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"Anything is Possible: On the Existence and Uniqueness of Equilibria in the Shleifer-Vishny Model of Limits of Arbitrage"

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Abstract

This paper gives a complete characterization of the equilibria in Shleifer and Vishny's (1997) model of "Limits of Arbitrage". We show that expected wealth (the arbitrageurs' objective function) is a possibly non-concave function of investment and that the relation between investment and prices is not necessarily continuous or single-valued or well-defined. As a result, "anything is possible": non-existence or multiplicity of equilibria may arise, and sunspots may govern the equilibrium selection in the latter case.

1 Introduction

In an ingenious paper, Shleifer and Vishny (1997)[1] (henceforth: SV) argue why arbitrage may fail when it is most profitable. Their argument is based on the idea that the amount of funds an arbitrageur manages responds positively to the returns he generates (performance-based arbitrage, PBA). To get an idea of the implications of PBA for the efficiency of arbitrage in bringing asset prices in line with fundamental values, suppose the price of an already undervalued asset falls further. Then money is withdrawn from the arbitrageurs invested in this asset, even though the widening gap between fundamental value and the actual price level implies that arbitrage becomes even more profitable. The SV model is a cornerstone of behavioral finance. This is because behavioral finance has identified empirically many kinds of irrational behavior in financial markets, but there are few models which help explain why arbitrage is limited, so that irrational behavior leads to mispricing (see, e.g., Barberis and Thaler, 2003, pp. 1056-1057).

Using numerical examples, SV derive a number of interesting implications from their apparently simple model. The aim of the present article is to show that the model's simplicity is more apparent than real. We provide a complete characterization of the equilibria in the SV model[2] and find that "anything is possible": an equilibrium may fail to exist, or if one exists, it may not be unique. To prove the former assertion, we have to investigate types of equilibria not considered by SV. In the latter case, sunspots may govern the selection of an equilibrium.

We show that the equilibrium conditions in the SV model can be summarized in a two-way relationship between (period-2) prices and (period-1) investment: prices and investment constitute an equilibrium when (a) the price clears the market given the individuals' demands and (b) the demands maximize expected wealth (the relevant objective function) given prices. The non-existence and multiplicity results are due to properties of this two-way-relationship which have not so far received appropriate attention. (a) Due to the fact that the amount of funds arbitrageurs may lose is constrained by the amount of funds under management, their objective function is possibly non-concave. (b) The correspondence relating (period-2) prices to (period-1) investment is possibly discontinuous or multi-valued or not even well-defined. This is due to a positive feedback effect: due to PBA, higher period-2 prices raise period-2 funds under control, thereby driving up period-2 prices.[3]

[1] See also Chapter 4 in Shleifer (2000).

[2] This follows Zwiebel's (2002, pp. 1219-1220) call for a rigorous treatment of the insightful models in Shleifer's (2000) *Inefficient Markets*.

[3] While (b) the latter problem is ruled out by the parameter assumptions made by SV, (a) the non-concavity problem can arise for parameters satisfying the SV assumptions.

(a) Our main result is that the non-concavity problem causes an existence problem. To get an idea why, notice that an increase in period-1 investment raises the arbitrageurs' payoffs if the price recovers at time 2, but decreases the amount of funds under management and, hence, wealth if the noise trader shock worsens. A necessary condition for the existence of an equilibrium with an interior solution to the problem of maximizing expected wealth is that these two effects balance out. Assume this is the case for marginal variations of period-1 investment. Assume further that the prices ensuing from the arbitrageurs' investment decisions are such that if an arbitrageur is fully invested in period 1 and the noise trader shock grows, then he loses all the funds under his management. This implies that expected wealth is increasing in the arbitrageur's investment for investment levels close to full investment, because the wealth-reducing effect of additional investment vanishes. It follows that expected wealth is a non-concave function of investment (flat for low investment levels and increasing for higher levels), and full investment maximizes wealth, violating the supposition that there is an equilibrium with less-than-full investment. (b) The fact that the relation between investment and prices is not well-behaved does not cause an existence problem of its own. However, it opens up the possibility of multiple equilibria, equilibrium selection via sunspots, and perverse comparative statics.

Section 2 briefly recapitulates the assumptions of the SV model. In Section 3, we define the relevant sorts of equilibria and state the conditions they satisfy. To motivate the subsequent analysis, we present a numerical example which leads to non-existence at the outset. The main results on existence and uniqueness are in Section 4. Section 5 concludes.

2 Model

This section gives a brief exposition of the SV model. Consider an asset market with three types of agents, noise traders, arbitrageurs, and investors in arbitrage funds. Time is discrete, and there are three time periods. The supply of the asset is inelastic and normalized to unity. The asset's fundamental value is V (> 0). So the asset is under-valued or valued correctly, depending on whether demand is less than or equal to V, respectively. At time 3, the asset is valued correctly.

The noise traders' demand for the asset in period 1 is $QN_1 = V - S_1$, where $0 < S_1$. At time 2, with probability q $(0 < q < 1)$, their demand is $QN_2 = V - S_2$, where $S_1 < S_2 < V$ ("noise trader misperceptions deepen"); with probability $1 - q$, on the other hand, $QN_2 = V$, and the asset price returns to its fundamental value.

There is a continuum of unit length of identical arbitrageurs. Their funds under management in periods 1 and 2 are denoted F_1 and F_2, respectively. F_1 is exogenous and satisfies $0 < F_1 < S_1$. Their investments in the asset at times $t = 1$ and $t = 2$ are denoted D_1 and D_2 respectively, where

$0 \leq D_t \leq F_t$, $t \in \{1, 2\}$. Non-invested funds are stored at zero interest. Let p_1 denote the period-1 price and p_2 the period-2 price in case of worsening noise trader expectations and x the gross return on F_1. If noise trader misperceptions deepen, then $x = 1 + (p_2/p_1 - 1)D_1/F_1$. Moreover, due to performance-based arbitrage, period-2 assets under control of the arbitrageurs are $F_2 = \max\{F_1(ax + 1 - a), 0\}$. Following SV, we focus on the case $a > 1$. Arbitrageurs maximize final wealth, W, in period 2 and expected final wealth, EW in period 1.

3 Equilibrium

3.1 Definition

In equilibrium,

$$p_1 = V - S_1 + D_1. \tag{1}$$

If noise trader expectations deepen:

$$p_2 = V - S_2 + D_2 \tag{2}$$

$$F_2 = \max\left\{F_1 + aD_1\left(\frac{p_2}{p_1} - 1\right), 0\right\} \tag{3}$$

$$W \equiv F_2 + \left(\frac{V}{p_2} - 1\right) D_2. \tag{4}$$

Equations (1)-(3) correspond to equations (3), (2), and (6), respectively, in SV (pp. 39-41).[4] If, on the other hand, noise trader expectations recover, then the period-2 price if V and $W = F_1 + aD_1(V/p_1 - 1)$. The arbitrageurs' investments maximize (expected) wealth:

$$D_1 = \arg\max_{D_1} : EW \text{ s.t.: } 0 \leq D_1 \leq F_1 \text{ and (3)} \tag{5}$$

$$D_2 = \arg\max_{D_2} : W \text{ s.t.: } 0 \leq D_2 \leq F_2 \tag{6}$$

(cf. SV, p. 42).

Definition: *An equilibrium is a tuple* $(p_1, p_2, D_1, D_2, F_2, W) \geq \mathbf{0}$ *that satisfies (1)-(6).*

[4] The only differences are that, other than SV, we do not impose $D_2 = F_2$ or $F_2 > 0$. We need the slightly more general formulation of the equations because, for one thing, it will turn out that equilibria with $D_2 \neq F_2$ may exist and, for another, to prove non-existence of an equilibrium we have to take into account allocations with $F_2 = 0$.

3.2 Preview

To motivate the subsequent analysis, we start with an example which illustrates that equilibria of the types considered by SV may fail to exist due to the non-concavity of the arbitrageurs' expected wealth function.

Example 1: Let $V = 1$, $F_1 = 0.1$, $a = 3$, $S_1 = 0.2$, $S_2 = 0.7$, and $q = 0.1$. We pose the following question: *is there an equilibrium with $p_2 < V$ and $F_2 > 0$ and with either full ($D_1 = F_1$) or partial ($D_1 < F_1$) investment?* The answer will be in the negative. The proof that other types of equilibria do not exist either is postponed to Section 4. $p_2 < V$ implies $D_2 = F_2$. Using the supposition $F_2 > 0$, (1)-(3) become

$$p_1 = 0.8 + D_1 \tag{7}$$

$$p_2 = 0.3 + F_2 \tag{8}$$

$$F_2 = 0.1 + 3D_1\left(\frac{p_2}{p_1} - 1\right). \tag{9}$$

Eliminating p_2 and F_1 from (7)-(9) yields

$$p_2 = \frac{(0.4 - 3D_1)(0.8 + D_1)}{0.8 - 2D_1}. \tag{10}$$

Suppose, to begin with, there is an equilibrium in which arbitrageurs are fully invested in period 1 (i.e., $D_1 = F_1$). From (7) and (10), $p_1 = 0.9$ and $p_2 = 0.15$. However, from (9), $F_2 = -0.15$, a contradiction.[5] Next, consider equilibria in which arbitrageurs hold back funds in period 1 (i.e., $D_1 < F_1$). Expected wealth is

$$EW \equiv 0.9\left[0.1 + 3D_1\left(\frac{1}{p_1} - 1\right)\right] + \frac{0.1}{p_2}\left[0.1 + 3D_1\left(\frac{p_2}{p_1} - 1\right)\right]. \tag{11}$$

In an equilibrium with $0 < D_1 < F_1$, EW must be constant in D_1, which implies $p_2 = p_1/(10 - 9p_1)$. Hence, using (7),

$$p_2 = \frac{0.8 + D_1}{2.8 - 9D_1}. \tag{12}$$

Solving (10) and (12) yields $D_1 = 0.0354$ and $p_2 = 0.3366$. From (7), $p_1 = 0.8354$. Now consider the expected wealth function. Given the equilibrium prices and taking the non-negativity of F_2 (cf. equation (3)) into account explicitly, (11) becomes

$$EW \equiv 0.9(0.1 + 0.5911D_1) + 0.2971\max\{0.1 - 1.7912D_1, 0\}.$$

For $D_1 < 0.0558$ ($= 1/1.7912$), the max-term is positive, and EW is in fact constant in D_1. However, for $D_1 > 0.0558$, the max-term becomes zero, so that $d(EW)/dD_1 = 0.5320 > 0$. Expected wealth

[5] It may be noted that SV's (p. 46) "stability condition", viz. that $aF_1 < p_1$ if $D_1 = F_1$, is satisfied (as $0.3 < 0.9$).

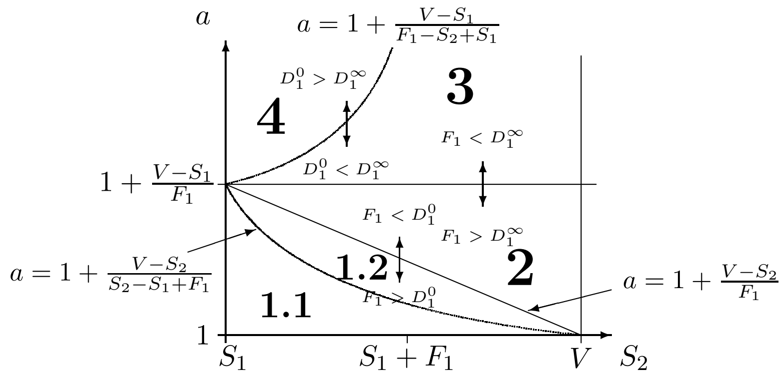

Figure 1: Case distinctions

is non-concave in D_1, and each arbitrageur's optimal choice is to be fully invested (i.e., $D_1 = 0.1$), a contradiction. This answers the question posed at the outset: an equilibrium with $p_2 < V$ and $F_2 > 0$ and with either full ($D_1 = F_1$) or partial ($D_1 < F_1$) investment does not exist.[6] In Section 4 we show that no other kind of equilibrium (e.g., with $p_2 = V$ or with $F_2 = 0$ or with $D_1 = 0$) exists either and that similar problems (and others as well) occur for a wide range of parameter values.

