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Abstract

The Cauchy problem is considered for the scalar wave equation in the Schwarzschild geometry. We derive an integral spectral representation for the solution and prove pointwise decay in time.

1 Introduction

Recently pointwise decay was proven for solutions of the scalar wave equation in the Kerr geometry [1, 2]. The main difficulties in this proof are due to the fact that the metric is only axisymmetric. In particular, the classical energy density may be negative inside the *ergosphere*, a region outside the event horizon in which the Killing vector corresponding to time translations becomes spacelike. This makes it necessary to apply special methods (spectral theory in Pontrjagin spaces, energy splitting estimates, causality arguments) which are technically demanding and not easily accessible. Therefore, it seems worthwile working out the special case of spherical symmetry (Schwarzschild geometry) separately. This is precisely the purpose of the present paper, where we derive an integral representation for the solution of the Cauchy problem and prove pointwise decay for the scalar wave equation in the Schwarzschild geometry. In this case, the classical energy density is positive everywhere outside the event horizon. This gives rise to a positive definite scalar product, making it possible to apply Hilbert space methods.

Recall that in Schwarzschild coordinates $(t, r, \vartheta, \varphi)$, the Schwarzschild metric takes the form

$$\begin{aligned} ds^2 &= g_{ij}\, dx^i dx^j \\ &= \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) \end{aligned} \tag{1.1}$$

with $r > 0$, $0 \le \vartheta \le \pi$, $0 \le \varphi < 2\pi$. We often use for the angular variables the short notation $x \in S^2$. Obviously, the metric has two singularities at $r = 0$ and $r = 2M$. The latter is called the *event horizon* and can be resolved by a simple coordinate transformation. In the following we consider only the region $r >$

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$2M$ outside the event horizon. The scalar wave equation in the Schwarzschild geometry is given by

$$\Box\phi := g^{ij}\nabla_i\nabla_j\phi = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}\, g^{ij}\frac{\partial}{\partial x^j}\right)\phi = 0 \tag{1.2}$$

where g denotes the determinant of the metric g_{ij}. We now state our main result.

Theorem 1.1. *Consider the Cauchy problem of the scalar wave equation in the Schwarzschild geometry*

$$\Box\phi = 0\ , \quad (\phi, i\partial_t\phi)(0,r,x) = \Phi_0(r,x)$$

for smooth initial data $\Phi_0 \in C_0^\infty((2M,\infty)\times S^2)^2$ which is compactly supported outside the event horizon. Then there exists a unique global solution $\Phi(t) = (\phi(t), i\partial_t\phi(t)) \in C^\infty(\mathbb{R}\times(2M,\infty)\times S^2)^2$ which is compactly supported for all times t. Moreover, for fixed (r,x) this solution decays as $t\to\infty$.

There has been considerable work on the wave equation in the Schwarzschild geometry. In 1957, Regge and Wheeler [3] investigated the linear stability of this geometry. Kay and Wald [4] proved boundedness for solutions of the Klein-Gordon equation in this space-time outside and on the event horizon. By heuristic arguments, Price [5] got evidence for polynomial decay of solutions of the scalar wave equation. More recently, Dafermos and Rodnianski [6] gave a mathematical proof for this decay for spherical symmetric initial data. For general initial data they derived decay rates [7], which are however not sharp. Furthermore, Morawetz and Strichartz-type estimates for a massless scalar field without charge in a Reissner Nordstrøm background with naked singularity are developed in [8]. And in [9] a Morawetz-type inequality was proven for the semi-linear wave equation in Schwarzschild.

The paper is organized as follows: First, we introduce the Regge-Wheeler variable and rewrite the wave equation as a first-order Hamiltonian system. The resulting Hamiltonian is a symmetric operator with respect to the scalar product arising from the conserved energy. Exploiting the spherical symmetry of the problem, we may consider the problem for fixed angular modes l and m. We then show that the corresponding Hamiltonian is essentially self-adjoint. More precisely, our goal is to apply Stone's formula, which relates the propagator to an integral over the resolvent. Thus in Section 4 we give an explicit construction for the resolvent. This construction is based on special solutions of the radial equation, which decay exponentially at $\pm\infty$. In Section 5 we prove the existence of these solutions via the formalism of the Jost equation. Moreover, we obtain appropriate regularity results for these solutions, which lead to an integral representation of the solution operators of the Cauchy problem for fixed l and m. According to the theory of symmetric hyperbolic systems, the Cauchy problem has a unique smooth solution. Thus, summing over the angular modes yields the desired representation of this solution. Combining this representation with a Sobolev imbedding argument, we obtain pointwise decay in time.

2 Preliminaries

In this section we reformulate the wave equation as a first order Hamiltonian system. This will make it possible to analyze the dynamics of the waves with

Hilbert space methods.

According to (1.1) and (1.2) the scalar wave equation in the Schwarzschild geometry with respect to Schwarzschild coordinates has the explicit form

$$\left[\frac{\partial^2}{\partial t^2} - \left(1 - \frac{2M}{r}\right) \frac{1}{r^2} \left(\frac{\partial}{\partial r}(r^2 - 2Mr)\frac{\partial}{\partial r} + \Delta_{S^2}\right)\right] \phi = 0 \,. \tag{2.1}$$

Here Δ_{S^2} denotes the standard Laplacian on the two sphere, which in the coordinates (ϑ, φ) is given by

$$\Delta_{S^2} = \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial}{\partial(\cos\vartheta)}\sin^2\vartheta\frac{\partial}{\partial(\cos\vartheta)} \,. \tag{2.2}$$

In order to bring the equation (2.1) into a more convenient form, we first introduce the Regge-Wheeler coordinate u by

$$u(r) := r + 2M \log\left(\frac{r}{2M} - 1\right) \,. \tag{2.3}$$

The variable u takes values in the whole interval $(-\infty, \infty)$ as r ranges over $(2M, \infty)$. It satisfies the relations

$$\frac{du}{dr} = \frac{1}{1 - \frac{2M}{r}} \,, \qquad \frac{\partial}{\partial u} = \left(1 - \frac{2M}{r}\right)\frac{\partial}{\partial r} \,. \tag{2.4}$$

In what follows the variable r is always implicitly given by u. Using the Regge-Wheeler coordinate, the wave equation (2.1) transforms to

$$\left[\frac{\partial^2}{\partial t^2} - \frac{1}{r}\frac{\partial^2}{\partial u^2} r + \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} - \frac{\Delta_{S^2}}{r^2}\right)\right] \phi = 0 \,. \tag{2.5}$$

To simplify this equation we multiply by r and substitute $\phi = r\psi$ This leads us to the Cauchy problem

$$\left.\begin{aligned} \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial u^2} + \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} - \frac{\Delta_{S^2}}{r^2}\right)\right] \psi(t, u, x) = 0 \\ (\psi, i\partial_t\psi)(0, u, x) = \Psi_0(u, x) \end{aligned}\right\} \tag{2.6}$$

where the initial data $\Psi_0 \in C_0^\infty(\mathbb{R} \times S^2)^2$ is smooth and compactly supported.

The equation in (2.6) can be reformulated as the Euler-Lagrange equation corresponding to the action

$$S = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} du \int_{-1}^{1} d(\cos\vartheta) \int_0^{2\pi} d\varphi\, \mathcal{L}(\psi, \nabla\psi) \,, \tag{2.7}$$

where the Lagrangian is given by

$$\begin{aligned} 2\,\mathcal{L} \;\;=\;\; & |\partial_t\psi|^2 - |\partial_u\psi|^2 - \left(1 - \frac{2M}{r}\right)\frac{2M}{r^3}|\psi|^2 - \\ & \left(1 - \frac{2M}{r}\right)\frac{1}{r^2}\left(\frac{1}{\sin^2\vartheta}|\partial_\varphi\psi|^2 + \sin^2\vartheta\,|\partial_{\cos\vartheta}\,\psi|^2\right) \,. \end{aligned} \tag{2.8}$$

As one sees immediately, the Lagrangian is invariant under time translations, and thus Noether's theorem gives rise to a conserved quantity, the energy E,

$$E[\psi] = \int_{-\infty}^{\infty} du \int_{-1}^{1} d(\cos\vartheta) \int_0^{2\pi} \frac{d\varphi}{\pi}\, \mathcal{E}, \tag{2.9}$$

where $\mathcal{E}$ is the energy density

$$2\mathcal{E} = 2\left(\frac{\partial \mathcal{L}}{\partial \psi_t}\, \psi_t - \mathcal{L}\right) = |\partial_t \psi|^2 + |\partial_u \psi|^2 +$$

$$+ \left(1 - \frac{2M}{r}\right) \left\{ \frac{2M}{r^3} |\psi|^2 + \frac{1}{r^2} \left(\frac{1}{\sin^2\vartheta} |\partial_\varphi \psi|^2 + \sin^2\vartheta\, |\partial_{\cos\vartheta}\, \psi|^2 \right) \right\}. \tag{2.10}$$

It is also easy to check directly that the above energy is conserved in time for all smooth solutions of the wave equation that are compactly supported for all times. Since we consider the wave equation outside the event horizon, i.e. $r > 2M$, it is clear that the energy density is positive everywhere.

Next we rewrite the Cauchy problem (2.6) in first-order Hamiltonian form. Letting

$$\Psi = \begin{pmatrix} \psi \\ i\partial_t \psi \end{pmatrix}, \tag{2.11}$$

the Cauchy problem takes the form

$$i\partial_t \Psi = H\Psi\,, \quad \Psi\big|_{t=0} = \Psi_0 \tag{2.12}$$

where H is the Hamiltonian

$$\begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix}. \tag{2.13}$$

Here A is the differential operator

$$A = -\partial_u^2 + \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} - \frac{1}{r^2} \Delta_{S^2}\right). \tag{2.14}$$

We use the energy E in order to introduce a scalar product such that the Hamiltonian H is symmetric with respect to it. More precisely, we endow the space $C_0^\infty(\mathbb{R} \times S^2)^2$ with the *energy scalar product* $\langle .,. \rangle$ by polarizing E, thus

$$\langle \Psi, \Phi \rangle := \int_{-\infty}^{\infty} du \int_{-1}^{1} d(\cos\vartheta) \int_0^{2\pi} \frac{d\varphi}{2\pi} \left\{ \overline{\partial_t \psi} \partial_t \phi + \overline{\partial_u \psi} \partial_u \phi + \left(1 - \frac{2M}{r}\right) \right.$$

$$\left. \times \left[\frac{2M}{r^3}\, \overline{\psi}\, \phi + \frac{1}{r^2} \left(\frac{1}{\sin^2\vartheta}\, \overline{\partial_\varphi \psi} \partial_\varphi \phi + \sin^2\vartheta\, \overline{\partial_{\cos\vartheta}\, \psi} \partial_{\cos\vartheta}\, \phi \right) \right] \right\}, \tag{2.15}$$

where again $\Psi = (\psi, i\partial_t \psi)^T$ and $\Phi = (\phi, i\partial_t \phi)^T$. Energy conservation implies that for a solution Ψ of the Cauchy problem (2.12) which is compactly supported for all times,

$$0 \;=\; \frac{d}{dt} E[\Psi] = \frac{d}{dt} \langle \Psi, \Psi \rangle =$$

$$= \quad \langle \dot\Psi, \Psi\rangle + \langle \Psi, \dot\Psi\rangle = i\langle H\Psi, \Psi\rangle - i\langle \Psi, H\Psi\rangle \; .$$

Since the initial data $\Psi_0 \in C_0^\infty(\mathbb{R}\times S^2)^2$ can be chosen arbitrarily, polarization yields

$$\langle H\Psi, \Phi\rangle = \langle \Psi, H\Phi\rangle \;, \quad \text{for all } \Psi, \Phi \in C_0^\infty(\mathbb{R}\times S^2)^2 \; . \tag{2.16}$$

Hence the operator H is symmetric on $C_0^\infty(\mathbb{R}\times S^2)^2$ with respect to $\langle ., .\rangle$.

We will now use the spherical symmetry to simplify the problem. More precisely, we make use of the fact that the angular dependence of the wave equation in the Schwarzschild geometry involves only the Laplacian on the two sphere. It is well-known that the spherical harmonics $\{Y_{lm}(\vartheta,\varphi)\}_{l\in\mathbb{N}_0, |m|\le l}$ are smooth eigenfunctions of Δ_{S^2} with the eigenvalues $-l(l+1)$. Moreover, they form an orthonormal basis of the space $L^2(S^2)$. Thus we can decompose an arbitrary $\Psi \equiv (\psi_1, \psi_2)^T \in C_0^\infty(\mathbb{R}\times S^2)^2$ in the following way,

$$\Psi(u,\vartheta,\varphi) = \sum_{l=0}^{\infty} \sum_{|m|\le l} \Psi^{lm}(u) Y_{lm}(\vartheta,\varphi) \; , \tag{2.17}$$

where for each component the sum converges for fixed u in $L^2(S^2)$. Since the $\Psi^{lm} \equiv (\psi_1^{lm}, \psi_2^{lm})^T$ are uniquely determined by $\psi_i^{lm}(u) = \langle Y_{lm}, \psi_i(u)\rangle_{L^2(S^2)}$ it is clear that $\Psi^{lm}(u) \in C_0^\infty(\mathbb{R})^2$ for all l, m. Using this decomposition, we rewrite the norm of Ψ corresponding to the energy scalar product as

$$\begin{aligned}
\langle \Psi, \Psi\rangle = & \int_{-\infty}^{\infty} du \int_1^1 d(\cos\vartheta) \int_0^{2\pi} \frac{d\varphi}{2\pi} \left\{ |\psi_2|^2 + |\partial_u \psi_1|^2 + \right. \\
& \qquad\qquad \left. + \overline{\psi_1} \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} - \frac{1}{r^2}\Delta_{S^2}\right)\psi_1 \right\} \\
= & \sum_{l=0}^{\infty}\sum_{|m|\le l} \int_{-\infty}^{\infty} du \Big\{ |\psi_2^{lm}(u)|^2 + |\partial_u \psi_1^{lm}(u)|^2 + \\
& \qquad \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{l(l+1)}{r^2}\right) |\psi_1^{lm}(u)|^2 \Big\} \; ,
\end{aligned} \tag{2.18}$$

where in the first equation we have integrated by parts with respect to (ϑ, φ). The second equation follows from the properties of the Y_{lm}. As one can immediately see, the integrand for every summand in (2.18) is positive. Hence again by polarizing we obtain for any angular mode l an scalar product $\langle ., .\rangle_l$ on $C_0^\infty(\mathbb{R})^2$ given by

$$\langle \Psi, \Phi\rangle_l = \int_{-\infty}^{\infty} \left\{ \overline{\psi_2}\phi_2 + \overline{\psi_1'}\phi_1' + V_l\, \overline{\psi_1}\phi_1 \right\} du \; , \tag{2.19}$$

with the potential $V_l(u)$ defined as

$$V_l(u) = \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{l(l+1)}{r^2}\right) \; . \tag{2.20}$$

