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CIRCLE-EQUIVARIANT CLASSIFYING SPACES AND THE RATIONAL EQUIVARIANT SIGMA GENUS

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ABSTRACT. We analyze the circle-equivariant spectrum $MString_{\mathbb{C}}$ which is the equivariant analogue of the cobordism spectrum $MU\langle 6\rangle$ of stably almost complex manifolds with $c_1 = c_2 = 0$. In [Gre05], the second author showed how to construct the ring $\mathbb{T}$-spectrum EC representing the $\mathbb{T}$-equivariant elliptic cohomology associated to a rational elliptic curve C. In the case that C is a complex elliptic curve, we construct a map of ring $\mathbb{T}$-spectra

$$MString_{\mathbb{C}} \to EC$$

which is the rational equivariant analogue of the sigma orientation of [AHS01]. Our method gives a proof of a conjecture of the first author in [And03b].

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1. Introduction.

In this article we construct a $\mathbb{T}$-equivariant version of the sigma orientation of Ando-Hopkins-Strickland [AHS01], taking values in the equivariant elliptic cohomology EC constructed by the second author in [Gre05], at least in the case of a complex elliptic curve $C = \mathbb{C}/\Lambda$. More precisely, let BU denote the classifying space for stable $\mathbb{T}$-equivariant complex vector bundles, and let BSU denote the classifying space for stable $\mathbb{T}$-equivariant complex vector bundles with trivial determinant. It turns out that BSU is the cover of BU trivializing the first Borel Chern class $c_1^{\mathcal{B}}$. Now let $BString_{\mathbb{C}}$ be the cover of BU trivializing the first and second Borel Chern classes $c_1^{\mathcal{B}}$ and $c_2^{\mathcal{B}}$; we call a virtual complex $\mathbb{T}$-vector bundle with a lift of its classifying map to $BString_{\mathbb{C}}$ a "$String_{\mathbb{C}}$ bundle." Let $MString_{\mathbb{C}}$ be the associated bordism spectrum. We construct a map of ring $\mathbb{T}$-spectra

$$MString_{\mathbb{C}} \longrightarrow EC$$

which specializes to the sigma orientation of Ando-Hopkins-Strickland [AHS01] in Borel-equivariant elliptic cohomology.

Our argument offers several improvements over the papers [AB02, And03b], which construct a canonical and natural Thom isomorphism for $String_{\mathbb{C}}$ bundles (and their real analogues) over compact $\mathbb{T}$-spaces in Grojnowski's equivariant elliptic cohomology [Gro07]. For one thing, our use of the spectrum EC of [Gre05] entitles us to work directly with the classifying spaces for equivariant bundles and their Thom spectra. More importantly, we are able for the first time to give a simple and conceptual formula for the Thom class of a $String_{\mathbb{C}}$-bundle. Briefly, for $a \in C$ and a $\mathbb{T}$-space X, let

$$X^a = \begin{cases} X^{\mathbb{T}[n]} & \text{if } a \text{ has finite order } n \\ X^{\mathbb{T}} & \text{otherwise.} \end{cases}$$

The long exact sequence (6.6) shows how to assemble $EC^*_{\mathbb{T}}(X)$ from the groups

$$H^*_{\mathbb{T}}(X^a; \mathcal{O}^{\wedge}_{C,a})$$

for $a \in C$, where the "coordinate data" of the elliptic curve C are used to give $\mathcal{O}^{\wedge}_{C,a}$ the structure of an $H^*(B\mathbb{T})$-algebra (see Section 6). If V is a virtual $\mathbb{T}$-equivariant vector bundle over X, then the groups relevant for $EC^*_{\mathbb{T}}(X^V)$ are

$$H^*_{\mathbb{T}}((X^V)^a; \mathcal{O}^{\wedge}_{C,a}).$$

The Thom class $\psi(V)$ near a must then be a unit multiple of $\mathrm{Thom}_{\mathbb{T}}(V^a)$, the Thom class of V^a in Borel-equivariant cohomology associated to the Weierstrass sigma function (see §7.C). In order to assemble a Thom class for V as a varies, we expect that

$$\psi(V)_a = \mathrm{Thom}_{\mathbb{T}}(V^a) e_{\mathbb{T}}(V/V^a), \tag{1.1}$$

where V/V^a is defined so that

$$V|_{X^a} \cong V^a \oplus (V/V^a),$$

and $e_{\mathbb{T}}$ denotes the Borel-equivariant Euler class associated to the Weierstrass sigma function.

One of the virtues of our approach is that our Thom class is given precisely by the formula (1.1); see (7.20) and Theorem 7.23. The reader is invited to compare these formulae with the formulae following Theorem 9.1 of [And03b] or (6.11), (6.17), and (6.23) of [AB02] to get an idea of the improvement (1.1) represents. It remains to show that, if V is a $String_{\mathbb{C}}$-bundle, then the proposed Thom class $\psi(V)$ has the necessary properties. We do this in Section 8. The argument uses characteristic classes which were, in some sense, the main discovery of [And03b], but by working universally we give a better account of them and so put them to more effective use.

In Part 3 we use our analysis of the equivariant elliptic cohomology of $BSU(d)$ to prove a conjecture in [And03b, And03a], giving a conceptual construction of the sigma orientation, for elliptic curves of the form $C = \mathbb{C}/\Lambda$. To a $\mathbb{T}$-space X we associate a sort of ringed space (actually a diagram of ringed spaces) $\mathbf{F}(X)$, which determines $EC^*_{\mathbb{T}}(X)$. Associated to a complex vector bundle V over X, we construct a line bundle $\mathcal{F}(X^V)$ over $\mathbf{F}(X)$, which determines $EC^*_{\mathbb{T}}(X^V)$.

Let T be the usual maximal torus of $SU(d)$, with Weyl group W, and let $\check{T} = \mathrm{Hom}(\mathbb{T}, T)$. Looijenga [Loo76] used the second Chern class to construct a line bundle $\mathcal{L} = \mathcal{L}(c_2)$ over $(\check{T} \otimes_{\mathbb{Z}} C)/W$. The Weierstrass sigma function determines a section σ of $\mathcal{L}$, and so a trivialization of $\mathcal{L} \otimes \mathcal{I}$, where $\mathcal{I}$ is the ideal sheaf of zeroes of σ.[1] We show that a $\mathbb{T}$-equivariant $SU(d)$-bundle V over X determines a pull-back diagram

$$\begin{tikzcd}
\mathcal{L}(V) \otimes \mathcal{F}(X^V) \arrow[r] \arrow[d, shift right] & \mathcal{L} \otimes \mathcal{I} \arrow[d, shift right] \\
\mathbf{F}(X) \arrow[r, "h"] \arrow[u, bend right, "\sigma(V)"'] & (\check{T} \otimes C)/W, \arrow[u, bend right, "\sigma"']
\end{tikzcd}$$

where $\mathcal{L}(V) = h^*\mathcal{L}$ and $\sigma(V) = h^*\sigma$. In particular $\sigma(V)$ is a trivialization of $\mathcal{L}(V) \otimes \mathcal{F}(X^V)$.

It turns out that if V^0 and V^1 are two $\mathbb{T}$-equivariant $SU(d)$-bundles over X such that

$$c_2^{\mathcal{B}}(V^0 - V^1) = 0,$$

then

$$\mathcal{L}(V^0) \cong \mathcal{L}(V^1),$$

so $\sigma(V^0) \otimes \sigma(V^1)^{-1}$ is a trivialization of

$$\frac{\mathcal{L}(V^0) \otimes \mathcal{F}(X^{V_0})}{\mathcal{L}(V^1) \otimes \mathcal{F}(X^{V^1})} \cong \mathcal{F}(X^{V^0 - V^1}).$$

This is our Thom class.

We note that Jacob Lurie [Lur05] has, independently of this paper and [And03b], announced a proof of the analogous integral result for his oriented derived elliptic spectra. Our results may be viewed as a classical analogue his work, relating it to the invariant theory of [Loo76] and highlighting the role of the Weierstrass sigma function. David Gepner has outlined to us the relationship between Lurie's equivariant derived elliptic spectra and the $\mathbb{T}$-equivariant elliptic spectra constructed by the second author in [Gre05]. Once his results become available, we expect to be able to show that these constructions of the equivariant sigma orientation are consistent.

Our work on this project has led us to a clearer understanding of the relationship between the $\mathbb{T}$-equivariant elliptic cohomology theories constructed by Grojnowski [Gro07] and Greenlees [Gre05]. In both cases, $EC^*_{\mathbb{T}}(X)$ is assembled from the groups

$$H^*_{\mathbb{T}}(X^a; \mathcal{O}^{\wedge}_{C,a})$$

[1] As explained in [And00], characters of representations of level k of the loop group $LSU(d)$ give sections of $\mathcal{L}^k$. Up to a normalization, σ corresponds to the unique irreducible representation of $LSU(d)$ of level 1

for $a \in C$. In order to make sense of this expression, one must give $\mathcal{O}^{\wedge}_{C,a}$ the structure of an $H^*(B\mathbb{T}) \cong \mathbb{Q}[z]$-algebra [2]. Grojnowski does this for a complex elliptic curve in the form $\mathbb{C}/\Lambda$, using the covering

$$\mathbb{C} \to \mathbb{C}/\Lambda, \tag{1.2}$$

the structure of $\mathcal{O}_{\mathbb{C}}$ as an $H^*(B\mathbb{T})$-algebra, and translation in the elliptic curve. One of the starting points of [Gre05] is the observation that, if $C\langle n\rangle$ denotes the divisor of points of C of order n and $k = \deg C\langle n\rangle$, then

$$C\langle n\rangle - k(0)$$

is the divisor of a function t_n on C, which is uniquely determined by its image in the appropriate power ω^{-k} of the cotangent bundle and serves as a coordinate at $a \in C\langle n\rangle$. We discuss this in Section 6, particularly before Lemma 6.3 and in Remark 6.7.

The paper is divided into three parts.

Part 1 is about equivariant classifying spaces and equivariant characteristic classes in general. In Section 2, we begin the study of the classifying spaces for equivariant vector bundles which arise in this work. In Sections 3 and 4 we discuss characteristic classes for these bundles. In Section 5 we use these characteristic classes to describe the Borel-equivariant ordinary cohomology of our classifying spaces. We make repeated use of a Universal Coefficient Theorem for Borel cohomology, which we discuss in the appendix.

Part 2 focuses on elliptic cohomology, introduces the sigma orientation and establishes the Thom isomorphism. In Section 6 we recall from [Gre05] the properties of equivariant elliptic cohomology which we need for our work. In Section 7 we recall the basic facts about the Weierstrass sigma function, and use it to give the formula for the Thom isomorphism over $\mathbb{T}$-fixed spaces. The behaviour for points with finite isotropy is given in Proposition 7.22, and proved in Section 8. Together, these give the Thom isomorphism: the main result is Theorem 7.23.

In Part 3 we reformulate the results of Part 2 in geometric terms. We explain our Thom isomorphism using the analytic geometry of the elliptic curve C and the invariant theory of [Loo76], proving the conjecture of [And03b] in this case. We also rephrase some of these ideas in terms of the *algebraic* geometry of C. We hope that these ideas will eventually lead to an algebraic version of our results. Section 9 gives a convenient sheaf theoretic formulation of the separation of behaviour over $\mathbb{T}$-fixed points and generic points of the curve from points with finite isotropy and torsion points on the curve. In Section 10 we give the geometric interpretation of the situation over the $\mathbb{T}$-fixed points of $BSU(d)$, and in Section 11 we extend this to all of $BSU(d)$. Finally in Section 12 we give a moduli interpretation in terms of divisors.

The appendix describes a universal coefficient theorem for Borel homology and cohomology.

Part 1. Equivariant classifying spaces and characteristic classes.

In this part we discuss equivariant classifying spaces and characteristic classes from several different points of view. In Section 2, we discuss the classifying spaces both via moduli and through specific models. In Section 3 we discuss characteristic classes via the splitting principle and formal roots. In Section 4 we apply the earlier sections to calculate the cohomology of the first few covers of BU.

[2]The generator of $H^2(B\mathbb{T})$ is denoted z here because we are thinking of it as a complex function on $\mathbb{C}$. When we think of it as the first Chern class of the canonical bundle we write c for the same generator

2. Classifying spaces for equivariant vector bundles.

Let G be a compact Lie group. In this section, we review various aspects of the classifying spaces for G-equivariant complex vector bundles. Initially we allow G to be an arbitrary compact Lie group, but our applications use the special case $G = \mathbb{T}$, and we will specialize to that case when it is convenient to do so. Much of this material is well-known; see for example [May96].

2.A. **The classifying space for equivariant complex vector bundles of finite rank.** Just as in the non-equivariant case, we may pass between $U(n)$-free $G \times U(n)$-spaces, or G-equivariant principal $U(n)$-bundles, and G-equivariant complex vector bundles of rank n. This gives two models for their classifying G-space $BU(n)$.

On the one hand, $BU(n)$ can be constructed as the quotient $EU(n)/U(n)$, where $EU(n)$ is a $G \times U(n)$ space with the property that, for all $K \subset G \times U(n)$,

$$\begin{aligned} EU(n)^K &\simeq * \text{ if } K \cap U(n) = 1 \\ EU(n)^K &= \emptyset \text{ otherwise.} \end{aligned}$$

On the other hand, $BU(n)$ can be modeled as the Grassmannian $\mathrm{Gr}_n(\mathcal{U})$ of n-dimensional subspaces of a complete complex G-universe. Thus

$$BU(n) = EU(n)/U(n) \simeq \mathrm{Gr}_n(\mathcal{U}).$$

We have omitted the G from the notation for $BU(n)$, because for $H \subseteq G$, the H-space underlying $BU(n)$ is the classifying H-space for H-equivariant $U(n)$-bundles, as one can check using either description of $BU(n)$.

Remark 2.1. Note that if X is a G-space, and H is a subgroup, then $N_G H/H$ acts on X^H. In particular, if $G = \mathbb{T}$ and $H = A$ is a finite subgroup, then $\mathbb{T}/A$ acts on $BU(n)^A$.

2.B. **The classifying space for stable bundles.** We will need to have a clear understanding of the stabilization process. For this, we let $\mathcal{U}$ denote a complete complex G-universe, and $U, V, W, X, \ldots$ denote finite dimensional subrepresentations of dimensions $u, v, w, x, \ldots$.

Let $BU(V) = \mathrm{Gr}_v(\mathcal{U} \oplus V)$, and let γ_V be the tautological bundle over this. These spaces form a direct system with structure maps

$$BU(V) = \mathrm{Gr}_v(\mathcal{U} \oplus V) \xrightarrow{\oplus W} \mathrm{Gr}_{v+w}(\mathcal{U} \oplus V \oplus W) = BU(V \oplus W).$$

Let

$$BU = \mathrm{colim}_W\, BU(W).$$

We let γ denote the universal bundle over BU, so that $\gamma|_{BU(W)} = \gamma_W - W$. The G-space BU classifies stable vector bundles of virtual dimension 0, and the G-space $BU \times \mathbb{Z}$ classifies arbitrary stable vector bundles.

2.C. **Fixed points.** It is straightforward to identify the H-fixed points of $BU(n)$, $BU(W)$, and BU. We do this two ways, by analyzing the Grassmannian model and by analyzing the homotopy functor represented by $BU(n)^H$.

If H is a compact Lie group, we write $H^\vee$ for the set of isomorphism classes of simple (complex) representations of H, so that if A is an abelian group then $A^\vee \cong \mathrm{Hom}(A, \mathbb{T})$ is its group of characters.

For a representation V of H we make the following definitions. We write $U(V)$ for the group of vector space automorphisms of V, and we write $Z(V)$ for the centralizer of H in $U(V)$, so

$$Z(V) = \mathrm{Aut}_H(V) = \{x \in U(V) | xhx^{-1} = h \text{ for all } h \in H\}.$$

For $S \in H^\vee$, we define V_S to be the S-isotypical summand, so

$$V \cong \bigoplus_{S \in H^\vee} V_S,$$

and we set

$$d_{S,V} = \dim \mathrm{Hom}(S, V).$$

By Schur's Lemma we have an isomorphism of H-modules

$$\mathrm{Hom}_H(S, V) \otimes S \cong V_S,$$

where H acts trivially on $\mathrm{Hom}_H(S, V)$, and an isomorphism of groups

$$\mathrm{Aut}_H(V) \cong \prod_{S \in H^\vee} \mathrm{Aut}(\mathrm{Hom}_H(S, V)) \cong \prod_{S \in H^\vee} U(d_{S,V}). \tag{2.2}$$

The set of isomorphism classes of n-dimensional representations of H is

$$\mathrm{Hom}(H, U(n))/\text{conjugacy}.$$

It is convenient to choose a set of representatives

$$\mathrm{Rep}_n(H) \subseteq \mathrm{Hom}(H, U(n)).$$

An H-fixed point of $\mathrm{Gr}_w(\mathcal{U} \oplus W)$ is an H-module of rank w, and so we have the function

$$BU(W)^H = \mathrm{Gr}(\mathcal{U} \oplus W)^H \to \mathrm{Rep}_w(H)$$

which sends a point to the representative of its isomorphism class. The function is surjective since $\mathcal{U}$ is complete, and the codomain is discrete, so for $V \in \mathrm{Rep}_w(H)$ we define

$$\mathrm{Gr}_V^H(\mathcal{U} \oplus W) \subseteq \mathrm{Gr}(\mathcal{U} \oplus W)^H$$

to be the component mapping to V. Specifying a point of $\mathrm{Gr}_V^H(\mathcal{U} \oplus W)$ is equivalent to specifying a point of $\mathrm{Gr}_{V_S}^H(\mathcal{U}_S \oplus W_S)$ for each $S \in H^\vee$.

Proposition 2.3. *For any compact Lie group H there is an equivalence of nonequivariant spaces*

$$BU(n)^H \simeq \coprod_{V \in \mathrm{Rep}_n(H)} BZ(V) = \coprod_{V \in \mathrm{Rep}_n(H)} \prod_{S \in H^\vee} BU(d_{S,V}). \tag{2.4}$$

For the Grassmannian $BU(W)^H$, we have

$$BU(W)^H = \mathrm{Gr}_w(\mathcal{U} \oplus W)^H = \coprod_{V \in \mathrm{Rep}_w(H)} \mathrm{Gr}_V^H(\mathcal{U} \oplus W)$$

and

$$\mathrm{Gr}_V^H(\mathcal{U} \oplus W) = \prod_{S \in H^\vee} \mathrm{Gr}_{V_S}^H(\mathcal{U}_S \oplus W_S) \simeq \prod_{S \in H^\vee} BU(\mathrm{Hom}_H(S, V)).$$

First proof. The displayed equalities for the Grassmannian model give proofs. The only equivalence which has not already been spelled out is the last. Over

$$\mathrm{Gr}_{V_S}^H(\mathcal{U}_S \oplus W_S)$$

we have, forgetting the action of H, a contractible principal $\mathrm{Aut}(V_S)$ bundle. The sub-group of automorphisms commuting with H is just $U(\mathrm{Hom}_H(S, V))$. □

Second proof. Since the H-space underlying $BU(n)$ classifies H-equivariant principal $U(n)$-bundles, it is clear that $BU(n)^H$ classifies H-equivariant principal $U(n)$-bundles over H-fixed spaces Z. It suffices to consider one component at a time, so let us suppose we are given such a bundle $\pi : P \to Z$, with Z connected.

Recall that

$$\mathrm{Aut}(P/Z) \cong \Gamma(P \times_{U(n)} U(n)^c \to Z),$$

where $U(n)^c$ denotes $U(n)$ with the adjoint action. It follows that an action of H on P/Z is given by a section s of

$$P \times_{U(n)} \mathrm{Hom}(H, U(n)^c) \to Z. \tag{2.5}$$

The set $\mathrm{Hom}(H, U(n)^c)$ is discrete, and so the function

$$m : Z \xrightarrow{s} P \times_{U(n)} \mathrm{Hom}(H, U(n)^c) \to * \times_{U(n)} \mathrm{Hom}(H, U(n)^c) \xleftarrow{\cong} \mathrm{Rep}_n(H)$$

is locally constant, and so constant.

Let $Z(m)$ be the centralizer

$$Z(m) = \{x \in U(n) | x m(h) x^{-1} = m(h) \text{ for all } h \in H\},$$

and let

$$Q = \{p \in P | s(\pi(p)) = \overline{(p, m)}\},$$

where $\overline{(p, m)}$ denotes the class in the Borel construction. Then

$$\pi|_Q : Q \to Z$$

is a principal $Z(m)$ bundle, classified by a map

$$f : Z \to BZ(m).$$

It follows that

$$BU(n)^H \simeq \coprod_{V \in \mathrm{Rep}_n(H)} BZ(V),$$

and the more detailed description in (2.4) follows from the isomorphism (2.2). □

Remark 2.6. In §4.A we use this analysis of $BU(n)^H$ to give a splitting principle for $\mathbb{T}$-equivariant vector bundles.

Passing to limits we find the fixed points in the stable case. We define $JU(H)$ to be the ideal of virtual representations of rank 0 in the complex representation ring $R(H)$.

Proposition 2.7. *There is an equivalence of nonequivariant spaces*

$$BU^H \simeq JU(H) \times \prod_{\alpha \in H^\vee} BU, \tag{2.8}$$

where the product is topologized as the direct limit of the finite products.

First proof. The stabilization map is

$$BU(V)^H = \coprod_{U \in \mathrm{Rep}_v(H)} \mathrm{Gr}^H_U(\mathcal{U} \oplus V) \xrightarrow{\oplus W} \coprod_{U' \in \mathrm{Rep}_{v+w}(H)} \mathrm{Gr}^H_{U'}(\mathcal{U} \oplus V \oplus W) = BU(V \oplus W)^H.$$

Thus the components of BU^H are labelled by virtual representations of rank 0, the component $\mathrm{Gr}^H_U(\mathcal{U} \oplus V)$ being labelled by $U - V$. The stabilization map is the product over $S \in H^\vee$ of

$$BU(d_{S,U}) \cong \mathrm{Gr}^H_{U_S}(\mathcal{U}_S \oplus V_S) \to \mathrm{Gr}_{U_S \oplus W_S}(\mathcal{U}_S \oplus V_S \oplus W_S) \cong BU(d_{S,U+W}).$$

For each factor, the colimit is BU. □

Second proof. We classify virtual H-equivariant bundles V on H-fixed spaces Z. It suffices to consider one component at a time, and so we suppose that Z is connected. Schur's Lemma provides a decomposition

$$V \cong \bigoplus_{\alpha \in H^\vee} \mathrm{Hom}_H(\alpha, V) \otimes \alpha$$

where now

$$\mathrm{Hom}_H(\alpha, V)$$

is a virtual bundle of rank $d_{\alpha,V}$ (say). We then have the map

$$f : Z \to \prod_{\alpha \in H^\vee} BU \tag{2.9}$$

which on the α factor classifies the virtual bundle of rank 0

$$\mathrm{Hom}_H(\alpha, V) - d_{\alpha,V}\epsilon,$$

where ϵ is the trivial complex line bundle of rank one, with trivial H-action.

If ξ_α denotes the tautological bundle of rank 0 over the α factor in (2.9), then

$$f^*\left(\sum \xi_\alpha \otimes \alpha\right) \cong V - \sum d_{\alpha,V}\alpha.$$

To recover V, then, we must add the element $\sum d_{\alpha,V}\alpha \in R(H)$. This shows that

$$(BU \times \mathbb{Z})^H \cong R(H) \times \prod_{\alpha \in H^\vee} BU,$$

with the universal bundle over the $(\sum_\alpha d_\alpha \alpha)$ summand being

$$\sum_\alpha (\xi_\alpha + d_\alpha) \otimes \alpha.$$

V has virtual dimension zero if and only if

$$\sum_\alpha d_{\alpha,V} \,\mathrm{rank}\, \alpha = 0,$$

and so

$$BU^H \cong JU(H) \times \prod_{\alpha \in H^\vee} BU.$$

$\square$

2.D. **Classifying spaces for SU-bundles.** Next we consider the classifying space $BSU(n)$ of n-dimensional bundles with determinant 1. This can be constructed as $ESU(n)/SU(n)$ where $ESU(n)$ is the universal $SU(n)$-free $G \times SU(n)$-space. Alternatively, there is a fibration

$$BSU(n) \longrightarrow BU(n) \longrightarrow BU(1)$$

of G-spaces, where the map $BU(n) \longrightarrow BU(1)$ classifies the determinant. The H-fixed points can be calculated in the same manner as in Proposition 2.3. Let

$$\mathrm{SRep}_n(H) \subset \mathrm{Hom}(H, SU(n))$$

be a set of representatives for $\mathrm{Hom}(H, SU(n))$/conjugacy; and for $V \in \mathrm{SRep}_n(H)$ let $Z(V)$ be its centralizer in $SU(n)$. The analysis leading to Proposition 2.3 gives the following.

Proposition 2.10. *For any compact Lie group H there is an equivalence*

$$BSU(n)^H \simeq \coprod_{V \in \mathrm{SRep}_n(H)} BZ(V). \tag{2.11}$$

□

Once again we may form the stable classifying space BSU as a direct limit

$$BSU \stackrel{\text{def}}{=} \operatorname{colim}_n BSU(n),$$

where the limit is now formed over addition of a cofinal collection of representations with determinant the trivial 1-dimensional representation ϵ, such as those of form $V \oplus V^*$. Again there is a fibration

$$BSU \longrightarrow BU \longrightarrow BU(1).$$

Taking H-fixed points we have

$$BSU^H \longrightarrow BU^H \longrightarrow BU(1)^H.$$

If $BU(1)^H_\epsilon$ is the component of $BU(1)^H$ corresponding to the trivial representation,

$$JU_2(G) = \{V \in JU(G) | \det V \cong \epsilon\}$$

is the subgroup of $JU(G)$ consisting of virtual representations with trivial determinant, and BU^G_S is the set of components of BU^H corresponding to representations with determinant ϵ, then we have a fibration

$$BSU^H \longrightarrow BU^H_S \longrightarrow BU(1)^H_\epsilon \tag{2.12}$$

with connected base, and an equivalence

$$BU^G_S \simeq JU_2(G) \times \prod_\alpha BU.$$

Again, the components of BSU^H are all equivalent, and, taking components of zero, there is a fibration

$$BSU^H_0 \longrightarrow BU^H_0 \longrightarrow BU(1)^H_\epsilon$$

of connected spaces.

2.E. **The tower over $BU \times \mathbb{Z}$.** In the next two sections we study characteristic classes for equivariant vector bundles, in light of the preceding analysis of their classifying spaces. One reason to do so is better to understand the spaces $BU\{2k\}$ over $BU \times \mathbb{Z}$ defined by the vanishing of the Borel Chern classes $c_0^{\mathcal{B}}, c_1^{\mathcal{B}}$, and $c_2^{\mathcal{B}}$. It is the Thom spectrum associated to $BU\{6\}$ which maps to elliptic cohomology.

It is perhaps surprising that the vanishing of Borel Chern classes plays such an important role in the relationship to elliptic cohomology. We note that the spaces $BU\{2k\}$ also occur as representing spaces for the equivariant version of connective K-theory constructed in [Gre04]; see §3.F. This equivariant version of connective K-theory is complex orientable, and its coefficient ring classifies multiplicative equivariant formal group laws for products of two topologically cyclic groups (and in particular for the circle and all its subgroups).

Nonequivariantly, $BU = BU\langle 2\rangle$ is the 1-connected cover of $BU \times \mathbb{Z}$, $BSU = BU\langle 4\rangle$ is the fibre of

$$BU \xrightarrow{c_1} K(\mathbb{Z}, 2),$$

and $BString_{\mathbb{C}} = BU\langle 6\rangle$ is the fibre of the second Chern class

$$BSU \xrightarrow{c_2} K(\mathbb{Z}, 4).$$

Borel cohomology classes, that is elements of $H^n(X \times_G EG)$, correspond to G-maps

$$f : X \longrightarrow \mathrm{map}(EG, K(\mathbb{Z}, n)).$$

We define spaces $BU\{2k\}$ by the following diagram, in which the indicated horizontal arrows are Borel Chern classes, and each vertical arrow is the fibre of the following horizontal arrow.

$$\begin{tikzcd}
BString_{\mathbb{C}} \arrow[r, equal] & BU\{6\} \arrow[d] & \\
BSU \arrow[r, "\simeq"] & BU\{4\} \arrow[r, "c_2^{\mathcal{B}}"] \arrow[d] & \mathrm{map}(EG, K(\mathbb{Z}, 4)) \\
BU \arrow[r, equal] & BU\{2\} \arrow[r, "c_1^{\mathcal{B}}"] \arrow[d] & \mathrm{map}(EG, K(\mathbb{Z}, 2)) \\
BU \times \mathbb{Z} \arrow[r, equal] & BU\{0\} \arrow[r, "c_0^{\mathcal{B}}"] & \mathrm{map}(EG, K(\mathbb{Z}, 0)).
\end{tikzcd} \tag{2.13}$$

We have used the notation $BU\{2k\}$ instead of $BU\langle 2k \rangle$ because the spaces in question are not equivariantly connected. We show in §3.E that BU and BSU occur as indicated in (2.13). We *define* $BString_{\mathbb{C}}$ to be $BU\{6\}$.

