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Admissible Semi-Linear Representations

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ADMISSIBLE SEMI-LINEAR REPRESENTATIONS

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Abstract. The category of admissible (in the appropriately modified sense of representation theory of totally disconnected groups) semi-linear representations of the automorphism group of an algebraically closed extension of infinite transcendence degree of the field of algebraic complex numbers is described.

Let k be a field of characteristic zero containing all ℓ-primary roots of unity for a prime ℓ, F be a universal domain over k, i.e., an algebraically closed extension of k of countable transcendence degree, and $G_{F/k}$ be the field automorphism group of F over k. We consider $G_{F/k}$ as a topological group with the base of open subgroups generated by $\{G_{F/k(x)} \mid x \in F\}$.

Denote by $\mathcal{C}$ the category of smooth (with open stabilizers) F-semi-linear representations of $G_{F/k}$, i.e., F-vector spaces V endowed with an additive semi-linear ($g(fv) = gf \cdot gv$ for any $f \in F$, $g \in G_{F/k}$ and $v \in V$) action $G_{F/k} \times V \to V$ of $G_{F/k}$.

Denote by $\mathcal{A}$ the full sub-category of $\mathcal{C}$ whose objects V are *admissible*: $\dim_{F^U} V^U < \infty$ for any open subgroup $U \subseteq G_{F/k}$. Clearly, $\mathcal{A}$ is an additive category and it is shown in [R] that it is a tensor (but not rigid) category. In the present paper one proves that the category $\mathcal{A}$ is abelian (Theorem 3.6), and F is its projective object (Proposition 3.4).

Let the ideal $\mathfrak{m} \subset F \otimes_{k_0} F$ be the kernel of the multiplication map $F \otimes_{k_0} F \xrightarrow{\times} F$, where $k_0 = k \cap \overline{\mathbb{Q}}$ is the number subfield of k. Consider the powers $\mathfrak{m}^s \subseteq F \otimes_{k_0} F$ of the ideal $\mathfrak{m}$ for all $s \geq 0$ as objects of $\mathcal{C}$ with the F-multiplication via $F \otimes_{k_0} k_0$.

In this paper we study the category $\mathcal{A}$ and describe it if k is a number field. Namely, in the case $k = \overline{\mathbb{Q}}$ we prove the following:

- The sum of the images of the F-tensor powers $\bigotimes_F^{\geq \bullet} \mathfrak{m}$ under all morphisms in $\mathcal{C}$ defines a decreasing filtration $W^\bullet$ on the objects of $\mathcal{A}$ such that its graded quotients gr_W^q are finite direct sums of direct summands of $\bigotimes_F^q \Omega_F^1$ (cf. §4.1, p.17 and Theorem 4.10). This filtration is evidently functorial and multiplicative: $(W^p V_1) \otimes_F (W^q V_2) \subseteq W^{p+q}(V_1 \otimes_F V_2)$ for any $p, q \geq 0$ and any $V_1, V_2 \in \mathcal{A}$.
- $\mathcal{A}$ is equivalent to the direct sum of the category of finite-dimensional k-vector spaces and its abelian full subcategory $\mathcal{A}^\circ$ with objects V such that $V^{G_{F/k}} = 0$ (Lemma 4.13).
- Any object V of $\mathcal{A}^\circ$ is a quotient of a direct sum of objects (of finite length) of type $\bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^s)$ for some $q, s \geq 1$ (Theorem 4.10).
- If $V \in \mathcal{A}$ is of finite type then it is of finite length and $\dim_k \mathrm{Ext}^j_{\mathcal{A}}(V, V') < \infty$ for any $j \geq 0$ and any $V' \in \mathcal{A}$; if $V \in \mathcal{A}$ is irreducible and $\mathrm{Ext}^1_{\mathcal{A}}(\mathfrak{m}/\mathfrak{m}^q, V) \neq 0$ for some $q \geq 2$ then $V \cong \mathrm{Sym}^q_F \Omega^1_F$ and $\mathrm{Ext}^1_{\mathcal{A}}(\mathfrak{m}/\mathfrak{m}^q, V) \cong k$ (Corollary 4.17).
- $\mathcal{A}^\circ$ has no projective objects (Corollary 4.14), but $\bigotimes_F^q \mathfrak{m}$ are its "projective pro-generators": the functor $\mathrm{Hom}_{\mathcal{C}}(\bigotimes_F^q \mathfrak{m}, -)$ is exact on $\mathcal{A}$ for any q (Corollary 4.16).

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To describe the objects of $\mathcal{A}$, one studies first their "restrictions" to projective groups ($\cong \mathrm{PGL}_m k$), considered as subquotients of $G_{F/k}$. It is known ([R]) that such semi-linear representations are related to homogeneous vector bundles on projective spaces.

Let $n \geq 1$ be an integer, $K_n = k(\mathbb{P}^n_k)$ be the function field of an n-dimensional projective k-space $\mathbb{P}^n_k$ and $G_n = \mathrm{Aut}(\mathbb{P}^n_k/k)$ be its automorphism group.

Fix a k-field embedding $K_n \hookrightarrow F$. We show, in particular, that if V is an admissible F-semi-linear representation of $G_{F/k}$ with no sub-objects isomorphic to F then any irreducible subquotient of the K_n-semi-linear representation $V^{G_{F/K_n}}$ of G_n is a direct summand of $\bigotimes^{\geq 1}_{K_n} \Omega^1_{K_n/k}$ (this is Theorem 2.4 and Proposition 3.4 below).

0.1. **Some motivation.** The study of semi-linear representations comes from the study of $\mathbb{Q}$-linear representations of $G_{F/k}$, that are related to geometry, cf. [R].

Let $\mathcal{S}m_G$ be the category of smooth representations of $G_{F/k}$ over k. Extending of coefficients to F gives a faithful functor $F\otimes_k : \mathcal{S}m_G \longrightarrow \mathcal{C}$. It is not full: if $U \subset G_{F/k}$ is an open subgroup and $\overline{f} \in (F^\times/k^\times)^U - \{1\}$ then $[\sigma] \mapsto \sigma f \cdot [\sigma]$ defines an element of $\mathrm{End}_{\mathcal{C}}(F[G_{F/k}/U])$ which is not in $\mathrm{End}_{\mathcal{S}m_G}(k[G_{F/k}/U])$. However, its restriction to the subcategory $\mathcal{I}_G \otimes k$ of "homotopy invariant" representations[1] is.

Lemma 0.1. *If $k = \overline{k}$ then the functor $\mathcal{I}_G \otimes k \xrightarrow{F\otimes_k} \mathcal{C}$ is fully faithful.*

Proof. More generally, let us show that $\mathrm{Hom}_{\mathcal{S}m_G}(W, W') = \mathrm{Hom}_{\mathcal{C}}(F\otimes_k W, F\otimes_k W')$ for any $W \in \mathcal{I}_G \otimes k$ and any $W' \in \mathcal{S}m_G$. Let $\varphi \in \mathrm{Hom}_G(W, F\otimes_k W')$ and $\varphi(w) = \sum_{j=1}^N f_j \otimes w_j$ for some $w \in W$, $w_j \in W'$, $f_j \in F$ and minimal possible $N \geq 1$. We have to show that $f_j \in k$.

Choose a smooth proper model X of $k(f_1, \dots, f_N)$ over k. If it is not a point, choose a generically finite rational dominant map π to a projective space Y over k which is

- well-defined at the generic points of the irreducible components D_α of the divisors of poles of $f_1, \dots, f_N$,
- induces on each D_α a birational map and
- separates D_α.

Then the trace $\pi_*\varphi(w)$ has poles. On the other hand, $\pi_*\varphi(w)$ is in the image of $W^{G_{F/k(Y)}} = W^{G_{F/k}}$, so $\pi_*\varphi(w) \in (F\otimes_k W')^{G_{F/k}} = k \otimes_k (W')^{G_{F/k}}$ by Lemma 7.5 of [R]. This contradiction implies that $f_j \in k$, and therefore, $\varphi(W) \subseteq k \otimes_k W' \subseteq F \otimes_k W'$. □

The $\mathbb{Q}$-linear representations of $G_{F/k}$ of particular interest are admissible representations, forming a full subcategory in $\mathcal{I}_G \otimes k$. Though tensoring with F does not transform them to admissible semi-linear representations,[2] there exists, at least if $k = \overline{\mathbb{Q}}$, a similar faithful functor in the opposite direction.

Namely, it is explained in Corollary 5.2 that, for any object V of $\mathcal{A}$ and any smooth k-variety Y, embedding of the generic points of Y into F determines a locally free coherent sheaf $\mathcal{V}_Y$ on Y. Any dominant morphism $X \xrightarrow{\pi} Y$ of smooth k-varieties induces an injection of coherent sheaves $\pi^*\mathcal{V}_Y \hookrightarrow \mathcal{V}_X$, which is an isomorphism if π is étale.

This gives an equivalence $\mathcal{S} : \mathcal{A} \xrightarrow{\sim} \{$"coherent" sheaves in the smooth topology$\}$, $V \longmapsto (Y \mapsto \mathcal{V}_Y(Y))$. More generally, the "coherent" sheaves are contained in the category $\mathcal{F}l$ of the flat "quasi-coherent" sheaves in the smooth topology, cf. §5, p.18. For any flat

[1] i.e. such that $W^{G_{F/L}} = W^{G_{F/L'}}$ for any purely transcendental extension L'/L of subfields in F

[2] and moreover, there are irreducible objects of $\mathcal{C}$ outside of $\mathcal{A}$ (Corollary 3.5).

"quasi-coherent" sheaf $\mathcal{V}$ in the smooth topology the space $\Gamma(Y, \mathcal{V}_Y)$ is a birational invariant of a proper Y (Lemma 5.3). Then we get a left exact functor $\mathcal{F}l \xrightarrow{\Gamma} \mathcal{S}m_G$ given by $V \mapsto \varinjlim \Gamma(Y, \mathcal{V}_Y)$, where Y runs over the smooth proper models of subfields in F of finite type over k.

The functor $\Gamma \circ \mathcal{S}$ is faithful, since $\Gamma(Y', \mathcal{V}_{Y'})$ generates the (generic fibre of the) sheaf $\mathcal{V}_{Y'}$ for appropriate finite covers Y' of Y (Lemma 5.3), if $\mathcal{V}$ is "coherent". But it is not full, and the objects in its image are highly reducible. If $\Gamma(Y, \mathcal{V}_Y)$ has the Galois descent property then $\Gamma(V)$ is admissible. However, there is no Galois descent property in general.

0.2. **Notation.** Let k, F and $G_{F/k}$ be as above. For a subfield L of F we denote by $\overline{L}$ its algebraic closure in F. We fix a transcendence basis $x_1, x_2, x_3, \dots$ of F over k.

For each $n \geq 1$ set $Y_n = \mathbf{Spec}k[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \subset \mathbb{A}_k^n = \mathbf{Spec}k[x_1, \dots, x_n] \subset \mathbb{P}_k^n = \mathbf{Proj}k[X_0, \dots, X_n]$ with $x_j = X_j/X_0$, $K_n = k(x_1, \dots, x_n)$, $G_n = \mathrm{Aut}(\mathbb{P}_k^n/k)$.

Let $\mathrm{Aff}_n = G_n \cap \mathrm{Aut}(\mathbb{A}_k^n/k)$ be the affine subgroup, and $(\mathrm{Aff}_n)_u$ be its unipotent radical, i.e., the translation subgroup. Let $H_n = \mathrm{Aff}_n \cap \mathrm{Aut}(\mathbb{A}_k^n/\mathbb{A}_k^{n-1})$ be the subgroup fixing the coordinates $x_1, \dots, x_{n-1}$ on $\mathbb{A}_k^n$. Let $T_n \subset G_n$ be the maximal torus acting freely on Y_n.

Denote by T_n^{tors} the torsion subgroup in T_n.

For a field extension L/K we denote by $\mathrm{Der}(L/K)$ the Lie K-algebra of derivations of L over K. For an integer $\ell \geq 2$, the group of ℓ-th roots of unity in $\overline{k}$ is denoted by μ_ℓ, and the corresponding cyclotomic number subfield in $\overline{k}$ is denoted by $\mathbb{Q}(\mu_\ell)$.

0.3. **Structure of the paper.** As it is mentioned above, we consider the projective groups G_n as subquotients of $G_{F/k}$. In §1 we identify irreducible subquotients of "restrictions" of objects of $\mathcal{A}$ to G_n with the generic fibres of the G_n-equivariant coherent sheaves on $\mathbb{P}_k^n$. Main ingredients there come from [BT] and [R]. In §2 we exclude some cases, thus showing that these irreducible subquotients are direct summands of $\bigotimes_{K_n}^{\bullet} \Omega_{K_n/k}^1$. In §3 we show that $\mathcal{A}$ is abelian and calculate Ext^1-groups between the irreducible objects of a tannakian category $\mathfrak{SL}_u^n$ (defined at the beginning of §1, p.3) of semi-linear representations of G_n, containing "restrictions" of objects of $\mathcal{A}$ to G_n. The latter part uses [LR]. After showing principal structural results on $\mathcal{A}$ (in §4) we identify (in §5) $\mathcal{A}$ with the category of "coherent" sheaves in smooth topology. Finally (in §6), we define a descending filtration $\mathcal{A}_{>\bullet}$ of $\mathcal{A}$ by Serre "ideal" subcategories. Then we localize the quotients $\mathcal{A}/\mathcal{A}_{>m}$ for each $m \geq 0$ to get a tannakian subcategory of finite-dimensional semi-linear representations of $G_{F'/k}$ over F' for an algebraically closed extension F' of k in F of transcendence degree m.

1. Equivariantness of irreducible PGL-sheaves

Let $\mathfrak{SL}_n^u$ be the category of finite-dimensional semi-linear representations of G_n over K_n whose restrictions to the maximal torus T_n in G_n are of type $K_n \otimes_k W$ for unipotent representations W of T_n (where T_n is considered as a discrete group).

Note that $V = V^{T_n^{\mathrm{tors}}} \otimes_k K_n$ for any $V \in \mathfrak{SL}_n^u$.

In [R], for $n > 1$, a fully faithful functor $\mathfrak{SL}_n^u \xrightarrow{\mathcal{S}} \{$coherent G_n-sheaves on $\mathbb{P}_k^n\}$ is constructed. (A G_n-sheaf is G_n-equivariant sheaf if G_n is considered as a discrete group. In other words, $\mathcal{V}$ is a G_n-sheaf if it is endowed with a collection of isomorphisms $\alpha_g : \mathcal{V} \xrightarrow{\sim} g^*\mathcal{V}$ for each $g \in G_n$ satisfying the chain rule: $\alpha_{hg} = g^*\alpha_h \circ \alpha_g$ for any $g, h \in G_n$. The term "G_n-equivariant" is reserved for G_n-vector bundles with algebraic G_n-action on their total spaces.) The composition of $\mathcal{S}$ with the generic fibre functor is the identical full embedding of $\mathfrak{S}\mathfrak{L}_n^u$ into the category of finite-dimensional K_n-semi-linear G_n-representations.

In this section we show that the category $\mathfrak{S}\mathfrak{L}_n^u$ is abelian and its irreducible objects are generic fibres of irreducible coherent G_n-*equivariant* sheaves on $\mathbb{P}_k^n$, i.e., direct summands of $\mathrm{Hom}_{K_n}((\Omega^n_{K_n/k})^{\otimes M}, \bigotimes^{\bullet}_{K_n}\Omega^1_{K_n/k})$ for appropriate integer $M \geq 0$.