This definition leads to an isometry

$$\left(C_0^\infty(\mathbb{R}\times S^2)^2, \langle ., .\rangle\right) \quad \longrightarrow \quad \bigoplus_{l=0}^{\infty} \bigoplus_{|m|\le l} \left(C_0^\infty(\mathbb{R})^2, \langle ., .\rangle_l\right)$$

$$\Psi \quad \mapsto \quad \Psi^{lm} \, . \tag{2.21}$$

Using (2.17), the Hamiltonian H also decomposes in the following way,

$$H\Psi(u,\vartheta,\varphi) = \sum_{l=0}^{\infty} \sum_{|m|\leq l} H_l \Psi^{lm}(u) Y_{lm}(\vartheta,\varphi) \, .$$

Here the H_l act on $C_0^\infty(\mathbb{R})^2$ and are given by

$$H_l = \begin{pmatrix} 0 & 1 \\ -\partial_u^2 + V_l(u) & 0 \end{pmatrix} . \tag{2.22}$$

Thus for fixed angular modes l and m the Cauchy problem (2.12) simplifies to

$$i\partial_t \Psi^{lm} = H_l \Psi^{lm} \, , \quad \Psi^{lm}\big|_{t=0} = \Psi_0^{lm} \, , \tag{2.23}$$

where the initial data is in $C_0^\infty(\mathbb{R})^2$. Moreover, the H_l are symmetric on $C_0^\infty(\mathbb{R})^2$ with respect to $\langle .,. \rangle_l$, because for any $\Psi, \Phi \in C_0^\infty(\mathbb{R})^2$ the functions $\Psi(u)Y_{lm}$ and $\Phi(u)Y_{lm}$ are in $C_0^\infty(\mathbb{R}\times S^2)^2$. Thus

$$\langle H_l\Psi, \Phi\rangle_l = \langle H(\Psi\, Y_{lm}), \Phi Y_{lm}\rangle = \langle \Psi Y_{lm}, H(\Phi\, Y_{lm})\rangle = \langle \Psi, H_l \Phi\rangle_l \, .$$

In particular, for solutions of (2.23) with compact support in u for all times, the norm with respect to $\langle .,. \rangle_l$ is constant. Therefore we again refer to $\langle .,. \rangle_l$ as the energy scalar product.

Our strategy is to solve for a given inital data $\Psi_0 \in C_0^\infty(\mathbb{R}\times S^2)^2$ the Cauchy problem (2.23) for fixed angular modes l and m, and to sum up the solutions afterwards. Therefore, in what follows we will fix the angular modes l, m and consider the problem (2.23). In order to avoid too many indices, we usually omit the subscript l in the Hamiltonian and energy scalar product.

3 Spectral Properties of the Hamiltonian

In the previous section we introduced the energy scalar product $\langle , \rangle$ on the space $C_0^\infty(\mathbb{R})^2$. Since we cannot expect $C_0^\infty(\mathbb{R})^2$ to be complete with respect to this inner product (and indeed it is not, because the energy scalar product in the second component is just the usual L^2-scalar product), we define the Hilbert space $H^1_{V_l 0}(\mathbb{R})$ as the completion of $C_0^\infty(\mathbb{R})$ within the Hilbert space

$$H^1_{V_l}(\mathbb{R}) = \Big\{ u \text{ with } u' \in L^2(\mathbb{R}) \text{ and } V_l^{1/2} u \in L^2(\mathbb{R}) \Big\}$$

endowed with the scalar product

$$\langle u, v\rangle_1 := (u', v')_{L^2} + (V_l\, u, v)_{L^2} \, .$$

Note that this coincides with the energy scalar product on the first component. Therefore, we choose $\mathscr{H} \equiv H^1_{V_l 0}(\mathbb{R}) \oplus L^2(\mathbb{R})$ endowed with the energy scalar product as the underlying Hilbert space for our Hamiltonian H.

In the previous section we have seen that the Hamiltonian H is symmetric on $C_0^\infty(\mathbb{R})^2$. Before we can use functional analytic methods, we need to construct a self-adjoint extension of H. In fact, we are able to prove the following lemma:

Lemma 3.1. *The operator H with domain $\mathscr{D}(H) = C_0^\infty(\mathbb{R})^2$ is essentially self-adjoint in the Hilbert space $\mathscr{H}$.*

In order to prove this lemma, we use the following version of Stone's theorem about strongly continuous one-parameter unitary groups. A proof of this theorem can be found in [10, Section VIII.4].

Theorem 3.2. *Let $U(t)$ be a strongly continuous one-parameter unitary group on a Hilbert space $\mathscr{H}$. Then there is a self-adjoint operator A on $\mathscr{H}$ such that $U(t) = e^{itA}$.*

Furthermore, let D be a dense domain which is invariant under $U(t)$ and on which $U(t)$ is strongly differentiable. Then i^{-1} times the strong derivative of $U(t)$ is essentially self-adjoint on D, and its closure is A.

Now we apply this theorem:

Proof of Lemma 3.1. According to the theory of symmetric hyperbolic systems (cf. [13, Section 5.3]), the Cauchy problem

$$\left.\begin{array}{c} \left(\partial_t^2 - \partial_u^2 + V_l(u)\right)\psi(t,u) = 0 \\ \psi|_{t=0} = f \ , \ i\partial_t\psi|_{t=0} = g \end{array}\right\}$$

with smooth, compactly supported initial data $f, g \in C_0^\infty(\mathbb{R})$ has a unique solution $\psi(t,u) \in C^\infty(\mathbb{R}\times\mathbb{R})$ which is also compactly supported in u for all times. Using this solution, we define for arbitrary $t \in \mathbb{R}$ the operators

$$\begin{array}{rcl} U(t) : C_0^\infty(\mathbb{R})^2 & \to & C_0^\infty(\mathbb{R})^2 \\ \begin{pmatrix} f \\ g \end{pmatrix} & \mapsto & \begin{pmatrix} \psi(t,.) \\ i\partial_t\psi(t,.) \end{pmatrix} , \end{array}$$

which leave the dense subspace $C_0^\infty(\mathbb{R})^2 \subseteq \mathscr{H}$ invariant for all times t.
Due to the energy conservation, the $U(t)$ are unitary with respect to the energy scalar product and hence extend to unitary operators on the entire Hilbert space $\mathscr{H}$. Furthermore, since the solution is uniquely determined by the initial data, the $U(t)$ have the following properties,

$$U(0) = Id, \quad U(t+s) = U(t)U(s) \quad \text{for all } t, s \in \mathbb{R} ,$$

and thus they form a one-parameter unitary group. Due to the fact that smooth initial data yields smooth solutions in t and u, this group is strongly continuous on $\mathscr{H}$ and strongly differentiable on the domain $C_0^\infty(\mathbb{R})^2$. Calculating i^{-1} times the strong derivative one gets

$$i^{-1} \lim_{h\searrow 0} \frac{1}{h}\left(U(h)\begin{pmatrix} f \\ g \end{pmatrix} - \begin{pmatrix} f \\ g \end{pmatrix}\right) = i^{-1}\begin{pmatrix} -ig \\ i(\partial_u^2 - V_l)f \end{pmatrix} = -H\begin{pmatrix} f \\ g \end{pmatrix}$$

for all $f, g \in C_0^\infty(\mathbb{R})$, and the lemma follows from Theorem 3.2. □

For the further investigations of the Hamiltonian H, we consider its self-adjoint closure which, for the sake of simplicity, we again denote by H. For our purposes, it is not important to know the exact domain of definition $\mathscr{D}(H)$ of the self-adjoint extension.

4 Construction of the Resolvent

Stone's formula for the spectral projections of a self-adjoint operator A (cf. [10] Theorem VII.13),

$$\frac{1}{2}\left[P_{[a,b]} + P_{(a,b)}\right] = \operatorname*{s-lim}_{\epsilon \searrow 0} \frac{1}{2\pi i} \int_a^b \left[(A-\omega-i\epsilon)^{-1} - (A-\omega+i\epsilon)^{-1}\right] d\omega \;, \quad (4.1)$$

relates the spectral projections to the resolvent (here s-lim denotes the strong limit of operators). In view of this relation, it is of interest to derive an explicit representation of the resolvent.

In the preceding section we have seen that there is a domain $\mathscr{D}(H)$ such that our Hamiltonian H is self-adjoint in the Hilbert space $(\mathscr{H}, \langle\, ,\rangle)$. From this it immediately follows that the spectrum $\sigma(H) \subseteq \mathbb{R}$ is on the real line and therefore the resolvent $(H-\omega)^{-1} : \mathscr{H} \to \mathscr{H}$ exists for every $\omega \in \mathbb{C} \setminus \mathbb{R}$.

Let us now fix $\omega \in \mathbb{C} \setminus \mathbb{R}$. We often denote the ω-dependence by a subscript $_\omega$ .We begin by reducing the eigenvalue equation $H\Psi = \omega\Psi$ by substituting the equation for the first component in the second equation. We thus obtain the Schrödinger-type equation

$$\left(-\partial_u^2 + V_\omega(u)\right)\phi(u) = 0 \quad (4.2)$$

with the potential

$$V_\omega(u) = -\omega^2 + V_l(u) = -\omega^2 + \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{l(l+1)}{r^2}\right) \;. \quad (4.3)$$

In what follows we refer to this equation simply as the Schrödinger equation. It can be regarded as the radial equation associated to the wave equation in (2.6). Our goal is to construct the resolvent $(H-\omega)^{-1}$ out of special solutions of this equation. We introduce fundamental solutions $\acute{\phi}_\omega$ and $\grave{\phi}_\omega$ of the Schrödinger equation (4.2) which satisfy asymptotic boundary conditions at $u = \pm\infty$ (the existence of these solutions will be proved in Section 5). More precisely, in the case $\operatorname{Im}(\omega) > 0$ we impose that

$$\lim_{u\to-\infty} e^{i\omega u}\acute{\phi}_\omega(u) = 1 \;, \qquad \lim_{u\to-\infty}\left(e^{i\omega u}\acute{\phi}_\omega(u)\right)' = 0 \quad (4.4)$$

$$\lim_{u\to+\infty} e^{-i\omega u}\grave{\phi}_\omega(u) = 1 \;, \qquad \lim_{u\to+\infty}\left(e^{-i\omega u}\grave{\phi}_\omega(u)\right)' = 0 \;, \quad (4.5)$$

whereas in the case $\operatorname{Im}(\omega) < 0$,

$$\lim_{u\to-\infty} e^{-i\omega u}\acute{\phi}_\omega(u) = 1 \;, \qquad \lim_{u\to-\infty}\left(e^{-i\omega u}\acute{\phi}_\omega(u)\right)' = 0 \quad (4.6)$$

$$\lim_{u\to+\infty} e^{i\omega u}\grave{\phi}_\omega(u) = 1 \;, \qquad \lim_{u\to+\infty}\left(e^{i\omega u}\grave{\phi}_\omega(u)\right)' = 0 \;. \quad (4.7)$$

Since the resolvent exists, the map $(H-\omega) : \mathscr{D}(H) \to \mathscr{H}$ is bijective and in particular the kernel is trivial. Hence the solutions $\acute{\phi}_\omega, \grave{\phi}_\omega$ are linearly independent (otherwise they would give rise to a vector in the kernel due to the exponential decay). Thus $\acute{\phi}_\omega$ and $\grave{\phi}_\omega$ are indeed a system of fundamental solutions with non-vanishing Wronskian

$$w(\acute{\phi}_\omega, \grave{\phi}_\omega) := \acute{\phi}_\omega(u)\grave{\phi}'_\omega(u) - \acute{\phi}'_\omega(u)\grave{\phi}_\omega(u) \ . \tag{4.8}$$

Note that the Wronskian is independent of the variable u, as is easily verified by differentiating with respect to u and substituting the Schrödinger equation.

In the next lemma, we use this fundamental system to derive the Green's function corresponding to (4.2).