3.3 Undervaluation equilibria

We now start our systematic analysis of equilibria. To begin with, we focus on the case $p_2 < V$. The discussion of equilibria is postponed until Subsection 3.4. We proceed in four steps. In step 1, we provide a convenient partition of the parameter space. Step 2 is concerned with the correspondence between aggregate period-1 investment, D_1, and the period-2 price level in the case of deepening noise trader expectations, p_2. Next, we solve the arbitrageurs' wealth maximization problem, which yields D_1 as a function of p_2 (step 3). Step 4 introduces different types of equilibria we have to distinguish in the subsequent section on existence and uniqueness.

[6] One might object that the non-existence result depends on the assumption that the funds under management are distributed symmetrically across arbitrageurs: the non-existence of an equilibrium with less-than-full investment is due to the fact that each arbitrageur prefers to invest all the funds under his management (i.e., $F_1 = 0.1$) when $p_1 = 0.8354$ and $p_2 = 0.3366$, since expected wealth is increasing in D_1 for $D_1 > 0.0558$. This raises the question of whether an equilibrium with aggregate investment $D_1 = 0.0354$, $p_1 = 0.8354$ and $p_2 = 0.3366$ prevails if the majority of arbitrageurs hold funds no greater than 0.0558 and choose to invest nothing and the remainder of the aggregate funds is concentrated in the hands of a few "big" and fully invested arbitrageurs. In the present example, the answer is: no. The mass of arbitrageurs with funds no greater than 0.0558 is bounded away from unity. So the funds under management of the "big" arbitrageurs is greater than $(0.1 - 0.0558 =)$ 0.0442. This is more than consistent with the stipulated equilibrium prices (i.e., with $D_1 = 0.0354$).

Step 1: Case distinctions

From (4) and $p_2 < V$, we have $dW/dD_2 = V/p_2 - 1 > 0$. Hence,

$$D_2 = F_2. \tag{13}$$

From (1)-(3) and (13),

$$p_2 = \max\left\{A(D_1) + F(D_1)p_2, V - S_2\right\}, \tag{14}$$

where

$$A(D_1) \equiv V - S_2 + F_1 - aD_1,\ \ F(D_1) \equiv \frac{aD_1}{V - S_1 + D_1}. \tag{15}$$

There is a positive feedback effect: higher period-2 prices, p_2, raise period-2 funds under control, F_2, thereby driving up p_2. $F(D_1)$ is measure of the strength of this positive feedback effect. We have

$$A(D_1) \gtreqless 0 \Leftrightarrow D_1 \lesseqgtr \frac{V - S_2 + F_1}{a} \equiv D_1^0 \tag{16}$$

and

$$F(D_1) \lesseqgtr 1 \Leftrightarrow D_1 \lesseqgtr \frac{V - S_1}{a - 1} \equiv D_1^\infty. \tag{17}$$

Notice

$$F_1 \lessgtr D_1^0 \Leftrightarrow a \lessgtr 1 + \frac{V - S_2}{F_1} \tag{18}$$

and

$$F_1 \lessgtr D_1^\infty \Leftrightarrow a \lessgtr 1 + \frac{V - S_1}{F_1}. \tag{19}$$

Furthermore,

$$D_1^0 \lessgtr D_1^\infty \Leftrightarrow a(F_1 - S_2 + S_1) \lessgtr F_1 - S_2 + V.$$

Therefore, $D_1^0 < D_1^\infty$ for $S_2 > S_1 + F_1$, and

$$D_1^0 \lessgtr D_1^\infty \Leftrightarrow a \lessgtr 1 + \frac{V - S_1}{F_1 - S_2 + S_1} \quad \text{for } S_2 < S_1 + F_1. \tag{20}$$

These inequalities can be used to divide the parameter space into four subspaces (see Figure 1). Let $\Omega \equiv (V, S_1, S_2, a, F_1)$ $(> \mathbf{0})$ denote the vector of model parameters.

Case 1: Let $\Omega_1 \equiv \{\omega \in \Omega |\, a < 1 + (V - S_2)/F_1\}$. Then, $F_1 < \min\{D_1^0, D_1^\infty\}$ for $\omega \in \Omega_1$.

Case 2: $D_1^0 < F_1 < D_1^\infty$ for $\omega \in \Omega_2 \equiv \{\omega \in \Omega | \, 1 + (V - S_2)/F_1 < a < 1 + (V - S_1)/F_1\}$.[7]

Case 3: $D_1^0 < D_1^\infty < F_1$ for $\omega \in \Omega_3$, where

$$\begin{aligned}\Omega_3 \quad \equiv \quad & \left\{\omega \in \Omega \,\middle|\, V < S_1 + F_1, 1 + \frac{V - S_1}{F_1} < a < 1 + \frac{V - S_1}{F_1 - S_2 + S_1}\right\} \\ & \cup \left\{\omega \in \Omega \,\middle|\, V > S_1 + F_1, S_2 < S_1 + F_1, 1 + \frac{V - S_1}{F_1} < a < 1 + \frac{V - S_1}{F_1 - S_2 + S_1}\right\} \\ & \cup \left\{\omega \in \Omega \,\middle|\, V > S_1 + F_1, S_2 > S_1 + F_1, a > 1 + \frac{V - S_1}{F_1}\right\}.\end{aligned}$$

Case 4: $D_1^\infty < D_1^0 < F_1$ for $\omega \in \Omega_4$, where

$$\begin{aligned}\Omega_4 \quad \equiv \quad & \left\{\omega \in \Omega \,\middle|\, V < S_1 + F_1, a > 1 + \frac{V - S_1}{F_1 - S_2 + S_1}\right\} \\ & \cup \left\{\omega \in \Omega \,\middle|\, V > S_1 + F_1, S_2 < S_1 + F_1, a > 1 + \frac{V - S_1}{F_1 - S_2 + S_1}\right\}.\end{aligned}$$

Step 2: The correspondence between D_1 and p_2

Let $p_2(D_1) : \mathbb{R}^+ \backslash D_1^\infty \to \mathbb{R}$ be given by

$$p_2(D_1) \equiv \frac{A(D_1)}{1 - F(D_1)} = \frac{(V - S_2 - aD_1 + F_1)(V - S_1 + D_1)}{V - S_1 - (a-1)D_1}. \tag{21}$$

Let $\mathcal{P}_2 = \{p_2 | \, p_2 > 0, p_2 \text{ solves (14)}\}$. Taking into account the relation between D_1^0 and D_1^∞, on the one hand, and $A(D_1)$ and $F(D_1)$, on the other hand, in (16) and (17), it can be seen from Figure 2 that

$$\mathcal{P}_2 = \begin{cases} \max\{p_2(D_1), V - S_2\}; & \text{for } D_1 < \min\{D_1^0, D_1^\infty\} \\ V - S_2; & \text{for } D_1^0 \le D_1 < D_1^\infty \\ \emptyset; & \text{for } D_1^\infty < D_1 < D_1^0 \\ \begin{cases} \{p_2(D_1), V - S_2\}; & \text{if } p_2(D_1) \ge V - S_2 \\ \emptyset; & \text{if } p_2(D_1) < V - S_2 \end{cases}; & \text{for } D_1 > \max\{D_1^0, D_1^\infty\} \end{cases}. \tag{22}$$

For the four cases distinguished above, we obtain for $D_1 \in [0, F_1]$ from (22) (see Figure 3):

Case 1:

$$\mathcal{P}_2 = \max\{p_2(D_1), V - S_2\}.$$

Case 2:

$$\mathcal{P}_2 = \begin{cases} \max\{p_2(D_1), V - S_2\}; & \text{for } D_1 \in [0, D_1^0) \\ V - S_2; & \text{for } D_1 \in [D_1^0, F_1] \end{cases}.$$

[7] Notice that $\Omega_1 \cup \Omega_2$ is the set of parameters which satisfy SV's (p. 46) stability condition, which states that $aF_1 < p_1$ if arbitrageurs are fully invested (i.e., $D_1 = F_1$).