Lemma 1.1. *The category $\mathfrak{S}\mathfrak{L}_n^u$ is closed under taking K_n-semi-linear subquotients.*

Proof. Let $V \in \mathfrak{S}\mathfrak{L}_n^u$ and $0 \to V_1 \to V \stackrel{\pi}{\to} V_2 \to 0$ be a short exact sequence of semi-linear representations of G_n over K_n. As the k-vector space $V^{T_n^{\mathrm{tors}}}$ (of the elements in V fixed by the torsion subgroup T_n^{tors} in T_n) spans the K_n-vector space V, the k-vector space $\pi(V^{T_n^{\mathrm{tors}}}) \subseteq V_2^{T_n^{\mathrm{tors}}}$ spans the K_n-vector space V_2.

This means that $V_2 = V_2^{T_n^{\mathrm{tors}}} \otimes_k K_n$ and $\pi(V^{T_n^{\mathrm{tors}}}) = V_2^{T_n^{\mathrm{tors}}}$.

In other words, the sequence of T_n^{tors}-invariants $0 \to V_1^{T_n^{\mathrm{tors}}} \to V^{T_n^{\mathrm{tors}}} \to V_2^{T_n^{\mathrm{tors}}} \to 0$ is exact, and extending its coefficients to K_n gives the exact sequence $0 \to V_1^{T_n^{\mathrm{tors}}} \otimes_k K_n \to V = V^{T_n^{\mathrm{tors}}} \otimes_k K_n \stackrel{\pi'}{\to} V_2 = V_2^{T_n^{\mathrm{tors}}} \otimes_k K_n \to 0$. As π coincides with π', we get $V_1 = V_1^{T_n^{\mathrm{tors}}} \otimes_k K_n$.

Clearly, any subquotient of a unipotent representation of T_n is again unipotent, and thus, $V_1, V_2 \in \mathfrak{S}\mathfrak{L}_n^u$. □

Lemma 1.2. *Let E be the total space of a vector bundle on $\mathbb{P}_k^n$, $\mathrm{Aut}_{\mathrm{lin}}(E)$ be the group of automorphisms of E over k inducing linear transforms between the fibres, and $\tau : G_n \to \mathrm{Aut}_{\mathrm{lin}}(E)$ be an irreducible G_n-structure on E, i.e., a discrete group homomorphism splitting the projection $\mathrm{Aut}_{\mathrm{lin}}(E) \to G_n$. Then the Zariski closure $\overline{\tau(G_n)}$ is reductive.*

Proof. Let Aut_τ be the kernel of the projection $\overline{\tau(G_n)} \stackrel{\pi}{\to} G_n$.

For each point $p \in \mathbb{P}_k^n$ let $\rho_p : R_p := \pi^{-1}(\mathrm{Stab}_p) \to \mathrm{GL}(E_p)$ be the natural representation.

As we suppose that E is an irreducible G_n-bundle, ρ_p is irreducible, since otherwise $B := \overline{\tau(G_n)}B_p \subset E$ is a G_n-subbundle for any proper R_p-invariant k-subspace $B_p \subset E_p$.

In particular, ρ_p is trivial on the unipotent radical of R_p. The unipotent radical of any algebraic group contains the unipotent radical of its arbitrary normal subgroup, so ρ_p is trivial on the unipotent radical of Aut_τ. As the action of Aut_τ on E is faithful, $\bigcap_p \ker \rho_p|_{\mathrm{Aut}_\tau} = \{1\}$, i.e., Aut_τ is reductive. As G_n is also reductive, so is $\overline{\tau(G_n)}$. □

For a commutative finite k-algebra A denote by $R_{A/k}$ the Weil functor of restriction of scalars on A-schemes, cf. [DG], I, §1, 6.6.

We need the following particular case of Théorème 8.16 of [BT].

Theorem 1.3. *Let G be a simply connected absolutely almost simple k-group, and G' be a reductive k-group. Let $\tau : G(k) \to G'(k)$ be a homomorphism with Zariski dense image. Let $G'_1, \dots, G'_m$ be the almost simple normal subgroups of G'.*

Then there exist finite field extensions k_i/k, field embeddings $\varphi_i : k \to k_i$, a special isogeny $\beta : \prod_{i=1}^m R_{k_i/k}{}^{\varphi_i}G \to G'$ (here ${}^{\varphi_i}G := G \times_{k,\varphi_i} k_i$) and a homomorphism $\mu : G(k) \to Z_{G'}(k)$ such that $\beta(R_{k_i/k}{}^{\varphi_i}G) = G'_i$ and $\tau(h) = \mu(h) \cdot \beta(\prod_{i=1}^m \varphi_i^\circ(h))$ for any $h \in G(k)$ (here $\varphi_i^\circ : G(k) \to (R_{k_i/k}{}^{\varphi_i}G)(k)$ is the canonical homomorphism). □

Corollary 1.4. *Under assumptions of Theorem 1.3, for any torus $T \subset G$ the Zariski closure of $\tau(T(k))$ is a torus in G'.* □

Proposition 1.5. *If $n \geq 2$ then any irreducible object of $\mathfrak{S}\mathfrak{L}_n^u$ is a direct summand of $\mathrm{Hom}_{K_n}((\Omega^n_{K_n/k})^{\otimes M}, \bigotimes^{\bullet}_{K_n}\Omega^1_{K_n/k})$ for an appropriate M.*

Proof. The functor $\mathcal{S}$, mentioned in the beginning of this §, associates to an irreducible object V of $\mathfrak{SL}_n^u$ a coherent G_n-sheaf $\mathcal{V}$ on $\mathbb{P}_k^n$ with generic fibre V.

Let, as before, T_n be a maximal torus in G_n and $Y_n \subset \mathbb{P}_k^n$ be the n-dimensional T_n-orbit. As $V^{T_n^{\rm tors}} = \Gamma(Y_n, \mathcal{V})^{T_n^{\rm tors}}$, cf. [R], is a unipotent representation of T_n, Lemma 1.2 and Corollary 1.4 imply that $\Gamma(Y_n, \mathcal{V})^{T_n^{\rm tors}}$ is a trivial representation of T_n.

In a k-basis of $V^{T_n^{\rm tors}}$ the G_n-action on V determines a 1-cocycle $(g_\sigma) \in Z^1(G_n, \mathrm{GL}_M K_n)$, where $M = \dim_{K_n} V$. There is an integer $N > n+2$ and elements $\alpha_1, \dots, \alpha_N \in G_n$ such that the morphism $(T_n)^N \xrightarrow{\pi} G_n$, given by $(h_1, \dots, h_N) \mapsto \alpha_1 h_1 \alpha_1^{-1} \cdots \alpha_N h_N \alpha_N^{-1}$, is surjective. Namely, using the Gauß elimination algorithm, one shows that any element of G_n is a product of $\le (n+1)^2$ elementary matrices and an element of T_n. On the other hand, it follows from the identity $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b-a & b \end{pmatrix}$ that for any elementary matrix α the product $T_n \cdot \alpha T_n \alpha^{-1} \cdot T_n$ contains all elementary matrices of the same type as α. This gives a surjection $(T_n \times \prod_{i \neq j} \alpha_{ij} T_n \alpha_{ij}^{-1})^{(n+1)^2} \times T_n \xrightarrow{\times} G_n$, where α_{ij} is the elementary matrix with 1 in the i-th row and j-th column.

Then

$$\begin{aligned} g_{\pi(h_1,\dots,h_N)} = {} & g_1(x) g_1'(\alpha_1 h_1(x)) g_2(\alpha_1 h_1 \alpha_1^{-1}(x)) g_2'(\alpha_1 h_1 \alpha_1^{-1} h_2(x)) \cdots \\ & g_N(\alpha_1 h_1 \alpha_1^{-1} \cdots \alpha_{N-1} h_{N-1} \alpha_{N-1}^{-1}(x)) g_N'(\alpha_1 h_1 \alpha_1^{-1} \cdots \alpha_{N-1} h_{N-1} \alpha_{N-1}^{-1} \alpha_N h_N(x)), \end{aligned}$$

where $g_j(x) := g_{\alpha_j}$ and $g_j'(x) := g_{\alpha_j^{-1}}$ for all $1 \le j \le N$. In other words, the lifting of the G_n-action on $\mathrm{tot}(\mathcal{V})$ to $(T_n)^N$-"coupling" via π determines a rational map $(T_n)^N \times \mathbb{P}_k^n \dashrightarrow \mathrm{GL}_M k$. Clearly, it corresponds to a regular morphism $(T_n)^N \times \mathrm{tot}(\mathcal{V}) \longrightarrow \mathrm{tot}(\mathcal{V})$ and factors through a regular morphism $G_n \times \mathrm{tot}(\mathcal{V}) \longrightarrow \mathrm{tot}(\mathcal{V})$ of k-varieties, i.e., we see that $\mathcal{V}$ is equivariant.

The generic fibres of irreducible G_n-equivariant sheaves on $\mathbb{P}_k^n$ are exactly of the desired type. □

Remark. There can exist, a priori, non-equivariant irreducible coherent G_n-sheaves on $\mathbb{P}_k^n$, e.g. the extension of coefficients to $\mathcal{O}_{\mathbb{P}_k^n}$ of a non-rational representation of G_n is seemingly of this type.

2. "Positivity"

In this section we show that for any admissible F-semi-linear representation V of $G_{F/k}$ any irreducible subquotient of the K_n-semi-linear representation $V^{G_{F/K_n}}$ of G is a direct summand of $\bigotimes_{K_n}^{\bullet} \Omega_{K_n/k}^1$.

It is shown in [R] that any finite-dimensional K_n-semi-linear G_n-representation extendable to $\mathrm{End}(K_n/k)$, e.g. $V^{G_{F/K_n}}$, is an object of $\mathfrak{SL}_n^u$. By Proposition 1.5, we only need to eliminate the negative twists by $\Omega_{K_n/k}^n$ in irreducible subquotients of $V^{G_{F/K_n}}$.

To do that we show first that the generic fibres of irreducible coherent G_n-equivariant sheaves are determined by their restrictions to the subgroup $\mathrm{Aff}_n = G_n \cap \mathrm{Aut}(\mathbb{A}_k^n/k)$.

Lemma 2.1. *Let Aff_n be the group of affine transformations of an affine space $\mathbb{A}_k^n$ with the function field K_n. Then the natural morphism*

(1) $\{\text{rational } k\text{-linear } \mathrm{Aff}_n\text{-representations}\} \xrightarrow{\otimes_k K_n} \{K_n\text{-semi-linear } \mathrm{Aff}_n\text{-representations}\}$

transforms isomorphism classes of irreducible k-representations of Aff_n *to isomorphism classes of irreducible K_n-semi-linear representations of* $\mathrm{Aff}_n^{(1)}\mathbb{Q}$*, the subgroup of* Aff_n *consisting of $\mathbb{Q}$-affine substitutions of $x_1, \dots, x_n$ with Jacobian equal to 1.*

Proof. Let W be an irreducible k-representations of Aff_n, and $U \subset W \otimes_k K_n$ a non-zero K_n-semi-linear subrepresentation of $\mathrm{Aff}_n^{(1)}\mathbb{Q}$. Let $\alpha = \sum_{j=1}^N w_j\alpha_j \in U$ be a non-zero element with minimal possible N, where $w_j \in W$ and $\alpha_j \in K_n$. Multiplying α by an element of K_n we may assume that all α_j are polynomials: $\alpha = \sum_I w'_I x^I$. Since W is irreducible, the elements of the unipotent radical $(\mathrm{Aff}_n)_u$ of Aff_n, i.e., σ such that $\sigma z - z \in k$ for any linear function z on $\mathbb{A}_k^n$, act trivially on W.

Applying an appropriate composition of difference operators $\sigma - \tau$ for some σ, τ in the unipotent radical of $\mathrm{Aff}_n^{(1)}\mathbb{Q}$ to α, we can lower the degrees of the polynomials α_j and eventually get a non-zero element of W. As $W = W_0 \otimes_{\mathbb{Q}} k$ for an irreducible representation W_0 of $\mathrm{Aff}_n^{(1)}\mathbb{Q}$, any non-zero element of W generates $W \otimes_k K_n$, which means that $U = W \otimes_k K_n$. □

Corollary 2.2. *Let* Aff_n, $(\mathrm{Aff}_n)_u$, $\mathbb{A}_k^n$ *and* K_n *be as in Lemma 2.1. Then* $V \mapsto V^{(\mathrm{Aff}_n)_u}$ *gives a natural bijection*

$$(2)\quad \left\{\begin{array}{c}\text{isomorphism classes of}\\ \text{irreducible } G_n\text{-subrepresentations in}\\ \bigoplus_M \mathrm{Hom}_K((\Omega^n_{K_n/k})^{\otimes M}, \bigotimes^{\bullet}_{K_n}\Omega^1_{K_n/k})\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\text{isomorphism classes of}\\ \text{irreducible rational } k\text{-linear}\\ \mathrm{Aff}_n\text{-representations}\end{array}\right\}$$

such that its composition with the morphism (1) is the inclusion map. □

Let W be an $(n+1)$-dimensional k-vector space, $L \subset W$ be a one-dimensional subspace, and $H_{\mathrm{lin}} = \ker[\mathrm{GL}(W,L) \to \mathrm{GL}(W/L)] \cong k^\times \ltimes \mathrm{Hom}(W/L, L)$ be the group preserving L and inducing the identity automorphism of W/L.

Lemma 2.3. *For any Young diagram λ with no columns of height $\geq n+1$ one has*

$$(S^\lambda W^\vee \otimes_k (\det W)^{\otimes s})^{H_{\mathrm{lin}}} = \begin{cases} S^\lambda(W/L)^\vee & \text{if } s = 0, \\ 0 & \text{otherwise} \end{cases}$$

Proof. Denote by $X = AF(W) \cong \mathrm{GL}(W)/R_u(B)$, the variety of complete affine flags in W. An affine flag is a filtration $W_\bullet = (0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{n+1} = W)$ with $\dim_k W_j = j$ and a collection of $l_j \in W_j/W_{j-1} - \{0\}$.

Let $Y \cong \mathrm{GL}(W)/B$ be the variety of complete linear flags in W. Then the natural projection $X \xrightarrow{\pi} Y$ is a principal $(\mathbb{G}_m)^{n+1}$-bundle, and there is a decomposition $\pi_*\mathcal{O}_X = \bigoplus_\mu \mathcal{M}(\mu)$ into a direct sum of invertible sheaves on Y, where μ runs over the group $\mathbb{Z}^{n+1}$ of characters of $(\mathbb{G}_m)^{n+1}$, so $\mathcal{O}(X) = \bigoplus_\mu \Gamma(Y, \mathcal{M}(\mu))$.

Set $X^\circ = \{(V_\bullet, l_\bullet) \in X \mid V_n \cap L = 0 \Leftrightarrow \text{“}l_{n+1} \in L\text{”}\}$. Then reduction modulo L defines a principal $L^{\oplus n} \times \mathbb{G}_m$-bundle $X^\circ \longrightarrow AF(W/L)$, $(l_1, \dots, l_{n+1}) \mapsto (l_1, \dots, l_n)$.

Let $X^\circ \xrightarrow{\text{“}l_{n+1}\text{”}} L - \{0\}$ be the natural H- (or $\mathbb{G}_m$-) equivariant map, and $\bar{l}_{n+1}$ be the composition of “l_{n+1}” with a fixed isomorphism $L - \{0\} \cong \mathbb{G}_m$.

Set $SH := H_{\mathrm{lin}} \cap \mathrm{SL}(W) = \mathrm{Hom}(W/L, L)$. Then $\mathcal{O}(X)^{SH} = \mathcal{O}(X) \cap \mathcal{O}(X^\circ)^{SH} = \mathcal{O}(X) \cap \mathcal{O}(AF(W/L))[\bar{l}_{n+1}, \bar{l}_{n+1}^{-1}]$, so $\mathcal{O}(X)(\mu)^{SH} = \mathcal{O}(X) \cap \mathcal{O}(AF(W/L))(\mu')\bar{l}_{n+1}^{\mu_{n+1}}$, where $\mu' \in \mathbb{Z}^n$ is the restriction of μ to the first n multiples of $(\mathbb{G}_m)^{n+1}$.