Lemma 4.1. *The function*

$$s_\omega(u,v) := -\frac{1}{w(\acute{\phi}_\omega, \grave{\phi}_\omega)} \cdot \begin{cases} \acute{\phi}_\omega(u)\grave{\phi}_\omega(v) \ , & \textit{if } u \le v \\ \acute{\phi}_\omega(v)\grave{\phi}_\omega(u) \ , & \textit{if } u > v \end{cases} \tag{4.9}$$

satisfies the distributional equations

$$\left(-\frac{\partial^2}{\partial u^2} + V_\omega(u)\right) s_\omega(u,v) = \delta(u-v) = \left(-\frac{\partial^2}{\partial v^2} + V_\omega(v)\right) s_\omega(u,v) \ .$$

Proof. By definition of the distributional derivative we have for every test function $\eta \in C_0^\infty(\mathbb{R})$,

$$\int_{-\infty}^{\infty} \eta(u) \left[-\partial_u^2 + V_\omega(u)\right] s_\omega(u,v) \, du = \int_{-\infty}^{\infty} \left[(-\partial_u^2 + V_\omega(u))\eta(u)\right] s(u,v) \, du \ .$$

It is obvious from its definition that the function $s(.,v)$ is smooth except at the point $u=v$, where its first derivative has a discontinuity. Thus, after splitting up the integral, we can integrate by parts twice to obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} \left[(-\partial_u^2 + V_\omega(u))\eta(u)\right] s(u,v) \, du = \\ &= \int_{-\infty}^{v} \eta(u)(-\partial_u^2 + V_\omega(u)) s(u,v) \, du + \lim_{u \nearrow v} \left[\eta(u)\partial_u s(u,v)\right] + \\ &\quad + \int_{v}^{\infty} \eta(u)(-\partial_u^2 + V_\omega(u)) s(u,v) \, du - \lim_{u \searrow v} \left[\eta(u)\partial_u s(u,v)\right] \ . \end{aligned}$$

Since for $u \neq v$, s is a solution of (4.2), the obtained integrals vanish. Computing the limits with the definition (4.9), we get

$$\begin{aligned} \int_{-\infty}^{\infty} \left[(-\partial_u^2 + V_\omega(u))\eta(u)\right] s(u,v) \, du &= \left(\lim_{u \nearrow v} - \lim_{u \searrow v}\right) \eta(u)\partial_u s(u,v) = \\ &= -\frac{1}{w(\acute{\phi}_\omega, \grave{\phi}_\omega)} \, \eta(v) \left[\acute{\phi}'_\omega(v)\grave{\phi}_\omega(v) - \grave{\phi}'_\omega(v)\acute{\phi}_\omega(v)\right] = \eta(v) \ , \end{aligned}$$

where in the last step we used the definition of the Wronskian (4.8). This yields the first equation. The second equation is proven exactly in the same way. ☐

With this function s we are now able to construct the resolvent. More precisely,

Proposition 4.2. *For every $\omega \in \mathbb{C} \setminus \mathbb{R}$, the resolvent $(H-\omega)^{-1} : \mathscr{H} \to \mathscr{H}$ can be represented as an integral operator with the integral kernel*

$$k_\omega(u,v) = \delta(u-v) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + s_\omega(u,v) \begin{pmatrix} \omega & 1 \\ \omega^2 & \omega \end{pmatrix} . \tag{4.10}$$

Proof. We introduce the integral operator S_ω with the integral kernel $k_\omega(u,v)$ on the domain

$$\mathscr{D}(S_\omega) := \left\{ (H-\omega)\Psi \,\middle|\, \Psi \in C_0^\infty(\mathbb{R})^2 \right\} .$$

Let us verify that $\mathscr{D}(S_\omega)$ is a dense subset of $\mathscr{H}$. Let $\phi \in \mathscr{H}$ be an arbitrary vector. Because of the existence of the resolvent, the operator $H-\omega : \mathscr{D}(H) \to \mathscr{H}$ is onto, and thus there is a vector $\psi \in \mathscr{D}(H)$ with $(H-\omega)\psi = \phi$. Then due to the definition of the closure of H, there is a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subseteq C_0^\infty(\mathbb{R})^2$ with $\psi_n \to \psi$ and $H\psi_n \to H\psi$ as $n \to \infty$. This shows that $\{(H-\omega)\psi_n\}_{n\in\mathbb{N}} \subseteq \mathscr{D}(S_\omega)$ converges to $(H-\omega)\psi = \phi$. We conclude that $\mathscr{D}(S_\omega)$ is dense. We now calculate the operator product $S_\omega\,(H-\omega)$ on $C_0^\infty(\mathbb{R})^2$. For an arbitrary $\Psi = (\psi_1,\psi_2)^T \in C_0^\infty(\mathbb{R})^2$ we have

$$\begin{aligned} (S_\omega(H-\omega)\psi)\,(u) &= \int_{-\infty}^{\infty} k_\omega(u,v)(H-\omega)\psi(v)\,dv = \\ &= \begin{pmatrix} 0 \\ -\omega\psi_1 + \psi_2 \end{pmatrix}(u) + \\ &\quad + \int_{-\infty}^{\infty} s_\omega(u,v) \begin{pmatrix} -\partial_v^2 + V_\omega(v) & 0 \\ \omega\left(-\partial_v^2 + V_\omega(v)\right) & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}(v)\,dv . \end{aligned}$$

Hence, according to Lemma 4.1,

$$S_\omega(H-\omega) = Id \quad \text{on } C_0^\infty(\mathbb{R})^2 .$$

This yields that $S_\omega = (H-\omega)^{-1}$ on the dense set $\mathscr{D}(S_\omega)$. Since $(H-\omega)^{-1}$ is a bounded operator, the claim follows. ☐

As mentioned at the beginning of this section, we can now apply Stone's formula for the spectral projections of H and get for every $\Psi \in \mathscr{H}$

$$\begin{aligned} &\frac{1}{2}\left[P_{[a,b]} + P_{(a,b)}\right]\Psi = \\ &\qquad = \lim_{\epsilon \searrow 0} \frac{1}{2\pi i} \int_a^b \left[(H-(\omega+i\epsilon))^{-1} - (H-(\omega-i\epsilon))^{-1} \right] \Psi\, d\omega , \end{aligned}$$

and this yields together with Proposition 4.2

$$= \lim_{\epsilon \searrow 0} \frac{1}{2\pi i} \int_a^b \left(\int_{\mathbb{R}} \left(k_{\omega+i\epsilon}(.,v) - k_{\omega-i\epsilon}(.,v)\right) \Psi(v) dv \right) d\omega , \tag{4.11}$$

where the limit is with respect to the norm in $\mathscr{H}$. It is therefore of special interest how the kernels $k_{\omega+i\epsilon}(u,v)$ and $k_{\omega-i\epsilon}(u,v)$ behave as $\epsilon \searrow 0$. Since these kernels are given explicitly in terms of the fundamental solutions $\acute{\phi}_{\omega\pm i\epsilon}$ and $\grave{\phi}_{\omega\pm i\epsilon}$, we will discuss their behavior in the next section.

5 The Jost Solutions of the Radial Equation

In this section we want to discuss the existence and the behavior of the solutions $\acute{\phi}_\omega, \grave{\phi}_\omega$ of the Schrödinger equation (4.2), which in Section 4 we used for the construction of the resolvent. We will prove the following theorem.

Theorem 5.1.

(i) For every $\omega \in D = \left\{\omega \in \mathbb{C} \mid \operatorname{Im} \omega \le \frac{1}{4M}\right\}$, there exists a unique solution $\phi_1(\omega, u)$ of the Schrödinger equation (4.2) satisfying the boundary conditions (4.6) such that for every fixed $u \in \mathbb{R}$ the function $\phi_1(\omega, u)$ is holomorphic in $\omega \in \mathring{D}$ and continuous in D.

(ii) For every angular momentum number l, the solutions $\grave{\phi}_\omega$ of the Schrödinger equation (4.2) with boundary conditions (4.7) are well-defined and uniquely determined on the set

$$E = \left\{\omega \in \mathbb{C} \;\middle|\; \operatorname{Im} \omega \le 0,\ \omega \neq 0\right\} .$$

For each fixed $u \in \mathbb{R}$, the function $\grave{\phi}_\omega(u)$ is holomorphic in $\omega \in \mathring{E}$ and continuous in E.

Furthermore, in the case $l = 0$, $\grave{\phi}_\omega(u)$ may be continuously extended to $\omega = 0$.

Once having proven this theorem, we simply set

$$\acute{\phi}_\omega(u) := \begin{cases} \overline{\phi_1(\overline{\omega}, u)} & , \text{if } \operatorname{Im} \omega > 0 \\ \phi_1(\omega, u) & , \text{if } \operatorname{Im} \omega \le 0 \end{cases} , \tag{5.1}$$

as well as

$$\grave{\phi}_\omega(u) := \overline{\grave{\phi}_{\overline{\omega}}(u)} \quad \text{if } \operatorname{Im} \omega > 0 , \tag{5.2}$$

to obtain the solutions of Section 4. For $\operatorname{Im} \omega < 0$ this is clear by definition. But in the case of $\operatorname{Im} \omega > 0$ the above defined $\acute{\phi}_\omega(u), \grave{\phi}_\omega(u)$ are indeed the unique solutions of the Schrödinger equation (4.2) with the desired boundary conditions (4.4) and (4.5), respectively. This follows immediately by complex conjugation of the Schrödinger equation due to the fact that our potential V_l is real.

For the proof of Theorem 5.1 we will formally manipulate the Schrödinger equation with boundary conditions (4.6, 4.7) in order to get an appropriate integral equation (which in different contexts is called the Jost or Lipman-Schwinger equation). Then we will perform a perturbation expansion and get estimates for all the terms of the expansion. A reference for this method can be found e.g. in [12, Section XI.8]. Since this reference contains only an outline of the proof, it seems worth working out the details.

To introduce the method, we begin with the solutions $\phi_1(\omega, u)$. First we write the Schrödinger equation (4.2) in the form

$$\left(-\frac{d^2}{du^2} - \omega^2\right) \phi_\omega(u) = -W(u)\phi_\omega(u) , \tag{5.3}$$

where W is a potential in $L^1(\mathbb{R})$ (later on, W will be replaced by V_l). Next we define for $\omega \in \mathbb{C}$ the function $G_\omega(u)$ by

$$G_\omega(u) := \begin{cases} 0 & \text{, if } u \leq 0 \\ -\dfrac{1}{\omega}\sin(\omega u) & \text{, if } u > 0 \text{ and } \omega \neq 0 \\ -u & \text{, if } u > 0 \text{ and } \omega = 0 \end{cases} \quad . \tag{5.4}$$

A simple computation shows that $G_\omega(u)$ defines a *Green's function* for the operator on the left hand side of the equation (5.3) in the sense that the distributional equation

$$\left(-\frac{d^2}{du^2} - \omega^2\right) G_\omega(u) = \delta(u)$$

holds. In order to build in the boundary condition (4.6), we make in equation (5.3) the substitution $\phi_\omega(u) = e^{i\omega u} + \tilde{\phi}_\omega(u)$ to obtain

$$\left(-\frac{d^2}{du^2} - \omega^2\right) \tilde{\phi}_\omega(u) = -W(u)\phi_\omega(u) \ .$$

Solving this equation formally by convoluting the right hand side with G_ω, we get the formal solution

$$\tilde{\phi}_\omega(u) = \big((-W\phi_\omega) * G_\omega\big)(u) \equiv -\int_{-\infty}^{\infty} G_\omega(u-v)W(v)\phi_\omega(v)\, dv \ .$$

Hence $\phi_\omega(u)$ satisfies the equation

$$\phi_\omega(u) = e^{i\omega u} - \int_{-\infty}^{u} G_\omega(u-v)W(v)\phi_\omega(v)\, dv \ , \tag{5.5}$$

which is referred to as the *Jost equation with boundary conditions at* $-\infty$. Its significance lies in the fact that we can now easily perform a perturbation expansion in the potential W. Namely, making for ϕ_ω the ansatz as the perturbation series

$$\phi_\omega = \sum_{k=0}^{\infty} \phi_\omega^{(k)} \ , \tag{5.6}$$

we are led to the iteration scheme

$$\left.\begin{array}{rcl} \phi_\omega^{(0)}(u) & = & e^{i\omega u} \\ & \vdots & \\ \phi_\omega^{(k+1)}(u) & = & -\displaystyle\int_{-\infty}^{u} G_\omega(u-v)W(v)\phi_\omega^{(k)}(v)\, dv \end{array}\right\} \ . \tag{5.7}$$

This iteration scheme can be used to construct solutions of the Jost equation.

We remark that under certain assumptions on W like continuity, the Jost equation is equivalent to the corresponding Schrödinger equation with appropriate boundary conditions. We will show this for our special case $W \equiv V_l$. A systematic method to rewrite second-order differential equations with boundary conditions as integral equations can be found e.g. in [12, Section XI.8 Appendix 2].

We now state a theorem about solutions of the Jost equation. We consider more general potentials W than we have in our case, because it might be of interest by itself.

Theorem 5.2. *Suppose that W is a measurable function obeying for a given $u_0 < 0$ the condition $\int_{-\infty}^{u_0} |W(v)|dv < \infty$. Define for $u \le u_0$ the function $P_\omega(u)$ by*

$$P_\omega(u) = \int_{-\infty}^{u} \frac{4|v|}{1+|\omega v|}\,|W(v)|e^{-(\operatorname{Im}\omega + |\operatorname{Im}\omega|)v}\,dv\,. \tag{5.8}$$

Then:

(i) For each $\omega \in E = \{\omega \in \mathbb{C} \,\big|\, \operatorname{Im}\omega \le 0, \omega \ne 0\}$ the Jost equation (5.5) has a unique solution $\phi_\omega(u)$ obeying $\lim_{u\to-\infty} \left|e^{-i\omega u}\phi_\omega(u)\right| < \infty$. Moreover, $\phi_\omega(u)$ is continuously differentiable in u on $(-\infty, u_0)$ with $\lim_{u\to-\infty} e^{-i\omega u}\phi_\omega(u) = 1$ and $\lim_{u\to-\infty} e^{-i\omega u}\,\phi'_\omega(u) = i\omega$. For each fixed u, the functions $\phi_\omega(u)$ and $\phi'_\omega(u)$ are holomorphic in $\mathring{E}$ and continuous in E. They satisfy the bounds

$$\begin{aligned} \left|\phi_\omega(u) - e^{i\omega u}\right| &\le e^{-u\,\operatorname{Im}\omega}\left|e^{P_\omega(u)} - 1\right| & (5.9)\\ \left|\phi'_\omega(u) - i\omega e^{i\omega u}\right| &\le e^{-u\,\operatorname{Im}\omega} e^{P_\omega(u)} \int_{-\infty}^{u} |W(v)|\,dv\,. & (5.10) \end{aligned}$$

(ii) If $\int_{-\infty}^{u_0} |v||W(v)|\,dv < \infty$, for every $u \le u_0$ the function $\phi_\omega(u)$ may be continuously extended to $\omega = 0$. Moreover, (5.9), (5.10) hold also at $\omega = 0$.

(iii) If $\int_{-\infty}^{u_0} e^{-mv}|W(v)|dv < \infty$, for every $u \le u_0$ the function $\phi_\omega(u)$ can be extended to a holomorphic function in $\left\{\omega \mid \operatorname{Im}\omega < \frac{1}{2}m\right\}$, continuous in $\left\{\omega \mid \operatorname{Im}\omega \le \frac{1}{2}m\right\}$. Moreover, in the interval $0 < \operatorname{Im}\omega < \frac{1}{2}m$ the inequalities (5.9),(5.10) are replaced by

$$\begin{aligned} \left|\phi_\omega(u) - e^{i\omega u}\right| &\le \frac{1}{|\omega|} e^{u\operatorname{Im}\omega} e^{P_\omega(u)} \int_{-\infty}^{u} e^{-2v\operatorname{Im}\omega}|W(v)|\,dv & (5.11)\\ \left|\phi'_\omega(u) - i\omega e^{i\omega u}\right| &\le e^{u\operatorname{Im}\omega} e^{P_\omega(u)} \int_{-\infty}^{u} e^{-2v\operatorname{Im}\omega}|W(v)|\,dv\,. & (5.12) \end{aligned}$$

In each case, ϕ obeys $\overline{\phi_\omega(u)} \equiv \overline{\phi(\omega, u)} = \phi(-\overline{\omega}, u)$.