To analyze H-fixed points, we use the equivalence

$$\mathrm{map}(EG, K(\mathbb{Z}, n))^H \simeq \mathrm{map}(BH, K(\mathbb{Z}, n)) \simeq \prod_{i=0}^{n} K(H^i(BH), n-i). \tag{2.14}$$

We are particularly interested in the case that G is the circle, and $H = A \subseteq \mathbb{T}$ is a closed subgroup. Such groups A have integral cohomology only in even degrees, so we obtain the following.

Proposition 2.15. *Taking A-fixed points in the diagram* (2.13) *yields a diagram*

$$\begin{tikzcd}
BString_{\mathbb{C}}^A \arrow[d] & \\
BSU^A \arrow[r, "\{c_2^0, c_2^2, c_2^4\}"] \arrow[d] & K(H^0(BA), 4) \times K(H^2(BA), 2) \times K(H^4(BA), 0) \\
BU^A \arrow[r, "\{c_1^0, c_1^2\}"] \arrow[d] & K(H^0(BA), 2) \times K(H^2(BA), 0) \\
BU^A \times \mathbb{Z} \arrow[r, "c_0^0"] & K(H^0(BA), 0)
\end{tikzcd}$$

of $\mathbb{T}/A$-spaces, in which again each vertical arrow is the fibre of the following horizontal one. □

We will describe the maps c_k^i as characteristic classes in Lemma 4.13.

3. Characteristic classes of equivariant bundles.

In this section we briefly discuss characteristic classes for a general compact Lie group of equivariance, and we show that $BSU \simeq BU\{4\}$. In the next section we analyze more closely the case of a circle.

3.A. **Nonequivariant Chern classes.** We write c_i for the usual Chern class in $H^{2i}BU$, so

$$H^*(BU) = \mathbb{Z}[c_1, c_2, \ldots].$$

We write

$$c_\bullet(V) = 1 + c_1(V) + c_2(V) + \cdots$$

for the total Chern class, and recall that it is exponential in the sense that

$$c_\bullet(V \oplus W) = c_\bullet(V) \cdot c_\bullet(W),$$

so we may extend $c_\bullet$ to virtual vector bundles by the formula

$$c_\bullet(V - W) = c_\bullet(V)c_\bullet(W)^{-1}.$$

Remark 3.1. It is convenient to record the behaviour of c_1 and c_2 on a difference of actual bundles. It is immediate that c_1 is additive, so that

$$c_1(U - V) = c_1(U) - c_1(V).$$

For c_2 there is a correction term:

$$c_2(U - V) = c_2(U) - c_2(V) - c_1(V)c_1(U - V),$$

but this simplifies to additivity when $c_1(U - V) = 0$.

3.B. **Chern classes assembled from isotypical summands.** Let X be a G-space, and suppose that ξ is a complex G-bundle ξ of rank n over X. If it happens that X is H-fixed, then we have an isomorphism of H-bundles

$$\xi \cong \bigoplus_{\alpha \in H^\vee} \mathrm{Hom}(\alpha, \xi) \otimes \alpha,$$

where on the one hand α is trivial as a non-equivariant bundle over X and on the other $\mathrm{Hom}(\alpha, \xi)$ carries a trivial H-action. We may define Chern classes $c_i^\alpha(\xi)$ by the formula

$$c_i^\alpha(\xi) \stackrel{\mathrm{def}}{=} c_i(\mathrm{Hom}(\alpha, \xi)). \tag{3.2}$$

If ξ is a virtual complex vector bundle of rank 0 over X, then it is classified by a map

$$[\xi] : X \to BU.$$

If H acts trivially on X, then we write

$$[\xi, H] : X \to BU^H$$

for the indicated factorization. The decomposition

$$BU^H \cong JU(H) \times \prod_{\alpha \in H^\vee} BU,$$

of Proposition 2.7 gives an isomorphism

$$H^*(BU^H) \cong \prod_{V \in JU(H)} \mathbb{Z}[\![c_1^\alpha, c_2^\alpha, \ldots \mid \alpha \in H^\vee]\!]. \tag{3.3}$$

The notation in (3.3) is consistent with the notation in (3.2) in the sense that

$$c_i^\alpha(\xi) = [\xi, H]^*(c_i^\alpha);$$

where on the right c_i^α is taken from the appropriate factor of BU^H. The double brackets in (3.3) refer to the completion arising from the fact (see Proposition 2.7) that the topology on the product in the description of BU^H is the direct limit of finite products. So

$$\sum_{\alpha \in \mathbb{T}^\vee} c_1^\alpha \in H^2(BU^{\mathbb{T}})$$

is allowed, but

$$\sum_n c_n^\alpha$$

is not.

3.C. **Borel Chern classes.** The G-Borel construction on ξ is a virtual complex vector bundle $\xi \times_G EG$ over $X \times_G EG$, classified by

$$[[\xi]] : X \times_G EG \longrightarrow BU. \tag{3.4}$$

The Borel Chern classes of ξ are defined to be

$$c_i^{\mathcal{B}}(\xi) \stackrel{\text{def}}{=} [[\xi]]^*(c_i).$$

If X is H-fixed, then there is a standard way to relate the Borel Chern classes to the c_i^α. Notice that there is an isomorphism

$$B : \mathrm{Rep}_1(H) = \mathrm{Hom}(H, \mathbb{T}) \xrightarrow{\cong} [BH, B\mathbb{T}] = H^2(BH), \tag{3.5}$$

via which we have, for $\alpha \in H^\vee$,

$$c_1^{\mathcal{B}}(\alpha) = B \det \alpha. \tag{3.6}$$

Lemma 3.7. *If ξ is an equivariant G-bundle over an H-fixed space X, then in $H^2(X \times BH)$ we have*

$$c_1^{\mathcal{B}}(\xi) = \sum_{\alpha \in H^\vee} \mathrm{rank}(\alpha) c_1^\alpha(\xi) + \mathrm{rank}(\mathrm{Hom}(\alpha, \xi)) B \det(\alpha). \tag{3.8}$$

Proof. By reducing to the universal case $X = BU(n)^H$, we may assume that $H^1 X = 0$. Recall that we have the isomorphism of H-bundles over X

$$\bigoplus_{\alpha \in H^\vee} \mathrm{Hom}(\alpha, \xi) \otimes \alpha \cong \xi.$$

This gives

$$\xi \times_H EH \cong \bigoplus_{\alpha \in H^\vee} \mathrm{Hom}(\alpha, \xi) \otimes (\alpha \times_H EH)$$

since H acts trivially on $\mathrm{Hom}(\alpha, \xi)$. Taking determinants gives

$$\det(\xi \times_H EH) \cong \prod_{\alpha \in H^\vee} \det(\mathrm{Hom}(\alpha, \xi) \otimes (\alpha \times_H EH)). \tag{3.9}$$

By definition, $c_1^\alpha(\xi) = c_1(\mathrm{Hom}(\alpha, \xi))$, so the result follows from (3.6), (3.9), and the formula

$$c_1(V \otimes W) = \mathrm{rank}\, V c_1 W + c_1 V \,\mathrm{rank}\, W.$$

□

3.D. BU **as a split** G**-space.** The Borel Chern classes have another less familiar description which will be useful in §3.F. Let $\overline{BU}$ be the stable Grassmannian associated to the trivial G-universe $\mathcal{U}^G$, so that $\mathbb{Z} \times \overline{BU}$ is a representing space for non-equivariant K-theory. The inclusion

$$\mathcal{U}^G \to \mathcal{U}$$

induces an equivariant map

$$\eta : \overline{BU} \to BU,$$

which is easily seen to be a non-equivariant weak equivalence: this is a space-level expression of the fact that the equivariant complex K-theory spectrum is a split ring spectrum. It follows that the induced map

$$\eta^* : H^*_G(BU) \to H^*_G(\overline{BU}) \cong H^*(\overline{BU} \times BG)$$

is an isomorphism.

If ξ denotes the tautological bundle over BU, then the map $[[\xi]]$ in (3.4) can be regarded as a map

$$[[\xi]] : BU \times_G EG \to \overline{BU},$$

and it is easy to check that the diagram

$$\begin{tikzcd} \overline{BU} \times BG \arrow[d] \arrow[dr, "\text{projection}"] & \\ BU \times_G EG \arrow[r] & \overline{BU} \end{tikzcd}$$

commutes up to homotopy. Thus we have the following.

Proposition 3.10. *The Borel Chern classes* $c^{\mathcal{B}}$ *are uniquely characterized by the fact that, under the splitting*

$$\eta : \overline{BU} \to BU,$$

they pull back to the ordinary Chern classes. That is, for each k *we have*

$$\eta^* c_k^{\mathcal{B}} = c_k$$

in $H^{2k}(\overline{BU} \times BG)$. □

3.E. **Comparison of** BSU **and** $BU\{4\}$**.** We explain how the G-spaces $BU \times \mathbb{Z}, BU$ and BSU fit into a diagram as displayed in §2.E. Since $\mathrm{map}(EG, K(\mathbb{Z}, 0)) \simeq K(\mathbb{Z}, 0)$, $c_0^{\mathcal{B}}$ is a bijection on components, and so the space BU is the fibre of $c_0^{\mathcal{B}}$.

For the next stage, observe that if L is the tautological line bundle over $BU(1)$, then

$$c_1^{\mathcal{B}}(L) \in H^2(BU(1) \times_G EG) \cong [BU(1), \mathrm{map}(EG, K(\mathbb{Z}, 2))].$$

The definition of the first Borel Chern class implies that the diagram

$$\begin{tikzcd} BU \arrow[r, "\det"] \arrow[dr, "c_1^{\mathcal{B}}"'] & BU(1) \arrow[d, "c_1^{\mathcal{B}}(L)"] \\ & \mathrm{map}(EG, K(\mathbb{Z}, 2)) \end{tikzcd} \tag{3.11}$$

commutes.

Proposition 3.12. [StrAfg] *The G-map*

$$BU(1) \longrightarrow \mathrm{map}(EG, K(\mathbb{Z}, 2))$$

corresponding to $c_1^{\mathcal{B}}(L)$ is a weak equivalence, and so induces a weak equivalence

$$BSU \simeq BU\{4\}.$$

Proof. Proposition 2.3 shows that, for each compact subgroup $H \subseteq G$,

$$BU(1)^H \simeq \mathrm{Rep}_1(H) \times K(\mathbb{Z}, 2).$$

At the same time, since H is compact, $H^1(BH) = 0$, and we have

$$\mathrm{map}(EG, K(\mathbb{Z}, 2))^H \simeq K(H^2(BH), 0) \times K(\mathbb{Z}, 2).$$

In terms of these isomorphisms, Lemma 3.7 shows that

$$(c_1^{\mathcal{B}})^H = B \times \mathrm{id} : \mathrm{Rep}_1(H) \times K(\mathbb{Z}, 2) \to K(H^2(BH), 0) \times K(\mathbb{Z}, 2),$$

where B is the isomorphism (3.5). □

Remark 3.13. This gives another proof that the natural map

$$\mathrm{Pic}_G(X) \to H^2_G(X; \mathbb{Z})$$

is an isomorphism, where $\mathrm{Pic}_G(X)$ is the group of equivariant line bundles over X (Atiyah and Segal [AS]).

3.F. **The spectrum** $MString_{\mathbb{C}}$**.** Associated to the spaces BU, BSU, and $BString_{\mathbb{C}}$ over BU we have Thom spectra MU, MSU and $MString_{\mathbb{C}}$. The spectrum MU is easily seen to be an E_∞ ring spectrum since it comes with an action of the linear isometries operad. We turn to MSU and $MString_{\mathbb{C}}$.

Proposition 3.14. *The spectra MSU and $MString_{\mathbb{C}}$ are E_∞-ring spectra.*

Proof. It suffices to show that BSU and $BString_{\mathbb{C}}$ are infinite loop spaces over BU, and so it suffices to show that the Borel Chern classes $c_1^{\mathcal{B}}$ and $c_2^{\mathcal{B}}$ arise from maps of spectra.

In [Gre04], the second listed author defined the G-equivariant connective K-theory spectrum ku to be the pull-back in the right square of the diagram

$$\begin{tikzcd} \overline{ku} \arrow[r] \arrow[dr] & ku \arrow[r] \arrow[d] & K \arrow[d] \\ & \mathrm{map}_*(EG_+, \overline{ku}) \arrow[r] & \mathrm{map}_*(EG_+, K). \end{tikzcd} \tag{3.15}$$

Here $\overline{ku}$ (respectively $\overline{K}$) is the inflation of the non-equivariant connective K-theory spectrum (respectively the equivariant periodic K-theory spectrum), and the bottom arrow is induced by the composition

$$\overline{ku} \to \overline{K} \to K,$$

obtained using the fact that periodic complex K-theory is split. Note that the construction implies that, as we have indicated, ku is split.

As explained in [Gre04], by looping down the diagram (3.15) we obtain diagrams of the form

$$\begin{tikzcd} \overline{BU} \arrow[r] \arrow[dr] & BU \arrow[r] \arrow[d, "\alpha_1"] & BU \arrow[d] \\ & \mathrm{map}_*(EG_+, \overline{BU}) \arrow[r] & \mathrm{map}_*(EG_+, BU) \end{tikzcd}$$

and

$$\begin{tikzcd}
\overline{BSU} \arrow[r] \arrow[dr] & BSU \arrow[r] \arrow[d, "\alpha_2"] & BU \arrow[d] \\
& \operatorname{map}_*(EG_+, \overline{BSU}) \arrow[r] & \operatorname{map}_*(EG_+, BU)
\end{tikzcd}$$

Already this exhibits BSU as an infinite loop space over BU, but to compare to the tower for $BU\{4\}$, we observe that the map

$$\overline{BU} \to BU \xrightarrow{\alpha_1} \operatorname{map}_*(EG_+, \overline{BU}) \to \operatorname{map}_*(EG_+, K(\mathbb{Z}, 2))$$

represents $c_1 \otimes 1$ in $H^2(\overline{BU} \times BG)$, and so Proposition 3.10 implies that the composition

$$BU \xrightarrow{\alpha_1} \operatorname{map}_*(EG_+, \overline{BU}) \to \operatorname{map}_*(EG_+, K(\mathbb{Z}, 2))$$

represents $c_1^{\mathcal{B}}$, as required. An analogous argument shows that the composition

$$BSU \xrightarrow{\alpha_2} \operatorname{map}_*(EG_+, \overline{BSU}) \to \operatorname{map}_*(EG_+, K(\mathbb{Z}, 4))$$

represents $c_2^{\mathcal{B}}$. In particular, this is an infinite loop map, and so exhibits $BString_{\mathbb{C}}$ as an infinite loop space over BSU. $\square$

4. Characteristic classes for $\mathbb{T}$-vector bundles.

In this section, we focus on the special case $G = \mathbb{T}$, and we write A for a general closed subgroup of $\mathbb{T}$. We have two goals. The first is to give an equivariant form of the splitting principle, so that in §8.B we can identify some characteristic classes of A-equivariant bundles over A-fixed spaces. The second is to record the calculation of the Borel Chern classes. These will be used throughout the remainder of the paper.

Because we use multiplicative notation in A^* and additive notation in $H^2(BA)$ we write

$$\log : A^* \xrightarrow{\cong} H^2(BA; \mathbb{Z})$$

for the isomorphism between them.

We write z for the generator of $H^2B\mathbb{T}$, and also for its restriction to H^*BA. Over a $\mathbb{T}$-fixed base, we always have

$$H^*(X \times_{\mathbb{T}} E\mathbb{T}) = H^*(X) \otimes H^*B\mathbb{T} \cong H^*(X)[z],$$

and our notation will reflect this. If $A = \mathbb{T}[n]$ and $X = X^A$, we still have

$$H^*(X \times_A EA) \cong H^*X[z]/nz,$$

provided H^*X is concentrated in even degrees.

4.A. **Reductions and the splitting principle.**

In this section we describe the cohomology rings $H^*(BU(n)^A)$ and $H^*(BSU(n)^A; \mathbb{Q})$ using the splitting principle. We start with $U(n)$. Since A is abelian, we may choose our representatives $m \in \operatorname{Rep}_n(A)$ of $\operatorname{Hom}(A, U(n)^c)/U(n)$ to be of the form

$$m : A \to T, \tag{4.1}$$

where T is the maximal torus of diagonal matrices. If m is such a homomorphism, then its centralizer

$$Z(m) = \{g \in U(n) | gmg^{-1} = m\} \cong \prod_{\alpha \in A^\vee} U(\operatorname{rank} \operatorname{Hom}(\alpha, m))$$

is a product of unitary matrices. In particular, it is connected, with maximal torus T. We define $W(m)$ to be the Weyl group of $Z(m)$ with respect to the torus T; it is a subgroup of the Weyl group W of T in $U(n)$. With these choices, Proposition 2.3 takes the following form.

Proposition 4.2.

$$BU(n)^A \simeq \coprod_{m \in \mathrm{Rep}_n(A)} BZ(m),$$

and so

$$H^*BU(n)^A \cong \prod_{m \in \mathrm{Rep}_n(A)} H^*(BT)^{W(m)}. \quad \square$$

Example 4.3. Any homomorphism $\mathbb{T} \to U(n)$ is conjugate to one of the form

$$z \mapsto m(z) = \mathrm{diag}(z^{m_1}, \ldots, z^{m_1}, z^{m_2}, \ldots, z^{m_2}, \ldots, z^{m_k}, \ldots, z^{m_k}),$$

where the m_i are integers, $m_i < m_j$ for $i < j$, m_i occurs d_i times, and

$$\sum d_i = n.$$

Then $Z(m)$ is the group of block-diagonal matrices $\prod U(d_i)$, with maximal torus T and Weyl group $\prod \Sigma_{d_i}$.

Recall that in the isomorphism of A-bundles

$$V \cong \bigoplus_{\alpha \in A^\vee} \mathrm{Hom}(\alpha, V) \otimes \alpha,$$

A acts trivially on $\mathrm{Hom}(\alpha, V)$, while the bundle underlying α is a topologically trivial line bundle. Thus if

$$\mathrm{Hom}(\alpha, V) \cong L_1 \oplus \cdots \oplus L_d$$

as a non-equivariant bundle, then

$$\mathrm{Hom}(\alpha, V) \otimes \alpha \cong L_1 \otimes \alpha \oplus \cdots \oplus L_d \otimes \alpha$$

as a bundle with A-action. Proposition 4.2 implies the following form of the splitting principle.

Lemma 4.4. *Let V be an A-equivariant vector bundle over an A-fixed space X. The splitting principle holds in the sense that there is another A-fixed space X' and a cohomology monomorphism $X' \longrightarrow X$ so that over X' we may write*

$$V \cong \bigoplus_{\alpha \in A^\vee} \bigoplus_{i=1}^{d_\alpha} L_{\alpha,i} \otimes \alpha,$$

where $L_{\alpha,i}$ is a line bundle with trivial action, and α describes the A-action. Moreover in the universal case, if $x_{\alpha,i} = c_1 L_{\alpha,i}$, then the image of H^*X in H^*X' consists of the expressions in the $x_{\alpha,i}$s which are invariant under the evident action of

$$\prod_\alpha \Sigma_{d_\alpha}.$$

$\square$

Proposition 4.2, like its parent, is phrased in terms of a choice of representatives for $\mathrm{Hom}(A, U(n)^c)/U(n)$. For us it will be important to have a more invariant expression, which by the way also applies to $SU(n)$. So let G stand for one of these groups, and let T be a maximal torus, with Weyl group W.

Suppose that

$$\pi : P \longrightarrow X$$

is an A-equivariant principal G-bundle, over a trivial A-space X. The action of A on P corresponds to a section

$$s : X \longrightarrow P \times_G \mathrm{Hom}(A, G^c).$$

giving a function

$$f : X \longrightarrow P \times_G \mathrm{Hom}(A, G^c) \longrightarrow \mathrm{Hom}(A, G^c)/\text{conjugacy}.$$

Definition 4.5. A *reduction* of the action of A on P/X is a function

$$m : \pi_0 X \to \mathrm{Hom}(A, T)$$

making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathrm{Hom}(A, G^c)/G \\ \downarrow & & \uparrow \\ \pi_0 X & \xrightarrow{m} & \mathrm{Hom}(A, T) \end{array}$$

commute. Note that a reduction always exists, because the right vertical arrow is a surjection of discrete spaces.

This definition is convenient for analyzing principal G-bundles over not-necessarily connected spaces. In the following discussion, though, we suppose that X is connected, leaving the modifications for general X to the reader.

Let $Z(m) \subseteq G$ be the centralizer of m in G. It is important to note the following.

Lemma 4.6. *For any $m : A \to T$, $Z(m)$ is connected, with maximal torus T.*

Proof. For $G = U(n)$ this is clear, since $Z(m)$ is a product of unitary groups (see Example 4.3). For $SU(n)$, it is a result of Bott and Samelson [BS58, BT89] that for any simply connected compact Lie group G, the centralizer of any element is connected. The maximal torus is T, since T is maximal in G. □

Let $W(m)$ be the Weyl group of $Z(m)$; it is a subgroup of W. Any other reduction $m' : A \to T$ is of the form

$$m' = wm$$

where $w \in W$, and

$$wm = m$$

if and only if $w \in W(m)$.

The reduction m determines a principal $Z(m)$-bundle $Q(m)$ over X, by the formula

$$Q(m) = \{p \in P | s\pi(p) = \overline{(p, m)}\}.$$

This is classified by a map

$$g_m : X \longrightarrow BZ(m).$$

By the splitting principle,

$$H^*(BZ(m); \mathbb{Q}) \cong H^*(BT; \mathbb{Q})^{W(m)},$$

and so an element Ξ of the right hand side gives an element

$$g_m^* \Xi \in H^*(X; \mathbb{Q}).$$

Proposition 4.7. *Let $G = U(n)$ or $SU(n)$ as above. Let A be a closed subgroup of $\mathbb{T}$. Then $H^*(BG^A; \mathbb{Q})$ is isomorphic to the ring*

$$\mathrm{Hom}_W(\mathrm{Hom}(A, T), H^*(BT; \mathbb{Q}))$$

of W-equivariant functions. More explicitly, it consists of functions

$$\Xi : \mathrm{Hom}(A, T) \to H^*(BT; \mathbb{Q})$$

such that

(1) *for each* $m \in \mathrm{Hom}(A, T)$, $\Xi(m) \in H^*(BT;\mathbb{Q})^{W(m)}$*; and*
(2) *for* $w \in W$,

$$\Xi(m) = w^*\Xi(wm) \in H^*(BT;\mathbb{Q})^{W(m)}.$$

In particular, any such function Ξ *determines a characteristic class of* A*-equivariant complex vector bundles over* A*-fixed spaces, by the formula*

$$\Xi(V) = g_m^*\Xi(m),$$

where $m : \pi_0 X \to \mathrm{Hom}(A, T)$ *is any choice of reduction of the action of* A *on* V/X*. For* $G = U(n)$*, the analogous statements for integral cohomology are true as well.*

Proof. Another choice of reduction m' determines $Z(m'), Q(m')$, and $g_{m'}$ as above, and there is an element

$$w \in W(m')\backslash W/W(m),$$

determined by the formula

$$m' = wm \in \mathrm{Hom}(A, T)$$

and making the diagram

$$\begin{tikzcd} X \arrow[r, "g_m"] \arrow[dr, "g_{m'}"'] & BZ(m) \arrow[d, "w"] \\ & BZ(m') \end{tikzcd}$$

commute. Thus if

$$\Xi(m) = w^*\Xi(m') \in H^*(BT)^{W(m)},$$

then

$$(g_{m'})^*\Xi(m') = g_m^*\Xi(m) \in H^*X.$$

□

Remark 4.8. The main ingredient in the argument is the splitting principle for $BZ(m)$, so one needs to know that $Z(m)$ is a connected compact Lie group. Thus the result of Bott and Samelson [BS58] implies that the Proposition holds rationally for any simply-connected compact Lie group.

Remark 4.9. The results of this section and of Proposition 2.3 say that the components of BG^A are labelled by elements of

$$\mathrm{Hom}(A, G^c)/G,$$

where G^c denotes G as a G-space with the conjugation action. A choice of representative $m : A \to G$ identifies the corresponding component with $BZ(m)$. One way to work with BG^A, then, is to fix a set of representatives. Elsewhere in this paper, particularly from Section 7 onwards, it is essential *not* to do so, because we must understand the behaviour of our characteristic classes under restriction

$$BG^{\mathbb{T}} \to BG^A,$$

which leads us to consider diagrams like

$$\begin{tikzcd} \mathbb{T} \arrow[r, "\tilde{m}"] & T \\ A. \arrow[u] \arrow[ur, "m"'] & \end{tikzcd}$$

Our approach is to give formulae which work for any $m : A \to T$ and which are compatible with the action of W by conjugation. Proposition 4.7 tells us how to do this. When we write that a homomorphism $m : A \to G$ "labels a component of BG^A", we mean that we use m to identify its component with $BZ(m)$.

4.B. **Chern classes of $\mathbb{T}$-bundles.** Our calculation of Chern classes uses the splitting principle (Lemma 4.4) to deduce the general case from the following result, which is a specialization of Lemma 3.7.

Lemma 4.10. *If L is a line bundle over an A-fixed space, and if $\alpha \in A^*$, then*

$$c_1^{\mathcal{B}}(L \otimes \alpha) = c_1(L) + \log(\alpha) \cdot z. \quad \square$$

Proof. The only point is to observe that, under the decomposition

$$BU^A \simeq JU(A) \times \prod_{\beta \in A^\vee} BU,$$

the map classifying $L \otimes \alpha$ maps to the α factor of BU as the map classifying L, and to the other factors trivially. That is,

$$c_1^\alpha(L \otimes \alpha) = c_1 L,$$

while

$$c_1^\beta L = 0$$

for $\beta \neq \alpha$. $\square$

Now suppose that V is an A-equivariant vector bundle over an A-fixed space, and that after pulling back along a cohomology monomorphism $X' \to X$ we have

$$V \cong L_1 \otimes \alpha_1 + \cdots + L_n \otimes \alpha_n.$$

Then

$$c_\bullet^{\mathcal{B}}(V) = \prod_i (1 + c_1(L_i) + \log(\alpha_i) z). \tag{4.11}$$

This gives a calculation of the first and second Borel Chern classes. In order to state the result, we introduce the following quantities. Suppose that $m = (m_1, \ldots, m_d)$ and $m' = (m'_1, \ldots, m'_d)$ are arrays of elements of $B \cong \mathbb{Z}$ or $\mathbb{Z}/n$ (in our applications, $m, m' \in \operatorname{Hom}(A, T)$). Let

$$\phi(m) \stackrel{\text{def}}{=} -\sum_{i<j} m_i m_j$$
$$I(m, m') \stackrel{\text{def}}{=} -\sum_{i \neq j} m_i m'_j.$$

Similarly, if $(x_1, \ldots, x_d)$ are elements of a B-module X, then

$$I(m, x) \stackrel{\text{def}}{=} -\sum_{i \neq j} m_i x_j.$$

We have chosen the signs of ϕ and I so that the right hand sides appear with positive sign in the following.

