

Semiclassical description of shell effects in finite fermion systems

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(Dated: May 29, 2006)

A short survey of the semiclassical periodic orbit theory, initiated by M. Gutzwiller and generalized by many other authors, is given. Via so-called semiclassical trace formulae, gross-shell effects in bound fermion systems can be interpreted in terms of a few periodic orbits of the corresponding classical systems. In integrable systems, these are usually the shortest members of the most degenerate families or orbits, but in some systems also less degenerate orbits can determine the gross-shell structure. Applications to nuclei, metal clusters, semiconductor nanostructures, and trapped dilute atom gases are discussed.

PACS numbers: 03.65.Sq

I. INTRODUCTION

The Nilsson model [1] gave rise to my first steps towards scientific research. My teacher in theoretical physics was Kurt Alder, who had been a member of the theoretical 'Coulomb excitation crew' [2] at the Niels Bohr Institute during the years when collective nuclear motion was explored and single-particle motion in deformed nuclei was studied. When I asked Alder about a possible subject for my diploma work, he gave me a copy of Sven Gösta's famous paper and asked me to write a program for computing the deformed single-particle wavefunctions; he had become weary of interpolating their expansion coefficients in Nilsson's tables. I set about to diagonalize the Nilsson Hamiltonian using, however, not the spherical basis but that of the deformed harmonic oscillator, i.e., the asymptotic Nilsson states. The results were bound to be the same as those of Nilsson and were therefore not published [3] – but through this exercise I became initiated into the shell structure of nuclei. I felt very privileged when I later came to know the great human being behind the famous model.

In this paper I want to review the semiclassical description of shell-effects in finite fermion systems using the periodic orbit theory. After a brief reminder about trace formulae, I will discuss some of their applications to systems in four different branches of physics. Since all results have been published elsewhere, I will not reproduce here any figures, but just discuss in words some of the most important results and conclusions.

II. PERIODIC ORBIT THEORY AND GROSS-SHELL EFFECTS

A. Semiclassical trace formulae

The periodic orbit theory (POT) was initiated by M. Gutzwiller in a series of publications culminating in his seminal paper in 1971 that contains the semiclassical trace formula [4]. It relates the quantum spectrum $\{E_i\}$ (which we here assume to be discrete, although the inclusion of a continuum is possible) of a hermitian Hamil-

tonian \hat{H} to the periodic orbits of the corresponding classical Hamiltonian $H(\mathbf{q}, \mathbf{p})$. The quantum-mechanical level density, defined as the sum of Dirac delta functions peaked at the levels E_i , can be decomposed into a smooth part $\tilde{g}(E)$ and an oscillating part $\delta g(E)$:

$$g(E) = \sum_i \delta(E - E_i) = \tilde{g}(E) + \delta g(E). \quad (1)$$

The smooth part contains by definition the average level density which usually is a monotonously increasing function of E and can be obtained in the extended Thomas-Fermi (ETF) model (see, e.g., [5], Ch. 4). The oscillating part can be expressed by the *semiclassical trace formula*

$$\delta g_{sc}(E) \simeq \sum_{po} \mathcal{A}_{po}(E) \cos [S_{po}(E)/\hbar - \sigma_{po}\pi/2]. \quad (2)$$

The sum is over all periodic orbits (po) of the classical system, $S_{po}(E) = \oint \mathbf{p} \cdot d\mathbf{q}$ are their action integrals, the amplitudes $\mathcal{A}_{po}(E)$ depend on their stabilities and degeneracies, and σ_{po} are called the Maslov indices. The sum in (2) is an asymptotic one, correct to leading order in $1/\hbar$, and in non-integrable systems it is hampered by convergence problems [6]. For isolated orbits, Gutzwiller expressed [4] the amplitudes $\mathcal{A}_{po}(E)$ in terms of their periods and stability matrices.

The trace formula (2) was later generalized to billiard systems [7] and to systems with continuous symmetries [8, 9, 10], including integrable systems. A relativistic trace formula for spin 1/2 particles was derived in [11], and a nonrelativistic trace formula for particles with arbitrary spin s in [12]. In all cases, the trace formula has the same general form (2), but the amplitudes $\mathcal{A}_{po}(E)$ take different forms. For isolated orbits, their \hbar dependence is given by a factor \hbar^{-1} , while for orbits appearing in f -fold degenerate families, the amplitudes go like $\hbar^{-(1+f/2)}$.

