Understanding Fixed Point Theorems

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Abstract

Fixed point theorems are the standard tool used to prove the existence of equilibria in mathematical economics. This paper shows how to prove a slight generalization of Brouwer's and Kakutani's fixed point theorems in R^n using the familiar techniques of drawing and shifting curves in the plane and is, therefore, intelligible without advanced knowledge of topology. This makes proofs of fixed point theorems accessible to a broader audience.

Keywords: Fixed points, existence of equilibrium.

JEL classification: C62.

1 Introduction

Fixed point theorems are the standard tool used to prove the existence of equilibria in mathematical economics. The strategy of existence proofs is to construct a mapping whose fixed points are solutions to the equations that characterize the equilibrium of the model considered and then apply a fixed point theorem. Sometimes this is not a difficult task, so that existence theorems are accessible to graduate students. The most important existence result in economics is a case in point: equipped with Brouwer's fixed point theorem, it is quite easy to prove the existence of a general equilibrium in a Walrasian system of continuous excess demand functions (see, e.g., Varian 1984, pp. 195-6). The "archetypical existence proof in game theory" (Fudenberg and Tirole, 1991, p. 29) is another good example: Nash's (1950) proof that every finite game has a mixed-strategy equilibrium is a straightforward application of Kakutani's fixed point theorem to the Cartesian product of the players' reaction correspondences (see, e.g., Fudenberg and Tirole, 1991, pp. 29-30). However, though the *application* of fixed point theorems is sometimes simple, *proofs* of fixed point theorems generally are not. Yet there is one relatively simple graphical approach to proving Brouwer's fixed point theorem set out on the web page www.mathpages.com. This proof builds upon the familiar technique of drawing and shifting curves in the plane. Its main virtue is that it is, therefore, intelligible without advanced knowledge of topology. The present paper serves two purposes. First, it extends the "curve shifting approach" to a fixed point theorem which is slightly more general than Brouwer's and Kakutani's because the continuity concept employed is weaker. Second, we clarify some considerations essential to dealing with fixed points on the boundary of the domain of definition and to generalizing the theorem from R^2 to R^n. In the proof at www.mathpages.com, the former issue is ignored and the proof for R^n with $n > 2$ is sketched only very roughly, though both issues are not easy to resolve.

Section 2 presents our simple proof of Brouwer's fixed point theorem. Section 3 is concerned with several extensions, such as Kakutani's fixed point theorem. Section 4 concludes.

2 Brouwer's fixed point theorem

Consider a mapping $\mathbf{f} : X \to X$, $X \in R^n$ $(n \geq 1)$.

Assumption 1: *X is compact (i.e., bounded and closed) and convex.*

Remark: Denote the projections of $\mathbf{x} \in X$ and X onto the $(x_i, \ldots, x_n)$-space as $\mathbf{x}_i$ and $X_i = \{\mathbf{x}_i | (x_1, \ldots, x_{i-1}) \in R^{i-1}, (x_1, \ldots, x_{i-1}, \mathbf{x}_i) \in X\}$, respectively $(i = 2, \ldots, n)$. Due to Assumption 1, X_i is compact (i.e., bounded and closed) and convex.

Boundedness is trivial. Since X is bounded, there are upper and lower bounds for all x_i $(i = 1, \ldots, n)$.

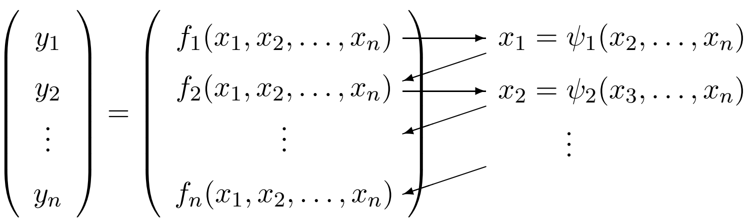

Figure 1: Idea behind the proof

The components x_j $(j = i, \dots, n)$ of vectors $\mathbf{x}_i \in X_i$ cannot take on values x_i outside of these bounds, so X_i is bounded.

To prove that X_i is closed, consider a boundary point $\mathbf{x}'_i$ of X_i. We show that to $\mathbf{x}'_i$ corresponds a point $(x'_1, \dots, x'_{i-1}, \mathbf{x}'_i)$ in X $((x'_1, \dots, x'_{i-1}) \in R^{i-1})$. It then follows that $\mathbf{x}'_i \in X_i$, i.e., that X_i is closed. Suppose, to the contrary, that no point $(x_1, \dots, x_{i-1}, \mathbf{x}'_i) \in R^n$ is in X $((x_1, \dots, x_{i-1}) \in R^{i-1})$. As $\mathbf{x}'_i$ is on the boundary of X_i, there are points $\mathbf{x}_i \in X_i$ close to $\mathbf{x}'_i$ and, hence, points $(x_1, \dots, x_{i-1}, \mathbf{x}_i) \in X$. As $\mathbf{x}_i$ goes to $\mathbf{x}'_i$, $(x_1, \dots, x_{i-1}, \mathbf{x}_i) \in X$ goes to $(x_1, \dots, x_{i-1}, \mathbf{x}'_i)$. Since X is closed, $(x_1, \dots, x_{i-1}, \mathbf{x}'_i)$ is in X, a contradiction.

To prove that X_i is convex, consider two points $\mathbf{x}'_i, \mathbf{x}''_i \in X_i$. By assumption, there exist vectors $(x'_1, \dots, x'_{i-1}), (x''_1, \dots, x''_{i-1}) \in R^{i-1}$ such that $(x'_1, \dots, x'_{i-1}, \mathbf{x}'_i)$, $(x''_1, \dots, x''_{i-1}, \mathbf{x}''_i) \in X$. Convexity of X implies that there exists a λ $(0 \leq \lambda \leq 1)$ such that $\lambda(x'_1, \dots, x'_{i-1}, \mathbf{x}'_i) + (1-\lambda)(x''_1, \dots, x''_{i-1}, \mathbf{x}''_i) \in X$. Convexity of X_i, i.e. $\lambda \mathbf{x}'_i + (1-\lambda)\mathbf{x}''_i \in X_i$, follows from the fact that there exists a vector $(x_1, \dots, x_{i-1}) \in R^{i-1}$ such that $(x_1, \dots, x_{i-1}, \lambda \mathbf{x}'_i + (1-\lambda)\mathbf{x}''_i) \in X$, namely $(x_1, \dots, x_{i-1}) = \lambda(x'_1, \dots, x'_{i-1}) + (1 - \lambda)(x''_1, \dots, x''_{i-1})$.

Remark: A property that will be used repeatedly is that the set $\{x_i | \mathbf{x}_{i+1} \in X_{i+1} \text{ given}, (x_i, \mathbf{x}_{i+1}) \in X_i\}$ is a closed interval $[\underline{x}_i, \overline{x}_i]$. The fact that the set is an interval follows from the observation that X_i is convex. The fact that the interval is closed follows from the fact that X_i is closed. For $i = 1, \dots, n-1$, the bounds of this interval, $\underline{x}_i$ and $\overline{x}_i$, may change when $\mathbf{x}_{i+1}$ changes (unless X is rectangular). For $i = n$, we have $X_n = \{x_n | (x_1, \dots, x_{n-1}) \in R^{n-1}, (x_1, \dots, x_{n-1}, x_n) \in X\} = \{x_n | \mathbf{x} \in X\}$, and $[\underline{x}_n, \overline{x}_n] = \{x_n | x_n \in X_n\}$ is uniquely determined.