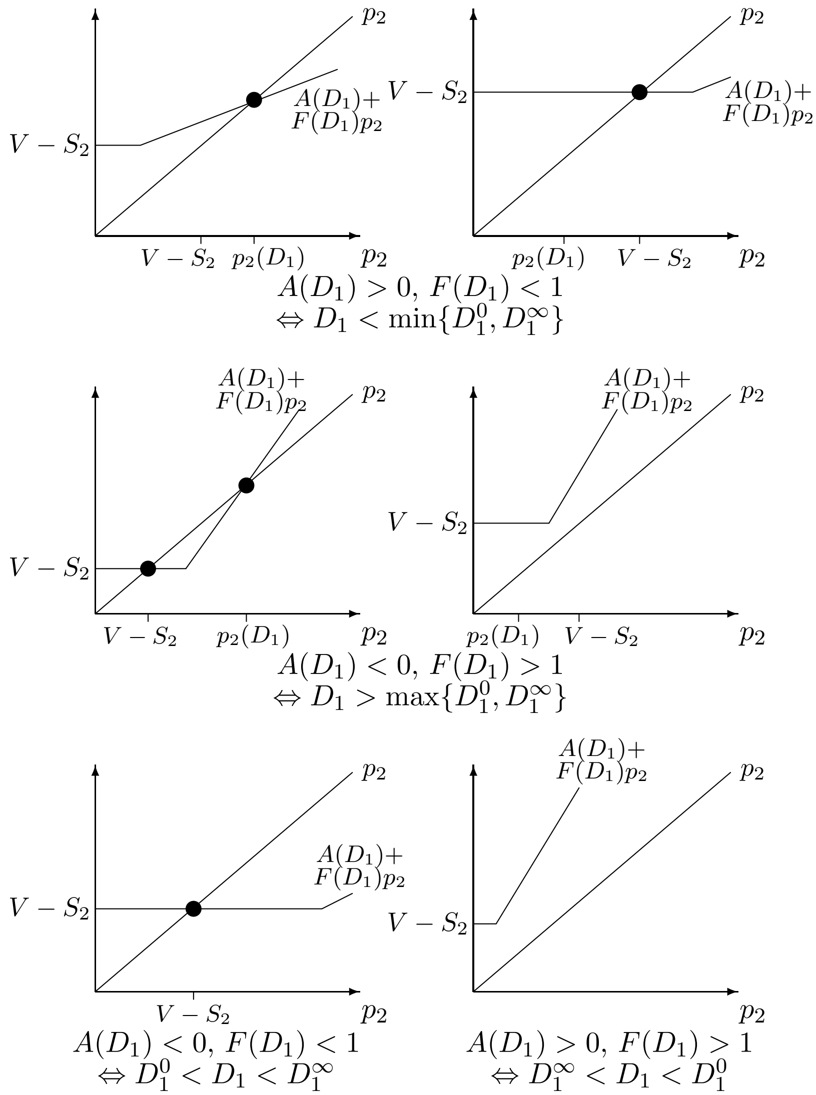

Figure 2: Equilibrium period-2 price level

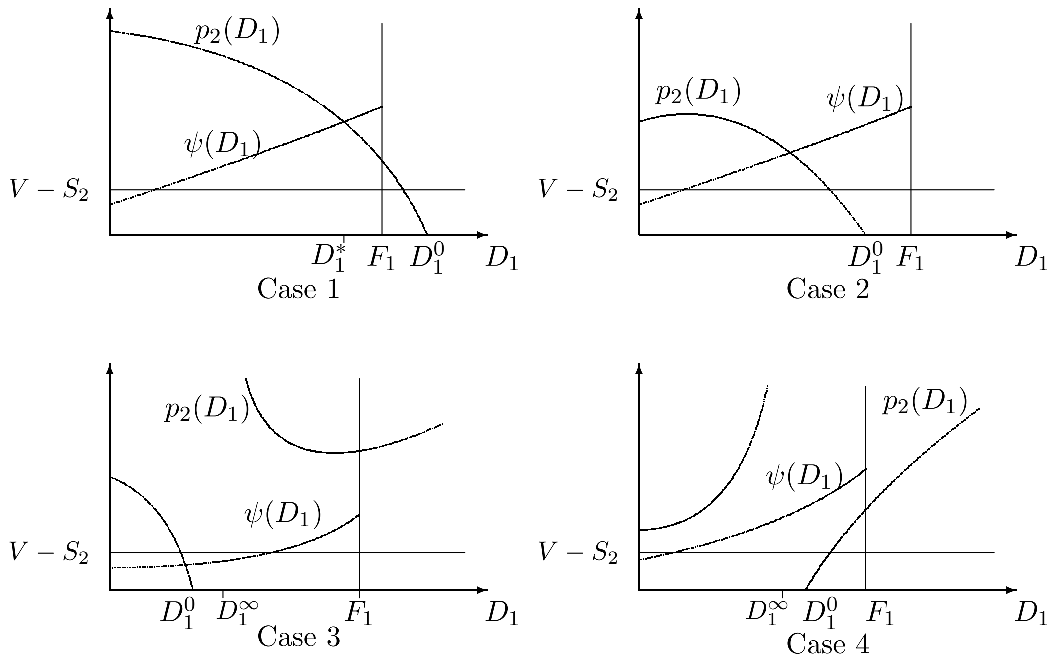

Figure 3: Relation between period-1 investment and period-2 prices

Case 3:

$$\mathcal{P}_2 = \begin{cases} \max\{p_2(D_1), V-S_2\}; & \text{for } D_1 \in [0, D_1^0) \\ V-S_2; & \text{for } D_1 \in [D_1^0, D_1^\infty) \\ \begin{cases} \{p_2(D_1), V-S_2\}; & \text{if } p_2(D_1) \geq V-S_2 \\ \emptyset; & \text{if } p_2(D_1) < V-S_2 \end{cases}; & \text{for } D_1 \in (D_1^\infty, F_1] \end{cases}.$$

Case 4:

$$\mathcal{P}_2 = \begin{cases} \max\{p_2(D_1), V-S_2\}; & \text{for } D_1 \in [0, D_1^\infty) \\ \emptyset; & \text{for } D_1 \in (D_1^\infty, D_1^0] \\ \begin{cases} \{p_2(D_1), V-S_2\}; & \text{if } p_2(D_1) \geq V-S_2 \\ \emptyset; & \text{if } p_2(D_1) < V-S_2 \end{cases}; & \text{for } D_1 \in (D_1^0, F_1] \end{cases}.$$

Notice that $p_2(0) = V - S_2 + F_1 > V - S_2$. Furthermore, notice that if $D_1 < D_1^\infty$, (21) implies

$$p_2(D_1) \underset{<}{\overset{>}{}} V - S_2 \Leftrightarrow a \underset{>}{\overset{<}{}} \frac{F_1}{D_1}\left(1 + \frac{V-S_2}{S_2 - S_1 + D_1}\right). \tag{23}$$

Hence,

$$p_2(F_1) \underset{<}{\overset{>}{}} V - S_2 \Leftrightarrow a \underset{>}{\overset{<}{}} 1 + \frac{V-S_2}{S_2 - S_1 + F_1}. \tag{24}$$

In case(s) 1 (and 2), we have $F_1 < D_1^\infty$. So the inequalities in (24) subdivide the parameter set Ω_1 into two subsets Ω_{11} and Ω_{12} (see Figure 1).

Case 1.1: For $\omega \in \Omega_{11} \equiv \{\omega \in \Omega | \, a < 1 + (V - S_2)/(S_2 - S_1 + F_1)\}$, we have $p_2(F_1) > V - S_2$. Since the term on the far right-hand side of (23) is decreasing in D_1, (23) then implies that $p_2(D_1) > V - S_2$ for all $D_1 \in [0, F_1]$.[8]

Case 1.2: For $\omega \in \Omega_{12} \equiv \{\omega \in \Omega | \, \omega \in \Omega_1, a > 1 + (V - S_2)/(S_2 - S_1 + F_1)\}$, we have $p_2(F_1) < V - S_2$.

In cases 3 and 4, the inequalities in (23) are reversed.

The upshot of step 2 is that the correspondence relating the period-2 asset price to aggregate period-1 investment, D_1, is not necessarily continuous or single-valued or defined over the entire interval $[0, F_1]$.

Step 3: Maximization of expected wealth

Expected wealth as of period 1 is

$$EW = (1-q)\left[F_1 + aD_1\left(\frac{V}{p_1} - 1\right)\right] + q\frac{V}{p_2}\max\left\{F_1 + aD_1\left(\frac{p_2}{p_1} - 1\right), 0\right\} \equiv EW(D_1). \tag{25}$$

Arbitrageurs maximize $EW(D_1)$ in D_1, given p_1 and p_2. An increase in D_1 has two effects on expected wealth. For one thing, it raises the return in case of a return to fundamental valuation in period 2 (see the term in square brackets in (25)). For another, it reduces the amount of funds under control at time 2 if $p_2 < p_1$, F_2, which yield a certain rate of return $V/p_2 - 1 > 0$ (see the max-expression in (25)). The latter effect vanishes when the first term in the max operator becomes zero, i.e. when $D_1 > \bar{D}_1$, where

$$\bar{D}_1 \equiv \frac{F_1}{a\left(1 - \frac{p_2}{p_1}\right)}. \tag{26}$$

For $D_1 < \bar{D}_1$, differentiating (25) yields

$$EW' = aV\left(\frac{1}{p_1} - \frac{1-q}{V} - \frac{q}{p_2}\right) \overset{>}{\underset{<}{=}} 0 \Leftrightarrow p_2 \overset{>}{\underset{<}{=}} \frac{q}{\frac{1}{p_1} - \frac{1-q}{V}}. \tag{27}$$

For $D_1 > \bar{D}_1$, $EW' = aV(1-q)(1/p_1 - 1/V) > 0$. Importantly, if $\bar{D}_1 < F_1$ and $EW' \leq 0$ for $D_1 < \bar{D}_1$, then expected wealth is a non-concave function of period-1 investment (see Figure 4).

Step 4: Types of equilibria

Having characterized how in the different cases we have to distinguish (step 1) period-1 investment, D_1, determines the period-2 price level, p_2, (step 2) and how p_2 affects the choice of D_1 (step 3),

[8] Evidently, the case $a = 1$ can be treated analogously to case 1.1.

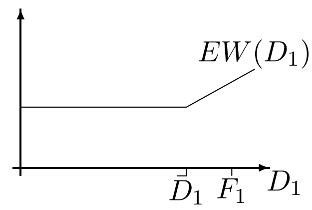

Figure 4: Expected wealth

we are now in a position to put the pieces together and characterize the equilibria of the model. Let $\psi(D_1) : [0, F_1] \to \mathbb{R}$ be given by

$$\psi(D_1) \equiv \frac{q}{\frac{1}{V-S_1+D_1} - \frac{1-q}{V}}. \tag{28}$$

$\psi(D_1)$ is continuous and increasing and satisfies $0 < \psi(D_1) < V - S_1 + F_1$ $(< V)$ for all $D_1 \in [0, F_1]$. From (1), (27), and (28), we have

$$EW' \gtreqless 0 \Leftrightarrow p_2 \gtreqless \psi(D_1) \tag{29}$$

for $D_1 < \bar{D}_1$.[9] The following four types of equilibria comprise an exhaustive list of equilibria with $p_2 < V$:

Partial-investment equilibrium (PIE): $0 < D_1 < F_1$, $p_2 \in \mathcal{P}_2$, $p_2 = \psi(D_1)$, $p_2 < V$, and $F_1 \leq \bar{D}_1$.

No-investment equilibrium (NIE): $D_1 = 0$, $p_2 = p_2(0)$, $p_2 \leq \psi(0)$, and $EW(0) \geq EW(F_1)$ if $F_1 > \bar{D}_1$.

Full-investment equilibrium 1 (FIE1): $D_1 = F_1$, $p_2 \in \mathcal{P}_2$, $p_2 \geq \psi(F_1)$, and $p_2 < V$.

Full-investment equilibrium 2 (FIE2): $D_1 = F_1$, $p_2 \in \mathcal{P}_2$, $p_2 \leq \psi(F_1)$, $\bar{D}_1 < F_1$, and $EW(F_1) \geq EW(0)$.

3.4 Early recovery

So far we have focused on the case $p_2 < V$. Clearly, p_2 cannot exceed V in equilibrium, as this requires $D_2 = S_2 > 0$, and arbitrageurs would do better by decreasing D_2. We now turn to "early recovery equilibria (EREs)" with $p_2 = V$.[10] $p_2 = V$ implies that no capital gains are possible in period 2, so

[9] We have to distinguish carefully the aggregate investment in the asset and an individual arbitrageur's investment, which both equal D_1. The aggregate D_1 determines the prices p_2 (via (21) and (22)) and $\psi(D_1)$ (via (28)). Equation (29) says that if, for instance, $p_2 > \psi(D_1)$, then $EW' > 0$ for each individual $D_1 < \bar{D}_1$.

[10] An implicit assumption of the model is that the rate of return required by investors is zero. It may be noted in passing that there is not a solvency problem for the arbitrageurs then. If the price recovers at time 2 (because noise

$D_1 = F_1$ (i.e., arbitrageurs are fully invested) and, using (3) and (1),

$$F_2 = F_1 + aF_1 \left(\frac{V}{V - S_1 + F_1} - 1 \right) > 0. \tag{30}$$

Moreover, $p_2 = V$ and (2) imply $D_2 = S_2$, which requires $F_2 \geq S_2$. Using (15) and (30), this condition becomes

$$A(F_1) + F(F_1)V \geq V. \tag{31}$$

From Figure 2, we can infer the following.

Case 1: For $\omega \in \Omega_1$, (31) is satisfied if, and only if, $p_2(F_1) \geq V$.

Case 2: For $\omega \in \Omega_2$, (31) is violated.

Cases 3 and 4: In these cases, (31) holds true if, and only if, $p_2(F_1) \leq V$.

4 Existence and uniqueness

This section analyzes the existence, uniqueness, and comparative-statics properties of the equilibrium.

4.1 Existence

In this subsection, we characterize the necessary and sufficient conditions for the existence of an equilibrium. It turns out that $\omega \in \Omega_1$ is sufficient for the existence of an equilibrium, while non-existence may arise in all other parameter subspaces.

Theorem 1: *For $\omega \in \Omega_{11}$, an equilibrium exists.*

Proof: Suppose there is a D_1^* $(0 < D_1^* < F_1)$ such that $p_2(D_1^*) = \psi(D_1^*)$. We assert that there is a PIE with $D_1 = D_1^*$ and $p_2 = p_2(D_1^*)$.