In integrable and mixed-dynamical systems, periodic orbits can change their stability under the variation of a control parameter (e.g., the energy E , a potential parameter, or an additional external field) and thereby undergo bifurcations. In such situations, the amplitudes \mathcal{A}_{po} diverge at the bifurcation points. The same happens also in limits where continuous symmetries are broken (or restored), since hereby the \hbar dependence of the \mathcal{A}_{po} changes

discontinuously. The remedy to remove these (unphysical!) divergences is to go beyond the stationary-phase approximation for the integration(s) used in the derivation of the semiclassical trace formula. This has, besides [7, 8, 9], been developed most systematically in [13] for symmetry breaking and bifurcations, and in [14] for symmetry breaking in weakly perturbed integrable systems, leading in all cases to local uniform approximations with finite amplitudes \mathcal{A}_{po} . Global uniform approximations which yield finite amplitudes at symmetry-breaking and bifurcation points, and far from them go over into the standard (extended) Gutzwiller trace formula, were developed for the breaking of U(1) symmetry in [15], for some cases of U(2) and SO(3) symmetry breaking in [16], for the symmetry breaking U(3) \rightarrow SO(3) in [17], and for various types of bifurcations in [18]. (Details and further references may be found in [5], Ch. 6.3.)

For interacting finite fermion systems described in the mean-field approximation (i.e., in Hartree-Fock or density functional theory), one can also obtain semiclassical trace formulae for the oscillating parts of the total binding energy E_b and the particle number N . Hereby one writes, similarly to (1), $E_b = \tilde{E}_b + \delta E$ and $N = \tilde{N} + \delta N$. The average quantities \tilde{E}_b and \tilde{N} are taken from the ETF model, and for the oscillating parts one finds [5, 8]

$$\begin{aligned} \delta E_{sc} &\simeq \sum_{po} \mathcal{A}_{po}(\lambda) \left(\frac{\hbar}{T_{po}}\right)^2 \cos\left[\frac{S_{po}(\lambda)}{\hbar} - \sigma_{po} \frac{\pi}{2}\right], \\ \delta N_{sc} &\simeq -\sum_{po} \mathcal{A}_{po}(\lambda) \left(\frac{\hbar}{T_{po}}\right) \sin\left[\frac{S_{po}(\lambda)}{\hbar} - \sigma_{po} \frac{\pi}{2}\right], \end{aligned} \quad (3)$$

both to be evaluated at the Fermi energy $\lambda(N)$ for a given number of particles N . The periodic orbits are ideally those of the classical counterpart of the self-consistent mean field, which for practical purposes often is taken as a shell-model type potential.

B. Coarse-graining and finite temperatures

Our present emphasis in the use of POT is not the full quantization of the spectra of finite fermion systems, but on the semiclassical description of their *gross-shell structure*. For this purpose we *coarse-grain* the quantum spectrum by a convolution of the level density (1) with a normalized Gaussian of width γ :

$$g_{qm}(E, \gamma) = \frac{1}{\gamma\sqrt{\pi}} \sum_i e^{-(E-E_i)^2/\gamma^2}. \quad (4)$$

The coarse graining of the trace formula (2) gives, using the stationary-phase approximation for the convolution integral, an extra exponential factor in the trace formula:

$$\begin{aligned} \delta g_{sc}(E, \gamma) &\simeq \sum_{po} \mathcal{A}_{po}(E) e^{-(\gamma T_{po}/2\hbar)^2} \times \\ &\cos\left[\frac{S_{po}(\lambda)}{\hbar} - \sigma_{po} \frac{\pi}{2}\right]. \end{aligned} \quad (5)$$

The same exponential factor appears in the trace formulae (3) for $\delta E_{sc}(\lambda, \gamma)$ and $\delta N_{sc}(\lambda, \gamma)$. It suppresses the contributions from orbits with larger periods T_{po} . A similar suppression of longer orbits and the overall amplitude of the shell effects occurs at *finite temperatures*. E.g. in a grand-canonical system at temperature T , the oscillating part of the free Helmholtz energy has the trace formula

$$\begin{aligned} \delta F_{sc}(\lambda, T) &\simeq \sum_{po} \mathcal{A}_{po}(\lambda) \left(\frac{\hbar}{T_{po}}\right)^2 \frac{\tau_{po}}{\text{Sinh}(\tau_{po})} \times \\ &\cos\left[\frac{S_{po}(\lambda)}{\hbar} - \sigma_{po} \frac{\pi}{2}\right]. \end{aligned} \quad (6)$$

Here $\tau_{po} = k_B T \pi T_{po}(\lambda)/\hbar$ and k_B is the Boltzmann constant. In both situations the *gross-shell effects* are dominated by the *shortest periodic orbits* of the system [8].