$\mathbf{y} = \mathbf{f}(\mathbf{x})$ is the set of points in X that corresponds to $\mathbf{x} \in X$. A *fixed point (FP)* is a point $\mathbf{x} \in X$ such that $\mathbf{x} \in \mathbf{f}(\mathbf{x})$. The question is what continuity conditions have to be placed on the mapping $\mathbf{f}$ in order to ensure the existence of a FP. Brouwer assumes that $\mathbf{f}$ is a continuous function. A continuous curve between two points in a given set is a curve in the set which connects the two points without jumps. Since it is our aim to prove a fixed point theorem using the "curve shifting approach", we state this assumption directly in terms of continuous curves (rather than closedness properties of the graph or

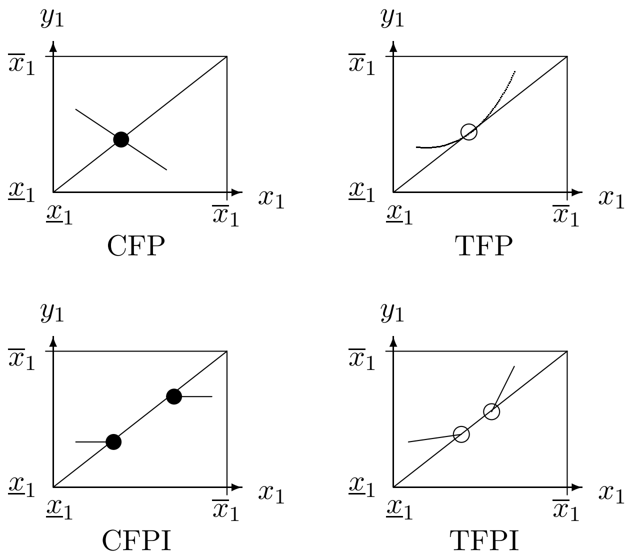

Figure 2: FPs $\psi_1(\mathbf{x}_2)$ of $y_1 = f_1(\mathbf{x})$

sequences):

Assumption 2: $\mathbf{f}(\mathbf{x})$ *is single-valued. For all* $\mathbf{x}', \mathbf{x}'' \in X$ *and for all continuous curves* $C \subset X$ *from* $\mathbf{x}'$ *to* $\mathbf{x}''$*, there exists a unique continuous curve in the graph* $\{(y_i, \mathbf{x}) | y_i = f_i(\mathbf{x}), \mathbf{x} \in C\}$ *which connects* $(f_i(\mathbf{x}'), \mathbf{x}')$ *and* $(f_i(\mathbf{x}''), \mathbf{x}'')$ *(*$i = 1, \ldots, n$*).*

Remark: We will repeatedly use three properties of $\mathbf{f}(\mathbf{x})$ arising from Assumption 2. (1) *Intermediate values:* Assume $f_i(\mathbf{x}') > y_i > f_i(\mathbf{x}'')$ for some y_i, $\mathbf{x}'$, and $\mathbf{x}''$ on a continuous curve C. Then $f_i(\mathbf{x})$ crosses y_i an odd number of times as $\mathbf{x}$ goes from $\mathbf{x}'$ to $\mathbf{x}''$. (2) *Continuous shifting:* Consider a continuous curve in the $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$-space. Holding x_i constant, this determines a continuous curve C. Assume $C \subset X$. Then $f_i(\mathbf{x})$ varies continuously as $\mathbf{x}$ moves along C $(i = 1, \ldots, n)$. Graphically, this means that for any closed interval $[\underline{x}_i, \overline{x}_i]$ such that $\mathbf{x} \in X$ for $x_i \in [\underline{x}_i, \overline{x}_i]$, the graph $\{(y_i, \mathbf{x}) | y_i = f_i(\mathbf{x}), x_i \in [\underline{x}_i, \overline{x}_i], x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \text{ given}\}$ in the (y_i, x_i)-plane shifts continuously as $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ changes. (3) *Projection:* Let C be a continuous curve in X. The projection of a continuous curve in the graph $\{(y_i, \mathbf{x}) | y_i = f_i(\mathbf{x}), \mathbf{x} \in C\}$ onto the (y_i, x_i)-plane is itself a continuous curve $(i = 1, \ldots, n)$. This follows from the fact that no component of a continuous curve must display jumps.

Remark: The idea behind our approach to finding a FP $\mathbf{x} \in \mathbf{f}(\mathbf{x})$ is to consider $\mathbf{f}$ component-wise (see Figure 1). We show first that for all admissible $\mathbf{x}_2$, the mapping $y_1 = f_1(x_1, \mathbf{x}_2)$ has a FP $x_1 = f_1(x_1, \mathbf{x}_2)$. The set of FPs of $f_1(x_1, \mathbf{x}_2)$ associated with $\mathbf{x}_2$ is denoted $\psi_1(\mathbf{x}_2)$. Next we turn to the composed mapping $y_2 = f_2(x_1, x_2, \mathbf{x}_3)$, $x_1 = \psi(x_2, \mathbf{x}_3)$ and show that for all admissible $\mathbf{x}_3$, the

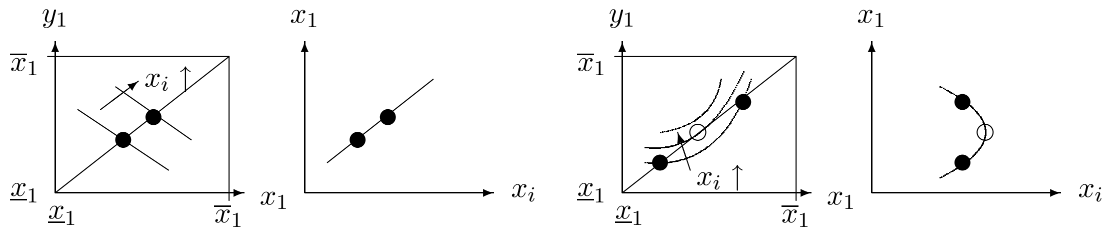

Figure 3: CFPs shift continuously or disappear in pairs

set of FPs $\psi_2(\mathbf{x}_3)$ is non-empty. An induction argument then proves the existence of a FP for the components $i = 3, \ldots, n$. The difficult part of the proof is to characterize how $\psi_i(\mathbf{x}_{i+1})$ depends on $\mathbf{x}_{i+1}$. Here we need the continuous shifting property stated in the previous remark. The existence of component-wise FPs then follows quite easily from the intermediate-value property stated there.

Brouwer's fixed point theorem: *Given Assumptions 1 and 2, the mapping* $\mathbf{f} : X \to X$ *has a FP.*

Lemma 1: *(i) For all* $\mathbf{x}_2 \in X_2$*, the set of FPs* $\psi_1(\mathbf{x}_2)$ *of the mapping* $f_1(x_1, \mathbf{x}_2) : [\underline{x}_1, \overline{x}_1] \to [\underline{x}_1, \overline{x}_1]$ *is non-empty. (ii) For all* $\mathbf{x}_2', \mathbf{x}_2'' \in X_2$ *and for all continuous curves* $C_2 \subset X_2$ *from* $\mathbf{x}_2'$ *to* $\mathbf{x}_2''$*, there exists a continuous curve in the graph* $\{(x_1, \mathbf{x}_2) | x_1 = \psi_1(\mathbf{x}_2), \mathbf{x}_2 \in C_2\}$ *which connects a point in* $(\psi_1(\mathbf{x}_2'), \mathbf{x}_2')$ *and a point in* $(\psi_1(\mathbf{x}_2''), \mathbf{x}_2'')$*.*

Proof: As mentioned in the second remark below Assumption 1, for a given $\mathbf{x}_2$, $[\underline{x}_1, \overline{x}_1]$ is a closed interval. Consider the mapping $f_1(x_1, \mathbf{x}_2) : [\underline{x}_1, \overline{x}_1] \to [\underline{x}_1, \overline{x}_1]$. We distinguish *interior FPs* $x_1 = f_1(x_1, \mathbf{x}_2)$ with $\underline{x}_1 < x_1 < \overline{x}_1$ and *corner FPs* $x_1 = f_1(x_1, \mathbf{x}_2)$ with $x_1 = \underline{x}_1$ or $x_1 = \overline{x}_1$. Interior and corner FPs for the other components x_i of $\mathbf{x}$ are defined analogously.

(1) To begin with, we assume that $f_1(\underline{x}_1, \mathbf{x}_2) > \underline{x}_1$ and $f_1(\overline{x}_1, \mathbf{x}_2) < \overline{x}_1$, so that there are no corner FPs.