We call this solution ϕ the **Jost solution**. For the proof of this theorem we need a good estimate for the Green's function G_ω.

Lemma 5.3. *For all $u \in \mathbb{C}$,*

$$|\sin u| \le \frac{2|u|}{1+|u|}\, e^{|\operatorname{Im} u|}\,. \tag{5.13}$$

In particular, if $\omega \ne 0$ and $v \le u \le 0$,

$$\left|\frac{1}{\omega}\sin(\omega(u-v))\right| \le \frac{4|v|}{1+|\omega v|}\, e^{-v|\operatorname{Im}\omega| - u\,\operatorname{Im}\omega}\,. \tag{5.14}$$

Proof. In the case $|u| \geq 1$, the inequality (5.13) follows directly from the Euler formula $\sin u = \frac{1}{2i}\left(e^{iu} - e^{-iu}\right)$ and the estimate

$$(1+|u|)|\sin u| \;\leq\; \frac{1}{2}(1+|u|)2e^{|\operatorname{Im} u|} \;\leq\; 2|u|e^{|\operatorname{Im} u|}\;.$$

In the remaining case $|u| < 1$, we again use the Euler formula to obtain

$$(1+|u|)|\sin u| = \frac{1}{2}(1+|u|)\left|e^{iu} - e^{-iu}\right| = \frac{1}{2}(1+|u|)\left|\int_{-1}^{1} iue^{iu\tau}\,d\tau\right|\,,$$

and hence

$$(1+|u|)|\sin u| \;\leq\; \frac{1}{2}(|u|+|u|^2)\int_{-1}^{1}\left|e^{iu\tau}\right|d\tau \;\leq\; \frac{1}{2}(|u|+|u|^2)2\,e^{|\operatorname{Im} u|}\;.$$

Now (5.13) follows by the assumption $|u| < 1$.

In order to show (5.14) we use the identity

$$\frac{1}{\omega}\sin(\omega(u-v)) = \frac{1}{\omega}\left(\sin(\omega u)e^{i\omega v} - \sin(\omega v)e^{i\omega u}\right)$$

and apply (5.13),

$$\begin{aligned} \left|\frac{1}{\omega}\sin(\omega(u-v))\right| \;&\leq\; \frac{1}{|\omega|}\left(\left|\sin(\omega u)e^{i\omega v}\right| + \left|\sin(\omega v)e^{i\omega u}\right|\right) \\ &\overset{(5.13)}{\leq}\; \frac{2|u|}{1+|\omega u|}\,e^{|u\operatorname{Im}\omega|}e^{-v\operatorname{Im}\omega} + \frac{2|v|}{1+|\omega v|}\,e^{|v\operatorname{Im}\omega|}e^{-u\operatorname{Im}\omega}\;. \end{aligned} \tag{5.15}$$

Due to the assumption $0 \geq u \geq v$, we know that $|v| \geq |u|$ and thus

$$\frac{2|u|}{1+|\omega u|} \leq \frac{2|v|}{1+|\omega v|}\,, \qquad -u|\operatorname{Im}\omega| - v\operatorname{Im}\omega \leq -v|\operatorname{Im}\omega| - u\operatorname{Im}\omega\;.$$

Using these inequalities in (5.15) the claim follows. ☐

Note that the estimate (5.14) remains valid in the limit $0 \neq \omega \to 0$, if one replaces $\frac{1}{\omega}\sin(\omega(u-v))$ by the function $u - v$.

Now we are ready to prove Theorem 5.2:

Proof of Theorem 5.2.
Using the perturbation expansion (5.6) together with the iteration scheme (5.7), one easily sees that we have already found a *formal* solution. So our goal is to show that this series is well-defined and has the desired properties. To this end, we shall prove inductively that

$$\left|\phi_\omega^{(k)}(u)\right| \leq e^{-u\,\operatorname{Im}\omega}\frac{1}{k!}P_\omega(u)^k \quad \text{for all } k\in\mathbb{N}_0,\ \text{for all } \omega, u \tag{5.16}$$

such that $P_\omega(u)$ is well-defined by (5.8). Due to the integrability conditions on the potential W in the statement of the theorem this is the case for $u \leq u_0$ and for all $\omega \in E$ (cf. (*i*)), $\omega \in \overline{E}$ (cf. (*ii*)), $\omega \in \{\operatorname{Im}\omega \leq \frac{1}{2}m\}$ (cf. (*iii*)), respectively. Furthermore, $P_\omega(u)$ is continuous in u as well as in ω in these domains. The first statement is obvious while the latter is due to the fact that the integrand in the definition (5.8) is continuous in ω and one directly finds an integrable dominating function such that one can apply Lebegue's Dominated Convergence Theorem.

We start the induction with the case $k = 0$ for which (5.16) certainly is satisfied. Thus assume that (5.16) holds for a given k. Then, estimating the integral equation in (5.7) using (5.14) and (5.8), we obtain

$$\begin{aligned}
\left|\phi_\omega^{(k+1)}(u)\right| &\leq \int_{-\infty}^{u} |G_\omega(u-v)||W(v)|\left|\phi_\omega^{(k)}(v)\right| dv \\
&\leq \int_{-\infty}^{u} \frac{4|v|}{1+|\omega v|}\, e^{-v|\operatorname{Im}\omega|-u\operatorname{Im}\omega}|W(v)|e^{-v\operatorname{Im}\omega}\frac{1}{k!}\,P_\omega(v)^k\,dv \\
&= e^{-u\operatorname{Im}\omega}\,\frac{1}{k!}\int_{-\infty}^{u}\frac{dP_\omega}{dv}(v)\,P_\omega(v)^k\,dv \\
&= e^{-u\operatorname{Im}\omega}\,\frac{1}{(k+1)!}P_\omega(u)^{k+1}\;,
\end{aligned}$$

where in the last step we used that $P_\omega(u)$ vanishes when u goes to $-\infty$. This concludes the proof of (5.16).

Summing over k, (5.16) yields the inequality

$$\sum_{k=0}^{\infty}\left|\phi_\omega^{(k)}(u)\right| \leq e^{-u\operatorname{Im}\omega}e^{P_\omega(u)}. \tag{5.17}$$

Because of the continuity of $P_\omega(u)$, the series (5.6) converges uniformly for u and ω in compact sets. Using the iteration scheme (5.7), this series can be written as

$$\sum_{k=0}^{\infty}\phi_\omega^{(k)}(u) = e^{i\omega u} - \sum_{k=1}^{\infty}\int_{-\infty}^{u} G_\omega(u-v)W(v)\phi_\omega^{(k-1)}(v)\,dv\;,$$

and the bound (5.17) allows us to apply Lebesgue's dominated convergence theorem and to interchange the sum and the integral. Hence the series is indeed a solution of the Jost equation (5.5).

Next we want to show that a solution of the Jost equation is continuously differentiable with respect to u. To this end, we first compute for an arbitrary $u < u_0$ the difference quotient,

$$\begin{aligned}
&\frac{1}{h}\left(\phi_\omega(u+h)-\phi_\omega(u)-e^{i\omega(u+h)}+e^{i\omega u}\right) \overset{(5.5)}{=} \\
&\quad\int_{-\infty}^{u+h}\frac{1}{h\omega}\left[\sin(\omega(u+h-v))-\sin(\omega(u-v))\right]W(v)\phi_\omega(v)\,dv\,+ \qquad (5.18)\\
&\quad+\frac{1}{h}\int_{u}^{u+h}\frac{1}{\omega}\sin(\omega(u-v))W(v)\phi_\omega(v)\,dv\;, \qquad (5.19)
\end{aligned}$$

where $h \neq 0$. We may restrict attention to the case $\operatorname{Im}\omega \leq 0$ and $h > 0$ (the other cases are analogous). Using the estimate

$$\left|\partial_u\left(\frac{1}{\omega}\sin(\omega(u-v))\right)\right| = \left|\cos(\omega(u-v))\right| \leq$$

$$\leq \frac{1}{2}\left(e^{-u\,\mathrm{Im}\,\omega+v\,\mathrm{Im}\,\omega}+e^{u\,\mathrm{Im}\,\omega-v\,\mathrm{Im}\,\omega}\right)$$

together with (5.17), we can apply the mean value theorem to the first integrand to obtain the dominating function

$$\frac{1}{2}\left(e^{-\xi(v)\,\mathrm{Im}\,\omega}e^{v\,\mathrm{Im}\,\omega}+e^{\xi(v)\,\mathrm{Im}\,\omega}e^{-v\,\mathrm{Im}\,\omega}\right)|W(v)|\,e^{-v\,\mathrm{Im}\,\omega}e^{P_\omega(v)}\ ,$$

where $\xi(v)\in[u,u+h]$. Due to the integrability conditions on W, it is clear that this function is integrable. Hence Lebesgue's dominated convergence theorem allows us to take the limit $h\to 0$ in (5.18). This gives

$$\int_{-\infty}^{u}\cos(\omega(u-v))W(v)\phi_\omega(v)\,dv\ .$$

In order to treat the second integral, we choose $h<h_0$, where h_0 is so small that

$$\max_{v\in[u,u+h]}\left|\frac{1}{\omega h}\sin(\omega(u-v))\right|\leq 2\quad\text{for all } h<h_0.$$

This is possible because $\lim_{h\to 0}\frac{1}{\omega h}\sin(\omega h)=1$. Thus we can estimate (5.19) by

$$\begin{aligned}&\left|\frac{1}{h}\int_u^{u+h}\frac{1}{\omega}\sin(\omega(u-v))W(v)\phi_\omega(v)\,dv\right|\leq\\&\qquad\leq 2\,e^{-(u+h)\,\mathrm{Im}\,\omega}e^{P_\omega(u+h)}\int_{(-\infty,u_0)}\left|1_{[u,u+h]}(v)W(v)\right|dv\ ,\end{aligned}$$

and the last integral goes to 0 as $h\to 0$ by Lebesgue's monotone convergence theorem using the fact that $W\in L^1(-\infty,u_0)$. Hence (5.19) vanishes.

Alltogether we conclude that $\phi_\omega(u)$ is differentiable with derivative

$$\phi'_\omega(u)=i\omega e^{i\omega u}+\int_{-\infty}^{u}\cos(\omega(u-v))W(v)\phi_\omega(v)dv\ ,\tag{5.20}$$

which is continuous on $(-\infty,u_0)$ because of the estimate (5.17).

The estimate (5.9) is a simple consequence of (5.16) together with the perturbation expansion (5.6). For the proof of (5.10) we use the representation of the derivative (5.20) together with the inequality (5.17):

$$\begin{aligned}\left|\phi'_\omega(u)-i\omega e^{i\omega u}\right|&\overset{(5.20)}{\leq}\int_{-\infty}^{u}|\cos(\omega(u-v))|\,|W(v)|\,|\phi_\omega(v)|dv\\&\overset{(5.17)}{\leq}\int_{-\infty}^{u}\frac{1}{2}\left(e^{-u\,\mathrm{Im}\,\omega}e^{v\,\mathrm{Im}\,\omega}+e^{u\,\mathrm{Im}\,\omega}e^{-v\,\mathrm{Im}\,\omega}\right)|W(v)|\,e^{-v\,\mathrm{Im}\,\omega}e^{P_\omega(v)}\,dv\\&\leq\ e^{-u\,\mathrm{Im}\,\omega}e^{P_\omega(u)}\int_{-\infty}^{u}|W(v)|\,dv\ ,\end{aligned}$$

where in the last step we used the fact that $P_\omega(v)$ and $e^{-v\,\mathrm{Im}\,\omega}$ (with $\mathrm{Im}\,\omega\leq 0$) are monotone increasing. The estimates (5.11) as well as (5.12) are shown in the same way.

Let us now verify that for any fixed u, the function $\phi_\omega(u)$ is holomorphic in ω, and continuous on the domains as specified in $(i), (ii)$ and (iii). Due to the locally uniform convergence of the perturbation series, it suffices to show that every $\phi_\omega^{(k)}(u)$ has the desired properties. We do this inductively, where the case $k = 0$ is trivial. Let us now assume that $\phi_\omega^{(k)}(u)$ is holomorphic in $\mathring{E}$ ($\{\operatorname{Im}\omega < \frac{1}{2}m\}$, respectively). In order to prove that $\phi_\omega^{(k+1)}$ is holomorphic, we want to apply Morera's theorem. Thus we must show that $\phi_\omega^{(k+1)}(u)$ is continuous in ω and that the integral

$$\oint_\gamma \phi_\omega^{(k+1)}(u)\, d\omega \overset{(5.7)}{=} \oint_\gamma \int_{-\infty}^u \frac{1}{\omega} \sin(\omega(u-v)) W(v) \phi_\omega^{(k)}(v)\, dv\, d\omega \tag{5.21}$$

vanishes for every closed contour γ in $\mathring{E}$ (or in case (iii), for every contour in $\{\operatorname{Im}\omega < \frac{1}{2}m\}$, respectively). Using the above estimates (5.14),(5.16) together with the monotonicity of $P_\omega(u)$ in u we get the following bound for the integrand

$$\begin{aligned}\Big| \frac{1}{\omega} \sin(\omega(u-v)) W(v) \phi_\omega^{(k)}(v) \Big| &\leq \\ &\leq |W(v)| \frac{4|v|}{1+|\omega v|} e^{-u \operatorname{Im}\omega - v|\operatorname{Im}\omega| - v \operatorname{Im}\omega} \frac{1}{k!} P_\omega(u)^k \;. \end{aligned} \tag{5.22}$$

Due to the induction hypothesis, the integrand is continuous in ω. Moreover, for a compact neighborhood $K(\omega_0)$ of a fixed ω_0 contained in the specified domains, (5.22) yields for the family $\frac{1}{\omega}\sin(\omega(u-v))W(v)\phi_\omega^{(k)}(v),\ \omega \in K(\omega_0)$ the uniformly dominating function

$$|W(v)| \frac{4|v|}{1+|v| \min|\omega|} e^{-u \operatorname{Im}\omega - v\ \max(|\operatorname{Im}\omega| + \operatorname{Im}\omega)} \frac{1}{k!} P_\omega(u)^k \;,$$

where the minimum and the maximum are taken in $K(\omega_0)$. This function is integrable for $K(\omega_0)$ chosen sufficiently small due to the integrability conditions on W. This lets us apply Lebesgue's dominated convergence theorem to show the continuity in ω for $\phi_\omega^{(k+1)}(u)$, which is given by the integral (5.7). Moreover, (5.22) together with the continuity in ω of $P_\omega(u)$ yield that the integral

$$\oint_\gamma \int_{-\infty}^u \Big| \frac{1}{\omega} \sin(\omega(u-v)) W(v) \phi_\omega^{(k)}(v) \Big|\, dv\, d\omega < \infty$$

exists for an arbitrary closed contour γ in $\mathring{E}$ (or $\{\operatorname{Im}\omega < \frac{1}{2}m\}$, respectively). By the theorem of Fubini, we may interchange the orders of integration in (5.21). Because of the induction hypothesis, the integrand of (5.21) on the right hand side is holomorphic. Thus the integral vanishes due to the Cauchy integral theorem. We conclude that $\phi_\omega^{(k)}$ is holomorphic for every k. Since $\phi_\omega(u)$ is holomorphic, the same argument together with equation (5.20) yields that ϕ'_ω is also holomorphic.