In mixed-dynamical and integrable systems, orbits with different degrees f of degeneracy can coexist. The gross-shell structure then results from a competition between the periods and the degeneracies of the shortest orbits [19]. Whereas their coarse-grained semiclassical amplitudes decrease with growing period T_{po} , they increase with growing degeneracy f due to their dependence $\propto \hbar^{-(1+f/2)}$ already mentioned above.

In arbitrary *spherical three-dimensional systems*, the most degenerate orbits undergo both radial and angular oscillations with rational frequency ratios (cf. [9, 17, 20]):

$$\omega_r : \omega_\phi = n : m, \quad |n|, |m| \in \mathbb{N}. \quad (7)$$

(For physical reasons, only pairs of integers n, m with equal signs are allowed; orbits with negative n, m correspond to the time-reversed of the orbits with positive n, m). The orientations of these 'rational tori' can be rotated about three Euler angles without changing their shapes, periods or actions; therefore they appear in three-fold degenerate families ($f = 3$). Their existence for arbitrary ratios $n:m$ depends, however, on the form of the radial potential $V(r)$ (cf. [17, 21]) and, in general, on the energy E . The special orbits with angular momentum $L = 0$ and $L = L_{max}$, corresponding to librating 'diameter' and rotating 'circle' orbits, respectively, form families with only two-fold degeneracy ($f = 2$), since one of the three Euler rotations does not change their orientations. The *Coulomb potential* $V(r) = -\alpha/r$ has an extra dynamical symmetry, leading to O(4); here all orbits have $n:m = 1:1$. *Spherical harmonic oscillators* $V(r) = ar^2$ also have an extra dynamical symmetry, leading in three dimensions to U(3); here all orbits have $n:m = 2:1$. In each of these two special potentials, *all* periodic orbits (including the diameters and circles) form one family with degeneracy $f = 4$. Quantum-mechanically, the dynamical symmetries reflect themselves in an accidental extra degeneracy of the eigenvalue spectrum $\{E_i\}$. The semiclassical trace formulae for $\delta g(E)$ of these systems, added to their ETF expressions for $\tilde{g}(E)$, reproduce the *exact* quantum-mechanical level densities according to (1) [see [5], Eqs. (3.144) and (3.69) for their explicit analytical expressions].

III. POT FOR FINITE FERMION SYSTEMS

A. Nuclei

1. *Ground-state deformations.* Strutinsky *et al.* [8, 22] were the first to extend the POT to systems with continuous symmetries and to apply it to the study of gross-shell effects in nuclei. In [22] they studied the periodic orbits in a spheroidal cavity with axis ratio η as a model for the mean field of a deformed nucleus. They plotted the shell-correction energy δE , obtained quantum-mechanically using Strutinsky's shell-correction method [23] with the spectra of realistic deformed Woods-Saxon potentials [24] with the same spheroidal deformations (including spin-orbit interaction), versus particle number N and deformation η . The slopes of the valleys in these deformation energy surfaces $\delta E(N, \eta)$ corresponding to the ground-state deformations could then be correctly reproduced by the condition that the actions S_{po} of the shortest and most degenerate periodic orbits in the spheroidal cavity (with $f = 2$) [25] be constant. Contributions of orbits with $f = 1$ were negligible. (The Fermi energies corresponding to the spherical magic numbers had to be adjusted, as no spin-orbit interaction was included in the cavity model.) Although this was only a qualitative result, it proved the correctness of the concept to interpret quantum-mechanical gross-shell effects semiclassically in terms of short periodic orbits of the corresponding classical system. The use of a cavity with infinitely steep walls for the mean field hereby justifies itself through the short range and the saturating property of the effective nucleon-nucleon interaction, leading to steep walls of the self-consistent Hartree-Fock potentials or their approximations by Woods-Saxon type shell-model potentials.

2. *Left-right asymmetry of fission barriers.* A prominent manifestation of shell effects is the 'double-humped' fission barrier of nuclei in the actinide region [26]. One particular aspect is that of the onset of a left-right asymmetry of the fissioning nuclear shapes which eventually leads to the asymmetric mass distributions of the fission fragments. Since Sven Gösta Nilsson and his group, and other scientists in Lund, were much involved in the study of this shell effect, I may dwell a little on its history. The mixing of pairs of single-particle states with opposite parities in a spheroidal harmonic-oscillator potential was studied earlier [27] as a possible mechanism leading to 'pear-shaped' nuclei. In 1962, S. A. E. Johansson [28] took this question up and investigated the possibility of octupole-deformed fission barriers. Since Strutinsky's shell-correction method [23] did not yet exist at that time, no realistic fission barriers could be obtained quantum-mechanically with the Nilsson model. Johansson showed, however, that at the typical deformations of the actinide fission barriers predicted by the liquid-drop model (LDM) [29], the mixing of single-particle orbits of the type used in [27] leads to an instability of the barrier against octupole shapes. Using the shell-correction method with the Nilsson model, P. Möller and S. G. Nils-