(1.i) Consider the continuous curve $\{(x_1, \mathbf{x}_2) | x_1 \in [\underline{x}_1, \overline{x}_1], \mathbf{x}_2 \, \text{given}\} \subset X$. By Assumption 2, there is a unique continuous curve in the graph $\{(y_1, x_1, \mathbf{x}_2) | y_1 = f_1(x_1, \mathbf{x}_2), x_1 \in [\underline{x}_1, \overline{x}_1], \mathbf{x}_2 \, \text{given}\}$, and by the projection property, the projection of this curve onto the (y_1, x_1)-plane is also a continuous curve. By the intermediate value property, this curve intersects the 45-degree line in the (y_1, x_1)-plane at least once. So $\psi_1(\mathbf{x}_2)$ is non-empty. If $f_1(x_1, \mathbf{x}_2) = x_1$ and the slope of $f_1(x_1, \mathbf{x}_2)$ is unity, there is a *FPs interval (FPI)* and $\psi_1(\mathbf{x}_2)$ is vertical. A single FP can be regarded as a degenerated FPI of length zero. A FP is either a *crossing FP (CFP)* with $f_1(x_1, \mathbf{x}_2)$ changing from one side of the 45-degree line to the other or a *tangential FP (TFP)* with $f_1(x_1, \mathbf{x}_2)$ staying on the same side of the 45-degree line. *Crossing fixed points intervals (CFPIs)* and *tangential fixed points intervals (TFPIs)* are defined analogously (the four possible sorts of FPs are illustrated in Figure 2 with filled circles for CFPs and

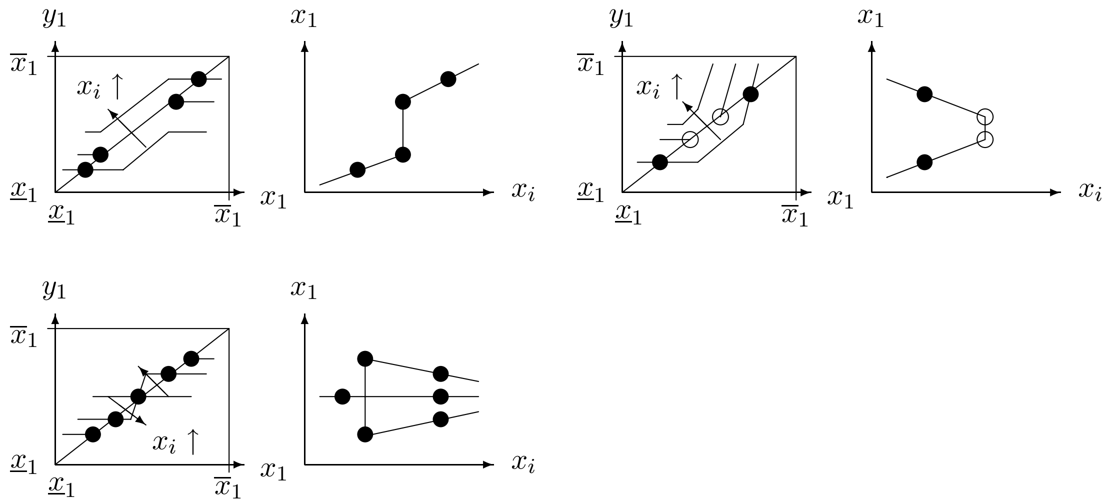

Figure 4: CFPIs shift continuously or disappear in pairs

unfilled circles for TFPs). Counting a CFPI as one CFP, it follows from the fact that $f_1(\underline{x}_1, \mathbf{x}_2) > \underline{x}_1$ and $f_1(\overline{x}_1, \mathbf{x}_2) < \overline{x}_1$ and from the intermediate value property that *for all* $\mathbf{x}_2$*, the number of CFPIs is odd.*

(1.ii) Consider a continuous curve C_2 that connects $\mathbf{x}_2'$ and $\mathbf{x}_2''$. By the continuous shifting property, small changes in $\mathbf{x}_2$ lead to small shifts of the graph $\{(y_1, x_1, \mathbf{x}_2) | y_1 = f_1(x_1, \mathbf{x}_2), x_1 \in [\underline{x}_1, \overline{x}_1], \mathbf{x}_2 \text{ given}\}$ in the (y_1, x_1)-plane. This alters the location of the FPs $\psi_1(\mathbf{x}_2)$ of $y_1 = f_1(x_1, \mathbf{x}_2)$ (see Figure 3, where it is assumed that only component i of $\mathbf{x}_2$ varies, a simplification which is not necessary for the validity of the arguments put forward).

(1.ii.a) Suppose for a moment that there are no FPIs. CFPs move slightly along the 45-degree line as $y_1 = f_1(x_1, \mathbf{x}_2)$ shifts slightly, so $\psi_1(\mathbf{x}_2)$ does not jump and CFPs do not disappear (see the left panel of Figure 3). There is only one way for CFPs to vanish: two CFPs collapse into one TFP (see the right panel of Figure 3). Conversely, there is only one way for new CFPs to emerge as $\mathbf{x}_2$ changes: a TFP splits into two CFPs. So *as* $\mathbf{x}_2$ *changes, CFPs move continuously, or they disappear and appear in pairs.*

(1.ii.b) Next, we allow for FPIs. As mentioned above, $\psi_1(\mathbf{x}_2)$ is vertical at a FPI. Three new possibilities arise of how $\psi_1(\mathbf{x}_2)$ can respond to changes in $\mathbf{x}_2$. One possibility is that a CFP turns into a CFPI, which in turn gives way to a CFP (see the upper left panel of Figure 4). $\psi_1(\mathbf{x}_2)$ does not jump or end in this case. Second, two CFPIs can collapse into one TFPI (see the upper right panel of Figure 4). In this case, CFPIs disappear in pairs. Conversely, pairs of CFPIs appear when a TFPI splits. Third, a CFPI can split into several FPIs (see the lower panel of Figure 4). In this case, two things are important. For one thing, the number of emerging CFPIs is odd. This is because $f_1(x_1, \mathbf{x}_2)$ changes

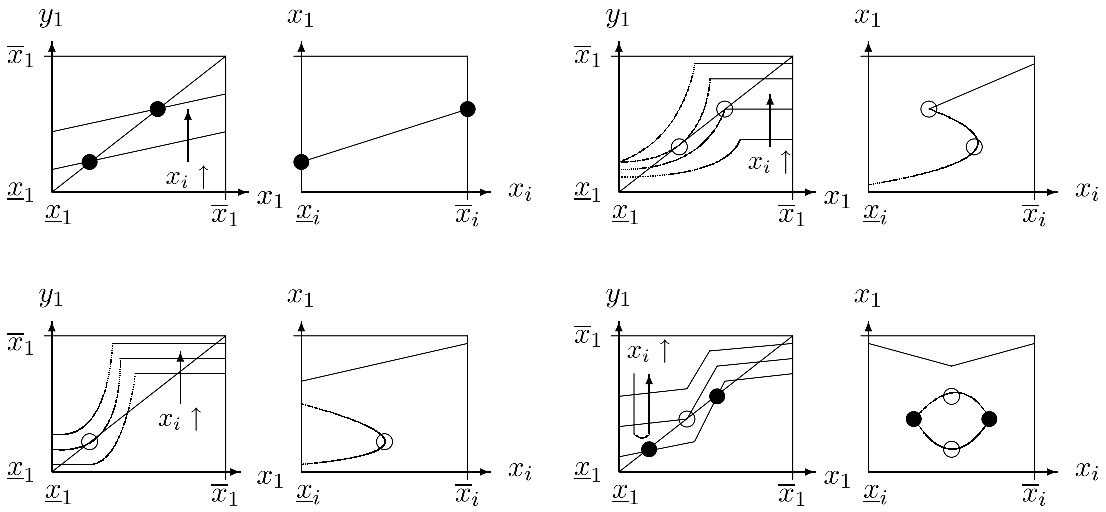

Figure 5: The graph $\{(x_1, \mathbf{x}_2)|x_1 = \psi_1(\mathbf{x}_2), \mathbf{x}_2 \in C_2\}$ contains a continuous curve

from one side of the 45-degree line to the other. TFPIs may obtain in addition, but these are inessential to the subsequent analysis. For another, the new CFPIs are connected to the splitting CFPI without jumps. Conversely, an odd number of CFPIs and several TFPIs can collapse into one CFPI. So we can generalize the statement made in paragraph (1.ii.a) about CFPs to CFPIs: *as* $\mathbf{x}_2$ *changes, CFPIs move continuously, or they disappear and appear in pairs.*