It remains to prove uniqueness. Let $\psi_\omega(u)$ be another solution of the Jost equation obeying $\lim_{u\to-\infty} \big| e^{-i\omega u} \psi_\omega(u) \big| < \infty$. Then we can find a $c > 0$ with $\big|\psi_\omega(u)\big| \leq c e^{-u \operatorname{Im}\omega}$ for all $u \leq u_0$. Then as above one shows inductively that

$$\Big| \psi_\omega(u) - \sum_{l=0}^N \phi_\omega^{(k)}(u) \Big| \leq c\, e^{-u \operatorname{Im}\omega} \frac{1}{(N+1)!} P_\omega(u)^{N+1} \;,$$

and taking $N \to \infty$ we obtain $\psi_\omega = \phi_\omega$.
The uniqueness also implies that $\overline{\phi(\omega, u)} = \phi(-\overline{\omega}, u)$, concluding the proof. $\square$

Remark 5.4. In order to treat the Schrödinger equation (4.2) with boundary conditions at infinity (4.7), we derive the corresponding Jost equation with boundary equations at $+\infty$ using the same procedure as on page 11:

$$\phi_\omega(u) = e^{-i\omega u} - \int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) W(v) \phi_\omega(v)\, dv \; . \tag{5.23}$$

It is obvious that the solution $\tilde{\phi}_\omega(u)$ of the Jost equation with boundary conditions at $-\infty$ with potential $W(-v)$ constructed in Theorem 5.2 gives rise to a solution ϕ_ω of (5.23) by defining $\phi_\omega(u) := \tilde{\phi}_\omega(-u)$.

With the results of Theorem 5.2 it is now easy to prove Theorem 5.1:

Proof of Theorem 5.1. Let us apply Theorem 5.2 to the potential $V_l(u)$ given by (2.20), which is obviously a smooth function in u. Furthermore, it vanishes on the event horizon $2M$ with the asymptotics $V_l = \mathcal{O}(r - 2M)$. Using the definition of the Regge-Wheeler coordinate u (2.3), this means that $V_l(u)$ decays exponentially as $u \to -\infty$. More precisely, there is a constant $c > 0$ such that

$$|V_l(u)| \le c\, e^{\frac{u}{2M}} \quad \text{for small } u \; .$$

Theorem 5.2 (iii) yields for $u \le u_0 < 0$ a solution $\phi_1(\omega, u)$ of the Jost equation (5.5) with the desired properties. It remains to show that ϕ_1 is also a solution of the Schrödinger equation (4.2) for $u \le u_0$. (Due to the Picard-Lindelöf theorem, this solution of the linear equation can be uniquely extended to $u \in \mathbb{R}$; the resulting function is analytic in ω due to the analytical dependence in ω from the coefficients and initial conditions.) But this follows immediately by differentiating equation (5.20) and using that $V_l \equiv W$ is smooth, so that the whole integrand is at least differentiable with respect to v. We have then proven the existence of $\acute{\phi}_\omega$. For the uniqueness, we show that in our special case every solution of (4.2) with boundary conditions (4.6) is a solution of (5.5). This can be done by integration by parts: For let $\psi_\omega(u)$ be such a solution. Then

$$\begin{aligned} &\int_{-\infty}^u \frac{1}{\omega} \sin(\omega(u-v)) V_l(v) \psi_\omega(v)\, dv = \\ &\qquad = \int_{-\infty}^u \frac{1}{\omega} \sin(\omega(u-v)) (\partial_v^2 + \omega^2) \psi_\omega(v)\, dv = \psi_\omega(u) - e^{i\omega u} \; , \end{aligned}$$

where the remaining terms vanish due to the boundary conditions. Since we know that the solution of the Jost equation is uniquely determined, this must be also the case for the solution of the Schrödinger equation. Thus we have proven part (i).

For the proof of (ii) we refer to Remark 5.4. In contrast to the exponential decay at $-\infty$, the potential $V_l(u)$ has only polynomial decay at $+\infty$. More precisely, according to the definition of u, $V_l(u) = \mathcal{O}(\frac{l(l+1)}{u^2})$ for $l \ge 1$, $V_0(u) = \mathcal{O}(\frac{2M}{u^3})$, respectively, as $u \to \infty$. Thus we can apply the analogs of Theorem 5.2 (i), (ii), respectively. This gives the existence and uniqueness of the solution $\grave{\phi}_\omega$ for the Schrödinger equation with the stated properties. $\square$

When taking the limit $\epsilon \searrow 0$ in Stone's formula (4.11), the behavior of $\grave{\phi}_\omega(u)$ at $\omega = 0$ still causes problems. While in the case $l = 0$ we know from Theorem 5.1 that $\grave{\phi}_\omega$ can be continuously extended there, we do not yet know what happens for $l \neq 0$. The following theorem settles this problem by showing that, after suitable rescaling, the solutions $\grave{\phi}_\omega$ have a well-defined limit at $\omega = 0$:

Theorem 5.5. *For every angular momentum number l, there is a solution ϕ_0 of the Schrödinger equation (4.2) for $\omega = 0$ with the asymptotics*

$$\lim_{u\to\infty} u^l \phi_0(u) = i^l \frac{2^l\sqrt{\pi}}{\Gamma(\frac{1}{2}-l)} = (-i)^l (2l-1)!! \,, \tag{5.24}$$

where

$$(2l-1)!! := \begin{cases} (2l-1)\cdot(2l-3)\cdot\ldots\cdot 3\cdot 1 & , \quad \textit{if } l \neq 0 \\ 1 & , \quad \textit{if } l = 0 \,. \end{cases}$$

This solution can be obtained as a limit of the solutions from Theorem 5.1, in the sense that for all $u \in \mathbb{R}$,

$$\phi_0(u) = \lim_{E\ni\omega\to 0} \omega^l \grave{\phi}_\omega(u) \quad \textit{and} \quad \phi_0'(u) = \lim_{E\ni\omega\to 0} \omega^l \grave{\phi}_\omega'(u) \,. \tag{5.25}$$

Note that the above properties of the solution ϕ_0 really coincide in the case $l = 0$ with that of the solution $\grave{\phi}_0$ already constructed in Theorem 5.1 (*ii*).

For the proof of this theorem we use the same method as in the proof for Theorem 5.1. However, the iteration scheme (5.7) does not work for $l \neq 0$ in the limit $\omega \to 0$, because the integral

$$\phi_0^{(1)}(u) = -\int_u^\infty (u-v) V_l(v)\, dv$$

diverges ($V_l(u)$ decays only quadratically at infinity for $l \neq 0$). We avoid this problem by adding the leading asymptotic term of the potential V_l to the unperturbed equation,

$$\left(-\frac{d^2}{du^2} - \omega^2 + \frac{(l+\frac{1}{2})^2 - \frac{1}{4}}{u^2}\right)\phi_\omega(u) = -W_l(u)\phi_\omega(u) \,. \tag{5.26}$$

Now the perturbation term $W_l(u) = V_l(u) - \frac{l(l+1)}{u^2}$ has the asymptotics $W_l(u) = \mathcal{O}(\frac{\log u}{u^3})$.

Fortunately, the unperturbed differential equation corresponding to (5.26) can still be solved exactly. The solutions can be expressed in terms of Bessel functions. For our further consideration, the two functions

$$h_1(l,\omega,u) = \sqrt{\frac{\pi\omega u}{2}}\, J_{l+\frac{1}{2}}(\omega u) \,, \quad h_2(l,\omega,u) = \sqrt{\frac{\pi\omega u}{2}}\, J_{-l-\frac{1}{2}}(\omega u) \tag{5.27}$$

play an important role. Here the function $J_\nu(u)$ is the Bessel function of the first kind (a good reference for the theory of the Bessel functions is [14]). It solves Bessel's differential equation

$$u^2 y''(u) + u y'(u) + (u^2 - \nu^2) y(u) = 0 \,.$$

In addition, it is an analytic function in ν and u for all values of ν and $u \neq 0$ (if $\operatorname{Re}\nu \geq 0$, it can be analytically extended even to $u = 0$). It has the series expansion

$$J_\nu(u) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(\nu+m+1)} \left(\frac{u}{2}\right)^{\nu+2m} \tag{5.28}$$

and the following asymptotics for $|u| \gg 1$ (cf. [14] 7.21):

$$\begin{aligned} J_\nu(u) \sim \sqrt{\frac{2}{\pi u}} \Bigg[&\cos\left(u - \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right) \cdot \sum_{m=0}^{\infty} \frac{(-1)^m (\nu, 2m)}{(2u)^{2m}} \\ &- \sin\left(u - \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right) \cdot \sum_{m=0}^{\infty} \frac{(-1)^m (\nu, 2m+1)}{(2u)^{2m+1}} \Bigg] , \end{aligned} \tag{5.29}$$

where we have used the notation

$$(\nu, m) := \frac{\Gamma\left(\nu + m + \frac{1}{2}\right)}{m!\Gamma\left(\nu - m + \frac{1}{2}\right)} .$$

Moreover, the derivatives satisfy the recurrence formulas

$$\begin{aligned} uJ'_\nu(u) &= uJ_{\nu-1}(u) - \nu J_\nu(u) \quad \text{and} \\ uJ'_\nu(u) &= \nu J_\nu(u) - uJ_{\nu+1}(u) . \end{aligned}$$

The Wronskian of the functions $J_\nu, J_{-\nu}$ (which both solve the same differential equation, since Bessel's differential equation is symmetric in ν) is given by the formula

$$w\left(J_\nu(u), J_{-\nu}(u)\right) = -\frac{2\sin(\nu\pi)}{\pi u} . \tag{5.30}$$

This yields that these functions form a fundamental system for Bessel's differential equation provided that ν is not an integer.

In our applications we choose $\nu = l + \frac{1}{2}$. Thus the functions $h_1(l, \omega, u)$ and $h_2(l, \omega, u)$ have the following asymptotics,

$$h_1(l,\omega,u) \sim \left\{ \begin{array}{ll} \cos\left(\omega u - (l+1)\frac{\pi}{2}\right) & , \ \text{if } |\omega u| \gg 1 \\ \frac{\sqrt{\pi}}{\Gamma(\frac{3}{2}+l)} \left(\frac{\omega u}{2}\right)^{l+1} & , \ \text{if } |\omega u| \ll 1 \end{array} \right\} \tag{5.31}$$

$$h_2(l,\omega,u) \sim \left\{ \begin{array}{ll} \cos\left(\omega u + l\frac{\pi}{2}\right) & , \ \text{if } |\omega u| \gg 1 \\ \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-l)} \left(\frac{\omega u}{2}\right)^{-l} & , \ \text{if } |\omega u| \ll 1 \end{array} \right\} . \tag{5.32}$$

Furthermore, the formula (5.30) for the Wronskian simplifies to

$$w\big(h_1(l,\omega,u), h_2(l,\omega,u)\big) = (-1)^{l+1}\omega , \qquad \text{if } l \text{ is an integer} , \tag{5.33}$$

and this yields that in the case $\omega \neq 0$ the solutions h_1, h_2 form a fundamental system.

Thus for $\omega \neq 0$ we take as the Green's function for the operator on the left hand side of (5.26) the standard formula

$$S_\omega(u,v) = \Theta(v-u)\frac{1}{w(h_1,h_2)}\big(h_1(v)h_2(u) - h_1(u)h_2(v)\big) \ , \tag{5.34}$$

where $h_{1/2}(u) \equiv h_{1/2}(l,\omega,u)$ and Θ denotes the Heaviside function defined by $\Theta(x) = 1$ if $x \geq 0$ and $\Theta(x) = 0$ otherwise. Note that S_ω is also well-defined in the limit $\omega \to 0$. For this we use the asymptotics and the value of the Wronskian and get for very small ω,

$$\begin{aligned}
&\lim_{\omega\to 0} S_\omega(u,v) = \lim_{\omega\to 0}\frac{(-1)^{l+1}}{\omega}\cdot\frac{\pi\omega}{2\Gamma\left(\frac{3}{2}+l\right)\Gamma\left(\frac{1}{2}-l\right)}\left(v^{l+1}u^{-l} - u^{l+1}v^{-l}\right)\\
&= \frac{(-1)^{l+1}\pi}{2\left(\frac{1}{2}+l\right)\Gamma\left(\frac{1}{2}+l\right)\Gamma\left(\frac{1}{2}-l\right)}\left(v^{l+1}u^{-l} - u^{l+1}v^{-l}\right) =\\
&= \frac{(-1)^{l+1}\pi\cos(\pi l)}{(2l+1)\pi}\left(v^{l+1}u^{-l} - u^{l+1}v^{-l}\right) =\\
&= -\frac{1}{2l+1}\left(v^{l+1}u^{-l} - u^{l+1}v^{-l}\right) \ ,
\end{aligned}$$

where we have used some elementary properties of the Gamma function. This also shows that the Green's function converges to the Green's function $S_0(u,v)$ given by the above formula for the solutions u^{l+1}, u^{-l} of the unperturbed differential operator on the left hand side of (5.26) for $\omega = 0$.
We now proceed with the perturbation series ansatz

$$\phi_\omega(u) = \sum_{m=0}^{\infty}\phi_\omega^{(m)}(u) \ , \tag{5.35}$$

which, as at the beginning of this section, leads to the iteration scheme

$$\phi_\omega^{(m+1)}(u) = -\int_u^\infty S_\omega(u,v)W_l(v)\phi_\omega^{(m)}(v)\,dv \ . \tag{5.36}$$