Since $p_2(D_1) > V - S_2$ (in case 1.1) and $\psi(D_1) < V$ for all $D_1 \in [0, F_1]$, we have $V - S_2 < p_2(D_1^*) < V$. To show that a PIE prevails, we have to show that $F_1 \leq \bar{D}_1$. In case 1, if a PIE prevails, we have $D_1 < D_1^\infty$ and, from (27), $p_2 < p_1$. Using (1), (26), and (21), it follows that $F_1 > \bar{D}_1$ if, and only if, $S_2 - S_1 - (F_1 - D_1) + D_1 > 0$ and

$$a > 1 + \frac{V - S_2 + (F_1 - D_1)}{S_2 - S_1 - (F_1 - D_1) + D_1}. \tag{32}$$

Suppose, contrary to what we want to prove, that $F_1 > \bar{D}_1$, so that (32) holds and both the numerator and the denominator in the fraction on the right-hand side are positive. Then the fraction is no less

---

trader pessimism disappears or an ERE prevails), then the return on investment is positive, so that arbitrageurs can pay F_1 to their investors. If not, then investors withdraw $F_1 - F_2$ in period 2. The amount of remaining claims against the arbitrageurs in period 3 is $F_1 - (F_1 - F_2) = F_2$. Since the rate of return on funds under management between periods 2 and 3 is $V/p_2 - 1$ (> 0) in equilibrium except in an ERE, $W > F_2$.

than $(V - S_2)/(S_2 - S_1 + F_1)$, which is obtained by subtracting $F_1 - D_1$ (≥ 0) in the numerator and adding $2(F_1 - D_1)$ (≥ 0) in the denominator. So the supposition made contradicts the case distinction made (viz., $a < 1 + (V - S_2)/(S_2 - S_1 + F_1)$ in case 1.1), and a PIE exists. Since $p_2(D_1)$ (in case 1) and $\psi(D_1)$ are both continuous on $[0, F_1]$, in order for there not to be an intersection, D_1^*, we must have either $p_2(0) \leq \psi(0)$ or $p_2(F_1) \geq \psi(F_1)$. In the former case, there is a NIE. $p_2 < V - S_2$ is satisfied, and $p_2 < V$ follows from $p_2(0) \leq \psi(0) < V$. $F_1 < \bar{D}_1$ follows from the same reasoning as above. In the latter case, there is a FIE1. Equation (21) ensures $p_2 = p_2(F_1) < V$. Q.E.D.

Example 2: Consider the example presented by SV (p. 44): $V = 1$, $F_1 = 0.2$, $a = 1.2$, $S_1 = 0.3$, and $S_2 = 0.4$. As $a = 1.2 < 4 = 1 + (V - S_2)/F_1$ and $a = 1.2 < 3 = (V - S_2)/(S_2 - S_1 + F_1)$, we have $\omega \in \Omega_{11}$, so that $\min\{D_1^0, D_1^\infty\} = \min\{0.6667, 3.5\} > 0.2 = F_1$. From (21) and (28),

$$p_2(D_1) = \frac{(0.8 - 1.2D_1)(0.7 + D_1)}{0.7 - 0.2D_1}$$

$$\psi(D_1) = \frac{q(0.7 + D_1)}{1 - (1 - q)(0.7 + D_1)}.$$

As $\psi(0) = 0.7q/(0.3 + 0.7q) < 0.7 < 0.8 = p_2(0)$, there is not a NIE. As shown by SV (p. 44), there is PIE for $q > 0.3590$ and a FIE1 for $q < 0.3590$.

Theorem 2: *There exist parameters $\omega \in \Omega_{12}$, $\omega \in \Omega_2$, $\omega \in \Omega_3$, and $\omega \in \Omega_4$ such that an equilibrium fails to exist.*

Proof: Numerical examples suffice to prove the theorem.[11]

Example 1: Recall the example introduced in Subsection 3.2: $V = 1$, $F_1 = 0.1$, $a = 3$, $S_1 = 0.2$, $S_2 = 0.7$, and $q = 0.1$. As $a = 3 < 4 = 1 + (V - S_2)/F_1$ and $a = 3 > 1.5 = (V - S_2)/(S_2 - S_1 + F_1)$, we have $\omega \in \Omega_{12}$ ($\min\{D_1^0, D_1^\infty\} = \min\{0.1333, 0.4\} > 0.1 = F_1$). The example is constructed such that an equilibrium fails to exist due to the non-concavity of the arbitrageurs' expected wealth function (cf. step 3 in Subsection 3.3). The functions $p_2(D_1)$ and $\psi(D_1)$ defined in (21) and (28) are given by the right-hand sides of (10) and (12), respectively. As shown in Subsection 3.2, a PIE does not exist (since $\bar{D}_1 = 0.0558 < 0.1 = F_1$).[12] Moreover, as $p_2(0) = 0.4 > 0.2857 = \psi(0)$, a NIE does not exist. As $V - S_2 = 0.3 < 0.4737 = \psi(F_1)$, a FIE1 does not exist. In a FIE2, $p_1 = 0.9$, $p_2 = p_2(F_1) = V - S_2 = 0.3$, and $\bar{D}_1 = 0.05 < 0.1 = F_1$.[13] From (25), $EW(F_1) = 0.12 < 0.1233 = EW(0)$, so a FIE2 does not exist. An ERE does not exist either, since $p_2(F_1) = 0.15 < 1 = V$, so that (31) is violated (alternatively, this can be deduced from the fact that $F_2 = 0.1333 < 0.7 = S_2$ for $p_1 = 0.9$, $p_2 = 1$).

[11] Notice that, as $\Omega_{12} \subset \Omega_1$, the non-existence result applies to parameters not ruled by SV's stability condition.

[12] And this holds true for any distribution of F_1 across arbitrageurs as well.

[13] Notice that different price levels, p_1 and p_2, yield different values for $\bar{D}_1$ (cf. (26)).

Example 3: Let $V = 1$, $F_1 = 0.2$, $a = 4.25$, $S_1 = 0.3$, $S_2 = 0.4$, and $q = 0.3$. $\omega \in \Omega_2$ as $1 + (V - S_2)/F_1 = 4 < a = 4 < 4.5 = 1 + (V - S_2)/F_1$. From (21) and (28),

$$p_2(D_1) = \frac{(0.8 - 4.25D_1)(0.7 + D_1)}{0.7 - 3.25D_1}$$

$$\psi(D_1) = \frac{2.1 + 3D_1}{5.1 - 7D_1}.$$

In a PIE, $D_1 = 0.1524$, $p_1 = 0.8524$, and $p_2 = 0.6341$. Then, however, $\bar{D}_1 = 0.1837 < 0.2 = F_1$, so a PIE does not exist. As $p_2(0) = 0.8 > 0.4118 = \psi(0)$, a NIE does not exist. As $V - S_2 = 0.6 < 0.7297 = \psi(F_1)$, a FIE1 does not exist. As $EW(F_1) = 0.2061 < 0.24 = EW(0)$ when $D_1 = F_1$ (so that $p_1 = 0.9$ and $p_2 = 0.9$), a FIE2 does not exist. As pointed out in Subsection 3.4, (31) is violated in case 2, so that an ERE does not exist either ($F_2 = 0.2944 < 0.4 = S_2$ for $p_1 = 0.9$, $p_2 = 1$).[14]

Example 4: Let $V = 1$, $F_1 = 0.2$, $a = 6$, $S_1 = 0.3$, $S_2 = 0.4$, and $q = 0.4$. Since $V = 1 > 0.7 = S_1 + F_1$, $S_2 = 0.4 < 0.5 = S_1 + F_1$, and $a = 6 < 8 = 1 + (V - S_1)/(F_1 - S_2 + S_1)$, we have $\omega \in \Omega_3$. This example highlights the importance of the discontinuity in the mapping from investment levels, D_1, to period-2 prices, p_2. Equations (21) and (28) become

$$p_2(D_1) = \frac{(0.8 - 6D_1)(0.7 + D_1)}{0.7 - 5D_1}$$

$$\psi(D_1) = \frac{2.8 + 4D_1}{5.8 - 6D_1}.$$

Since $p_2(0) = 0.8 > 0.4828$, there is not a NIE. $p_2(D_1) = \psi(D_1)$ for $D_1 = 0.1206$, which implies $p_1 = 0.8206$ and $p_2 = 0.6466$. That this is not a PIE follows from the fact that $\bar{D}_1 = 0.1572 < 0.2 = F_1$. For $D_1 \in (0.14, 0.2)$, we have $p_2(D_1) > 1$. This implies that there is not a PIE in this interval (since $p_2(D_1) > 1 > \psi(D_1)$) and that there is not a FIE1 either (since $p_2(F_1) > 1 = V$). As $EW(F_1) = 0.20 < 0.2533 = EW(0)$ when $D_1 = F_1$, a FIE2 does not exist. Finally, the condition for the existence of an ERE in case 3 is violated: $p_2(F_1) = 1.2 > 1 = V$ ($F_2 = 0.3333 < 0.4 = S_2$ for $p_1 = 0.9$, $p_2 = 1$).

Example 5: Let $V = 1$, $F_1 = 0.2$, $a = 6$, $S_1 = 0.3$, $S_2 = 0.35$, and $q = 0.4$. $\omega \in \Omega_4$ because $V = 1 > 0.5 = S_1 + F_1$, $S_2 = 0.35 < 0.5 = S_1 + F_1$, and $a = 6 > 5.3333 = 1 + (V - S_1)/(F_1 - S_2 + S_1)$. Equation (21) becomes

$$p_2(D_1) = \frac{(0.85 - 6D_1)(0.7 + D_1)}{0.7 - 5D_1}$$

[14] In this example an equilibrium exists for different distributions of F_1 across arbitrageurs. Suppose 90% of the arbitrageurs have funds 0.18, and 10% "big" arbitrageurs have 0.38 (notice that $0.9 \cdot 0.18 + 0.1 \cdot 0.38 = 0.2 = F_1$). Suppose the former invest 0.1271 and the latter 0.38, so that $D_1 = 0.1524$ ($= 0.9 \cdot 0.1271 + 0.1 \cdot 0.38$). This yields $p_1 = 0.8524$, $p_2 = 0.6341$, and $\bar{D}_1 = 0.1837$. An equilibrium prevails because any investment between 0 and 0.18 yields the same level of expected wealth for the former arbitrageurs and being fully invested is the optimal choice for "big" arbitrageurs.

The equation for $\psi(D_1)$ (equation (28)) reads as in Example 4. Since $p_2(0) = 0.85 > 0.4828$, there is not a NIE. $p_2(D_1) = \psi(D_1)$ if, and only if, $D_1 = 0.1451$, $p_1 = 0.8451$, and $p_2 = 0.6858$. As $\bar{D}_1 = 0.1768 < 0.2 = F_1$, this is not an equilibrium. As shown in Subsection 3.3 (step 2), both $p_2(D_1)$ and $V - S_2$ are potential equilibrium price levels for $D_1 \in (D_1^\infty, F_1]$. However, $p_2(F_1) = 1.05$ ($> 1 = V$) contradicts the definition of a FIE1. For $p_2 = V - S_2 = 0.65$ and $D_1 = F_1 = 0.2$, we have $p_1 = 0.9$, $\bar{D}_1 = 0.12 < 0.2 = F_1$, but $EW(F_1) = 0.20 < 0.2431 = EW(0)$, so that a FIE2 does not exist either. As in the previous example, the fact that $p_2(F_1) = 1.05 > 1 = V$ rules out an ERE.
This proves Theorem 2. Q.E.D.