We can now describe the shape of the graph $\{(x_1, \mathbf{x}_2)|x_1 = \psi_1(\mathbf{x}_2), \mathbf{x}_2 \in C_2\}$ that obtains when $\mathbf{x}_2$ goes along the continuous curve C_2 (Figure 5 illustrates the case where only one component i of $\mathbf{x}_2$ varies). The simplest case is that for all $\mathbf{x}_2$, all FPs are CFPs. In this case, $\{(x_1, \mathbf{x}_2)|x_1 = \psi_1(\mathbf{x}_2), \mathbf{x}_2 \in C_2\}$ consists of continuous curves that connect points in $(\psi_1(\mathbf{x}_2'), \mathbf{x}_2')$ and points in $(\psi_1(\mathbf{x}_2''), \mathbf{x}_2'')$ (see the upper left panel of Figure 5). If, by contrast, there are also TFPs, then the graph $\{(x_1, \mathbf{x}_2)|x_1 = \psi_1(\mathbf{x}_2), \mathbf{x}_2 \in C_2\}$ contains backward-bending portions (see the upper right panel of Figure 5), and two new possibilities arise. First, curves in the graph $\{(x_1, \mathbf{x}_2)|x_1 = \psi_1(\mathbf{x}_2), \mathbf{x}_2 \in C_2\}$ that start at $\mathbf{x}_2'$ or $\mathbf{x}_2''$ may return to $\mathbf{x}_2'$ or $\mathbf{x}_2''$, respectively (see the lower left panel of Figure 5). Second, curves in the graph $\{(x_1, \mathbf{x}_2)|x_1 = \psi_1(\mathbf{x}_2), \mathbf{x}_2 \in C_2\}$ may form closed loops (see the lower right panel of Figure 5). Evidently, continuous curves in $\{(x_1, \mathbf{x}_2)|x_1 = \psi_1(\mathbf{x}_2), \mathbf{x}_2 \in C_2\}$ from a point in $(\psi_1(\mathbf{x}_2'), \mathbf{x}_2')$ to a point in $(\psi_1(\mathbf{x}_2''), \mathbf{x}_2'')$ represent an odd number of CFPIs for all $\mathbf{x}_2$, whereas curves in $\{(x_1, \mathbf{x}_2)|x_1 = \psi_1(\mathbf{x}_2), \mathbf{x}_2 \in C_2\}$ which start at and return to $\mathbf{x}_2'$ or $\mathbf{x}_2''$ or form closed loops represent an even number of CFPIs for all $\mathbf{x}_2$. Suppose there is no continuous curve in $\{(x_1, \mathbf{x}_2)|x_1 = \psi_1(\mathbf{x}_2), \mathbf{x}_2 \in C_2\}$ from a point in $(\psi_1(\mathbf{x}_2'), \mathbf{x}_2')$ to a point in $(\psi_1(\mathbf{x}_2''), \mathbf{x}_2'')$. Then *the number of CFPIs must be even for all* $\mathbf{x}_2$. But this contradicts the fact that *for all* $\mathbf{x}_2$*, there is an odd number of CFPIs.* So the supposition that the

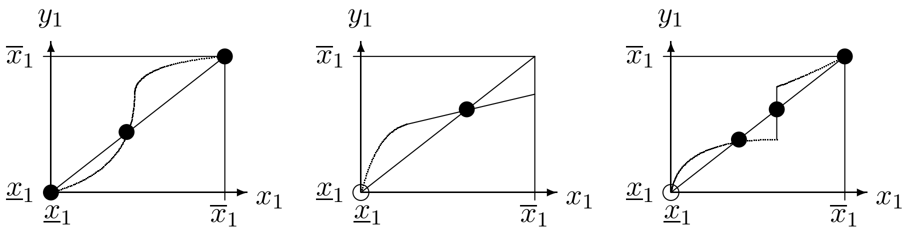

Figure 6: The number of RFPIs is odd

set of continuous curves in $\{(x_1, \mathbf{x}_2)|x_1 = \psi_1(\mathbf{x}_2), \mathbf{x}_2 \in C_2\}$ is empty must be wrong. This completes the proof of Lemma 1 for the case of no corner FPs. Changes in $\mathbf{x}_2$ lead to continuous variations in $\underline{x}_1$ and $\overline{x}_1$. But this is inessential as long as only interior FPs can occur.

(2) Next we allow for $f_1(\underline{x}_1, \mathbf{x}_2) = \underline{x}_1$ or $f_1(\overline{x}_1, \mathbf{x}_2) = \overline{x}_1$, i.e. for corner FPs (*CoFPs*). A *corner FPI (CoFPI)* prevails when $f_1(\underline{x}_1, \mathbf{x}_2) = \underline{x}_1$ and $f_1(x_1, \mathbf{x}_2) = x_1$ for x_1 slightly greater than $\underline{x}_1$ or when $f_1(\overline{x}_1, \mathbf{x}_2) = \overline{x}_1$ and $f_1(x_1, \mathbf{x}_2) = x_1$ for x_1 slightly less than $\overline{x}_1$. As before, a CoFP can be regarded as a degenerated CoFPI of length zero. We say that a FP or a FPI at $\underline{x}_1$ is *robust* (i.e., a *RCoFP* or a *RCoFPI*, respectively) if $f_1(x_1, \mathbf{x}_2)$ first goes to the area below the 45-degree line as x_1 rises. Similarly a FP at $\overline{x}_1$ is a RCoFP or a RCoFPI if $f_1(x_1, \mathbf{x}_2)$ first goes to the area above the 45-degree line as x_1 falls. A FP or FPI at $\underline{x}_1$ is called *fragile* (i.e., a *FCoFP* or a *FCoFPI*, respectively) if $f_1(x_1, \mathbf{x}_2)$ first goes to the area above the 45-degree line as x_1 rises. Similarly a FP or FPI at $\overline{x}_1$ is called fragile if $f_1(x_1, \mathbf{x}_2)$ first goes to the area below the 45-degree line as x_1 falls.

(2.i) Trivially, if there is a CoFP, $\psi_1(\mathbf{x}_2)$ is non-empty. Denote CFPIs and RCoFPIs as *robust FPIs (RFPIs)*. Then *the number of RFPs is odd* (see Figure 6). To see this, notice that if both $\underline{x}_1$ and $\overline{x}_1$ (left panel of Figure 6) or neither of the two (middle panel of Figure 6) are RCoFPIs, $f_1(x_1, \mathbf{x}_2)$ changes from one side of the 45-degree line to the other an odd number of times. So, by the intermediate value property, the number of CFPIs is odd. If there is exactly one RCoFPI (right panel of Figure 6), $f_1(x_1, \mathbf{x}_2)$ changes from one side of the 45-degree line to the other an even number of times, and the number of CFPIs is even (see Figure 4). In both cases, the number of RFPIs is odd.

(2.ii) Consider a continuous curve C_2 in X_2 from $\mathbf{x}_2'$ to $\mathbf{x}_2''$. By the continuous shifting property, small changes in $\mathbf{x}_2$ move the graph $\{(y_1, x_1, \mathbf{x}_2)|y_1 = f_1(x_1, \mathbf{x}_2), x_1 \in [\underline{x}_1, \overline{x}_1], \mathbf{x}_2 \text{ given}\}$ in the (y_1, x_1)-plane by small amounts, thereby altering the location of the FPs. Moreover, changes in $\mathbf{x}_2$ possibly alter $\underline{x}_1$ and $\overline{x}_1$, which is essential when dealing with CoFPs. This opens up new possibilities for CFPs to disappear and appear. To see this, it suffices to consider FPs at $\underline{x}_1$. FPs at $\overline{x}_1$ can be handled symmetrically.