For any μ this is an irreducible representation of $\mathrm{GL}(W/L)$, and thus, $\mathcal{O}(X)(\mu)^{SH} = \mathcal{O}(AF(W/L))(\mu')\bar{l}_{n+1}^{\mu_{n+1}}$ if $\mathcal{O}(X)(\mu) \neq 0$.

As any irreducible representation of $\mathrm{SL}(W)$ coincides with $\mathcal{O}(X)(\mu)$ for some μ, this implies that $(S^\lambda W^\vee)^{SH} = S^\lambda(W/L)^\vee$. □

Theorem 2.4. *For any F-semi-linear $G_{F/k}$-representation $V \in \mathcal{A}$ any irreducible subquotient of the K_n-semi-linear G_n-representation $V^{G_{F/K_n}}$ is a direct summand of $\bigotimes^\bullet_{K_n}\Omega^1_{K_n/k}$.*

Proof. Let $W = \mathbb{A}^{n+1}_k$ be the vector space with coordinates $x_1, \dots, x_{n+1}$, so $k(W) = K_{n+1}$. By Proposition 1.5 and Corollary 2.2, the restrictions to $\mathrm{Aff}_{n+1} = \mathrm{Aff}(W)$ of irreducible subquotients of the K_{n+1}-semi-linear G_{n+1}-representation $V^{G_{F/K_{n+1}}}$ are of type $(S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K_{n+1}$ for a Young diagram λ with no columns of height $n+1$ and some integer s, where Aff_{n+1} acts on W via its reductive quotient $\mathrm{GL}(W)$.

Let $H \subset \mathrm{Aff}_{n+1}$ be the subgroup fixing the functionals $x_1, \dots, x_n$ in $W^\vee$ vanishing on L. Let Aff_u be the unipotent radical of Aff_{n+1}, i.e. the group of translations of W.

Then the restrictions to Aff_n of the irreducible subquotients of the K_n-semi-linear G_n-representation $V^{G_{F/K_n}}$ are contained in $((S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K_{n+1})^H$. As $H \cap \mathrm{Aff}_u = \langle 1\rangle_k \cong k$, we get $((S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K_{n+1})^{H\cap\mathrm{Aff}_u} = (S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K_n$, so $((S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K_{n+1})^H = (S^\lambda W^\vee \otimes (\det W)^{\otimes s})^H \otimes_k K_n$.

By Lemma 2.3, $(S^\lambda W^\vee \otimes (\det W)^{\otimes s})^H$ coincides with $S^\lambda(W/L)^\vee$ if $s = 0$, and vanishes otherwise. This means that any representation of Aff_n obtained this way is a direct summand of the tensor algebra of the representation $(W/L)^\vee = (\Omega^1_{K_n/k})^{\{\text{translations}\}}$ of $\mathrm{GL}_n k$. As any irreducible subquotient U of the K_n-semi-linear G_n-representation $V^{G_{F/K_n}}$ is determined by its restriction $U|_{\mathrm{Aff}_n}$ to Aff_n and $U|_{\mathrm{Aff}_n}$ is a direct summand of $\bigotimes^\bullet_{K_n}\Omega^1_{K_n/k}$, the same holds for U. □

3. Extensions in $\mathfrak{S}\mathfrak{L}^u_n$ and in $\mathcal{A}$

For an integer $\ell \geq 2$ such that $\mu_\ell \subset k$ (see §0.2), denote by $\mathrm{Aff}^{(\ell)}_n\mathbb{Q}$ the subgroup of Aff_n consisting of the $\mathbb{Q}(\mu_\ell)$-affine substitutions of $x_1, \dots, x_n$ with Jacobian in μ_ℓ: $x_i \mapsto \sum_{j=1}^n a_{ij}x_j + b_i$, where $a_{ij}, b_i \in \mathbb{Q}(\mu_\ell) \subset k$ and $\det(a_{ij}) \in \mu_\ell$; and by $\mathrm{SAff}^{(\ell)}_n\mathbb{Q}$ the subgroup of index ℓ consisting of elements with Jacobian equal to 1: $\det(a_{ij}) = 1$.

Lemma 3.1. *Let $n, \ell \geq 2$ be integers. Assume that $\mu_\ell \subset k$. Let U_0 be the unipotent radical of $\mathrm{SAff}^{(\ell)}_n\mathbb{Q}$. Then for any object $V \in \mathfrak{S}\mathfrak{L}^u_n$ there is a rational representation W of the reductive quotient $\mathrm{SL}_n\mathbb{Q}(\mu_\ell) = \mathrm{SAff}^{(\ell)}_n\mathbb{Q}/U_0$ of $\mathrm{SAff}^{(\ell)}_n\mathbb{Q}$, and an isomorphism of semi-linear $\mathrm{SAff}^{(\ell)}_n\mathbb{Q}$-modules $W \otimes_{\mathbb{Q}(\mu_\ell)} K_n \xrightarrow{\sim} V$.*

Irreducible rational representations of $\mathrm{SL}_n\mathbb{Q}(\mu_\ell)$ with coefficients extended to K_n are irreducible semi-linear representations of $\mathrm{SAff}^{(\ell)}_n\mathbb{Q}$ over K_n. In particular, any extension in $\mathfrak{S}\mathfrak{L}^u_n$ splits as an extension of K_n-semi-linear representations of $\mathrm{SAff}^{(\ell)}_n\mathbb{Q}$.

Proof. It is shown in Lemma 6.3 (1) of [R] that $H^0(U_0, -)$ is a fibre functor on $\mathfrak{S}\mathfrak{L}^u_n$ independent of ℓ, so $V = V^{U_0} \otimes_k K_n$, i.e, the restriction of V to $\mathrm{SAff}^{(\ell)}_n\mathbb{Q}$ is a k-linear representation V^{U_0} of $\mathrm{SL}_n\mathbb{Q}(\mu_\ell)$ with coefficients extended to K_n, for any $V \in \mathfrak{S}\mathfrak{L}^u_n$.

As it follows from Proposition 1.5, the irreducible subquotients V_α of V restricted to $\mathrm{SAff}^{(\ell)}_n\mathbb{Q}$ are of the form $W_\alpha \otimes_{\mathbb{Q}(\mu_\ell)} K_n$, where W_α are rational irreducible representations of $\mathrm{SL}_n\mathbb{Q}(\mu_\ell)$. Then the irreducible subquotients of V^{U_0} are $V^{U_0}_\alpha = W_\alpha \otimes_{\mathbb{Q}(\mu_\ell)} k$, and V_α are irreducible semi-linear representations of $\mathrm{SAff}^{(\ell)}_n\mathbb{Q}$ by Lemma 2.1.

If V^{U_0} is not semi-simple then it admits a non-semi-simple subquotient W of length 2. Let in Theorem 3.9 $\kappa = \mathbb{Q}(\mu_\ell)$, $K = k$, $G = \mathrm{SL}_n$, $\mathcal{G}$ be the Zariski closure of the image of $\mathrm{SL}_n\mathbb{Q}(\mu_\ell)$ in $\mathrm{GL}_k(W)$ and let τ be given by the $\mathrm{SL}_n\mathbb{Q}(\mu_\ell)$-action on W. Then the unipotent radical of $\mathcal{G}$ is commutative. As the derivations of $\kappa = \mathbb{Q}(\mu_\ell)$ are zero, we see that the k-linear representation W of $\mathrm{SL}_n\mathbb{Q}(\mu_\ell)$ is semi-simple. $\square$

REMARKS. 1. Using Theorems 1.3 and 3.9 it is not hard to show that any representation of SL_nK over any field of characteristic zero is semi-simple for any number field K.

2. Let $V = \Omega^1_{K_n}/\Lambda \otimes_k K_n$, where $\Lambda \subset \Omega^1_k$ is a proper k-subspace. Let the extension $0 \to V \to U \to K_n \to 0$ be given by the cocycle $(\omega_\sigma = d\log\frac{\sigma\omega}{\omega}) \in Z^1(G_n, V)$, where $\omega = dx_1 \wedge \cdots \wedge dx_n \in \Omega^n_{K_n/k}$. Then the restriction of (ω_σ) to GL_nk is non-trivial.

3. The convolution with the Euler vector field $\sum_{j=1}^n x_j\partial/\partial x_j$ defines a GL_nk-equivariant morphism $\Omega^1_{K_n/k} \to K_n$ given by $dx_j \mapsto x_j$. It is non-split for $n \geq 3$, since $(\Omega^1_{K_n/k})^{\mathrm{SL}_n\mathbb{Q}} = 0$.

Lemma 3.2. *Let $n, \ell \geq 2$ and s be some integers, and λ be a Young diagram with columns of height $< n$ such that ℓ does not divide $s + \frac{|\lambda|}{n-1}$, if λ is rectangular of height $n-1$ and non-empty. Let $V = S^\lambda_{K_n}\Omega^1_{K_n/k} \otimes_{K_n} (\Omega^n_{K_n/k})^{\otimes s}$.*

Then $(V^{\mathcal{H}^{(\ell)}_n})^{\mathrm{Aff}^{(\ell)}_{n-1}\mathbb{Q}} = V^{\mathrm{Aff}^{(\ell)}_n\mathbb{Q}}$, where $\mathcal{H}^{(\ell)}_n := G_{K_n/K_{n-1}} \cap \mathrm{Aff}^{(\ell)}_n\mathbb{Q}$.

Proof. Let W be the k-span of $dx_1, \dots, dx_n$ in $\Omega^1_{K_n/k}$. Then $S^\lambda_{K_n}\Omega^1_{K_n/k} = S^\lambda_k W \otimes_k K_n$ and $\Omega^n_{K_n/k} = \det_k W \otimes_k K_n$. Set $S\mathcal{H}^{(\ell)}_n = \mathcal{H}^{(\ell)}_n \cap \mathrm{SAff}^{(\ell)}_n\mathbb{Q}$. Then $\mathcal{H}^{(\ell)}_n \cong \mu_\ell \ltimes S\mathcal{H}^{(\ell)}_n$, and therefore, $V^{\mathcal{H}^{(\ell)}_n} = (V^{S\mathcal{H}^{(\ell)}_n})^{\mu_\ell}$.

One has $V^{S\mathcal{H}^{(\ell)}_n} = (S^\lambda_k W \otimes_k K_n)^{S\mathcal{H}^{(\ell)}_n} \otimes_k (\det_k W)^{\otimes s}$. As the intersection of the unipotent radical of $\mathrm{Aff}^{(\ell)}_n\mathbb{Q}$ with $S\mathcal{H}^{(\ell)}_n$ (i.e. the $\mathbb{Q}(\mu_\ell)$-translations of x_n) acts trivially on $S^\lambda_k W$ and fixes exactly K_{n-1} in K_n, if $n \geq 1$, we get

$$V^{S\mathcal{H}^{(\ell)}_n} = (S^\lambda_k W)^{S\mathcal{H}^{(\ell)}_n} \otimes_k (\det_k W)^{\otimes s} \otimes_k K_{n-1} = S^\lambda_k(W^{S\mathcal{H}^{(\ell)}_n}) \otimes_k (\det_k W)^{\otimes s} \otimes_k K_{n-1}.$$

Then

$$V^{\mathcal{H}^{(\ell)}_n} = S^\lambda_k(W^{S\mathcal{H}^{(\ell)}_n}) \otimes_k ((\det_k W)^{\otimes s})^{\mu_\ell} \otimes_k K_{n-1}$$
$$= \begin{cases} S^\lambda_{K_{n-1}}\Omega^1_{K_{n-1}/k} \otimes_{K_{n-1}} (\Omega^{n-1}_{K_{n-1}/k})^{\otimes s} & \text{if } \ell | s \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, $V^{\mathrm{Aff}^{(\ell)}_n\mathbb{Q}} = (S^\lambda_k W \otimes_k (\det_k W)^{\otimes s} \otimes_k K_n)^{\mathrm{Aff}^{(\ell)}_n\mathbb{Q}}$ coincides with $(S^\lambda_k W \otimes_k (\det_k W)^{\otimes s})^{\mathrm{Aff}^{(\ell)}_n\mathbb{Q}}$, since the unipotent radical of $\mathrm{Aff}^{(\ell)}_n\mathbb{Q}$ acts trivially on $S^\lambda_k W \otimes_k (\det_k W)^{\otimes s}$ and fixes exactly k in K_n. Thus, for $n \geq 1$, we get $V^{\mathrm{Aff}^{(\ell)}_n\mathbb{Q}} = \begin{cases} k & \text{if } \lambda = 0 \text{ and } \ell | s, \\ 0 & \text{otherwise.} \end{cases}$ This implies that $(V^{\mathcal{H}^{(\ell)}_n})^{\mathrm{Aff}^{(\ell)}_{n-1}\mathbb{Q}} = \begin{cases} k & \text{if } \lambda \text{ is rectangular of height } n-1 \text{ and } \ell | (s + \frac{|\lambda|}{n-1}), \\ 0 & \text{otherwise} \end{cases}$ for $n \geq 2$ (assuming that empty λ is $(0 \times (n-1))$-rectangular). $\square$

Lemma 3.3 ([R], Lemma 7.1)**.** *Let $n > m \geq 0$ be integers and H be a subgroup of $G_{F/k}$ preserving K_n and projecting onto a subgroup of $G_{K_n/k}$ containing the permutation group of the set $\{x_1, \dots, x_n\}$. Then the subgroup in $G_{F/k}$ generated by G_{F/K_m} and H is dense.* $\square$

We note that $\mathrm{Aff}^{(\ell)}_n\mathbb{Q} \subset G_n \subset G_{K_n/k}$ does indeed contain the permutation group of the set $\{x_1, \dots, x_n\}$ for any even $\ell \geq 2$.

For any $U \in \mathcal{A}$ and $m \geq 0$ set $U_m = U^{G_{F/K_m}}$. Using smooth cochains, one defines the smooth cohomology $H^j_{\mathrm{smooth}}(G_{F/k}, -) := \mathrm{Ext}^j_{\mathcal{S}m_{G_{F/k}}}(\mathbb{Q}, -)$.

Proposition 3.4. *If $U \in \mathcal{A}$ and there is a subquotient of $U_n \in \mathfrak{SL}^u_n$ isomorphic to K_n then there is an embedding $F \hookrightarrow U$ in $\mathcal{A}$. One has $H^1_{\mathrm{smooth}}(G_{F/k}, V) = 0$ for any $V \in \mathcal{A}$.*

Proof. By Lemma 3.3, $U^{G_{F/k}} = U_{n+1}^{\mathrm{Aff}^{(\ell)}_{n+1}\mathbb{Q}} \cap U_n$ for any even $\ell \geq 2$. By Theorem 2.4, Lemma 3.1 and Lemma 3.2, $(U_{n+1}^{\mathcal{H}^{(\ell)}_{n+1}})^{\mathrm{Aff}^{(\ell)}_n\mathbb{Q}} = U_{n+1}^{\mathrm{Aff}^{(\ell)}_{n+1}\mathbb{Q}}$ for any $n \geq 1$ and any sufficiently big ℓ (where $\mathcal{H}^{(\ell)}_n$ is defined in Lemma 3.2). Then, as $U_n \subseteq U_{n+1}^{\mathcal{H}^{(\ell)}_{n+1}}$, one has $U^{G_{F/k}} = U_n^{\mathrm{Aff}^{(\ell)}_n\mathbb{Q}}$ for any sufficiently big even ℓ, and thus, $U^{G_{F/k}} \neq 0$ if there is a subquotient of $U_n \in \mathfrak{SL}^u_n$ isomorphic to K_n.

Clearly, $\mathrm{Ext}^j_{\mathcal{S}m_{G_{F/k}}}(\mathbb{Q}, -) = \mathrm{Ext}^j_{\mathcal{C}}(F, -)$ on $\mathcal{C}$ for any $j \geq 0$,[3] so we have to show that any smooth F-semi-linear extension $0 \longrightarrow V \longrightarrow U \longrightarrow F \longrightarrow 0$ splits.