Remark: The discontinuity of the relation between D_1 and p_2 alone is not sufficient to obtain a non-existence result. To see this, suppose the condition $F_1 \leq \bar{D}_1$ is satisfied for any prices, p_1 and p_2, which potentially occur in equilibrium. In cases 1 and 2, $\max\{p_2(D_1), V - S_2\} \geq \psi(D_1)$ for all $D_1 \in [0, F_1]$, or $\max\{p_2(D_1), V - S_2\} \leq \psi(D_1)$ for all $D_1 \in [0, F_1]$, or $\max\{p_2(D_1), V - S_2\} = \psi(D_1)$ for some $D_1 \in (0, F_1)$, so that a NIE or a FIE1 or a PIE, respectively, exists. Replacing $[0, F_1]$ with $[0, D_1^\infty)$, the same argument holds true for case 3 (recall that $p_2 = V - S_2$ for $D_1 \in [D_1^0, D_1^\infty)$). In case 4, if $\psi(F_1) \leq V - S_2$, then there is a FIE 1 with $p_1 = V - S_1 + F_1$ and $p_2 = V - S_2$. If $\psi(F_1) > V - S_2$, then $\psi(D_1^*) = V - S_2$ for some $D_1^* \in (D_1^0, F_1)$, and there is a PIE with $p_1 = V - S_1 + D_1^*$ and $p_2 = V - S_2$.

4.2 Uniqueness

In this subsection, we show that an equilibrium, if one exists, is not necessarily unique.

Theorem 3: *There exist parameter values such that the equilibrium is not unique.*

Proof: Again it suffices to construct an example.
Example 6: Let $V = 1$, $F_1 = 0.2$, $a = 10$, $S_1 = 0.4$, $S_2 = 0.6$, and $q = 0.8$ (so that $\omega \in \Omega_3$). There are, then, three equilibria. First, there is a PIE with $D_1 = 0.0289$ ($< 0.06 = D_1^0$), $p_1 = 0.6289$, $p_2 = 0.5755$, and $\bar{D}_1 = 0.2356$ ($> 0.2 = F_1$). Second, as $p_2(F_1) = 0.9333$ and $\psi(F_1) = 0.7619$, there is a FIE1 with $p_1 = 0.8$ and $p_2 = 0.9333$. For $p_2 = V - S_2 = 0.4$ and $D_1 = F_1 = 0.2$, we have $p_1 = 0.8$, $\bar{D}_1 = 0.04 < 0.2 = F_1$, but $EW(F_1) = 0.14 < 0.44 = EW(0)$, so that a FIE2 does not exist. Third, since $p_2(F_1) \leq V$, there is an ERE with $p_1 = 0.8$, $p_2 = 1$, $F_2 = 0.7$, and $D_2 = 0.6$.
Example 7: As another example, let $V = 1$, $F_1 = 0.2$, $a = 35$, $S_1 = 0.4$, $S_2 = 0.5$, and $q = 0.8$ (so that $\omega \in \Omega_4$). As $p_2(D_1) > \psi(D_1)$ for all $D_1 \in [0, D_1^\infty)$, there is not a NIE or a PIE with $D_1 < D_1^\infty$. $p_2(D_1) = \psi(D_1)$ for $D_1 = 0.0392$. This, however, implies $\bar{D}_1 = 0.0691 < 0.2 = F_1$, so a PIE does not prevail. As $1 = V > p_2(F_1) = 0.8129 > 0.7619 = \psi(F_1)$, there is a FIE1 with $p_1 = 0.8$ and $p_2 = 0.8129$. Interestingly, suppose $D_1 = F_1$, such that $p_1 = 0.8$, and $p_2 = V - S_2 = 0.5$. Then, $\bar{D}_1 = 0.0152$ and $EW(F_1) = 0.39 > 0.36 = EW(0)$, so that a FIE2 prevails. Finally, as $p_2(F_1) = 0.8129 \leq 1 = V$, there

is also an ERE (with $F_2 = 1.95 > 0.5 = S_2$). So as in Example 6, three equilibria exist, but here all three equilibria entail that arbitrageurs are fully invested. Q.E.D.[15]

4.3 Sunspots

A sunspot equilibrium is an equilibrium such that at least two different period-2 price levels, p_2, possibly occur, with given non-zero probabilities. Multiplicity of equilibria naturally gives rise to sunspot equilibria.

Theorem 4: *There exist parameter values such that sunspot equilibria exist.*

Proof: As usual, an example suffices to prove the theorem. Reconsider Example 7. Let $p_{21} = 0.8129$, $p_{22} = 0.5$, and $p_{23} = 1$. These are the equilibrium price levels, p_2, in the FIE1, the FIE2, and the ERE, respectively. Furthermore, let $p_1 = 0.8$. Suppose (conditional on worsening noise trader expectations) arbitrageurs expect the period-2 price p_{2i} to prevail with probability π_i, where $\sum_{i=1}^{3} \pi_i = 1$ and $\pi_i \geq 0$ for all $i \in \{1, 2, 3\}$, with strict inequality for at least two $i \in \{1, 2, 3\}$. Analogously to (25), expected wealth as of period 1 is

$$EW(D_1) = \sum_{i=1}^{3} \pi_i \left(0.2 \left[0.2 + 35 D_1 \left(\frac{1}{0.8} - 1 \right) \right] + 0.8 \frac{1}{p_{2i}} \max \left\{ 0.2 + 35 D_1 \left(\frac{p_{2i}}{0.8} - 1 \right), 0 \right\} \right). \quad (33)$$

As $D_1 = 0.2$ maximizes expected wealth for each $i \in \{1, 2, 3\}$, it also maximizes (33). In period 2, arbitrageurs choose $D_2 = F_2 = 0.3129$, $D_2 = F_2 = 0$, or $D_2 = S_2 = 0.5$, depending on whether p_{21}, p_{22}, or p_{23} is realized. Q.E.D.

4.4 Comparative statics

This subsection shows that, as one would expect, the fact that the functions which determine the equilibrium are non-well-behaved possible gives rise to perverse comparative statics properties. For the sake of brevity, we restrict attention to the impact of changes in the period-2 noise trader shock, S_2, on the period-2 price, p_2 (cf. SV, Proposition 2 and 4, pp. 44, 46).

Theorem 5: $dp_2/dS_2 < 0$ *for* $\omega \in \Omega_1 \cup \Omega_2$. *There exist parameters* $\omega \in \Omega_3 \cup \Omega_4$ *such that* $dp_2/dS_2 > 0$.

Proof: Consider a full-investment equilibrium with $p_2 = p_2(F_1)$. Differentiating (21) and evaluating the derivative at $D_1 = F_1$ yields

$$\frac{dp_2}{dS_2} = -\frac{V - S_1 + F_1}{V - S_1 - (a-1)F_1}. \quad (34)$$

[15] It may be noted that SV (p. 46) presume that if their stability condition is violated, then the only period-2 equilibrium price level is $V - S_2$. As the stability condition is violated in cases 3 and 4, Examples 6 and 7 show that other equilibria may emerge as well.

In cases 1 and 2 (where $F_1 < D_1^\infty$), the denominator on the right-hand side of (34) is positive, so that $dp_2/dS_2 < -1$. Moreover, in cases 1 and 2, a PIE satisfies $p_2 = \max\{p_2(D_1), V - S_2\} = \psi(D_1)$, where $p_2(D_1)$ and $\psi(D_1)$ are given by (21) and (28), respectively. As an increase in S_2 decreases $\max\{p_2(D_1), V - S_2\}$ and leaves (28) unaffected, p_2 falls. This proves $dp_2/dS_2 < 0$ in cases 1 and 2. In cases 3 and 4, the denominator in (34) is negative, so that $dp_2/dS_2 > 0$ in a full-investment equilibrium with $p_2 = p_2(F_1)$. Q.E.D.

Contrary to what one might expect, the comparative statics are not necessarily perverse for PIEs with $D_1 > D_1^\infty$. To see this, assume in case 4, $\psi(D_1)$ intersects $p_2(D_1)$ from above for $D_1 \in (D_1^\infty, F_1)$. An increase in S_2 shifts $p_2(D_1)$ upward, such that p_2 falls.

Example 8: Let $V = 1$, $F_1 = 0.2$, $a = 10$, $S_1 = 0.4$, $S_2 = 0.5$, and $q = 0.8$ ($\omega \in \Omega_4$). There is a PIE with $D_1 = 0.0868$ ($> 0.0667 = D_1^\infty$), $p_1 = 0.6868$, $p_2 = p_2(D_1) = 0.6370$, and $\bar{D}_1 = 0.2754$ ($> 0.2 = F_1$). If S_2 rises to 0.51, the period-2 price falls to $p_2 = 0.6304$ ($D_1 = 0.0807$, $p_1 = 0.6807$, $\bar{D}_1 = 0.2705$).

5 Conclusion

A thorough analysis of equilibria in SV's seminal model of limits of arbitrage that "anything is possible": non-existence, multiplicity, and sunspot equilibria. Thus, given limits of arbitrage, non-rational behavior in the stock market may give rise, not only to mispricing, but also to non-well-behaved demand and supply correspondences which possibly lead to more fundamental allocation problems.

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Appendix

Proof of (23)

$$
\begin{aligned}
p_2(D_1) &\gtrless V - S_2 \\
\underbrace{\frac{V - S_2 + F_1 - aD_1}{V - S_1 - (a-1)D_1}}_{>0}(V - S_1 + D_1) &\gtrless V - S_2 \\
(V - S_2 + F_1 - aD_1)(V - S_1 + D_1) &\gtrless (V - S_2)[V - S_1 - (a-1)D_1] \\
F_1(V - S_1 + D_1) &\gtrless aD_1 \underbrace{(S_2 - S_1 + D_1)}_{>0} \\
a &\lessgtr \frac{F_1}{D_1}\left(1 + \frac{V - S_2}{S_2 - S_1 + D_1}\right).
\end{aligned}
$$

Proof of (31)

$$
\begin{aligned}
F_1 + aF_1\left(\frac{V}{V - S_1 + F_1} - 1\right) &\geq S_2 \\
\underbrace{V - S_2 + F_1 - aF_1}_{\equiv A(F_1)} + \underbrace{\frac{aF_1}{V - S_1 + F_1}}_{\equiv F(F_1)} V &\geq V \\
A(F_1) + F(F_1)V &\geq V.
\end{aligned}
$$

Proof of (32)

$$
\begin{aligned}
F_1 &\gtrless \bar{D}_1 \equiv \frac{F_1}{\underbrace{a\left(1-\dfrac{p_2}{p_1}\right)}_{>0}} \\
a\left(1-\frac{p_2}{p_1}\right) &\gtrless 1 \\
a\left[1-\frac{V-S_2-aD_1+F_1}{V-S_1-(a-1)D_1}\right] &\gtrless 1 \\
a\frac{S_2-S_1-(F_1-D_1)}{\underbrace{V-S_1-(a-1)D_1}_{>0}} &\gtrless 1 \\
a[S_2-S_1-(F_1-D_1)] &\gtrless V-S_1-(a-1)D_1 \\
a\underbrace{[S_2-S_1-(F_1-D_1)+D_1]}_{>0} &\gtrless V-S_1+D_1 \\
a &\gtrless \frac{V-S_1+D_1}{S_2-S_1-(F_1-D_1)+D_1} \\
a &\gtrless 1+\frac{V-S_2+(F_1-D_1)}{S_2-S_1-(F_1-D_1)+D_1}.
\end{aligned}
$$