Lemma 4.12. (1) *ϕ is quadratic, I is symmetric and bilinear, and*

$$\phi(m + m') = \phi(m) + I(m, m') + \phi(m').$$

(2) *If $\sum m_i = 0$, then*

$$I(m, x) = \sum_i m_i x_i.$$

(3) *If* $\sum m_i = 0$ *then, then*

$$2\phi(m) = \sum_i m_i^2.$$

$\square$

Lemma 4.13. *Writing* $m_i = \log(\alpha_i)$ *and* $x_i = c_1(L_i)$*, we have*

$$c_1^B(\xi) = c_1(\xi) + (\sum_i m_i) \cdot z$$

and

$$c_2^B(\xi) = c_2(\xi) - I(m,x)z - \phi(m)z^2.$$

In particular,

$$\begin{aligned}
c_1^0(\xi) &= c_1(\xi) \\
c_1^2(\xi) &= \sum_i m_i \\
c_2^0(\xi) &= c_2(\xi) \\
c_2^2(\xi) &= -I(m,x) \\
c_2^4(\xi) &= -\phi(m).
\end{aligned}$$

If $c_1^B(\xi) = 0$ *then*

$$c_2^2(\xi) = -\sum m_i x_i. \tag{4.14}$$

If $c_1^B(\xi) = 0$ *and* $A = \mathbb{T}$*, then*

$$c_2^4(\xi) = -\frac{1}{2}\sum_i m_i^2. \tag{4.15}$$

Proof. The expressions for $c_1^{\mathcal{B}}$ and $c_2^{\mathcal{B}}$ follow easily from the product formula (4.11). If $c_1^{\mathcal{B}}(\xi) = 0$, then $\sum_i m_i = 0$, and the formula (4.14) follows from Lemma 4.12. Finally, if we note that in the universal case 2 is not a zero divisor, (4.15) also follows from Lemma 4.12. $\square$

Applying Lemma 4.13 to the universal bundle ξ over

$$BU^A \simeq JU(A) \times \prod_{\beta \in A^\vee} BU, \tag{4.16}$$

we have the following, which will be useful in Section 5.

Proposition 4.17. *Let*

$$V = \sum_{\alpha \in A^\vee} d_\alpha \alpha$$

be an element of $JU(A)$*. In the* V *factor of*

$$H^*(BU^A) \cong \prod_{V \in JU(A)} \mathbb{Z}[\![c_1^\alpha, c_2^\alpha, \dots | \alpha \in A^\vee]\!],$$

we have

$$\begin{aligned}
c_1^0(\xi) &= \sum_\alpha c_1^\alpha \\
c_1^2(\xi) &= \sum_\alpha d_\alpha \log(\alpha).
\end{aligned}$$

If $V \in JU_2(A)$, then in the V factor of BSU^A, we have (for any fixed ordering on $A^\vee$)

$$c_2^0(\xi) = \sum_\alpha c_2^\alpha + \sum_{\alpha<\beta} c_1^\alpha c_1^\beta$$

$$c_2^2(\xi) = \sum_\alpha \log(\alpha) c_1^\alpha.$$

Proof. Let ξ_α denote the universal bundle (of rank 0) over the α factor of BU in (4.16). Then the universal bundle over the V component of BU^A is (see the second proof of Proposition 2.7)

$$\xi = \sum_\alpha \xi_\alpha \otimes \alpha + \sum_\alpha d_\alpha \alpha.$$

The formulae in the Proposition follow from this and Lemma 4.13. □

5. The cohomology of covers of $BU \times \mathbb{Z}$.

The long exact sequence (6.4) we use to calculate the $\mathbb{T}$-equivariant elliptic cohomology of X involves the (rational) Borel (co)homology of X^A. In this section we carry out the calculation for $BU\{2k\}$ with $k = 0, 1, 2, 3$. The main point is that the ordinary (rational in case $k = 3$) cohomology of the fixed set is concentrated in even degrees, so the Serre spectral sequence for the Borel cohomology collapses.

5.A. **Components and simple connectivity.** Let G be a compact Lie group. We recall that $JU(G)$ is the augmentation ideal of representations of virtual dimension zero in the complex representation ring $RU(G)$ of G.

Lemma 5.1. *If A is finite cyclic or the circle group then*

$$\pi_0(BU\{2k\}^A) = JU^k(A) \text{ for } k = 0, 1, 2, 3.$$

All components of $BU\{2k\}^A$ are homotopy equivalent.

Remark 5.2. For general groups G it is more natural to expect

$$\pi_0(BU\{2k\}^G) = JU_k(G) \text{ for } k = 0, 1, 2, 3,$$

where $JU_k(G)$ is the ideal generated by the representation-theoretic Chern classes $c_l(V)$ for $l \geq k$. If G is abelian then $JU_k(G) = JU(G)^k$.

Proof. The equivalence between the components comes from the H-space structures. We carry out the π_0 calculations. Since all homotopy groups of $\mathrm{map}(BA, K(\mathbb{Z}, 2n))$ are in even degree, we have

$$0 \longrightarrow \pi_0(BU\{2k+2\}) \longrightarrow \pi_0(BU\{2k\}) \xrightarrow{c_k^{2k}} H^{2k}(BA)$$

as in the diagram of Proposition 2.15. By definition c_0^0 is the dimension and surjective, and so $\pi_0(BU^A) = JU(A)$.

Lemma 4.13 implies that

$$c_1^2 : JU(A) = \pi_0(BU^A) \longrightarrow H^2(BA) = A^*$$

is the determinant. This is surjective, and $\pi_0(BSU^A)$ is the ideal $JU_2'(A)$ consisting of elements of $JU(A)$ with determinant 1. It is easy to check

$$JU(A)^2 \subseteq JU_2(A) \subseteq JU_2'(A),$$

and it remains to show that $JU_2'(A) \subseteq JU(A)^2$.

We may do this explicitly as follows (the argument is due to Neil Strickland). First note that an arbitrary element x of $JU_2'(A)$ is of the form

$$x = \alpha_1 \oplus \cdots \oplus \alpha_s - \beta_1 \oplus \cdots \oplus \beta_s,$$

where the α_i and β_i are the classes of one-dimensional representations, and

$$\prod \beta_i = \prod \alpha_i.$$

Notice that

$$(1-\beta_1)(1-\beta_2) + (1-\beta_1\beta_2)(1-\beta_3) = 2 - \beta_1 - \beta_2 - \beta_3 + \beta_1\beta_2\beta_3$$

This has the generalization

$$\sum_{j=1}^{s-1}(1 - \prod_{k=1}^{j}\beta_k)(1-\beta_{j+1}) = (s-1) - \beta_1 - \cdots - \beta_s + \prod_{j=1}^{s}\beta_j.$$

Let us write $n(\beta_1, \ldots, \beta_s)$ for this element of $JU(A)^2$. Then

$$x = n(\beta_1, \ldots, \beta_s) - n(\alpha_1, \ldots, \alpha_s) \in JU(A)^2.$$

Finally we have

$$c_2^4 : JU^2(A) = \pi_0(BSU^A) \longrightarrow H^4(BA) = \mathrm{Symm}^2(A^*). \tag{5.3}$$

When A is a compact abelian group, the isomorphism

$$RU(A) \cong \mathbb{Z}[A^*],$$

identifies $JU(A)$ with the augmentation ideal $I(A^*)$, and it is not difficult to check that the map

$$c_2^4 : JU(A)^2 \to H^4BA$$

factors as

$$\begin{tikzcd} I(A^*)^2 \arrow[r, "c_2^4"] \arrow[dr, "\text{can.}"'] & \mathrm{Symm}^2 A^* \arrow[d] \\ & I(A^*)^2/I(A^*)^3, \end{tikzcd} \tag{5.4}$$

where the vertical map is the one induced by the fact that, for any abelian group B, the map

$$B \times B \to I(B)^2/I(B)^3$$

sending (x, y) to the class of $(1-x)(1-y)$ is symmetric and bilinear. For any abelian group, this vertical map is an isomorphism [Pas79, Theorem 8.6], and so the kernel of the horizontal map is $I(A^*)^3 \cong JU(A)^3$. $\square$

Remark 5.5. For general G and $k = 2$ we may prove the result indicated in Remark 5.2 as follows. Using the fact JU_2 is an ideal we may assume $\det U = \det V = 1$ and hence $x = (U-n) + (n-V)$ is a sum of two elements of JU_2'. Now $U - n$ is the pullback from $SU(n)$ of $\tilde{U} - n$ where $\tilde{U}$ is the natural representation, and this is the pullback of $\tilde{U} - (\delta + n - 1)$ from $U(n)$, where δ is the determinant of $\tilde{U}$. This universal case follows from the calculation of $ku_*^{U(n)}$ in [Gre04].

Lemma 5.6. *If A is finite cyclic or the circle group, all components of $BU\{2k\}^A$ are simply connected.*

Proof. The non-equivariant simple connectivity of BU is well known, and implies that of the components of $BU^A \times \mathbb{Z}$ and BU^A. For BSU^A it follows from the surjectivity of

$$c_1^0 : BU^A \to K(H^0(BA), 2)$$

in π_2, which is a consequence of Lemma 4.10 and the well-known non-equivariant case. For $BU\{6\}^A$ it suffices to show the surjectivity of

$$\pi_2(c_2^2) : \pi_2 BSU^A \to \pi_2 K(H^2(BA), 2) = H^2(BA).$$

Notice that the map

$$B\mathbb{T} \to K(H^2 B\mathbb{T}, 2)$$

corresponding to a generator of $H^2 B\mathbb{T}$ is an equivalence, and that the natural map

$$K(H^2 B\mathbb{T}, 2) \to K(H^2 BA, 2)$$

is an epimorphism in π_2. In particular, for every element $x \in \pi_2 K(H^2 BA, 2) = H^2 BA$, there is a line bundle L over S^2 such that the map

$$S^2 \xrightarrow{L} B\mathbb{T} \to K(H^2 BA, 2)$$

represents $-x$.

Recall that we have chosen a generator z of $H^2 B\mathbb{T}$, and let $\alpha \in A^*$ be a generator, so that

$$B\alpha^* z \in H^2 BA$$

is a generator (which we will also call z). Now consider the A-bundle

$$\xi = (1 - L)(1 - \alpha) = 1 - L - \alpha + L \otimes \alpha.$$

over S^2. Its Borel Chern class is

$$c_\bullet^{\mathcal{B}}(\xi) = \frac{1 - x + z}{(1 - x)(1 + z)} = 1 + xz + \text{degree } 6.$$

In particular, it is an SU bundle whose c_2^2 component is x. Thus we have a commutative diagram

$$\begin{tikzcd}
BSU^A \arrow[r, "c_2^2"] & K(H^2(BA), 2), \\
S^2 \arrow[u, "\xi"] \arrow[ur, "x"'] &
\end{tikzcd}$$

showing that c_2^2 is surjective in π_2. ☐

5.B. **Homology and cohomology of fixed points of BSU.** The cohomology ring of BU is well known to be polynomial on the Chern classes, so that

$$H^*(BU^A) = \prod_{V \in JU(A)} \mathbb{Z}[c_1^\alpha, c_2^\alpha, \ldots \mid \alpha \in A^*].$$

We note that the usual calculation of $H_* BU$ in the nonequivariant case generalizes to give

$$H_*(BU^A) = \operatorname{Symm} H_*(BU(1)^A) = \mathbb{Z}[\beta_1^\alpha, \beta_2^\alpha, \ldots \mid \alpha \in A^*][JU(A)],$$

where β_i^α is the basis dual to $(c_1^\alpha)^i$ in $H^* BU(1)^A$.

For BSU we consider the fibre sequence $BSU^A \longrightarrow BU_S^A \longrightarrow BU(1)_\epsilon^A$ of (2.12), and note that the generator of $H^*(BU(1)_\epsilon^A)$ acts as $\Sigma_\alpha c_1^\alpha$. It follows that $H^*(BU_V^A)$ is flat as a module over $H^*(BU(1)_\epsilon^A)$ for each V and hence the Eilenberg-Moore spectral sequence gives

$$H^*(BSU_V^A) = \mathbb{Z} \otimes_{H^*(BU(1)_\epsilon^A)} H^*(BU_V^A).$$

This means that each component of BSU^A has polynomial cohomology in even degrees. Dually, in homology we may deal with all components at once to find

$$H_*(BSU^A) = \mathrm{Hom}_{H^*(BU(1)^A_\epsilon)}(\mathbb{Z}, H_*(BU^A_S)).$$

The Serre spectral sequence gives exactly the same calculation if we consider the fibration $K(\mathbb{Z},1) \longrightarrow BSU^A \longrightarrow BU^A_S$.

5.C. **Homology and cohomology of fixed points of** $BU\{6\}$**.** To continue, we suppose that A is either a finite cyclic group C or the circle group $\mathbb{T}$, so that $H^3(BA) = 0 = H^1(BA)$. Thus have a fibration

$$BU\{6\}^A \longrightarrow BSU^A \longrightarrow K(H^0(BA),4) \times K(H^2(BA),2) \times K(H^4(BA),0).$$

As for BSU, we may trim away the component group in the base by letting BSU^A_S consist of the components indexed by representations in

$$JU(A)^3 = \mathrm{Ker}\, \pi_0 BSU^A \longrightarrow H^4BA.$$

Then we have a fibration

$$BU\{6\}^A \longrightarrow BSU^A_S \longrightarrow K(H^0(BA),4) \times K(H^2(BA),2)$$

with connected base. All components are equivalent, and we have a fibration

$$BU\{6\}^A_0 \longrightarrow BSU^A_0 \longrightarrow K(H^0(BA),4) \times K(H^2(BA),2)$$

of connected spaces, where the subscript 0 indicates the component of the 0 bundle.

For the purposes of this paper, it is sufficient to work over the rationals, so that $K(H^0(BA),4)$ and $K(H^2(BA),2)$ have polynomial cohomology.

We deal separately with the case $A = C$ is finite and the case $A = \mathbb{T}$. In the first we even have $H^2(BC;\mathbb{Q}) = 0$, and so the rational fibration

$$BU\{6\}^C \longrightarrow BSU^C_S \longrightarrow K(\mathbb{Q},4).$$

Now $H^*(K(\mathbb{Q},4);\mathbb{Q}) = \mathbb{Q}[c_2^0]$, where c_2^0 acts as its name suggests by

$$\sum_\alpha c_2^\alpha + \sum_{\alpha<\beta} c_1^\alpha c_1^\beta = \sum_\alpha c_2^\alpha - \frac{1}{2}\sum_\alpha (c_1^\alpha)^2$$

for any fixed ordering on A^* (see Proposition 4.17). Since c_2^0 can be chosen as a polynomial generator of $H^*(BSU^C_S)$ we obtain

$$H^*(BU\{6\}^C) = \mathbb{Q} \otimes_{\mathbb{Q}[c_2^0]} H^*(BSU^C_S) = H^*(BSU^C_S)/(c_2^0)$$
$$H^*(BU\{6\}^C_0) = \mathbb{Q} \otimes_{\mathbb{Q}[c_2^0]} H^*(BSU^C_0) = H^*(BSU^C_0)/(c_2^0)$$
$$H_*(BU\{6\}^C_0) = \mathrm{Hom}_{\mathbb{Q}[c_2^0]}(\mathbb{Q}, H_*(BSU^C_0)).$$

We note that the cohomology ring of each component is polynomial and in even degree.

When $A = \mathbb{T}$ we have

$$BU\{6\}^{\mathbb{T}} \longrightarrow BSU^{\mathbb{T}}_S \longrightarrow K(\mathbb{Q},4) \times K(\mathbb{Q},2),$$

and note that $H^*(K(\mathbb{Q},4) \times K(\mathbb{Q},2);\mathbb{Q}) = \mathbb{Q}[c_2^0, c_2^2]$ where c_2^0 and c_2^2 act as indicated in Proposition 4.17. Since c_2^0 and c_2^2 generate a tensor factor $\mathbb{Q}[c_2^0, c_2^2]$ we obtain

$$H^*(BU\{6\}^{\mathbb{T}}_V) = \mathbb{Q} \otimes_{\mathbb{Q}[c_2^0,c_2^2]} H^*(BSU^{\mathbb{T}}_V) = H^*(BSU^{\mathbb{T}}_S)/(c_2^0, c_2^2)$$

and

$$H_*(BU\{6\}^{\mathbb{T}}_V) = \mathrm{Hom}_{\mathbb{Q}[c_2^0,c_2^2]}(\mathbb{Q}, H_*(BSU^{\mathbb{T}}_S)).$$

Once again, the cohomology ring of each component is polynomial and in even degrees.

5.D. **Borel homology and cohomology.** The long exact sequence we use to calculate the elliptic cohomology of $BU\{2k\}$ involves the rational Borel (co)homology of $BU\{2k\}^A$. We continue the convention that A is a finite cyclic group or the circle, and we continue to work with rational coefficients.

Proposition 5.7. *For $k \leq 3$, the A-fixed point spaces of $X = BU\{2k\}$ have cohomology in even degrees. Each component has polynomial cohomology. As for Borel homology and cohomology, there are isomorphisms (non-canonical unless $A = \mathbb{T}$)*

$$H^*_{\mathbb{T}}(X^A) = H^*(X^A)[z]$$

and

$$H^{\mathbb{T}}_*(X^A) = H_*(X^A) \otimes H_*(B\mathbb{T}).$$

Moreover, the natural map

$$H^*_{\mathbb{T}}(X^A) \to H^*_{\mathbb{T}}(X^{\mathbb{T}}) \tag{5.8}$$

is injective.

Proof. All the spaces $X^A = BU\{2k\}^A$ for $k \leq 3$ have components whose cohomology is polynomial and in even degrees. Accordingly the Serre spectral sequence calculating the $\mathbb{T}$-equivariant Borel cohomology of one component collapses and shows the Borel cohomology is isomorphic to a tensor product of $H^*(B\mathbb{T})$ and the polynomial cohomology ring. When A is finite, this involves *choosing* lifts of the polynomial generators of $H^*(X^A_V)$ to $H^*_{\mathbb{T}}(X^A_V)$. Since the map $X^{\mathbb{T}} \longrightarrow X^A$ is injective in cohomology, it follows that (5.8) is as well. Similar arguments show that the Borel homology spectral sequence also collapses to give the isomorphism of $H^*(B\mathbb{T})$-modules

$$H^{\mathbb{T}}_*(X^A) \cong H_*(X^A) \otimes H_*(B\mathbb{T}).$$

$\square$

Part 2. Elliptic cohomology and the sigma orientation.

In this part we turn towards elliptic cohomology and the sigma genus. First, in Section 6 we introduce notation for discussing the geometry of an elliptic curve, before summarizing the relevant properties of equivariant elliptic cohomology from [Gre05]. The sigma genus is most easily introduced for $\mathbb{T}$-fixed spaces and generic points on the curve, because the role of the topology and geometry is largely unlinked: we discuss this in Section 7. Finally, in Section 8 we turn to the more subtle question of how to deal with points with finite isotropy and torsion points on the elliptic curve.

6. Properties of equivariant elliptic cohomology.

6.A. **Geometry of the elliptic curve.** In this section we summarize the relevant properties of the $\mathbb{T}$-equivariant elliptic cohomology defined in [Gre05]. We begin by introducing notation to describe the elliptic curve. Let C be a rational elliptic curve

$$\begin{tikzcd} C \arrow[r, "p"'] & S \arrow[l, bend right, "0"'] \end{tikzcd}$$

with identity 0 and structure map p, over an affine $\mathbb{Q}$-scheme S.

We write $\mathcal{O}$ for the structure sheaf of C, and for a divisor D, the sheaf $\mathcal{O}(D)$ consists of functions with $\mathrm{Div}(f)+D \geq 0$. We write $\mathcal{K}$ for the constant sheaf of meromorphic functions on C with poles only at points of finite order.

For any $n \geq 1$ we write $C[n] = \ker(n : C \longrightarrow C)$ for the subgroup of points of order dividing n and $C\langle n\rangle$ for the scheme of points of exact order n. It is convenient to index certain divisors by representations of $\mathbb{T}$. Given a representation V with $V^{\mathbb{T}} = 0$ we write $V = \sum_n a_n z^n$, and take $D(V) = \sum_n a_n C[n]$. Thus we have

$$\mathcal{K} = \mathrm{colim}_{V^{\mathbb{T}}=0}\, \mathcal{O}(D(V)).$$

Next, we write $\mathcal{K}_n$ for the functions regular on $C\langle n\rangle$[3] and $T_nC = \mathcal{K}/\mathcal{K}_n$ for the sheaf of principal parts of functions on $C\langle n\rangle$; this can also be described as the local cohomology group $H^1_{C\langle n\rangle}(C)$. We set

$$TC = \bigoplus_n T_nC.$$

For a finite subgroup $A = \mathbb{T}[n]$, it is the topology of X^A which controls the behaviour of the equivariant elliptic cohomology of X near $C\langle n\rangle$. As a consequence we adopt the convention that

$$\begin{aligned} C\langle A\rangle &= C\langle |A|\rangle \\ \mathcal{K}_A &= \mathcal{K}_n \\ T_AC &= T_nC \\ \mathcal{O}^{\wedge}_A &= \mathcal{O}^{\wedge}_{C\langle A\rangle}. \end{aligned}$$

We write $\Omega = \Omega_C$ for the sheaf of Kähler differentials, and Ω^d_C for its d^{th} *tensor* power. We write

$$\underline{\omega} = p_*\Omega \cong 0^*\Omega$$

for the $\mathcal{O}_S$-module of invariant differentials. Our analysis will involve expressions of the form $f(Dt)^k$, where f is a meromorphic function on C and Dt is an invariant differential; this may be regarded as a section of

$$\mathcal{K} \otimes_{\mathcal{O}} p^*\underline{\omega}^k \cong \mathcal{K} \otimes_{\mathcal{O}} \Omega^k_C.$$

We will identify the constant sheaf $\mathcal{K}$ and its twists by differentials with their modules of global sections. That is, we will generally not distinguish in our notation between

$$\mathcal{K} \otimes_{\mathcal{O}} \Omega^*_C \cong \mathcal{K} \otimes_{\mathcal{O}} p^*\underline{\omega}^*$$

and

$$\Gamma(\mathcal{K} \otimes \Omega^*_C) \cong \Gamma(\mathcal{K} \otimes_{\mathcal{O}} p^*\underline{\omega}^*).$$

6.B. **Coordinate data.** We recall that to give an $\mathbb{T}$-equivariant elliptic spectrum we specify not only an elliptic curve C but also a section t_1 of $\mathcal{K}_C$ which is a coordinate at the identity of C. Note that it is equivalent to specify a pair $(\mathcal{D}, \omega)$ where ω is an invariant differential and $\mathcal{D}$ is a divisor satisfying

(1) $\deg \mathcal{D} = 0$
(2) $\sum^C [n_P](P) = 0$
(3) $n_P = 0$ unless P is a point of finite order of C
(4) $n_0 = 1$.

[3]This is also the local ring $\mathcal{O}_{C\langle n\rangle}$.

The first three conditions imply that there is a meromorphic function $t_1 \in \mathcal{K}$ with $\operatorname{Div} t_1 = \mathcal{D}$; the last condition implies that t_1 is a coordinate at the identity.

In this paper we work with a complex elliptic curve $C \cong \mathbb{C}/\Lambda$, in which case, if $\bar{P}$ is a choice of lifts to $\mathbb{C}$ of the points of $\mathcal{D}$, then we can take

$$t_1(z) = \prod_P \sigma(z - \bar{P})^{n_P}.$$

Next, for $s > 1$ we define t_s to be the meromorphic function with the properties

(1) $\operatorname{Div}(t_s) = C\langle s\rangle - |C\langle s\rangle|(0)$
(2) $(t_1^{|C\langle s\rangle|} t_s)(0) = 1$.

In our complex case, if $\overline{C\langle s\rangle} \subset \mathbb{C}$ is a set of lifts of the points of $C\langle s\rangle$, then

$$t_s(z) = \lambda_s \frac{\prod_{p \in \overline{C\langle s\rangle}} \sigma(z-p)}{\sigma(z)^{|C\langle s\rangle|}},$$

where λ_s is a constant easily expressed in terms of values of the sigma function.

We use t_s/Dt to make $T_sC \otimes \omega_C^*$ into a torsion $\mathbb{Q}[c]$-module, where Dt is the invariant differential agreeing with dt_1 at the identity: for $f \otimes \omega \in T_sC \otimes \omega_C^*$,

$$c^k f \otimes \omega = t_s^k f \otimes (Dt)^{-k} \omega.$$

6.C. **Spheres and line bundles.** Let V be a virtual complex representation of $\mathbb{T}$, and suppose that $V^{\mathbb{T}} = 0$. The spectrum EC is constructed so that

$$EC^i_{\mathbb{T}}(S^V) = H^i(C; \mathcal{O}(-D(V)))$$

for $i = 0, 1$. Twisting by a trivial representation of rank 1 is equivalent to a double suspension, and as in the non-equivariant case this introduces a twist by the Kähler differentials, giving

$$EC^{i-2d}_{\mathbb{T}}(S^V) = H^i(C; \mathcal{O}(-D(V)) \otimes (\Omega^1_C)^{\otimes d})$$
$$EC^{\mathbb{T}}_{2d-i}(S^V) = H^i(C; \mathcal{O}(D(V)) \otimes (\Omega^1_C)^{\otimes d}).$$

6.D. **Localization and completion.** Elliptic cohomology satisfies a localization theorem and a completion theorem.

Let $\mathcal{F}$ be the family of finite subgroups of $\mathbb{T}$. Recall that there is a universal space $E\mathcal{F}$ for $\mathbb{T}$-spaces with isotropy in $\mathcal{F}$, characterized by the fact that its fixed points under finite subgroups are contractible, whereas $E\mathcal{F}^{\mathbb{T}} = \emptyset$. This is related to the join $\tilde{E}\mathcal{F} = S^0 * E\mathcal{F}$ by the cofibre sequence

$$E\mathcal{F}_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathcal{F}.$$

It is convenient to use the models

$$E\mathcal{F} = \bigcup_{V^{\mathbb{T}}=0} S(V) \text{ and } \tilde{E}\mathcal{F} = \bigcup_{V^{\mathbb{T}}=0} S^V,$$

where $S(V)$ is the unit sphere in V and $S^V \cong S^0 * S(V)$ is the one-point compactification of V. The usefulness of these spaces arises since for any based $\mathbb{T}$-space X, the inclusion $X^{\mathbb{T}} \to X$ induces a weak equivalence

$$X^{\mathbb{T}} \wedge \tilde{E}\mathcal{F} \xrightarrow{\simeq} X \wedge \tilde{E}\mathcal{F}.$$

The corresponding statement holds for spectra if we use geometric fixed points, but we restrict to spaces so we can retain familiar notation.

Lemma 6.1. *For any $\mathbb{T}$-space X we have*

$$EC_*^{\mathbb{T}}(X \wedge \tilde{E}\mathcal{F}) = H_*(X^{\mathbb{T}}; \mathcal{K} \otimes \Omega_C^*).$$

Similarly in cohomology for finite complexes X. The corresponding statement holds for spectra if we use geometric fixed points.

Proof. Since $\tilde{E}\mathcal{F} = \mathrm{colim}_{V^{\mathbb{T}}=0} S^V$ and $X \wedge \tilde{E}\mathcal{F} \simeq X^{\mathbb{T}} \wedge \tilde{E}\mathcal{F}$ we easily deduce this from the values on spheres. Indeed, $\mathrm{colim}_V \mathcal{O}(D(V)) = \mathcal{K}$, so that

$$EC_{2d}^{\mathbb{T}}(\tilde{E}\mathcal{F}) = \mathcal{K} \otimes (\Omega_C^1)^{\otimes d}.$$

□

Before stating the completion theorem, we pause briefly to summarize the relationship between Borel homology and cohomology, which is described in more detail in Appendix A. Given a graded module M over $H^*B\mathbb{T} = k[c]$ for a field k, we can form the Borel cohomology $H_{\mathbb{T}}^*(X; M)$, and in certain cases there are simple descriptions. (This notation means the cohomology theory represented by the module over the Borel spectrum, and not a Brown-Comenetz type theory as in [Gre05]; the distinction is explained further in Remark A.3). If M is flat and X is a finite $\mathbb{T}$-CW complex, we have

$$H_{\mathbb{T}}^*(X; M) \cong H_{\mathbb{T}}^*(X) \otimes_{H^*B\mathbb{T}} M.$$

As usual, homological and cohomological gradings of the same module are related by $M_k = M^{-k}$, and c is of cohomological degree 2 and homological degree -2. It is also useful to consider the torsion $H^*B\mathbb{T}$-module $(M[c^{-1}]/M)$. If c is not a zero-divisor in M, there is a natural map

$$\kappa : H_{\mathbb{T}}^p(X; M) \to \mathrm{Hom}_{H^*B\mathbb{T}}^p(H_*^{\mathbb{T}}X; \Sigma^{-2}(M[c^{-1}]/M)). \tag{6.2}$$

If M is a free module, then $M[c^{-1}]/M$ is injective, so we have a natural transformation of cohomology theories, and it is easy to check then that the map is completion. Since completion of M does not affect $M[c^{-1}]/M$, we see that if M is the completion of a free module then κ is an isomorphism.