Fix $u \in U$ in the preimage of $1 \in F$. The stabilizer of u contains a subgroup of type $G_{F/L}$ such that the elements of L are algebraic over K_m for some $m > 1$. Then the normalized trace $\mathrm{tr}_{/K_m} u \in U_m$ belongs again to the preimage of $1 \in K_m$, so U_m surjects onto K_m.

By Theorem 2.4 and Lemma 3.1 the semi-linear representation U_m of $\mathrm{Aff}^{(\ell)}_m\mathbb{Q}$ over K_m splits as $K_m \oplus V_m$, and thus, $U^{G_{F/k}}$ projects onto k. Then sending $1 \in k \subset F$ to one of its preimages in $U^{G_{F/k}}$ extends to a splitting of $U \longrightarrow F$. $\square$

Corollary 3.5. *For an integer $n \geq 1$ let $H \subseteq G_{F/k}$ be a subgroup containing G_{F/K_n} such that $G_{F/\overline{K_n}}$ is a normal subgroup in H. Consider $H/G_{F/K_n}$ as a subset in the set $\{K_n \overset{/k}{\hookrightarrow} \overline{K_n}\}$ of field embeddings of K_n into its algebraic closure in F over k. Suppose that $H/G_{F/K_n}$ contains Aff_n. Let $V = F[G_{F/k}/H]^\circ \in \mathcal{C}$ consist of formal degree-zero F-linear combinations of elements in $G_{F/k}/H$.*[4] *Then any quotient of V which lies in $\mathcal{A}$ is zero.*

Proof. V is generated by $\alpha = [1] - [\sigma] \in V_{2n}^{\langle(\mathrm{Aff}_{2n})_u, T_{2n}\rangle}$, where σ sends x_j to x_{2n+1-j} for each $1 \leq j \leq 2n$. Any admissible semi-linear quotient of V is generated by the image of α, which is, by Propositions 1.5 and 3.4, fixed by the whole $G_{F/k}$. On the other hand, $\sigma\alpha = -\alpha$, so any admissible semi-linear quotient of V is zero. $\square$

Theorem 3.6. *The category $\mathcal{A}$ is abelian. The functor $H^0(G_{F/L}, -)$ is exact on $\mathcal{A}$ for any subfield L in F containing k.*

Proof. We have to check that $\mathcal{A}$ is stable under taking quotients. Let $V \in \mathcal{A}$ and $V \overset{\pi}{\to} V'$ be a surjection of F-semi-linear representations of $G_{F/k}$. By Proposition 3.4, for any $K \subset F$ of finite type over k and any $v \in (V')^{G_{F/\overline{K}}} - \{0\}$, the extension $0 \to \ker\pi \to \pi^{-1}(F \cdot v) \to F \to 0$ of F-semi-linear representations of $G_{F/\overline{K}}$ splits. This implies that

[3] Any class in $\mathrm{Ext}^j_{\mathcal{C}}(F, V)$ represented by $0 \to V \to V_j \to \cdots \to V_1 \to F \to 0$ is sent to the class of $0 \to V \to V_j \to \cdots \to V_2 \to V_1 \times_F \mathbb{Q} \to \mathbb{Q} \to 0$ in $\mathrm{Ext}^j_{\mathcal{S}m_{G_{F/k}}}(\mathbb{Q}, V)$. Conversely, the class of $0 \to V \to U_j \to \cdots \to U_1 \to \mathbb{Q} \to 0$ in $\mathrm{Ext}^j_{\mathcal{S}m_{G_{F/k}}}(\mathbb{Q}, V)$ is sent to the class of $0 \to V \to (U_j \otimes F)/K \to U_{j-1} \otimes F \to \cdots \to U_1 \otimes F \to F \to 0$, where K is the kernel of the surjection $\mathrm{forget}(V) \otimes F \to V$ and $\mathrm{forget} : \mathcal{C} \longrightarrow \mathcal{S}m_{G_{F/k}}$ is the forgetful functor.

[4] In particular, if $H = G_{\{F, \overline{K_n}\}/k}$ then $V \cong F[\{L \subset F \mid L \cong \overline{K_n}\}]^\circ$ consists of formal degree-zero F-linear combinations of algebraically closed subfields in F of transcendence degree n over k.

the natural projection $V^{G_{F/K}} \xrightarrow{\pi_K} (V')^{G_{F/K}}$ is surjective, and thus, V' is also an admissible semi-linear representation.

The functor $H^0(G_{F/L}, -)$ on $\mathcal{A}$ is the composition of the forgetful functor $\Phi : \mathcal{A}_k \to \mathcal{C}_L$, the functor $H^0(G_{F/\overline{L}}, -)$ on $\mathcal{C}_L$ and the exact functor $H^0(G_{\overline{L}/L}, -)$ on $\mathcal{S}m_{G_{\overline{L}/L}}$. If L is of finite transcendence degree over k then the forgetful functor Φ factors through $\mathcal{A}_{\overline{L}}$, so the composition $H^0(G_{F/\overline{L}}, -) \circ \Phi$ is exact. If L is of infinite transcendence degree over k then $H^0(G_{F/\overline{L}}, -)$ induces an equivalence of categories $\mathcal{S}m_{G_{F/k}} \xrightarrow{\sim} \mathcal{S}m_{G_{\overline{L}/k}}$, so $H^0(G_{F/\overline{L}}, -)$ is also exact. □

Corollary 3.7. $H^1_{\text{smooth}}(G_{F/k}, \Omega^\bullet_{F/k,\text{closed}}) = H^1_{\text{smooth}}(G_{F/k}, \Omega^\bullet_{F/k,\text{exact}}) = 0.$

Proof. By Proposition 3.4, $H^1_{\text{smooth}}(G_{F/k}, \Omega^\bullet_{F/k}) = 0$. Then a piece of the long cohomological sequence of the short exact sequence $0 \to \Omega^q_{F/k,\text{closed}} \to \Omega^q_{F/k} \xrightarrow{d} \Omega^{q+1}_{F/k,\text{exact}} \to 0$ looks as $H^0(G_{F/k}, \Omega^{q+1}_{F/k,\text{exact}}) \to H^1_{\text{smooth}}(G_{F/k}, \Omega^q_{F/k,\text{closed}}) \to H^1_{\text{smooth}}(G_{F/k}, \Omega^q_{F/k}) = 0$. Evidently, $H^0(G_{F/k}, \Omega^{q+1}_{F/k,\text{exact}}) = 0$, so $H^1_{\text{smooth}}(G_{F/k}, \Omega^\bullet_{F/k,\text{closed}}) = 0$.

Clearly, $H^0(G_{F/k}, H^q_{\text{dR}/k}(F)) = 0$.[5] A piece of the long cohomological sequence of short exact sequence $0 \to \Omega^q_{F/k,\text{exact}} \to \Omega^q_{F/k,\text{closed}} \to H^q_{\text{dR}/k}(F) \to 0$ looks as

$$H^0(G_{F/k}, H^q_{\text{dR}/k}(F)) \longrightarrow H^1_{\text{smooth}}(G_{F/k}, \Omega^q_{F/k,\text{exact}}) \longrightarrow H^1_{\text{smooth}}(G_{F/k}, \Omega^q_{F/k,\text{closed}}) = 0,$$

so $H^1_{\text{smooth}}(G_{F/k}, \Omega^\bullet_{F/k,\text{exact}}) = 0$. □

3.1. **Extensions in $\mathfrak{GL}^u_n$.** Now we need the following particular case of Bott's theorem.

Theorem 3.8 ([B], cf. also [D]). *If $\mathcal{V}$ is an irreducible G_n-equivariant coherent sheaf on $\mathbb{P}^n_k$ then there exists at most one $j \geq 0$ such that $H^j(\mathbb{P}^n_k, \mathcal{V}) \neq 0$. If $H^j(\mathbb{P}^n_k, \mathcal{V})^{G_n} \neq 0$ then $\mathcal{V} \cong \Omega^j_{\mathbb{P}^n_k/k}$.* □

We also need the following explicit description of the homomorphisms in the case of commutative unipotent radicals of the target groups. It confirms general expectations, sketched in Remark 8.19 of [BT] and in [T], §5.1.

Theorem 3.9 ([LR], Theorem 3). *Let G be a simple simply connected Chevalley group over a field κ of characteristic zero. Let $\mathcal{G}$ be a connected algebraic group over a field extension K of κ. Let $\tau : G(\kappa) \to \mathcal{G}(K)$ be a homomorphism with Zariski dense image. Assume that the unipotent radical $\mathcal{G}_u$ of $\mathcal{G}$ is commutative and the composition $G(\kappa) \xrightarrow{\tau} \mathcal{G}(K) \to G'(K)$, where $G' = \mathcal{G}/\mathcal{G}_u$, is induced by a rational K-homomorphism $\lambda : G \times_\kappa K \longrightarrow G'$.*

Then $\mathcal{G}_u$ splits over a finite field extension L/K into a direct sum of r copies of the adjoint representation of G', so $r = \dim \mathcal{G}_u / \dim G'$.

Let $A = \kappa[\varepsilon_1, \dots, \varepsilon_r]/(\varepsilon_1^2, \dots, \varepsilon_r^2)$ and $\mathcal{H} = R_{A/\kappa}(G \times_\kappa A) \cong G \ltimes \mathfrak{g}^{\oplus r}$, where $\mathfrak{g} = \text{Lie}(G)$ is the adjoint representation of G.

Then there exist derivations $\delta_1, \dots, \delta_r : \kappa \to L$ and an L-isogeny $\mu : \mathcal{H} \times_\kappa L \longrightarrow \mathcal{G} \times_\kappa L$ such that $\tau = \mu \circ \eta_\delta$, where $\eta_\delta : G(\kappa) \longrightarrow \mathcal{H}(L)$ is induced by the ring embedding $id + \sum_{j=1}^r \varepsilon_j \delta_j : \kappa \to A \otimes_\kappa L$. □

[5] Let $\omega \in \Omega^q_{A/k} \subset \Omega^q_{F/k}$ represent a $G_{F/k}$-fixed element for a smooth finitely generated k-subalgebra $A \subset F$. Fix $\sigma \in G_{F/k}$ such that A and $\sigma(A)$ are algebraically independent over k. Then $\omega - \sigma\omega = d\eta$ for some $\eta \in \Omega^{q-1}_{B/k}$, where $B \subset F$ is a smooth finitely generated $(A \otimes_k \sigma(A))$-subalgebra. Fix a k-algebra homomorphism $\varphi : \sigma(A) \longrightarrow \overline{k} \subset F$ and extend $id \cdot \varphi : A \otimes_k \sigma(A) \longrightarrow A \otimes_k \overline{k} \subset F$ to $\psi : B \longrightarrow F$. Then ψ induces a morphism of differential graded k-algebras $\psi_* : \Omega^\bullet_{B/k} \longrightarrow \Omega^\bullet_{F/k}$ identical on $\Omega^\bullet_{A/k}$, so $\omega = d\psi_*(\eta)$.

Lemma 3.10. *Let $n \geq 2$. Suppose that $\mathrm{Ext}^1_{\mathfrak{S}\mathfrak{L}^u_n}(K_n, V_\circ) \neq 0$ for some irreducible object $V_\circ$ of $\mathfrak{S}\mathfrak{L}^u_n$. Then either $V_\circ \cong \Omega^1_{K_n/k}$, or $V_\circ \cong \mathrm{Der}(K_n/k)$. One has $\mathrm{Ext}^1_{\mathfrak{S}\mathfrak{L}^u_n}(K_n, \Omega^1_{K_n/k}) = k$ and $\mathrm{Ext}^1_{\mathfrak{S}\mathfrak{L}^u_n}(K_n, \mathrm{Der}(K_n/k)) = \mathrm{Der}(k)$.*

Proof. Let $\mathcal{V} = \mathcal{S}(V_\circ)$ be the irreducible coherent G_n-equivariant sheaf on $\mathbb{P}(Q) = \mathbb{P}^n_k$ with the generic fibre $V_\circ$, and let $0 \longrightarrow V_\circ \longrightarrow V \longrightarrow K_n \longrightarrow 0$ be an extension in $\mathfrak{S}\mathfrak{L}^u_n$.

Suppose that the short exact sequence $0 \to \mathcal{V} \to \mathcal{S}(V) \to \mathcal{O} \to 0$ of coherent sheaves on $\mathbb{P}(Q)$ splits. Let E be the total space of $\mathcal{S}(V) \cong \mathcal{O} \oplus \mathcal{V}$. Then, as $\mathrm{Aut}(\mathcal{S}(V), \mathcal{V}) \cong (\mathbb{G}_m \times \mathbb{G}_m) \ltimes \Gamma(\mathbb{P}^n_k, \mathcal{V})$, the G_n-structure on V corresponds to a splitting of the sequence

$$1 \longrightarrow (\mathbb{G}_m \times \mathbb{G}_m) \ltimes \Gamma(\mathbb{P}(Q), \mathcal{V}) \longrightarrow \mathrm{Aut}_{\mathrm{lin}}(E, \mathrm{tot}(\mathcal{V})) \longrightarrow G_n \longrightarrow 1. \tag{3}$$

As $H^1(G_n, \mathbb{G}_m \times \mathbb{G}_m) = 1$, Theorem 3.9 (with $G = \mathrm{SL}_{n+1}k$, $G' = G_n$ and $\mathcal{G}_u \subseteq \Gamma(\mathbb{P}^n_k, \mathcal{V})$) implies that a non-standard splitting of (3) can exist only if $\Gamma(\mathbb{P}^n_k, \mathcal{V})$ is isomorphic to the adjoint representation of G_n, i.e., if $\mathcal{V} \cong \mathcal{T}_{\mathbb{P}^n_k/k}$. The identity $\mathrm{Ext}^1_{\mathfrak{S}\mathfrak{L}^u_n}(K_n, \mathrm{Der}(K_n/k)) = \mathrm{Der}(k)$ follows also from Theorem 3.9.