Maple output for the examples

Example 1

```
> V:=1; S_1:=0.2;a:=3;F_1:=0.1;S_2:=0.7;q:=0.1;
```

$$V := 1$$

$$S\_1 := 0.2$$

$$a := 3$$

$$F\_1 := 0.1$$

$$S\_2 := 0.7$$

$$q := 0.1$$

```
> Dinfty:=(V-S_1)/(a-1);
```

$$Dinfty := 0.4000000000$$

```
> D0=(V-S_2+F_1)/a;
```

$$\mathrm{D0} = 0.1333333333$$

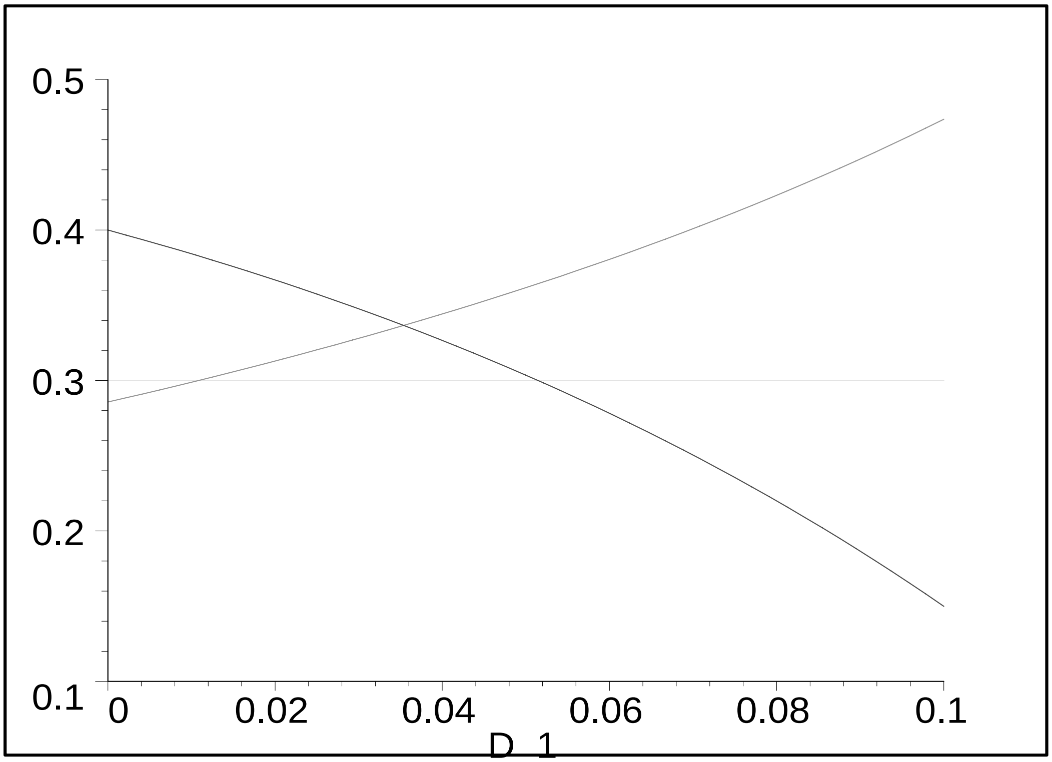
```
> p_2:=(V-S_2+F_1-a*D_1)*(V-S_1+D_1)/(V-S_1-(a-1)*D_1);
```

$$p\_2 := \frac{(0.4 - 3\,D\_1)\,(0.8 + D\_1)}{0.8 - 2\,D\_1}$$

```
> psi:=q/(1/(V-S_1+D_1)-(1-q)/V);
```

$$\psi := \frac{0.1}{\dfrac{1}{0.8 + D\_1} - 0.9}$$

```
> plot([p_2,psi,V-S_2],D_1=0..F_1,y=0.1..0.5,discont =
> true,labels=[D_1,``]);
```




```
> solve(p_2=psi,D_1);
```

$$-0.8000000000,\ 0.3349907290,\ 0.03537964136$$

```
> D_1star:=.3537964136e-1; p_1star=V-S_1+D_1star;
> p_2star=eval(psi,D_1=D_1star);
```

$$D\_1star := 0.03537964136$$

$$p\_1star = 0.8353796414$$

$$p\_2star = 0.3366317250$$

```
> bard1:=F_1/(a*(1-.3366317250/.8353796414));
```

$$bard1 := 0.05583178823$$

Example 2

```
> V:=1; S_1:=0.3;a:=1.2;F_1:=0.2;S_2:=0.4;q:=0.5;
```

$$V := 1$$

$$S\_1 := 0.3$$

$$a := 1.2$$

$$F\_1 := 0.2$$

$$S\_2 := 0.4$$

$$q := 0.5$$

```
> Dinfty:=(V-S_1)/(a-1);
```

$$Dinfty := 3.500000000$$

```
> D0=(V-S_2+F_1)/a;
```

$$\mathrm{D0} = 0.6666666667$$

```
> p_1:=V-S_1+D_1;
> p_2:=(V-S_2+F_1-a*D_1)*(V-S_1+D_1)/(V-S_1-(a-1)*D_1);
```

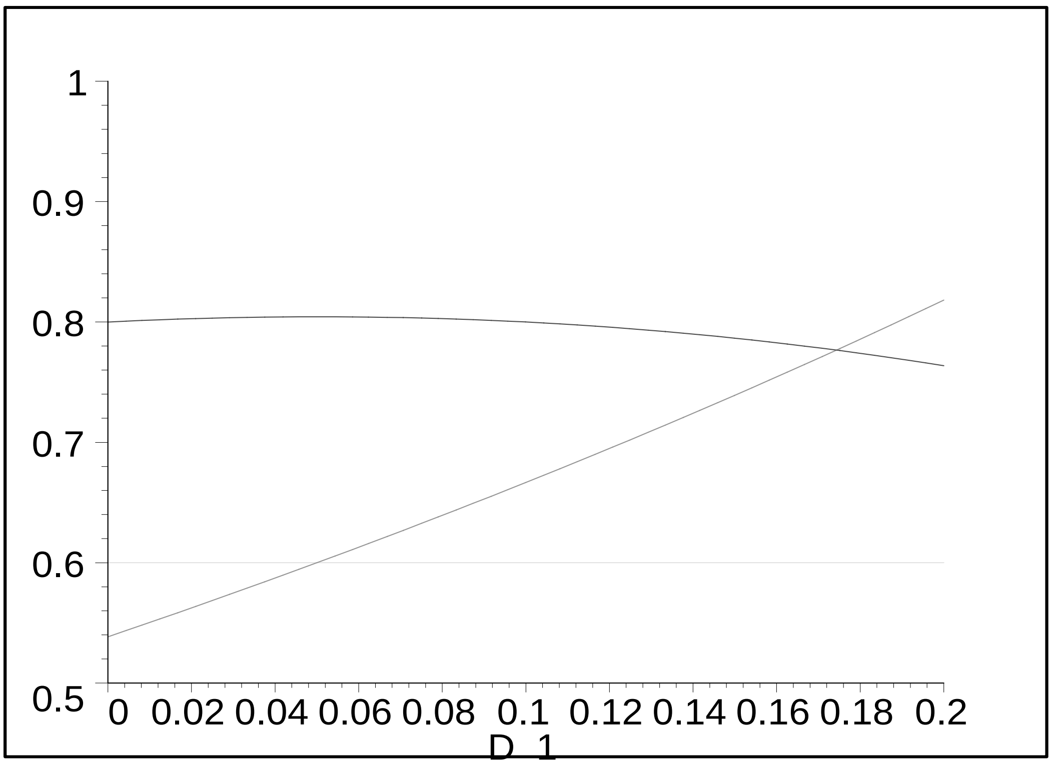
$$p\_1 := 0.7 + D\_1$$

$$p\_2 := \frac{(0.8 - 1.2\,D\_1)\,(0.7 + D\_1)}{0.7 - 0.2\,D\_1}$$

```
> psi:=q/(1/(V-S_1+D_1)-(1-q)/V);
```

$$\psi := \frac{0.5}{\dfrac{1}{0.7 + D\_1} - 0.5}$$

```
> plot([p_2,psi,V-S_2],D_1=0..F_1,y=0.5..V,discont =
> true,labels=[D_1,``]);
```




```
> solve(p_2=psi,D_1);
```

$$-0.7000000000,\ 1.625718035,\ 0.1742819648$$

```
> D_1star:=.1742819648; p1=V-S_1+D_1star; p2=eval(psi,D_1=D_1star);
```

$$D\_1star := 0.1742819648$$

$$p1 = 0.8742819648$$

$$p2 = 0.7766438285$$

```
> bard1:=F_1/(a*(1-.7766438285/.8742819648));
```

$$bard1 := 1.492384701$$

Example 3

```
> V:=1; S_1:=0.3;a:=4.25;F_1:=0.2;S_2:=0.4;q:=0.3;
```

$$V := 1$$

$$S\_1 := 0.3$$

$$a := 4.25$$

$$F\_1 := 0.2$$

$$S\_2 := 0.4$$

$$q := 0.3$$

```
> Dinfty:=(V-S_1)/(a-1);
```

$$Dinfty := 0.2153846154$$

```
> D0=(V-S_2+F_1)/a;
```

$$\mathrm{D0} = 0.1882352941$$

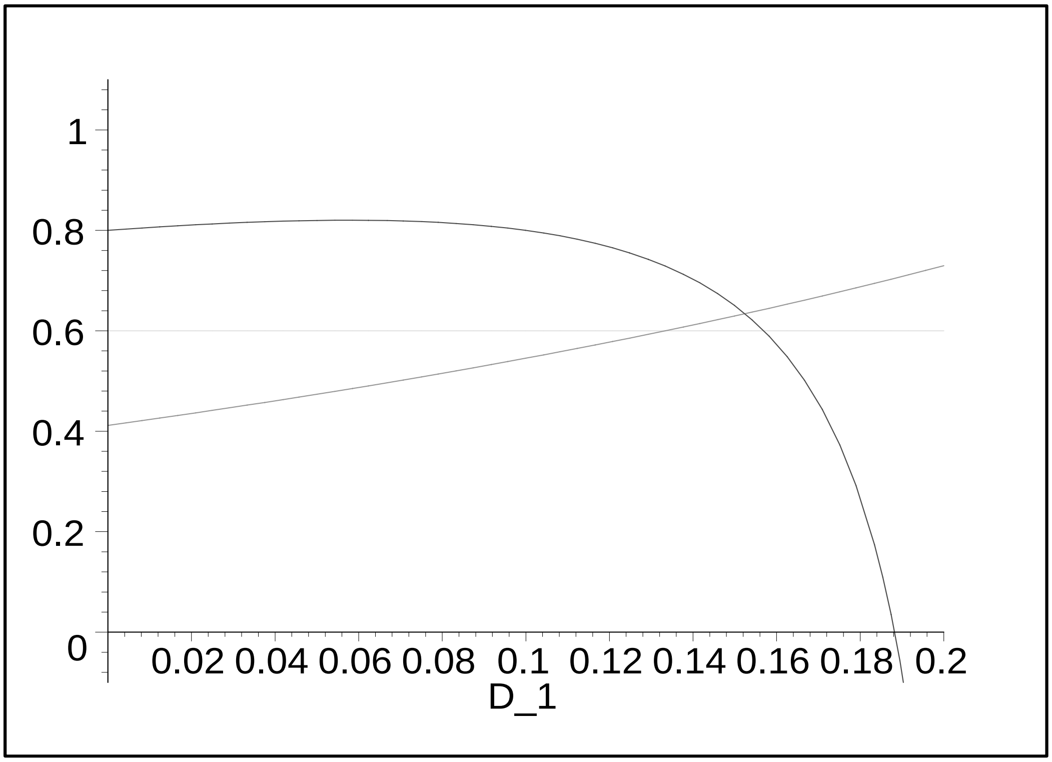
```
> p_2:=(V-S_2+F_1-a*D_1)*(V-S_1+D_1)/(V-S_1-(a-1)*D_1);
```

