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Reconstructing the Thermal Green Functions at Real Times from Those at Imaginary Times

Gianaurelio Cuniberti[1], **Enrico De Micheli**[2], **Giovanni Alberto Viano**[3]

[1] Max–Planck–Institut für Physik komplexer Systeme, Nöthnitzer Straße 38, 01187 Dresden, Germany.
E-mail: cunibert@mpipks-dresden.mpg.de
[2] Istituto di Cibernetica e Biofisica - Consiglio Nazionale delle Ricerche, Via De Marini 6, 16149 Genova, Italy.
E-mail: demic@icb.ge.cnr.it
[3] Dipartimento di Fisica - Università di Genova and Istituto Nazionale di Fisica Nucleare
Via Dodecaneso, 33 - 16146 Genova, Italy. E-mail: viano@ge.infn.it

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Abstract: By exploiting the analyticity and boundary value properties of the thermal Green functions that result from the KMS condition in both time and energy complex variables, we treat the general (non–perturbative) problem of recovering the thermal functions at real times from the corresponding functions at imaginary times, introduced as primary objects in the Matsubara formalism. The key property on which we rely is the fact that the Fourier transforms of the retarded and advanced functions in the energy variable have to be the "unique Carlsonian analytic interpolations" of the Fourier coefficients of the imaginary–time correlator, the latter being taken at the discrete Matsubara imaginary energies, respectively in the upper and lower half–planes. Starting from the Fourier coefficients regarded as "data set", we then develop a method based on the Pollaczek polynomials for constructing explicitly their analytic interpolations.

1. Introduction

In the standard imaginary–time formalism of quantum statistical mechanics (tracing back to Matsubara [15]) and, later on, of quantum field theory at finite temperature (see e.g. [14] and references therein), there arises the a–priori non–trivial problem of recovering the "physical" correlations at real times starting from data at imaginary times. More specifically, the correlations at imaginary–time of observables (or, more generally, of boson or fermion fields) in a thermal equilibrium state at temperature $T = \beta^{-1}$ are defined as periodic (or antiperiodic) functions of period β, and therefore they are equivalently characterized by their *discrete* mode expansion $\frac{1}{\beta}\sum_n \mathcal{G}_n \exp(-i\zeta_n\tau)$ in terms of the so–called "Matsubara energies" $i\zeta_n$, where $\zeta_n = 2n\pi/\beta$ (or $(2n+1)\pi/\beta$).

The problem of recovering the correlations at real time, or equivalently the retarded and advanced Green's functions at real energies, from the previous sequence of Fourier coefficients $\{\mathcal{G}_n\}$ admits a unique and well–defined theoretical solution in terms of the notion of "Carlsonian analytic interpolation of this sequence". This can be achieved [5], and will be recalled below in Sect. 2, if the imaginary–time formalism is embedded in the (conceptually more satisfactory) general description of quantum thermal states as KMS states [11]. However, as suggested by the lattice approach of the imaginary–time formalism, it may be interesting to have a *concrete* procedure for constructing satisfactory approximate solutions of this problem when one starts from incomplete data sets.

In this paper we give a precise algorithm for the previous reconstruction problem; this mathematical method is presented in Sect. 3. Moreover, in the subsequent Sect. 4, the method is applied to the case when the data are finite in number and affected by noise.

Let us consider the algebra $\mathcal{A}$ generated by the observables of a quantum system. Denoting by $A, B, \ldots$ arbitrary elements of $\mathcal{A}$ and by $A \to A(t)$ $(A = A(0))$ the action of the (time–evolution) group of automorphisms on this algebra, we now recall the KMS analytic structure of two–point correlation functions $\langle A(t_1)B(t_2)\rangle_{\Omega_\beta}$, in a thermal equilibrium state Ω_β of the system at temperature $T = \beta^{-1}$.

By time–translation invariance, these quantities only depend on $t = t_1 - t_2$, and we shall put

$$\mathcal{W}_{AB}(t) = \langle A(t)\, B\rangle_{\Omega_\beta}\,, \tag{1}$$

$$\mathcal{W}'_{AB}(t) = \langle B\, A(t)\rangle_{\Omega_\beta}\,. \tag{2}$$

In finite volume approximations, the time–evolution is represented by a unitary group $e^{iH't}$, so that

$$A(t) = e^{iH't} A\, e^{-iH't}, \tag{3}$$

where $H' = H - \mu N$, H being the Hamiltonian, μ the chemical potential, and N the particle number; under general conditions, the operators $e^{-\beta H'}$ have finite traces for all $\beta > 0$ (see e.g. [11]). Then the correlation functions are given, correspondingly, by the formulae

$$\mathcal{W}_{AB}(t) = \frac{1}{Z_\beta} \mathrm{Tr}\left\{e^{-\beta H'} A(t)\, B\right\}, \tag{4}$$

$$\mathcal{W}'_{AB}(t) = \frac{1}{Z_\beta} \mathrm{Tr}\left\{e^{-\beta H'} B A(t)\right\}, \tag{5}$$

where $Z_\beta = \mathrm{Tr}\, e^{-\beta H'}$.

One then introduces the following holomorphic functions of the complex time variable $t + i\gamma$:

$$G_{AB}(t + i\gamma) = \frac{1}{Z_\beta} \mathrm{Tr}\left\{e^{-(\beta+\gamma)H'} A(t)\, e^{\gamma H'} B\right\}, \tag{6}$$

analytic in the strip $\{t + i\gamma;\, t \in \mathbb{R},\, -\beta < \gamma < 0\}$, and

$$G'_{AB}(t + i\gamma) = \frac{1}{Z_\beta} \mathrm{Tr}\left\{e^{-(\beta-\gamma)H'} B\, e^{-\gamma H'} A(t)\right\}, \tag{7}$$

analytic in the strip $\{t + i\gamma;\, t \in \mathbb{R},\, 0 < \gamma < \beta\}$, which are such that:

$$\lim_{\substack{\gamma\to 0\\ \gamma<0}} G_{AB}(t + i\gamma) = \mathcal{W}_{AB}(t), \tag{8}$$

$$\lim_{\substack{\gamma\to 0\\ \gamma>0}} G'_{AB}(t + i\gamma) = \mathcal{W}'_{AB}(t). \tag{9}$$

From (6), (7) and the cyclic property of Tr, we then obtain the KMS relation

$$\mathcal{W}_{AB}(t) = \mathrm{Tr}\, e^{-\beta H'} A(t)\, B = \mathrm{Tr}\, B\, e^{-\beta H'} A(t) = G'_{AB}(t + i\beta), \tag{10}$$

which implies the identity of holomorphic functions (in the strip $0 < \gamma < \beta$)

$$G_{AB}(t + i(\gamma - \beta)) = G'_{AB}(t + i\gamma). \tag{11}$$

According to the analysis of [11] in the Quantum Mechanical framework and of [8] in the Field–theoretical framework, this KMS analytic structure is preserved by the thermodynamic limit under rather general conditions.

In the case when the algebra $\mathcal{A}$ is generated by smeared–out bosonic or fermionic field operators (field theory at finite temperature), the principle of relativistic causality of the theory implies additional relations for the corresponding pairs of analytic functions (G, G'). In fact, this principle of relativistic causality is expressed by the *commutativity* (resp. *anticommutativity*) *relations* for the boson field $\Phi(\mathbf{x})$ (resp. fermion field $\Psi(\mathbf{x})$) at space–like separation:

$$[\Phi(t,\boldsymbol{x}), \Phi(t',\boldsymbol{x'})] = 0 \qquad (\text{resp. } \{\Psi(t,\boldsymbol{x}), \Psi(t',\boldsymbol{x'})\} = 0) \quad \text{for} \quad (t-t')^2 < (\boldsymbol{x} - \boldsymbol{x'})^2. \tag{12}$$

In this field–theoretical case, we can choose as suitable operators A the "smeared–out field operators" of the form $A = \int \Phi(y_0, \boldsymbol{y}) f(y_0, \boldsymbol{y}) dy_0\, d\boldsymbol{y}$ (resp. $\int \Psi(y_0, \boldsymbol{y}) f(y_0, \boldsymbol{y}) dy_0\, d\boldsymbol{y}$), where f is any smooth test–function with (arbitrary small) compact support around the origin in space–time variables. For the observable B, we can then choose any operator $A_{\boldsymbol{x}}$ obtained from A by the action of the space–translation group (which amounts to replace the test–function $f(y_0, \boldsymbol{y})$ by $f(y_0, \boldsymbol{y}) = f(y_0, \boldsymbol{y} - \boldsymbol{x})$). It then follows from (12) that the corresponding analytic functions $G_{AA_{\boldsymbol{x}}}(t + i\gamma)$ and $G'_{AA_{\boldsymbol{x}}}(t + i\gamma)$ (satisfying (11)) have real boundary values $\mathcal{W}_{AA_{\boldsymbol{x}}}(t)$ and $\mathcal{W}'_{AA_{\boldsymbol{x}}}(t)$ which satisfy, on some interval $|t| < t(\boldsymbol{x}, f)$, *coincidence relations* of the following form:

$$\mathcal{W}_{AA_{\boldsymbol{x}}}(t) = \mathcal{W}'_{AA_{\boldsymbol{x}}}(t) \qquad \text{in the boson case,} \tag{13}$$

$$\mathcal{W}_{AA_{\boldsymbol{x}}}(t) = -\mathcal{W}'_{AA_{\boldsymbol{x}}}(t) \qquad \text{in the fermion case.} \tag{14}$$

Then, in view of identity (11), the coincidence relations (13) and (14) imply the existence of a single analytic function $\mathcal{G}_{AA_{\boldsymbol{x}}}(t + i\gamma)$ which is such that:

a) in the boson case:

$$\mathcal{G}_{AA_{\boldsymbol{x}}} = G_{AA_{\boldsymbol{x}}} \qquad \text{for } -\beta < \gamma < 0, \tag{15}$$

$$\mathcal{G}_{AA_{\boldsymbol{x}}} = G'_{AA_{\boldsymbol{x}}} \qquad \text{for } 0 < \gamma < \beta; \tag{16}$$

b) in the fermion case:

$$\mathcal{G}_{AA_{\boldsymbol{x}}} = G_{AA_{\boldsymbol{x}}} \qquad \text{for } -\beta < \gamma < 0, \tag{17}$$

$$\mathcal{G}_{AA_{\boldsymbol{x}}} = -G'_{AA_{\boldsymbol{x}}} \qquad \text{for } 0 < \gamma < \beta. \tag{18}$$

Correspondingly, it follows that $\mathcal{G}_{AA_{\boldsymbol{x}}}$ is either periodic or antiperiodic with period $i\beta$ in the full complex plane minus periodic cuts along the half–lines $\{t + i\gamma;\, t > t(\boldsymbol{x}, f),\, \gamma = k\beta,\, k \in \mathbb{Z}\}$ and $\{t + i\gamma;\, t < -t(\boldsymbol{x}, f),\, \gamma = k\beta,\, k \in \mathbb{Z}\}$.