If $\mathcal{V} \cong \Omega^1_{\mathbb{P}^n_k/k}$ then the target of the homomorphism $\mathrm{Ext}^1_{\mathfrak{S}\mathfrak{L}^u_n}(K_n, \mathcal{V}_\circ) \xrightarrow{\alpha} \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}_{\mathbb{P}^n_k}, \mathcal{V}) = k$ induced by the functor $\mathcal{S}$ is generated by the class of the Euler extension $0 \to \Omega^1_{\mathbb{P}(Q)/k} \to Q^\vee \otimes_k \mathcal{O}_{\mathbb{P}(Q)}(-1) \to \mathcal{O}_{\mathbb{P}(Q)} \to 0$. Let E be the total space of the vector bundle with the sheaf of sections $Q^\vee \otimes_k \mathcal{O}(-1)$. Any G_n-structure on the middle term of this extension corresponding to an element of $\mathrm{Ext}^1_{\mathfrak{S}\mathfrak{L}^u_n}(K_n, \Omega^1_{K_n/k})$ is a splitting of the short exact sequence

$$1 \longrightarrow \mathrm{Aut}(Q^\vee \otimes_k \mathcal{O}(-1), \Omega^1_{\mathbb{P}(Q)/k}) \longrightarrow \mathrm{Aut}_{\mathrm{lin}}(E, \mathrm{tot}(\Omega^1_{\mathbb{P}(Q)/k})) \longrightarrow G_n \longrightarrow 1. \tag{4}$$

As any point of $\mathbb{P}(Q)$ determines a hyperplane in $Q^\vee$, the group $\mathrm{Aut}(Q^\vee \otimes_k \mathcal{O}(-1), \Omega^1_{\mathbb{P}(Q)})$ coincides with the subgroup of $\mathrm{GL}(Q)$ stabilizing all hyperplanes in $Q^\vee$, i.e., with the centre $\mathbb{G}_m$ of $\mathrm{GL}(Q)$. Then $\mathrm{Aut}_{\mathrm{lin}}(E, \mathrm{tot}(\Omega^1_{\mathbb{P}(Q)/k}))$ is a central $\mathbb{G}_m$-extension of G_n, so the splitting of (4) is unique and corresponds to the usual G_n-equivariant structure. ◻

Corollary 3.11. *If $k = \overline{\mathbb{Q}}$ then $\mathfrak{S}\mathfrak{L}^u_n$ is equivalent to the category of G_n-equivariant vector bundles on $\mathbb{P}^n_k$.*

Proof. The category of G_n-equivariant vector bundles on $\mathbb{P}^n_k$ is a full sub-category of $\mathfrak{S}\mathfrak{L}^u_n$ with the same irreducible objects. As it is mentioned at the beginning of §1, p.3, the objects of $\mathfrak{S}\mathfrak{L}^u_n$ are generic fibres of coherent G_n-sheaves on $\mathbb{P}^n_k$. Suppose that $V \in \mathfrak{S}\mathfrak{L}^u_n$ is the generic fibre of a non-equivariant vector bundle on $\mathbb{P}^n_k$ of minimal possible rank. Then it fits into an exact sequence $0 \to B \to V \to A \to 0$, where A, B are the generic fibres of G_n-equivariant vector bundles on $\mathbb{P}^n_k$ and A is irreducible. Let $0 \neq C \subseteq B$ be an irreducible sub-object and $D = B/C$. Then the rows in the following commutative diagram are exact:

$$\begin{array}{ccccccccc} \mathrm{Hom}_u(A,D) & \to & \mathrm{Ext}^1_u(A,C) & \to & \mathrm{Ext}^1_u(A,B) & \to & \mathrm{Ext}^1_u(A,D) & \to & \mathrm{Ext}^2_u(A,C) \\ \| & & \|\text{Lemma 3.10} & & \bigcup & & \|\text{minimality of } V & & \uparrow \xi \\ \mathrm{Hom}_{\mathrm{eq}}(A,D) & \to & \mathrm{Ext}^1_{\mathrm{eq}}(A,C) & \to & \mathrm{Ext}^1_{\mathrm{eq}}(A,B) & \to & \mathrm{Ext}^1_{\mathrm{eq}}(A,D) & \to & \mathrm{Ext}^2_{\mathrm{eq}}(A,C) \end{array}$$

where subscript u refers to the category $\mathfrak{S}\mathfrak{L}^u_n$ and eq refers to the category of equivariant vector bundles.

Let us show that ξ is injective for any irreducible A and C. As $C \otimes_{K_n} A^\vee$ is semi-simple (as it follows from Proposition 1.5) and $\mathrm{Ext}^2_?(A, C) = \mathrm{Ext}^2_?(K_n, C \otimes_{K_n} A^\vee)$, where $? = u$ or eq, we may assume that $A = K_n$ and C is still irreducible. By Bott's Theorem 3.8, $\dim_k \mathrm{Ext}^2_{\mathrm{eq}}(K_n, C) \leq 1$ with equality only if $C \cong \Omega^2_{K_n/k}$, so we assume further that

$C = \Omega^2_{K_n/k}$. The forgetful functor from $\mathfrak{S}\mathfrak{L}^u_n$ to the category of coherent sheaves on $\mathbb{P}^n_k$ induces a homomorphism $\mathrm{Ext}^2_u(A, C) \to H^2(\mathbb{P}^n_k, \Omega^2_{\mathbb{P}^n_k/k})$. Clearly, its composition with ξ is an isomorphism.

Then the 5-lemma implies that $\mathrm{Ext}^1_u(A, B) = \mathrm{Ext}^1_{\mathrm{eq}}(A, B)$, and thus, V is equivariant. $\square$

4. The category $\mathcal{A}$ in the case $k = \overline{\mathbb{Q}}$

In this section we determine (in Theorem 4.10) the structure of the objects of $\mathcal{A}$ in the case $k = \overline{\mathbb{Q}}$, the field of algebraic numbers. The objects V of $\mathcal{A}$ are quotients of sums of representations of G over k induced by rational representations of $\mathrm{GL}_m k$'s (considered as subquotients of G) with coefficients extended to F (cf. Lemma 4.1). Then we find (in Lemma 4.2) a supply of elements in the induced representations vanishing in V, and use them in Lemmas 4.3–4.7 to show that the objects of $\mathcal{A}$ are sums of quotients of $\bigotimes^{\bullet}_F \mathfrak{m}$. In §4.1 we study extensions in $\mathcal{A}$.

Let $V \in \mathcal{A}$ and $m \geq 0$ be such that $V_m \neq 0$. Then there is a non-zero morphism $F[G_{F/k}/G_{F/K_m}] \otimes_{K_m[\mathrm{PGL}_{m+1}k]} V_m \longrightarrow V$ in $\mathcal{C}$. The object V_m of $\mathfrak{S}\mathfrak{L}^u_m$ admits an irreducible sub-object $A \neq 0$. By Theorem 2.4, $A \cong S^{\lambda}_{K_m} \Omega^1_{K_m/k}$ for a Young diagram λ. Then $F[G_{F/k}/G_{F/K_m}] \otimes_{K_m[\mathrm{PGL}_{m+1}k]} A \longrightarrow V$ is also non-zero. Clearly, $A = B \otimes_k K_m$, where $B := A^{(\mathrm{Aff}_m)_u} \cong (S^{\lambda}_{K_m} \Omega^1_{K_m/k})^{(\mathrm{Aff}_m)_u} \cong S^{\lambda}_k(k^m)$ is a rational irreducible representation of $\mathrm{GL}_m k := \mathrm{Aff}_m/(\mathrm{Aff}_m)_u$.

This implies that there is a non-zero morphism $U := F[W^\circ] \otimes_{k[\mathrm{GL}_m k]} B \xrightarrow{\varphi} V$ and a surjection $U \longrightarrow S^{\lambda}_F \Omega^1_{F/k}$, where $W^\circ := \{K_m \overset{/k}{\hookrightarrow} F\}/(\mathrm{Aff}_m)_u$ is considered as a $G_{F/k}$-set.

As any embedding $K_m \overset{/k}{\hookrightarrow} F$ is determined by the images of $x_1, \dots, x_m$, one can consider W° as a subset of $(F/k)^m$ consisting of m-tuples with entries algebraically independent over k. More invariantly, let $W := \mathrm{Hom}_k((K_m/k)^{(\mathrm{Aff}_m)_u}, F/k) \cong (F/k)^m$ be the group (a k-vector space) generated by W°. The isomorphism is given by restriction of the homomorphisms to the basis $\{\overline{x_1}, \dots, \overline{x_m}\}$ of $(K_m/k)^{(\mathrm{Aff}_m)_u}$. Define a homogeneous map $\varkappa : W \longrightarrow \Omega^m_{F/k} \otimes_k \det_k \mathrm{Hom}_G(F/k, W)$ of degree m by inverting the first isomorphism in the sequence $W \overset{\sim}{\longleftarrow} (F/k) \otimes_k \mathrm{Hom}_G(F/k, W) \overset{d \otimes id}{\hookrightarrow} \Omega^1_{F/k} \otimes_k \mathrm{Hom}_G(F/k, W) \longrightarrow \mathrm{Sym}^m_F(\Omega^1_{F/k} \otimes_k \mathrm{Hom}_G(F/k, W)) \longrightarrow \Omega^m_{F/k} \otimes_k \det_k \mathrm{Hom}_G(F/k, W)$. Then $W^\circ = \{w \in W \mid \varkappa(w) \neq 0\}$.

Let $(y_1, \dots, y_m) \mapsto [y_1, \dots, y_m]$ be the map $(F/k)^m \longrightarrow \{0\} \cup W^\circ$ sending $(y_1, \dots, y_m)$ to $[x_j \mapsto y_j]$ if $y_1, \dots, y_m$ are algebraically independent over k, and to 0 otherwise. Then $[\mu y_1, \dots, \mu y_m] \otimes b = \mu^{|\lambda|}[y_1, \dots, y_m] \otimes b$ in U for any $\mu \in k$. If $y_1, \dots, y_m$ belong to the k-linear envelope of $x_1, \dots, x_M$ for some integer $M \geq 1$ then $[y_1, \dots, y_m] \otimes b \in U_M^{(\mathrm{Aff}_M)_u}$ is a weight $|\lambda|$ eigenvector of the centre of $\mathrm{GL}_M k$.

Let $U^!_M \subseteq U_M^{(\mathrm{Aff}_M)_u}$ be the $k[\mathrm{GL}_M k]$-envelope of $[x_1, \dots, x_m] \otimes b$ for some $b \neq 0$ (which is the same as k-envelope of all $[y_1, \dots, y_m] \otimes c$ for algebraically independent $y_1, \dots, y_m$ in the k-linear envelope of $x_1, \dots, x_M$ and all c). Clearly, $U^!_m \cong B$ as $k[\mathrm{GL}_m k]$-modules, and any non-zero morphism $U \longrightarrow V$ induces an embedding $U^!_m \hookrightarrow V$.

Lemma 4.1. *If $k = \overline{\mathbb{Q}}$ then for any $V \in \mathfrak{S}\mathfrak{L}^u_M$ the representation $V^{(\mathrm{Aff}_M)_u}$ of $\mathrm{GL}_M k$ is rational semi-simple, and $V^{(\mathrm{Aff}_M)_u} \otimes_k K_M = V$.*

Proof. By Corollary 3.11, V is the generic fibre of a $\mathrm{PGL}_{M+1}k$-equivariant vector bundle.

Then, by Lemma 6.3 (1) of [R], $V = V^{U_0} \otimes_k K_M$, where U_0 is a $\mathbb{Q}$-lattice in $(\mathrm{Aff}_M)_u$. The group $(\mathrm{Aff}_M)_u$ acts rationally on V^{U_0}. As the action of the $\mathbb{Q}$-lattice U_0 is trivial, the action of the entire $(\mathrm{Aff}_M)_u$ is trivial, i.e., $V = V^{(\mathrm{Aff}_M)_u} \otimes_k K_M$. Then the action of $\mathrm{GL}_M k$ on $V^{(\mathrm{Aff}_M)_u}$ is rational, and thus, semi-simple. □

REMARK. There is no semisimplicity if k contains a transcendental element.

Indeed, let the underlying K_n-vector space of $V \in \mathfrak{S}\mathfrak{L}_n^u$ be $K_n \oplus \Omega^1_{K_n}/(\Lambda \otimes_k K_n)$ for a proper k-vector subspace $\Lambda \subset \Omega^1_k$ of finite codimension, and the G_n-action be given by $\sigma(f, \omega) = (\sigma f, \sigma\omega + \sigma f \cdot d\log(\sigma\eta/\eta))$ for any $\sigma \in G_n$, where $\eta = dx_1 \wedge \cdots \wedge dx_n \in \Omega^n_{K_n/k}$. (So V fits into a non-split exact sequence $0 \to \Omega^1_{K_n}/(\Lambda \otimes_k K_n) \to V \to K_n \to 0$.) Then $\sigma(f, \omega) = (\sigma f, \sigma\omega)$ if $\sigma \in (\mathrm{Aff}_n)_u$, and therefore, $V^{(\mathrm{Aff}_n)_u} = k \oplus \Omega^1_k/\Lambda$ is a non-trivial extension of trivial representations of $\mathrm{GL}_n k$.

Lemma 4.2. *The kernel of $U^!_M \longrightarrow S^\lambda_F \Omega^1_{F/k}$ is contained in the kernel of $U \xrightarrow{\varphi} V$.*

Proof. By Lemma 4.1, the image $\overline{U^!_M}$ of $U^!_M$ in V is isomorphic to $\bigoplus_{|\nu|=|\lambda|}(S^\nu_k(k^M))^{m_\nu}$. As $\overline{U^!_m} \subseteq \overline{U^!_M}^{\mathrm{GL}_M k \cap G_{K_M/K_m}} \cong \bigoplus_{|\nu|=|\lambda|}(S^\nu_k(k^m))^{m_\nu}$ and $U^!_M$ is generated by $U^!_m \xrightarrow{\sim} S^\lambda_k(k^m) \subseteq S^\lambda_k(k^M)$, we see that $\overline{U^!_M}$ is isomorphic to $S^\lambda_k(k^M)$. Then the kernel of $U^!_M \longrightarrow S^\lambda_F \Omega^1_{F/k}$ is contained in the kernel of the morphism $U \xrightarrow{\varphi} V$. □

Let $W^{M\circ} \subset W^M$ be the subset consisting of M-tuples $(y_1, \dots, y_M)$ such that $\sum_{i\in I} y_i \in W^\circ$ for any non-empty subset $I \subseteq \{1, \dots, M\}$.

Let $k[W^{M\circ}] \longrightarrow k[W^\circ] \otimes_{k[k^\times]} k(M)$ be the k-linear map sending $(y_1, \dots, y_M)$ to

$$\langle y_1, \dots, y_M \rangle := \sum_{I \subseteq \{1,\dots,M\}} (-1)^{\#I}[\sum_{i\in I} y_i] \in k[W^\circ] \otimes_{k[k^\times]} k(M).$$

Here $k(M)$ denotes a one-dimensional k-vector space with $k^\times$-action by M-th powers. As $(y, \dots, y)$ is sent to

$$\sum_{j\geq 0}(-1)^j \binom{M}{j} j^M[y] = (t\frac{d}{dt})^M(1-t)^M|_{t=1} \cdot [y] = (-1)^M M! \cdot [y],$$

it is surjective. Clearly, $\langle y_1, \dots, y_M \rangle = \langle y_{\theta(1)}, \dots, y_{\theta(M)} \rangle$ for any permutation $\theta \in \mathfrak{S}_M$. Let $\tilde{U} := F[W^{|\lambda|\circ}] \longrightarrow U$ be the F-linear surjection sending $(y_1, \dots, y_{|\lambda|})$ to $\langle y_1, \dots, y_{|\lambda|} \rangle \otimes b$.

Lemma 4.3. *Let the k-linear map $\alpha : k[W^\circ] \longrightarrow \bigotimes^M_k W$ be given by $[w] \mapsto w^{\otimes M}$. Then α factors through $k[W^\circ] \otimes_{k[k^\times]} k(M)$ and $\langle y_1, \dots, y_M \rangle \mapsto (-1)^M \sum_{\theta\in\mathfrak{S}_M} y_{\theta(1)} \otimes \cdots \otimes y_{\theta(M)}$ if $(y_1, \dots, y_M) \in W^{M\circ}$.*

Proof. The element $\langle y_1, \dots, y_M \rangle$ is sent to

$$\sum_{I \subseteq \{1,\dots,M\}} (-1)^{\#I}(\sum_{i\in I} y_i)^{\otimes M} = \sum_{1\leq i_1,\dots,i_M \leq M} A_{i_1,\dots,i_M} y_{i_1} \otimes \cdots \otimes y_{i_M}.$$

If $S = \{1, \dots, M\} \backslash \{i_1, \dots, i_M\}$ then $A_{i_1,\dots,i_M} = \sum_{J\subseteq S}(-1)^{M-\#J}$, so $A_{i_1,\dots,i_M} = 0$ if S is non-empty, and $A_{i_1,\dots,i_M} = (-1)^M$ if $\{1, \dots, M\} = \{i_1, \dots, i_M\}$. □

Lemma 4.4. *If $M = |\lambda|$, $\mu \in k$, $y_0, y_1, y_0 + y_1 \in W^\circ$ and all coordinates of $t_2, \dots, t_M \in W$ are algebraically independent over $k(y_0, y_1)$ then*

$$\langle y_0 + y_1, t_2, \dots, t_M \rangle \otimes b \equiv \langle y_0, t_2, \dots, t_M \rangle \otimes b + \langle y_1, t_2, \dots, t_M \rangle \otimes b \bmod \ker \varphi,$$

and $\langle \mu y_1, t_2, \dots, t_M\rangle \otimes b \equiv \mu\langle y_1, t_2, \dots, t_M\rangle \otimes b \bmod \ker\varphi$.