son [30] obtained in 1970 the instability of the outer fission barrier against a suitable mixture of ϵ_3 and ϵ_5 deformations. Thus, the onset of the fission mass asymmetry was clearly a quantum-mechanical shell effect that could not be explained by the classical LDM model. In a detailed microscopical study, C. Gustafsson, P. Möller, and S. G. Nilsson [31] showed a year later that those pairs of single-particle states, which are most sensitive to the left-right asymmetric shapes and hence responsible for their onset, have their wavefunction nodes and extrema on parallel planes at and near the waist-line of the fissioning nucleus perpendicular to its symmetry axis.

30 years later, in a Lund-Regensburg-Dresden collaboration [32], this effect was studied semiclassically. The POT had been used in the same collaboration [33] for cavities with the (c, h, α) shapes of [24], for which the mass asymmetry of the outer fission barrier of actinide nuclei had also been obtained quantum-mechanically in [34]. The shortest periodic orbits here are families with $f = 1$, having the axial U(1) symmetry; they are simply the diagonal, triangular and square-shaped orbits in the circular planes perpendicular to the nuclear symmetry axis. The semiclassical trace formula (with a uniform approximation for the bifurcation occurring at the onset of the neck, where the orbits in the central plane become unstable and give birth to two new parallel planes with stable orbits) was shown [33] to yield realistic deformation energy surfaces $\delta E_{sc}(c, h, \alpha)$ in the region of the outer fission barrier, predicting its instability against the asymmetry parameter α in good agreement with the old quantum-mechanical results [34]. (Again, the spin-orbit interaction was omitted; the Fermi energy as the only parameter was adjusted to yield the fission isomer minimum at the correct deformation.) Similarly to [22], the valley of steepest descent through the deformation energy surface, leading over a left-right asymmetric outer saddle, is obtained by the stationary condition $\delta S_{po} = 0$ for the shortest orbits. In [32], it was shown that these orbits are situated in a very small regular island of a dominantly chaotic phase space. An approximate Einstein-Brillouin-Keller (EBK) quantization of the linearized classical motion in these regular islands reproduced rather precisely the quantum-mechanical energies of those diabatic single-particle states with opposite parity which are most sensitive to the α deformations and hence quantum-mechanically responsible for the α instability of the outer barrier. Furthermore, their wavefunctions were found to have their nodes and extrema precisely in the planes near the nuclear waist-line that contain the shortest periodic orbits responsible semiclassically for the asymmetry effect.

This application of the POT represents an interesting example for the classical-to-quantum correspondence of the interplay between chaos and order: a tiny regular island in an almost chaotic phase space causes a quantum shell effect in an interacting many-body system with observable consequences in the form of the mass asymmetry of the fission fragments.

B. Metal clusters

Metal clusters are interesting finite fermion systems which allow one to study the transition from atoms over molecules towards condensed matter [35]. In the simplest theoretical description, the so-called 'jellium model', the ions are replaced by a structureless but deformable positive background and the systems of N interacting valence electrons in the external jellium potential are studied [36]. In neutral clusters, the electrostatic long-range forces cancel and the valence electrons are only bound by the short-ranged exchange and correlation effects. Neutral metal clusters therefore have much in common with nuclei. One difference to nuclei is that there is no measurable spin-orbit interaction in most metal clusters. The magic numbers N_i of the smallest spherically stable clusters correspond to those of a harmonic oscillator ($N_i = 2, 8, 20, 40$) [35]. For the analysis of early experimental abundance spectra of small sodium clusters, the Nilsson model without spin-orbit term was successfully employed [37] to interpret the regions between the spherical shell closures in terms of prolate and oblate deformations. In a self-consistent mean-field description [38, 39], the average potential of clusters with $N \gtrsim 80$ valence electrons has steep walls like heavy nuclei, and therefore cavity models provide again a good approximation.

One early result of POT was the observation by Balian and Bloch [7] that the coarse-grained level density of a *spherical cavity* exhibits a pronounced *beating pattern*: a rapid regular oscillation, reflecting the shell structure of the spectrum, modulated by a slow oscillation reaching over some 13 - 14 shells. From the trace formula derived in [7] one sees that the beat comes about by the interference of the shortest periodic orbits of highest degeneracy ($f = 3$), which here are the triangle ($n : m = 3 : 1$) and square ($4 : 1$) orbits. (The diameter orbit with degeneracy $f = 2$ can be neglected.) The rapid shell oscillations are determined by the average length of these two orbits, while the period of their amplitude modulation, the so-called 'super-shell' oscillation, is given by the difference of their lengths.