For example, if $M = H^*B\mathbb{T}$, then

$$\Sigma^{-2}(M[c^{-1}]/M) \cong H_*B\mathbb{T},$$

and this is injective, so we have the isomorphism

$$H_{\mathbb{T}}^*(X; H^*(B\mathbb{T})) \cong \mathrm{Hom}_{H^*B\mathbb{T}}^*(H_*^{\mathbb{T}}X; H_*B\mathbb{T}).$$

For example, let $\mathcal{O}_A^{\wedge}$ be the formal completion of $\mathcal{O}$ at $C\langle A\rangle$, and let $M = \mathcal{O}_A^{\wedge} \otimes \omega^*$ be considered as an $H^*B\mathbb{T}$-algebra via

$$c \mapsto t_A \otimes (Dt)^{-1}.$$

Then

$$(M[c^{-1}]/M) \cong T_A C \otimes \omega^*,$$

and this is an injective $\mathbb{Q}[c]$-module. Thus Example (A.10) shows that we have

$$H_{\mathbb{T}}^*(X; \mathcal{O}_A^{\wedge} \otimes \omega^*) \cong \mathrm{Hom}_{H^*(B\mathbb{T})}(H_*^{\mathbb{T}}(X), T_A C \otimes \omega^*).$$

Lemma 6.3. *For any $\mathbb{T}$-space X*

$$EC_{\mathbb{T}}^*(X \wedge E\mathcal{F}_+) \cong \prod_A H_{\mathbb{T}}^*(X^A; \mathcal{O}_A^{\wedge} \otimes \omega_C^*).$$

If $H_*^{\mathbb{T}}(X^A) = H_*(X^A) \otimes H_*(B\mathbb{T})$ for all finite $A \subset \mathbb{T}$ then

$$EC_{\mathbb{T}}^*(X \wedge E\mathcal{F}_+) \cong \prod_A H^*(X^A; \mathcal{O}_A^{\wedge} \otimes \omega_C^*).$$

The corresponding statement holds for spectra if we use geometric fixed points.

Proof. The first statement amounts to the fact that $EC \wedge \Sigma E\mathcal{F}_+$ is injective, with coefficients $TC \otimes \omega_C^*$. Now we use the fact that there is a rational splitting $E\mathcal{F}_+ \simeq \bigvee_A E\langle A\rangle$ corresponding to $TC \simeq \bigoplus_A T_A C$, and that $[X, E\langle A\rangle \wedge Y]^{\mathbb{T}} = [X^A, E\langle A\rangle \wedge Y]^{\mathbb{T}}$. Passing to the summand corresponding to A, the $H^*(B\mathbb{T})$-module structure on rings of functions is through $t_{|A|}/Dt$. The second statement follows since the short exact sequence

$$0 \longrightarrow \mathcal{K}_A \longrightarrow \mathcal{K} \longrightarrow T_A C \longrightarrow 0$$

gives an isomorphism

$$\mathrm{Hom}_{H^*(B\mathbb{T})}(H_*(B\mathbb{T}), T_A C \otimes \omega_C^*) = \mathrm{Ext}_{H^*(B\mathbb{T})}(H_*(B\mathbb{T}), \mathcal{K}_A \otimes \omega_C^*) = \mathcal{O}_A^\wedge \otimes \omega_C^*.$$

(See Appendix A for further details.) $\square$

6.E. **Periodicity.** It is sometimes convenient to define the "periodic ordinary cohomology spectrum" by the formula

$$HP = \bigvee_{k\in\mathbb{Z}} \Sigma^{2k} H$$

This spectrum has the feature that

$$\mathrm{spf}\, HP^0\mathbb{C}P^\infty \cong \widehat{\mathbb{G}}_a,$$

while

$$HP^0 S^2 \cong \pi_2 HP \cong \Gamma(\omega_{\widehat{\mathbb{G}}_a}).$$

Note that the coordinate data used to construct EC determine an isomorphism

$$\widehat{C} \cong \widehat{\mathbb{G}}_a$$

carrying Dt_1 to the standard generator of $\widehat{\mathbb{G}}_a$, and so inducing an isomorphism

$$H^*(X; R \otimes \omega^*) \cong HP^*(X; R).$$

We shall find it convenient simply to *define* ω^*-periodic cohomology as

$$HP^*(X; R) \stackrel{\mathrm{def}}{=} H^*(X; R \otimes \omega_C^*).$$

With this notation, the localization and completion isomorphisms above become

$$EC_*^{\mathbb{T}}(X \wedge \tilde{E}\mathcal{F}) \cong HP_*(X^{\mathbb{T}}; \mathcal{K})$$
$$EC_{\mathbb{T}}^*(X \wedge E\mathcal{F}_+) \cong \prod_A HP_{\mathbb{T}}^*(X^A; \mathcal{O}_A^\wedge).$$

6.F. **The Hasse square.** The localization and completion theorems combine to give an extremely useful long exact sequence, relating equivariant elliptic cohomology to Borel cohomology and the elliptic curve. The idea is to take (i) information from the $\mathbb{T}$-fixed point space, generic on the curve and (ii) information from the A-fixed point space in a neighbourhood of the points of order $|A|$ on the curve and to splice them together. The idea that points with isotropy of order n in topology are associated to points of order n on the curve is a recurrent central theme. Topological and geometric information interacts very little over $\mathbb{T}$-fixed spaces, but much more over points with finite isotropy.

This sequence is [Gre05, 15.3], but we have used Remark A.10 to give it in a more geometrically transparent form.

Proposition 6.4. *For any $\mathbb{T}$-space X there is a long exact sequence*

$$\cdots \longrightarrow EC^n_{\mathbb{T}}(X) \longrightarrow H^n(X^{\mathbb{T}};\mathcal{K}\otimes\Omega^*_C) \times \prod_A H^n_{\mathbb{T}}(X^A;\mathcal{O}^\wedge_A\otimes\omega^*_C) \longrightarrow$$
$$H^n(X^{\mathbb{T}};\mathcal{K}^\wedge_{\mathcal{F}}\otimes\omega^*_C) \longrightarrow EC^{n+1}_{\mathbb{T}}(X) \longrightarrow \cdots, \quad (6.5)$$

natural in X, where $\mathcal{K}^\wedge_{\mathcal{F}} = \prod_A \mathcal{O}^\wedge_A \otimes \mathcal{K}$. If we are given an isomorphism $H^{\mathbb{T}}_*(X^A) \cong H_*(X^A) \otimes H_*(B\mathbb{T})$, then we obtain an isomorphism

$$H^n_{\mathbb{T}}(X^A;\mathcal{O}^\wedge_A\otimes\omega^*_C) \cong H^n(X^A;\mathcal{O}^\wedge_A\otimes\omega^*_C). \quad (6.6)$$

The corresponding statement holds for spectra if we use geometric fixed points.

Remark 6.7. (1) The first displayed map is a ring homomorphism when X is a space.

(2) For the spaces we care most about, $H^*(X^{\mathbb{T}})$ and $H^*_{\mathbb{T}}(X^A)$ are in even degrees for all A, so that this degenerates to give a pullback square of rings for $EC^*_{\mathbb{T}}(X)$.

(3) We remark that we may arrange that the map

$$H^n(X^{\mathbb{T}};\mathcal{K}\otimes\Omega^*_C) \times \prod_A H^n_{\mathbb{T}}(X^A;\mathcal{O}^\wedge_A\otimes\omega^*_C) \to H^n(X^{\mathbb{T}};\mathcal{K}^\wedge_{\mathcal{F}}\otimes\omega^*_C)$$

from the long exact sequence is the obvious one. On the $X^{\mathbb{T}}$ factor, it is induced by the natural map

$$\mathcal{K} \to \mathcal{K}^\wedge_{\mathcal{F}}.$$

On the X^A factor, we may arrange that it is restriction along

$$X^{\mathbb{T}} \to X^A,$$

composed with the natural map

$$\mathcal{O}^\wedge_A \to \mathcal{K}^\wedge_{\mathcal{F}}.$$

To make sense of this, we arrange that both remaining terms may be interpreted as Borel cohomology. Indeed, we may make $\mathcal{K}^\wedge_{\mathcal{F}}\otimes\omega^*_C$ into a module over $H^*(B\mathbb{T})$ by letting c act through $t_{|A|}/Dt$ in the A-factor, and as such we have

$$H^n(X^{\mathbb{T}};\mathcal{K}^\wedge_{\mathcal{F}}\otimes\omega^*_C) \cong H^n_{\mathbb{T}}(X^{\mathbb{T}};\mathcal{K}^\wedge_{\mathcal{F}}\otimes\omega^*_C).$$

It will appear from the proof that the map is as stated by naturality of the completion theorem.

The exact sequence (6.5) suggests that $EC^*_{\mathbb{T}}(X)$ is related to the cohomology of a sheaf on C: the $H^n(X^{\mathbb{T}};\mathcal{K}\otimes\Omega^*_C)$ factor concerns the behaviour of a section generically on C, while the $H^n_{\mathbb{T}}(X^A;\mathcal{O}^\wedge_A\otimes\omega^*_C)$ factors concern the behaviour in small neighborhoods of the points of finite order. We shall study the string orientation from this point of view in Part 3.

(4) Indeed, the Borel cohomology groups which appear in (6.5) are essentially those which describe Grojnowski's sheaf-valued theory (in the case of a finite complex). Note that Grojnowski treats the case of an elliptic curve of the form $\mathbb{C}/\Lambda$, and uses the projection

$$\mathbb{C} \to \mathbb{C}/\Lambda$$

and translation in the elliptic curve to give $\mathcal{O}^\wedge_A$ the structure of an $H^*B\mathbb{T}$-algebra. One of the innovations of [Gre05] is to handle the algebraic case, using the functions $t_{|A|}$ to make $\mathcal{O}^\wedge_A$ into an $H^*B\mathbb{T}$-algebra.

Proof. Any $\mathbb{T}$-spectrum E occurs in the Tate homotopy pullback square

$$\begin{array}{ccc} E & \longrightarrow & E \wedge \tilde{E}\mathcal{F} \\ \downarrow & & \downarrow \\ F(E\mathcal{F}_+, E) & \longrightarrow & F(E\mathcal{F}_+, E) \wedge \tilde{E}\mathcal{F} \end{array}$$

where $\mathcal{F}$ is the family of proper subgroups, and applying $F(X, \cdot)$ we obtain the homotopy pullback square

$$\begin{array}{ccc} F(X, E) & \longrightarrow & F(X, E \wedge \tilde{E}\mathcal{F}) \\ \downarrow & & \downarrow \\ F(X \wedge E\mathcal{F}_+, E) & \longrightarrow & F(X, F(E\mathcal{F}_+, E) \wedge \tilde{E}\mathcal{F}). \end{array}$$

Note that

$$[X, Y \wedge \tilde{E}\mathcal{F}]^{\mathbb{T}}_* = [\Phi^{\mathbb{T}} X, \Phi^{\mathbb{T}} Y]_* = [X^{\mathbb{T}}, \Phi^{\mathbb{T}} Y]_*,$$

so that both the right hand terms can be expressed in terms of the geometric fixed points of X. In the case that E is elliptic cohomology, we apply the localization theorem to see that $\pi^{\mathbb{T}}_*(F(X, EC \wedge \tilde{E}\mathcal{F})) = H^*(X^{\mathbb{T}}; \mathcal{K} \otimes \omega^*_C)$ and the completion theorem to see that

$$\pi^{\mathbb{T}}_*(F(X \wedge E\mathcal{F}_+, EC)) = EC^*_{\mathbb{T}}(X \wedge E\mathcal{F}_+) = \prod_A H^*_{\mathbb{T}}(X^A; \mathcal{O}^\wedge_A).$$

$\square$

7. The sigma orientation.

In this section we describe the construction of our Thom class for the tautological bundle over $BString_{\mathbb{C}}$. We implement the strategy for bundles over $\mathbb{T}$-fixed spaces, by showing how to use the Weierstrass sigma function to construct a Thom class. Details for spaces which are not fixed are deferred to Section 8.

7.A. **The sigma function.**

First of all, we write σ for the expression

$$\sigma(w, q) = (w^{1/2} - w^{-1/2}) \prod_{n \geq 1} \frac{(1 - q^n w)(1 - q^n w^{-1})}{(1 - q^n)^2}.$$

We can consider σ as a function of $(z, \tau) \in \mathbb{C} \times \mathfrak{h}$ by setting

$$w^r = e^{rz}$$
$$q^r = e^{2\pi i r \tau}$$

for $r \in \mathbb{Q}$. It is convenient to consider σ sometimes as a function of w, writing the first argument multiplicatively, and sometimes as a function of z, writing the first argument additively. We'll adopt the convention that the second argument (τ or q) indicates the form of the first argument.

The function σ is holomorphic, vanishes only at lattice points, and has the following properties.

$$\sigma(z, \tau) = z + o(z^2) \tag{7.1}$$
$$\sigma(-z, \tau) = -\sigma(z, \tau) \tag{7.2}$$
$$\sigma(z + 2\pi i l + 2\pi i k\tau, \tau) = (-1)^{l+k} e^{-kz - \pi i k^2 \tau} \sigma(z, \tau) \tag{7.3}$$
$$\sigma(wq^k, q) = (-1)^k w^{-k} q^{-\frac{k^2}{2}} \sigma(w, q). \tag{7.4}$$

7.B. **The Witten genus and the sigma orientation.** We use the expansion of σ in terms of z in (7.1) to determine an exponential orientation for complex vector bundles. More precisely, if V is a complex vector bundle over X, then there is a Thom class

$$\mathrm{Thom}(V) = \mathrm{Thom}^{\sigma}(V) \in H^*(X^V; \mathbb{C}), \tag{7.5}$$

characterized by the property that, if

$$c_{\bullet}(V) = \prod (1 + x_i),$$

then the Euler class associated to $\mathrm{Thom}(V)$ is

$$e(V, \tau) \stackrel{\text{def}}{=} \prod \sigma(x_i, \tau). \tag{7.6}$$

As explained in [HBJ92] (see also [Wit87] and [AHS01]), the q-form of σ is the K-theory characteristic series of a multiplicative orientation

$$\sigma : MSU \to K[\![q]\!]$$

for SU bundles in integral K-theory, with coefficients in $\mathbb{Z}[\![q]\!]$. The Euler class of $V = L_1 \oplus \cdots \oplus L_d$ is

$$e(V, q) = \prod_i \sigma(L_i, q) = \Delta_{-1}(V) \otimes \bigotimes \Lambda_{-q^n}(V - \operatorname{rank} V) \bigotimes \Lambda_{-q^n}(\bar{V} - \operatorname{rank} V).$$

That is, the orientation given by σ is a twist of the $\widehat{A}$ orientation of Atiyah-Bott-Shapiro. The associated genus is of an SU-manifold M is

$$\widehat{A}(M; \bigotimes_{n \geq 1} \mathrm{S}_{q^n}(V - \operatorname{rank} V) \otimes \mathrm{S}_{q^n}(\bar{V} - \operatorname{rank} V)),$$

which is known as the *Witten genus*.

In [AHS01], the authors define an *elliptic spectrum* to be a triple (E, C, t), where E is an even periodic ring spectrum (and so complex-orientable), C is an elliptic curve over $\pi_0 E$, and t is an isomorphism of formal groups

$$t : \operatorname{spf} E^0 \mathbb{C}P^{\infty} \cong \widehat{C}.$$

They show that the data of an elliptic spectrum determine a map of (non-equivariant) ring spectra

$$\sigma(E, C, t) : MU\langle 6 \rangle \to E.$$

called the *sigma orientation*.

The Tate curve is an elliptic curve C_{Tate} over $\mathbb{Z}[\![q]\!]$ which provides an arithmetic model for the multiplicative uniformization of a complex elliptic curve as

$$C \cong \mathbb{C}/\Lambda \cong \mathbb{C}^{\times}/q^{\mathbb{Z}},$$

where $\Lambda = 2\pi i \mathbb{Z} + 2\pi i \tau \mathbb{Z}$ and $q = e^{2\pi i \tau}$. It comes with an isomorphism of formal groups

$$t : \widehat{\mathbb{G}}_m \cong \widehat{C}_{\mathrm{Tate}},$$

so $K_{\mathrm{Tate}} \stackrel{\text{def}}{=} (K[\![q]\!], C_{\mathrm{Tate}}, t)$ is an elliptic spectrum. It turns out [AHS01, §2.6,2.7] that the sigma orientation of K_{Tate} is just the restriction to $MU\langle 6 \rangle$ of the orientation above: that is, the diagram

$$\begin{tikzcd} MU\langle 6 \rangle \arrow[d] \arrow[dr, "\sigma(K_{\mathrm{Tate}})"] & \\ MSU \arrow[r, "\sigma"'] & K[\![q]\!] \end{tikzcd}$$

commutes.

7.C. **The Borel equivariant sigma orientation.** If V is a G-equivariant SU-bundle, then there is similarly an equivariant Thom class

$$\operatorname{Thom}_G(V) \stackrel{\text{def}}{=} \operatorname{Thom}(V \times_G EG) \in H^*_G(X^V), \tag{7.7}$$

and we write

$$e_G(V) \stackrel{\text{def}}{=} \zeta^* \operatorname{Thom}_G(V) \in H^*_G(X) \tag{7.8}$$

for the associated Euler class. In this section we record some formulae for this class and some related characteristic classes, in the case of the circle group. In Sections 7 and 8, we use these formulae to construct the equivariant sigma orientation.

Suppose that V is an $\mathbb{T}$-bundle over an $\mathbb{T}$-fixed space, given as

$$V \cong L_1 \otimes \alpha_1 \oplus \cdots \oplus L_d \otimes \alpha_d, \tag{7.9}$$

where L_i is a complex line bundle with Chern class x_i and $\alpha_i \in \mathbb{T}^\vee$. Let $m_i = \log \alpha_i \in \mathbb{Z}$. Then

$$e_{\mathbb{T}}(V, \tau) = \prod_i \sigma(x_i + m_i z, \tau), \tag{7.10}$$

where $z = c_1 L \in H^2(B\mathbb{T})$. Considering z to be a complex number defines maps

$$H^*(B\mathbb{T}) = \mathbb{C}[z] \to \mathcal{O}_{\mathbb{C}} \to \mathbb{C}[\![z]\!] = HP^0(B\mathbb{T})$$

and we observe that $e_{\mathbb{T}}(V)$ defines an element of

$$HP^*(X; \mathcal{O}_{\mathbb{C}}) \subseteq HP^*(X; \mathbb{C}[\![z]\!]) \cong H^*(X)[\![z]\!] \cong HP^*(X \times B\mathbb{T}).$$

When working multiplicatively, we set $w = e^z$.

The manipulations that follow are more manageable if we adopt vector notation, and abbreviate

$$x = (x_1, \ldots, x_d).$$

Similarly we'll write $u_i = e^{x_i}$ and

$$u = (e^{x_1}, \ldots, e^{x_d}).$$

If x is such a vector, we define

$$\sigma(x, \tau) \stackrel{\text{def}}{=} \prod_j \sigma(x_j, \tau)$$

and similarly for $\sigma(u, q)$, so $\sigma(u, q) = \sigma(x, \tau)$ as in the "scalar" case. Then if

$$V \cong L_1 \oplus \cdots \oplus L_d$$

with

$$x_i = c_1 L_i$$

and

$$x = (x_1, \ldots, x_d),$$

then

$$e(V, \tau) = \sigma(x, \tau).$$

If

$$u = (L_1, \ldots, L_d)$$

then the corresponding K-theory Euler class is

$$e(V, q) = \sigma(u, q);$$

and these are related by

$$e(V, \tau) = \operatorname{ch} e(V, q),$$

where ch is the Chern character.

This notation is particularly helpful when we have to deal with the equivariant Euler class. Let $T \subset SU(d) \subset U(d)$ be the standard maximal torus, and let

$$\check{T} = \mathrm{Hom}(\mathbb{T}, T)$$

be its lattice of cocharacters: so

$$T = \{\mathrm{diag}(w_1, \ldots, w_d) | \prod w_i = 1\},$$

and

$$\check{T} \cong \{m \in \mathbb{Z}^d | \sum m_i = 0\}.$$

Define

$$\begin{aligned} I :&\check{T} \times \check{T} \to \mathbb{Z} \\ \phi :&\check{T} \to \mathbb{Z} \end{aligned}$$

by the formulae

$$\begin{aligned} \phi(m) &= \frac{1}{2} \sum m_i^2 \\ I(m, m') &= \sum_i m_i m_i'. \end{aligned}$$

The important points about ϕ and I are

$$\begin{aligned} \phi(0) &= 0 \\ \phi(km) &= k^2 \phi(m) \\ \phi(m + m') &= \phi(m) + I(m, m') + \phi(m') \\ \phi(wm) &= \phi(m) \\ I(km, m') &= kI(m, m') = kI(m', m) \text{ etc.} \\ I(wm, wm') &= I(m, m') \end{aligned} \tag{7.11}$$

for $m, m' \in \check{T}$, $k \in \mathbb{Z}$, and $w \in W$.

Remark 7.12. Note that Lemma 4.12 shows that, for $SU(d)$, the formulae for ϕ and I here agree with those in §4.B.

As above, we continue to suppose that $x = (x_1, \ldots, x_d)$, and $u = (u_1, \ldots, u_d) = (e^{x_1}, \ldots, e^{x_d})$. We define

$$I(x, m) = \sum m_j x_j.$$

and

$$u^{I(m)} = \prod u_i^{m_i},$$

so that

$$u^{I(m)} = e^{I(x,m)}.$$

If b is a scalar, then the meaning of

$$mb = bm = (m_1 b, \ldots, m_d b)$$

is clear. Its multiplicative analogue is

$$\beta^m = (\beta^{m_1}, \ldots, \beta^{m_d});$$

again these are related by

$$e^{mb} = (e^b)^m.$$

With these notations, the functional equations for σ imply the following.

Lemma 7.13. *Suppose that $\lambda = 2\pi i l + 2\pi i k\tau$, that $x = (x_1, \dots, x_d)$, and $u = e^x$. Suppose that $m \in \check{T}$. Then*

$$\sigma(x + m\lambda, \tau) = e^{-kI(m,x) - 2\pi i \tau k^2 \phi(m)} \sigma(x, \tau)$$
$$\sigma(uq^{km}, q) = u^{-kI(m)} q^{-k^2\phi(m)} \sigma(u, q). \quad \square$$

Remark 7.14. The factor of $(-1)^{l+k}$ in (7.3) contributes 1, because it becomes

$$(-1)^{(l+k)\sum m_i} = 1.$$

Remark 7.15. To work with virtual vector bundles, we may as well extend our abbreviations by using *super*-vector notation, and so let

$$x = (x^0, x^1), u = (u^0, u^1), m = (m^0, m^1),$$

etc. stand for *ordered pairs* of quantities as above. So for example

$$\sigma(x, \tau) = \frac{\sigma(x^0, \tau)}{\sigma(x^1, \tau)}.$$

Our first use of all this notation is to give the following result about the equivariant Euler class associated to the sigma orientation. It shows that that characteristic class restriction $c_1^{\mathcal{B}} = 0 = c_2^{\mathcal{B}}$ suffices to ensure the Euler class descends to a meromorphic function on the elliptic curve. This observation goes back at least to [Wit87, BT89]. Note that if V is a $\mathbb{T}$-vector bundle over a $\mathbb{T}$-fixed space, then it admits a decomposition

$$V \cong V^{\mathbb{T}} \oplus V',$$

where $V' \cong V/V^{\mathbb{T}}$.

Proposition 7.16. *Let $m = (m_1, \dots, m_d) : \mathbb{T} \to T$ be a cocharacter, corresponding to a component $BZ(m)$ of $BSU(d)^{\mathbb{T}}$, and let ξ be the tautological $\mathbb{T}$-equivariant vector bundle over this space. Let $x = (x_1, \dots, x_d)$ be the roots of the total Chern class of ξ. Then*

$$e_{\mathbb{T}}(\xi)(z, \tau) = \sigma(x + mz, \tau) = \sigma(uw^m, q) \in H^*_{\mathbb{T}}(BSU(d)^{\mathbb{T}}; \mathcal{O}_{\mathbb{C}})$$

and

$$e_{\mathbb{T}}(\xi/\xi^{\mathbb{T}})(z, \tau) = \prod_{m_j \neq 0} \sigma(x_j + m_j z, \tau) = \prod_{m_j \neq 0} \sigma(u_j w^{m_j}, q) \in H^*_{\mathbb{T}}(BSU(d)^{\mathbb{T}}; \mathcal{O}_{\mathbb{C}})$$

These elements satisfy

$$e_{\mathbb{T}}(\xi)(z + \lambda, \tau) = \exp\left(-kI(x, m) - kI(m, m)z - 2\pi i k^2 \phi(m)\tau\right) e_{\mathbb{T}}(\xi)(z, \tau)$$
$$e_{\mathbb{T}}(\xi/\xi^{\mathbb{T}})(z + \lambda, \tau) = \exp\left(-kI(x, m) - kI(m, m)z - 2\pi i k^2 \phi(m)\tau\right) e_{\mathbb{T}}(\xi/\xi^{\mathbb{T}})(z, \tau)$$

if $\lambda = 2\pi i l + 2\pi i k\tau$; in q-notation this is

$$e_{\mathbb{T}}(\xi)(wq^k, q) = u^{-kI(m)} w^{-kI(m,m)} q^{-k^2\phi(m)} e_{\mathbb{T}}(\xi)(w, q)$$
$$e_{\mathbb{T}}(\xi/\xi^{\mathbb{T}})(wq^k, q) = u^{-kI(m)} w^{-kI(m,m)} q^{-k^2\phi(m)} e_{\mathbb{T}}(\xi/\xi^{\mathbb{T}})(w, q)$$

In particular, if $V = V_0 - V_1$ is a $\mathbb{T}$-equivariant bundle over a $\mathbb{T}$-fixed space X, with $c_1^{\mathcal{B}} V = 0 = c_2^{\mathcal{B}} V$, then

$$e_{\mathbb{T}}(V/V^{\mathbb{T}})(z, \tau) \in H^*_{\mathbb{T}}(X; \mathcal{K}_C^{\times})$$

Proof. The formula for the equivariant Euler class is just (7.10). The transformation formula follows from Lemma 7.13. Note that the entries $m_j = 0$ make no contribution to $I(x, m)$ or $\phi(m)$. Lemma 4.13 then shows that if $V = V_0 - V_1$ is a $BU\{6\}$-bundle, then $e_{\mathbb{T}}(V/V^{\mathbb{T}})$ descends to $\mathcal{K}_C$; it remains to show that it is non-zero. We have

$$\begin{aligned} e_{\mathbb{T}}(V/V^{\mathbb{T}}) &= \frac{e_{\mathbb{T}}(V_0/V_0^{\mathbb{T}})}{e_{\mathbb{T}}(V_1/V_1^{\mathbb{T}})} \\ &= \frac{\prod_{m_j^0 \neq 0} \sigma(x_j^0 + m_j^0 z, \tau)}{\prod_{m_j^1 \neq 0} \sigma(x_j^1 + m_j^1 z, \tau)}. \end{aligned}$$

Each factor of the product is of the form $\sigma(x + mz, \tau)$. This is holomorphic, and the cohomology class x takes integer values on homology classes, so it has zeroes only at points of finite order (i.e., when a multiple of z is a lattice point). Accordingly, the product takes values which are invertible meromorphic functions.

□

7.D. **The Thom class.** In this section we give the formula for our Thom class, although the proof that it works as we say depends on some results in Section 8.