Proof. It follows from Lemmas 4.2 and 4.3, that $\langle z_0+z_1, z_2, \dots, z_M\rangle\otimes b-\langle z_0, z_2, \dots, z_M\rangle\otimes b - \langle z_1, \dots, z_M\rangle \otimes b$ and $\langle \mu z_1, z_2, \dots, z_M\rangle \otimes b - \mu\cdot\langle z_1, \dots, z_M\rangle\otimes b$ are sent to zero by φ, where the coordinates of z_j are $x_{jm+1}, \dots, x_{jm+m}$. As the G-orbits of these elements are also sent to zero by φ, for some $u, v \in W^\circ$ with coordinates algebraically independent over the subfield in F generated over k by $y_0, y_1, t_2, \dots, t_M$, one has the following congruences modulo the kernel of φ:

$$\langle y_0+y_1, t_2, \dots, t_M\rangle \otimes b \equiv \langle y_0+y_1+u, t_2, \dots, t_M\rangle\otimes b - \langle u, t_2, \dots, t_M\rangle \otimes b, \tag{5}$$

$$\langle y_0, t_2, \dots, t_M\rangle \otimes b \equiv \langle y_0+u-v, t_2, \dots, t_M\rangle\otimes b - \langle u-v, t_2, \dots, t_M\rangle \otimes b, \tag{6}$$

$$\langle y_1, t_2, \dots, t_M\rangle \otimes b \equiv \langle y_1+v, t_2, \dots, t_M\rangle\otimes b - \langle v, t_2, \dots, t_M\rangle \otimes b. \tag{7}$$

As $\langle y_0+y_1+u, t_2, \dots, t_M\rangle\otimes b \equiv \langle y_0+u-v, t_2, \dots, t_M\rangle\otimes b + \langle y_1+v, t_2, \dots, t_M\rangle \otimes b$, and $\langle u, t_2, \dots, t_M\rangle\otimes b \equiv \langle u-v, t_2, \dots, t_M\rangle \otimes b + \langle v, t_2, \dots, t_M\rangle\otimes b$, the left hand side of the congruence (5) is congruent to the sum of the left hand sides of the congruences (6) and (7) modulo $\ker\varphi$. ☐

Lemma 4.5. *Let* $(y_1, \dots, y_M) \in \{0\}\times W^{(M-1)\circ}\cup W^{M\circ}$ *and let the coordinates of* $t_{ij}\in W^\circ$ *be algebraically independent over* $k(y_1, \dots, y_M)$*, where* $1\le i\le M$ *and* $2\le j\le M$*. Set* $[0]:=0$ *and* $\langle 0, y_2, \dots, y_M\rangle := 0$*. Then*

$$\langle y_1, \dots, y_M\rangle\otimes b \equiv \sum_{J\subseteq\{2,\dots,M\}}(-1)^{\#J}\langle y_1, \sum_{s\in\{1\}\cup J} t_{s2}, \dots, \sum_{s\in\{1\}\cup J} t_{sM}\rangle\otimes b$$
$$- \sum_{\emptyset\neq I\subseteq\{2,\dots,M\}}(-1)^{\#I}\langle y_1, y_2+\sum_{i\in I}t_{2i}, \dots, y_M+\sum_{i\in I}t_{Mi}\rangle\otimes b \bmod \ker\varphi. \tag{8}$$

Proof. It follows from the identities

$$[\sum_{s\in J} y_s] = \sum_{\emptyset\neq I\subseteq\{2,\dots,M\}}(-1)^{\#I}\left([\sum_{s\in J}\sum_{i\in I}t_{si}] - [\sum_{s\in J}(y_s+\sum_{i\in I}t_{si})]\right)$$
$$- \langle \sum_{s\in J} y_s, \sum_{s\in J} t_{s2}, \dots, \sum_{s\in J} t_{sM}\rangle$$

that

$$\langle y_1, \dots, y_M\rangle = \sum_{\emptyset\neq I\subseteq\{2,\dots,M\}}(-1)^{\#I}\left(\langle\sum_{i\in I}t_{1i}, \dots, \sum_{i\in I}t_{Mi}\rangle\right.$$
$$\left.-\langle y_1+\sum_{i\in I}t_{1i}, \dots, y_M+\sum_{i\in I}t_{Mi}\rangle\right) - \sum_{J\subseteq\{1,\dots,M\}}(-1)^{\#J}\langle\sum_{s\in J}y_s, \sum_{s\in J}t_{s2}, \dots, \sum_{s\in J}t_{sM}\rangle. \tag{9}$$

Then Lemma 4.4, applied to the summands containing y_1, implies that

$$\langle y_1, \dots, y_M\rangle\otimes b \equiv \sum_{J\subseteq\{2,\dots,M\}}(-1)^{\#J}\langle y_1, \sum_{s\in\{1\}\cup J} t_{s2}, \dots, \sum_{s\in\{1\}\cup J} t_{sM}\rangle\otimes b$$
$$- \sum_{\emptyset\neq I\subseteq\{2,\dots,M\}}(-1)^{\#I}\langle y_1, y_2+\sum_{i\in I}t_{2i}, \dots, y_M+\sum_{i\in I}t_{Mi}\rangle\otimes b + \langle 0, y_2, \dots, y_M\rangle\otimes b,$$

so we get (8). ☐

Lemma 4.6. *If* $M=|\lambda|$, $\mu\in k$ *and* $(z_j,y_2,\dots,y_M),(\sum_{i=1}^{N}z_i,y_2,\dots,y_M),(\mu z_1,y_2,\dots,y_M)\in W^{M\circ}$ *for all* $1\le j\le N$ *then*

$$\langle\sum_{j=1}^{N}z_j,y_2,\dots,y_M\rangle\otimes b\equiv\sum_{j=1}^{N}\langle z_j,y_2,\dots,y_M\rangle\otimes b \bmod \ker\varphi, \tag{10}$$

and $\langle\mu z_1,y_2,\dots,y_M\rangle\otimes b\equiv\mu\langle z_1,y_2,\dots,y_M\rangle\otimes b \bmod \ker\varphi$.

Proof. If $N=2$ then (10) follows from Lemmas 4.4 and 4.5. If $N\ge 3$ then

$$\langle\sum_{j=1}^{N}z_j,y_2,\dots,y_M\rangle\otimes b\equiv\langle\sum_{j=3}^{N}z_j-u,y_2,\dots,y_M\rangle\otimes b+\langle z_1+z_2+u,y_2,\dots,y_M\rangle\otimes b$$

for any sufficiently general $u\in W^\circ$. By the induction assumption, this is congruent to $\sum_{j=3}^{N}\langle z_j,y_2,\dots,y_M\rangle\otimes b-\langle u,y_2,\dots,y_M\rangle\otimes b+\langle z_1,y_2,\dots,y_M\rangle\otimes b+\langle z_2+u,y_2,\dots,y_M\rangle\otimes b\equiv\sum_{j=1}^{N}\langle z_j,y_2,\dots,y_M\rangle\otimes b$. □

Lemma 4.7. *The* k*-linear map* $k[W^{M\circ}]\longrightarrow\bigotimes_k^M W$, *given by* $[(y_1,\dots,y_M)]\mapsto y_1\otimes\cdots\otimes y_M$, *is surjective and its kernel is spanned over* k *by* $[(y_0,\dots,y_{j-1}+y_j,\dots,y_M)]-[(y_0,\dots,\widehat{y_{j-1}},\dots,y_M)]-[(y_0,\dots,\widehat{y_j},\dots,y_M)]$ *and* $\mu[(y_1,\dots,y_M)]-[(y_1,\dots,\mu y_j,\dots,y_M)]$ *for all* $y_0,\dots,y_M\in W^\circ$ *and all* $\mu\in k^\times$.

Proof. By Zorn's lemma, there exists a maximal subset S in W° consisting of k-linear independent elements. If S does not generate W then the k-linear envelope of S does not contain W°, i.e., an element $y\in W^\circ$ k-linear independent over S, so $S\cup\{y\}$ is a bigger subset in W° consisting of k-linear independent elements. This contradiction shows that S is a k-basis of W.

For any $y\in W^\circ$ and any $z\in W$ there exist at most m values of $\mu\in k$ such that $y+\mu z\notin W^\circ$, since this condition is equivalent to vanishing of the $\Omega^m_{F/k}$-valued polynomial $(dy_1+\mu dz_1)\wedge\cdots\wedge(dy_m+\mu dz_m)$ of degree $\le m$ in μ with non-zero constant term. Let us show that the map $(k^\times S)^M\cap W^{M\circ}\longrightarrow S^M$ given by the projectivization is surjective. Indeed, let $(s_1,\dots,s_M)\in S^M$. For all but $\le m$ values of $\mu\in k^\times$ one has $s_1+\mu s_2\in W^\circ$. Fix one of such μ and set $s_2':=\mu s_2$. For all but $\le 3m$ values of $\mu\in k^\times$ one has $s_1+\mu s_3, s_2'+\mu s_3, s_1+s_2'+\mu s_3\in W^\circ$. Fix one of such μ and set $s_3':=\mu s_3$. Proceeding further this way, we get an element $(s_1,s_2',\dots,s_M')\in((k^\times S)^M)\cap W^{M\circ}$ projecting onto $(s_1,\dots,s_M)$.

Fix a section of the projection $(k^\times S)^M\cap W^{M\circ}\longrightarrow S^M$. Denote by $\widetilde{S^M}$ the image of S^M under this section. Then $\widetilde{S^M}$ considered as a subset in $k[W^{M\circ}]$ maps to a basis of $\bigotimes_k^M W$, which shows the surjectivity.

For the injectivity is suffices to check that $k[\widetilde{S^M}]$ maps onto $k[W^{M\circ}]$ modulo the relations. For an element $w=(w_1,\dots,w_M)\in W^{M\circ}$ set $l(w):=\sum_{j=1}^{M}l_j(w)\ge M$, where $l_j(w)$ is the number of non-zero coordinates of w_j in the basis S.

By induction on $l(w)$ we are going to show that $[w]$ is in the image of $k[\widetilde{S^M}]$.

If $l(w) = M$ then $w_j = \mu_j s_j$ for all $1 \le j \le M$, where $(s_1, \dots, s_M) \in \widetilde{S^M}$. For any sufficiently general $\nu_2, \dots, \nu_M \in k^\times$ one has

$$[w] \equiv \nu_2^{-1} \cdots \nu_M^{-1}[(w_1, \nu_2 w_2, \dots, \nu_M w_M)] \equiv \mu_1 \nu_2^{-1} \cdots \nu_M^{-1}[(s_1, \nu_2 w_2, \dots, \nu_M w_M)]$$
$$\equiv \mu_1 \mu_2 \nu_3^{-1} \cdots \nu_M^{-1}[(s_1, s_2, \nu_3 w_3, \dots, \nu_M w_M)] \equiv \cdots \equiv \mu_1 \cdots \mu_M[(s_1, \dots, s_M)].$$

The induction step: if, for instance, $l_1(w) \ge 2$ then for all but $\le l_1(w)m + m$ values of $\mu \in k^\times$ one has $[w] \equiv \mu^{-1}[(\mu w_1, w_2, \dots, w_M)] \equiv \mu^{-1} \sum_{s \in S}[(\mu \mu_s s, w_2, \dots, w_M)]$, where $w_1 = \sum_{s \in S} \mu_s s$ is a finite sum. By the induction assumption, the summands $[(\mu\mu_s s, w_2, \dots, w_M)]$ are in the image of $k[\widetilde{S^M}]$, and thus, $[w]$ is also there. □

Let $\mathfrak{m}$ be the kernel of the multiplication map $F \otimes_k F \xrightarrow{\times} F$. The map $F \otimes_k (F/k) \longrightarrow \mathfrak{m}$, given by $\sum_j z_j \otimes \overline{y_j} \mapsto \sum_j z_j \otimes y_j - (\sum_j z_j y_j) \otimes 1$ is clearly an isomorphism, so we can use the notation $\mathfrak{m}$ instead of $F \otimes_k (F/k)$, and the multiplicative structure of the ideal $\mathfrak{m}$.

Lemma 4.8. *The element* $\alpha_q := (x_1 \otimes 1 - 1 \otimes x_1)^{s_1} \otimes \cdots \otimes (x_q \otimes 1 - 1 \otimes x_q)^{s_q} \in \bigotimes_F^q \mathfrak{m}$ *generates the sub-object* $\mathfrak{m}^{s_1} \otimes_F \cdots \otimes_F \mathfrak{m}^{s_q}$.

Proof. We need to show that for any collection of $\beta_i \in \mathfrak{m}^{s_i}$ the element $\beta_1 \otimes \cdots \otimes \beta_q$ belongs to the $F[G_{F/k}]$-submodule generated by α_q. Set $\alpha := x_1 \otimes 1 - 1 \otimes x_1$. Then the $G_{F/k}$-orbit of α^s contains $(\sum_{j=1}^s a_j(y_j \otimes 1 - 1 \otimes y_j))^s$ for any $a_j \in k$ and $y_j \in F$ such that $\sum_{j=1}^s a_j y_j \notin k$. The k-span of such elements with fixed $y_1, \dots, y_s$ contains $\prod_{j=1}^s (y_j \otimes 1 - 1 \otimes y_j)$. Such products generate $\mathfrak{m}^s$ as an ideal. Moreover, they generate $\mathfrak{m}^s$ as a $F \otimes_k k$-module: $(1 \otimes b) \prod_{j=1}^s (y_j \otimes 1 - 1 \otimes y_j) = ((by_1 \otimes 1 - 1 \otimes by_1) - (y_1 \otimes 1)(b \otimes 1 - 1 \otimes b)) \prod_{j=2}^s (y_j \otimes 1 - 1 \otimes y_j) = (by_1 \otimes 1 - 1 \otimes by_1) \prod_{j=2}^s (y_j \otimes 1 - 1 \otimes y_j) - (y_1 \otimes 1)(b \otimes 1 - 1 \otimes b) \prod_{j=2}^s (y_j \otimes 1 - 1 \otimes y_j)$.

This implies that $\beta_i = \sum_{j=1}^{s_i} f_{ij} \cdot \sigma_{ij} \alpha^{s_i}$ for some $\sigma_{ij} \in G_{F/k}$ and $f_{ij} \in F$. The $G_{F/k}$-orbit of α_q contains $\alpha' := (z_1 \otimes 1 - 1 \otimes z_1)^{s_1} \otimes \cdots \otimes (z_q \otimes 1 - 1 \otimes z_q)^{s_q}$, where $z_1, \dots, z_q \in F$ are algebraically independent over the subfield in F generated over k by all $f_{ij}, \sigma_{ij} x_1$.

For each pair (i, j) such that $1 \le i \le q$ and $1 \le j \le s_i$ there exists an element $\xi_{ij} \in G_{F/k}$ fixing all $f_{\lambda\mu}, \sigma_{\lambda\mu} x_1$ and the elements $z_{i+1}, \dots, z_q$, such that $\xi_{ij} z_\mu = z_\mu + \sigma_{ij} x_1$. Then $(\sum_{j=1}^{s_q} f_{qj}(\xi_{qj} - 1)^{s_q}) \dots (\sum_{j=1}^{s_1} f_{1j}(\xi_{1j} - 1)^{s_1})(\alpha') = \beta_1 \otimes \cdots \otimes \beta_q$. □

Corollary 4.9. *Any homomorphism* $F \otimes_k \bigotimes_k^M (F/k) \longrightarrow V$ *factors through* $\bigotimes_F^M (\mathfrak{m}/\mathfrak{m}^s)$ *for some* $s \ge 1$.