The numbers of fermions needed to reach the first super-shell node is of the order of $\sim 800 - 1000$. Super-shells can therefore not be seen in nuclei. Neutral metal clusters, however, can be made arbitrarily large. This inspired Nishioka *et al.* [40] to study the super-shell structure in Woods-Saxon potentials fitted to self-consistent mean fields [38, 42], and to predict that it should be observable in metal clusters. Indeed, the super-shells were experimentally observed, for the first time in supersonic beams of hot sodium clusters [43]. Their abundance in an adiabatically expanding beam is dominated by their stability against evaporation of single atoms and exhibits pronounced peaks at the spherically magic numbers. The larger ones, $N_i = 92, 138, 192, 264, \dots$, are almost exactly those of a spherical cavity. By a suitable extraction of the oscillating part of the abundance spectra, the super-shell beat could clearly be exhibited [43], with its first

node appearing around $N \sim 900$.

Plotting the cube roots of the magic numbers, $N_i^{1/3}$ which are proportional to the r.m.s. radii of the magic clusters, against the shell number i , one obtains a straight line with a slope $s = N_{i+1}^{1/3} - N_i^{1/3}$. The experimental value of this slope is $s_{exp} = 0.61 \pm 0.01$; it was confirmed by later experiments, also with other types of metal clusters (their different Wigner-Seitz radii do not affect the value of s , as long as they are single-valenced) [35, 36]. Around $i \simeq 14$, there is a slight discontinuity in the plot $s(i)$ before it continues again with the same slope. This is due to the phase change of the rapid shell oscillations when passing through the first super-shell node. The value of this slope predicted by the POT of [7, 40] is $s_{pot} = 0.603$, that of the self-consistent quantum-mechanical mean-field calculations is $s_{mf} = 0.61$, both in perfect quantitative agreement with experiment.

This provides another example of the good agreement of POT with both experiment and quantum mechanics. An easily readable account of the super-shells in metal clusters is given in [44].

Unfortunately, it is very difficult to extract spectroscopic data on the electronic single-particle states in metallic clusters. The most direct access to their shapes is given by the so-called 'Mie plasmons', the collective dipole oscillations of the valence electrons against the ions [41, 42] (which are the origin of the colors in stained-glass windows [45]). Similarly to the giant-dipole resonances in nuclei, their splitting gives evidence of the average deformations of the clusters. This gave rise to a series of theoretical investigations of cluster deformations using deformed shell models (see [36]). For applications of the POT one employs hereby most comfortably the spheroidal cavity model used also in [8]. I refer to [46] for a detailed account containing also a nice application of the POT to the semiclassical interpretation of moments of inertia. The effects of weak magnetic fields on spherical metal clusters were studied in [47]. In [48], spheroidal cavities with the lowest multipole deformations ϵ_2 , ϵ_3 , and ϵ_4 were studied, and the perturbative trace formula of [14] (cf. also [49]) was used to predict their ground-state deformations. The results were in very good agreement with those of quantum-mechanical shell-correction calculations using the spectra of the same cavities.

C. Semiconductor nanostructures

Semiconductor heterostructures can be used to construct two-dimensional systems of quasi-free electrons on the nanometer scale. With the help of external metallic gates or lithography, the electrons can further be laterally confined to form so-called quantum dots, quantum channels, quantum wires, antidot superlattices, etc. [50, 51]. Applying a perpendicular magnetic field B , one can measure the magneto-resistance of such devices. Under suitable experimental circumstances, both the mean free path and the phase coherence length can be made larger than the sizes of these structures, so that quan-

tum interference still takes place while the dimensions are large enough to allow for a semiclassical description. Nanostructures are therefore ideal tools to study the interplay between classical and quantum mechanics.

Weiss *et al.* [52] measured the resistance of antidot superlattices and found oscillations which can be explained classically by the commensurability of cyclotron orbits with the superdot lattice: when an electron is trapped in a cyclotron orbit that fits around 1, 2, 4, 9, etc. antidots, it does not contribute to the conductance and hence a peak is seen in the magneto-resistance (see [53] for an easily readable account). In weak B fields at very low temperatures, some rapid B -periodic oscillations could be observed. They could be interpreted semiclassically by the interference of different trapped periodic orbits of comparable lengths; the linear response of the system to the B field was hereby described by a semiclassical version of the Kubo theory yielding a trace formula for the conductance [54]. These so-called 'Aharonov-Bohm (AB) oscillations' can be easily understood in a perturbative approach [14] in which the effect of the magnetic field is taken into account to lowest order only in the actions $S_{po} = \oint \mathbf{p} \cdot d\mathbf{q}$, while the shapes of the orbits are left unchanged. Under the canonical substitution $\mathbf{p} \rightarrow \mathbf{p} - (e/c)\mathbf{A}$, where \mathbf{A} is the vector potential with $\mathbf{B} = \nabla \times \mathbf{A}$, the action of a periodic orbit changes like