(2.ii.a) Suppose first that as $\mathbf{x}_2$ changes, $\underline{x}_1$ is constant (see Figure 7). There are three new possibilities

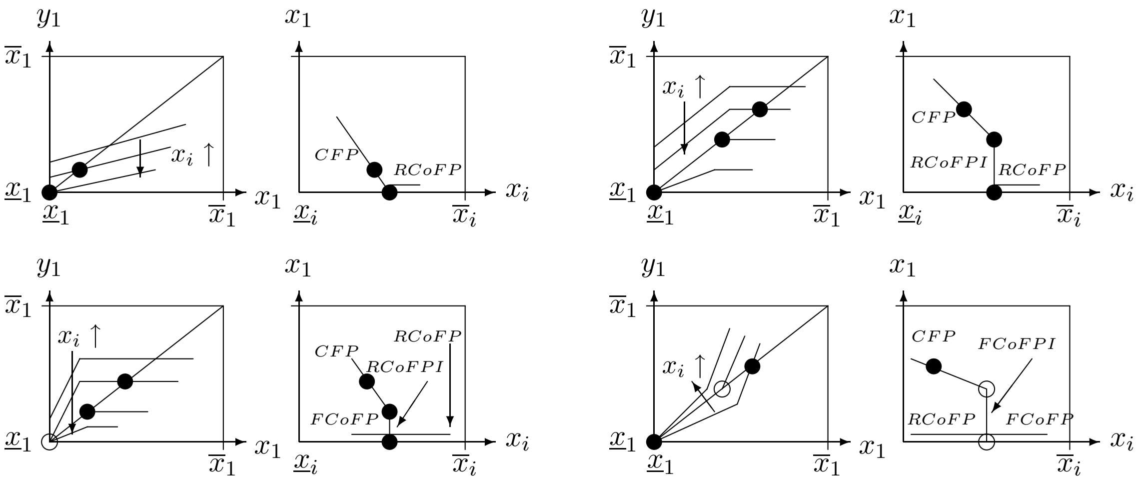

Figure 7: RFPs shift continuously or disappear in pairs when $f_1(x_1, \mathbf{x}_2)$ shifts

for CFPs to disappear and appear. First, a CFP turns into a RCoFP (see the upper left panel of Figure 7). This case arises if the slope of $f_1(x_1, \mathbf{x}_2)$ is less than one at $\underline{x}_1$ when the CFP disappears. Conversely, a CFP can appear when a RCoFP vanishes. Second, a CFP turns into a RCoFPI, which turns into a RCoFP (see the upper right and the lower left panels of Figure 7). This case arises if the slope of $f_1(x_1, \mathbf{x}_2)$ is equal to one at $\underline{x}_1$ and $f_1(x_1, \mathbf{x}_2)$ goes to below the 45-degree line when the CFP vanishes. There may (lower left panel of Figure 7) or may not be (upper right panel of Figure 7) a FCoFP before the CFP vanishes. Conversely, a RCoFPI can turn into a RCoFP, which then turns into a CFP. Third, a CFP and a RCoFP can jointly give way for a FCoFPI (see the lower right panel of Figure 7). This case arises if the slope of $f_1(x_1, \mathbf{x}_2)$ is equal to one at $\underline{x}_1$ and $f_1(x_1, \mathbf{x}_2)$ goes to above the 45-degree line when the CFP disappears. Conversely, a CFP and a RCoFP can appear jointly. Evidently, a RCoFP cannot appear if the slope of $f_1(x_1, \mathbf{x}_2)$ is greater than one at $\underline{x}_1$.

(2.ii.b) Next, we allow for changes in $\underline{x}_1$ (see Figure 8). This gives rise to three further possibilities for CFPs to disappear and appear, which are analogous to the three possibilities described in the preceding paragraph (2.ii.a). First, a CFP turns into a RCoFP (see the upper left panel of Figure 8) if the slope of $f_1(x_1, \mathbf{x}_2)$ is less than one at $\underline{x}_1$ when the CFP disappears. Conversely, a CFP can appear when a RCoFP vanishes. Second, a CFP turns into a RCoFPI, which turns into a RCoFP (see the upper right and the lower left panels of Figure 8). This case arises if the slope of $f_1(x_1, \mathbf{x}_2)$ is equal to one at $\underline{x}_1$ and $f_1(x_1, \mathbf{x}_2)$ goes to below the 45-degree line when the CFP vanishes. There may (lower left panel of Figure 8) or may not be (upper right panel of Figure 8) a FCoFP before the CFP vanishes. Conversely, a RCoFPI can turn into a RCoFP, which then turns into a CFP. Third, a CFP and a RCoFP can jointly give way for a FCoFPI (see the lower right panel of Figure 8). This case

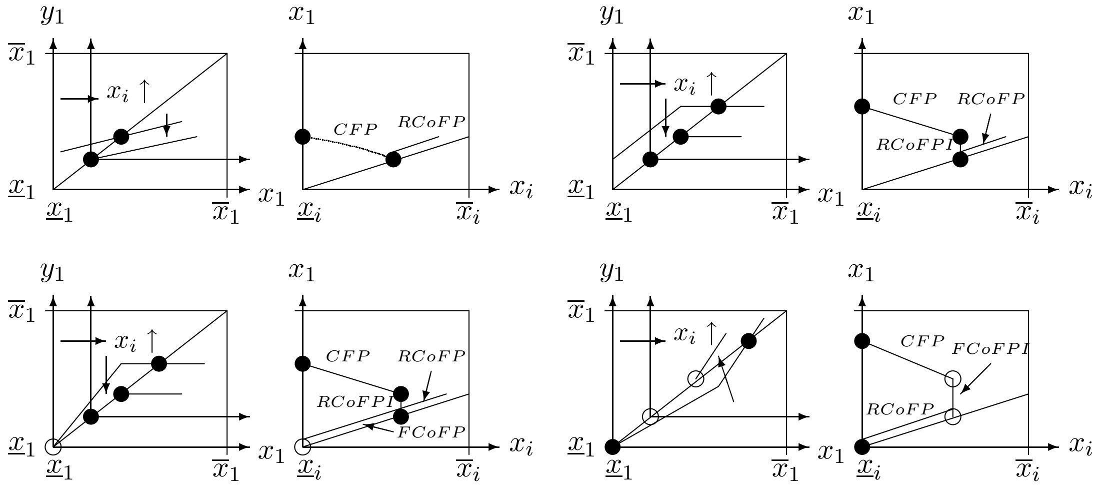

Figure 8: RFPs shift continuously or disappear in pairs when $f_1(x_1, \mathbf{x}_2)$ and $\underline{x}_1$ shift

arises if the slope of $f_1(x_1, \mathbf{x}_2)$ is equal to one at $\underline{x}_1$ and $f_1(x_1, \mathbf{x}_2)$ goes to above the 45-degree line when the CFP disappears. Conversely, a CFP and a RCoFP can appear jointly. Evidently, a RCoFP cannot appear if the slope of $f_1(x_1, \mathbf{x}_2)$ is greater than one at $\underline{x}_1$.