Proof. For any integer $s \ge 1$ the element $\alpha_s := (x_1 \otimes 1 - 1 \otimes x_1)^s \otimes x_2 \otimes \cdots \otimes x_M = \sum_{j=0}^s (-1)^j \binom{s}{j} x_1^{s-j} \otimes x_1^j \otimes x_2 \otimes \cdots \otimes x_M \in (\mathfrak{m}^s \otimes_k \bigotimes_k^{M-1} (F/k))_M^{(\mathrm{Aff}_M)_u}$ is homogeneous of degree $s + M - 1$. As V_M is finite-dimensional, the image of α_s in V_M is zero for all sufficiently big s. Note that α_s generates $\mathfrak{m}^s \otimes_k \bigotimes_k^{M-1} (F/k)$ as an F-semi-linear representation of G.

This implies that the image of U is a quotient of $\mathfrak{m}/\mathfrak{m}^s \otimes_k \bigotimes_k^{M-1} (F/k)$ for some $s \ge 1$, and therefore, any homomorphism $F \otimes_k \bigotimes_k^M (F/k) = \bigotimes_F^M \mathfrak{m} \longrightarrow V$ factors through $\bigotimes_F^M (\mathfrak{m}/\mathfrak{m}^s)$ for some $s \ge 1$. □

Theorem 4.10. *Any (finitely generated) object* V *of* $\mathcal{A}$ *is a quotient of a (finite) direct sum of objects of type* $\bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^s)$ *for some* $q, s \ge 1$ *and* F, *if* $k = \overline{\mathbb{Q}}$. *In particular, any irreducible object of* $\mathcal{A}$ *is a direct summand of the tensor algebra* $\bigotimes_F^\bullet \Omega_{F/k}^1$.

Proof. V is generated by V_m for some $m \ge 0$. By Lemma 4.1, $V_m^{(\mathrm{Aff}_m)_u}$ is a semi-simple $\mathrm{GL}_m k$-module generating V. As it is explained at the beginning of this section, V is a quotient of a direct sum of U's corresponding to irreducible direct summands of $V_m^{(\mathrm{Aff}_m)_u}$. By Lemmas 4.6 and 4.7, V is a quotient of a direct sum of $F\otimes_k\bigotimes_k^M(F/k)$ for some $M\geq 0$. Then the conclusion follows from Corollary 4.9 and the identities $\mathfrak{m}^j/\mathfrak{m}^{j+1}=\mathrm{Sym}^j_F(\mathfrak{m}/\mathfrak{m}^2)$ and $\mathfrak{m}/\mathfrak{m}^2=\Omega^1_{F/k}$. □

Corollary 4.11. *Any finitely generated object of $\mathcal{A}$ is of finite length, if $k=\overline{\mathbb{Q}}$.* □

4.1. **Ext's in $\mathcal{A}$.**

Lemma 4.12. $\mathrm{Hom}_{\mathcal{C}}(\bigotimes^r_F\mathfrak{m},\bigotimes^q_F(\mathfrak{m}/\mathfrak{m}^{N+1}))=\mathrm{Hom}_{\mathcal{A}}(\bigotimes^r_F(\mathfrak{m}/\mathfrak{m}^{N+1}),\bigotimes^q_F(\mathfrak{m}/\mathfrak{m}^{N+1}))$ *admits a natural k-basis identified with the set $P=P(q,r,N)$ of the surjections $\{1,\dots,r\}\longrightarrow\{1,\dots,q\}$ with fibres of cardinality $\leq N$, if $N\geq 1$, $q,r\geq 0$ and $q+r\geq 1$. In particular (take $r\geq q=1$), any subobject of $\mathfrak{m}/\mathfrak{m}^{N+1}$ is of type $\mathfrak{m}^r/\mathfrak{m}^{N+1}$.*

EXAMPLE. $P=\emptyset$ if $q>r$, or if $r>qN$; $\#P=q!$ if $q=r$; $\#P=1$ if $N\geq r\geq q=1$.

Proof. By Lemma 4.8, $\mathfrak{m}^{s_1}\otimes_F\cdots\otimes_F\mathfrak{m}^{s_r}$ is generated by the element $\otimes_{j=1}^r(x_j\otimes 1-1\otimes x_j)^{s_j}\in(\bigotimes_{j=1}^r\mathfrak{m}^{s_j})_r^{(\mathrm{Aff}_r)_u}$ of weight $(s_1,\dots,s_r)$ with respect to $(k^\times)^r\subseteq\mathrm{GL}_rk:=\mathrm{Aff}_r/(\mathrm{Aff}_r)_u$. The central weights of $(\bigotimes^q_F(\mathfrak{m}/\mathfrak{m}^{N+1}))_r^{(\mathrm{Aff}_r)_u}$ are contained in the interval $[q,qN]$, so $\bigotimes_{j=1}^r\mathfrak{m}^{s_j}$ is mapped to 0 if $\sum_{j=1}^r s_j\notin[q,qN]$. In particular, the morphisms factor through $\bigotimes^r_F(\mathfrak{m}/\mathfrak{m}^{qN-r+2})$, and are zero if $r>qN$.

The elements $\pi_\varphi:=\otimes_{i=1}^q\prod_{\varphi(u)=i}(x_u\otimes 1-1\otimes x_u)$ for all surjections $\varphi\in P$ span the $(\underbrace{1,\dots,1}_r)$-eigenspace of $(\bigotimes^q_F(\mathfrak{m}/\mathfrak{m}^{N+1}))_r^{(\mathrm{Aff}_r)_u}$. Any morphism of $\bigotimes^r_F\mathfrak{m}$ is determined by the image $\sum_{\varphi\in P}\lambda_\varphi\pi_\varphi$ of the generator $\otimes_{j=1}^r(x_j\otimes 1-1\otimes x_j)$ for some collection of $\lambda_\varphi\in k$. □

Lemma 4.13. *$\mathcal{A}$ splits as $\mathcal{V}ec_k\oplus\mathcal{A}^\circ$, where $\mathcal{V}ec_k$ is the category of finite-dimensional k-vector spaces and $\mathcal{A}^\circ$ is the full subcategory of $\mathcal{A}$ with objects V such that $V^{G_{F/k}}=0$.*

Proof. For any $V\in\mathcal{A}$ set $V^\circ:=\bigcap_{\varphi\in\mathrm{Hom}_{\mathcal{C}}(V,F)}\ker\varphi$. It follows from Theorem 4.10 and Lemma 4.12 that $V=(V^{G_{F/k}}\otimes_k F)\oplus V^\circ$, and $\mathrm{Ext}^*_{\mathcal{A}}(F,\mathcal{A}^\circ)=\mathrm{Ext}^*_{\mathcal{A}}(\mathcal{A}^\circ,F)=0$. The equivalence is given by $V\mapsto(V^{G_{F/k}},V^\circ)$. □

Define the following decreasing "weight" filtration on the objects V of $\mathcal{A}$: W^qV is the sum of the images of all morphisms to V from $\bigotimes_F^{\geq q}\mathfrak{m}$. Clearly, $W^\bullet$ is functorial and multiplicative. By Theorem 4.10, gr^q_WV is a finite direct sum of direct summands of $\bigotimes^q_F\Omega^1_F$.

Corollary 4.14. *$\mathcal{A}^\circ$ has no non-zero projective objects.*

Proof. Let $P\in\mathcal{A}^\circ$ be a projective object and $\xi_2:P\longrightarrow S^\lambda_F\Omega^1_F$ be its irreducible quotient for a Young diagram λ, where $|\lambda|$ is minimal such that $W^{|\lambda|+1}P\neq P$. Then, for any $s\geq 2$, there is a lifting $\xi_s:P\longrightarrow S^\lambda_F(\mathfrak{m}/\mathfrak{m}^s)$ of ξ_2. By Theorem 4.10, there exist $q,a\geq 1$ and a morphism $\bigotimes^q_F(\mathfrak{m}/\mathfrak{m}^a)\longrightarrow P$ such that its composition with ξ_2 is non-zero. Then its composition with any ξ_s is also non-zero. By Lemma 4.12, $\mathrm{Hom}_{\mathcal{A}}(\bigotimes^q_F(\mathfrak{m}/\mathfrak{m}^a),S^\lambda_F(\mathfrak{m}/\mathfrak{m}^N))=0$ for any $N\geq a+q$, leading to contradiction. □

Lemma 4.15. *One has $\mathrm{Ext}^1_{\mathcal{A}}(\bigotimes^q_F(\mathfrak{m}/\mathfrak{m}^N),V)=0$ for any $V\in\mathcal{A}$ of finite length, $q\geq 1$ and $N>$ the maximal weight of V.*

Proof. Induction on the length of V reduces the problem to the case of irreducible V. Let $0\longrightarrow V\longrightarrow E\xrightarrow{\pi}\bigotimes^q_F(\mathfrak{m}/\mathfrak{m}^N)\longrightarrow 0$ be an extension. By Theorem 4.10, there is a surjection of a direct sum of objects of type $\bigotimes_F^p(\mathfrak{m}/\mathfrak{m}^a)$ onto E. By Lemma 4.12, $\mathrm{Hom}_{\mathcal{A}}(\bigotimes_F^{\neq q}(\mathfrak{m}/\mathfrak{m}^a), \bigotimes_F^q(\mathfrak{m}/\mathfrak{m}^2)) = 0$, so there is a morphism of a direct sum of objects of type $\bigotimes_F^q(\mathfrak{m}/\mathfrak{m}^a)$ to E surjective over $\bigotimes_F^q(\mathfrak{m}/\mathfrak{m}^2)$. As the latter is semi-simple, there is a morphism of $\bigoplus_{|\lambda|=q} S_F^\lambda(\mathfrak{m}/\mathfrak{m}^a)$ to E surjective over $\bigotimes_F^q(\mathfrak{m}/\mathfrak{m}^2)$. By Lemma 4.12, its composition with π is surjective, and therefore, the weights of its kernel are $\geq N$, so it does not intersect V. In other words, the extension splits. □

Corollary 4.16. *The following pro-representable functor on $\mathcal{A}$*

$$\mathrm{Hom}_{\mathcal{C}}(\mathfrak{m}^{s_1} \otimes_F \cdots \otimes_F \mathfrak{m}^{s_q}, -) = \varinjlim \mathrm{Hom}_{\mathcal{A}}((\mathfrak{m}^{s_1}/\mathfrak{m}^N) \otimes_F \cdots \otimes_F (\mathfrak{m}^{s_q}/\mathfrak{m}^N), -)$$

is exact if and only if $s_1 = \cdots = s_q = 1$.

Proof. Let $V \longrightarrow V'$ be a surjection in $\mathcal{A}$ and $\xi : \bigotimes_F^q \mathfrak{m} \longrightarrow V'$ be a morphism in $\mathcal{C}$. We have to show that ξ factors through V. By Lemma 4.8, the image of ξ is cyclic. Let V'' be the cyclic sub-object of V generated by a pre-image of a generator of the image of ξ. Then the kernel K of $V'' \longrightarrow \mathrm{Im}(\xi)$ is of finite length. As ξ factors through $\bigotimes_F^q(\mathfrak{m}/\mathfrak{m}^N)$ for some $N \gg 0$, and Lemma 4.15 implies that $\mathrm{Ext}^1_{\mathcal{A}}(\bigotimes_F^q(\mathfrak{m}/\mathfrak{m}^N), K) = 0$, ξ factors through V.

The rest follows from the fact that the projection $\mathfrak{m}^s \longrightarrow \mathfrak{m}^s/\mathfrak{m}^{N+s}$ does not lift to $\mathfrak{m}^s \longrightarrow \bigotimes_F^s(\mathfrak{m}/\mathfrak{m}^{N+1})$, if $s \geq 2$: neither non-zero morphism $\bigotimes_F^s \mathfrak{m} \longrightarrow \bigotimes_F^s(\mathfrak{m}/\mathfrak{m}^{N+1})$ factors through $\mathfrak{m}^s$, if $N \geq 2$. □

Corollary 4.17. *If $V \in \mathcal{A}$ is of finite type then $\dim_k \mathrm{Ext}^j_{\mathcal{A}}(V, V') < \infty$ for any $j \geq 0$ and any $V' \in \mathcal{A}$. If $V \in \mathcal{A}$ is irreducible and $\mathrm{Ext}^1_{\mathcal{A}}(\mathfrak{m}/\mathfrak{m}^q, V) \neq 0$ for some $q \geq 2$ then $V \cong \mathfrak{m}^q/\mathfrak{m}^{q+1}$ and $\mathrm{Ext}^1_{\mathcal{A}}(\mathfrak{m}/\mathfrak{m}^q, V) \cong k$.*

Proof. If $V \in \mathcal{A}$ is of finite type then, by Theorem 4.10, it admits a resolution $\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0$ whose terms are finite direct sums of objects of type $\bigotimes_F^s \mathfrak{m}$. By Lemma 4.8, the terms of the complex $\mathrm{Hom}_{\mathcal{C}}(P_\bullet, V')$ are finite-dimensional over k and, by Corollary 4.16, it calculates $\mathrm{Ext}^\bullet_{\mathcal{A}}(V, V')$. □

Corollary 4.18. *The filtration $W^\bullet$ is strictly compatible with the surjections.*

Proof. Let $V \longrightarrow V'$ be a surjection in $\mathcal{A}$. Then, by Corollary 4.16, any morphism $\bigotimes_F^q \mathfrak{m} \longrightarrow V'$ factors through V. □

5. "Coherent" sheaves in smooth topology

Let $\mathfrak{S}m_k$ be the category of locally dominant morphisms of smooth k-schemes. Consider on $\mathfrak{S}m_k$ the (pre-)topology, where the covers are surjective smooth morphisms. Clearly, the covers are stable under the base changes.

By definition, the structure presheaf $\mathcal{O}$ of $\mathfrak{S}m_k$ associates to any $Y \in \mathfrak{S}m_k$ its k-algebra of regular functions $\mathcal{O}(Y)$. Clearly, $\mathcal{O}$ is a sheaf in this topology.

A sheaf $\mathcal{F}$ on $\mathfrak{S}m_k$ is "*(quasi-)coherent*" if its values $\mathcal{F}(Y)$ are endowed with $\mathcal{O}(Y)$-module structures and its restriction to the small étale site of Y is a (quasi-)coherent sheaf for any $Y \in \mathfrak{S}m_k$.

Lemma 5.1. *Let $X \longrightarrow Y$ be an étale morphism of smooth varieties over k sending a point $q \in X$ to a point $p \in Y$. Then $\mathfrak{m}_q^s/\mathfrak{m}_q^N = \mathcal{O}_q \otimes_{\mathcal{O}_p} (\mathfrak{m}_p^s/\mathfrak{m}_p^N)$ for any $s \leq N$, where $\mathfrak{m}_q := \ker(\mathcal{O}_q \otimes_k \mathcal{O}_q \xrightarrow{\times} \mathcal{O}_q)$.*

Proof. One has $\mathfrak{m}_q/\mathfrak{m}_q^2 = \mathcal{O}_q \otimes_{\mathcal{O}_p} (\mathfrak{m}_p/\mathfrak{m}_p^2)$, so applying $\mathrm{Sym}^s_{\mathcal{O}_q}$ we get $\mathfrak{m}_q^s/\mathfrak{m}_q^{s+1} = \mathrm{Sym}^s_{\mathcal{O}_q}(\mathfrak{m}_q/\mathfrak{m}_q^2) = \mathcal{O}_q \otimes_{\mathcal{O}_p} \mathrm{Sym}^s_{\mathcal{O}_p}(\mathfrak{m}_p/\mathfrak{m}_p^2) = \mathcal{O}_q \otimes_{\mathcal{O}_p} (\mathfrak{m}_p^s/\mathfrak{m}_p^{s+1})$. The induction on $N - s$ gives the conclusion:

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{m}_q^{s+1}/\mathfrak{m}_q^N & \to & \mathfrak{m}_q^s/\mathfrak{m}_q^N & \to & \mathfrak{m}_q^s/\mathfrak{m}_q^{s+1} & \to & 0 \\ & & \| & & \bigcup & & \| & & \\ 0 & \to & \mathcal{O}_q \otimes_{\mathcal{O}_p} (\mathfrak{m}_p^{s+1}/\mathfrak{m}_p^N) & \to & \mathcal{O}_q \otimes_{\mathcal{O}_p} (\mathfrak{m}_p^s/\mathfrak{m}_p^N) & \to & \mathcal{O}_q \otimes_{\mathcal{O}_p} (\mathfrak{m}_p^s/\mathfrak{m}_p^{s+1}) & \to & 0 \end{array}$$

□

Corollary 5.2. *The category $\mathcal{A}$ is equivalent to the category of "coherent" sheaves on $\mathfrak{Sm}_k$, if $k = \overline{\mathbb{Q}}$.*

Proof. Fix an embedding over k of the function field of each connected component of each smooth k-variety into F. Then, for any $V \in \mathcal{A}$, $Y \in \mathfrak{Sm}_k$ and a point $q \in Y$ define an $\mathcal{O}_q$-lattice $\mathcal{V}_q \subset V^{G_{F/k(Y)}}$ as follows. Let $\mathcal{O}_p \subseteq \mathcal{O}_q$ be an étale extension of a local subring in F of a closed point p of a projective space.