Let $X = BString_{\mathbb{C}}$, and let ξ be the tautological bundle over X, so

$$MString_{\mathbb{C}} = X^{\xi}.$$

We will use the exact sequence of Proposition 6.4 to specify a class in $EC_{\mathbb{T}}(X^{\xi})$, taking advantage of the fact, proven in Proposition 5.7, that $H^*_{\mathbb{T}}X^A$ is concentrated in even degrees, so that we have an exact sequence

$$0 \to EC^{2n}_{\mathbb{T}}(X) \longrightarrow H^{2n}(X^{\mathbb{T}}; \mathcal{K} \otimes \Omega^*_C) \times \prod_A H^{2n}_{\mathbb{T}}(X^A; \mathcal{O}^\wedge_A \otimes \omega^*_C) \to \\ H^{2n}(X^{\mathbb{T}}; \mathcal{K}^\wedge_{\mathcal{F}} \otimes \omega^*_C) \to EC^{2n+1}_{\mathbb{T}}(X) \to 0. \quad (7.17)$$

Thus we must specify

$$\psi_{\mathbb{T}}(\xi) \in H^*_{\mathbb{T}}((X^{\mathbb{T}})^{\xi^{\mathbb{T}}}; \mathcal{K}_C)$$

and, for each finite $A \subseteq \mathbb{T}$, an element

$$\psi_A(\xi) \in H^*_{\mathbb{T}}((X^A)^{\xi^A}; \mathcal{O}^\wedge_A),$$

with the property that

$$\psi_A(\xi)|_{(X^{\mathbb{T}})^{\xi^{\mathbb{T}}}} = \psi_{\mathbb{T}}(\xi) \quad (7.18)$$

in

$$H^*_{\mathbb{T}}((X^{\mathbb{T}})^{\xi^{\mathbb{T}}}; \mathcal{K}^\wedge_{\mathcal{F}}).$$

The obvious way to produce such a class is to start with

$$\mathrm{Thom}_{\mathbb{T}}(\xi) \in H^*_{\mathbb{T}}(X^{\xi}), \quad (7.19)$$

and then for each A to pull back along

$$(X^A)^{\xi^A} \to (X^A)^{\xi} \to X^{\xi}.$$

Thus for all A, finite or not, the formula for ψ_A is

$$\psi_A(\xi) = \mathrm{Thom}_{\mathbb{T}}(\xi^A) e_{\mathbb{T}}(\xi/\xi^A). \quad (7.20)$$

Here $\mathrm{Thom}_{\mathbb{T}}(\xi^A)$, as defined in (7.7), is the Thom class using σ of the $\mathbb{T}$-Borel construction of ξ^A, and $e_{\mathbb{T}}(\xi/\xi^A)$ is the Euler class, again using σ, of the $\mathbb{T}$-Borel construction of the "complement" $\xi - \xi^A \cong \xi/\xi^A$.

First we note $\psi_{\mathbb{T}}$ behaves as it should.

Lemma 7.21. *The class $\psi_{\mathbb{T}}(\xi)$ gives an element of*

$$H^*_{\mathbb{T}}((X^{\mathbb{T}})^{\xi^{\mathbb{T}}}; \mathcal{K}_C),$$

and multiplication by $\psi_{\mathbb{T}}(\xi)$ is an isomorphism

$$H^*_{\mathbb{T}}(X^{\mathbb{T}}; \mathcal{K}_C) \xrightarrow[\cong]{\psi_{\mathbb{T}}} H^*_{\mathbb{T}}((X^{\mathbb{T}})^{\xi^{\mathbb{T}}}; \mathcal{K}_C).$$

Proof. It is an isomorphism because $\mathrm{Thom}_{\mathbb{T}}(\xi^{\mathbb{T}}) = \mathrm{Thom}(\xi^{\mathbb{T}})$ is a Thom class, and in Proposition 7.16 it was shown that $e_{\mathbb{T}}(\xi/\xi^{\mathbb{T}})$ is a unit of $H^*_{\mathbb{T}}(X^{\mathbb{T}}; \mathcal{K}_C)$. □

We turn now to ψ_A for $A \subset \mathbb{T}$ a finite subgroup. Our formula gives an element of

$$H^*_{\mathbb{T}}((X^A)^{\xi^A}; \mathcal{K}_{\mathbb{C}})$$

and we produce from it an element of

$$H^*_{\mathbb{T}}((X^A)^{\xi^A}, \mathcal{K}_A)$$

simply by choosing, for each point a of order n, a lift $\tilde{a}$, and then electing to evaluate our element of $\mathcal{K}_{\mathbb{C}}$ near $\tilde{a}$. Of course the apparent dependence on arbitrary choices is not satisfactory. In Section 8, we shall prove the following result.

Proposition 7.22. *The value of $\psi_A(\xi)$ at $\tilde{a} \in \mathbb{C}$ depends only on the image a of $\tilde{a}$ in C. As such, multiplication by $\psi_A(\xi)$ is an isomorphism*

$$H^*_{\mathbb{T}}(X^A; \mathcal{O}_A) \xrightarrow[\cong]{\psi_A} H^*_{\mathbb{T}}((X^A)^{\xi^A}; \mathcal{O}_A).$$

Thus we have the following.

Theorem 7.23. *Let ξ be the tautological $\mathbb{T}$-equivariant complex vector bundle over $X = BString_{\mathbb{C}}$. The classes $\psi_A(\xi)$ for $A \subseteq \mathbb{T}$ assemble to give a class $\psi(\xi) \in EC_{\mathbb{T}}(X^{\xi})$, and multiplication by $\psi(\xi)$ is an isomorphism*

$$EC^*_{\mathbb{T}}(X) \xrightarrow[\cong]{\psi(\xi)} EC^*_{\mathbb{T}}(X^{\xi}).$$

Proof. Lemma 7.21 and Proposition 7.22 together with the exact sequence (7.17) show that we have assembled an element of $\psi(\xi)$ of $EC_{\mathbb{T}}(X^{\xi})$. Moreover, using the exactness of (7.17), its analogue for $EC_{\mathbb{T}}(X^{\xi})$, the Thom isomorphism in ordinary cohomology, and the Five Lemma, we may conclude that multiplication by $\psi(\xi)$ is an isomorphism. □

7.E. **Multiplicativity.** Theorem 7.23 gives a map of spectra

$$\psi : MString_{\mathbb{C}} \longrightarrow EC.$$

By Proposition 3.14 $MString_{\mathbb{C}}$ is an E_∞ ring spectrum, and we would like to know that map is multiplicative.

Theorem 7.24. *The map $\psi : MString_{\mathbb{C}} \longrightarrow EC$ is a ring map up to homotopy.*

Proof. The product on $MString_{\mathbb{C}}$ arises from the formula

$$(X \times Y)^{V \oplus W} \cong X^V \wedge Y^W,$$

and the map ψ is multiplicative because it arises from the exponential class $\mathrm{Thom}_{\mathbb{T}}$. □

8. Translation of the Thom class by a point of order n.

In this section we assemble a proof of Proposition 7.22. The formula (7.20) for ψ_A gives an element of

$$H^*_{\mathbb{T}}((X^A)^{\xi^A}; \mathcal{K}_{\mathbb{C}}),$$

and we must show that this element descends to $\mathcal{K}_A$ and is holomorphic near $a \in C\langle A\rangle$.

8.A. **The strategy via translation.** The cohomology ring $H^*B\mathbb{T}$ is the ring of functions on the completion of $\mathbb{C}$ at the origin, so to study ψ_A near a point $\tilde{a} \in \mathbb{C}\langle A\rangle = \pi^{-1}(C\langle A\rangle)$, we study $T^*_{\tilde{a}}\psi_A$ near zero. The essential problem is to understand the Euler class

$$\zeta^* T^*_{\tilde{a}} \psi_A \in H^*_{\mathbb{T}}(X^A)$$

near zero. Here ζ denotes the zero section, and T will denote translation in $\mathbb{C}$ or C. Using the description of $H^*BSU(d)^A$ in §4.A, we introduce a characteristic class

$$\delta_A(V) : \mathbb{C}\langle A\rangle \to H^*_{\mathbb{T}}(BSU^A; \mathcal{K}_{\mathbb{C}}),$$

with the property that

$$\delta_A(V, \tilde{a}) = \zeta^* T^*_{\tilde{a}} \psi_A(V).$$

The explicit formula for δ_A makes it possible to prove Proposition 7.22. For example, we show that if ξ is a $String_{\mathbb{C}}$-bundle, then $\delta_A(\xi, \tilde{a})$ depends only on $\pi(\tilde{a}) \in C\langle A\rangle$: that is, we have a factorization

$$\begin{tikzcd}
C\langle A\rangle \arrow[r, dashed, "\delta_A"] & H^*_{\mathbb{T}} BString^A_{\mathbb{C}} \\
\mathbb{C}\langle A\rangle \arrow[u, "\pi"] \arrow[r, "\delta_A"] & H^*_{\mathbb{T}} BSU^A. \arrow[u]
\end{tikzcd}$$

This argument by translation was introduced by [BT89], and its use in Grojnowski's equivariant elliptic cohomology goes back to Rosu [Ros01]. The class δ_A was introduced in [And03b], to show that the translation argument could be made independent of the complicated choices in [BT89, Ros01].

As we note in the introduction, earlier treatments of the Thom class required the translation argument even to give a formula for ψ_A. Our formula (7.20) (essentially, (7.19)) does not involve the translation argument, and so is much simpler than earlier formulae. What we must do is adapt the translation argument to show that our ψ_A has the required properties.

8.B. **The class δ_A.** Suppose that $A = \mathbb{T}[n]$ is a finite subgroup of the circle. We define an element of $H^*(BSU(d)^A; \mathcal{O}_{\mathbb{C}})$ as follows. Let V be the tautological over $BSU(d)$. Suppose that

$$m = (m_1, \ldots, m_d) : A \to T$$

is a homomorphism, corresponding to a component of $BSU(d)^A$. Thus $m_i \in A^* \cong \mathbb{Z}/n$, and

$$\sum m_i \equiv 0 \mod n.$$

Suppose that x_j are the Chern roots of V with respect to this decomposition: that is, we suppose that we have a splitting

$$V \cong L_1 \otimes \mathbb{C}(m_1) \oplus \cdots L_d \otimes \mathbb{C}(m_d),$$

where

$$m_j = \log \alpha_j \text{ and } x_j = c_1 L_j.$$

Let a be a point of C of order n, and let $\tilde{a}$ be a lift of a to $\mathbb{C}$. Define integers k and l by the formula

$$n\tilde{a} = \lambda = 2\pi i l + 2\pi i k\tau,$$

and let $\lambda = 2\pi i l + 2\pi i k \tau$. Let $\tilde{m}$ be any factorization

$$\begin{tikzcd} \mathbb{T} \arrow[r, "\tilde{m}"] & T. \\ A \arrow[u] \arrow[ur, "m"'] & \end{tikzcd}$$

That is, $\tilde{m}_j$ is an integer lift of m_j.

In order to give q-expansion formulae we also set

$$\begin{aligned} u^r &= e^{rx} \\ \alpha^r &= e^{ra} \end{aligned}$$

for $r \in \mathbb{Q}$. Finally, let $\delta_A = \delta_A(x, \tilde{m}, \tilde{a})$ be the expression

$$\begin{aligned} \delta_A(x, \tilde{m}, \tilde{a}) &\overset{\text{def}}{=} \exp\big(\tfrac{k}{n} I(\tilde{m}, x) + \tfrac{k}{n}\tilde{a}\phi(\tilde{m})\big)\sigma(x + \tilde{m}\tilde{a}, \tau) \\ &= \exp\big(\tfrac{k}{n}\sum_j \tilde{m}_j x_j + \tfrac{k}{n}\tfrac{\tilde{a}}{2}\sum_j \tilde{m}_j^2\big)\prod_j \sigma(x_j + \tilde{m}_j\tilde{a}, \tau) \\ &= u^{\frac{k}{n}I(\tilde{m})}\alpha^{\frac{k}{n}\phi(\tilde{m})}\sigma(u\alpha^{\tilde{m}}, q). \end{aligned} \tag{8.1}$$

Lemma 8.2. *The expression δ_A is independent of the choice of lift $\tilde{m}$. Moreover, it is invariant under the action of $W(m)$, the Weyl group of $Z(m)$. As such, it defines a characteristic class of principal $Z(m)$-bundles.*

Proof. Suppose that $\tilde{m}'$ is another lift. Then

$$\tilde{m}' = \tilde{m} + n\Delta,$$

where $\Delta \in \mathrm{Hom}(\mathbb{T}, T)$. Then

$$\begin{aligned} \delta_A(x, \tilde{m}', \tilde{a}) =& \exp\big(\tfrac{k}{n} I(\tilde{m} + n\Delta, x) + \tfrac{k}{n}\tilde{a}\phi(\tilde{m} + n\Delta)\big)\sigma(x + (\tilde{m} + n\Delta)\tilde{a}, \tau) \\ =& \exp\big(\tfrac{k}{n} I(\tilde{m}, x) + kI(\Delta, x) + \tfrac{k}{n}\tilde{a}\phi(\tilde{m}) + k\tilde{a}I(\tilde{m}, \Delta) + kn\tilde{a}\phi(\Delta)\big)\sigma(x + \tilde{m}\tilde{a} + \Delta\lambda, \tau) \\ =& \exp\big(\tfrac{k}{n} I(\tilde{m}, x) + kI(\Delta, x) + \tfrac{k}{n}\tilde{a}\phi(\tilde{m}) + k\tilde{a}I(\tilde{m}, \Delta) + k\lambda\phi(\Delta)\big) \\ & \exp\big(-I(x, k\Delta) - I(\tilde{m}, k\Delta)\tilde{a} - 2\pi i\tau\pi\phi(n\Delta)\big) \\ & \sigma(x + \tilde{m}\tilde{a}, \tau) \\ =& \delta_A(x, \tilde{m}, \tilde{a}). \end{aligned}$$

Now suppose that $w \in Z(m)$ so $wm = m$: then

$$w\tilde{m} = \tilde{m} + n\Delta$$

for some $\Delta \in \check{T}$, and a similar argument to the one just given shows that

$$\delta_A(x, \tilde{m}, \tilde{a}) = \delta_A(wx, w\tilde{m}, \tilde{a}) = \delta_A(wx, \tilde{m} + n\Delta, \tilde{a}) = \delta_A(wx, \tilde{m}, \tilde{a}).$$

$\square$

As a related matter, it is easy to understand the action of a general $w \in W$ (i.e. one which does not necessarily fix m).

Proposition 8.3. *For $w \in W$, the Weyl group of $SU(d)$, we have*

$$\delta_A(wx, wm, \tilde{a}) = \delta_A(x, m, \tilde{a}).$$

That is, the family of expressions $\delta(x, m, \tilde{a})$ for $m \in \mathrm{Hom}(A, T)$ satisfies the hypothesis of Proposition 4.7, and so assemble to give an element of $H^*BSU(d)^A$. $\square$

Remark 8.4. As noted in Remark 7.15, the appropriate extension to virtual vector bundles is given by the same formulae, provided we admit $\mathbb{Z}/2$-graded notation. Thus if $x = (x^0, x^1)$, $m = (m^0, m^1)$, and $\tilde{m} = (\tilde{m}^0, \tilde{m}^1)$, then

$$\delta_A(x, \tilde{m}, \tilde{a}) = \delta(x^0, \tilde{m}^0, \tilde{a})/\delta(x^1, \tilde{m}^1, \tilde{a}).$$

Definition 8.5. If V is a virtual $\mathbb{T}$-equivariant vector bundle over an $A = \mathbb{T}[n]$-fixed space X, with

$$c_1^{\mathcal{B}}(V) = 0,$$

and if $\tilde{a}$ is a point of $\mathbb{C}$ over a point a of order n in C, then we write

$$\delta_A(V, \tilde{a})$$

for the class in $H^*(X)$ provided by Proposition 8.3.

Now we investigate the dependence of δ_A on the lift $\tilde{a}$. Suppose $\tilde{a}'$ is another lift of a. Then there are $\epsilon, \delta \in \mathbb{Z}$ such that

$$\tilde{a}' = \tilde{a} + 2\pi i\epsilon + 2\pi i\delta\tau,$$

so

$$e^{\tilde{a}'} = e^{\tilde{a}} q^{\delta},$$

and

$$e^{n\tilde{a}'} = q^{k+n\delta}.$$

Let $k' = k + n\delta$. Then

$$w(a, q^{\frac{1}{n}}) \stackrel{\text{def}}{=} e^{-\tilde{a} + \frac{k}{n}\tau} = e^{-\tilde{a}' + \frac{k'}{n}\tau}$$

is an n^{th} root of unity which does not depend on the choice of lift $\tilde{a}$; in fact it is the Weil pairing of a with $q^{1/n}$ [KM85, p. 90]. Because it is an n^{th} root of unity, the quantity

$$w(a, q^{\frac{1}{n}})^{\phi(m)} \stackrel{\text{def}}{=} w(a, q^{\frac{1}{n}})^{\phi(\tilde{m})}$$

does not depend on the lift $\tilde{m}$ of m. The dependence of δ_A on the choice of lift $\tilde{a}$ is given by the following.

Lemma 8.6.

$$\delta_A(V, m, \tilde{a}') = w(a, q^{\frac{1}{n}})^{\delta\phi(m)} \delta_A(V, m, \tilde{a}).$$

Proof. Let $\alpha = e^{\tilde{a}}$. In q-notation,

$$\begin{aligned}
\delta_A(V, \tilde{m}, \tilde{a}') &= u^{\frac{k'}{n} I(\tilde{m})} (\alpha q^{\delta})^{\frac{k'}{n}\phi(\tilde{m})} \sigma(u\alpha^{\tilde{m}} q^{\delta\tilde{m}}, q) \\
&= u^{\frac{k}{n} I(\tilde{m})} u^{\delta I(\tilde{m})} \alpha^{\frac{k}{n}\phi(\tilde{m})} \alpha^{\delta\phi(\tilde{m})} q^{\delta\frac{k}{n}\phi(\tilde{m})} q^{\delta^2\phi(\tilde{m})} u^{-\delta I(\tilde{m})} \alpha^{-\delta I(\tilde{m}, \tilde{m})} q^{-\delta^2\phi(\tilde{m})} \sigma(u\alpha^{\tilde{m}}, q) \\
&= \delta_A(V, \tilde{m}, \tilde{a}) \alpha^{-\delta\phi(\tilde{m})} q^{\delta\frac{k}{n}\phi(\tilde{m})} \alpha^{-\delta I(\tilde{m}, \tilde{m})}.
\end{aligned}$$

Noting that

$$\phi(\tilde{m}) - I(\tilde{m}, \tilde{m}) = -\phi(\tilde{m}),$$

the last expression becomes

$$\delta_A(V,\tilde{m},\tilde{a})\alpha^{-\delta\phi(\tilde{m})}q^{\delta\frac{k}{n}\phi(\tilde{m})}\alpha^{-\delta I(\tilde{m},\tilde{m})} = w(a,q^{\frac{1}{n}})^{\delta\phi(m)}\delta_A(V,\tilde{m},\tilde{a}).$$

□

Proposition 8.7. *If V is a virtual $\mathbb{T}$-equivariant SU-bundle with $c_2^{\mathcal{B}}(V)=0$, then class $\delta_A(V,\tilde{a})$ does not depend on the choice of lift $\tilde{a}$ of a. Equivalently, for any two lifts $\tilde{a}$ and $\tilde{a}'$ of a,*

$$\delta_A(\xi,\tilde{a})|_{BU\{6\}^A} = \delta_A(\xi,\tilde{a}')|_{BU\{6\}^A},$$

where ξ is the universal bundle over $BU\{6\}^A$.

Proof. Lemma 4.13 implies that, if m is any reduction of the action of A on V, and if $\tilde{m}$ is a lift of m, then

$$\phi(\tilde{m}) \equiv 0 \mod n,$$

so $w(a,q^{1/n})^{\phi(m)}=1$. □

It is important that the class δ_A has a Borel-equivariant version as well. For if V is a $\mathbb{T}$-equivariant bundle over an A-fixed space X, then the $\mathbb{T}$-action preserves the decomposition into isotypical summands for the A-action

$$V \cong \bigoplus_{\alpha\in A^\vee} \alpha\otimes \mathrm{Hom}(\alpha,V).$$

and so the reduction m determines a $\mathbb{T}$-*equivariant* principal $Z(m)$-bundle over X. Put another way, the $\mathbb{T}$-action on $BSU(d)^A$ determines one on the component $BZ(m)$, and as such the map classifying the Borel construction of the tautological bundle factors as

$$\begin{tikzcd} E\mathbb{T}\times_{\mathbb{T}} BZ(m) \arrow[r] \arrow[dr] & BZ(m) \arrow[d] \\ & BSU(d)^A. \end{tikzcd}$$

We write $\delta_A^{\mathcal{B}}(V,m,\tilde{a})$ for the resulting Borel class, in $H^*_{\mathbb{T}}(X;\mathcal{O}_{\mathbb{C}})$.

It is important to understand the restriction of $\delta_A^{\mathcal{B}}$ to the fixed subspace $Y=X^{\mathbb{T}}$. Since $Y\subseteq X$, any reduction

$$\tilde{m}:\pi_0 Y\to \mathrm{Hom}(\mathbb{T},T)$$

of the action of $\mathbb{T}$ on $V|_Y$ is a lift of m. If $x=(x_1,\ldots,x_d)$ are the Chern roots of $V|_Y$, then

$$\delta_A^{\mathcal{B}}(V,\tilde{a})|_Y = \delta_A^{\mathcal{B}}(x,\tilde{m},\tilde{a}) = \exp\big(\tfrac{k}{n}I(\tilde{m},x+\tilde{m}z)+\tfrac{k}{n}\tilde{a}\phi(\tilde{m})\big)\sigma(x+\tilde{m}z+\tilde{m}\tilde{a},\tau) \tag{8.8}$$

As promised, we can now show that $\delta_A^{\mathcal{B}}$ is the translation of the equivariant Euler class associated to σ.

Proposition 8.9. *Let ξ be the tautological bundle over $BString_{\mathbb{C}}^A$. Then*

$$\delta_A^{\mathcal{B}}(\xi,\tilde{a}) = T_{\tilde{a}}^* e_{\mathbb{T}}(\xi)\in H^*(BString_{\mathbb{C}}^A;\mathcal{K}_{\mathbb{C}}).$$

Proof. Let $X=BString_{\mathbb{C}}$. We showed in Proposition 5.7 that $H^*(X^A;\mathbb{Q})$ is concentrated in even degrees and that

$$H^*_{\mathbb{T}}(X^A)=H^*(X^A)[z].$$

In Proposition 5.7, we showed that

$$H^*_{\mathbb{T}}(X^A;\mathcal{K}_C)\to H^*_{\mathbb{T}}(X^{\mathbb{T}};\mathcal{K}_C)$$

is injective, and so letting $Y = X^{\mathbb{T}}$, it suffices to prove that

$$\delta_A^{\mathcal{B}}(\xi, \tilde{a})|_Y = T_{\tilde{a}}^* e_{\mathbb{T}}(\xi)|_Y.$$

But under the indicated characteristic class restrictions, we have $I(\tilde{m}, x) = 0$ and $\phi(\tilde{m}) = 0$ by Lemma 4.13, and so equation (8.8) becomes

$$\delta_A^{\mathcal{B}}(\xi, \tilde{a})|_Y = \sigma(x + \tilde{m}z + \tilde{m}\tilde{a}, \tau) = T_{\tilde{a}}^* e_{\mathbb{T}}(\xi)|_Y.$$

□

8.C. **Variants.** We need two variants of δ_A, corresponding to the decomposition

$$V \cong V^A \oplus V',$$

where $V' \cong V/V^A$. To give formulae we introduce some restricted sums and products.

Let

$$\begin{aligned}
\sum_i{}' \tilde{m}_i y_i &= \sum_{\tilde{m}_i \not\equiv 0 \mod n} \tilde{m}_i y_i \\
\prod_i{}' f(y_i) &= \prod_{\tilde{m}_i \not\equiv 0 \mod n} f(y_i) \\
I'(m, x) &\stackrel{\text{def}}{=} \sum_i{}' \tilde{m}_i y_i \\
\phi'(\tilde{m}) &= \frac{1}{2} \sum_i{}' \tilde{m}_i \tilde{m}_i \\
\sum_i{}'' \tilde{m}_i y_i &= \sum_{\tilde{m}_i \equiv 0 \mod n} \tilde{m}_i y_i \\
I''(m, x) &= \sum_i{}'' \tilde{m}_i y_i \\
\phi''(\tilde{m}) &= \frac{1}{2} \sum_i{}'' \tilde{m}_i \tilde{m}_i \\
\prod_i{}'' f(y_i) &= \prod_{\tilde{m}_i \equiv 0 \mod n} f(y_i),
\end{aligned}$$

and let

$$\begin{aligned}
\sigma'(y, \tau) &= \prod_i{}' \sigma(y_i, \tau) \\
\sigma''(y, \tau) &= \prod_i{}'' \sigma(y_i, \tau) \\
\delta'_A(x, \tilde{m}, \tilde{a}) &= \exp(\frac{k}{n} I'(\tilde{m}, x) + \frac{k}{n} \tilde{a} \phi'(\tilde{m})) \sigma'(x + \tilde{m}\tilde{a}, \tau) \\
\delta''_A(x, \tilde{m}, \tilde{a}) &= \exp(\frac{k}{n} I''(\tilde{m}, x) + \frac{k}{n} \tilde{a} \phi''(\tilde{m})) \sigma''(x + \tilde{m}\tilde{a}, \tau).
\end{aligned}$$

Notice that

$$
\begin{aligned}
I &= I' + I'' \\
\phi &= \phi' + \phi'' \\
\delta_A &= \delta'_A \delta''_A.
\end{aligned}
$$

Our analysis of δ_A applies to δ'_A and δ''_A to give the following.

Proposition 8.10. *The classes $\delta'_A(x, \tilde{m}, \tilde{a})$ and $\delta''_A(x, \tilde{m}, \tilde{a})$ are independent of the choice of lift $\tilde{m}$, and are invariant under the action of $W(m)$. Moreover if $w \in W$, then*

$$\delta'_A(wx, wm, \tilde{a}) = \delta'_A(x, m, \tilde{a}),$$

and similarly for δ''_A. As such, they assemble to give elements of $H^*(BU(d)^A)$. As d varies, they define stable exponential classes $\delta'_A(\xi, \tilde{a})$ and $\delta''_A(\xi, \tilde{a})$ in $H^*(BSU^A)$.

Proof. The arguments for δ_A in §8.B decouple in this way. The main point is, if V is an A-bundle or $\mathbb{T}$-bundle over an A-fixed space, then with respect to the equivariant decomposition

$$V \cong V^A \oplus V',$$

the "prime" parts above correspond to V', while the "prime-prime" parts correspond to V^A. □

The behaviour of δ'_A and δ''_A with respect to change from $\tilde{a}$ to $\tilde{a}'$ is similar, but there is an additional subtlety. First of all, note the following.

Lemma 8.11. *For any lift $\tilde{m}$ of m, we have*

$$\phi''(\tilde{m}) \equiv 0 \mod n.$$

Proof. Recall that ϕ'' corresponds to restriction to V^A, where each $\tilde{m}_i \equiv 0 \mod n$. We have

$$\phi''(\tilde{m}) = -\sum_{i<j}'' \tilde{m}_i \tilde{m}_j.$$

□

Proposition 8.12. *If $\tilde{a}'$ is another lift of a, and δ is defined by*

$$e^{\tilde{a}'} = e^{\tilde{a}} q^{\delta},$$

then

$$
\begin{aligned}
\delta'_A(V, m, \tilde{a}') &= w(a, q^{\frac{1}{n}})^{\delta\phi'(m)} \delta'_A(V, m, \tilde{a}) \\
&= w(a, q^{\frac{1}{n}})^{\delta\phi(m)} \delta'_A(V, m, \tilde{a})
\end{aligned}
$$

and

$$\delta''_A(V, m, \tilde{a}') = \delta''_A(V, m, \tilde{a}).$$

In particular, we have a well-defined characteristic class $\delta'_A(\xi, a)$ of $BU\{6\}$-bundles, and a well-defined characteristic class δ''_A of BSU-bundles. □

The fact that, even for an SU-bundle, $\delta''_A(V, \tilde{a})$ does not depend on the choice of lift $\tilde{a}$ is striking, until it is discovered to be trivial. Recall from (7.6) that, if V is a (virtual) vector bundle, then $e(V)$ is its Euler class with respect to the orientation given by σ.

