\
&= \mathbf{wx}\left[\frac{\sum_{k\in\mathcal{K}^d} h_k^{-\frac{1-\alpha_k}{\alpha_k}} d_k(\tilde{\mathbf{c}}(\mathbf{w})) \sum_{l=1}^{L} w_l a_{kl}(\mathbf{w}) + \sum_{k\in\mathcal{K}^h} d_k(\tilde{\mathbf{c}}(\mathbf{w})) \sum_{l=1}^{L} w_l a_{kl}(\mathbf{w})}{1 - \sum_{k\in\mathcal{K}^d}(1-\alpha_k)\tilde{c}_k(\mathbf{w}) d_k(\tilde{\mathbf{c}}(\mathbf{w}))} - 1\right] \\
&= \mathbf{wx}\left[\frac{\sum_{k\in\mathcal{K}^d} h_k^{-\frac{1-\alpha_k}{\alpha_k}} d_k(\tilde{\mathbf{c}}(\mathbf{w})) c_k(\mathbf{w}) + \sum_{k\in\mathcal{K}^h} d_k(\tilde{\mathbf{c}}(\mathbf{w})) c_k(\mathbf{w})}{1 - \sum_{k\in\mathcal{K}^d}(1-\alpha_k)\tilde{c}_k(\mathbf{w}) d_k(\tilde{\mathbf{c}}(\mathbf{w}))} - 1\right] \\
&= \mathbf{wx}\left[\frac{\sum_{k\in\mathcal{K}^d} \alpha_k d_k(\mathbf{p}) p_k + \sum_{k\in\mathcal{K}^h} d_k(\mathbf{p}) p_k}{1 - \sum_{k\in\mathcal{K}^d}(1-\alpha_k) p_k d_k(\mathbf{p})} - 1\right] \\
&= \mathbf{wx}\left[\frac{\sum_{k=1}^{K} d_k(\mathbf{p}) p_k - \sum_{k\in\mathcal{K}^d}(1-\alpha_k) d_k(\mathbf{p}) p_k}{1 - \sum_{k\in\mathcal{K}^d}(1-\alpha_k) p_k d_k(\mathbf{p})} - 1\right] \\
&= \mathbf{wx}\left[\frac{\mathbf{pd}(\mathbf{p}) - \sum_{k\in\mathcal{K}^d}(1-\alpha_k) d_k(\mathbf{p}) p_k}{1 - \sum_{k\in\mathcal{K}^d}(1-\alpha_k) p_k d_k(\mathbf{p})} - 1\right] \\
&= \mathbf{wx}\left[\frac{1 - \sum_{k\in\mathcal{K}^d}(1-\alpha_k) p_k d_k(\mathbf{p})}{1 - \sum_{k\in\mathcal{K}^d}(1-\alpha_k) p_k d_k(\mathbf{p})} - 1\right] \\
&= 0.
\end{aligned}$$

A.3 Theorem 4.1

Using the definitions of $c_k(\mathbf{w})$ and $\bar{c}_k(\mathbf{w})$, $h_k^{-(1-\alpha_k)/\alpha_k} c_k(\mathbf{w}) = \alpha_k p_k$ for $k \in \mathcal{K}^d$, $p_k = c_k(\mathbf{w})$ for $k \in \mathcal{K}^h$, the free entry condition $h_k \bar{c}_k(\mathbf{w}) = (1-\alpha_k) p_k d_k(\mathbf{p})\mathbf{wx}$ for all $k \in \mathcal{K}^d$, and $\mathbf{pd}(\mathbf{p}) = 1$, we obtain:

$$\begin{aligned}
\sum_{l=1}^{L} w_l z_l(\mathbf{w}) &= \sum_{k\in\mathcal{K}^d} \left[h_k^{-\frac{1-\alpha_k}{\alpha_k}} d_k(\mathbf{p})\mathbf{wx} \sum_{l=1}^{L} w_l a_{kl}(\mathbf{w}) + h_k \sum_{l=1}^{L} w_l \bar{a}_{kl}(\mathbf{w}) \right] \\
&\quad + \sum_{k\in\mathcal{K}^h} d_k(\mathbf{p})\mathbf{wx} \sum_{l=1}^{L} w_l a_{kl}(\mathbf{w}) - \sum_{l=1}^{L} w_l x_l \\
&= \mathbf{wx} \sum_{k\in\mathcal{K}^d} h_k^{-\frac{1-\alpha_k}{\alpha_k}} d_k(\mathbf{p}) c_k(\mathbf{w}) + \sum_{k\in\mathcal{K}^d} h_k \bar{c}_k(\mathbf{w}) + \mathbf{wx} \sum_{k\in\mathcal{K}^h} d_k(\mathbf{p}) c_k(\mathbf{w}) - \mathbf{wx} \\
&= \mathbf{wx} \sum_{k\in\mathcal{K}^d} d_k(\mathbf{p}) \alpha_k p_k + \sum_{k\in\mathcal{K}^d} (1-\alpha_k) p_k d_k(\mathbf{p})\mathbf{wx} + \mathbf{wx} \sum_{k\in\mathcal{K}^h} d_k(\mathbf{p}) p_k - \mathbf{wx} \\
&= \mathbf{wx} \left[\sum_{k\in\mathcal{K}^d} \alpha_k d_k(\mathbf{p}) p_k + \sum_{k\in\mathcal{K}^d} (1-\alpha_k) p_k d_k + \sum_{k\in\mathcal{K}^h} d_k(\mathbf{p}) p_k - 1 \right] \\
&= \mathbf{wx} \left[\sum_{k\in\mathcal{K}^d} d_k(\mathbf{p}) p_k + \sum_{k\in\mathcal{K}^h} d_k(\mathbf{p}) p_k - 1 \right] \\
&= \mathbf{wx} \left[\sum_{k=1}^{K} d_k(\mathbf{p}) p_k - 1 \right] \\
&= \mathbf{wx} \left[\mathbf{pd}(\mathbf{p}) - 1 \right] \\
&= 0.
\end{aligned}$$