Any object V of $\mathcal{A}$ is a quotient of a direct sum of objects of type $\bigotimes^s_F(\mathfrak{m}/\mathfrak{m}^N)$. Then, as it is true for $\bigotimes^s_F(\mathfrak{m}/\mathfrak{m}^N)$ (Lemma 5.1), it follows that the module $\mathcal{V}_p \subset V$ provided by the exact functor $\mathcal{S}$, cf. §1, is independent of the choice of the projective space, and $\mathcal{V}_q := \mathcal{O}_q \otimes_{\mathcal{O}_p} \mathcal{V}_p \subset V$ is independent of $\mathcal{O}_p$.

This determines a locally free coherent sheaf $\mathcal{V}_Y$ on Y with the generic fibre $V^{G_{F/k(Y)}}$.

It follows also that, for any dominant morphism $X \xrightarrow{\pi} Y$ of smooth k-varieties, the inclusion of the generic fibres $k(X) \otimes_{k(Y)} V^{G_{F/k(Y)}} \subseteq V^{G_{F/k(X)}}$ induces an injection of the coherent sheaves $\pi^*\mathcal{V}_Y \hookrightarrow \mathcal{V}_X$ on X, which is an isomorphism if π is étale.

To check that $\mathcal{V}$ is a sheaf on $\mathfrak{Sm}_k$, we need to show that for any surjective smooth morphism $X \longrightarrow Y$ the sequence $0 \longrightarrow \mathcal{V}(Y) \xrightarrow{\beta} \mathcal{V}(X) \xrightarrow{p_1^*-p_2^*} \mathcal{V}(X \times_Y X)$ is exact. As $\mathcal{V}_X$ is a sheaf in Zariski topology on X, it suffices to treat the case of affine X and Y. In the case $V = \bigotimes^s_F(\mathfrak{m}/\mathfrak{m}^N)$, which is sufficient by Theorem 4.10 and Lemma 4.12, this amounts to the exactness of the sequence $0 \longrightarrow \bigotimes^s_B(\mathfrak{m}_B/\mathfrak{m}_B^N) \longrightarrow \bigotimes^s_A(\mathfrak{m}_A/\mathfrak{m}_A^N) \longrightarrow \bigotimes^s_{A\otimes_B A}(\mathfrak{m}_{A,B}/\mathfrak{m}_{A,B}^N)$, where B is a smooth k-algebra of finite type, A is a smooth B-algebra of finite type, $\mathfrak{m}_C := \ker(C \otimes_k C \xrightarrow{\times} C)$ for any k-algebra C, and $\mathfrak{m}_{A,B} := \mathfrak{m}_{A\otimes_B A}$. But this is clear.

Conversely, a "coherent" sheaf $\mathcal{V}$ on $\mathfrak{Sm}_k$ is sent to the object $\varinjlim \mathcal{V}(U)$, where U runs over the spectra of regular subalgebras in F of finite type over k. (As F is the union of its regular subalgebras of finite type over k, $\varinjlim \mathcal{V}(U)$ is an $(F = \varinjlim \mathcal{O}(U))$-module. The action of an element $\sigma \in G$ comes as the limit of isomorphisms $\sigma^* : \mathcal{V}(U) \xrightarrow{\sim} \mathcal{V}(U')$, where $U = \mathbf{Spec}(A)$ and $U = \mathbf{Spec}(\sigma(A))$ induced by the isomorphism $U' \xrightarrow{\sim} U$.) □

Lemma 5.3. *For any "quasi-coherent" flat (as $\mathcal{O}$-module) sheaf $\mathcal{V}$ on $\mathfrak{Sm}_k$ the k-space $\mathcal{V}(Y)$ is a birational invariant of proper Y. If $\mathcal{V}$ is "coherent" then $\mathcal{V}(Y')$ generates the (generic fibre of the) sheaf $\mathcal{V}_{Y'}$ for appropriate finite covers Y' of Y.*

Proof. According to Hironaka, for any pair of smooth proper birational k-varieties Y, Y'' there is a smooth proper k-variety Y' and birational k-morphisms $Y' \xrightarrow{\pi} Y$ and $Y' \longrightarrow Y''$. Let $Z \subset Y$ be the subset consisting of points z such that $\pi : \pi^{-1}(z) \to z$ is not an isomorphism. It is a subvariety of codimension ≥ 2. As $\mathcal{V}$ is torsion-free, one has $\mathcal{V}(Y) \longrightarrow \mathcal{V}(Y') \overset{i^*}{\hookrightarrow} \mathcal{V}(U)$,[6] where $U := Y - Z \overset{i}{\hookrightarrow} Y'$ is the section of π. It suffices

[6]To show that i^* is also injective, choose an affine covering $\{U_j\}$ of Y', and a dense affine subset $U' \subseteq U$. As sum of ample divisors is ample, any intersection of open affine subsets is again affine, so $\{U_j \cap U'\}$ is an

to check that for any affine Y one has $\mathcal{V}(Y) \xrightarrow{\sim} \mathcal{V}(U)$. Choose an affine covering $\{U_j\}$ of U. Then $0 \longrightarrow \mathcal{V}(U) \longrightarrow \bigoplus_j \mathcal{O}(U_j) \otimes_{\mathcal{O}(Y)} \mathcal{V}(Y) \longrightarrow \bigoplus_{i,j} \mathcal{O}(U_i \cap U_j) \otimes_{\mathcal{O}(Y)} \mathcal{V}(Y)$ is exact, so, as $0 \longrightarrow \mathcal{O}(U) = \mathcal{O}(Y) \longrightarrow \bigoplus_j \mathcal{O}(U_j) \longrightarrow \bigoplus_{i,j} \mathcal{O}(U_i \cap U_j)$ is also exact, we get $\mathcal{V}(Y) = \mathcal{V}(U)$. □

REMARK. If $\mathcal{V} : Y \mapsto \Omega^j_{k(Y)}/\Omega^j(Y)$ then the sequence $0 \longrightarrow \Omega^j(Y) \longrightarrow \Omega^j_{k(Y)} \longrightarrow \mathcal{V}(Y) \longrightarrow H^1(Y, \Omega^j_Y) \longrightarrow 0$ is exact, so $\mathcal{V}(Y)$ is birationally invariant if and only if $j = 0$: for any closed smooth $Z \subset \mathbb{P}^{j+1} = Y$ of codimension 2 such that $\Omega^{j-1}(Z) \neq 0$ one has $H^1(Y', \Omega^j_{Y'}) \cong H^1(Y, \Omega^j_Y) \oplus \Omega^{j-1}(Z)$, where Y' is the blow-up of Y along Z.

Then, using Lemma 5.3, we get a left exact (non faithful) functor (with faithful restriction to the subcategory of "coherent" sheaves)

$$\{\text{flat "quasi-coherent" sheaves on } \mathfrak{Sm}_k\} \xrightarrow{\Gamma} \{\text{smooth representations of } G_{F/k} \text{ over } k\}$$

given by $\mathcal{V} \mapsto \varinjlim \Gamma(Y, \mathcal{V}_Y)$, where Y runs over the smooth proper models of subfields in F of finite type over k. This functor is not full, and the objects in its image are highly reducible, e.g., $\Gamma(\Omega^1_{/k}) \cong \bigoplus_A (A(F)/A(k)) \otimes_{\mathrm{End}(A)} \Gamma(A, \Omega^1_{A/k})$, where A runs over the set of isogeny classes of simple abelian varieties over k. If $\mathcal{V}$ is "coherent" and $\Gamma(Y, \mathcal{V}_Y)$ has the Galois descent property then $\Gamma(\mathcal{V})$ is admissible. However, there is no Galois descent property in general.

EXAMPLE. Let Y' be a smooth projective hyperelliptic curve $y^2 = P(x)$, considered as a 2-fold cover of the projective line Y. Then, for $\mathcal{V}_Y = (\Omega^1_{Y/k})^{\otimes 2}$, the section $y^{-2}(dx)^2 = P(x)^{-1}(dx)^2$ is a Galois invariant element of $\Gamma(Y', \mathcal{V}_{Y'})$, which is not in $\Gamma(Y, \mathcal{V}_Y) = 0$.

6. $\mathcal{A}/\mathcal{A}_{>m}$

The only finite-dimensional objects of $\mathcal{A}$ are direct sums of copies of F, so the category $\mathcal{A}$ is far from being tannakian. However, $\mathcal{A}$ admits a decreasing filtration by Serre subcategories $\mathcal{A}_{>m}$ such that all $\mathcal{A}/\mathcal{A}_{>m}$ are again abelian tensor categories and their objects are finite-dimensional. The category $\mathcal{A}/\mathcal{A}_{>m}$ is not rigid.

Let $\mathcal{A}_{>m}$ be the full subcategory of $\mathcal{A}$ with objects V such that $V_m = 0$. Clearly, $\mathcal{A}_{>m}$ is a Serre subcategory of $\mathcal{A}$. Moreover, it is an "ideal" in $\mathcal{A}$ in the sense that the tensor product functor $\mathcal{A}_{>m} \times \mathcal{A} \longrightarrow \mathcal{A}$ factors through $\mathcal{A}_{>m}$, so the quotient abelian category $\mathcal{A}/\mathcal{A}_{>m}$ carries a tensor structure.

By definition, the objects of $\mathcal{A}/\mathcal{A}_{>m}$ are the objects of $\mathcal{A}$, but the morphisms are defined by $\mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V, V') = \mathrm{Hom}_{\mathcal{A}}(\langle V_m \rangle, V'/(V')_{>m}) = \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V, \langle V'_m \rangle)$, where $\langle V_m \rangle$ denotes the semi-linear subrepresentation of V generated by V_m and $(V')_{>m}$ is the maximal subobject of V' in $\mathcal{A}_{>m}$.[7] In particular, $V \cong \langle V_m \rangle$ in $\mathcal{A}/\mathcal{A}_{>m}$.

EXAMPLE. $\mathcal{A}/\mathcal{A}_{>0}$ is equivalent to the category of finite-dimensional k-vector spaces.

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affine covering of U'. Then the diagram

$$\begin{array}{ccccc} & & \mathcal{V}(Y') & \hookrightarrow & \bigoplus_i \mathcal{V}(U_i) \\ & i^*_{\swarrow} & \downarrow & & \downarrow \varphi \\ \mathcal{V}(U) & \to & \mathcal{V}(U') & \hookrightarrow & \bigoplus_i \mathcal{V}(U_i \cap U') \end{array}$$

is commutative, and φ is injective since $\mathcal{V}(U_i \cap U') = \mathcal{O}(U_i \cap U') \otimes_{\mathcal{O}(U_i)} \mathcal{V}(U_i)$ and $\mathcal{V}$ is torsion-free.

[7]The functor $\mathcal{A} \longrightarrow \mathcal{A}_{>m}$, $V \mapsto (V)_{>m}$ is right adjoint to inclusion functor $\mathcal{A}_{>m} \longrightarrow \mathcal{A}$. In particular, it is left exact.

The functor $\mathcal{A}/\mathcal{A}_{>m} \longrightarrow \mathfrak{SL}^u_m$, $V \mapsto V_m$ is exact, faithful and tensor. Note also that the objects of $\mathcal{A}/\mathcal{A}_{>m}$ are *finite-dimensional*. Namely, $\bigwedge^{\dim_{K_m} V_m+1} V = 0$.

Let Φ be a monoid of one-dimensional objects of $\mathcal{A}/\mathcal{A}_{>m}$, such as $(\Omega^m_{F/k})^{\otimes N}$ for any $N \geq 0$. The set Φ is partially ordered: $\omega \leq \eta$ if there is $\xi \in \mathcal{A}/\mathcal{A}_{>m}$ such that $\eta \cong \omega \otimes \xi$. In particular, $\omega \leq \omega \otimes \eta$ and $\eta \leq \omega \otimes \eta$. If $k = \overline{\mathbb{Q}}$ then Φ consists of some (symmetric) F-tensor powers of $\Omega^m_{F/k}$.

Lemma 6.1. *The k-vector space $\mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V \otimes \omega, V' \otimes \omega)$ is finite-dimensional and independent of $\omega \in \Phi$ for ω sufficiently big.*

Proof. For any $\omega, \eta \in \Phi$ such that $\omega \leq \eta$ (i.e., $\eta \cong \omega \otimes \xi$ for some $\xi \in \mathcal{A}/\mathcal{A}_{>m}$) the twist by ξ defines a canonical inclusion $\mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V \otimes \omega, V' \otimes \omega) \subseteq \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V \otimes \eta, V' \otimes \eta)$. The k-vector spaces

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V, V') = \mathrm{Hom}_{\mathcal{A}}(\langle V_m \rangle, V'/(V')_{>m}) \longrightarrow \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V \otimes U, V' \otimes U)\\ &= \mathrm{Hom}_{\mathcal{A}}(\langle V_m \otimes_{K_m} U_m \rangle, (V' \otimes U)/(V' \otimes U)_{>m}) \subseteq \mathrm{Hom}_{K_m\langle G_{K_m/k}\rangle}(V_m \otimes_{K_m} U_m, V'_m \otimes_{K_m} U_m)\end{aligned}$$

are finite-dimensional. On the other hand,

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}}(\langle V_m \rangle, V'/(V')_{>m}) &\subseteq \mathrm{Hom}_{K_m\langle G_{K_m/k}\rangle}(V_m, V'_m)\\ &\subseteq \mathrm{Hom}_{K_m\langle G_{K_m/k}\rangle}(V_m \otimes_{K_m} U_m, V'_m \otimes_{K_m} U_m)\end{aligned}$$

for any $U \in \mathcal{C}$ with $U_m \neq 0$, where the second equality takes place if and only if $\dim_{K_m} U_m = 1$, e.g., for $U \in \Phi$. □

Let $Ob(\mathcal{A}^+_{\Phi,m}) := Ob(\mathcal{A})$ and $\mathrm{Hom}_{\mathcal{A}^+_{\Phi,m}}(V, V') := \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V \otimes \omega, V' \otimes \omega)$ for sufficiently big $\omega \in \Phi$. Then $\otimes\omega : \mathcal{A}^+_{\Phi,m} \longrightarrow \mathcal{A}^+_{\Phi,m}$ is a fully faithful functor, so we can invert objects in Φ to get a category $\mathcal{A}_{\Phi,m} := \mathcal{A}^+_{\Phi,m}[\Phi^{-1}]$. If Φ is the set of all one-dimensional objects of $\mathcal{A}/\mathcal{A}_{>m}$ then $\mathcal{A}_m := \mathcal{A}_{\Phi,m}$ is tannakian.

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