Proposition 8.13. *Let V be the tautological bundle over $BSU(d)$. For any choice $\tilde{a}$ of lift of a,*

$$\delta''_A(V,\tilde{a}) = e(V^A).$$

and

$$(\delta''_A)^{\mathcal{B}}(V,\tilde{a}) = e_{\mathbb{T}}(V^A)$$

Proof. Let $\alpha = e^{\tilde{a}}$, so

$$\alpha^n = q^k.$$

Let $\tilde{m}$ be a lift of $m : A \to T$, and define integers Δ_i by the rule

$$\Delta_i = \begin{cases} \frac{m_i}{n} & m_i \equiv 0 \mod n \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$u = (e^{x_1}, \dots, e^{x_d}).$$

We use q-notation.

$$\begin{aligned} \delta''_A(x,\tilde{m},\tilde{a}) &= u^{\frac{k}{n}I''(\tilde{m})}\alpha^{\frac{k}{n}\phi''(\tilde{m})}\sigma''(u\alpha^{\tilde{m}}) \\ &= u^{\frac{k}{n}I''(n\Delta)}\alpha^{\frac{k}{n}\phi''(n\Delta)}\sigma''(u\alpha^{n\Delta}) \\ &= u^{kI''(\Delta)}\alpha^{kn\phi''(\Delta)}\sigma''(uq^{k\Delta}) \\ &= \sigma''(u), \end{aligned}$$

as required. The equivariant case is similar. □

Corollary 8.14. *We have*

$$T^*_{\tilde{a}} \operatorname{Thom}_{\mathbb{T}}(V) = \operatorname{Thom}_{\mathbb{T}}(V) \in H^*_{\mathbb{T}}((BSU(d)^A)^{V^A}; \mathcal{O}_{\mathbb{C}}).$$

That is, the Thom class is invariant under translation by $\tilde{a}$.

Proof. Proposition 8.13 gives the result for the corresponding Euler class. In this universal case, the cohomology of the base $BSU(d)^A$ is a domain, and cohomology of the Thom space is a principal ideal, so the result for the Euler class gives the result for the Thom class. □

For our analysis, a crucial feature of δ'_A is that it has no zeros or poles near 0.

Proposition 8.15. *The class $(\delta'_A)^{\mathcal{B}}(V,\tilde{a})$ gives an element of $H^*_{\mathbb{T}}(BSU(d)^A; \mathcal{O}^{\times}_{\mathbb{C},0})$. Moreover, if ξ is the tautological bundle over $BString^A_{\mathbb{C}}$, then*

$$(\delta'_A)^{\mathcal{B}}(V,\tilde{a}) = T^*_a e_{\mathbb{T}}(\xi/\xi^A) \in H^*_{\mathbb{T}}(BString^A_{\mathbb{C}}; \mathcal{O}^{\times}_{\mathbb{C},0})$$

Proof. For the first part, it suffices as in the proof of Proposition 8.9 to check that $(\delta'_A)^{\mathcal{B}}(V,\tilde{a})$ restricts to an element of $H^*_{\mathbb{T}}(BSU(d)^{\mathbb{T}}; \mathcal{O}^{\times}_{\mathbb{C},0})$. We have

$$(\delta'_A)^{\mathcal{B}}(V,\tilde{m},\tilde{a})(z,\tau) = \exp\bigl(\tfrac{k}{n}I'(\tilde{m}, x+\tilde{m}z) + \tfrac{k}{n}\tilde{a}\phi'(\tilde{m})\bigr) \prod_{m_j \neq 0} \sigma(x_j + \tilde{m}_j z + \tilde{m}_j\tilde{a}, \tau).$$

Recall that $\tilde{a}$ is a lift of a point a of order n. If $\tilde{m}_j$ is not divisible by n, then there is a small neighborhood U of 0 such that $\tilde{m}_j(z+\tilde{a}) \notin \Lambda$ for $z \in U$: so $\sigma(\tilde{m}_j z + \tilde{m}_j\tilde{a})$ is a unit of $\mathcal{O}_{\mathbb{C},0}$.

Now consider the Taylor series expansion

$$\sigma(x + \tilde{m}z + \tilde{a}, \tau) = \sigma(\tilde{m}z + \tilde{m}\tilde{a}, \tau) + o(x).$$

Since x is a power series variable, this is a unit provided that $\sigma(\tilde{m}+\tilde{m}\tilde{a},\tau)$ is a unit. This proves the first statement. The proof of the second part proceeds exactly as for the case of $\delta_A^{\mathcal{B}}$ in Proposition 8.9. $\square$

8.D. **Proof of Proposition 7.22.** As in the statement of the Proposition, we let $X = BString_{\mathbb{C}}$, and we write ξ for the tautological bundle over X.

To illustrate the argument we give it first for the Euler class associated to ψ_A. Let $\zeta : X^A \to (X^A)^{\xi^A}$ be the zero section. Let $T_{\tilde{a}}$ denote translation by $\tilde{a}$ in $\mathbb{C}$. Then, as we showed in Proposition 8.9,

$$T_{\tilde{a}}^*\zeta^*\psi_A(z) = \delta_A^{\mathcal{B}}(\xi,\tilde{a})(z),$$

so to understand the behaviour of ψ_A near $\tilde{a}$, it suffices to understand the behaviour of $\delta_A^{\mathcal{B}}$ near 0. But we have shown in Proposition 8.7 that if ξ is a $BU\{6\}$-bundle, then the class $\delta_A^{\mathcal{B}}$ does not depend on the choice of lift $\tilde{a}$ of a. So

$$T_{\tilde{a}}^*\zeta^*\psi_A(z) = \delta_A^{\mathcal{B}}(\xi,\tilde{a})(z) = \delta_A^{\mathcal{B}}(\xi,\tilde{a}')(z) = T_{\tilde{a}'}^*\zeta^*\psi_A(z).$$

The refinement to ψ_A itself is clear, given the preceding discussion and the fact that, by definition,

$$\psi_A = \mathrm{Thom}_{\mathbb{T}}(\xi^A)e_{\mathbb{T}}(\xi/\xi^A).$$

Corollary 8.14 shows that

$$T_{\tilde{a}}^*\,\mathrm{Thom}_{\mathbb{T}}(\xi^A) = \mathrm{Thom}_{\mathbb{T}}(\xi^A)$$

and so this quantity is independent of $\tilde{a}$ and for that matter of a. Meanwhile by Proposition 8.15,

$$T_{\tilde{a}}^* e_{\mathbb{T}}(\xi/\xi^A) = (\delta_A')^{\mathcal{B}}(\xi,\tilde{a}),$$

and Proposition 8.12 shows that this quantity is independent of the lift $\tilde{a}$. Thus we have shown that $T_{\tilde{a}}^*\psi_A$ depends only on a, and not the choice of lift $\tilde{a}$.

Finally we must show that multiplication by ψ_A is an isomorphism

$$H_{\mathbb{T}}^*(X^A;\mathcal{O}_A) \xrightarrow[\cong]{\psi_A} H_{\mathbb{T}}^*((X^A)^{\xi^A};\mathcal{O}_A).$$

Certainly $\mathrm{Thom}_{\mathbb{T}}(\xi^A)$ is an isomorphism

$$H_{\mathbb{T}}^*(X^A) \to H_{\mathbb{T}}^*((X^A)^{\xi^A}).$$

In Proposition 8.15 we showed that

$$T_{\tilde{a}}^* e_{\mathbb{T}}(\xi/\xi^A) = (\delta_A')^{\mathcal{B}}(\xi,\tilde{a})$$

is a unit of $H_{\mathbb{T}}^*(X^A;\mathcal{O}_{C,0})$. As a ranges over the points of $C\langle A\rangle$, we find that

$$e_{\mathbb{T}}(\xi/\xi^A) \in H_{\mathbb{T}}^*((X^A);\mathcal{O}_A^{\wedge})^{\times},$$

as required.

This completes the proof of Proposition 7.22. $\square$

Part 3. Analytic and algebraic geometry of the sigma orientation

In this part, we give an account of the string orientation in terms of the analytic geometry of the curve $C = \mathbb{C}/\Lambda$. In Section 9, we associate to a $\mathbb{T}$-spectrum X a sort of sheaf $\mathcal{F}(X)$ on C, whose sections are calculated by an exact sequence like (6.5). If X is a space, this is a sheaf of rings, and so gives rise to a ringed space $\mathbf{F}(X)$.

We then turn to the analysis of $\mathbf{F}(BSU(d))$ and the line bundle $\mathcal{F}(BSU(d)^V)$ over it. As before, we begin in Section 10 by dealing with $\mathbb{T}$-fixed spaces and generic points on the curve and in Section 11, we turn to the analysis of points with finite isotropy and torsion points on the curve. We show that our Thom class gives a trivialization of the line bundle $\mathcal{F}(BString_{\mathbb{C}}(d)^V)$ over $\mathbf{F}(BString_{\mathbb{C}}(d))$. Our argument gives a proof of the conjecture in [And03b, And03a] in this setting. We conclude in Section 12 by rephrasing the situation in algebraic terms; we hope that this will eventually lead to an algebraic proof for equivariant elliptic cohomology theories associated to arithmetic elliptic curves.

9. Elliptic cohomology and sheaves of $\mathcal{O}_C$-modules.

The sequence (6.5) suggests that $EC^*_{\mathbb{T}}(X)$ is approximately the cohomology of a sheaf on C: the $H^n(X^{\mathbb{T}}; \mathcal{K} \otimes \Omega^*_C)$ factor concerns the behaviour of a section generically on C, while the $H^n_{\mathbb{T}}(X^A; T_A C \otimes \omega^*_C)$ factors concern the behaviour in small neighborhoods of the points of finite order.

In Sections 19–22 of [Gre05], the second author constructs such a sheaf, which we describe in §9.A, however this does not have the formal properties we need, so in §9.B we construct a more suitable variant.

9.A. The Grojnowski sheaf.

We briefly recall some of the properties of the sheaf $\mathcal{M}(X)$, (which is denoted $\mathcal{M}_C F(X, EC)$. in [Gre05]).

Proposition 9.1. *There is a functor $\mathcal{M}$ from $\mathbb{T}$-spectra to ω^*-periodic $\mathcal{O}_C$-modules enjoying the following properties.*

(1) *If W is a virtual complex representation of $\mathbb{T}$, then*

$$\mathcal{M}(S^W) \cong \mathcal{O}_C(-D(W)) \otimes \omega^*.$$

(2) *There is a short exact sequence*

$$0 \to \Sigma H^1(C; \mathcal{M}(X)) \to EC^*_{\mathbb{T}}(X) \to H^0(C; \mathcal{M}(X)) \to 0.$$

(3)

$$\mathcal{M}(X^{\mathbb{T}}) \cong H^*(X^{\mathbb{T}}; \mathcal{O}_C \otimes \omega^*)$$

(4) *Let a be a point of C of exact order n, and let $A \subseteq \mathbb{T}$ be the subgroup of order n ($A = \mathbb{T}$ if $n = \infty$). For a finite $\mathbb{T}$-space X*

$$\mathcal{M}(X)_a \cong EC^*_{\mathbb{T}}(X^A) \otimes \mathcal{O}_{C,a}.$$

In the case that

$$H^{\mathbb{T}}_*(X^A) \cong H_*(X^A) \otimes H_*(B\mathbb{T}),$$

then

$$\mathcal{M}(X)_a \cong H^*(X^A; \mathcal{O}_{C,a} \otimes \omega^*).$$

In the case of finite X and an elliptic curve of the form $C = \mathbb{C}/\Lambda$, the sheaf $\mathcal{M}(X)$ is equivalent to that of [Gro07]; see [Gre05, §22]. As we have already observed, one of the important innovations of [Gre05] is to realize that this sheaf can be constructed in the case of a rational elliptic curve, by using the function $t_{|A|}$ rather than the covering $\mathbb{C} \to \mathbb{C}/\Lambda$ to make $\mathcal{O}_A^\wedge$ into an $H^*B\mathbb{T}$-algebra.

We shall be interested in taking $X = BSU(d)$ or $BString_{\mathbb{C}}$, in which case it is more difficult to describe $\mathcal{M}(X)$ explicitly. Instead, we introduce a variant of $\mathcal{M}$, which amounts to working with the sheaf $\mathcal{M}(X^{\mathbb{T}})$ together with the collection of stalks $\mathcal{M}(X)_a$ for a of finite order.

The long exact sequence (6.5) shows that the difference between $EC^*_{\mathbb{T}}(X)$ and $EC^*_{\mathbb{T}}(X^{\mathbb{T}})$ is local on the elliptic curve: the "meromorphic" sections of $EC^*_{\mathbb{T}}(X)$ or $EC^*_{\mathbb{T}}(X^{\mathbb{T}})$ are just the meromorphic functions on $\operatorname{spec}(H^*(X^{\mathbb{T}})) \times C$, and the question of whether a meromorphic function s is holomorphic can be checked one point at a time. For $X^{\mathbb{T}}$, this is a question of checking whether for each a of finite order, s_a lives in

$$HP^*(X^{\mathbb{T}}; \mathcal{O}_a^\wedge) \subset HP^*(X^{\mathbb{T}}; \mathcal{K}_{\mathcal{F}}^\wedge)$$

but for X on which $\mathbb{T}$ acts non-trivially one must specify for each finite $A \subset \mathbb{T}$ an element s_A of

$$HP^*_{\mathbb{T}}(X^A; \mathcal{O}_A^\wedge)$$

which restricts to s in

$$H^*(X^{\mathbb{T}}; \mathcal{O}_A^\wedge).$$

To display this situation systematically, we introduce the following.

Definition 9.2. The category of *$\mathcal{E}$-sheaves* is the category in which an object $\mathcal{F}$ consists of

(1) an ω^*-periodic $\mathcal{O}_C$-module $\mathcal{F}_{\mathbb{T}}$;
(2) for each finite $A \subset \mathbb{T}$, an ω^*-periodic module $\mathcal{F}_A$ over the local ring $\mathcal{O}_A$, and a map of $\mathcal{O}_A^\wedge$-modules

$$r_A : \mathcal{F}_A^\wedge \to (\mathcal{F}_{\mathbb{T}})^\wedge_{C\langle A\rangle}. \tag{9.3}$$

A morphism $\mathcal{F} \to \mathcal{F}'$ consists of maps

$$\mathcal{F}_A \to \mathcal{F}'_A$$

for $A \subseteq \mathbb{T}$, which intertwine the maps (9.3).

Definition 9.4. If $\mathcal{F}$ is an $\mathcal{E}$-sheaf, then a section s of $\mathcal{F}$ consists of sections s_A of $\mathcal{F}_A$ for $A \subseteq \mathbb{T}$, such that for each finite A,

$$r_A s_A = s_{\mathbb{T}}.$$

We write $\Gamma\mathcal{F}$ for the group of sections of $\mathcal{F}$.

We formalize the motivating example.

Example 9.5. If X is a $\mathbb{T}$-spectrum, then $\mathcal{M}(X)$ defines an $\mathcal{E}$-sheaf $\mathcal{N}(X)$ by taking

$$\mathcal{N}(X)_{\mathbb{T}} := \mathcal{M}(X^{\mathbb{T}})$$

and

$$\mathcal{N}(X)_A := \mathcal{M}(X)_{C\langle A\rangle}$$

with structure map

$$\mathcal{M}(X)^\wedge_{C\langle A\rangle} \to \mathcal{M}(X^{\mathbb{T}})^\wedge_{C\langle A\rangle}$$

9.B. **The completed Grojnowski $\mathcal{E}$-sheaf of a $\mathbb{T}$-space.** Our main example, motivated by Proposition 6.4, is designed to highlight the regularity of various sections we later construct. It turns out to be a type of completion of the Grojnowski sheaf with convenient formal properties.

Proposition 9.6. *There is a functor*

$$\mathcal{F} : \mathbb{T}\text{-spaces} \longrightarrow \mathcal{E}\text{-sheaves}.$$

It is defined by the formulae

$$\begin{aligned} \mathcal{F}(X)_{\mathbb{T}} &= HP^*(X^{\mathbb{T}}; \mathcal{O}_C) \\ \mathcal{F}(X)_A &= HP^*_{\mathbb{T}}(X^A; \mathcal{O}^{\wedge}_A) \text{ for } A \text{ finite,} \end{aligned}$$

with structure map

$$\mathcal{F}(X)_A = HP^*_{\mathbb{T}}(X^A; \mathcal{O}^{\wedge}_A) \to HP^*_{\mathbb{T}}(X^{\mathbb{T}}; \mathcal{O}^{\wedge}_A) \cong HP^*(X^{\mathbb{T}}; \mathcal{O}^{\wedge}_A) = (\mathcal{F}(X)_{\mathbb{T}})^{\wedge}_{C\langle A\rangle}.$$

The functor is a naive 2-periodic $\mathbb{T}$-equivariant cohomology theory in the sense that it is homotopy invariant, exact and satisfies excision and the wedge axiom.

There is a map

$$\mathcal{N}(X) \longrightarrow \mathcal{F}(X)$$

natural in X, and it is an isomorphism at $\mathbb{T}$ in that $\mathcal{F}(X)_{\mathbb{T}} = \mathcal{M}(X^{\mathbb{T}}) = \mathcal{N}(X)_{\mathbb{T}}$. If X is finite, then it is completion at A in that

$$\mathcal{F}(X)_A \cong HP^*(X^A; \mathcal{O}^{\wedge}_A) \cong \mathcal{M}(X)^{\wedge}_{C\langle A\rangle}.$$

Proof. For behaviour at $\mathbb{T}$, we need only note that $\mathcal{M}(X)$ is associated to the free EC-module $F(X^{\mathbb{T}}, EC)$ of the same rank as $HP^*(X^{\mathbb{T}}; \mathcal{O}_C)$.

For behaviour at A we need to recall that the stalk of the sheaf $\mathcal{M}(X)$ at A is defined to be the homotopy groups of $F(X, EC) \wedge \tilde{E}\langle\neg A\rangle$, where $\tilde{E}\langle\neg A\rangle$ is defined by a cofibre sequence

$$E\langle\neg A\rangle \longrightarrow S^0 \longrightarrow \tilde{E}\langle\neg A\rangle$$

and

$$E\langle\neg A\rangle = \bigvee_{B\neq A} E\langle B\rangle.$$

Now we maps

$$F(X, EC) \wedge \tilde{E}\langle\neg A\rangle \longrightarrow F(X^A, EC) \wedge \tilde{E}\langle\neg A\rangle \longrightarrow F(X^A \wedge E\langle A\rangle, EC) \wedge \tilde{E}\langle\neg A\rangle.$$

The homotopy of the left-hand side is $\mathcal{M}(X)_A$, and the homotopy of the right-hand side is

$$HP^*_{\mathbb{T}}(X^A; \mathcal{O}^{\wedge}_A) = \mathcal{F}(X)_A$$

by the completion theorem. The first of the maps is an equivalence when X is finite since X/X^A is built from basic cells corresponding to finite subgroups other than A. The second of the maps is a completion. □

If X is a space, then it will also be convenient to reverse the arrows and consider the ringed spaces over C given by

$$\begin{aligned} \mathbf{F}(X)_{\mathbb{T}} &\stackrel{\text{def}}{=} (C, \mathcal{F}(X)_{\mathbb{T}}) \\ \mathbf{F}(X)_A &\stackrel{\text{def}}{=} (C\langle A\rangle, \mathcal{F}(X)_A). \end{aligned} \tag{9.7}$$

In this guise the structure map of $\mathcal{E}$ is a map of ringed spaces over $C^{\wedge}_{C\langle A\rangle}$

$$(\mathbf{F}(X)_{\mathbb{T}})^{\wedge}_{C\langle A\rangle} \to \mathbf{F}(X)^{\wedge}_A. \tag{9.8}$$

Definition 9.9. We shall refer to a collection of spaces $\mathbf{F}_A$ for $A \subseteq \mathbb{T}$, equipped with maps (9.8), as an *$\mathcal{E}$-space.* If $\mathbf{F}$ is a $\mathcal{E}$-space, then we write $\mathcal{O}_{\mathbf{F}}$ for its associated $\mathcal{E}$-sheaf.

Proposition 9.10. *Suppose that X is an* even *$\mathbb{T}$-space, that is, for each $A \subseteq \mathbb{T}$, $H^*(X^A)$ is concentrated in even degrees. Then there is a natural monomorphism*

$$\Gamma(\mathcal{O}_{\mathbf{F}(X)}) \to EC^0_{\mathbb{T}}(X).$$

If X is $\mathbb{T}$-fixed, this is an isomorphism.

Proof. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(\mathcal{F}(X)) & \longrightarrow & H^0(X^{\mathbb{T}}; \mathcal{O}_C) \times \prod_A H^0_{\mathbb{T}}(X^A; \mathcal{O}^{\wedge}_A) & \longrightarrow & H^0(X^{\mathbb{T}}; \mathcal{K}^{\wedge}_{\mathcal{F}}) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & EC^0_{\mathbb{T}}(X) & \longrightarrow & H^0(X^{\mathbb{T}}; \mathcal{K}) \times \prod_A H^0_{\mathbb{T}}(X^A; \mathcal{O}^{\wedge}_A) & \longrightarrow & H^0(X^{\mathbb{T}}; \mathcal{K}^{\wedge}_{\mathcal{F}}) & \longrightarrow & EC^1_{\mathbb{T}}(X) \longrightarrow 0 \end{array}$$

(In the interest of space, we have consistently omitted terms of the form $\otimes\omega^*_C$ from the coefficients). The exactness of the top row describes $\Gamma(\mathcal{F}(X))$. With our hypotheses, the bottom row is exact. The only difference between the middle terms is the $\mathcal{O}_C$, mapping to $\mathcal{K}$ in the bottom. The middle vertical arrow is the obvious map, and it is injective. It is clear that the left vertical arrow exists, and is injective.

If $s_{\mathbb{T}} \in H^0(X^{\mathbb{T}}; \mathcal{K})$ participates in an element s of the kernel of

$$H^0(X^{\mathbb{T}}; \mathcal{K}) \times \prod_A H^0_{\mathbb{T}}(X^A; \mathcal{O}^{\wedge}_A) \to H^0(X^{\mathbb{T}}; \mathcal{K}^{\wedge}_{\mathcal{F}}),$$

and X is $\mathbb{T}$-fixed then it clearly is in the image of

$$H^0(X^{\mathbb{T}}; \mathcal{O}_C) \to H^0(X^{\mathbb{T}}; \mathcal{K}).$$

and so s gives an element of $\Gamma(\mathcal{F}(X))$, showing that the left arrow is also surjective, as required. □

10. Analytic geometry of the sigma orientation I: $\mathbb{T}$-fixed spaces.

In this section we concentrate on $\mathbb{T}$-fixed spaces and generic points on the curve. As before this is the easiest part since the topology and the geometry are largely unlinked. We begin the analysis of the Thom space of the tautological bundle V over $BSU(d)$, by considering its retriction to the fixed point space $BSU(d)^{\mathbb{T}}$. In this section we describe $\mathcal{F}(BSU(d)^{\mathbb{T}})$ and $\mathcal{F}((BSU(d)^{\mathbb{T}})^V)$. In Section 11, we include the points with finite isotropy and torsion points on the curve, thereby extending our analysis to a description of $\mathcal{F}(BSU(d))$ and $\mathcal{F}(BSU(d)^V)$.

10.A. **The $\mathcal{E}$-space associated to the $\mathbb{T}$-fixed points of $BSU(d)$.** For brevity we write $Y = BSU(d)^{\mathbb{T}}$. Since $\mathbb{T}$ acts trivially on Y, we have

$$H^*_{\mathbb{T}}(Y^A) = H^*(Y \times B\mathbb{T})$$

and so $\Gamma(\mathcal{F}(Y))$ is the kernel in the diagram

$$0 \to \Gamma(\mathcal{F}(Y)) \to HP^0(Y; \mathcal{O}_C) \times \prod_A HP^0(Y; \mathcal{O}^{\wedge}_A) \to HP^0(Y; \mathcal{K}^{\wedge}_{\mathcal{F}}), \tag{10.1}$$

while $\Gamma(\mathcal{F}(Y^V))$ is the kernel in the diagram

$$0 \to \Gamma(\mathcal{F}(Y^V)) \to HP^0(Y^{V^{\mathbb{T}}}; \mathcal{O}_C) \times \prod_A HP^0_{\mathbb{T}}(Y^{V^A}; \mathcal{O}^{\wedge}_A) \to HP^0(Y^{V^{\mathbb{T}}}; \mathcal{K}^{\wedge}_{\mathcal{F}}). \tag{10.2}$$

For each component Z of Y, $HP^*(Z)$ is a domain, and so each factor in the right of (10.2) is a principal ideal of the corresponding factor in (10.1), generated by an Euler class. Our goal is to understand these ideals.

Recall from Proposition 4.7 that elements $\Xi \in HP^0(Y) = HP^0(BSU(d)^{\mathbb{T}})$ are given by compatible elements

$$\Xi(m) \in HP^0(BT)^{W(m)}$$

where m ranges over $\check{T} = \mathrm{Hom}(\mathbb{T}, T)$. Now

$$\mathrm{spf}\, HP^0 B\mathbb{T} \cong \widehat{\mathbb{G}}_a,$$

and the projection

$$\mathbb{C} \to C$$

gives an isomorphism

$$\widehat{\mathbb{G}}_a \cong \widehat{C},$$

so we have

$$\mathrm{spf}\, HP^0 BT \cong \check{T} \otimes \widehat{C}.$$

Thus we may view an element of $HP^0(Y; \mathcal{O}_C)$ as a family of functions

$$(\check{T} \otimes \widehat{C})/W(m) \times C \to \mathbb{C}. \tag{10.3}$$

This suggests that we make the following definitions.

Definition 10.4. For $m : \mathbb{T} \to T$, let $\mathfrak{X}_m$ be the space

$$\mathfrak{X}_m \stackrel{\mathrm{def}}{=} (\check{T} \otimes \widehat{C})/W(m) \times C.$$

Note that this is a ringed space over C, so $\mathcal{O}_{\mathfrak{X}_m}$ is an $\mathcal{O}_C$-algebra.

For $w \in W$, there is an evident isomorphism

$$w : \mathfrak{X}_m \to \mathfrak{X}_{wm},$$

which is the identity if $w \in W(m)$ so that $w = wm$. Thus let

$$\mathfrak{X}_{\mathbb{T}} = \left(\coprod_{m:\mathbb{T}\to T} \mathfrak{X}_m \right) /W.$$

Proposition 10.5. *For each $m : \mathbb{T} \to T$, there is a canonical isomorphism of $\mathcal{O}_C$-algebras*

$$\mathcal{F}(BZ(m))_{\mathbb{T}} \cong \mathcal{O}_{\mathfrak{X}_m},$$

or equivalently of ringed spaces over C

$$\mathbf{F}(BZ(m))_{\mathbb{T}} \cong \mathfrak{X}_m.$$

These assemble to an isomorphism

$$\mathcal{F}(BSU(d)^{\mathbb{T}})_{\mathbb{T}} \cong \left(\prod_{m:\mathbb{T}\to T} \mathcal{O}_{\mathfrak{X}_m} \right)^W,$$

or equivalently

$$\mathbf{F}(BSU(d)^{\mathbb{T}})_{\mathbb{T}} \cong \mathfrak{X}_{\mathbb{T}}.$$

$\square$

Remark 10.6. For any space Z on which $\mathbb{T}$ acts trivially,

$$\mathbf{F}(Z)_A \cong (\mathbf{F}(Z)_{\mathbb{T}})^{\wedge}_{C\langle A\rangle}.$$

Thus Proposition 10.5 implies that

$$\mathbf{F}(BSU(d)^{\mathbb{T}})_A \cong (\mathfrak{X}_{\mathbb{T}})^{\wedge}_{C\langle A\rangle}$$

for finite A, and we have given a complete description of $\mathbf{F}(BSU(d)^{\mathbb{T}})$.

Now we turn to the Thom space Y^V. Suppose that $m = (m_1, \dots, m_d) \in \check{T} \subset \mathbb{Z}^d$ labels a component $BZ(m)$ of $BSU(d)^{\mathbb{T}}$, and $x_i \in H^2BT$ are the corresponding generators. The equivariant Euler class of V in the orientation of ordinary cohomology given by the sigma function is

$$f_m(x,z) \overset{\text{def}}{=} e_{\mathbb{T}}(V)|_{BZ(m)} = \prod_i \sigma(x_i + m_i z, \tau). \tag{10.7}$$

This defines a holomorphic function

$$f_m : (\check{T} \otimes \widehat{C})/W(m) \times \mathbb{C} \to \mathbb{C},$$

but it does *not* descend to $\mathfrak{X}_m = (\check{T} \otimes \widehat{C})/W(m) \times C$ as in (10.3). Instead, it is a holomorphic section of a line bundle $\mathcal{L}_m$, as we now explain.

10.B. **The Loojienga line bundle.** There is a line bundle over

$$\check{T} \otimes C = \check{T} \otimes (\mathbb{C}^{\times}/q^{\mathbb{Z}})$$

given by the formula

$$\frac{(\check{T} \otimes \mathbb{C}^{\times}) \times \mathbb{C}}{(u,\lambda) \sim (uq^m, \lambda u^{-I(m)} q^{-\phi(m)})}, \tag{10.8}$$

and the Weyl-invariance of I and ϕ (7.11) imply that this line bundle descends to a line bundle on $(\check{T} \otimes C)/W$, which we call $\mathcal{L}$; as far as we know it was introduced by Looijenga [Loo76]. The functional equation

$$\sigma(uq^n, q) = z^{-I(m)} q^{-\phi(m)} \sigma(u,q)$$

of Lemma 7.13 implies that the product of sigma functions

$$\sigma(u,q) = \prod_i \sigma(u_i, q)$$

descends to a holomorphic section of $\mathcal{L}$.