These analytic functions $\mathcal{G}_{AA_{\boldsymbol{x}}}(t + i\gamma)$ are smeared–out forms (corresponding to various test–functions f) of the thermal two–point function of the fields Φ (or Ψ) in the complex time variable. In other words, this thermal two–point function can be fully characterized in terms of an analytic function $\mathcal{G}(t + i\gamma, \boldsymbol{x})$ (with regular dependence in the space variables) enjoying the following properties:

a) $\mathcal{G}(t + i\gamma, \boldsymbol{x}) = \epsilon\, \mathcal{G}(t + i(\gamma - \beta), \boldsymbol{x})$, where $\epsilon = +$ for a boson field, and $\epsilon = -$ for a fermion field;
b) for each $\boldsymbol{x}$, the domain of $\mathcal{G}$ in the complex variable t is $\mathbb{C} \setminus \{t + i\gamma;\, |t| > |\boldsymbol{x}|;\, \gamma = k\beta,\, k \in \mathbb{Z}\}$;

c) the boundary values of $\mathcal{G}$ at real times are the thermal correlations of the field, namely:

$$\lim_{\substack{\gamma\to 0\\ \gamma<0}} \mathcal{G}(t+i\gamma,\boldsymbol{x}) = \mathcal{W}(t,\boldsymbol{x}), \tag{19}$$

$$\lim_{\substack{\gamma\to 0\\ \gamma>0}} \mathcal{G}(t+i\gamma,\boldsymbol{x}) = \mathcal{W}'(t,\boldsymbol{x}), \tag{20}$$

where in finite volume regions, $\mathcal{W}$ and $\mathcal{W}'$ can be formally expressed as follows (a rigorous justification of the trace–operator formalism in the appropriate Hilbert space being given in [8]):

$$\mathcal{W}(t,\boldsymbol{x}) = \frac{1}{Z_\beta}\,\mathrm{Tr}\, e^{-\beta H}\,\Phi(t,\boldsymbol{x})\,\Phi(0,\mathbf{0}), \tag{21}$$

$$\mathcal{W}'(t,\boldsymbol{x}) = \frac{1}{Z_\beta}\,\mathrm{Tr}\, e^{-\beta H}\,\Phi(0,\mathbf{0})\,\Phi(t,\boldsymbol{x}), \tag{22}$$

for the boson case, and similarly in terms of $\Psi(t,\boldsymbol{x})$ for the fermion case.

In this analytic structure, we shall distinguish two quantities that play an important role:

i) the restriction $\mathcal{G}(i\gamma,\boldsymbol{x})$ of the function $\mathcal{G}$ to the imaginary axis is a β–periodic (or antiperiodic) function of γ which must be identified with the "*time–ordered product at imaginary times*", considered in the Matsubara approach of imaginary–time formalism. In the latter, this quantity or its set of Fourier coefficients plays the role of *initial data*.

ii) The "retarded" and "advanced" two–point functions

$$\mathsf{R}(t,\boldsymbol{x}) = i\,\theta(t)[\mathcal{W}(t,\boldsymbol{x}) - \epsilon\,\mathcal{W}'(t,\boldsymbol{x})], \tag{23}$$

$$\mathsf{A}(t,\boldsymbol{x}) = -i\,\theta(-t)[\mathcal{W}(t,\boldsymbol{x}) - \epsilon\,\mathcal{W}'(t,\boldsymbol{x})], \tag{24}$$

which are respectively the "jumps" of the function $\mathcal{G}$ across the real cuts $\{t;\, t \geq |\boldsymbol{x}|\}$ and $\{t;\, t < -|\boldsymbol{x}|\}$. These kernels have an important causal interpretation; in particular, R describes the "response of the system" to small perturbations of the equilibrium state. The knowledge of R and A and, consequently, of $\mathcal{W}-\mathcal{W}' = -i\,(\mathsf{R}-\mathsf{A})$ allows one to reconstruct $\mathcal{W}$ and $\mathcal{W}'$ by the application of the Bose–Einstein factor $1/(1-e^{\mp\beta\omega})$ to their Fourier transforms $\widetilde{\mathcal{W}}(\omega)$, $\widetilde{\mathcal{W}'}(\omega)$ (this procedure being an implementation of the KMS property in the energy variable ω).

The rest of the paper is devoted to the problem of *recovering the "real–time quantities"* R *and* A*, starting from the "time–ordered product at imaginary times" as initial data*. This will require the conjoint use of the analytic structure of $\mathcal{G}$ in complex time and of its Fourier–Laplace transform in the complex energy variable. In fact, the key property on which our reconstruction of real–time quantities relies is the following one: the Fourier–Laplace transforms $\widetilde{\mathsf{R}}$ and $\widetilde{\mathsf{A}}$ of the functions R and A, which are defined and analytic respectively in the upper and lower half–planes of the energy variable ω, are *analytic interpolations of the set of Fourier coefficients* $\{\mathcal{G}_n\}$ of the function $\mathcal{G}$ at imaginary times, the latter being taken at the Matsubara energies $\omega = i\,\zeta_n$. Moreover, the uniqueness of this interpolation is ensured by global bounds on $\widetilde{\mathsf{R}}$ and $\widetilde{\mathsf{A}}$, according to a standard theorem by Carlson [3]. The basic equalities that relate $\widetilde{\mathsf{R}}(i\,\zeta_n)$ and $\widetilde{\mathsf{A}}(i\,\zeta_n)$ to the corresponding coefficients $\mathcal{G}_n$ will be called "Froissart–Gribov–type equalities" for the following historical reason. A general n–dimensional mathematical study of the type of double–analytic structure encountered here has been performed in [6] in connection with the theory of complex angular momentum, where the original Froissart–Gribov equalities had been first discovered (in the old framework of S-matrix theory). The fact that this structure is relevant (in its simplest one–dimensional form) in the analysis of thermal quantum states has been already presented in [5] in the framework of Quantum Field Theory at finite temperature.

2. Double Analytic Structure of the Thermal Green Function and Froissart–Gribov–type Equalities

In the following mathematical study we replace the complex time variable $t+i\gamma$ of the introduction by $\tau = i(t+i\gamma)$ in such a way that, in our "reconstruction problem" treated in Sects. 3 and 4, the *initial data* of the function $\mathcal{G}(\tau,\cdot)$ considered below correspond to *real values of* τ. Up to this change of notation, this general analytic function $\mathcal{G}(\tau,\cdot)$ can play the role of the previously described two–point function of a boson or fermion field at fixed $\boldsymbol{x}$. However, since the only variables involved in the forthcoming study are τ and its Fourier–conjugate variable ζ, the extra "spectator variables", denoted by the point $(\cdot)$, may as well represent a fixed momentum (after Fourier transformation with respect to the space variables) or the action on a test–function f (as for the correlations of field observables $A = A(f)$ described in the introduction).

Let us summarize the analytic structure that we want to study.

Hypotheses. *The function* $\mathcal{G}(\tau,\cdot)$, $(\tau = u+iv,\ u,v\in\mathbb{R})$, *satisfies the following properties:*

a) *it is analytic in the open strips* $k\beta < u < (k+1)\beta$ $(v\in\mathbb{R},\ k\in\mathbb{Z},\ \beta=1/T)$ *and continuous at the boundaries;*

b) *it is periodic* (*antiperiodic*) *for bosons* (*fermions*) *with period* β, *i.e.*

$$\mathcal{G}(\tau+\beta,\cdot) = \begin{cases} \mathcal{G}(\tau,\cdot) & \textit{for bosons,} \qquad (\tau\in\mathbb{C}\,), \\ -\mathcal{G}(\tau,\cdot) & \textit{for fermions,} \qquad (\tau\in\mathbb{C}\,); \end{cases}$$

c)
$$\sup_{-k\beta<u<(k+1)\beta} |\mathcal{G}(u+iv,\cdot)| \leq C|v|^{\alpha}, \qquad (v\in\mathbb{R};\ C,\alpha\ \textit{constants}). \tag{25}$$

We shall treat both the boson and fermion field cases at the same time by exploiting the 2β–periodicity of the function $\mathcal{G}(\tau,\cdot)$. To this purpose, we take the Fourier series (in the sense of $L^2[-\beta,\beta]$) of $\mathcal{G}(\tau,\cdot)$, which we write

$$\mathcal{G}(\tau,\cdot) = \frac{1}{2\beta}\sum_{n=-\infty}^{+\infty}\mathcal{G}_n(\cdot)e^{-i\zeta_n\tau}, \quad \zeta_n = \frac{\pi}{\beta}\,n\,, \tag{26}$$

and whose Fourier coefficients are given by

$$\mathcal{G}_n(\cdot) = \int_{-\beta}^{\beta}\mathcal{G}(\tau,\cdot)\,e^{i\zeta_n\tau}\,d\tau. \tag{27}$$

It is convenient to split expansion (26) into two terms as follows:

$$\mathcal{G}^{(+)}(\tau,\cdot) = \frac{1}{2\beta}\sum_{n=0}^{+\infty}\mathcal{G}_n^{(+)}(\cdot)\,e^{-i\zeta_n\tau}, \quad \mathcal{G}_n^{(+)}(\cdot)\equiv\mathcal{G}_n(\cdot), \quad (n=0,1,2,\ldots), \tag{28}$$

$$\mathcal{G}^{(-)}(\tau,\cdot) = \frac{1}{2\beta}\sum_{n=-1}^{-\infty}\mathcal{G}_n^{(-)}(\cdot)\,e^{-i\zeta_n\tau}, \quad \mathcal{G}_n^{(-)}(\cdot)\equiv\mathcal{G}_n(\cdot), \quad (n=-1,-2,\ldots), \tag{29}$$

then,

$$\mathcal{G}_n^{(+)}(\cdot) = \int_{-\beta}^{\beta}\mathcal{G}^{(+)}(\tau,\cdot)\,e^{i\zeta_n\tau}\,d\tau, \qquad (n=0,1,2,\ldots), \tag{30}$$

$$\mathcal{G}_n^{(-)}(\cdot) = \int_{-\beta}^{\beta}\mathcal{G}^{(-)}(\tau,\cdot)\,e^{i\zeta_n\tau}\,d\tau, \qquad (n=-1,-2,\ldots). \tag{31}$$

We now introduce in the complex plane of the variable $\tau = u+iv$ ($u, v \in \mathbb{R}$) the following domains: the half–planes $\mathcal{I}_\pm = \{\tau \in \mathbb{C} \,|\, \mathrm{Im}\,\tau \gtrless 0\}$; the "cut–domain" $\mathcal{I}_+ \setminus \Xi_+$, where the cuts Ξ_+ are given by $\Xi_+ = \{\tau \in \mathbb{C} \,|\, \tau = k\beta + iv,\, v \geq 0,\, k \in \mathbb{Z}\}$, and $\mathcal{I}_- \setminus \Xi_-$, where $\Xi_- = \{\tau \in \mathbb{C} \,|\, \tau = k\beta + iv,\, v \leq 0,\, k \in \mathbb{Z}\}$. Moreover, we denote by $\mathring{A}$ any subset A of $\mathbb{C}$ which is invariant under the translation by $k\beta$, $k \in \mathbb{Z}$ (e.g. $\mathring{\Xi}_\pm$, $\mathcal{I}_\pm \setminus \mathring{\Xi}_\pm$, etc.) (see Ref. [6]I). Accordingly, the periodic cut–τ–plane $\mathbb{C} \setminus (\mathring{\Xi}_+ \cup \mathring{\Xi}_-)$ will be denoted by $\mathring{\Pi}_\tau$. We now introduce the jump functions $J^{(+)}_{(k\beta)}(v,\cdot)$ and $J^{(-)}_{(k\beta)}(v,\cdot)$ that represent the discontinuities of $\mathcal{G}^{(+)}(\tau,\cdot)$ and $\mathcal{G}^{(-)}(\tau,\cdot)$ across the cuts located respectively at $\mathrm{Re}\,\tau \equiv u = k\beta$, $v \geq 0$, and at $\mathrm{Re}\,\tau \equiv u = k\beta$, $v \leq 0$, ($k \in \mathbb{Z}$):

$$J^{(+)}_{(k\beta)}(v,\cdot) = +i \lim_{\substack{\epsilon\to 0\\ \epsilon>0}} \left\{ \mathcal{G}^{(+)}(k\beta+\epsilon+iv,\cdot) - \mathcal{G}^{(+)}(k\beta-\epsilon+iv,\cdot) \right\}, \qquad (v \geq 0,\ k \in \mathbb{Z}), \tag{32}$$

$$J^{(-)}_{(k\beta)}(v,\cdot) = -i \lim_{\substack{\epsilon\to 0\\ \epsilon>0}} \left\{ \mathcal{G}^{(-)}(k\beta+\epsilon+iv,\cdot) - \mathcal{G}^{(-)}(k\beta-\epsilon+iv,\cdot) \right\}, \qquad (v \leq 0,\ k \in \mathbb{Z}). \tag{33}$$

Let us note that these definitions are well–posed and appropriate because, as we shall see in the following theorem, $\mathcal{G}^{(+)}(\tau,\cdot)$ and $\mathcal{G}^{(-)}(\tau,\cdot)$ are holomorphic in the cut–domains $\mathcal{I}_- \cup [\mathcal{I}_+ \setminus \mathring{\Xi}_+]$ and $\mathcal{I}_+ \cup [\mathcal{I}_- \setminus \mathring{\Xi}_-]$, respectively. Moreover, we suppose hereafter that the slow–growth condition (25) extends to the discontinuities $J^{(\pm)}_{(k\beta)}(v,\cdot)$, that turn out to be "tempered functions" [4]. Finally, in view of the periodicity properties of $\mathcal{G}(\tau,\cdot)$, it is sufficient to consider only the strip, in the τ–plane, defined by $-a \leq u \leq 2\beta - a$ ($0 < a < \beta$), $v \in \mathbb{R}$ (see Fig. 1).