As initial function we take

$$\phi_\omega^{(0)}(u) = \omega^l e^{-i(l+1)\frac{\pi}{2}}\sqrt{\frac{\pi\omega u}{2}}H^{(2)}_{l+\frac{1}{2}}(\omega u) \ ,$$

where $H^{(2)}_\nu$ is another solution of Bessels equation (called Bessel function of the third kind or second Hankel function). It is related to J_ν by

$$H^{(2)}_\nu(u) = \frac{J_{-\nu}(u) - e^{\nu\pi i}J_\nu(u)}{-i\sin(\nu\pi)} \ ,$$

and has for large $|u|$ the asymptotics

$$H^{(2)}_\nu(u) \sim \sqrt{\frac{2}{\pi u}}e^{-i\left(u-\frac{1}{2}\pi\left(\nu+\frac{1}{2}\right)\right)}\sum_{m=0}^{\infty}\frac{(\nu,m)}{(2iu)^m} \ . \tag{5.37}$$

Thus our intial function $\phi_\omega^{(0)}(u)$ solves the unperturbed equation, and we have the relation

$$\phi_\omega^{(0)}(u) = \omega^l\left((-i)^{l+1}h_1(l,\omega,u) + i^l h_2(l,\omega,u)\right) \tag{5.38}$$

together with the asymptotics

$$\phi_\omega^{(0)}(u) = \omega^l e^{-i\omega u} \left(1 + \mathcal{O}\left(\frac{1}{u}\right)\right) \quad , \text{ if } |u| \gg 1 \; . \tag{5.39}$$

Moreover, the function $\phi_\omega^{(0)}$ converges in the limit $\omega \to 0$ pointwise for all $u \geq u_0 > 0$:

$$\lim_{\omega \to 0} \phi_\omega^{(0)}(u) = i^l \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2} - l\right)} \left(\frac{u}{2}\right)^{-l} \; . \tag{5.40}$$

Since we are interested in statements for $\omega = 0$, it is convenient in what follows to restrict ω to the domain

$$F := \{\omega \in \mathbb{C} \,\big|\, \operatorname{Im}\omega \leq 0, |\omega| \leq 1\} \; .$$

The following lemma yields that our perturbation series (5.35) is well-defined.

Lemma 5.6. *For every $u_0 > 0$, the iteration scheme (5.35),(5.36),(5.38) converges locally uniformly for all $u \geq u_0$ and $\omega \in F$. In particular, the functions $\phi_\omega(u)$ are for fixed u a continuous family in $\omega \in F$. They satisfy the integral equation*

$$\phi_\omega(u) = \phi_\omega^{(0)}(u) - \int_u^\infty S_\omega(u,v) W_l(v) \phi_\omega(v)\, dv \; . \tag{5.41}$$

Proof. In order to prove the lemma, we need to derive good bounds for the initial function $\phi_\omega^{(0)}(u)$ as well as for the Green's function $S_\omega(u,v)$. To this end, we exploit the asymptotics of h_1, h_2. We thus obtain the bound

$$\frac{1}{C_1} \leq |\phi_\omega^{(0)}(u)| e^{-u \operatorname{Im}\omega} \left(\frac{u}{1+|\omega|u}\right)^l \leq C_1 \; . \tag{5.42}$$

Likewise, for the Green's function we have (note that $v \geq u > 0$),

$$|S_\omega(u,v)| \;\leq\; C_2 \left(\frac{u}{1+|\omega|u}\right)^{-l} \left(\frac{v}{1+|\omega|v}\right)^{l+1} \quad , \quad \text{if } |\omega v| \ll 1$$

and

$$|S_\omega(u,v)| \;\leq\; C_3 \frac{v}{1+|\omega|v} e^{v|\operatorname{Im}\omega| + u \operatorname{Im}\omega} \; , \qquad \text{if } |\omega u| \gg 1.$$

The last inequality follows from the asymptotics

$$|S_\omega(u,v)| \;\sim\; \left|\frac{1}{\omega} \sin(\omega(u-v))\right|, \quad \text{if } |\omega u| \gg 1 \; ,$$

in the same way as the second inequality of Lemma 5.3. Combining these cases we find a constant such that

$$|S_\omega(u,v)| \leq C_4 \left(\frac{u}{1+|\omega|u}\right)^{-l} \left(\frac{v}{1+|\omega|v}\right)^{l+1} e^{v|\operatorname{Im}\omega| + u \operatorname{Im}\omega} \; . \tag{5.43}$$

Hence defining the function Q_ω by

$$Q_\omega(u) := C_4 \int_u^\infty \frac{v}{1+|\omega|v} \, |W_l(v)| \, dv \; , \tag{5.44}$$

which is well-defined for all $\omega \in F$ and $u \geq u_0 > 0$ due to the asymptotic of W_l, it is straightforward to show inductively (cf. proof of Theorem 5.2) that for all $m \in \mathbb{N}$

$$\left|\phi_\omega^{(m)}(u)\right| \leq C_1 \left(\frac{u}{1+|\omega| u}\right)^{-l} e^{u \operatorname{Im} \omega} \frac{Q_\omega(u)^m}{m!} \ . \tag{5.45}$$

Now we proceed exactly as in the proof of Theorem 5.2, where the inequality (5.45) can be considered as the analogue of (5.16). It follows that the series (5.35) converges locally uniformly in ω and u and satisfies the integral equation (5.41). Furthermore, one shows inductively applying Lebesgue's dominated convergence theorem, that for fixed u each $\phi_\omega^{(m)}(u)$ depends continuously of $\omega \in F$. It follows that the same is true for the series due to local uniform convergence. $\square$

We are now ready to prove Theorem 5.5:

Proof of Theorem 5.5. According to Lemma 5.6, our perturbation series (5.35) satisfies the integral equation (5.41). Using the recurrence formulas for the derivatives of $J_\nu(u)$, one obtains

$$\begin{aligned} h_1'(l,\omega,u) &= -\frac{l}{u} h_1(l,\omega,u) + \omega h_1(l-1,\omega,u), \\ h_2'(l,\omega,u) &= -\frac{l}{u} h_2(l,\omega,u) - \omega h_2(l-1,\omega,u) \ , \text{ respectively.} \end{aligned}$$

This allow us to estimate the behavior of $\partial_u S_\omega(u,v)$. Exactly as for $S_\omega(u,v)$, we obtain the following asymptotic formulas,

$$|\partial_u S_\omega(u,v)| \leq C_5 \left(\frac{u}{1+|\omega| u}\right)^{-l-1} \left(\frac{v}{1+|\omega| v}\right)^{l+1} e^{v|\operatorname{Im}\omega| + u \operatorname{Im}\omega} \ .$$

Following the same arguments of the proofs of Theorems 5.1 and 5.2, and combining them with the above estimates and asymptotic formulas we now have the following results:

1) One can differentiate $\phi_\omega(u)$ with respect to u, and $\phi_\omega'(u)$ is given by

$$\phi_\omega'(u) = \left(\phi_\omega^{(0)}\right)'(u) - \int_u^\infty \partial_u S_\omega(u,v) W_l(v) \phi_\omega(v)\, dv \ .$$

In particular, Lebesgue's dominated convergence theorem yields that for fixed u, $\phi_\omega'(u)$ is continuous in $\omega \in F$.

2) $\phi_\omega(u)$ and $\phi_\omega'(u)$ obey the following estimates,

$$|\phi_\omega(u) - \phi_\omega^{(0)}(u)| \leq C_1 \left(\frac{u}{1+|\omega| u}\right)^{-l} e^{u \operatorname{Im}\omega} \left(e^{Q_\omega(u)} - 1\right)$$

$$\left|\phi_\omega'(u) - \left(\phi_\omega^{(0)}\right)'(u)\right| \leq C_5 \left(\frac{u}{1+|\omega| u}\right)^{-l-1} e^{Q_\omega(u_0)} e^{u \operatorname{Im}\omega} \int_u^\infty v |W_l(v)| dv \ .$$

Thus $\phi_\omega(u) \sim \omega^l e^{-i\omega u}$ and $\phi_\omega'(u) \sim -i\omega^{l+1} e^{-i\omega u}$ as $u \to \infty$.

3) Differentiating $\phi_\omega(u)$ twice with respect to u shows that $\phi_\omega(u)$ is a solution of the Schrödinger equation (4.2) for all $u \geq u_0$. Furthermore, from the asymptotics at infinity combined with the uniqueness statement in Theorem 5.1, we know that

$$\phi_\omega(u) = \omega^l \grave{\phi}_\omega(u) \quad , \text{ if } \omega \neq 0, u \geq u_0 \; . \tag{5.46}$$

Obviously, this extends to all $u \in \mathbb{R}$.

Thus we have proven the continuity statement (5.25) for all $u \geq u_0$. On the other hand, we know from the Picard-Lindelöf theorem that for u on compact intervals, the solutions depend continuously on ω. This yields (5.25) for all $u \in \mathbb{R}$.

Finally, the asymptotics (5.24) is a simple consequence of (5.40). $\square$

6 An Integral Spectral Representation

In the previous section we derived some regularity results for the solutions $\acute{\phi}_\omega$ and $\grave{\phi}_\omega$. We already know (cf. Section 4) that these solutions are a system of fundamental solutions of the Schrödinger equation (4.2) in the cases $\operatorname{Im}\omega < 0$ and $\operatorname{Im}\omega > 0$, respectively. Thus the Wronskian $w(\acute{\phi}_\omega, \grave{\phi}_\omega)$ is non-vanishing in these regions, which implies that the integral kernel $k_\omega(u, v)$ of the resolvent is well defined. Since our next goal is to get the limit in (4.11), we prove in the next lemma that the continuous extension of the solutions $\acute{\phi}_\omega, \grave{\phi}_\omega$ to the real axis again yields a system of fundamental solutions. More precisely,

Lemma 6.1. *The Wronskian $w(\acute{\phi}_\omega, \grave{\phi}_\omega)$ does not vanish for $\omega \in \mathbb{R} \setminus \{0\}$. In particular, $\acute{\phi}_\omega, \grave{\phi}_\omega$ are fundamental solutions for the Schrödinger equation (4.2). In addition, this remains true for the solutions $\acute{\phi}_0$ and ϕ_0 in the case $\omega = 0$.*

Proof. Let us begin with the statement for $\acute{\phi}_0, \phi_0$:
For $\omega = 0$, the solutions $\acute{\phi}_0(u), \phi_0(u)$ have the asymptotics

$$\lim_{u \to -\infty} \acute{\phi}_0(u) = 1 \quad \text{and} \quad \lim_{u \to \infty} u^l \phi_0(u) = (-i)^l (2l-1)!! \; .$$

Looking at the construction of these solutions, one sees that $\acute{\phi}_0$ is a real solution, while ϕ_0 is either purely real or imaginary (depending on the value of l). The Schrödinger equation for $\omega = 0$ reduces to $\phi''(u) = V_l(u)\phi(u)$ with a everywhere positive potential V_l. Hence, exploiting the special asymptotics, the solution $\acute{\phi}_0$ is convex and $\operatorname{Re}\phi_0$ ($\operatorname{Im}\phi_0$, respectively) is either convex or concave depending on l. In any case, we see that $\acute{\phi}_0$ and ϕ_0 are linearly independent, and thus $w(\acute{\phi}_0, \phi_0) \neq 0$.

In order to prove the main part of the Lemma, we consider a complex solution $z = z_1 + i z_2$ of the Schrödinger equation, where $\{z_1, z_2\}$ is a fundamental system of real solutions, especially $w(z_1, z_2) \equiv c \neq 0$. Setting $y = \frac{z'}{z}$, a simple computation shows that

$$\operatorname{Im} y = \frac{w(z_1, z_2)}{|z|^2} \; ,$$

where the right hand side is well defined because $w(z_1, z_2) \neq 0$ implies that $|z| \neq 0$ everywhere. As a consequence, we have $\operatorname{Im} y \neq 0$ everywhere. Thus it follows that for all u either $\operatorname{Im} y(u) > 0$ or < 0, due to the continuity of the solution z in u.

Applying this result to the solutions $\acute{\phi}_\omega$ and $\grave{\phi}_\omega$, respectively, and exploiting their asymptotics, one sees that $\operatorname{Im} \acute{y}_\omega(u)$ and $\operatorname{Im} \grave{y}_\omega(u)$ have different signs for all u. Therefore,

$$w(\acute{\phi}_\omega, \grave{\phi}_\omega) = \acute{\phi}_\omega(u)\grave{\phi}'_\omega(u) - \acute{\phi}'_\omega(u)\grave{\phi}_\omega(u) = \acute{\phi}_\omega(u)\grave{\phi}_\omega(u)\big(\grave{y}_\omega(u) - \acute{y}_\omega(u)\big) \neq 0 \,.$$

□

As a consequence we have the following

Corollary 6.2. *The function $s_\omega(u,v)$ given by (4.9) is continuous in (ω, u, v) for $\omega \in \{\operatorname{Im}\omega \leq 0\}$, $(u,v) \in \mathbb{R}^2$.*

Proof. We already know that for fixed $u_0 < 0$, $\acute{\phi}_\omega(u_0)$ is continuous in ω on $\{\operatorname{Im}\omega \leq 0\}$. Thus as solutions of the linear differential equation (4.2), which depends analytically on ω and smooth on u, the family $\acute{\phi}_\omega(u)$ is (at least) continuous in (ω, u) in the region $\{\operatorname{Im}\omega \leq 0\} \times \mathbb{R}$. Analogously this holds for $\omega^l \grave{\phi}_\omega(u)$ according to Theorems 5.1 and 5.5. Since $s_\omega(u,v)$ is invariant if we substitute $\omega^l \grave{\phi}_\omega(u)$ for $\grave{\phi}_\omega(u)$, the preceding lemma yields the claim. □

Note that the corollary is also true if ω is in the upper half plane. The essential statement in this corollary is that one can extend $s_\omega(u,v)$ continuously in ω up to the real axis.