Addition in the abelian group $\check{T} \otimes C$ induces a map

$$\mu_m : \mathfrak{X}_m = (\check{T} \otimes \widehat{C})/W(m) \times C \xrightarrow{i \times m} (\check{T} \otimes C)/W(m) \times (\check{T} \otimes C)^{W(m)} \to (\check{T} \otimes C)/W(m) \to (\check{T} \otimes C)/W; \tag{10.9}$$

if $(a_1, \dots, a_d) \in \check{T} \otimes \widehat{C}$ represents a point of $(\check{T} \otimes \widehat{C})/W(m)$ and $z \in C$, then

$$\mu_m(a_1, \dots, a_d, z) = (a_1 + m_1 z, \dots, a_d + m_d z).$$

This has the following relationship to topology. The Borel construction of V is classified by a map

$$BZ(m) \times B\mathbb{T} \to BSU(d),$$

and so provides a map

$$\mu_m^{\mathrm{top}} : (\check{T} \otimes \widehat{C})/W(m) \times \widehat{C} \cong \operatorname{spf} HP^0 BZ(m) \times B\mathbb{T} \to \operatorname{spf} HP^0 BSU \cong (\check{T} \otimes \widehat{C})/W.$$

Lemma 10.10. *The diagram*

$$\begin{array}{ccc}
\operatorname{spf} HP^0(BZ(m)\times B\mathbb{T}) & \xrightarrow{\mu_m^{\mathrm{top}}} & \operatorname{spf} HP^0 BSU(d) \\
\big\downarrow & & \big\downarrow \\
(\check{T}\otimes\widehat{C})/W(m)\times C & \xrightarrow{\mu_m} & (\check{T}\otimes C)/W
\end{array}$$

commutes.

Proof. This is an expression of the fact (see (4.11)) that

$$c_{\bullet}^{\mathcal{B}}(V) = \prod_i (1 + x_i + m_i z).$$

□

Definition 10.11. Let $\mathcal{L}_m$ be the line bundle

$$\mathcal{L}_m \overset{\text{def}}{=} \mu_m^* \mathcal{L}$$

over $\mathfrak{X}_m$. Explicitly, $\mathcal{L}_m$ is obtained from the line bundle

$$\frac{(\check{T}\otimes\mathbb{C}^\times)\times\mathbb{C}^\times\times\mathbb{C}}{(u,z,\lambda)\sim(u,zq^k,\lambda u^{-I(m)}z^{-kI(m,m)}q^{-k^2\phi(m)})} \tag{10.12}$$

over $(\check{T}\otimes\mathbb{C}^\times)\times C$ by restriction to

$$(\check{T}\otimes\widehat{C})\times C$$

and then descent to $\mathfrak{X}_m$.

It is easy to check that the functions f_m of (10.7) descend to the holomorphic sections

$$f_m = \mu_m^* \sigma$$

of $\mathcal{L}_m$. By construction, these are compatible as m varies.

Proposition 10.13. *For $w \in W$, the diagram*

$$\begin{tikzcd}
\mathfrak{X}_m \arrow[rr, "w"] \arrow[dr, "\mu_m"'] & & \mathfrak{X}_{wm} \arrow[dl, "\mu_{wm}"] \\
& (\check{T}\otimes C)/W &
\end{tikzcd}$$

commutes, and so there are natural isomorphisms

$$w^*\mathcal{L}_{wm} \cong \mathcal{L}_m,$$

with respect to which

$$w^* f_{wm} = f_m.$$

□

By the Proposition, the $\mathcal{L}_m$ descend to a line bundle

$$\mathcal{L}_{\mathbb{T}} \to \mathfrak{X}_{\mathbb{T}},$$

equipped with a section $f_{\mathbb{T}}$. Because the sigma function has zeroes, $f_{\mathbb{T}}$ is not a trivialization of $\mathcal{L}_{\mathbb{T}}$. Instead, let $\mathcal{I}(f_{\mathbb{T}})$ be the ideal of zeroes of $f_{\mathbb{T}}$.

Corollary 10.14. *The section $f_{\mathbb{T}}$ is a trivialization of the line bundle $\mathcal{L}_{\mathbb{T}}\otimes\mathcal{I}(f_{\mathbb{T}})$.* □

10.C. **Trivializing line bundles of string bundles I: $\mathbb{T}$-fixed spaces.** We now turn towards relating this to the sigma orientation, retaining the abbreviation $Y = BSU(d)^{\mathbb{T}}$.

Proposition 10.15. *After identifying*
$$\mathbf{F}(Y)_{\mathbb{T}} \cong \mathfrak{X}_{\mathbb{T}},$$
the map
$$\mathcal{F}(Y^V) \to \mathcal{F}(Y)$$
induced by the zero section $\zeta : Y \to Y^V$ induces an isomorphism of line bundles over $\mathfrak{X}_{\mathbb{T}}$
$$\gamma_{\mathbb{T}} : \mathcal{I}(f_{\mathbb{T}}) \cong \mathcal{F}(Y^V)_{\mathbb{T}}.$$
For each finite $A \subset \mathbb{T}$, we have
$$\mathcal{F}(Y^V)_A \cong (\mathcal{F}(Y^V)_{\mathbb{T}})^{\wedge}_{C\langle A\rangle},$$
and so $\gamma_{\mathbb{T}}$ induces an isomorphism
$$\mathcal{F}(Y^V)_A \cong \mathcal{I}(f_{\mathbb{T}})^{\wedge}_{C\langle A\rangle}.$$

Proof. Let a be a point of C, of order n with $1 \leq n \leq \infty$. Let
$$V^a = V^{\mathbb{T}[n]}.$$
Then
$$(\mathcal{F}(Y)_{\mathbb{T}})^{\wedge}_a \cong HP^0(Y; \mathcal{O}^{\wedge}_a),$$
while
$$(\mathcal{F}(Y^V)_{\mathbb{T}})^{\wedge}_a \cong HP^0_{\mathbb{T}}(Y^{V^a}; \mathcal{O}^{\wedge}_a).$$
Let
$$m = (m_1, \dots, m_d) \in \check{T} \subset \mathbb{Z}^d$$
be a cocharacter, labelling a component $Z = BZ(m)$ of Y. Then $H^*_{\mathbb{T}}(Z^{V^a})$ is the ideal in $H^*_{\mathbb{T}}(Z)$ generated by its Euler class
$$e_{\mathbb{T}}(V^a) = \prod_{m_j \equiv 0 \mod n} \sigma(x_j + m_j z).$$
But
$$f_m(V) = e_{\mathbb{T}}(V) = e_{\mathbb{T}}(V^a) e_{\mathbb{T}}(V/V^a),$$
where
$$e_{\mathbb{T}}(V/V^a) = \prod_{m_j \not\equiv 0 \mod n} \sigma(x_j + m_j z).$$
The argument in the proof of Proposition 8.15 applies: the x_j are topologically nilpotent, and so $e_{\mathbb{T}}(V/V^a)$ is a unit of $\mathcal{M}(Z)^{\wedge}_a$, and $f_m(V)$ generates the same ideal as $e_{\mathbb{T}}(V^a)$. □

We can display the situation described by the Proposition in the following diagram, in which each square is a pull-back, and the curved arrows are trivializations of the indicated sheaves.

$$\begin{tikzcd}
\mathcal{L}(V) \otimes \mathcal{F}(Y^V) \arrow[r, "\cong"] \arrow[d] & \mathcal{L}_{\mathbb{T}} \otimes \mathcal{I}(f_{\mathbb{T}}) \arrow[r] \arrow[d] & \mathcal{L} \otimes \mathcal{I}(\sigma) \arrow[d] \\
\mathbf{F}(Y)_{\mathbb{T}} \arrow[r, "\cong"] \arrow[u, bend right, "f(V)"'] & \mathfrak{X}_{\mathbb{T}} \arrow[r, "\mu"] \arrow[u, bend right, "f_{\mathbb{T}}"'] & (\check{T} \otimes C)/W \arrow[u, bend right, "\sigma"']
\end{tikzcd}$$

Thus a $\mathbb{T}$-equivariant $SU(d)$ bundle V over a $\mathbb{T}$-fixed space Z gives rise to a map
$$h : \mathbf{F}(Z) \to \mathfrak{X}_{\mathbb{T}},$$

and so we can form the line bundles

$$\mathcal{L}(V) = h^*\mathcal{L}_{\mathbb{T}}$$
$$\mathcal{I}(V) = h^*\mathcal{I}(f_{\mathbb{T}})$$

over $\mathbf{F}(Z)$. The section

$$f(V) = h^* f_{\mathbb{T}}$$

is a trivialization of $\mathcal{L}(V) \otimes \mathcal{I}(V)$, and we have an canonical isomorphism of line bundles

$$\mathcal{F}(Z^V) \cong \mathcal{I}(V).$$

In Section 11, we explain how to handle the full space $BSU(d)$, and so spaces Z on which $\mathbb{T}$ acts non-trivially. Before doing so, we discuss the sigma orientation for $\mathbb{T}$-fixed spaces Z.

Suppose that for $i = 0, 1$, V^i is a $\mathbb{T}$-equivariant $BSU(d)$-bundle over a $\mathbb{T}$-fixed space Z, and let $\xi = V^0 - V^1$. We then have two maps

$$h_i : \mathbf{F}(Z) \to \mathfrak{X}_{\mathbb{T}},$$

and we can form the line bundles $\mathcal{L}(V^i) = h_i^*\mathcal{L}_{\mathbb{T}}$ with sections $f^i = f(V^i)$ as above. The ratio

$$\psi = \frac{f^0}{f^1}$$

is a trivialization of

$$\frac{\mathcal{L}(V^0) \otimes \mathcal{I}(V^0)}{\mathcal{L}(V^1) \otimes \mathcal{I}(V^1)} \cong \frac{\mathcal{L}(V^0) \otimes \mathcal{F}(Y^{V^0})}{\mathcal{L}(V^1) \otimes \mathcal{F}(Y^{V^1})}.$$

Proposition 10.16. *If* $c_2^{\mathcal{B}}(\xi) = 0$ *then*

$$\mathcal{L}(V^0) \cong \mathcal{L}(V^1).$$

Thus if $c_2^{\mathcal{B}}(\xi) = 0$, then ψ gives a trivialization of

$$\mathcal{F}(Z^{V^0}) \otimes \mathcal{F}(Z^{V^1})^{-1}$$

as a $\mathcal{F}(Z)$-module. This is precisely our Thom class ψ from Theorem 7.23.

Proof of Proposition 10.16. We can factor the maps

$$h_i : Z \to BSU(d)$$

as

$$h_i : Z \to BZ(m_i),$$

where $m_i : \mathbb{T} \to T$ are cocharacters. Then the formula (10.12) shows that $\mathcal{L}_m = \mathcal{L}_{m_0} \otimes \mathcal{L}_{m_1}^{-1}$ is obtained from the line bundle

$$\frac{(\check{T} \otimes \mathbb{C}^\times)^2 \times \mathbb{C}^\times \times \mathbb{C}}{(u_0, u_1, z, \lambda) \sim (u_0, u_1, zq^k, \lambda u_0^{-kI(m_0)} u_1^{kI(m_1)} z^{-2k\phi(m_0)+2k\phi(m_1)} q^{-k^2\phi(m_0)+k^2\phi(m_1)})}$$

over

$$(\check{T} \otimes \mathbb{C}^\times)^2 \times C.$$

by restricting to $(\check{T} \otimes \widehat{C})^2$ and then descending to

$$((\check{T} \otimes \widehat{C})/W(m_0)) \times ((\check{T} \otimes \widehat{C})/W(m_1)) \times C.$$

By Lemma 4.13, if $c_2^{\mathcal{B}}(\xi) = 0$, then

$$\phi(m_0) = \phi(m_1),$$

and over Z,

$$u_0^{I(m_0)} = u_1^{I(m_1)},$$

and so this line bundle is trivial. □

11. Analytic geometry of the sigma orientation II: finite isotropy.

In Section 10, we constructed a model

$$\mathbf{F}(BSU(d)^{\mathbb{T}}) \cong \mathfrak{X}_{\mathbb{T}}$$

and used it to construct a map

$$\mu : \mathbf{F}(BSU(d)^{\mathbb{T}}) \to (\check{T} \otimes C)/W.$$

Over $(\check{T} \otimes C)/W$ we have a line bundle $\mathcal{L}$, equipped with a holomorphic section σ defined by products of the Weierstrass sigma function. We showed that

$$\mathcal{F}((BSU(d)^{\mathbb{T}})^V)$$

is the ideal sheaf on $\mathbf{F}(BSU(d)^{\mathbb{T}})$ of zeroes of $\mu^*\sigma$.

Since we always have $\mathbf{F}(X)_{\mathbb{T}} \cong \mathbf{F}(X^{\mathbb{T}})_{\mathbb{T}}$, the analysis so far describes the Thom isomorphism for the $\mathbf{F}(BSU(d))_{\mathbb{T}}$ piece of $\mathbf{F}(BSU(d))$. In this section, we give a similar analysis of $\mathbf{F}(BSU(d))_A$ and $\mathcal{F}(BSU(d)^V)_A$ for finite A.

11.A. **Overview.** It is not hard to describe the basic idea. Let $m : A \to T$ label a component $BZ(m)$ of $BSU(d)^A$. Since $H^*BZ(m)$ is concentrated in even degrees, Lemma 6.3 implies that there is an isomorphism

$$H^*_{\mathbb{T}}(BZ(m); T_A C) \cong H^*_{\mathbb{T}}(BZ(m); \mathcal{O}^{\wedge}_A) \cong H^*(BZ(m); \mathcal{O}^{\wedge}_A) \tag{11.1}$$

(the first isomorphism is natural and the second involves choices). The right-hand side is the ring of functions on

$$(\check{T} \otimes \widehat{C})/W(m) \times C^{\wedge}_A.$$

The idea is to construct a map

$$\mu_m : (\check{T} \otimes \widehat{C})/W(m) \times C^{\wedge}_A \to (\check{T} \otimes C)/W, \tag{11.2}$$

like the map in (10.9). Then we can form the line bundle

$$\mathcal{L}_m = \mu_m^* \mathcal{L}$$

with section

$$f_m = \mu_m^* \sigma,$$

and identify $\mathcal{F}(X^V)_A$ with $\mathcal{I}(f_m)$, as in the $\mathbb{T}$-fixed case.

There are two related problems. The first is that the isomorphism (11.1) is not canonical, and we must be able to construct the map μ_m compatibly with restriction to $BSU(d)^{\mathbb{T}}$ in order to extend the analysis of Section 10. The second is that the homomorphism

$$m = (m_1, \dots, m_d) : A \to T$$

does not quite determine μ_m as in (11.2). We do get a homomorphism

$$C[A] \to (\check{T} \otimes C)^{W(m)},$$

and so a map

$$\mu_m^{\text{weak}} : (\check{T} \otimes C)/W(m) \times C[A] \xrightarrow{\text{id} \times m} (\check{T} \otimes C)/W(m) \times (\check{T} \otimes C)^{W(m)} \to \check{T} \otimes C/W, \tag{11.3}$$

analogous to (10.9). The problem is to extend this map to the formal neighborhood $C^{\wedge}_A$ of $C\langle A \rangle$.

11.B. **The $\mathcal{E}$-space associated to $BSU(d)$.** The $\mathbb{T}$-action on $BZ(m) \subset BSU(d)$ (see Remark 2.1) provides the extra information we need. The Borel construction

$$V \times_{\mathbb{T}} E\mathbb{T} \to BZ(m) \times_{\mathbb{T}} E\mathbb{T}$$

is classified by a map

$$BZ(m) \times_{\mathbb{T}} E\mathbb{T} \to BZ(m),$$

inducing a map

$$\mu_{m,0} : \operatorname{spf} HP^0(BZ(m) \times_{\mathbb{T}} E\mathbb{T}) \to (\check{T} \otimes \widehat{C})/W(m). \tag{11.4}$$

A choice of isomorphism

$$HP^0(BZ(m) \times_{\mathbb{T}} E\mathbb{T}) \cong HP^0(BZ(m) \times B\mathbb{T}) \tag{11.5}$$

permits us to view $\mu_{m,0}$ as a map

$$(\check{T} \otimes \widehat{C})/W(m) \times \widehat{C} \to (\check{T} \otimes \widehat{C})/W(m) \to (\check{T} \otimes C)/W(m),$$

giving us the desired map (11.2) in a formal neighborhood of 0. We can then define μ_m at a point a of exact order n by translation, noting that the diagram

$$\begin{tikzcd}
(\check{T} \otimes \widehat{C})/W(m) \times C[A] \arrow[r, "\mu_m^{\mathrm{weak}}"] & (\check{T} \otimes C)/W(m) \\
(\check{T} \otimes \widehat{C})/W(m) \times C[A] \arrow[r, "\mu_m^{\mathrm{weak}}"] \arrow[u, "1\times T_a"] & (\check{T} \otimes C)/W(m) \arrow[u, "T_{m(a)}"']
\end{tikzcd} \tag{11.6}$$

commutes.

In doing so, there is little to be gained by choosing the isomorphism (11.5). Instead, we note that $\operatorname{spf} HP^0_{\mathbb{T}}(BZ(m))$ is in any case a formal scheme over $\widehat{C} \cong \operatorname{spf} HP^0 B\mathbb{T}$, and so for $m : A \to \check{T}$ labelling a component $BZ(m)$ of $BSU(d)^A$, we define $\mathfrak{X}_m$ to be the formal scheme over $C_A^\wedge$ given by

$$\mathfrak{X}_m \stackrel{\mathrm{def}}{=} \coprod_{a \in C\langle A\rangle} T_{-a}^* \operatorname{spf} HP^0_{\mathbb{T}}(BZ(m)).$$

By construction, for $w \in W$ we have natural isomorphisms

$$\mathfrak{X}_m \to \mathfrak{X}_{wm},$$

and so setting

$$\mathfrak{X}_A \stackrel{\mathrm{def}}{=} \left(\coprod_{m:A\to\mathbb{T}} \mathfrak{X}_m \right)/W,$$

we have an isomorphism

$$\mathbf{F}(BSU(d))_A \cong \mathfrak{X}_A.$$

Since by Proposition 10.5,

$$(\mathbf{F}(BSU(d))_{\mathbb{T}})^\wedge_{C\langle A\rangle} \cong \mathfrak{X}_{\mathbb{T}},$$

we get a map

$$\mathfrak{X}_A \cong \mathbf{F}(BSU(d))_A \to (\mathbf{F}(BSU(d))_{\mathbb{T}})^\wedge_{C\langle A\rangle} \cong (\mathfrak{X}_{\mathbb{T}})^\wedge_{C\langle A\rangle},$$

and so $\mathfrak{X}$ is a $\mathcal{E}$-space. Let

$$\mu_m : \mathfrak{X}_m \to (\check{T} \otimes C)/W(m)$$

be the map which on the a component of $\mathfrak{X}_m$ is given by

$$\operatorname{spf} HP^0_{\mathbb{T}}(BZ(m)) \xrightarrow{\mu_{m,0}} (\check{T} \otimes C)/W(m) \xrightarrow{T_{m(a)}} (\check{T} \otimes C)/W(m).$$

It is easy to check that the μ_m for $m : A \to T$ induce a map

$$\mu_A : \mathfrak{X}_A \to (\check{T} \otimes C)/W.$$

Then we have the following.

Proposition 11.7. *For $A \subset \mathbb{T}$, the diagram*

$$\begin{tikzcd} (\mathfrak{X}_{\mathbb{T}})^{\wedge}_{C\langle A\rangle} \arrow[r] \arrow[d] & \mathfrak{X}_A \arrow[d, "\mu_A"] \\ \mathfrak{X}_{\mathbb{T}} \arrow[r, "\mu_{\mathbb{T}}"] & (\check{T} \otimes C)/W \end{tikzcd}$$

commutes. Thus we have an isomorphism

$$\mathbf{F}(BSU(d)) \cong \mathfrak{X}$$

of $\mathcal{E}$-spaces over C, and a map

$$\mu : \mathfrak{X} \to (\check{T} \otimes C)/W.$$

□

Proof. We consider what is happening over a particular point $a \in C$ of exact order $|A|$. We also work one component at a time: fix a pair of homomorphisms

$$\begin{tikzcd} \mathbb{T} \arrow[r, "\tilde{m}"] & T \\ A, \arrow[u] \arrow[ur, "m"'] & \end{tikzcd}$$

labelling a component $BZ(m)$ of $BSU(d)^A$, and a component $BZ(\tilde{m})$ of $BZ(m)^{\mathbb{T}}$. We must show that the diagram

$$\begin{tikzcd} (\mathfrak{X}_{\tilde{m}})^{\wedge}_{C\langle A\rangle} \arrow[r] \arrow[d] & \mathfrak{X}_m \arrow[d, "\mu_m"] \\ \mathfrak{X}_{\tilde{m}} \arrow[r, "\mu_{\tilde{m}}"] & (\check{T} \otimes C)/W \end{tikzcd} \tag{11.8}$$

commutes.

Note that the diagram

$$\begin{tikzcd} (\check{T} \otimes C)/W(\tilde{m}) \times C_0^{\wedge} \arrow[r, "1\times T_a"] \arrow[d, "\mu_{\tilde{m}}"'] & (\check{T} \otimes C)/W(\tilde{m}) \times C_a^{\wedge} \arrow[d, "\mu_{\tilde{m}}"] \\ (\check{T} \otimes C)/W(\tilde{m}) \arrow[r, "T_{\tilde{m}(a)}"] & (\check{T} \times C)/W(\tilde{m}) \end{tikzcd}$$

commutes: if $\tilde{m} = (m_1, \ldots, m_d) \in \check{T} \subset \mathbb{Z}^d$, then either composition sends the element of $(\check{T} \otimes C)/W(\tilde{m}) \times C_0^{\wedge}$ represented by $(x_1, \ldots, x_d, z) \in (\check{T} \otimes C) \times C_0^{\wedge}$ to the class of

$$(x_1 + m_1(z + a), \ldots, x_d + m_d(z + a))$$

in $(\check{T} \otimes C)/W(\tilde{m})$. Thus the counterclockwise composition in the diagram (11.8) at a may be replaced by the top row in the diagram

$$\begin{tikzcd} (\mathfrak{X}_{\tilde{m}})^{\wedge}_a \arrow[r, "1\times T_{-a}"] \arrow[d] & (\mathfrak{X}_{\tilde{m}})^{\wedge}_0 \arrow[r, "\mu_{\tilde{m},0}"] \arrow[d] & (\check{T} \otimes C)/W(\tilde{m}) \arrow[r, "T_{\tilde{m}(a)}"] \arrow[d] & (\check{T} \otimes C)/W(\tilde{m}) \arrow[d] \\ (\mathfrak{X}_m)_a \arrow[r, "T^*_{-a}"] & (\mathfrak{X}_m)_0 \arrow[r, "\mu_{m,0}"] & (\check{T} \otimes C)/W(m) \arrow[r, "T_{m(a)}"] & (\check{T} \otimes C)/W(m). \end{tikzcd}$$

The commutativity of the first square and third squares is straightforward. The commutativity of the second square follows from Lemma 10.10 and the commutativity of the diagram

$$\begin{tikzcd} BZ(\tilde{m}) \times_{\mathbb{T}} E\mathbb{T} \arrow[r] \arrow[dr] & BZ(m) \times_{\mathbb{T}} E\mathbb{T} \arrow[d] \\ & BZ(m). \end{tikzcd}$$

□

11.C. **Building the line bundle of the Thom space over** $BSU(d)$**.** Having described the $\mathcal{E}$-space $\mathbf{F}(BSU(d))$, we turn to the $\mathcal{E}$-sheaf $\mathcal{F}(BSU(d)^V)$. As in Section 10, we can define a line bundle

$$\mathcal{L}_m \stackrel{\text{def}}{=} \mu_m^* \mathcal{L}$$

with section

$$f_m \stackrel{\text{def}}{=} \mu_m^* \sigma$$

over $\mathfrak{X}_m$, and

$$\mathcal{L}_A \stackrel{\text{def}}{=} \mu_A^* \mathcal{L}$$

$$f_A \stackrel{\text{def}}{=} \mu_A^* \sigma$$

over $\mathfrak{X}_A$.

Unlike $\mathfrak{X}_{\mathbb{T}}$, $\mathfrak{X}_A$ is affine, and so $\mathcal{L}_A$ is trivializable, and it is easy to check that a trivialization will induce an isomorphism

$$\gamma_A : \mathcal{L}_A \otimes \mathcal{I}(f_A) \cong \mathcal{F}(BSU(d)^V)_A \tag{11.9}$$

of ideals in $\mathcal{F}(BSU(d))_A$. The problem is to arrange things so that, in the case of a $BString_{\mathbb{C}}$-bundle, $\gamma_A f_A$ coincides with $\gamma_{\mathbb{T}} f_{\mathbb{T}}$.

Suppose that $\tilde{a}$ is a lift of a to $\mathbb{C}$, and $\alpha = e^{\tilde{a}}$ is the corresponding lift of a to $\mathbb{C}^\times$. Let $\tilde{m}$ be a cocharacter making the diagram

$$\begin{tikzcd} \mathbb{T} \arrow[r, "\tilde{m}"] & T \\ A \arrow[u] \arrow[ur, "m"'] & \end{tikzcd}$$

commute. The isomorphism

$$\mathcal{L} \cong \mathbf{1}$$

of line bundles over

$$\check{T} \otimes \mathbb{C}^\times$$

induces an isomorphism

$$g(\tilde{m}, \alpha) : T_{\alpha^{\tilde{m}}}^* \mathcal{L} \cong \mathbf{1}$$

over $\check{T} \otimes \mathbb{C}^\times$. To be precise, we mean that if $f : \check{T} \otimes \mathbb{C}^\times \to \mathbb{C}$ is a function, considered as a section of $\mathcal{L}$, then $g(\tilde{m}, \alpha)(T_{\alpha^{\tilde{m}}}^* f)$ is the section of $\mathbf{1}$ given by

$$g(\tilde{m}, \alpha)(T_{\alpha^{\tilde{m}}}^* f)(u) = (u, f(u\alpha^{\tilde{m}})).$$

In particular

$$g(\tilde{m}, \alpha)(T_{\alpha^{\tilde{m}}}^* \sigma)(u) = (u, \sigma(u\alpha^{\tilde{m}})).$$

Now $g(\tilde{m}, \alpha)$ does *not* induce an isomorphism

$$\mathcal{L} \otimes \mathcal{I}(\sigma) \cong H^*(BZ(m)^V),$$

because $\sigma(u\alpha^{\tilde{m}})$ depends on the choice of $\tilde{m}$, which is not $W(m)$ invariant, and so $\sigma(u\alpha^{\tilde{m}})$ does not give an element of $H^*(BZ(m))$.