We then introduce the Laplace transforms of the jump functions across the cuts located at $\mathrm{Re}\,\tau = 0$, and at $\mathrm{Re}\,\tau = \beta$; i.e.

$$\tilde{J}^{(+)}_{(0)}(\zeta,\cdot) = \int_0^{+\infty} J^{(+)}_{(0)}(v,\cdot)\, e^{-\zeta v}\, dv, \qquad (\zeta = \xi + i\eta,\ \mathrm{Re}\,\zeta > 0), \tag{34}$$

$$\tilde{J}^{(-)}_{(0)}(\zeta,\cdot) = \int_{-\infty}^{0} J^{(-)}_{(0)}(v,\cdot)\, e^{-\zeta v}\, dv, \qquad (\mathrm{Re}\,\zeta < 0), \tag{35}$$

$$\tilde{J}^{(+)}_{(\beta)}(\zeta,\cdot) = \int_0^{+\infty} J^{(+)}_{(\beta)}(v,\cdot)\, e^{-\zeta v}\, dv, \qquad (\mathrm{Re}\,\zeta > 0), \tag{36}$$

$$\tilde{J}^{(-)}_{(\beta)}(\zeta,\cdot) = \int_{-\infty}^{0} J^{(-)}_{(\beta)}(v,\cdot)\, e^{-\zeta v}\, dv, \qquad (\mathrm{Re}\,\zeta < 0). \tag{37}$$

We can state the following theorem.

Theorem 1. *If the functions $\mathcal{G}(\tau,\cdot)$ and $J^{(\pm)}_{(k\beta)}(v,\cdot)$ satisfy the slow–growth condition (25) uniformly in $\mathring{\Pi}_\tau = \mathbb{C} \setminus (\mathring{\Xi}_+ \cup \mathring{\Xi}_-)$ up to the closure, the following properties hold true:*
i) The function $\mathcal{G}^{(+)}(\tau,\cdot)$ (respectively $\mathcal{G}^{(-)}(\tau,\cdot)$) is holomorphic in the cut–domain $\mathcal{I}_- \cup [\mathcal{I}_+ \setminus \mathring{\Xi}_+]$ (respectively $\mathcal{I}_+ \cup [\mathcal{I}_- \setminus \mathring{\Xi}_-]$).
ii-a) The Laplace transforms $\tilde{J}^{(+)}_{(0)}(\zeta,\cdot)$ and $\tilde{J}^{(+)}_{(\beta)}(\zeta,\cdot)$ are holomorphic in the half–plane $Re\,\zeta > 0$. The Laplace transforms $\tilde{J}^{(-)}_{(0)}(\zeta,\cdot)$ and $\tilde{J}^{(-)}_{(\beta)}(\zeta,\cdot)$ are holomorphic in the half–plane $Re\,\zeta < 0$.
ii-b) $\tilde{J}^{(+)}_{(0)}(\zeta,\cdot)$ and $\tilde{J}^{(+)}_{(\beta)}(\zeta,\cdot)$ belong to the Hardy space $\mathbb{H}^2\left(\mathbb{C}^{(+)}_{(\delta)}\right)$, where $\mathbb{C}^{(+)}_{(\delta)} = \{\zeta \in \mathbb{C} \,|\, Re\,\zeta > \delta,\, \delta \geq \epsilon > 0\}$. $\tilde{J}^{(-)}_{(0)}(\zeta,\cdot)$ and $\tilde{J}^{(-)}_{(\beta)}(\zeta,\cdot)$ belong to the Hardy space $\mathbb{H}^2\left(\mathbb{C}^{(-)}_{(\delta)}\right)$, where $\mathbb{C}^{(-)}_{(\delta)} = \{\zeta \in \mathbb{C} \,|\, Re\,\zeta < \delta,\, \delta \geq \epsilon > 0\}$.

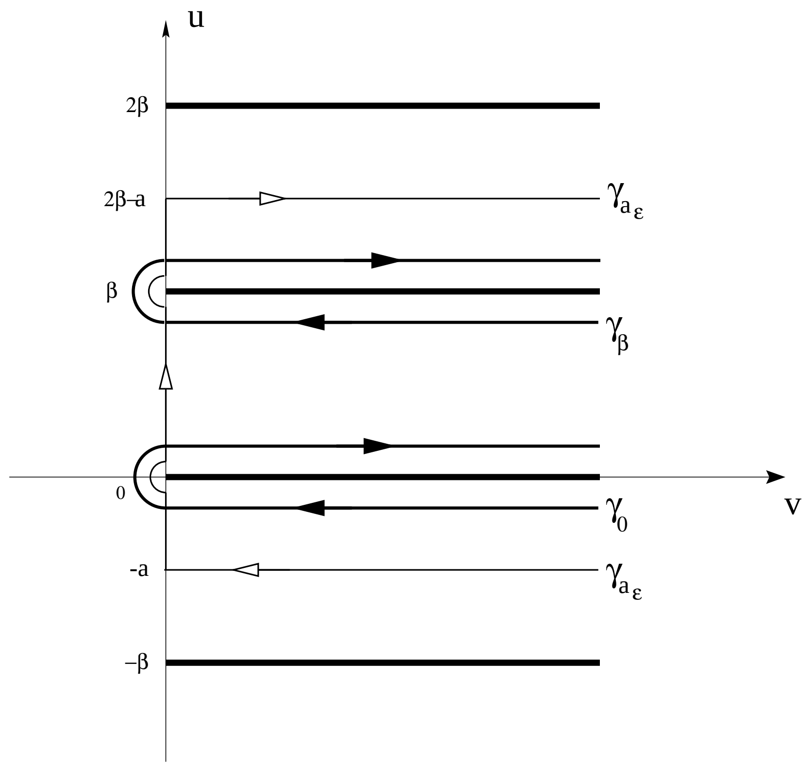

Fig. 1. Integration paths used in the proof of Theorem 1

iii-a) In the case of the boson statistics the symmetric combinations $\tilde{\mathcal{G}}^{(+,b)}(\zeta,\cdot) :\equiv \tilde{J}^{(+)}_{(0)}(\zeta,\cdot) + \tilde{J}^{(+)}_{(\beta)}(\zeta,\cdot)$ *and* $\tilde{\mathcal{G}}^{(-,b)}(\zeta,\cdot) :\equiv \tilde{J}^{(-)}_{(0)}(\zeta,\cdot) + \tilde{J}^{(-)}_{(\beta)}(\zeta,\cdot)$ *interpolate uniquely the Fourier coefficients* $\mathcal{G}^{(+)}_{2m}(\cdot)$ *and* $\mathcal{G}^{(-)}_{2m}(\cdot)$ *respectively (hereafter the superscript (b) stands for the boson statistics). Let* $\zeta_m = 2m\pi/\beta$*, then the following Froissart–Gribov–type equalities hold:*

$$\tilde{\mathcal{G}}^{(+,b)}(\zeta_m,\cdot) = \tilde{J}^{(+)}_{(0)}(\zeta_m,\cdot) + \tilde{J}^{(+)}_{(\beta)}(\zeta_m,\cdot) = \mathcal{G}^{(+)}_{2m}(\cdot), \qquad (m = 1,2,3,\ldots), \tag{38}$$

$$\tilde{\mathcal{G}}^{(-,b)}(\zeta_m,\cdot) = \tilde{J}^{(-)}_{(0)}(\zeta_m,\cdot) + \tilde{J}^{(-)}_{(\beta)}(\zeta_m,\cdot) = \mathcal{G}^{(-)}_{2m}(\cdot), \qquad (m = -1,-2,-3,\ldots). \tag{39}$$

iii-b) In the case of the fermion statistics the antisymmetric combinations $\tilde{\mathcal{G}}^{(+,f)}(\zeta,\cdot) :\equiv \tilde{J}^{(+)}_{(0)}(\zeta,\cdot) - \tilde{J}^{(+)}_{(\beta)}(\zeta,\cdot)$ *and* $\tilde{\mathcal{G}}^{(-,f)}(\zeta,\cdot) :\equiv \tilde{J}^{(-)}_{(0)}(\zeta,\cdot) - \tilde{J}^{(-)}_{(\beta)}(\zeta,\cdot)$ *interpolate uniquely the Fourier coefficients* $\mathcal{G}^{(+)}_{2m+1}(\cdot)$ *and* $\mathcal{G}^{(-)}_{2m+1}(\cdot)$ *respectively (hereafter the superscript (f) stands for the fermion statistics). Let* $\zeta_m = (2m+1)\pi/\beta$*, then the following Froissart–Gribov–type equalities hold:*

$$\tilde{\mathcal{G}}^{(+,f)}(\zeta_m,\cdot) = \tilde{J}^{(+)}_{(0)}(\zeta_m,\cdot) - \tilde{J}^{(+)}_{(\beta)}(\zeta_m,\cdot) = \mathcal{G}^{(+)}_{2m+1}(\cdot), \qquad (m = 0,1,2,3,\ldots), \tag{40}$$

$$\tilde{\mathcal{G}}^{(-,f)}(\zeta_m,\cdot) = \tilde{J}^{(-)}_{(0)}(\zeta_m,\cdot) - \tilde{J}^{(-)}_{(\beta)}(\zeta_m,\cdot) = \mathcal{G}^{(-)}_{2m+1}(\cdot), \qquad (m = -1,-2,-3,\ldots). \tag{41}$$