¿From the definitions (5.1) and (5.2), we have for $\omega \in \{\operatorname{Im}\omega \neq 0\}$ the relations

$$\overline{s_\omega(u,v)} = s_{\overline{\omega}}(u,v) \,, \quad \text{hence} \quad \overline{k_\omega(u,v)} = k_{\overline{\omega}}(u,v) \,.$$

This allows us to simplify the expression (4.11). Evaluating for fixed u the right hand side of (4.11) we obtain for any $\Psi \in \mathscr{H}$ as well as for any bounded interval $[a,b] \subseteq \mathbb{R}$

$$\lim_{\epsilon \searrow 0} -\frac{1}{\pi} \int_a^b \left(\int_{\mathbb{R}} \operatorname{Im}\big(k_{\omega - i\epsilon}(u,v)\big)\Psi(v)dv \right) d\omega \,.$$

According to the above corollary, we know that $\operatorname{Im} k_\omega(u,v)$ is continuous in (ω, u, v) for $\omega \in \{\operatorname{Im}\omega \leq 0\}$, $(u,v) \in \mathbb{R}^2$. Thus, if we restrict Ψ to the dense set $C_0^\infty(\mathbb{R})^2$, we integrate a continuous integrand over a compact interval. Hence, considering the limit as a pointwise limit for any u, we may interchange the limit and integration. Thus for any $\Psi \in C_0^\infty(\mathbb{R})^2$, $[a,b] \subset \mathbb{R}$ bounded and u the right hand side of (4.11) converges pointwise to

$$-\frac{1}{\pi} \int_a^b \left(\int_{\operatorname{supp}\psi} \operatorname{Im}\big(k_\omega(u,v)\big)\psi(v)dv \right) d\omega \,.$$

Hence, together with the norm convergence in (4.11), the spectral projections of H are for every u described by the formula

$$\frac{1}{2}\big(P_{[a,b]} + P_{(a,b)}\big)\Psi(u) = -\frac{1}{\pi} \int_a^b \left(\int_{\operatorname{supp}\psi} \operatorname{Im}\big(k_\omega(u,v)\big)\psi(v)dv \right) d\omega \,. \tag{6.1}$$

In particular, this representation yields that $P_{[a,b]} \equiv P_{(a,b)}$.

As an immediate consequence we have the following

Corollary 6.3. *The spectrum $\sigma(H)$ of the operator H is absolutely continuous, i.e. $\sigma(H) \equiv \sigma_{ac}(H)$.*

Proof. The corollary is equivalent to the statement that the spectral measure $\langle \Psi, dP_\omega \Psi \rangle$ of any $\Psi \in \mathscr{H}$ is absolutely continuous. This is clear by (6.1) for any $\Psi \in C_0^\infty(\mathbb{R})^2$. But since this subset is dense, this also holds on the whole Hilbert space $\mathscr{H}$. $\square$

Next we want to write the integrand in (6.1), i.e. $\int_{\mathrm{supp}\Psi} \ldots dv$, in a more compact way. We first note that for real ω the complex conjugates of $\acute{\phi}_\omega$ and $\grave{\phi}_\omega$ are again solutions of (4.2). Hence, for any $\omega \in \mathbb{R} \setminus \{0\}$ the pair $\left\{ \acute{\phi}_\omega, \overline{\acute{\phi}_\omega} \right\}$ forms a fundamental system for this equation due to the boundary conditions. Thus we can express $\grave{\phi}_\omega$ as a linear combination of $\acute{\phi}_\omega$ and $\overline{\acute{\phi}_\omega}$,

$$\grave{\phi}_\omega(u) = \lambda(\omega) \acute{\phi}_\omega(u) + \mu(\omega) \overline{\acute{\phi}_\omega(u)} \quad (\omega \in \mathbb{R} \setminus \{0\}) \,,$$

where λ and μ are referred to as *transmission coefficients*. The Wronskian of $\acute{\phi}_\omega$ and $\grave{\phi}_\omega$ can be expressed by

$$w\left(\acute{\phi}_\omega, \grave{\phi}_\omega \right) = \mu(\omega) w \left(\acute{\phi}_\omega, \overline{\acute{\phi}_\omega} \right) = -2i\omega\mu(\omega) \,,$$

where in the last step we used the asymptotics (4.6). Moreover, we introduce the real fundamental solutions

$$\phi^1_\omega(u) = \operatorname{Re} \acute{\phi}_\omega(u) \,, \quad \phi^2_\omega(u) = \operatorname{Im} \acute{\phi}_\omega(u)$$

and denote the corresponding eigenvectors of the Hamiltonian H by $\Phi^a_\omega(u) = (\phi^a_\omega(u), \omega\phi^a_\omega(u))^T$.

Using the above definitions, a short calculation shows that for $\omega \neq 0$ we can express the imaginary part of the Green's function $s_\omega(u,v)$ by

$$\operatorname{Im} s_\omega(u,v) = -\frac{1}{2\omega} \sum_{a,b=1}^{2} t_{ab}(\omega) \phi^a_\omega(u) \phi^b_\omega(v) \,, \tag{6.2}$$

where the coefficients $t_{ab}(\omega)$ are given by

$$\begin{aligned} t_{11}(\omega) &= 1 + \operatorname{Re}\left(\frac{\lambda}{\mu}(\omega) \right), \qquad t_{12}(\omega) = t_{21}(\omega) = -\operatorname{Im}\left(\frac{\lambda}{\mu}(\omega) \right) \quad , \\ t_{22}(\omega) &= 1 - \operatorname{Re}\left(\frac{\lambda}{\mu}(\omega) \right) \,. \end{aligned} \tag{6.3}$$

Since we know that $\operatorname{Im} s_\omega(u,v)$ is continuous for $\omega \in \mathbb{R}$ and the expression (6.2) holds for all $\omega \in \mathbb{R} \setminus \{0\}$, it extends to $\omega = 0$. With (6.2), the integrand in (6.1) can be written as

$$\begin{aligned} &-\frac{1}{2\omega} \Big(\int_{\mathrm{supp}\Psi} \sum_{a,b=1}^{2} t_{ab}(\omega) \phi^a_\omega(u) \phi^b_\omega(v) \begin{pmatrix} \omega & 1 \\ \omega^2 & \omega \end{pmatrix} \Psi(v) dv \Big) = \\ &= -\frac{1}{2\omega^2} \sum_{a,b=1}^{2} t_{ab}(\omega) \Phi^a_\omega(u) \Big(\int_{\mathrm{supp}\Psi} \omega^2 \phi^b_\omega(v) \psi_1(v) + \omega \phi^b_\omega(v) \psi_2(v) \, dv \Big) \,, \end{aligned}$$

where the ψ_i denote the two components of Ψ.

Since $\phi^b_\omega(u)$ solves the Schrödinger equation (4.2), it satisfies the relation $\big(-\partial_v^2 + V_l(v)\big)\phi^b_\omega(v) = \omega^2\phi^b_\omega(v)$. Using this and integrating by parts, this simplifies to

$$-\frac{1}{2\omega^2}\sum_{a,b=1}^{2} t_{ab}(\omega)\Phi^a_\omega(u)\big\langle\Phi^b_\omega,\Psi\big\rangle\ . \tag{6.4}$$

(Note that in this case the energy scalar product of Φ^b_ω and Ψ is well defined, because Ψ has compact support. Whereas in general this does not exist for arbitrary $\Psi \in \mathscr{H}$, due to the fact that $\Phi^b_\omega \notin \mathscr{H}$.)

With (6.4), we now obtain a more compact representation for the spectral projections. Moreover, we can use (6.4) to express the solution operators e^{-itH}.

Proposition 6.4. *Consider the Cauchy Problem (2.23) for compactly supported smooth initial data $\Psi_0 \in C_0^\infty(\mathbb{R})^2$. Then the solution has the integral representation*

$$\begin{aligned}\Psi(t) &= e^{-itH}\Psi_0 = \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega t}\frac{1}{\omega^2}\sum_{a,b=1}^{2} t_{ab}(\omega)\Phi^a_\omega\left\langle\Phi^b_\omega,\Psi_0\right\rangle d\omega\ .\end{aligned} \tag{6.5}$$

Here the integral converges in norm in the Hilbert space $\mathscr{H}$.

Proof. We use the following variation of Stone's formula to obtain for any bounded interval $(c,d) \subseteq \mathbb{R}$

$$\begin{aligned}&\frac{1}{2}e^{-itH}\left(P_{[c,d]} + P_{(c,d)}\right)\Psi \\ &\qquad = \lim_{\epsilon\searrow 0}\int_c^d e^{-i\omega t}\left[(H-\omega-i\epsilon)^{-1} - (H-\omega+i\epsilon)^{-1}\right]\Psi\, d\omega\ ,\end{aligned}$$

where the limit is with respect to the norm of $\mathscr{H}$. Since we know that $P_{[c,d]} \equiv P_{(c,d)}$, it follows that this expression is equal to $e^{-itH}P_{(c,d)}\Psi$. Using the explicit formula for the resolvent, for every $u \in \mathbb{R}$ the right hand side is equal to

$$\lim_{\epsilon\searrow 0} -\frac{1}{\pi}\int_c^d e^{-i\omega t}\left(\int_{\mathbb{R}} \mathrm{Im}\big(k_{\omega-i\epsilon}(u,v)\big)\Psi(v)dv\right)d\omega\ . \tag{6.6}$$

Due to the continuity of the imaginary part of the kernel $k_\omega(u,v)$, we may take for $\Psi_0 \in C_0^\infty(\mathbb{R})^2$ and (c,d) bounded the pointwise limit for any $u \in \mathbb{R}$. Hence, using (6.4) we can simplify (6.6) to

$$\frac{1}{2\pi}\int_c^d e^{-i\omega t}\frac{1}{\omega^2}\sum_{a,b=1}^{2} t_{ab}(\omega)\Phi^a_\omega(u)\big\langle\Phi^b_\omega,\Psi_0\big\rangle\, d\omega\ ,$$

and together with the norm convergence it follows that this term is equal to $e^{-itH}P_{(c,d)}\Psi_0(u)$. Using the abstract spectral theorem and that e^{-itH} is a unitary operator, it is clear that $e^{-itH}P_{(-n,n)}\Psi_0 \to e^{-itH}\Psi_0$ in norm as $n \to \infty$. $\square$

This proposition extends to the following theorem.

Theorem 6.5. *For any fixed $u \in \mathbb{R}$ the integrand in the representation (6.5) is in $L^1(\mathbb{R}, \mathbb{C}^2)$ as a function of ω. In particular, the representation (6.5) of the solutions holds pointwise for every $u \in \mathbb{R}$, i.e.*

$$\Psi(t,u) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega t} \frac{1}{\omega^2} \sum_{a,b=1}^{2} t_{ab}(\omega) \Phi_\omega^a(u) \left\langle \Phi_\omega^b, \Psi_0 \right\rangle d\omega \,. \tag{6.7}$$

Moreover, for u fixed, the function $\Psi(t,u)$ vanishes as $t \to \infty$.

Proof. Since we know that the integrand is continuous in ω, it is in $L^1([a,b], \mathbb{C}^2)$ for any bounded interval $[a,b]$. Thus it remains to analyze the integrand for large $|\omega|$.

To this end, we must investigate the asymptotic behavior of the fundamental solutions $\acute{\phi}_\omega$ and $\grave{\phi}_\omega$ in ω. We constructed these solutions with the iteration scheme (5.7) as solutions of the Jost equation. For the proof of this, the estimate (5.14) played an essential role. Since in this case we consider real ω with $|\omega| \gg 1$, we can use the simple estimate $\left|\frac{1}{\omega}\sin(\omega(u-v))\right| \leq \frac{1}{|\omega|}$ instead of (5.14), which now holds for every $u, v \in \mathbb{R}$. Thus, proceeding exactly in the same way as in the proof of Theorem 5.2, we now obtain the following estimates for the several terms in the series expansion (5.6)

$$\left|\phi_\omega^{(k)}(u)\right| \leq \frac{1}{k!} \hat{P}_\omega(u)^k \,, \quad \text{where } \hat{P}_\omega(u) := \int_{-\infty}^{u} \frac{1}{|\omega|} V_l(v) dv \,,$$

for any $k \in \mathbb{N}$ and $u \in \mathbb{R}$. Thus the solution $\acute{\phi}_\omega(u)$ ($\omega \neq 0$) is given *for all* $u \in \mathbb{R}$ by this series expansion and obeys the uniform bound

$$\left|\acute{\phi}_\omega(u) - e^{i\omega u}\right| \;\leq\; e^{\hat{P}_\omega(u)} - 1 \;\leq\; e^{\frac{1}{|\omega|}\|V_l\|_{L^1}} - 1 \,,$$

since $V_l \in L^1(\mathbb{R})$. In particular,

$$\left|\acute{\phi}_\omega(u)\right| \leq 1 + \mathcal{O}\left(\frac{1}{|\omega|}\right) \quad \text{for } |\omega| \gg 1 \,. \tag{6.8}$$

Next we investigate the ω-dependence of $\left\langle \Phi_\omega^b, \Psi_0 \right\rangle$. We integrate by parts to obtain

$$\left\langle \Phi_\omega^b, \Psi_0 \right\rangle = \int\limits_{\mathrm{supp}\Psi_0} \phi_\omega^b(v) \big(\omega \psi_2(v) - \psi_1(v)'' + V_l(v)\psi_1(v)\big) dv \,,$$

where $\Psi_0 = (\psi_1, \psi_2)^T$ (note that the boundary terms drop out, because $\Psi_0 \in C_0^\infty(\mathbb{R})^2$). Since $\phi_\omega^b(u)$ is a solution of the Schrödinger equation (4.2), we substitute $\frac{1}{\omega^2}\left(-\phi_\omega^b(u)'' + V_l(u)\phi_\omega^b(u)\right)$ for $\phi_\omega^b(u)$ and integrate by parts twice,

$$= \frac{1}{\omega^2} \int\limits_{\mathrm{supp}\Psi_0} \phi_\omega^b \big(-(\omega\psi_2 - \psi_1'' + V_l\psi_1)'' + V_l(\omega\psi_2 - \psi_1'' + V_l\psi_1)\big) dv \,.$$

We can now iterate this procedure as often as we like due to the fact that $\Psi_0 \in C_0^\infty(\mathbb{R})^2$ and $V_l \in C^\infty(\mathbb{R})$. Thus using the bound (6.8), we obtain arbitrary polynomial decay in ω for $\left\langle \Phi_\omega^b, \Psi_0 \right\rangle$.