$$p\_2 := \frac{(0.8 - 4.25\,D\_1)\,(0.7 + D\_1)}{0.7 - 3.25\,D\_1}$$

```
> psi:=q/(1/(V-S_1+D_1)-(1-q)/V);
```

$$\psi := \frac{0.3}{\dfrac{1}{0.7 + D\_1} - 0.7}$$

```
> plot([p_2,psi,V-S_2],D_1=0..F_1,y=-0.1..1.1,discont =
> true,labels=[D_1,``]);
```




```
> solve(p_2=psi,D_1);
```

$$-0.7000000000,\ 0.4366571407,\ 0.1524184896$$

```
> D_1star:=.1524184896; p_1star=V-S_1+D_1star;
> p_2star=eval(psi,D_1=D_1star);
```

$$D\_1star := 0.1524184896$$

$$p\_1star = 0.8524184896$$

$$p\_2star = 0.6340715895$$

```
> bard1:=F_1/(a*(1-.6340715895/.8524184896));
```

$$bard1 := 0.1837159641$$

Example 4

```
> V:=1; S_1:=0.3;a:=6;F_1:=0.2;S_2:=0.4;q:=0.4;
```

$$V := 1$$

$$S\_1 := 0.3$$

$$a := 6$$

$$F\_1 := 0.2$$

$$S\_2 := 0.4$$

$$q := 0.4$$

```
> Dinfty:=(V-S_1)/(a-1);
```

$$Dinfty := 0.1400000000$$

```
> D0=(V-S_2+F_1)/a;
```

$$\mathrm{D0} = 0.1333333333$$

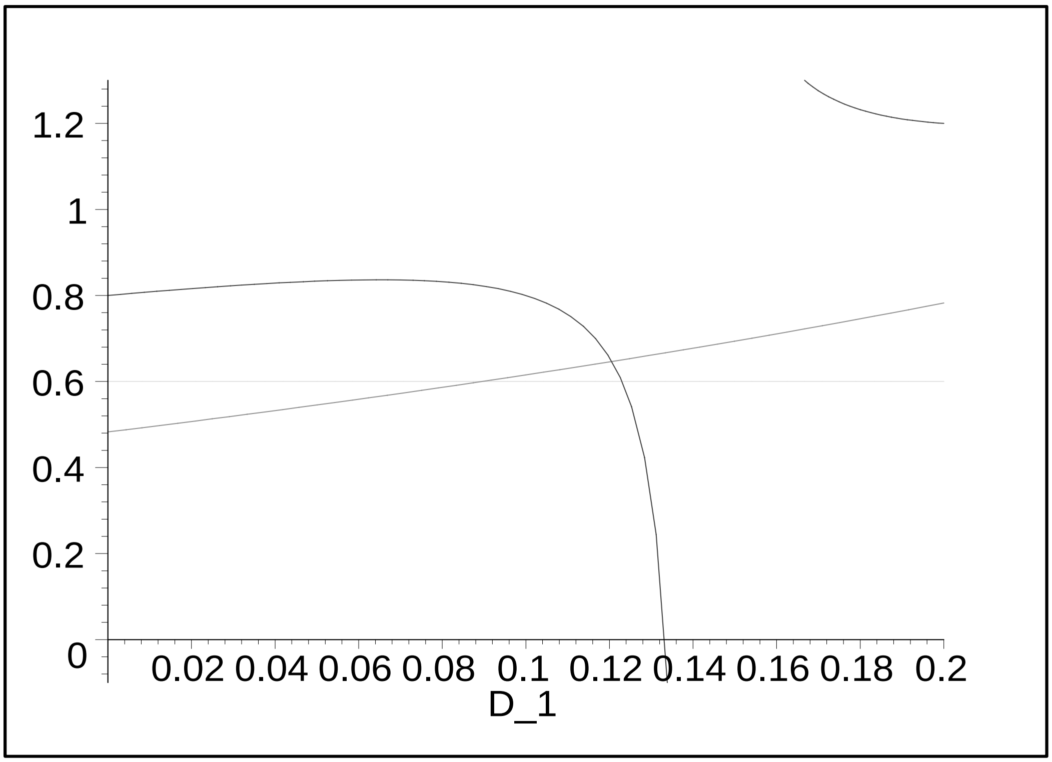
```
> p_2:=(V-S_2+F_1-a*D_1)*(V-S_1+D_1)/(V-S_1-(a-1)*D_1);
```

$$p\_2 := \frac{(0.8 - 6\,D\_1)\,(0.7 + D\_1)}{0.7 - 5\,D\_1}$$

```
> psi:=q/(1/(V-S_1+D_1)-(1-q)/V);
```

$$\psi := \frac{0.4}{\dfrac{1}{0.7 + D\_1} - 0.6}$$

```
> plot([p_2,psi,V-S_2],D_1=0..F_1,y=-0.1..1.3,discont =
> true,labels=[D_1,``]);
```




```
> solve(p_2=psi,D_1);
```

$$-0.7000000000,\ 0.4238593785,\ 0.1205850660$$

```
> D_1star:=.1205850660; p_1star=V-S_1+D_1star;
> p_2star=eval(psi,D_1=D_1star);
```

$$D\_1star := 0.1205850660$$

$$p\_1star = 0.8205850660$$

$$p\_2star = 0.6465767720$$

```
> bard1:=F_1/(a*(1-.6465767720/.8205850660));
```

$$bard1 := 0.1571927114$$

Example 5

```
> V:=1; S_1:=0.3;a:=6;F_1:=0.2;S_2:=0.35;q:=0.4;
```

$$V := 1$$

$$S\_1 := 0.3$$

$$a := 6$$

$$F\_1 := 0.2$$

$$S\_2 := 0.35$$

$$q := 0.4$$

```
> Dinfty:=(V-S_1)/(a-1);
```

$$Dinfty := 0.1400000000$$

```
> D0=(V-S_2+F_1)/a;
```

$$\mathrm{D0} = 0.1416666667$$

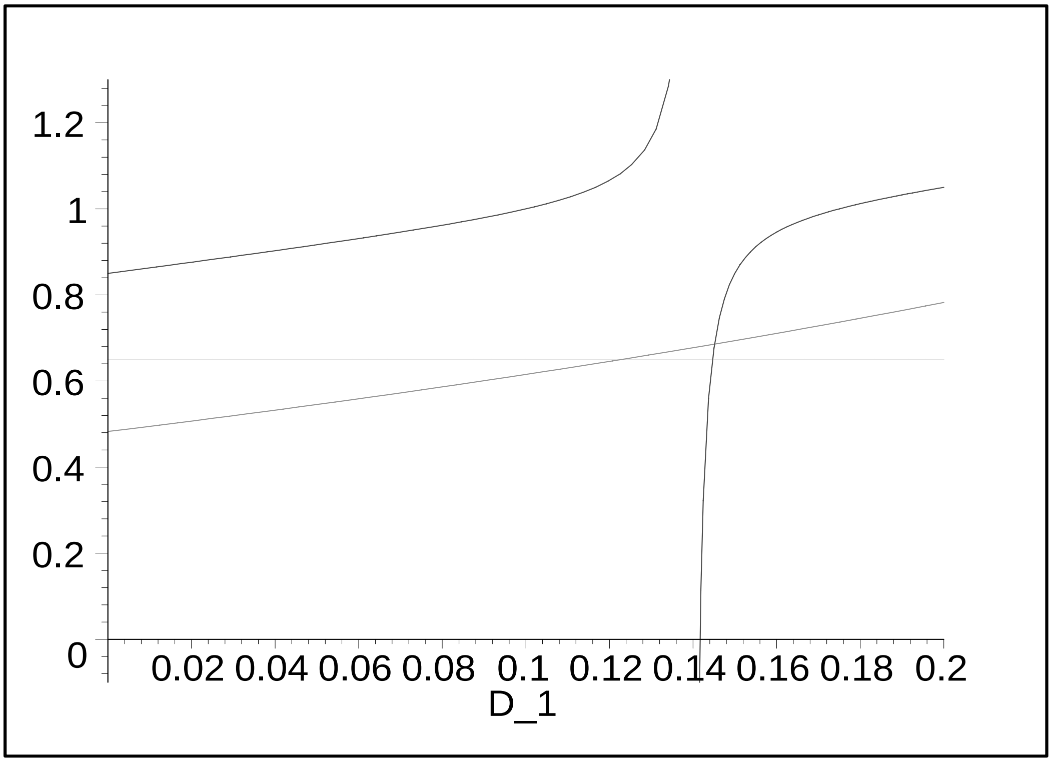
```
> p_2:=(V-S_2+F_1-a*D_1)*(V-S_1+D_1)/(V-S_1-(a-1)*D_1);
```

$$p\_2 := \frac{(0.85 - 6\,D\_1)\,(0.7 + D\_1)}{0.7 - 5\,D\_1}$$

```
> psi:=q/(1/(V-S_1+D_1)-(1-q)/V);
```

$$\psi := \frac{0.4}{\dfrac{1}{0.7 + D\_1} - 0.6}$$

```
> plot([p_2,psi,V-S_2],D_1=0..F_1,y=-0.1..1.3,discont =
> true,labels=[D_1,‘‘]);
```




```
> solve(p_2=psi,D_1);
```

$$-0.7000000000,\ 0.4076297028,\ 0.1451480750$$

```
> D_1star:=.1451480750; p_1star=V-S_1+D_1star;
> p_2star=eval(psi,D_1=D_1star);
```

$$D\_1star := 0.1451480750$$

$$p\_1star = 0.8451480750$$

$$p\_2star = 0.6858421172$$

```
> bard1:=F_1/(a*(1-.6858421172/.8451480750));
```

$$bard1 := 0.1768396040$$

Example 6

```
> V:=1; S_1:=0.4;a:=10;F_1:=0.2;S_2:=0.6;q:=0.8;
```

$$V := 1$$

$$S\_1 := 0.4$$

$$a := 10$$

$$F\_1 := 0.2$$

$$S\_2 := 0.6$$

$$q := 0.8$$

```
> Dinfty:=(V-S_1)/(a-1);
```

$$\mathit{Dinfty} := 0.06666666667$$

```
> D0=(V-S_2+F_1)/a;
```

$$\mathrm{D0} = 0.06000000000$$

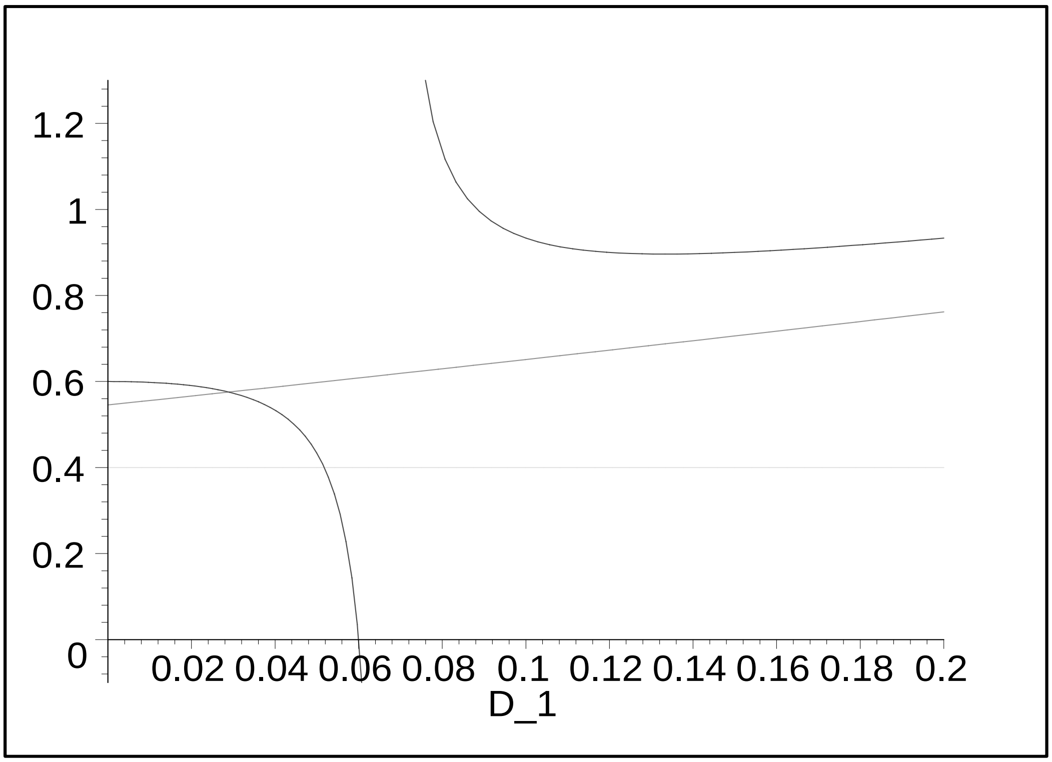
```
> p_2:=(V-S_2+F_1-a*D_1)*(V-S_1+D_1)/(V-S_1-(a-1)*D_1);
```