Proof. (i) In view of the Riemann–Lebesgue theorem, and since $\mathcal{G}^{(+)}(\tau,\cdot) \in L_1[-\beta,\beta]$, the Fourier coefficients $\mathcal{G}_n^{(+)}(\cdot)$ tend to zero as $n \to \infty$. From expansion (28) we have for all $\tau = u + iv$, with $v < 0$:

$$|\mathcal{G}^{(+)}(\tau,\cdot)| = \left| \frac{1}{2\beta} \sum_{n=0}^{+\infty} \mathcal{G}_n^{(+)}(\cdot) e^{-i\zeta_n \tau} \right| \leq K \sum_{n\geq 0} e^{\zeta_n v}, \tag{42}$$

where $K = \int_{-\beta}^{\beta} |\mathcal{G}(\tau,\cdot)|\, d\tau$. The series $\sum_{n\geq 0}^{+\infty} e^{\zeta_n v}$ converges uniformly in any domain compactly contained in the half–plane $\operatorname{Im}\tau < 0$. In view of the Weierstrass theorem on the uniformly convergent series of analytic functions, we can conclude that $\mathcal{G}^{(+)}(\tau,\cdot)$ is holomorphic in the half–plane $\operatorname{Im}\tau < 0$. By using analogous arguments we can prove that $\mathcal{G}^{(-)}(\tau,\cdot)$ is holomorphic in the half–plane $\operatorname{Im}\tau > 0$. Furthermore, we know from Hypothesis a) that $\mathcal{G}(\tau,\cdot) = \mathcal{G}^{(+)}(\tau,\cdot) + \mathcal{G}^{(-)}(\tau,\cdot)$ is holomorphic in the strips $k\beta < u < (k+1)\beta$ ($k \in \mathbb{Z}$, $v \in \mathbb{R}$), and continuous at the boundaries of the strips. We can conclude that $\mathcal{G}^{(+)}(\tau,\cdot)$ is holomorphic in the cut–domain $\mathcal{I}_- \cup [\mathcal{I}_+ \setminus \overset{\circ}{\Xi}_+]$, and $\mathcal{G}^{(-)}(\tau,\cdot)$ is holomorphic in the cut–domain $\mathcal{I}_+ \cup [\mathcal{I}_- \setminus \overset{\circ}{\Xi}_-]$.
(ii) Property (ii, a) follows easily from the assumption of "temperateness" of the jump functions [4]. For what concerns property (ii, b) we limit ourselves to prove that $\tilde{J}_{(0)}^{(+)}(\zeta,\cdot)$ belongs to the Hardy space $\mathbb{H}^2\left(\mathbb{C}_{(\delta)}^{(+)}\right)$, since the remaining part of the statement can be proved analogously. To this purpose, we rewrite the Laplace transform (34) in the following form:

$$\int_0^{+\infty} \left(J_{(0)}^{(+)}(v,\cdot) e^{-\delta v} \right) e^{-\zeta' v}\, dv :\equiv \tilde{J}_{(0)(\delta)}^{(+)}(\zeta',\cdot), \qquad (\operatorname{Re}\zeta' > 0), \tag{43}$$

where $\operatorname{Re}\zeta' = \operatorname{Re}\zeta - \delta$ ($\delta \geq \epsilon > 0$). In view of the slow–growth property of $J_{(0)}^{(+)}(v,\cdot)$, we can then say that the function $J_{(0)}^{(+)}(v,\cdot)\exp(-\delta v)$ belongs to the intersection $L^1[0,+\infty) \cap L^2[0,+\infty)$. Then, thanks to the Paley–Wiener theorem, we can conclude (returning to the variable ζ) that $\tilde{J}_{(0)}^{(+)}(\zeta,\cdot)$ belongs to the Hardy space $\mathbb{H}^2\left(\mathbb{C}_{(\delta)}^{(+)}\right)$ (see Ref. [12]). Accordingly, $\tilde{J}_{(0)}^{(+)}(\zeta,\cdot)$ tends uniformly to zero as ζ tends to infinity inside any fixed half–plane $\operatorname{Re}\zeta \geq \delta' > \delta$. In particular, $\tilde{J}_{(0)}^{(+)}(\zeta_n,\cdot)$, with $\zeta_n = n\pi/\beta$ ($n = 1, 2, \ldots$), tends to zero as $n \to \infty$.
(iii) We introduce the integral $I_\gamma^{(+)}$ defined as follows (this method has been introduced by Bros and Buchholz [5], and will be developed in a more detailed form in [7] within the general framework of QFT):

$$I_\gamma^{(+)}(\zeta,\cdot) = \int_\gamma \mathcal{G}^{(+)}(\tau,\cdot)\, e^{i\zeta\tau}\, d\tau, \tag{44}$$

where the path γ encloses both the cuts located at $u = 0$, $v \geq 0$ and at $u = \beta$, $v \geq 0$ (see Fig. 1). In view of the slow–growth condition (25), this integral is well–defined. By choosing as integration path a pair of contours (γ_0, γ_β) enclosing respectively the cuts at $u = 0$, $v \geq 0$ and at $u = \beta$, $v \geq 0$, and then flattening them (in a folded way) onto the cuts (see Fig. 1), we obtain:

$$I_{(\gamma_0\cup\gamma_\beta)}^{(+)}(\zeta,\cdot) = \int_0^{+\infty} J_{(0)}^{(+)}(v,\cdot)\, e^{-\zeta v}\, dv + e^{i\zeta\beta} \int_0^{+\infty} J_{(\beta)}^{(+)}(v,\cdot)\, e^{-\zeta v}\, dv = \tilde{J}_{(0)}^{(+)}(\zeta,\cdot) + e^{i\zeta\beta} \tilde{J}_{(\beta)}^{(+)}(\zeta,\cdot) \tag{45}$$

Next, we choose the path γ_{a_ϵ}, whose support is: $]-a+i\infty,-a]\cup[-a,-\epsilon]\cup[\gamma_\epsilon^{(0)}]\cup[\epsilon,\beta-\epsilon]\cup[\gamma_\epsilon^{(\beta)}]\cup[\beta+\epsilon,2\beta-a]\cup[2\beta-a,2\beta-a+i\infty[$, where $\gamma_\epsilon^{(0)}$ and $\gamma_\epsilon^{(\beta)}$ are half–circles turning around the points $\tau=0$ and $\tau=\beta$, respectively (see Fig. 1). By taking into account the 2β–periodicity of $\mathcal{G}^{(+)}(\tau,\cdot)$, we get, for $\zeta=\zeta_n=n\pi/\beta$, $(n=1,2,\dots)$:

$$\lim_{\epsilon\to 0} I^{(+)}_{\gamma_{a_\epsilon}}(\zeta_n,\cdot)=\int_{-a}^{2\beta-a}\mathcal{G}^{(+)}(\tau,\cdot)\,e^{i\zeta_n\tau}\,d\tau=\mathcal{G}^{(+)}_n(\cdot). \tag{46}$$

Then, from the Cauchy distortion argument, we have $I^{(+)}_{\gamma_0\cup\gamma_\beta}(\zeta_n,\cdot)=\lim_{\epsilon\to 0}I^{(+)}_{\gamma_{a_\epsilon}}(\zeta_n,\cdot)$, that is

$$\tilde{J}^{(+)}_{(0)}(\zeta_n,\cdot)+e^{i\zeta_n\beta}\,\tilde{J}^{(+)}_{(\beta)}(\zeta_n,\cdot)=\mathcal{G}^{(+)}_n(\cdot). \tag{47}$$

We now distinguish two cases:

1) n even: $n=2m$, $\zeta_m=2m\pi/\beta$ $(m=1,2,\dots)$; then from (47) we obtain equalities (38).
2) n odd: $n=2m+1$, $\zeta_m=(2m+1)\pi/\beta$ $(m=0,1,2,\dots)$; then from (47) we obtain equalities (40).

We have thus obtained two combinations (symmetric and antisymmetric, respectively) that interpolate the Fourier coefficients $\mathcal{G}^{(+)}_n(\cdot)$. The uniqueness of the interpolation is guaranteed by the Carlson theorem [3] that can be applied since $\tilde{J}^{(+)}_{(0)}(\zeta,\cdot)$ and $\tilde{J}^{(+)}_{(\beta)}(\zeta,\cdot)$ belong to the Hardy space $\mathbb{H}^2\left(\mathbb{C}^{(+)}_{(\delta)}\right)$. Proceeding with analogous arguments applied to $\mathcal{G}^{(-)}(\tau,\cdot)$ equalities (39) and (41) are obtained. □

In conclusion, we can say that the thermal Green functions present a double analytic structure involving the analyticity properties in the $\tau=u+iv$ and $\zeta=\xi+i\eta$ planes. The 2β–periodic function $\mathcal{G}^{(+)}(\tau,\cdot)$ (resp. $\mathcal{G}^{(-)}(\tau,\cdot)$) is analytic in the cut–domain $\mathcal{I}_-\cup[\mathcal{I}_+\setminus\mathring{\Xi}_+]$ (resp. $\mathcal{I}_+\cup[\mathcal{I}_-\setminus\mathring{\Xi}_-]$); its Fourier coefficients can be uniquely interpolated (in the sense of the Carlson theorem), and are the restriction to the appropriate Matsubara energies of a function $\tilde{\mathcal{G}}^{(+,b-f)}(\zeta,\cdot)$ (resp. $\tilde{\mathcal{G}}^{(-,b-f)}(\zeta,\cdot)$), analytic in the half–plane $\mathrm{Re}\,\zeta>0$ (resp. $\mathrm{Re}\,\zeta<0$). It is straightforward to verify that the jump function $J^{(+)}_{(0)}(v,\cdot)$ coincides with the retarded Green function, and $J^{(-)}_{(0)}(v,\cdot)$ coincides with the advanced one; analogously, putting $i\,\zeta=\omega$, we can identify $\tilde{\mathcal{G}}^{(+,b-f)}(\zeta,\cdot)$ and $\tilde{\mathcal{G}}^{(-,b-f)}(\zeta,\cdot)$ respectively with the retarded and advanced Green functions in the energy variable ω conjugate to the real time t.

3. Representation of the Jump Function in Terms of an Infinite Set of Fourier Coefficients

First let us consider a system of bosons; since n is even, i.e. $n=2m$, $\zeta_m=(2m\pi)/\beta$, $(m=0,1,2,\dots)$, we have:

$$\tilde{\mathcal{G}}^{(+,b)}\left(\frac{2m\pi}{\beta},\cdot\right)=2\int_0^\beta\mathcal{G}^{(+)}(\tau,\cdot)\,e^{i\frac{2m\pi}{\beta}\tau}\,d\tau\,. \tag{48}$$

Next, recalling that $\mathcal{G}^{(+)}(\tau,\cdot)$ is β–periodic, we can write also the following Fourier expansion:

$$\mathcal{G}^{(+)}(\tau,\cdot)=\frac{1}{\beta}\sum_{m=0}^{\infty}\tilde{\mathcal{G}}^{(+,b)}_{(\beta)}\left(\frac{2m\pi}{\beta},\cdot\right)e^{-i\frac{2m\pi}{\beta}\tau}, \tag{49}$$

$$\tilde{\mathcal{G}}^{(+,b)}_{(\beta)}\left(\frac{2m\pi}{\beta},\cdot\right)=\int_0^\beta\mathcal{G}^{(+)}(\tau,\cdot)\,e^{i\frac{2m\pi}{\beta}\tau}\,d\tau=\frac{1}{2}\,\tilde{\mathcal{G}}^{(+,b)}\left(\frac{2m\pi}{\beta},\cdot\right). \tag{50}$$

Finally, putting $\beta = 2\pi$, formulae (49), (50) can be rewritten in the more convenient form:

$$\mathcal{G}^{(+)}(\tau,\cdot) = \frac{1}{2\pi}\sum_{m=0}^{\infty}\tilde{\mathcal{G}}^{(+,b)}_{(2\pi)}(m,\cdot)e^{-im\tau}, \tag{51}$$

$$\tilde{\mathcal{G}}^{(+,b)}_{(2\pi)}(m,\cdot) = \int_0^{2\pi}\mathcal{G}^{(+)}(\tau,\cdot)\,e^{im\tau}\,d\tau = \frac{1}{2}\,\tilde{\mathcal{G}}^{(+,b)}(m,\cdot). \tag{52}$$

Recalling once again the β–periodicity of the function $\mathcal{G}^{(+)}(\tau,\cdot)$, we write now the Froissart–Gribov equalities (38) as

$$\tilde{\mathcal{G}}^{(+,b)}(m,\cdot) = \tilde{J}^{(+,b)}_{(0)}(m,\cdot) + \tilde{J}^{(+,b)}_{(2\pi)}(m,\cdot) = 2\tilde{J}^{(+,b)}_{(0)}(m,\cdot) = 2\tilde{\mathcal{G}}^{(+,b)}_{(2\pi)}(m,\cdot), \qquad (m = 1,2,3,\ldots). \tag{53}$$

It is now convenient to introduce an auxiliary function $J^{(b)}_{*}(v,\cdot)$, defined as follows:

$$J^{(b)}_{*}(v,\cdot) = e^{-v}\,J^{(+,b)}_{(0)}(v,\cdot), \qquad (v \in \mathbb{R}^{+}), \tag{54}$$

and the corresponding Laplace transform:

$$\tilde{J}^{(b)}_{*}(\zeta,\cdot) = \int_0^{+\infty} J^{(b)}_{*}(v,\cdot)\,e^{-\zeta v}\,dv, \qquad (\zeta = \xi + i\eta, \mathrm{Re}\,\zeta > -1+\delta,\, \delta \geq \epsilon > 0). \tag{55}$$

It is straightforward to prove, via the Paley–Wiener theorem, that $\tilde{J}^{(b)}_{*}(\zeta,\cdot)$ belongs to the Hardy space $\mathbb{H}^2\left(\mathbb{C}^{(+)}_{(-1+\delta)}\right)$, where $\mathbb{C}^{(+)}_{(-1+\delta)} = \{\zeta \in \mathbb{C} \mid \mathrm{Re}\,\zeta > -1+\delta,\, \delta \geq \epsilon > 0\}$. Next, the Froissart–Gribov equalities (53) can be rewritten as

$$\tilde{J}^{(b)}_{*}(m,\cdot) = \tilde{\mathcal{G}}^{(+,b)}_{(2\pi)}(m+1,\cdot)\ , \qquad (m = 0,1,2,\ldots). \tag{56}$$

Then we can prove the following lemma.