It follows that the statement made in paragraph (1.ii.b) about how CFPIs move or disappear and appear extends to the case with CoFPs as follows: *as* $\mathbf{x}_2$ *changes, RFPIs move continuously, or they disappear and appear in pairs.* From here on, the analysis of the case without CoFPs applies. The graph $\{(x_1, \mathbf{x}_2)|x_1 = \psi_1(\mathbf{x}_2), \mathbf{x}_2 \in C_2\}$ consists of continuous curves from points in $(\psi_1(\mathbf{x}_2'), \mathbf{x}_2')$ to points in $(\psi_1(\mathbf{x}_2''), \mathbf{x}_2'')$, curves that start at $\mathbf{x}_2'$ or $\mathbf{x}_2''$ and return to $\mathbf{x}_2'$ or $\mathbf{x}_2''$, respectively, and closed loops. Continuous curves from a point in $(\psi_1(\mathbf{x}_2'), \mathbf{x}_2')$ to a point in $(\psi_1(\mathbf{x}_2''), \mathbf{x}_2'')$ represent an odd number of RFPIs for all $\mathbf{x}_2$. The other two kinds of curves represent an even number of RFPIs for all $\mathbf{x}_2$. If there was no continuous curve in $\{(x_1, \mathbf{x}_2)|x_1 = \psi_1(\mathbf{x}_2), \mathbf{x}_2 \in C_2\}$ from a point in $(\psi_1(\mathbf{x}_2'), \mathbf{x}_2')$ to a point in $(\psi_1(\mathbf{x}_2''), \mathbf{x}_2'')$, then *the number of RFPIs would be even for all* $\mathbf{x}_2$. This contradicts the fact that *for all* $\mathbf{x}_2$*, there is an odd number of RFPIs.* This completes the proof of Lemma 1. $\diamond \diamond \diamond$

Remark: Lemma 1 (i) establishes the Theorem for the case $n = 1$.

Remark: We have not established that there is a *unique* continuous curve in $\{(x_1, \mathbf{x}_2)|x_1 = \psi_1(\mathbf{x}_2), \mathbf{x}_2 \in C_2\}$. Counterexamples can be easily constructed.

Lemma 2: *Suppose it holds true that for all* $\mathbf{x}_2', \mathbf{x}_2'' \in X_2$ *and for all continuous curves* $C_2 \subset X_2$ *from* $\mathbf{x}_2'$ *to* $\mathbf{x}_2''$*, there exists a continuous curve in the graph* $\{(x_1, \mathbf{x}_2)|x_1 = \psi_1(\mathbf{x}_2), \mathbf{x}_2 \in C_2\}$ *which connects a point in* $(\psi_1(\mathbf{x}_2'), \mathbf{x}_2')$ *and a point in* $(\psi_1(\mathbf{x}_2''), \mathbf{x}_2'')$*. Then: (i) For all* $\mathbf{x}_3 \in X_3$*, the set of FPs* $\psi_2(\mathbf{x}_3)$

of the composed mapping $F_2 : [\underline{x}_2, \overline{x}_2] \to [\underline{x}_2, \overline{x}_2]$ *with*

$$y_2 = f_2(\mathbf{x}),\ x_1 = \psi_1(\mathbf{x}_2),$$

is non-empty. (ii) For all $\mathbf{x}_3', \mathbf{x}_3'' \in X_3$ *and for all continuous curves* $C_3 \subset X_3$ *from* $\mathbf{x}_3'$ *to* $\mathbf{x}_3''$*, there exists a continuous curve in the graph* $\{(x_2, \mathbf{x}_3) | x_2 = \psi_2(\mathbf{x}_3), \mathbf{x}_3 \in C_3\}$ *which connects a point in* $(\psi_2(\mathbf{x}_3'), \mathbf{x}_3')$ *and a point in* $(\psi_2(\mathbf{x}_3''), \mathbf{x}_3'')$*.*

Proof: (i) Hold $\mathbf{x}_3$ constant. $[\underline{x}_2, \overline{x}_2]$ is a closed interval. $\{(x_2, \mathbf{x}_3)|x_2 \in [\underline{x}_2, \overline{x}_2], \mathbf{x}_3 \,\text{given}\} \equiv C_2$ is a continuous curve in X_2. The lemma hypothesizes that there is a continuous curve in the graph $\{(x_1, \mathbf{x}_2)|x_1 = \psi_1(\mathbf{x}_2), \mathbf{x}_2 \in C_2\} = \{\mathbf{x}|x_1 = \psi_1(\mathbf{x}_2), x_2 \in [\underline{x}_2, \overline{x}_2], \mathbf{x}_3 \,\text{given}\} \subset X$. So by Assumption 2, there is a unique continuous curve in the graph $\{(y_2, \mathbf{x})|y_2 = f_2(\mathbf{x}), x_1 = \psi_1(\mathbf{x}_2), x_2 \in [\underline{x}_2, \overline{x}_2], \mathbf{x}_3 \,\text{given}\}$. By the projection property, the projection of this graph onto the (y_2, x_2)-plane is a continuous curve. By the same reasoning as in Lemma 1 (i), it follows that the set of FPs $\psi_2(\mathbf{x}_3)$ of the composed mapping $y_2 = f_2(\mathbf{x}),\ x_1 = \psi_1(\mathbf{x}_2)$ is non-empty. (ii) Hold x_2 fixed, and consider a continuous curve C_3 from $\mathbf{x}_3'$ to $\mathbf{x}_3''$. $C_2 = \{(x_2, \mathbf{x}_3)|x_2 \,\text{given}, \mathbf{x}_3 \in C_3\}$ is a continuous curve in X_2. The lemma hypothesizes that there is a continuous curve in the graph $\{(x_1, \mathbf{x}_2)|x_1 = \psi_1(\mathbf{x}_2), x_2 \,\text{given}, \mathbf{x}_3 \in C_3\} \subset X$. By the continuous shifting property, as $\mathbf{x}_3$ and, hence, $(\psi_1(x_2, \mathbf{x}_3), \mathbf{x}_3)$ change, the graph $\{(y_2, \mathbf{x})|y_2 = f_2(\mathbf{x}), x_1 = \psi_1(x_2, \mathbf{x}_3), x_2 \in [\underline{x}_2, \overline{x}_2], \mathbf{x}_3 \,\text{given}\}$ in the (y_2, x_2)-plane shifts continuously. By the same reasoning as in Lemma 1 (ii), it follows that there is a continuous curve in $\{(x_2, \mathbf{x}_3)|x_2 = \psi_2(\mathbf{x}_3), \mathbf{x}_3 \in C_3\}$. ⋄ ⋄ ⋄

Remark: Lemma 2 (i) establishes the Theorem for the case $n = 2$.

Lemma 3: *Suppose it holds true for all* $j = 1, \ldots, i-1$ *that for all* $\mathbf{x}_{j+1}', \mathbf{x}_{j+1}'' \in X_{j+1}$ *and for all continuous curves* $C_{j+1} \subset X_{j+1}$ *from* $\mathbf{x}_{j+1}'$ *to* $\mathbf{x}_{j+1}''$*, there exists a continuous curves in the graph* $\{(x_j, \mathbf{x}_{j+1})|x_j = \psi_j(\mathbf{x}_{j+1}), \mathbf{x}_{j+1} \in C_{j+1}\}$ *which connects a point in* $(\psi_j(\mathbf{x}_{j+1}'), \mathbf{x}_{j+1}')$ *and a point in* $(\psi_j(\mathbf{x}_{j+1}''), \mathbf{x}_{j+1}'')$*. Then: (i) For all* $\mathbf{x}_{i+1}$ *in* X_{i+1}*, the set of FPs* $\psi_i(\mathbf{x}_{i+1})$ *of the composed mapping* $F_i : [\underline{x}_i, \overline{x}_i] \to [\underline{x}_i, \overline{x}_i]$ *with*

$$y_i = f_i(\mathbf{x}),\ x_j = \psi_j(\mathbf{x}_{j+1}) \text{ for } j = 1, \ldots, i-1,$$

is non-empty. (ii) For all $\mathbf{x}_{i+1}', \mathbf{x}_{i+1}'' \in X_{i+1}$ *and for all continuous curves* C_{i+1} *from* $\mathbf{x}_{i+1}'$ *to* $\mathbf{x}_{i+1}''$*, there exists a continuous curve in the graph* $\{(x_i, \mathbf{x}_{i+1})|x_i = \psi_i(\mathbf{x}_{i+1}), \mathbf{x}_{i+1} \in C_{i+1}\}$ *which connects a point in* $(\psi_i(\mathbf{x}_{i+1}'), \mathbf{x}_{i+1}')$ *and a point in* $(\psi_i(\mathbf{x}_{i+1}''), \mathbf{x}_{i+1}'')$*.*