$$S_{po} \rightarrow S_{po} - (e/c) \oint \mathbf{A} \cdot d\mathbf{q} = S_{po} - (e/c)\Phi_{po}, \quad (8)$$

where $\Phi_{po} = \oint \mathbf{B} \cdot d\mathbf{F}_{po}$ is the magnetic flux through the area F_{po} surrounded by the orbit. Consequently, the perturbed semiclassical trace formula is modified only by a factor $\cos(e\Phi_{po}/\hbar c)$ containing the Aharonov-Bohm phase which causes the B -periodic oscillations.

Similar AB oscillations have also been measured in the magneto-conductance of a circular quantum dot containing some ~ 1200 to 2000 electrons [55]. They could be qualitatively well explained [56] by the perturbed level density $\delta g(E, B)$ of a two-dimensional circular billiard. This is an integrable system whose trace formula is analytically known [57]. The conductance oscillations as functions of the radius of the quantum dot (regulated experimentally by the applied gate voltage) were well reproduced by the average length of the shortest orbits

(here: diameters and triangles), while the period of the AB oscillations according to (8) was well reproduced by the area enclosed by the triangular orbit [56].

An analytical trace formula for the circle billiard in arbitrarily strong transverse magnetic fields B was given in [58], and the magnetization of quantum dots was studied semiclassically in [59].

In a mesoscopic semiconductor channel with two antidots, the magneto-conductance was measured in [60] and also found to exhibit AB oscillations. These could be well explained [61] using the semiclassical Kubo formula [54] with a suitably modeled two-dimensional confinement potential for the channel (with antidots), which represents a mixed-dynamical system with a rather chaotic phase space. Plotting the maxima of the experimental AB oscillations versus magnetic field B and gate voltage (which regulates the radii of the antidots), one obtains a grid of lines exhibiting some characteristic displacements [60]. At first sight, these might be attributed to missing flux units. Quantum-mechanical calculations [62] reproduced these displacements but could not explain the physics behind them. In the semiclassical calculations the displacements could, in fact, be attributed to bifurcations of some of the trapped periodic orbits [61, 63].

D. Trapped dilute atomic gases

I mention only briefly the finite fermion systems produced by confining diluted fermionic atom gases, e.g. in magneto-optic traps [64]. Hartree-Fock calculations for N harmonically trapped atoms with a short-ranged repulsive interaction [65] yielded shell effects $\delta E(N)$ in their total binding energies $E_b(N)$ which remind about the super-shells discussed above. More details are given in [66]; let me just emphasize here that the origin of the beating shell structure is different here from that in a spherical cavity [7]. The self-consistent mean field can be modeled by a perturbed harmonic oscillator $V(r) = ar^2 + \epsilon r^4$. For such potentials it was shown recently [17] that the gross-shell structure is dominated by the two-fold degenerate diameter and circle orbits, whose interference explains the super-shells found in [65]. The shortest three-fold degenerate orbits have frequency ratios $n : m \geq 7 : 3$ and contribute only to finer details of the quantum spectrum at relatively high energies.

[1] S. G. Nilsson, Mat.-Fys. Medd. Dan. Vid. Selsk. **29**, no. 16 (1955).
 [2] K. Alder, A. Bohr, T. Huus, B. Mottelson, and A. Winther, Rev. Mod. Phys. **28**, 432 (1956).
 [3] M. Brack: *Das Nilsson-Modell in der asymptotischen Darstellung* (diploma thesis, Basel University, 1968).
 [4] M. C. Gutzwiller, J. Math. Phys. **12**, 343 (1971); and earlier references quoted therein
 [5] M. Brack and R. K. Bhaduri: *Semiclassical Physics* (revised edition, Westview Press, Boulder, USA, 2003).
 [6] M. C. Gutzwiller: *Chaos in Classical and Quantum Me-*

chanics (Springer Verlag, New York, 1990);
 H.-J. Stöckmann: *Quantum Chaos: an Introduction* (Cambridge University Press, Cambridge, UK, 1999);
 F. Haake: *Quantum Signatures of Chaos* (Springer, 2nd edition, 2001).
 [7] R. Balian and C. Bloch, Ann. Phys. (N.Y.) **69**, 76 (1972).
 [8] V. M. Strutinsky, Nukleonika (Poland) **20**, 679 (1975);
 V. M. Strutinsky and A. G. Magner, Sov. J. Part. Nucl. **7**, 138 (1976).
 [9] M. V. Berry and M. Tabor, Proc. R. Soc. Lond. A **349**, 101 (1976); J. Phys. A **10**, 371 (1977).