Lemma 1. *The function $\tilde{J}^{(b)}_{*}(-1/2+i\eta,\cdot)$, $(\eta \in \mathbb{R})$ can be represented by the following series, that converges in the sense of the L^2–norm:*

$$\tilde{J}^{(b)}_{*}\left(-\frac{1}{2}+i\eta,\cdot\right) = \sum_{\ell=0}^{\infty} c_\ell \psi_\ell(\eta), \tag{57}$$

$\psi_\ell(\eta)$ denoting the Pollaczek functions defined by

$$\psi_\ell(\eta) = \frac{1}{\sqrt{\pi}}\,\Gamma\left(\frac{1}{2}+i\eta\right)P_\ell(\eta), \tag{58}$$

Γ being the Euler gamma function, and P_ℓ the Pollaczek polynomials [2,16]. The coefficients c_ℓ are given by:

$$c_\ell = 2\sqrt{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\tilde{\mathcal{G}}^{(+,b)}_{(2\pi)}(m+1,\cdot)P_\ell\left[-i\left(m+\frac{1}{2}\right)\right]. \tag{59}$$

Proof. The Pollaczek polynomials $P_\ell^{(\alpha)}(\eta)$, $(\eta \in \mathbb{R})$, are orthogonal in $L^2(-\infty,+\infty)$ with weight function (see Refs. [2,16]):

$$w(\eta) = \frac{1}{\pi}\, 2^{(2\alpha-1)} \left|\Gamma(\alpha+i\eta)\right|^2. \tag{60}$$

For $\alpha = 1/2$, the orthogonality property reads:

$$\int_{-\infty}^{+\infty} w(\eta) P_\ell^{(1/2)}(\eta) P_{\ell'}^{(1/2)}(\eta)\, d\eta = \delta_{\ell,\ell'}\ , \qquad \left(w(\eta) = \frac{1}{\pi}\left|\Gamma\left(\frac{1}{2}+i\eta\right)\right|^2\right), \tag{61}$$

(in the following, when $\alpha = 1/2$, we omit the index α in the notation). Next, we introduce the following functions, that will be called Pollaczek functions (of index $\alpha = 1/2$):

$$\psi_\ell(\eta) = \frac{1}{\sqrt{\pi}}\, \Gamma\left(\frac{1}{2}+i\eta\right) P_\ell(\eta), \tag{62}$$

which form a complete basis in $L^2(-\infty,+\infty)$ [13]. Since $\tilde{J}_*^{(b)}(\zeta,\cdot)$ belongs to the Hardy space $\mathbb{H}^2\left(\mathbb{C}^{(+)}_{(-1+\delta)}\right)$, then $\tilde{J}_*^{(b)}(-1/2+i\eta,\cdot)$ $(\eta\in\mathbb{R})$ belongs to $L^2(-\infty,+\infty)$. Therefore, we may expand $\tilde{J}_*^{(b)}(-1/2+i\eta,\cdot)$ in terms of Pollaczek functions as follows:

$$\tilde{J}_*^{(b)}\left(-\frac{1}{2}+i\eta,\cdot\right) = \sum_{\ell=0}^{\infty} c_\ell \psi_\ell(\eta), \tag{63}$$

where the series at the r.h.s. of (63) converges to $\tilde{J}_*^{(b)}(-1/2+i\eta,\cdot)$ in the sense of the L^2–norm. From (63) we get

$$c_\ell = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \tilde{J}_*^{(b)}\left(-\frac{1}{2}+i\eta,\cdot\right) \Gamma\left(\frac{1}{2}-i\eta\right) P_\ell(\eta)\, d\eta. \tag{64}$$

The integral at the r.h.s. of (64) can be evaluated by the contour integration method along the path shown in Fig. 2, and taking into account the asymptotic behaviour of the gamma function given by the Stirling formula. We obtain:

$$c_\ell = 2\sqrt{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\, \tilde{J}_*^{(b)}(m,\cdot) P_\ell\left[-i\left(m+\frac{1}{2}\right)\right]. \tag{65}$$

Finally, from (56), (63) and (65) the proof of the lemma follows. □

From (55), when $\zeta = -1/2+i\eta$ $(\eta\in\mathbb{R})$, we have:

$$\tilde{J}_*^{(b)}\left(-\frac{1}{2}+i\eta,\cdot\right) = \int_0^{+\infty} J_*^{(b)}(v,\cdot)\, e^{v/2} e^{-i\eta v}\, dv. \tag{66}$$

The r.h.s. of (66) is the Fourier transform of $J_*^{(b)}(v,\cdot)e^{v/2}$. Noting that $\tilde{J}_*^{(b)}(-1/2+i\eta,\cdot)$ belongs to $L^2(-\infty,+\infty)$, but not necessarily to $L^1(-\infty,+\infty)$, the inversion of the Fourier transform (66) holds only as a limit in the mean order two, and can be written as follows:

$$J_*^{(b)}(v,\cdot)\, e^{v/2} = \underset{\eta_0\to+\infty}{\mathrm{l.i.m.}} \left(\frac{1}{2\pi}\int_{-\eta_0}^{\eta_0} \tilde{J}_*^{(b)}\left(-\frac{1}{2}+i\eta,\cdot\right) e^{i\eta v}\, d\eta\right), \qquad (v\in\mathbb{R}^+). \tag{67}$$

Then, we can prove the following lemma.

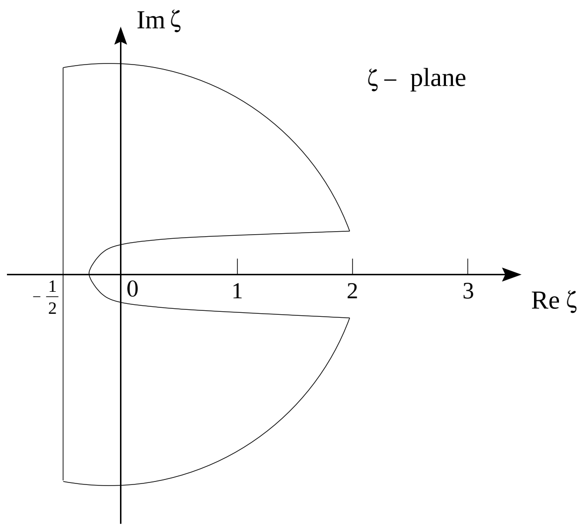

Fig. 2. Integration path for the evaluation of integral (64)

Lemma 2. *The function* $J_*^{(b)}(v,\cdot)e^{v/2}$ *can be represented by the following expansion that converges in the sense of the* L^2*–norm:*

$$e^{v/2}J_*^{(b)}(v,\cdot)=\sum_{\ell=0}^{\infty}a_\ell\Phi_\ell(v),\qquad (v\in\mathbb{R}^+),\tag{68}$$

where the coefficients a_ℓ *are given by:*

$$a_\ell=\sqrt{2}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\,\tilde{\mathcal{G}}^{(+,b)}_{(2\pi)}(m+1,\cdot)\,P_\ell\left[-i\left(m+\frac{1}{2}\right)\right],\tag{69}$$

P_ℓ *being the Pollaczek polynomials, and the functions* $\Phi_\ell(v)$ *are given by*

$$\Phi_\ell(v)=i^\ell\sqrt{2}\,L_\ell(2e^{-v})\,e^{-e^{-v}}e^{-v/2},\tag{70}$$

L_ℓ *being the Laguerre polynomials.*

Proof. Let us observe that

$$\Gamma\left(\frac{1}{2}+i\eta\right)=\int_0^{+\infty}e^{-t}\,t^{(i\eta-1/2)}\,dt=\int_{-\infty}^{+\infty}e^{-e^{-v}}e^{-v/2}\,e^{-i\eta v}\,dv=\mathcal{F}\left\{e^{-e^{-v}}e^{-v/2}\right\},\tag{71}$$

where $\mathcal{F}$ denotes the Fourier integral operator. Let us note that the function $\exp(-e^{-v})e^{-v/2}$ belongs to $S^\infty(\mathbb{R})$, i.e. the Schwartz space of the $C^\infty(\mathbb{R})$ functions that, together with all their derivatives, tend to zero, for $|v|$ tending to $+\infty$, faster than any negative power of $|v|$. Therefore, we can write (see formula (62)):

$$\psi_\ell(\eta)=\frac{1}{\sqrt{\pi}}\,\mathcal{F}\left\{P_\ell\left(-i\frac{d}{dv}\right)\left[e^{-e^{-v}}e^{-v/2}\right]\right\}.\tag{72}$$

Substituting in expansion (57) to the Pollaczek functions their representation (72), we obtain:

$$\tilde{J}_*^{(b)}(-\frac{1}{2}+i\eta,\cdot) = \sum_{\ell=0}^{\infty} c_\ell \left\{ \frac{1}{\sqrt{\pi}}\,\mathcal{F}\left[P_\ell\left(-i\frac{d}{dv}\right)\left(e^{-e^{-v}}e^{-v/2}\right)\right]\right\}. \tag{73}$$

Let us now apply the operator $\mathcal{F}^{-1}$ to the r.h.s. of (73). If we exchange the integral operator $\mathcal{F}^{-1}$ with the sum, and this is legitimate within the L^2–norm convergence, we obtain:

$$\mathcal{F}^{-1}\sum_{\ell=0}^{\infty} c_\ell \left\{ \frac{1}{\sqrt{\pi}}\,\mathcal{F}\left[P_\ell\left(-i\frac{d}{dv}\right)\left(e^{-e^{-v}}e^{-v/2}\right)\right]\right\} = \sum_{\ell=0}^{\infty} c_\ell \left\{ \frac{1}{\sqrt{\pi}}\left[P_\ell\left(-i\frac{d}{dv}\right)\left(e^{-e^{-v}}e^{-v/2}\right)\right]\right\} \tag{74}$$

Finally, recalling formula (67), we obtain the following expansion for the function $J_*^{(b)}(v,\cdot)e^{v/2}$:

$$e^{v/2}J_*^{(b)}(v,\cdot) = \sum_{\ell=0}^{\infty} \frac{c_\ell}{\sqrt{\pi}}\,P_\ell\left(-i\frac{d}{dv}\right)\left(e^{-e^{-v}}e^{-v/2}\right), \tag{75}$$

whose convergence is in the sense of the L^2–norm. It can be easily verified that [9]

$$\sqrt{2}P_\ell\left(-i\frac{d}{dv}\right)\left(e^{-e^{-v}}e^{-v/2}\right) = i^\ell\sqrt{2}\,L_\ell\left(2e^{-v}\right)e^{-e^{-v}}e^{-v/2}, \tag{76}$$

where L_ℓ denotes the Laguerre polynomials.