Proof: (i) Hold $\mathbf{x}_{i+1}$ constant. $[\underline{x}_i, \overline{x}_i]$ is a closed interval. $\{(x_i, \mathbf{x}_{i+1})|x_i \in [\underline{x}_i, \overline{x}_i], \mathbf{x}_{i+1} \,\text{given}\} \equiv C_i$ is a continuous curve in X_i. The lemma hypothesizes that there is a continuous curve in the graph $\{(x_{i-1}, \mathbf{x}_i)|x_{i-1} = \psi_{i-1}(\mathbf{x}_i), \mathbf{x}_i \in C_i\} = \{\mathbf{x}_{i-1}|x_{i-1} = \psi_{i-1}(\mathbf{x}_i), x_i \in [\underline{x}_i, \overline{x}_i], \mathbf{x}_{i+1} \,\text{given}\} \equiv C_{i-1} \subset$

X_{i-1}. Applying the hypothesis of the lemma repeatedly, we find that there exist continuous curves in the graph $\{(x_{i-2}, \mathbf{x}_{i-1}) | x_{i-2} = \psi_{i-2}(\mathbf{x}_{i-1}), \mathbf{x}_{i-1} \in C_{i-1}\} = \{\mathbf{x}_{i-2} | x_j = \psi_j(\mathbf{x}_{j+1})\,(j = i-2, i-1), x_i \in [\underline{x}_i, \overline{x}_i], \mathbf{x}_{i+1}\,\text{given}\} \equiv C_{i-2} \subset X_{i-2}$ and a fortiori in $\{\mathbf{x} | x_j = \psi_j(\mathbf{x}_{j+1})\,(j = 1, \ldots, i-1), x_i \in [\underline{x}_i, \overline{x}_i], \mathbf{x}_{i+1}\,\text{given}\} \subset X$. From Assumption 2, it then follows that there is a unique continuous curve in the graph $\{(y_i, \mathbf{x}) | y_i = f_i(\mathbf{x}), x_j = \psi_j(\mathbf{x}_{j+1})\,(j = 1, \ldots, i-1), x_i \in [\underline{x}_i, \overline{x}_i], \mathbf{x}_{i+1}\,\text{given}\}$. By the projection property, the projection of this graph onto the (y_i, x_i)-plane is a continuous curve. By the same reasoning as in Lemma 1 (i), it follows that the set of FPs $\psi_i(\mathbf{x}_{i+1})$ of the composed mapping $y_i = f_i(\mathbf{x})$, $x_j = \psi_j(\mathbf{x}_{j+1})$ $(j = 1, \ldots, i-1)$ is non-empty. (ii) Hold x_i fixed, and consider a continuous curve C_{i+1} from $\mathbf{x}'_{i+1}$ to $\mathbf{x}''_{i+1}$. $\{(x_i, \mathbf{x}_{i+1}) | x_i\,\text{given}, \mathbf{x}_{i+1} \in C_{i+1}\} \equiv C_i$ is a continuous curve in X_i. From the hypothesis of the lemma, it follows that there exist continuous curves in the graph $\{(x_{i-1}, \mathbf{x}_i) | x_{i-1} = \psi_{i-1}(\mathbf{x}_i), x_i \in C_i\} = \{\mathbf{x}_{i-1} | x_{i-1} = \psi_{i-1}(\mathbf{x}_i), x_i\,\text{given}, \mathbf{x}_{i+1} \in C_{i+1}\} \equiv C_{i-1} \subset X_{i-1}$, in the graph $\{(x_{i-2}, \mathbf{x}_{i-1}) | x_{i-2} = \psi_{i-2}(\mathbf{x}_{i-1}), x_{i-1} \in C_{i-1}\} = \{\mathbf{x}_{i-2} | x_j = \psi_j(\mathbf{x}_{j+1})\,(j = i-2, i-1), x_i\,\text{given}, \mathbf{x}_{i+1} \in C_{i+1}\} \equiv C_{i-2} \subset X_{i-2}$, and a fortiori in $\{\mathbf{x} | x_j = \psi_j(\mathbf{x}_{j+1})\,(j = 1, \ldots, i-1), x_i\,\text{given}, \mathbf{x}_{i+1} \in C_{i+1}\} \subset X$. By the continuous shifting property, as $\mathbf{x}_{i+1}$ and, hence, $(x_1, \ldots, x_{i-1}, \mathbf{x}_{i+1})$ change, the graph $\{(y_i, \mathbf{x}) | y_i = f_i(\mathbf{x}), x_j = \psi_j(\mathbf{x}_{j+1})\,(j = 1, \ldots, i-1), x_i \in [\underline{x}_i, \overline{x}_i], \mathbf{x}_{i+1}\,\text{given}\}$ in the (y_i, x_i)-plane shifts continuously. By the same reasoning as in Lemma 1 (ii), it thus follows that there is a continuous curve in $\{(x_i, \mathbf{x}_{i+1}) | x_i = \psi_i(\mathbf{x}_{i+1}), \mathbf{x}_{i+1} \in C_{i+1}\}$. ⋄ ⋄ ⋄

Proof of Brouwer's fixed point theorem: Lemma 1 proves that there exists a continuous curve $\psi_1(\mathbf{x}_2)$ of FPs of $y_1 = f_1(\mathbf{x})$. Lemma 2 then proves the existence of a continuous curve $\psi_2(\mathbf{x}_3)$ of FPs of the composed mapping $y_2 = f_2(\mathbf{x})$, $x_1 = \psi_1(\mathbf{x}_2)$. Finally, Lemma 3 yields the Theorem by induction: Lemma 3 (ii) proves that for all $i = 3, \ldots, n-1$, there exists a continuous curve $\psi_i(\mathbf{x}_{i+1})$ of FPs of the composed mapping $y_i = f_i(\mathbf{x})$, $x_j = \psi_j(\mathbf{x}_{j+1})$ $(j = 1, \ldots, i-1)$. And Lemma 3 (i) proves the existence of a FP of $y_n = f_n(\mathbf{x})$, $x_j = \psi_j(\mathbf{x}_{j+1})$ $(j = 1, \ldots, n-1)$, which maps $[\underline{x}_n, \overline{x}_n]$ onto itself. ⋄ ⋄ ⋄

3 Extensions

We now turn from functions to multi-valued mappings $\mathbf{y} = \mathbf{f}(\mathbf{x})$. In terms of continuous curves, the crucial continuity requirement we impose throughout this section is:

Assumption 2': *For all $\mathbf{x}', \mathbf{x}'' \in X$ and for all continuous curves $C \subset X$ from $\mathbf{x}'$ to $\mathbf{x}''$, there exists a continuous curve in the graph $\{(y_i, \mathbf{x}) | y_i = f_i(\mathbf{x}), \mathbf{x} \in C\}$ which connects a point in $(f_i(\mathbf{x}'), \mathbf{x}')$ and a point in $(f_i(\mathbf{x}''), \mathbf{x}'')$ $(i = 1, \ldots, n)$.*

In contrast to Section 2, we do not assume that $\mathbf{f}$ is single-valued. Before dealing with Kakutani's fixed point theorem, we prove a somewhat weaker theorem:

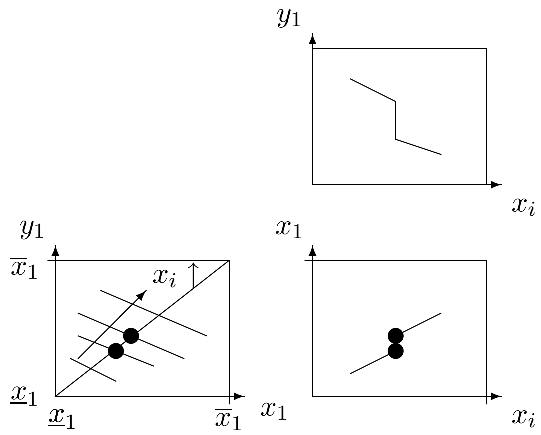

Figure 9: CFP of a multi-valued mapping

Assumption 3: $f_i(\mathbf{x})$ *($i = 1, \ldots, n$) is single-valued except where a vertical portion "convexifies" the graph.*