Thus it remains to control the coefficients $t_{ab}(\omega)$ for large $|\omega|$. According to the definition of the transmission coefficients $\lambda(\omega)$ and $\mu(\omega)$, they satisfy the following relations,

$$w(\grave{\phi}_\omega, \acute{\phi}_\omega) = 2i\mu(\omega) \quad \text{and} \quad w(\grave{\phi}_\omega, \overline{\acute{\phi}_\omega}) = 2i\lambda(\omega)\ .$$

In order to calculate the Wronskians, we substitute the Jost integral equations (5.5),(5.23) for $\acute{\phi}_\omega$ and $\grave{\phi}_\omega$, respectively, as well as the corresponding integral equations for the derivatives (for instance (5.20) in the case $\acute{\phi}_\omega$) and obtain immediately

$$\mu(\omega) = 1 + \mathcal{O}\left(\frac{1}{\omega}\right)\ , \quad \lambda(\omega) = \mathcal{O}\left(\frac{1}{\omega}\right).$$

Hence the coefficients $t_{ab}(\omega)$ remain (at least) bounded, according to their definition (6.3).

We conclude that the integrand in (6.7) is in $L^1(\mathbb{R}, \mathbb{C}^2)$ as a function of ω. Thus the right hand side in the integral representation (6.5) converges also pointwise and, together with the norm convergence, (6.7) follows.

Since for u fixed, $\Psi(t,u)$ is a Fourier transform of a L^1 -function, the Riemann-Lebesgue lemma applies. Hence $\Psi(t,u)$ vanishes as $t \to \infty$. ◻

In the next step we extend this proposition to the Cauchy problem (2.6).

Theorem 6.6. *Consider the Cauchy problem (2.6) for smooth and compactly supported initial data. Then there exists a unique smooth solution, which is compactly supported for all times.*

Moreover, decomposing the initial data Ψ_0 into spherical harmonics, the solution has the representation

$$\Psi(t,u,\vartheta,\varphi) = \sum_{l=0}^{\infty} \sum_{|m|\le l} e^{-itH_l}\, \Psi_0^{lm}(u) Y_{lm}(\vartheta,\varphi)\ . \tag{6.9}$$

Proof. For the existence and uniqueness of such a solution we apply the theory of linear symmetric hyperbolic systems (cf. [13]). Since the equation in (2.6) is defined on $\mathbb{R} \times \mathbb{R} \times S^2$ we have to work in local coordinates for S^2. We demonstrate the idea in the chart $\big(U, (\vartheta,\varphi)\big)$, where U is an open, relative compact subset of S^2 such that (ϑ,φ) are well defined on $\overline{U}$.

Letting $\Phi = (\partial_t\psi, \partial_u\psi, \partial_{\cos\vartheta}\psi, \partial_\varphi\psi, \psi)^T$ we can write the equation as the first order system

$$A^0\partial_t\Phi + A^1\partial_u\Phi + A^2\partial_{\cos\vartheta}\Phi + A^3\partial_\varphi\Phi + B\Phi = 0\ ,$$

where the matrices A^i, B are given by

$$A^0 = \operatorname{diag}\left(1, 1, \left(1 - \frac{2M}{r}\right)\frac{1}{r^2}\sin^2, \left(1 - \frac{2M}{r}\right)\frac{1}{r^2}\frac{1}{\sin^2\vartheta}, 1\right)\ ,$$

$$A^1 = (a^1_{ij})\ , \quad \text{with} \quad a^1_{12} = a^1_{21} = -1\ ,$$

$$A^2 = (a^2_{ij})\ , \quad \text{with} \quad a^2_{13} = a^2_{31} = -\left(1 - \frac{2M}{r}\right)\frac{1}{r^2}\sin^2\vartheta\ ,$$

$$A^3 = (a^3_{ij}) \ , \quad \text{with} \quad a^3_{14} = a^3_{41} = -\left(1 - \frac{2M}{r}\right) \frac{1}{r^2} \frac{1}{\sin^2 \vartheta} \ ,$$

$$B = (b_{ij}) \ , \quad \text{with} \quad b_{13} = \left(1 - \frac{2M}{r}\right) \frac{1}{r^2} 2\cos\vartheta \ ,$$

$$b_{15} = \left(1 - \frac{2M}{r}\right) \frac{2M}{r^3}, \ b_{51} = -1 \ ,$$

and all other coefficients vanish. By multiplying this system with $\left(1 - \frac{2M}{r}\right)^{-1} r^2$, we obtain a linear symmetric hyperbolic system on $\mathbb{R} \times \mathbb{R} \times U$ in the sense that the A^i are symmetric and A^0 is uniformly positive definite on $\mathbb{R} \times \mathbb{R} \times U$. Since the initial data Ψ_0 has compact support, we can restrict the system to $\mathbb{R} \times V \times U$, where $V \subseteq \mathbb{R}$ is an open, relative compact set with $\text{supp}\Psi_0 \subseteq V$. Considering the system on this domain, the matrices A^i, B remain uniformly bounded. Since we can cover S^2 by a finite number of such charts, the theory of symmetric hyperbolic systems yields an $\epsilon_1 > 0$ such that there is unique and smooth solution $\psi(t, u, x)$ for all $t < \epsilon_1$ on $\mathbb{R} \times V \times S^2$ with initial data Ψ_0.

Moreover, this solution has finite propagation speed, which is independent of u (this is physically clear from causality; it follows mathematically by considering lens-shaped regions for our symmetric hyperbolic systems). Thus there exists an $\epsilon > 0$ (possibly smaller than ϵ_1) such that the solution $\psi(t, u, x)$ has compact support in $V \times S^2$ for all times $t \leq \epsilon$. Iterating this argument for the Cauchy problem with initial data $(\psi(\epsilon, u, x), i\partial_t \psi(\epsilon, u, x))$ (and choosing $V \subseteq \mathbb{R}$ sufficiently large), we get a unique and smooth solution $\psi(t, u, x)$ with compact support for all $t \leq 2\epsilon$ and so forth. Thus we have proven the existence of a global solution $\psi(t, u, x) \in C^\infty(\mathbb{R} \times \mathbb{R} \times S^2)$ which is unique and compactly supported for all times t.

In order to prove the representation (6.9), we consider the restriction of the solution $\Psi(t, u, x) = (\psi(t, u, x), i\partial_t \psi(t, u, x))^T$ of the Cauchy problem (2.12) in Hamiltonian form to fixed modes l, m

$$\Psi^{lm}(t, u) Y_{lm}(\vartheta, \varphi) = \langle \Psi(t, u), Y_{lm} \rangle_{L^2(S^2)} Y_{lm}(\vartheta, \varphi) \ .$$

Then $\Psi^{lm}(t, u) Y_{lm}(\vartheta, \varphi)$ is a solution of (2.12) with initial data $\Psi^{lm}_0(u) Y_{lm}(\vartheta, \varphi)$, which is smooth and compactly supported. Thus $\Psi^{lm}(t, u)$ is a solution of the Cauchy problem (2.23), and due to the uniqueness of such solutions

$$\Psi^{lm}(t, u) = e^{-itH_l} \Psi^{lm}_0(u) \ .$$

Now the uniqueness of the decomposition into spherical harmonics yields (6.9). □

We are now ready to prove our main theorem.

Proof of Theorem 1.1. The existence and uniqueness of solutions of the Cauchy problem follow directly from Theorem 6.6 after the substitution $\phi = r\psi$. Thus it remains to show the pointwise decay.

The conserved energy for solutions which are compactly supported for all times t implies that for every t

$$\|\Psi(t, u, \vartheta, \varphi)\|^2 = \|\Psi_0(u, \vartheta, \varphi)\|^2 = \sum_{l=0}^{\infty} \sum_{|m| \leq l} \|\Psi^{lm}_0(u)\|^2_l \ ,$$

where for the second equation we used the isometry (2.21). Hence, defining

$$\Psi^L(t,u,\vartheta,\varphi) := \sum_{l=L}^{\infty} \sum_{|m|\le l} \Psi^{lm}(t,u) Y_{lm}(\vartheta,\varphi) ,$$

we can find for every $\epsilon > 0$ a number L_0 such that

$$\|\Psi^{L_0}(t,u,\vartheta,\varphi)\|^2 = \sum_{l=L_0}^{\infty} \sum_{|m|\le l} \|\Psi_0^{lm}(u)\|_l^2 < \epsilon .$$

Let us now consider the Cauchy problem (2.6) with initial data

$$H\Psi_0 = \sum_{l=0}^{\infty} \sum_{|m|\le l} (H_l \Psi_0^{lm}) Y_{lm} .$$

Obviously, this data is also smooth and compactly supported and thus gives rise to the solution

$$\sum_{l=0}^{\infty} \sum_{|m|\le l} \left(e^{-itH_l} H_l \Psi_0^{lm}\right) Y_{lm} = \sum_{l=0}^{\infty} \sum_{|m|\le l} \left(H_l e^{-itH_l} \Psi_0^{lm}\right) Y_{lm} = H\Psi ,$$

where in the second equation we again used the uniqueness of the decomposition into spherical harmonics. Thus for every $\epsilon > 0$ there is a L_1 (without restriction $\ge L_0$) such that

$$\|H\Psi^{L_1}(t)\| < \epsilon , \quad \text{for all times } t .$$

Proceeding inductively, we find for every number N and for every $\epsilon > 0$ a number L_N such that

$$\|H^n \Psi^{L_N}(t)\| < \epsilon , \quad \text{for all } t \text{ and } n \le N .$$

Let $K \subseteq \mathbb{R} \times S^2$ be an arbitrary compact subset with smooth boundary. Then, due to the definition of the energy, there exists a constant $C_0(K) > 0$ such that for $\Psi^{L_N} = (\psi_1^{L_N}, \psi_2^{L_N})^T$,

$$\|\psi_1^{L_N}\|_{H^1(K)} + \|\psi_2^{L_N}\|_{L^2(K)} \;\le\; C_0(K) \|\Psi^{L_N}\| .$$

Applying the same argument to $H\Psi^{L_N} = (\psi_2^{L_N}, A\psi_1^{L_N})^T$, where A is the differential operator given by (2.14), there is a $C_1(K) > 0$ such that

$$\|A\psi_1^{L_N}\|_{L^2(K)} + \|\psi_2^{L_N}\|_{H^1(K)} \;\le\; C_1(K) \|H\Psi^{L_N}\| .$$

Since the differential operator A is of the form $A = -\Delta + X$, where X is a first order differential operator, it is in particular a second order elliptic partial differential operator. Thus, for $u \in C^\infty(\mathbb{R} \times S^2)$ and for each $U \subset\subset V \subset\subset \mathbb{R} \times S^2$ ($\subset\subset$ denotes relative compact) there is an estimate (cf. [15, p.379 (11.3)])

$$\|u\|_{H^{k+2}(U)} \le C\|Au\|_{H^k(V)} + C\|u\|_{H^{k+1}(V)} \quad \text{for all } k \ge 0.$$

It follows that there exist new constants $C_0(K), C_1(K)$ such that

$$\|\psi_1^{L_N}\|_{H^2(K)} + \|\psi_2^{L_N}\|_{H^1(K)} \;\le\; C_0(K)\|\Psi^{L_N}\| + C_1(K)\|H\Psi^{L_N}\| .$$

Iterating this inequality, we obtain constants $C_0(K), ..., C_k(K)$ such that

$$\|\psi_1^{L_N}\|_{H^{k+1}(K)} + \|\psi_2^{L_N}\|_{H^k(K)} \;\leq\; \sum_{n=0}^{k} C_n(K) \|H^n \Psi^{L_N}\| \; .$$

In particular, for every $\epsilon > 0$ there is a number L such that

$$\|\Psi^L(t)\|_{H^2(K)} < \epsilon \quad \text{for all } t \; .$$

Thus the Sobolev embedding theorem yields (possibly after enlarging L)

$$\|\Psi^L(t)\|_{L^\infty(K)} < \epsilon \quad \text{for all } t \; .$$

Furthermore, due to the pointwise decay for fixed modes l, m which was shown in Theorem 6.5, we can find for any $\epsilon > 0$ and $(u, x) \in \mathbb{R} \times S^2$ a time t_0 and a number L such that for the solution $\Psi(t, u, x)$ of the Cauchy problem (2.6),

$$|\Psi(t, u, x)| \leq \sum_{l=0}^{L-1} \sum_{|m| \leq l} |\Psi^{lm}(t, u) Y_{lm}(x)| + |\Psi^L(t, u, x)| < \epsilon \quad \text{for all } t \geq t_0 \; .$$

Since $\phi = r\psi$, this concludes the proof. $\square$

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References

[1] F. Finster, N. Kamran , J. Smoller, S-T. Yau, *An Integral Spectral Representation of the Propagator for the Wave Equation in the Kerr Geometry.* Commun. Math. Phys. **260** (2005) no.2, 257-298.

[2] F. Finster, N. Kamran , J. Smoller, S-T. Yau, *Decay of Solutions of the Wave Equation in the Kerr Geometry.* gr-qc/0504047, to appear in Commun. Math. Phys. (2006)

[3] T. Regge, J. A. Wheeler, *Stability of a Schwarzschild Singularity.* Phys. Rev. (2) **108** (1957) 1063-1069.

[4] B. Kay, R. Wald, *Linear Stability of Schwarzschild under the perturbations which are non-vanishing on the bifurcation 2-sphere.* Class. Quantum Grav. **4** (1987) 893-898.

[5] R. H. Price, *Nonspherical Perturbations of Relativistic Gravitational Collaps. I. Scalar and Gravitational Perturbations.* Phys. Rev. D (3) **5** (1972) 2419-2438.

[6] M. Dafermos, I. Rodnianski, *A proof of Price's law for the collapse of a self-gravitating scalar field.* Invent. Math. **162** (2005), 381-457.

[7] M. Dafermos, I. Rodnianski, *The red-shift effect and radiation decay on black hole spacetimes.* gr-qc/0512119

[8] J. G. Stalker, A. Shadi Tahvildar-Zadeh, *Scalar waves on a naked-singularity background.* Class. Quantum Grav. **21** (2004) 2831-2848.

[9] P. Blue, J. Sterbenz, *Uniform Decay of Local Energy and the Semi-Linear Wave Equation on Schwarzchild Space.* math.AP/ 0510315.

[10] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, Vol. I: Functional Analysis.* Academic Press (1980).

[11] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, Vol. II: Fourier Analysis, Self-Adjointness.* Academic Press (1975).

[12] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, Vol. III: Scattering Theory.* Academic Press (1979).

[13] F. John, *Partial Differential Equations.* Springer-Verlag New York (1982)

[14] G. N. Watson, *Theory Of Bessel Functions.* Cambridge University Press (1966).

[15] M. E. Taylor, *Partial Differential Equations I.* Springer-Verlag New York (1996)

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