It can be checked that the polynomials $\mathcal{L}_\ell(v) = i^\ell\sqrt{2}\,L_\ell(2e^{-v})$ are a set of polynomials orthonormal on the real line with weight function $w(v) = \exp(-v)\exp(-2e^{-v})$, and, consequently, the set of functions $\Phi_\ell(v)$, defined by formula (70), forms an orthonormal basis in $L^2(-\infty,+\infty)$.

Finally, from (75) we obtain:

$$e^{v/2}J_*^{(b)}(v,\cdot) = \sum_{\ell=0}^{\infty} a_\ell\left\{i^\ell\sqrt{2}\,L_\ell(2e^{-v})\,e^{-e^{-v}}e^{-v/2}\right\} = \sum_{\ell=0}^{\infty} a_\ell\Phi_\ell(v), \qquad (v\in\mathbb{R}^+), \tag{77}$$

where $a_\ell = c_\ell/\sqrt{2\pi}$, and the functions $\Phi_\ell(v)$ are given by formula (70). □

We now introduce the weighted L^2–space $L^2_{(w)}[0,+\infty)$, whose norm is defined by:

$$\|f\|_{L^2_{(w)}[0,+\infty)} = \left(\int_0^{+\infty} w(v)\,|f(v)|^2\,dv\right)^{1/2}, \tag{78}$$

$w(v)$ being a weight function which will be specified in the following. Then we can prove the following result.

Theorem 2. *The jump function $J_{(0)}^{(+,b)}(v,\cdot)$ can be represented by the following expansion:*

$$J_{(0)}^{(+,b)}(v,\cdot) = e^{v/2}\sum_{\ell=0}^{\infty} a_\ell\Phi_\ell(v), \qquad (v\in\mathbb{R}^+), \tag{79}$$

which converges in the sense of the $L^2_{(w)}[0,+\infty)$–norm, with weight function $w(v) = e^{-v}$, $(v\in\mathbb{R}^+)$.

Proof. We can write:

$$\left\| J_{(0)}^{(+,b)}(v,\cdot) - e^{v/2}\sum_{\ell=0}^{L} a_\ell \Phi_\ell(v)\right\|_{L^2_{(w)}[0,+\infty)} = \left(\int_0^{+\infty} e^{-v}\left| J_{(0)}^{(+,b)}(v,\cdot) - e^{v/2}\sum_{\ell=0}^{L} a_\ell \Phi_\ell(v)\right|^2 dv\right)^{1/2}$$
$$= \left(\int_0^{+\infty} \left| e^{v/2} J_*^{(b)}(v,\cdot) - \sum_{\ell=0}^{L} a_\ell \Phi_\ell(v)\right|^2 dv\right)^{1/2}. \quad (80)$$

In view of Lemma 2 we can thus state that:

$$\lim_{L\to\infty}\left\| J_{(0)}^{(+,b)}(v,\cdot) - e^{v/2}\sum_{\ell=0}^{L} a_\ell \Phi_\ell(v)\right\|_{L^2_{(w)}[0,+\infty)} = 0, \quad (81)$$

that proves the statement. □

Consider now a system of fermions. In this case the function $\mathcal{G}^{(+)}(\tau,\cdot)$ is antiperiodic with period β. Then, if we put $\zeta_m = (2m+1)\pi/\beta$ ($m = 0, 1, 2, \ldots$) and $\beta = 2\pi$, we have the following expansion:

$$\mathcal{G}^{(+)}(\tau,\cdot) = \frac{1}{2\pi}\sum_{m=0}^{\infty} \tilde{\mathcal{G}}_{(2\pi)}^{(+,f)}\left(m+\frac{1}{2},\cdot\right) e^{-i(m+1/2)\tau}, \quad (82)$$

$$\tilde{\mathcal{G}}_{(2\pi)}^{(+,f)}\left(m+\frac{1}{2},\cdot\right) = \int_0^{2\pi} \mathcal{G}^{(+)}(\tau,\cdot)\, e^{i(m+1/2)\tau}\, d\tau = \frac{1}{2}\,\tilde{\mathcal{G}}^{(+,f)}\left(m+\frac{1}{2},\cdot\right). \quad (83)$$

Recalling once again the antiperiodicity of $\mathcal{G}^{(+)}(\tau,\cdot)$, we write the Froissart–Gribov equalities (40) in the following form:

$$\tilde{\mathcal{G}}^{(+,f)}\left(m+\frac{1}{2},\cdot\right) = \tilde{J}_{(0)}^{(+,f)}\left(m+\frac{1}{2},\cdot\right) - \tilde{J}_{(2\pi)}^{(+,f)}\left(m+\frac{1}{2},\cdot\right)$$
$$= 2\,\tilde{J}_{(0)}^{(+,f)}\left(m+\frac{1}{2},\cdot\right) = 2\,\tilde{\mathcal{G}}_{(2\pi)}^{(+,f)}\left(m+\frac{1}{2},\cdot\right), \qquad (m = 0, 1, 2, \ldots). \quad (84)$$

We can now proceed in a way strictly analogous to that followed in the case of bosons. We put: $J_*^{(f)}(v,\cdot) = e^{-v} J_{(0)}^{(+,f)}(v,\cdot)$ and, accordingly, $\tilde{J}_*^{(f)}(\zeta,\cdot) = \int_0^{+\infty} J_*^{(f)}(v,\cdot) e^{-\zeta v} dv$ ($\zeta = \xi + i\eta, \mathrm{Re}\,\zeta \equiv \xi > -1+\delta$, $\delta \geq \epsilon > 0$). Then, the Froissart–Gribov equalities (84) now read:

$$\tilde{J}_*^{(f)}\left(m+\frac{1}{2},\cdot\right) = \tilde{\mathcal{G}}_{(2\pi)}^{(+,f)}\left(m+\frac{3}{2},\cdot\right), \qquad (m = 0, 1, 2, \ldots). \quad (85)$$

We can now state the following theorem.

Theorem 3. *i) The function* $\tilde{J}_*^{(f)}(i\eta,\cdot)$, $(\eta \in \mathbb{R})$ *can be represented by the following series, that converges in the sense of the* L^2*–norm:*

$$\tilde{J}_*^{(f)}(i\eta,\cdot) = \sum_{\ell=0}^{\infty} d_\ell \psi_\ell(\eta), \quad (86)$$

where $\psi_\ell(\eta)$ are the Pollaczek functions defined by formula (58)*, and the coefficients d_ℓ are given by:*

$$d_\ell = 2\sqrt{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\,\tilde{\mathcal{G}}^{(+,f)}_{(2\pi)}\left(m+\frac{3}{2},\cdot\right)P_\ell\left[-i\left(m+\frac{1}{2}\right)\right], \tag{87}$$

P_ℓ denoting the Pollaczek polynomials.

ii) The function $J_*^{(f)}(v,\cdot)$ can be represented by the following expansion that converges in the sense of L^2–norm:

$$J_*^{(f)}(v,\cdot) = \sum_{\ell=0}^{\infty} b_\ell \Phi_\ell(v), \qquad (v\in\mathbb{R}^+), \tag{88}$$

where the coefficients b_ℓ are given by $b_\ell = d_\ell/\sqrt{2\pi}$, and the functions $\Phi_\ell(v)$ are defined by formula (70)*.*

iii) The function $J_{(0)}^{(+,f)}(v,\cdot)$ can be represented by the following expansion:

$$J_{(0)}^{(+,f)}(v,\cdot) = e^v\sum_{\ell=0}^{\infty} b_\ell \Phi_\ell(v), \qquad (v\in\mathbb{R}^+), \tag{89}$$

that converges in the sense of the $L^2_{(w)}[0,+\infty)$–norm with weight function $w(v)=e^{-2v}$, $(v\in\mathbb{R}^+)$.

Proof. The proof runs exactly as in the case of the boson statistics, with the only remarkable difference that we use the Froissart–Gribov equalities (85) instead of (56). □

We can reconstruct, by the use of this method, the function $\tilde{J}_*^{(f)}(i\eta,\cdot)$ but not the function $\tilde{J}_{(0)}^{(+,f)}(i\eta,\cdot)$, which is much more interesting from the physical viewpoint. In order to recover the function $\tilde{J}_{(0)}^{(+,f)}(i\eta,\cdot)$ we must introduce a more restrictive assumption, requiring the function $\tilde{J}_{(0)}^{(+,f)}(\zeta,\cdot) = \int_0^{+\infty} J_{(0)}^{(+,f)}(v,\cdot)e^{-\zeta v}\,dv$ to be holomorphic in the half–plane $\mathrm{Re}\,\zeta > -\gamma$ $(\gamma>0)$. Accordingly, in place of the temperateness condition (25) we assume that $J_{(0)}^{(+,f)}(v,\cdot)$ belongs to $L^1[0,+\infty)\cap L^2[0,+\infty)$. Here, for the sake of simplicity, we treat only the case of fermions; analogous considerations hold true also in the case of the boson statistics. We can thus suppose that the singularities of $\tilde{J}_{(0)}^{(+,f)}(\zeta,\cdot)$, corresponding to the excited states, all lie in the half–plane $\mathrm{Re}\,\zeta < -\gamma$, γ being the smallest damping factor of the spectrum (see Refs. [1,10]). If this is the case, $\tilde{J}_{(0)}^{(+,f)}(i\eta,\cdot)$ is analytic, and, moreover, belongs also to $L^2(-\infty,+\infty)$. We can thus state the following result.

Theorem 4. *Let us assume that $\tilde{J}_{(0)}^{(+,f)}(\zeta,\cdot)$ is a function holomorphic in the half–plane $\mathrm{Re}\,\zeta > -\gamma$ $(\gamma>0)$; then $\tilde{J}_{(0)}^{(+,f)}(i\eta,\cdot)$ can be represented by the following expansion that converges in the sense of the L^2–norm:*

$$\tilde{J}_{(0)}^{(+,f)}(i\eta,\cdot) = \sum_{\ell=0}^{\infty} d'_\ell \psi_\ell(\eta), \tag{90}$$

where $\psi_\ell(\eta)$ are the Pollaczek functions defined by formula (58), and the coefficients d'_ℓ are given by:

$$d'_\ell = 2\sqrt{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\,\tilde{\mathcal{G}}^{(+,f)}_{(2\pi)}\left(m+\frac{1}{2},\cdot\right)P_\ell\left[-i\left(m+\frac{1}{2}\right)\right], \tag{91}$$

P_ℓ denoting the Pollaczek polynomials.

Proof. The proof is strictly analogous to the one followed for proving equality (57), and successively adapted to the fermion statistics in order to obtain expansion (86). The only remarkable difference is that now in the expression of the coefficients d'_ℓ we have the terms $\tilde{\mathcal{G}}^{(+,f)}_{(2\pi)}\left(m+\frac{1}{2},\cdot\right)$ instead of $\tilde{\mathcal{G}}^{(+,f)}_{(2\pi)}\left(m+\frac{3}{2},\cdot\right)$; therefore all the coefficients corresponding to $m=0,1,2,\ldots$, are involved in the determination of the function $\tilde{J}^{(+,f)}_{(0)}(i\eta,\cdot)$. $\square$

Analogous methods and results can be worked out for the function $\tilde{J}^{(-,f)}_{(0)}(i\eta,\cdot)$, assuming that $\tilde{J}^{(-,f)}_{(0)}(\zeta,\cdot)$ is holomorphic in the half–plane $\mathrm{Re}\,\zeta<\gamma$ $(\gamma>0)$. We are then able to reconstruct the difference $\tilde{J}^{(+,f)}_{(0)}(i\eta,\cdot)-\tilde{J}^{(-,f)}_{(0)}(i\eta,\cdot)$ which leads to the determination of the "spectral density" [17].