- [10] S. C. Creagh and R. G. Littlejohn, Phys. Rev. A **44**, 836 (1991); J. Phys. A **25**, 1643 (1992); S. C. Creagh, J. Phys. A **26**, 95 (1993).
- [11] J. Bolte and S. Keppeler, Phys. Rev. Lett. **81** 1987 (1998); Ann. Phys. (N.Y.) **274** 125 (1999).
- [12] M. Pletyukhov, Ch. Amann, M. Mehta, and M. Brack, Phys. Rev. Lett. **89**, 116601 (2002); M. Pletyukhov and O. Zeitsev, J. Phys. A **36**, 5181 (2003).
- [13] A. M. Ozorio de Almeida and J. H. Hannay, J. Phys. A **20**, 5873 (1987).
- [14] S. C. Creagh, Ann. Phys. (N. Y.) **248**, 60 (1996).
- [15] S. Tomsovic, M. Grinberg, and D. Ullmo, Phys. Rev. Lett. **75**, 4346 (1995).
- [16] M. Brack, P. Meier, and K. Tanaka, J. Phys. A **32**, 331 (1999).
- [17] M. Brack, M. Ögren, Y. Yu, and S. M. Reimann, preprint arXiv:cond-mat/060609 (2005).
- [18] M. Sieber, J. Phys. A **29**, 4715 (1996); H. Schomerus and M. Sieber, J. Phys. A **30**, 4537 (1997); M. Sieber and H. Schomerus, J. Phys. A **31**, 165 (1998).
- [19] The situation is different when one wants to obtain full quantization in integrable systems by summing over *all* periodic orbits. Then, the most degenerate families of orbits can be shown to dominate over those of lower degeneracy; see A. M. Ozorio de Almeida, C. H. Lewenkopf, and S. Tomsovic, J. Phys. A **35**, 10629 (2002).
- [20] A. Bohr and B. R. Mottelson: *Nuclear Structure* Vol. I (World Scientific, New York, 1975).
- [21] K. Arita, Int. J. Mod. Phys. E **13**, 191 (2004), and these proceedings.
- [22] V. M. Strutinsky, A. G. Magner, S. R. Ofengenden, and T. Døssing, Z. Phys. A **283**, 269 (1977).
- [23] V. M. Strutinsky, Sov. J. Nucl. Phys. **3**, 449 (1966) [Yad. Fiz. **3**, 614 (1966)]; Ark. Fys. **36**, 629 (1966); Nucl. Phys. A **95**, 420 (1967); Nucl. Phys. A **122**, 1 (1968).
- [24] M. Brack, J. Damgård, A. S. Jensen, H. C. Pauli, V. M. Strutinsky, and C. Y. Wong, Rev. Mod. Phys. **44**, 320 (1972).
- [25] These orbits are the families of polygonal orbits with $n \geq 3$ reflections per period. In [22], only the rhombus-type orbits ($n=4$) were considered, but the deformation dependence of their actions is the same for all n . For more recent applications of the POT to spheroidal cavities, see A. G. Magner, S. N. Fedotkin, F. A. Ivanyuk, P. Meier, M. Brack, S. M. Reimann, and H. Koizumi, Ann. Phys. (Leipzig) **6**, 555 (1997), and [49].
- [26] W. J. Swiatecki and S. Bjørnholm, Phys. Rep. **4C** No. 6 (1972); S. Bjørnholm and J. E. Lynn, Rev. Mod. Phys. **52**, 725 (1980).
- [27] K. Lee and D. R. English, Phys. Rev. **108**, 774 (1957).
- [28] S. A. E. Johansson, Nucl. Phys. **22**, 529 (1962).
- [29] N. Bohr and J. A. Wheeler, Phys. Rev. **56**, 426 (1939); see also L. Willets: *Theories of nuclear fission* (Clarendon Press, Oxford, 1964).
- [30] P. Möller and S. G. Nilsson, Phys. Lett. **31 B**, 283 (1970).
- [31] C. Gustafsson, P. Möller, and S. G. Nilsson, Phys. Lett. **34 B**, 349 (1971).
- [32] M. Brack, M. Sieber, and S. M. Reimann, in: *Nobel Symposium on Quantum Chaos*, eds. K.-F. Berggren and S. Åberg; Physica Scripta **T90**, 146 (2001).