Theorem: *Given Assumptions 1, 2', and 3, the mapping* $\mathbf{f} : X \to X$ *has a FP.*

Proof: By the same reasoning as in Lemma 1 (i), for all $\mathbf{x}_2 \in X_2$, the set of FPs $\psi_1(\mathbf{x}_2)$ of the mapping $f_1(x_1, \mathbf{x}_2) : [\underline{x}_1, \overline{x}_1] \to [\underline{x}_1, \overline{x}_1]$ is non-empty, and FPs can be of the same four varieties as above. Lemma 1 (ii) also remains valid: for all $\mathbf{x}_2', \mathbf{x}_2'' \in X_2$ and for all continuous curves $C_2 \subset X_2$ which connect $\mathbf{x}_2'$ and $\mathbf{x}_2''$, there exists a continuous curve in the graph $\{(x_1, \mathbf{x}_2) | x_1 = \psi_1(\mathbf{x}_2), \mathbf{x}_2 \in C_2\}$ which connects a point in $(\psi_1(\mathbf{x}_2'), \mathbf{x}_2')$ and a point in $(\psi_1(\mathbf{x}_2''), \mathbf{x}_2'')$. The proof given in Section 2 has to be modified in only one respect: as $\mathbf{x}_2$ takes on a value where $f_1(x_1, \mathbf{x}_2)$ is multi-valued for a FP $\psi_1(\mathbf{x}_2) \in [\underline{x}_1, \overline{x}_1]$, the FP turns into a FPI. But, as illustrated in Figures 9 and 10, it remains true that $\psi_1(\mathbf{x}_2)$ does not simply end, but proceeds forward at CFPs and bends backward at TFPs. Lemmas 2 and 3 go through without modification. $\diamond \diamond \diamond$

Kakutani's theorem is obtained by replacing Assumption 3 with:

Assumption 3': $f_i(\mathbf{x})$ *is convex-valued for all* $\mathbf{x}$ *($i = 1, \ldots, n$).*

Remark: Kakutani's theorem assumes upper hemi-continuity and 3'. Upper hemi-continuity is more general than Assumption 2' because it does not rule out jumps. But upper hemi-continuity and Assumption 3' together do rule out jumps and thus imply the validity of Assumption 2'.

Kakutani's fixed point theorem: *Given Assumptions 1, 2', and 3', the mapping* $\mathbf{f} : X \to X$ *has a FP.*

Proof: This is a direct corollary of the previous theorem. Consider a mapping $\mathbf{y} = \mathbf{f}(\mathbf{x})$ which satisfies

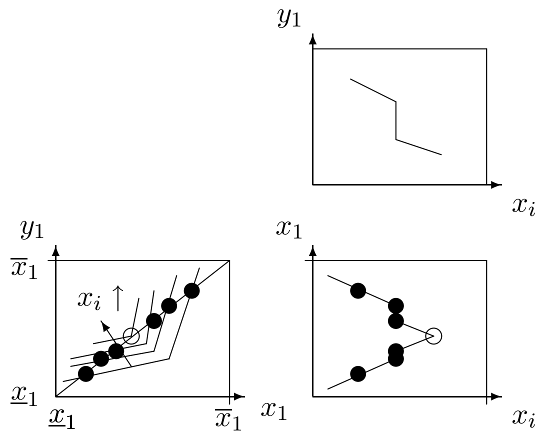

Figure 10: TFP of a multi-valued mapping

Assumptions 1, 2', and 3'. It is possible to eliminate points from $\{(y_i, \mathbf{x})|y_i = f_i(\mathbf{x}), \mathbf{x} \in X\}$ $(i = 1, \ldots, n)$ in such a way that the resulting graph satisfies Assumptions 2' and 3. By virtue of the previous theorem, the mapping thus obtained has a FP. Since its graph is a subset of the graph $\{(\mathbf{y}, \mathbf{x})|\mathbf{y} = \mathbf{f}(\mathbf{x}), \mathbf{x} \in X\}$, a FP of this mapping is a FP of $\mathbf{y} = \mathbf{f}(\mathbf{x})$. $\diamond \diamond \diamond$

It is possible to generalize Kakutani's theorem by dispensing with the convexity requirement in Assumption 3' altogether:

Theorem: *Given Assumptions 1 and 2', the mapping* $\mathbf{f} : X \to X$ *has a FP.*

Proof: Assumption 2' assures the existence of a continuous curve in the graph $\{(y_i, \mathbf{x})|y_i = f_i(\mathbf{x}), \mathbf{x} \in C\}$ which connects a point in $(f_i(\mathbf{x}'), \mathbf{x}')$ and a point in $(f_i(\mathbf{x}''), \mathbf{x}'')$ $(i = 1, \ldots, n)$ for all $\mathbf{x}', \mathbf{x}'' \in X$ and for all continuous curves $C \subset X$ from $\mathbf{x}'$ to $\mathbf{x}''$.

(1) Assume to begin with that there is exactly one such curve. Lemma 1 (i) is obtained by the same reasoning as above. The graph $\{(y_1, x_1, \mathbf{x}_2)|y_1 = f_1(x_1, \mathbf{x}_2), x_1 \in [\underline{x}_1, \overline{x}_1], \mathbf{x}_2 \,\text{given}\}$ can now have backward-bending portions, but this is inessential for the argument. Lemma 1 (ii) also remains valid: for all $\mathbf{x}_2', \mathbf{x}_2'' \in X_2$ and for all continuous curves $C_2 \subset X_2$ which connect $\mathbf{x}_2'$ and $\mathbf{x}_2''$, there exists a continuous curve in the graph $\{(x_1, \mathbf{x}_2)|x_1 = \psi_1(\mathbf{x}_2), \mathbf{x}_2 \in C_2\}$ which connects a point in $(\psi_1(\mathbf{x}_2'), \mathbf{x}_2')$ and a point in $(\psi_1(\mathbf{x}_2''), \mathbf{x}_2'')$. The proof of the first theorem in the present section has to be modified in two respects. First, for x_1-values where $y_1 = f_1(x_1, \mathbf{x}_2)$ is multi-valued, all points (y_1, x_1) change as $\mathbf{x}_2$ changes. Second, starting at $\mathbf{x}'$, it might be necessary to go back and forth on $C \subset X$ in order to arrive at $\mathbf{x}''$. Both considerations are inessential for the argument. So it remains true that for all $\mathbf{x}_2', \mathbf{x}_2'' \in X_2$ and for all continuous curves $C_2 \subset X_2$ which connect $\mathbf{x}_2'$ and $\mathbf{x}_2''$, there exists a continuous curve in the graph $\{(x_1, \mathbf{x}_2)|x_1 = \psi_1(\mathbf{x}_2), \mathbf{x}_2 \in C_2\}$ which connects a point in $(\psi_1(\mathbf{x}_2'), \mathbf{x}_2')$ and a point

in $(\psi_1(\mathbf{x}_2''), \mathbf{x}_2'')$. (2) By the same reasoning as in the proof of Kakutani's fixed point theorem, the same holds true if the there are more than one continuous curves in the graph $\{(y_i, \mathbf{x})|y_i = f_i(\mathbf{x}), \mathbf{x} \in C\}$. Lemmas 2 and 3 go through without modification. $\diamond \diamond \diamond$

4 Summary

Fixed point theorems play a central role in existence proofs in mathematical economics. This paper provides a way of understanding fixed point theorems by means of the familiar methodology of drawing and shifting curves in the plane. Because a number of case distinctions have to be made, the argument put forward is quite long. However, its main benefit is that it is intelligible without any advanced knowledge of topology. So it makes the proofs of the standard fixed point theorems used in mathematical economics accessible to a broader audience.

References

Fudenberg, Drew, and Jean Tirole (1991), *Game Theory*, Cambridge, MA: MIT Press.

Nash, John F. (1950), "Equilibrium points in n-person games", *Proceedings of the National Academy of Sciences* 36, 48-9.

Varian, Hal (1984), *Microeconomic Analysis*, New York: W. W. Norton.

www.mathpages.com/home/kmath262/kmath262.htm, "Reconstructing Brouwer".