4. Reconstruction of the Jump Function in Terms of a Finite Number of Fourier Coefficients

Up to now we have assumed that all the Fourier coefficients are known, and, in addition, that they are noiseless; but this assumption is clearly unrealistic. We now suppose that only a finite number of coefficients are known within a certain degree of approximation. We focus our attention on the case of the boson statistics, and specifically on the results contained in Lemmas 1 and 2, and Theorem 2. The case of the fermion statistics can be treated similarly. We can simplify the notation, without ambiguity, by putting: $\tilde{\mathcal{G}}^{(+,b)}_{(2\pi)}(m+1,\cdot)=g_m$, $e^{v/2}J^{(b)}_*(v,\cdot)=F_*(v)$, and $J^{(+,b)}_{(0)}(v,\cdot)=F(v)$. Then, we denote by $g^{(\epsilon)}_m$ the Fourier coefficients $\tilde{\mathcal{G}}^{(+,b)}_{(2\pi)}(m+1,\cdot)$ when they are perturbed by noise. We now assume that only $(N+1)$ Fourier coefficients are known within an approximation error of order ϵ: i.e. $|g^{(\epsilon)}_m-g_m|\le\epsilon$ $(m=0,1,2,\ldots,N)$.

We consider the following finite sums:

$$a^{(\epsilon,N)}_\ell=\sqrt{2}\sum_{m=0}^{N}\frac{(-1)}{m!}\,g^{(\epsilon)}_m P_\ell\left[-i\left(m+\frac{1}{2}\right)\right]. \tag{92}$$

Accordingly, we have $a^{(0,\infty)}_\ell=a_\ell$ (see (69)). We can then prove the following lemma.

Lemma 3. *The following statements hold true:*

i) $$\sum_{\ell=0}^{\infty}\left|a^{(0,\infty)}_\ell\right|^2=\|F_*\|^2_{L^2[0,\infty)}=C,\qquad (C=\mathit{constant}). \tag{93}$$

ii) $$\sum_{\ell=0}^{\infty}\left|a^{(\epsilon,N)}_\ell\right|^2=+\infty. \tag{94}$$

iii) $$\lim_{\substack{N\to\infty\\ \epsilon\to 0}}a^{(\epsilon,N)}_\ell=a^{(0,\infty)}_\ell=a_\ell,\qquad (\ell=0,1,2,\ldots). \tag{95}$$

iv) If $k_0(\epsilon, N)$ *is defined as*

$$k_0(\epsilon, N) = \max\left\{ k \in \mathbb{N} \,:\, \sum_{\ell=0}^{k} |a_\ell^{(\epsilon,N)}|^2 \leq C \right\}, \tag{96}$$

i.e. it is the largest integer such that $\sum_{\ell=0}^{k} \left| a_\ell^{(\epsilon,N)} \right|^2 \leq C$*, then*

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0}} k_0(\epsilon, N) = +\infty. \tag{97}$$

v) The sum

$$M_k^{(\epsilon,N)} = \sum_{\ell=0}^{k} \left| a_\ell^{(\epsilon,N)} \right|^2, \qquad (k \in \mathbb{N}), \tag{98}$$

satisfies the following properties:
a) it increases for increasing values of k*;*
b) the following relationships hold true:

$$M_k^{(\epsilon,N)} \geq \left| a_k^{(\epsilon,N)} \right|^2 \underset{k\to\infty}{\sim} \frac{1}{(N!)^2}\,(2k)^{2N}, \qquad (N\ \textit{fixed}). \tag{99}$$

Proof. (i) Equality (93) follows from the Parseval theorem applied to expansion (68), and recalling that $F_*(v)$ belongs to $L^2(-\infty, +\infty)$.
(ii) Let us rewrite the sums $a_\ell^{(\epsilon,N)}$ as follows:

$$a_\ell^{(\epsilon,N)} = \sum_{m=0}^{N} b_m^{(\epsilon)} P_\ell\left[-i\left(m+\frac{1}{2}\right)\right], \tag{100}$$

where $b_m^{(\epsilon)} = \sqrt{2}(-1)^m g_m^{(\epsilon)}/m!$. Now, we can write the following inequality:

$$\begin{aligned} \left| a_\ell^{(\epsilon,N)} \right| &= \left| \sum_{m=0}^{N} b_m^{(\epsilon)} P_\ell\left[-i\left(m+\frac{1}{2}\right)\right] \right| \\ &\geq \left| b_N^{(\epsilon)} P_\ell\left[-i\left(N+\frac{1}{2}\right)\right] \right| \cdot \left| 1 - \frac{\left|\sum_{m=0}^{N-1} b_m^{(\epsilon)} P_\ell\left[-i\left(m+\frac{1}{2}\right)\right]\right|}{\left| b_N^{(\epsilon)} P_\ell\left[-i\left(N+\frac{1}{2}\right)\right] \right|} \right|. \end{aligned} \tag{101}$$

Let us now recall that in the Appendix of Ref. [9] the asymptotic behaviour of the Pollaczek polynomials $P_\ell[-i(m+1/2)]$ for large values of l (at fixed m) is proved to be:

$$P_\ell\left[-i\left(m+\frac{1}{2}\right)\right] \underset{\ell\to\infty}{\sim} \frac{(-1)^\ell i^\ell}{m!}\,(2\ell)^m. \tag{102}$$

Therefore, we have:

$$\frac{\left|\sum_{m=0}^{N-1} b_m^{(\epsilon)} P_\ell\left[-i\left(m+\frac{1}{2}\right)\right]\right|}{\left| b_N^{(\epsilon)} P_\ell\left[-i\left(N+\frac{1}{2}\right)\right] \right|} \leq \frac{\sum_{m=0}^{N-1} \left| b_m^{(\epsilon)} P_\ell\left[-i\left(m+\frac{1}{2}\right)\right]\right|}{\left| b_N^{(\epsilon)} P_\ell\left[-i\left(N+\frac{1}{2}\right)\right] \right|} \underset{\ell\to\infty}{\sim} \sum_{m=0}^{N-1} \left| \frac{b_m^{(\epsilon)}}{b_N^{(\epsilon)}} \right| \frac{N!}{m!} (2\ell)^{m-N} \underset{\ell\to\infty}{\longrightarrow} 0. \tag{103}$$

From (101), (102) and (103) it follows that for ℓ sufficiently large:

$$\left|a_\ell^{(\epsilon,N)}\right| \underset{\ell\to\infty}{\sim} \frac{\left|b_N^{(\epsilon)}\right|}{N!}\,(2\ell)^N. \tag{104}$$

Therefore, $\lim_{\ell\to\infty}\left|a_\ell^{(\epsilon,N)}\right| = +\infty$, and statement (ii) follows.
(iii) We can write the difference $a_\ell^{(0,\infty)} - a_\ell^{(\epsilon,N)}$ as follows:

$$\begin{aligned} a_\ell^{(0,\infty)} - a_\ell^{(\epsilon,N)} = \sqrt{2}\Bigg\{ &\sum_{m=0}^{N}\frac{(-1)^m}{m!}\,(g_m - g_m^{(\epsilon)})P_\ell\left[-i\left(m+\frac{1}{2}\right)\right] \\ &+ \sum_{m=N+1}^{\infty}\frac{(-1)^m}{m!}g_m P_\ell\left[-i\left(m+\frac{1}{2}\right)\right]\Bigg\}. \end{aligned} \tag{105}$$

In view of the fact that the series $\sqrt{2}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}g_m P_\ell[-i(m+\frac{1}{2})]$ converges to $a_\ell^{(0,\infty)}$, it follows that the second term in bracket (105) tends to zero as $N\to\infty$. Concerning the first term, we may write the inequality:

$$\left|\sum_{m=0}^{N}\frac{(-1)^m}{m!}(g_m - g_m^{(\epsilon)})P_\ell\left[-i\left(m+\frac{1}{2}\right)\right]\right| \leq \epsilon\sum_{m=0}^{N}\frac{1}{m!}\left|P_\ell\left[-i\left(m+\frac{1}{2}\right)\right]\right|, \tag{106}$$

where the inequalities $\left|g_m - g_m^{(\epsilon)}\right| \leq \epsilon$, $(m=0,1,2,\ldots,N)$ have been used. Next, by rewriting the Pollaczek polynomials $P_\ell[-i(m+1/2)]$ as

$$P_\ell\left[-i\left(m+\frac{1}{2}\right)\right] = \sum_{j=0}^{\ell} p_j^{(\ell)}\left(m+\frac{1}{2}\right)^j, \tag{107}$$

and, substituting this expression in inequality (106), we obtain:

$$\epsilon\sum_{m=0}^{N}\frac{1}{m!}\left[\sum_{j=0}^{l}\left|p_j^{(\ell)}\right|\left(m+\frac{1}{2}\right)^j\right]. \tag{108}$$

Next, we perform the limit for $N\to\infty$. In view of the fact that $\sum_{j=0}^{l} p_j^{(\ell)}(m+1/2)^j$ is finite, and the series $\sum_{m=0}^{\infty}(m+1/2)^j/m!$ converges, we can exchange the order of the sums and write:

$$\epsilon\sum_{j=0}^{l}\left|p_j^{(\ell)}\right|\sum_{m=0}^{\infty}\frac{1}{m!}\left(m+\frac{1}{2}\right)^j. \tag{109}$$

Finally, performing the limit for $\epsilon\to 0$, and recalling equality (105), statement (iii) is obtained.
(iv) From definition (96) it follows, for $k_1 = k_0 + 1$, that:

$$\sum_{\ell=0}^{k_1}\left|a_\ell^{(\epsilon,N)}\right|^2 > C. \tag{110}$$

Statement (iv) (formula (97)) is proved if we can show that $\lim_{\substack{N\to\infty\\ \epsilon\to 0}} k_1(\epsilon,N) = +\infty$. Let us suppose that $\lim_{\substack{N\to\infty\\ \epsilon\to 0}} k_1(\epsilon,N)$ is finite. Then there should exist a finite number K (independent of ϵ and N) such that, for N tending to ∞ and ϵ tending to zero, $k_1(\epsilon,N) \leq K$. Then, from inequality (110) we have:

$$C < \sum_{\ell=0}^{k_1(\epsilon,N)} \left| a_\ell^{(\epsilon,N)} \right|^2 \leq \sum_{\ell=0}^{K} \left| a_\ell^{(\epsilon,N)} \right|^2 . \tag{111}$$

But as $N\to\infty, \epsilon\to 0$ we have (recalling also statement (iii) formula (95)):

$$C < \sum_{\ell=0}^{K} \left| a_\ell^{(0,\infty)} \right|^2 \leq \sum_{\ell=0}^{\infty} \left| a_\ell^{(0,\infty)} \right|^2 = C, \tag{112}$$

which leads to a contradiction. Then statement (iv) follows.

(v) Concerning statement (a), it follows obviously from definition (98) of $M_k^{(\epsilon,N)}$. Finally, the first relationship in (99) is obvious; the second one follows from the asymptotic behavior of $P_\ell[-i(m+1/2)]$ at large ℓ (for fixed m), i.e. formula (102). $\square$

Remark 1. From statement (v) and formula (97) it follows that the sum $M_k^{(\epsilon,N)}$ presents, for large values of N and small values of ϵ, a plateau for $k \sim k_0$.