- [33] M. Brack, S. M. Reimann, and M. Sieber, Phys. Rev. Lett. **79**, 1817 (1997); M. Brack, P. Meier, S. M. Reimann, and M. Sieber, in: *Similarities and Differences between Atomic Nuclei and Clusters*, eds. Y. Abe *et al.* (Am. Inst. of Phys., 1998) p. 17.
- [34] H. C. Pauli, T. Ledergerber, and M. Brack, Phys. Lett. **34 B**, 264 (1971).
- [35] W. A. de Heer, Rev. Mod. Phys. **65**, 611 (1993).
- [36] M. Brack, Rev. Mod. Phys. **65**, 677 (1993).
- [37] K. Clemenger, Phys. Rev. B **32**, 1359 (1985).
- [38] W. Ekardt, Phys. Rev. B **29**, 1558 (1984).
- [39] O. Genzken and M. Brack, Phys. Rev. Lett. **67**, 3286 (1991); M. Brack, O. Genzken, and K. Hansen, Z. Phys. D **21**, 65 (1991).
- [40] H. Nishioka, K. Hansen, and B. R. Mottelson, Phys. Rev. B **42**, 9377 (1990).
- [41] W. Ekardt, Phys. Rev. B **31**, 6360 (1985).
- [42] M. Brack, Phys. Rev. B **39**, 3533 (1989).
- [43] J. Pedersen, S. Bjørnholm, J. Borggren, K. Hansen, T. P. Martin, and H. D. Rasmussen, Nature **353**, 733 (1991).
- [44] M. Brack, The Scientific American, Dec. 1997, p. 50.
- [45] U. Kreibitz and M. Vollmer: *Optical Properties of Metal Clusters* (Springer-Verlag, Berlin, 1995).
- [46] S. Frauendorf, these proceedings.
- [47] K. Tanaka, S. C. Creagh, and M. Brack, Phys. Rev. B **53**, 16050 (1996).
- [48] V. V. Pashkevich, P. Meier, M. Brack, and A. V. Unzhakova, Phys. Lett. A **294**, (2002).
- [49] P. Meier, M. Brack, and S. C. Creagh, Z. Phys. D **41**, 281 (1997).
- [50] T. Heinzel: *Mesoscopic Electronics in Solid State Nanostructures* (Wiley-VCH, Weinheim, 2003).
- [51] S. M. Reimann and M. Manninen, Rev. Mod. Phys. **74**, 1283 (2002).
- [52] D. Weiss, K. Richter, A. Menschig, R. Bergmann, H. Schweizer, K. von Klitzing, and G. Weimann, Phys. Rev. Lett. **70**, 4118 (1993); and earlier refs. quoted therein.
- [53] D. Weiss and K. Richter, Physica **D 83**, 290 (1995).
- [54] K. Richter, Europhys. Lett. **29**, 7 (1995); G. Hackenbroich and F. von Oppen, Europhys. Lett. **29**, 151 (1995).
- [55] P. E. Lindelof, P. Hullmann, P. Bøggild, M. Persson, and S. M. Reimann, in: *Large Clusters of Atoms and Molecules*, ed. T. P. Martin (Kluwer, Dordrecht, 1996), p. 89; and earlier references quoted therein.
- [56] S. M. Reimann, M. Persson, P. E. Lindelof, and M. Brack, Z. Phys. **B 101**, 377 (1996).
- [57] S. M. Reimann, M. Brack, A. G. Magner, J. Blaschke, and M. V. N. Murthy, Phys. Rev. A **53**, 39 (1996).
- [58] J. Blaschke and M. Brack, Phys. Rev. A **56**, 182 (1997).
- [59] K. Tanaka, Ann. Phys. (N.Y.) **268**, 31 (1998).
- [60] C. Gould, A. S. Sachrajda, Y. Feng, A. Delage, P. J. Kelly, K. Leung, and P. T. Coleridge, Phys. Rev. B **51**, 11213 (1995).
- [61] J. Blaschke, Ph.D. thesis (Regensburg University, 1999) available at <http://www.joachim-blaschke.de>
- [62] G. Kirczenov, B. L. Johnson, P. J. Kelly, C. Gould, A. S. Sachrajda, Y. Feng, and A. Delage, Phys. Rev. B **56**, 7503 (1997).
- [63] J. Blaschke and M. Brack, Europhys. Lett. **50**, 294 (2000).
- [64] see, e.g., R. Grimm, M. Weidermüller and Y. B. Ovchinnikov, in: *Advances in Atomic, Molecular and Optical Physics* (Academic Press, 2000), p. 95.
- [65] Y. Yu, M. Ögren, S. Åberg, S. M. Reimann, and M. Brack, preprint arXiv:cond-mat/0502096 (2005).
- [66] M. Ögren *et al.*, these proceedings.