$$p\_2 := \frac{(0.6 - 10\,D\_1)\,(0.6 + D\_1)}{0.6 - 9\,D\_1}$$

```
> psi:=q/(1/(V-S_1+D_1)-(1-q)/V);
```

$$\psi := \frac{0.8}{\dfrac{1}{0.6 + D\_1} - 0.2}$$

```
> plot([p_2,psi,V-S_2],D_1=0..F_1,y=-0.1..1.3,discont =
> true,labels=[D_1,``]);
```




```
> solve(p_2=psi,D_1);
```

$$-0.6000000000,\ 0.8311234224,\ 0.02887657760$$

```
> D_1star:=.2887657760e-1; p_1star=V-S_1+D_1star;
> p_2star=eval(psi,D_1=D_1star);
```

$$D\_1star := 0.02887657760$$

$$p\_1star = 0.6288765776$$

$$p\_2star = 0.5754827918$$

```
> bard1:=F_1/(a*(1-.5754827918/.6288765776));
```

$$bard1 := 0.2355617114$$

Example 7

```
> V:=1; S_1:=0.4;a:=35;F_1:=0.2;S_2:=0.5;q:=0.8;
```

$$V := 1$$

$$S\_1 := 0.4$$

$$a := 35$$

$$F\_1 := 0.2$$

$$S\_2 := 0.5$$

$$q := 0.8$$

```
> Dinfty:=(V-S_1)/(a-1);
```

$$Dinfty := 0.01764705882$$

```
> D0=(V-S_2+F_1)/a;
```

$$\mathrm{D0} = 0.02000000000$$

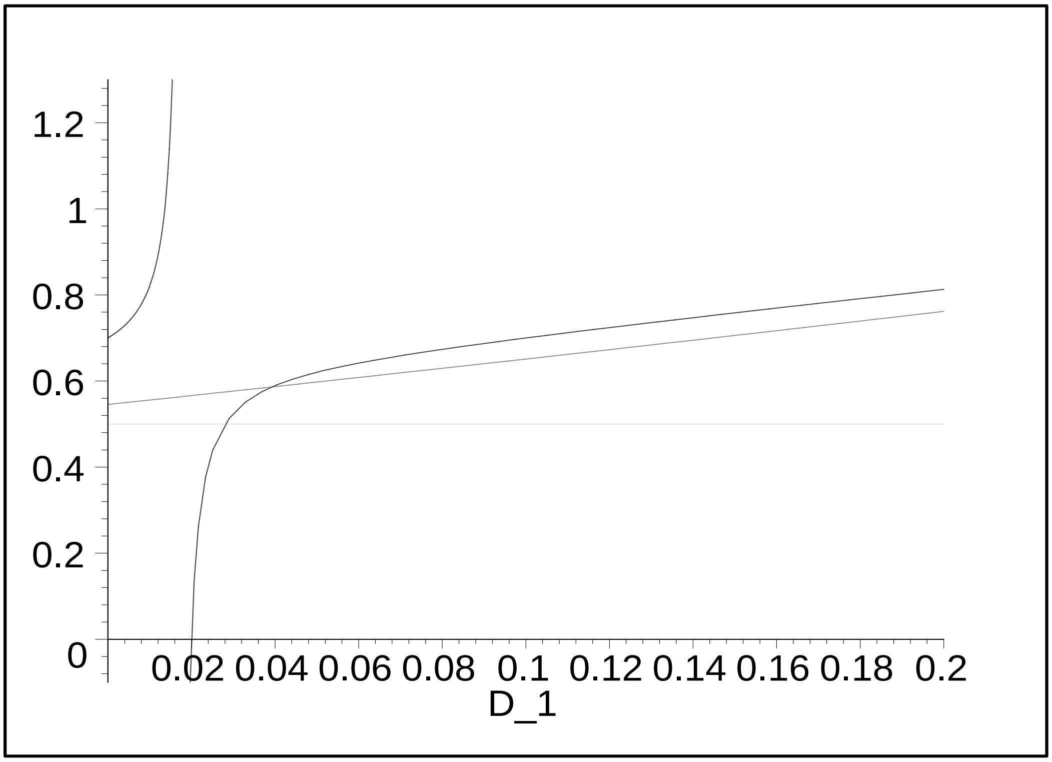
```
> p_2:=(V-S_2+F_1-a*D_1)*(V-S_1+D_1)/(V-S_1-(a-1)*D_1);
```

$$p\_2 := \frac{(0.7 - 35\,D\_1)\,(0.6 + D\_1)}{0.6 - 34\,D\_1}$$

```
> psi:=q/(1/(V-S_1+D_1)-(1-q)/V);
```

$$\psi := \frac{0.8}{\frac{1}{0.6 + D\_1} - 0.2}$$

```
> plot([p_2,psi,V-S_2],D_1=0..F_1,y=-0.1..1.3,discont =
> true,labels=[D_1,``]);
```




```
> solve(p_2=psi,D_1);
```

$$-0.6000000000,\ 0.4950391816,\ 0.03924653270$$

```
> D_1star:=.3924653270e-1; p_1star=V-S_1+D_1star;
> p_2star=eval(psi,D_1=D_1star);
```

$$D\_1star := 0.03924653270$$

$$p\_1star = 0.6392465327$$

$$p\_2star = 0.5863633774$$

```
> bard1:=F_1/(a*(1-.5863633774/.6392465327));
```

$$bard1 := 0.06907374023$$

```
> EW_1:=(1-q)*(F_1+a*F_1*(V/(V-S_1+F_1)-1));
```

$$EW\_1 := 0.3900000000$$

```
> EW_0:=(1-q)*F_1+q*(V/(V-S_2))*F_1;
```

$$EW\_0 := 0.3600000000$$

Example 8

```
> V:=1; S_1:=0.4;a:=10;F_1:=0.2;S_2:=0.5;q:=0.8;
```

$$V := 1$$

$$S\_1 := 0.4$$

$$a := 10$$

$$F\_1 := 0.2$$

$$S\_2 := 0.5$$

$$q := 0.8$$

```
> Dinfty:=(V-S_1)/(a-1);
```

$$Dinfty := 0.06666666667$$

```
> D0=(V-S_2+F_1)/a;
```

$$\mathrm{D0} = 0.07000000000$$

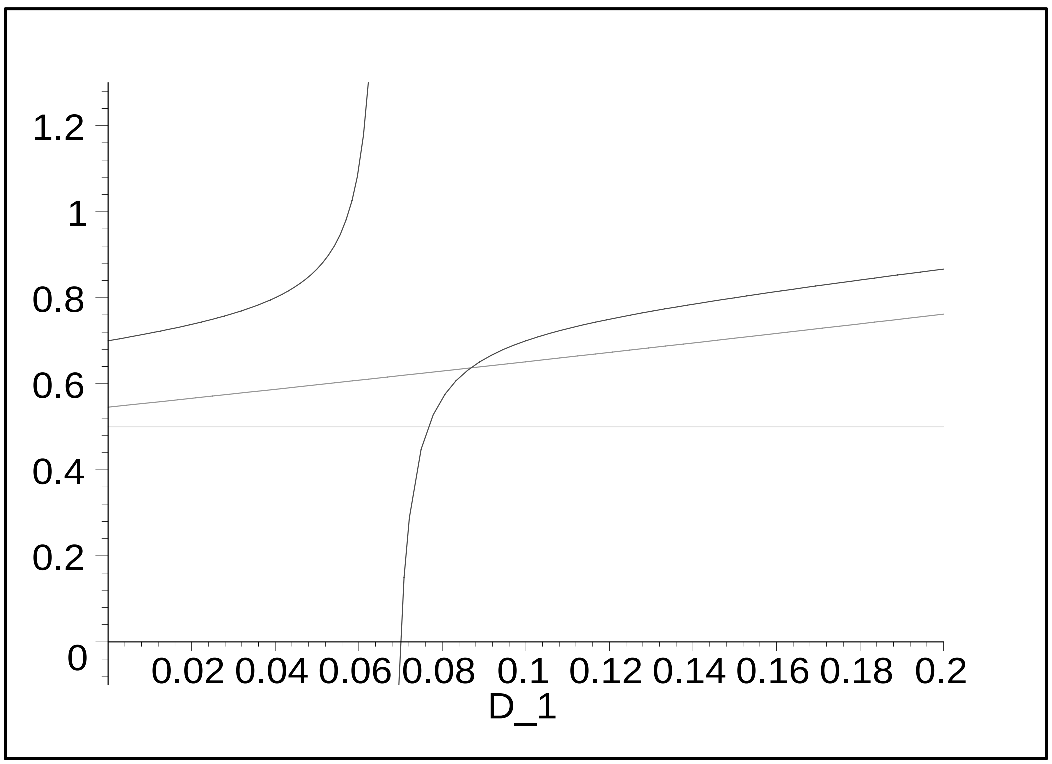
```
> p_2:=(V-S_2+F_1-a*D_1)*(V-S_1+D_1)/(V-S_1-(a-1)*D_1);
```

$$p\_2 := \frac{(0.7 - 10\,D\_1)\,(0.6 + D\_1)}{0.6 - 9\,D\_1}$$

```
> psi:=q/(1/(V-S_1+D_1)-(1-q)/V);
```

$$\psi := \frac{0.8}{\frac{1}{0.6 + D\_1} - 0.2}$$

```
> plot([p_2,psi,V-S_2],D_1=0..F_1,y=-0.1..1.3,discont =
> true,labels=[D_1,``]);
```



```
> solve(p_2=psi,D_1);
```

$$-0.6000000000,\ 0.7831738072,\ 0.08682619283$$

```
> D_1star:=.8682619283e-1; p_1star=V-S_1+D_1star;
> p_2star=eval(psi,D_1=D_1star);
```

$$D\_1star := 0.08682619283$$

$$p\_1star = 0.6868261928$$

$$p\_2star = 0.6369566573$$

```
> bard1:=F_1/(a*(1-.6369566573/.6868261928));
```

$$bard1 := 0.2754492040$$