By truncating expansion (68) we may now introduce an approximation of the function $F_*(v)$ of the following type:

$$F_*^{(\epsilon,N)}(v) = \sum_{\ell=0}^{k_0(\epsilon,N)} a_\ell^{(\epsilon,N)} \Phi_\ell(v), \qquad (v \in \mathbb{R}^+). \tag{113}$$

Approximation $F_*^{(\epsilon,N)}(v)$ is defined through the truncation number $k_0(\epsilon,N)$; the latter can be numerically determined by plotting the sum $M_k^{(\epsilon,N)}$ versus k, and exploiting properties (a) and (b), proved in statement (v) of the previous lemma and the property stated in the remark above (see also Ref. [9]).

Now, we want to prove that the approximation $F_*^{(\epsilon,N)}(v)$ converges asymptotically to $F_*(v)$ in the sense of the L^2–norm, as $N\to\infty$ and $\epsilon\to 0$. We can prove the following theorem.

Theorem 5. *The equality*

$$\lim_{\substack{N\to\infty\\ \epsilon\to 0}} \left\| F_* - F_*^{(\epsilon,N)}(v) \right\|_{L^2[0,+\infty)} = 0 \tag{114}$$

holds true.

Proof. From the Parseval equality it follows that:

$$\left\| F_* - F_*^{(\epsilon,N)} \right\|^2_{L^2[0,+\infty)} = \left\{ \sum_{\ell=k_0+1}^{\infty} \left| a_\ell^{(0,\infty)} \right|^2 + \sum_{\ell=0}^{k_0} \left| a_\ell^{(\epsilon,N)} - a_\ell^{(0,\infty)} \right|^2 \right\} . \tag{115}$$

Since $\sum_{\ell=0}^{\infty} \left| a_\ell^{(0,\infty)} \right|^2 = C$ and $\lim_{\substack{N\to\infty \\ \epsilon\to 0}} k_0(\epsilon, N) = +\infty$, it follows that $\lim_{\substack{N\to\infty \\ \epsilon\to 0}} \sum_{\ell=k_0+1}^{\infty} \left| a_\ell^{(\epsilon,N)} \right|^2 = 0$. It is convenient to rewrite the second term of the r.h.s. of (115) as follows. Let us define:

$$h_\ell^{(0,\infty)} = \begin{cases} a_\ell^{(0,\infty)} & \text{if } \ell \text{ is even,} \\ -i a_\ell^{(0,\infty)} & \text{if } \ell \text{ is odd,} \end{cases} \tag{116}$$

$$h_\ell^{(\epsilon,N)} = \begin{cases} a_\ell^{(\epsilon,N)} & \text{if } \ell \text{ is even,} \\ -i a_\ell^{(\epsilon,N)} & \text{if } \ell \text{ is odd.} \end{cases} \tag{117}$$

Notice that $h_\ell^{(0,\infty)}$ and $h_\ell^{(\epsilon,N)}$ are real, and $\sum_{\ell=0}^{k_0} \left| a_\ell^{(\epsilon,N)} - a_\ell^{(0,\infty)} \right|^2 = \sum_{\ell=0}^{k_0} \left(h_\ell^{(\epsilon,N)} - h_\ell^{(0,\infty)} \right)^2$. Next, we introduce the following functions:

$$H^{(0,\infty)}(v) = \sum_{\ell=0}^{\infty} h_\ell^{(0,\infty)} \mathbf{1}_{[\ell,\ell+1[}(v), \tag{118}$$

$$H^{(\epsilon,N)}(v) = \sum_{\ell=0}^{\infty} h_\ell^{(\epsilon,N)} \mathbf{1}_{[\ell,\ell+1[}(v), \tag{119}$$

where $\mathbf{1}_E$ is the characteristic function of the set E. From statements (i), (ii) and (iii) of the previous lemma (formulae (93), (94) and (95)) we obtain:

$$\int_0^{+\infty} \left(H^{(0,\infty)}(v) \right)^2 dv = \sum_{\ell=0}^{\infty} \left(h_\ell^{(0,\infty)} \right)^2 = C, \tag{120}$$

$$\int_0^{+\infty} \left(H^{(\epsilon,N)}(v) \right)^2 dv = \sum_{\ell=0}^{\infty} \left(h_\ell^{(\epsilon,N)} \right)^2 = +\infty, \tag{121}$$

$$H^{(\epsilon,N)}(v) \underset{\substack{N\to\infty \\ \epsilon\to 0}}{\longrightarrow} H^{(0,\infty)}(v), \qquad (v \in [0, +\infty)). \tag{122}$$

Hereafter, we assume, for the sake of simplicity and without loss of generality, that every term $h_\ell^{(\epsilon,N)}$ is different from zero. Next, let $V(\epsilon, N)$ be the unique root of equation $\int_0^V \left(H^{(\epsilon,N)}(v) \right)^2 dv = C$. Let us indeed observe that $\int_0^V \left(H^{(\epsilon,N)}(v) \right)^2 dv$ is a continuous non–decreasing function which is zero for $V = 0$, and $+\infty$ for $V \to +\infty$. Furthermore, from statement (iv) of the previous lemma (formula (97)) we have $\lim_{\substack{N\to\infty \\ \epsilon\to 0}} V(\epsilon, N) = +\infty$.

Then we can write:

$$\int_0^{V(\epsilon,N)} \left[H^{(\epsilon,N)}(v) - H^{(0,\infty)}(v) \right]^2 dv =$$
$$= \int_{V(\epsilon,N)}^{+\infty} \left(H^{(0,\infty)}(v) \right)^2 dx - 2 \int_0^{V(\epsilon,N)} H^{(0,\infty)}(v) \left[H^{(\epsilon,N)}(v) - H^{(0,\infty)}(v) \right] dv. \tag{123}$$

Next, we perform the limit for $N \to \infty$ and $\epsilon \to 0$. Concerning the first term at the r.h.s. of (123) we have:

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0}} \int_{V(\epsilon,N)}^{+\infty} \left(H^{(0,\infty)}(v)\right)^2 dv = 0. \tag{124}$$

For what concerns the second term, we introduce the following function:

$$B^{(\epsilon,N)}(v) = \begin{cases} H^{(\epsilon,N)}(v) - H^{(0,\infty)}(v) & \text{if } 0 \le v \le V(\epsilon,N), \\ 0 & \text{if } v > V(\epsilon,N). \end{cases} \tag{125}$$

Then, we have by the use of the Schwarz inequality

$$\int_0^{+\infty} \left|B^{(\epsilon,N)}(v)\right|^2 dv \le 4C, \qquad (N < \infty, \epsilon > 0). \tag{126}$$

Moreover, from (122) we have:

$$B^{(\epsilon,N)}(v) \underset{\substack{N\to\infty \\ \epsilon\to 0}}{\longrightarrow} 0\ , \qquad v \in [0,+\infty). \tag{127}$$

The family of functions $\{B^{(\epsilon,N)}(v)\}$ is bounded in $L^2[0,+\infty)$, therefore it has a subsequence which is weakly convergent in $L^2[0,+\infty)$. The limit of this subsequence is zero. In fact, let us observe that $|B^{(\epsilon,N)}(v)| \le 2C$; then we consider the function $B^{(\epsilon,N)}(v)\phi(v)$, where ϕ is an arbitrary element of the class of functions $C_c^\infty(\mathbb{R}^+)$. We then have $|B^{(\epsilon,N)}(v)\phi(v)| \le 2C|\phi(v)|$, and this inequality does not depend on N and ϵ. In view of the Lebesgue dominated convergence theorem we can then write (see also limit (127)):

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0}} \sup \left|\int_0^{+\infty} B^{(\epsilon,N)}(v)\phi(v)\,dv\right| = 0. \tag{128}$$

Since the set of functions $C_c^\infty(\mathbb{R}^+)$ is everywhere dense in $L^2[0,+\infty)$, given an arbitrary function $\psi \in L^2[0,+\infty)$ and an arbitrary number $\eta > 0$, there exists a function $\phi_k \in C_c^\infty(\mathbb{R}^+)$ such that $\|\psi - \phi_k\|_{L^2[0,+\infty)} < \eta$. Furthermore, through the Schwarz inequality we have:

$$\begin{aligned} \int_0^{+\infty} \left|B^{(\epsilon,N)}(v)[\phi_k(v) - \psi(v)]\right| dv &\le \left(\int_0^{+\infty} \left|B^{(\epsilon,N)}(v)\right|^2 dv\right)^{1/2} \left(\int_0^{+\infty} |\phi_k(v) - \psi(v)|^2\, dv\right)^{1/2} \\ &\le 2\sqrt{C}\,\eta. \end{aligned} \tag{129}$$

From (128) and (129) we can conclude that

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0}} \sup \left|\int_0^{+\infty} B^{(\epsilon,N)}(v)\psi(v)\,dv\right| = 0, \tag{130}$$

for any $\psi \in L^2[0,+\infty)$.

Next, by using the same type of arguments, we can state that if there is an arbitrary subsequence belonging to the family $\{B^{(\epsilon,N)}\}$ that weakly converges in $L^2[0,+\infty)$, then the weak limit of this subsequence is necessarily zero. Finally, from the uniqueness of the (weak) limit point, it follows that the whole family $\{B^{(\epsilon,N)}\}$ converges weakly to zero in $L^2[0,+\infty)$.

We can thus write:

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0}} \int_0^{+\infty} H^{(0,\infty)}(v) B^{(\epsilon,N)}(v)\, dv = 0, \tag{131}$$

and from equality (123) we have

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0}} \int_0^{V(\epsilon,N)} \left[H^{(\epsilon,N)}(v) - H^{(0,\infty)}(v) \right]^2 dv = 0. \tag{132}$$

Since $\sum_{\ell=0}^{k_0} \left| a_\ell^{(\epsilon,N)} - a_\ell^{(0,\infty)} \right|^2 \leq \int_0^{V(\epsilon,N)} \left[H^{(\epsilon,N)}(v) - H^{(0,\infty)}(v) \right]^2 dv$, we have:

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0}} \sum_{\ell=0}^{k_0} \left| a_\ell^{(\epsilon,N)} - a_\ell^{(0,\infty)} \right|^2 = 0, \tag{133}$$

and, in view of equality (115), the theorem is proved. □

We can then prove the following corollary.

Corollary 1. *The following equality holds true:*

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0}} \left\| F(v) - e^{v/2} \sum_{\ell=0}^{k_0(\epsilon,N)} a_\ell^{(\epsilon,N)} \Phi_\ell(v) \right\|_{L^2_{(w)}[0,+\infty)} = 0, \tag{134}$$

$L^2_{(w)}[0,+\infty)$ *being the weighted* L^2*–space with weight function* $w(v) = e^{-v}$, $(v \in \mathbb{R}^+)$, *and the functions* $\Phi_\ell(v)$ *are defined by formula* (70).

Proof. The statement follows immediately from Theorem 5 by noting that:

$$\int_0^{+\infty} \left| F_*(v) - \sum_{\ell=0}^{k_0(\epsilon,N)} a_\ell^{(\epsilon,N)} \Phi_\ell(v) \right|^2 dv = \int_0^{+\infty} e^{-v} \left| F(v) - e^{v/2} \sum_{\ell=0}^{k_0(\epsilon,N)} a_\ell^{(\epsilon,N)} \Phi_\ell(v) \right|^2 dv$$
$$= \left\| F(v) - e^{v/2} \sum_{\ell=0}^{k_0(\epsilon,N)} a_\ell^{(\epsilon,N)} \Phi_\ell(v) \right\|^2_{L^2_{(w)}[0,+\infty)} . \tag{135}$$

We can thus conclude that the jump function $J_{(0)}^{(+,b)}(v,\cdot) = F(v)$ can be approximated by the truncated expansion

$$J_{(0)}^{(+,b)}(v,\cdot) \sim e^{v/2} \sum_{\ell=0}^{k_0(\epsilon,N)} a_\ell^{(\epsilon,N)} \Phi_\ell(v), \qquad (v \in \mathbb{R}^+). \qquad \Box \tag{136}$$

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