To fix this, let

$$\gamma(\tilde{m},\alpha) = u^{\frac{k}{n}I(\tilde{m})}\alpha^{\frac{k}{n}\phi(\tilde{m})}g(\tilde{m},\alpha) : T^*_{\alpha^{\tilde{m}}}\mathcal{L} \cong \mathbf{1}.$$

Then

$$\gamma(\tilde{m},\alpha)(T^*_{\alpha^{\tilde{m}}}\sigma)(u) = (u, u^{\frac{k}{n}I(\tilde{m})}\alpha^{\frac{k}{n}\phi(\tilde{m})}\sigma(u\alpha^{\tilde{m}})) = (u, \delta_A(u,\tilde{m},\tilde{a})). \tag{11.10}$$

We showed in the work leading to Definition 8.5 that this *does* give an element of $H^*(BZ(m))$. In the present setting, this means that we have a well-defined isomorphism

$$\gamma(m,\alpha) : T^*_{\alpha^m}\mathcal{L}\otimes\mathcal{I}(\sigma) \cong H^*(BZ(m)^V)$$

which, by pulling back along

$$BZ(m)\times_{\mathbb{T}} E\mathbb{T} \to BZ(m),$$

gives an isomorphism

$$\gamma(m,\alpha) : (\mathcal{L}_m\otimes\mathcal{I}(f_m))_a \cong (\mathcal{F}(BZ(m)^V)_A)_a.$$

If we assemble the $\gamma(m,\alpha)$ for various m, we get an isomorphism

$$\gamma_\alpha : (\mathcal{L}_A\otimes\mathcal{I}(f_A))_a \cong (\mathcal{F}(BSU(d)^V)_A)_a.$$

To codify the dependence of γ on the lift $\alpha = e^{\tilde{a}}$ and assemble the γ for various $a\in C\langle A\rangle$, let ℓ be a lift as in the diagram

$$\begin{tikzcd} & \mathbb{C} \arrow[d] \\ C\langle A\rangle \arrow[ur, dashed, "\ell"] \arrow[r, tail] & C. \end{tikzcd} \tag{11.11}$$

We then can view the γ_α for various α as giving an isomorphism

$$\gamma_\ell : \mathcal{L}_A\otimes\mathcal{I}(f_A) \cong \mathcal{F}(BZ(m)^V)_A.$$

If ℓ and ℓ' are two such sections, let

$$d(\ell',\ell) : C\langle A\rangle \to \mathbb{Z}$$

be the function such that

$$e^{\ell'} = e^{\ell}q^{d(\ell',\ell)} : C\langle A\rangle \to \mathbb{C}^\times.$$

Let

$$c(\ell',\ell) : \mathfrak{X}_A \to \{\zeta\in\mathbb{C}^\times | \zeta^{|A|} = 1\}$$

be the function given by the formula

$$c(\ell',\ell,m,a) = w(a,q^{1/n})^{d(\ell',\ell,a)\phi(\tilde{m})},$$

where $m : A\to T$ labels a component of $BSU(d)^A$, $\tilde{m} : \mathbb{T}\to T$ is any lift of m, and $w(a,q^{1/n})$ is the Weil pairing of the indicated elements of $C[A]$, as in Lemma 8.6. This quantity does not depend on $\tilde{m}$, because $w(a,q^{1/n})^n = 1$. It does not depend on the choice of m, because

$$\phi(\tilde{m}) = \phi(w\tilde{m})$$

for $w\in W$.

Lemma 11.12. *If ℓ' and ℓ are two lifts, then*

$$\gamma_{\ell'}\gamma_\ell^{-1}$$

is multiplication by

$$c(\ell',\ell) : \mathfrak{X}_A \to \mathbb{C}^\times.$$

Proof. This is just a formulation of Lemma 8.6. $\square$

Let $BString_{\mathbb{C}}(d)$ be the pull-back in the diagram

$$\begin{array}{ccc} BString_{\mathbb{C}}(d) & \longrightarrow & BString_{\mathbb{C}} \\ \big\downarrow & & \big\downarrow \\ BSU(d) \times BSU(d) & \xrightarrow{V_0 - V_1} & BSU. \end{array}$$

Proposition 8.7 and Lemma 11.12 together imply

Proposition 11.13. $c(\ell, \ell')|_{\mathbf{F}(BString_{\mathbb{C}}(d))_A} \equiv 1.$ $\square$

11.D. **Trivializing line bundles of string bundles II: the global case.** Finally, we may draw the threads together, describing the line bundle given by the $\mathcal{E}$-sheaf of a Thom space and showing that there is a canonical trivialization for $String_{\mathbb{C}}$-bundles.

Theorem 11.14. *After identifying the $\mathcal{E}$-spaces*

$$\mathbf{F}(BSU(d)) \cong \mathfrak{X},$$

pull-back along the zero section identifies $\mathcal{F}(BSU(d)^V)$ with an ideal sheaf of $\mathcal{F}(BSU(d))$. For $A \subseteq \mathbb{T}$ we have given maps

$$\mu_A : \mathfrak{X}_A \to (\check{T} \otimes C)/W,$$

and so we have line bundles $\mathcal{L}_A$ over $\mathfrak{X}_A$, equipped with sections

$$f_A = \mu_A^* \sigma.$$

We have a canonical isomorphism

$$\gamma_{\mathbb{T}} : \mathcal{I}(f_{\mathbb{T}}) \cong \mathcal{F}(BSU(d)^V)_{\mathbb{T}}$$

and, for each lift ℓ as in (11.11)*, an isomorphism*

$$\gamma_\ell : \mathcal{L}_A \otimes \mathcal{I}(f_A) \cong \mathcal{F}(BSU(d)^V)_A.$$

$\square$

If X is a $\mathbb{T}$-space, and V is a $\mathbb{T}$-equivariant $SU(d)$-bundle over X, then the map

$$X \to BSU(d)$$

classifying V induces maps

$$h : \mathbf{F}(X) \to \mathfrak{X}$$

of $\mathcal{E}$-spaces, so we can form the line bundle

$$\mathcal{L}(V) \overset{\text{def}}{=} h^* \mathcal{L}$$

over $\mathbf{F}(X)$, equipped with the section

$$f(V) \overset{\text{def}}{=} h^* f.$$

Now suppose that V^0 and V^1 are two $\mathbb{T}$-equivariant $SU(d)$-bundles over X, (universally, we can take $X = BString_{\mathbb{C}}(d)$). Let $V = V^0 - V^1$ and set

$$\mathcal{L}(V) = \mathcal{L}(V^0) \otimes \mathcal{L}(V^1)^{-1},$$

and similarly for f and γ.

Theorem 11.15. *If $V = V^0 - V^1$ is a difference of $SU(d)$-bundles over X and $c_2^{\mathcal{B}}(V) = 0$ then the following conclusions hold.*

(1) *There is a canonical isomorphism trivialization of $\mathcal{L}(V)_{\mathbb{T}}$, so $f(V)_{\mathbb{T}}$ may be viewed as trivialization of $\mathcal{I}(f(V)_{\mathbb{T}})$, and $\gamma_{\mathbb{T}}(f(V)_{\mathbb{T}})$ as a trivialization of $\mathcal{F}(X^V)_{\mathbb{T}}$.*
(2) *For each finite A, the isomorphism*

$$\gamma_\ell : \mathcal{L}(V)_A \otimes \mathcal{I}(f(V)_A) \cong \mathcal{F}(X^V)_A,$$

is independent of ℓ; we call it γ_A.
(3) *Over $(\mathbf{F}(X)_{\mathbb{T}})^\wedge_{C\langle A\rangle}$, we have*

$$\mathcal{L}(V)_{\mathbb{T}} \cong \mathcal{L}(V)_A. \tag{11.16}$$

Using the trivialization of $\mathcal{L}(V)_{\mathbb{T}}$ to regard $\gamma_A f(V)_A$ as a section of $\mathcal{I}(f(V)_A)$, we have

$$\gamma_{\mathbb{T}} f(V)_{\mathbb{T}} = \gamma_A f(V)_A$$

in $(\mathcal{F}(X)_{\mathbb{T}})^\wedge_{C\langle A\rangle}$

The resulting section $\gamma f(V)$ of the $\mathcal{E}$-sheaf $\mathcal{F}(X^V)$ coincides with the Thom class $\psi(V)$ provided by Theorem 7.23.

Proof. The proof of the first part is essentially the same as the proof of Proposition 10.15. The second part follows from Proposition 11.13. For the third part, the isomorphism (11.16) follows from Proposition 11.7, and the rest is equivalent to the fact that

$$\delta^{\mathcal{B}}_A(V,a)|_{X^{\mathbb{T}}} = T_a^* e_{\mathbb{T}}(V),$$

which we proved as Corollary 8.14 and Proposition 8.15. The formulae for the sections $\gamma f(V)$ are the same as the formulae we have already given for $\psi(V)$. □

12. Cohomology of unitary groups and moduli spaces of divisors.

We give an account in terms of divisors of the analysis in Section 10. It is illuminating to do so for its own sake, and it indicates an approach to the $\mathbb{T}$-equivariant sigma orientation for an *algebraic* elliptic curve.

12.A. The classical non-equivariant description. The starting point is the observation that if $BU(d)$ denotes the *nonequivariant* classifying space for $U(d)$-bundles, then

$$\operatorname{spf} HP^0 BU(d) \cong \widehat{C}^d/\Sigma_d \cong \operatorname{Div}^d_+(\widehat{C})$$

is the scheme of effective divisors of degree d on $\widehat{C}$, the formal group of C. (We continue to work with an elliptic curve in the form

$$C = \mathbb{C}/\Lambda,$$

so the projection $\mathbb{C} \to C$ induces an isomorphism of formal groups $\widehat{C} \cong \widehat{\mathbb{G}}_a$.) Moreover, the determinant

$$BU(d) \to \mathbb{C}P^\infty$$

corresponds to the map

$$\operatorname{Div}^d_+(\widehat{C}) \to \widehat{C}$$

which sends a divisor $\sum(P)$ to $\sum^{\widehat{C}} P$, so

$$\operatorname{spf} HP^0 BSU(d) \cong \operatorname{Div}^d_+(\widehat{C})_0$$

is the scheme of effective divisors which sum to zero in $\widehat{C}$. (See Strickland [Str99]).

12.B. **Centralizers.** Now we consider the $\mathbb{T}$-equivariant classifying space. Let T be the maximal torus of $U(d)$, and let

$$m = (m_1, \ldots, m_d) : \mathbb{T} \to T$$

be a cocharacter, corresponding to a component $BZ(m)$ of $BU(d)$. Let us suppose that we have arranged m nondecreasing order, so it is of the form

$$\begin{aligned} m_{e_0+1} = \cdots = m_{e_1} &< m_{e_1+1} = \cdots = m_{e_2} \\ &< \cdots \\ &< m_{e_{r-1}+1} = \cdots = m_{e_r}, \end{aligned} \tag{12.1}$$

with $0 = e_0 < e_1 < \cdots < e_r = d$. It is convenient also to number this partition by setting $d_i = e_i - e_{i-1}$ for $1 \leq i \leq r$, so

$$\begin{gathered} d_i \geq 1 \\ \sum_{i=1}^{r} d_i = d. \end{gathered}$$

It is clear that every $m : \mathbb{T} \to T$ is conjugate to exactly one of this form, and so these suffice to describe $BU(d)^{\mathbb{T}}$. It is also easy to see that

$$Z(m) \cong U(d_1) \times \cdots \times U(d_r),$$

and so

$$\operatorname{spf} HP^0 BZ(m) \cong \operatorname{Div}_+^{d_1}(\widehat{C}) \times \cdots \operatorname{Div}_+^{d_r}(\widehat{C}) :$$

this is the scheme of r effective divisors $D_1, \ldots, D_r$, with $\deg D_i = d_i$. Another way to say this is that if we write the tautological divisor D over $\operatorname{Div}_+^d(\widehat{C}) \times C$ as

$$D = \sum_i P_i,$$

then the array m labels each point P_i with an integer m_i, and $\operatorname{spf} HP^0 BZ(m)$ is the scheme of effective divisors labelled with integers in this way.

Definition 12.2. Let $m : \mathbb{T} \to T$ be a cocharacter as in (12.1). We define

$$\operatorname{Div}_+^m(\widehat{C}) \overset{\text{def}}{=} \operatorname{Div}_+^{d_1}(\widehat{C}) \times \cdots \operatorname{Div}_+^{d_r}(\widehat{C}),$$

and we write D_m for the tautological divisor over $\operatorname{Div}_+^m(\widehat{C}) \times C$. If P is a point of D_m, then we write m_P for its integer label. We write

$$\operatorname{Div}_+^m(\widehat{C})_0 \overset{\text{def}}{=} \operatorname{Div}_+^m(\widehat{C}) \cap \operatorname{Div}_+^d(\widehat{C})_0$$

for the subscheme consisting of divisors which sum to zero in $\widehat{C}$.

Strickland's ideas, applied to our calculation of $H^* BSU(d)^{\mathbb{T}}$ in Proposition 4.7, imply the following.

Proposition 12.3. *Let T be the maximal torus of diagonal matrices in $SU(d)$, and let*

$$m : \mathbb{T} \to T$$

be a cocharacter, corresponding to a component $BZ(m)$ of $BSU(d)^{\mathbb{T}}$. Then

$$\operatorname{spf} HP^0 BZ(m) \cong \operatorname{Div}_+^m(\widehat{C})_0.$$

□

Definition 12.4. If P is a point of C and n is an integer, let $D(P, n)$ be the divisor

$$D(P, n) \stackrel{\text{def}}{=} \sum_{\{a \in C | na+P=0\}} (a).$$

Proposition 12.5. *Let*

$$m = (m_1, \ldots, m_d) : \mathbb{T} \to T$$

be a cocharacter of the form (12.1)*, labelling a component* $BZ(m)$ *of* $BSU(d)^{\mathbb{T}}$*. Let* V *be the tautological bundle over* $BZ(m)$*. If we write the tautological divisor over*

$$\operatorname{spf} HP^0 BZ(m) \times C \cong Div_+^m(\widehat{C}) \times C$$

as

$$D_m = \sum_i (P_i),$$

numbered so that $m_i = m_{P_i}$*, then* $EC^*_{\mathbb{T}}(BZ(m)^V)$ *is the cohomology of the the ideal sheaf of the divisor*

$$\sum_i D(P_i, m_i).$$

Proof. By Proposition 10.15, it is equivalent to show that

$$\operatorname{Div} f_m = \sum_i D(P_i, m_i).$$

The reduction m corresponds to decomposing V into isotypical summands according to the action of $\mathbb{T}$. The choice of P_i corresponds to using the splitting principle to decompose the tautological bundle V over $BSU(d)$ as

$$V|_{BZ(m)} \cong L_1 \otimes \mathbb{C}(m_1) \oplus \cdots L_d \otimes \mathbb{C}(m_d).$$

If $x_i = c_1 L_i$, then

$$f_m(x, z) = \prod_i \sigma(x_i + m_i z, \tau).$$

The result follows from the fact that σ vanishes to first order at the points of the lattice, and nowhere else. $\square$

12.C. **The global equivariant picture.** We now ask, what is the failure of $D_m = \operatorname{Div} f_m$ to be the divisor of a function on C? The Riemann-Roch Theorem gives two conditions.

Proposition 12.6. *Let* $m : \mathbb{T} \to T$ *be a cocharacter, corresponding to a component of* $BSU(d)^{\mathbb{T}}$*. Then*

$$\deg f_m = 2\phi(m),$$

and if we write

$$D_m = \sum_i P_i$$

as in Proposition 12.5, then D_m *sums to*

$$\sum^C m_i P_i$$

in the elliptic curve.

Proof. Let us examine a typical summand $D(P,n)$ of f_m. If Q is any point of C such that $nQ = P$, then

$$D(P,n) = \sum_{nb=0} (Q+b).$$

This shows that

$$\deg D(P,n) = n^2,$$

and it follows that

$$\deg f_m = 2\phi(m).$$

Meanwhile,

$$\sum_{nb=0}^{C} Q + b = nP + \sum_{nb=0}^{C} b = nP,$$

and so

$$\sum_{i}^{C} D(P_i, m_i) = \sum_{i}^{C} m_i P_i.$$

$\square$

Now suppose that for $i = 0, 1$, V^i is a $\mathbb{T}$-equivariant $BSU(d^i)$-bundle over a $\mathbb{T}$-fixed space X, and let $\xi = V^0 - V^1$. Suppose for simplicity that $H^*(X)$ is concentrated in even degrees, and let

$$\mathfrak{D} = \operatorname{spf} HP^0X.$$

Let D^i be the divisor on $\mathfrak{D} \times C$ which is obtained by pulling back along the map

$$\mathfrak{D} \times C \to (\check{T}^i \otimes \widehat{C})/W^i \times C.$$

Then we have the following.

Theorem 12.7. *If $c_1^{\mathcal{B}}(\xi) = 0 = c_2^{\mathcal{B}}(\xi)$, then $D^0 - D^1$ is the divisor of a meromorphic function on the elliptic curve $\mathfrak{D} \times C$ over $\mathfrak{D}$, and this meromorphic function is a trivialization of $EC^*_{\mathbb{T}}(X^V)$ as an $EC^*_{\mathbb{T}}(X)$-module.*

Proof. Let m^i be a reduction of the action of $\mathbb{T}$ on V^i. By Lemma 4.13, the characteristic class restrictions imply that

$$\phi(m^0) = \phi(m^1)$$

and that

$$\sum_{P\in D^0}^{C} m_P^0 P = \sum_{Q\in D^1}^{C} m_Q^1 Q.$$

$\square$

Appendix A. On the relationship between Borel homology and cohomology.

In the appendix we work with coefficents in k, so that the coefficient ring of Borel cohomology is $H^*(B\mathbb{T}) = H^*(B\mathbb{T}; k) \cong k[c]$. In our applications, k will be a field, but we make this assumption explicitly where necessary.

The naive Kronecker pairing

$$H^p_{\mathbb{T}}(X; N) \to \operatorname{Hom}(H_p^{\mathbb{T}} X, N)$$

relating $\mathbb{T}$-equivariant Borel homology and cohomology with coefficients in a k-module N fails to take account of the coefficient ring $H^*(B\mathbb{T}) = H^*(B\mathbb{T}; k) \cong k[c]$. In this section we construct a Kronecker pairing which does reflect this structure. To see what such a sequence might look like, we note that the generator $c \in H^2 B\mathbb{T}$ lowers degree in $H_*^{\mathbb{T}} X$, and so all of $H_*^{\mathbb{T}} X$ is c-torsion. Thus in order to get a reasonable answer, one might hope for a map

$$\kappa : H^p_{\mathbb{T}}(X) \to \mathrm{Hom}^p_{H^* B\mathbb{T}}(H_*^{\mathbb{T}} X, H_* B\mathbb{T}),$$

and we construct such a map.

1. **Some algebra.** Suppose that M is a (graded) $H^* B\mathbb{T}$-module. Let

$$\Gamma_{(c)} M = \{r \in M | c^k r = 0 \text{ for some } k\}$$

be the subgroup of c-power torsion in M. The *local cohomology groups* of M are defined by the exact sequence

$$0 \longrightarrow H^0_{(c)}(M) \longrightarrow M \longrightarrow M[c^{-1}] \longrightarrow H^1_{(c)}(M) \longrightarrow 0,$$

and Grothendieck observed [Gro67] that they calculate the right derived functors of c-power torsion:

$$H^*_{(c)}(M) = R^* \Gamma_{(c)} M.$$

For example, if M is torsion free,

$$H^1_{(c)}(M) = M[1/c]/M,$$

and $H^0_{(c)} M = 0$. A special case is instructive.

Example A.1. For $M = H^*(B\mathbb{T})$ we have

$$H^1_{(c)}(H^*(B\mathbb{T})) = k[c, c^{-1}]/k[c] \cong \Sigma^2 H_* B\mathbb{T}.$$

However, note that the second isomorphism is not natural for ring automorphisms. The natural statement, given by the residue, involves the Kähler differentials:

$$H^1_{(c)}(H^*(B\mathbb{T})) \otimes_{H^*(B\mathbb{T})} \Omega^1_{H^*(B\mathbb{T})/k} \cong H_*(B\mathbb{T}).$$

In more concrete terms, if α is the automorphism multiplying c by λ, α multiplies $H_{2s}(B\mathbb{T})$ by λ^{-s}, and the part of $H^1_{(c)}(H^*(B\mathbb{T}))$ in the corresponding degree (viz $2+2s$) by λ^{-s-1}.

Note too that, if k is a field, then graded $H^*(B\mathbb{T})$-modules are injective if and only if they are divisible. If in addition M is torsion free, then the sequence

$$0 \longrightarrow M \longrightarrow M[1/c] \longrightarrow H^1_{(c)}(M) \longrightarrow 0$$

is an injective resolution, giving the following calculation.

Lemma A.2. *If the coefficient ring k is a field, L is a torsion module and M is torsion free, we have*

$$\mathrm{Ext}^1_{H^* B\mathbb{T}}(L, M) \cong \mathrm{Hom}_{H^* B\mathbb{T}}(L, H^1_{(c)}(M)). \quad \square$$

2. **The construction.** Let Hk denote the inflation (in the sense of Elmendorf-May [EM97]) of the nonequivariant spectrum representing ordinary cohomology with coefficients in the commutative ring k, and let $Hb = F(E\mathbb{T}_+, Hk)$ be the spectrum representing Borel cohomology with coefficients in k. These are both strictly commutative ring spectra, so we may consider the triangulated homotopy category of modules over them. Let HM be an Hb-module spectrum with $\pi_*^{\mathbb{T}}(HM) = M$; existence and uniqueness are easily checked when k is a field, from the fact that the coefficient ring is of injective dimension 1 and in even degrees. Borel cohomology with coefficients in M is defined as usual by

$$H^p_{\mathbb{T}}(X; M) = [Hb \wedge X_+, \Sigma^p HM]_{Hb,\mathbb{T}} \cong [X_+, \Sigma^p HM]_{\mathbb{T}}.$$

Remark A.3. To avoid confusion, we highlight some distinctions. Firstly, if N is a k-module one may consider the usual cohomology groups

$$H^*_{\mathbb{T}}(X;N) = H^*(E\mathbb{T} \times_{\mathbb{T}} X;N)$$

of the Borel construction. This does not coincide with $H^*_{\mathbb{T}}(X;\epsilon^*N)$ where $\epsilon : k[c] \longrightarrow k$ is the augmentation, but in practice no confusion should arise.

The second distinction is more important. If I is an injective $k[c]$-module we may define a Brown-Comenetz type cohomology theory

$$H^*_{\mathbb{T}}(X;I)_{BC} = \mathrm{Hom}_{H^*(B\mathbb{T})}(H_*^{\mathbb{T}}(X), I).$$

Note that this is quite different from $H^*_{\mathbb{T}}(X;I)$. For example

$$H^*_{\mathbb{T}}(pt;H_*(B\mathbb{T})) = H_*(B\mathbb{T}) \ncong H^*(B\mathbb{T}) = H^*_{\mathbb{T}}(pt;H_*(B\mathbb{T}))_{BC}.$$

Our present notation conflicts with that used in [Gre05], where the Brown-Comenetz type theory was used without the subscript BC.

Given a map of Hb-module $\mathbb{T}$-spectra

$$f : Hb \wedge X_+ \to \Sigma^p HM,$$

we form

$$Hb \wedge X_+ \wedge ET_+ \xrightarrow{f \wedge 1} \Sigma^p HM \wedge ET_+. \tag{A.4}$$

Now apply $\pi_*^{\mathbb{T}}$. The association $f \mapsto \pi_*^{\mathbb{T}}(f \wedge 1)$ gives a function

$$H^p_{\mathbb{T}}(X;M) \to \mathrm{Hom}_{H^*B\mathbb{T}}(\pi_*^{\mathbb{T}}(Hb \wedge X_+ \wedge ET_+), \pi_*^{\mathbb{T}}(\Sigma^p HM \wedge ET_+)).$$

To interpret the homotopy of the domain of (A.4), use the Adams isomorphism [Ada84, LMSM86]: if A is $\mathbb{T}$-fixed and B is $\mathbb{T}$-free, we have

$$[A,B]^{\mathbb{T}} = [A, \Sigma B/\mathbb{T}]. \tag{A.5}$$

Remark A.6. The suspension in (A.5) arises as smashing with S^{ad}, the Thom space of the adjoint representation of $\mathbb{T}$ on its Lie algebra.

In our setting, the Adams isomorphism gives

$$\pi_*^{\mathbb{T}}(Hb \wedge X_+ \wedge E\mathbb{T}_+) \cong H^{\mathbb{T}}_{*-1}(X) \cong \Sigma H_*^{\mathbb{T}}(X).$$

To understand the homotopy of the target of (A.4), note that, since c is the Euler class of the natural representation, applying $\pi_*^{\mathbb{T}}$ to the cofibre sequence

$$HM \wedge E\mathbb{T}_+ \to HM \wedge S^0 \to HM \wedge \widetilde{E}\mathbb{T}$$

gives a triangle

$$\pi_*^{\mathbb{T}}(HM \wedge E\mathbb{T}_+) \to M \to M[c^{-1}].$$

Thus, if M is in even degrees we find

$$\pi_*^{\mathbb{T}}(HM \wedge E\mathbb{T}_+) \cong H^*_{(c)}M,$$

and we have a map

$$\kappa : H^p_{\mathbb{T}}(X;M) \to \mathrm{Hom}_{H^*B\mathbb{T}}(\Sigma H_*^{\mathbb{T}}X, H^*_{(c)}M).$$

It is easiest to sort out the gradings by example. If c is regular in M, then

$$\pi_k^{\mathbb{T}}(\Sigma^p HM \wedge E\mathbb{T}_+) \cong (M[c^{-1}]/M)_{k-p+1},$$

and we have given a map

$$H^p_{\mathbb{T}}(X;M) \to \mathrm{Hom}_{H^*B\mathbb{T}}(\Sigma H_*^{\mathbb{T}}X, \Sigma^{p-1}(M[c^{-1}]/M))$$

or

$$H^p_{\mathbb{T}}(X;M) \to \mathrm{Hom}^p_{H^*B\mathbb{T}}(H^{\mathbb{T}}_* X, \Sigma^{-2}(M[c^{-1}]/M)). \tag{A.7}$$

3. **Isomorphisms.** Suppose now that k is a field. Since $M[c^{-1}]/M$ is an injective $H^*B\mathbb{T}$-module, both the left and right sides of (A.7) are cohomology theories in X, and we have the following.

Proposition A.8. *If M is torsion free and complete, the Kronecker pairing is an isomorphism*

$$\kappa : H^p_{\mathbb{T}}(X;M) \xrightarrow{\cong} \mathrm{Hom}^p_{H^*B\mathbb{T}}(H^{\mathbb{T}}_* X, \Sigma^{-2}(M[c^{-1}]/M)).$$

Proof. We must show that the map is an isomorphism when $X = S^k \wedge \mathbb{T}/A_+$ for all subgroups A and all integers k. First, note that the map is an isomorphism when $X = S^0$: here we have the map

$$M \longrightarrow \mathrm{Hom}_{H^*B\mathbb{T}}(H_*(B\mathbb{T}), \Sigma^{-2}(M[c^{-1}]/M)).$$

The fact that this is an isomorphism when M is complete is essentially local duality, but can be seen directly since the codomain is

$$\begin{array}{rcl} \mathrm{Hom}_{H^*B\mathbb{T}}(H_*(B\mathbb{T}), \Sigma^{-2}M[c^{-1}]/M) & \cong & \mathrm{Ext}^1_{H^*B\mathbb{T}}(H_*(B\mathbb{T}), \Sigma^{-2}M) \\ & \cong & \mathrm{Ext}^1_{H^*B\mathbb{T}}(\mathrm{colim}_s(\mathrm{ann}(c^s, \Sigma^2 H_*(B\mathbb{T})), M) \\ & \cong & \lim_s M/(c^s)M \end{array}$$

since $\mathrm{ann}(c^s, \Sigma^2 H_*(B\mathbb{T})) \cong \Sigma^{2s} k[c]/(c^s)$. Using suspension isomorphisms on both sides, we obtain the result for all spectra $X = S^k$.

Now, moving into topology, the Thom isomorphism implies that we have an isomorphism when $X = S^k \wedge S^V$, where V is a complex representation of $\mathbb{T}$. If A is cyclic of finite order n, then we have the cofibre sequence

$$\mathbb{T}/A_+ \to S^0 \to S^{z^n},$$

and so an isomorphism for cells of the form $S^k \wedge \mathbb{T}/A_+$ for general A. □

The simplest example is when $M = H^*(B\mathbb{T})$.

Example A.9. We have the isomorphism

$$H^*_{\mathbb{T}}(X; H^*(B\mathbb{T})) \cong \mathrm{Hom}_{H^*B\mathbb{T}}(H^{\mathbb{T}}_*(X), H_*(B\mathbb{T}) \otimes_{H^*(B\mathbb{T})} (\Omega^1_{H^*(B\mathbb{T})|k})^{-1})$$

where the Kähler differentials can be omitted if only the $H^*(B\mathbb{T})$-module structure is relevant. □

Finally, we specialize to the case of interest to us, arising from the geometry of an elliptic curve over a field of characteristic 0.

Example A.10. Now suppose that, as in Section 6, we have an elliptic curve C over a field k of characteristic 0, and coordinate data t_1. For $n \geq 1$, the function t_n vanishes to the first order on points of exact order n (and nowhere else), so that if we let c act on $\mathcal{O}^{\wedge}_A \otimes \omega^*$ via t_n/Dt we have

$$H^1_{(c)}(\mathcal{O}^{\wedge}_A \otimes \omega^*) \cong H^1_{(t_n)}(\mathcal{O}^{\wedge}_A) \otimes \omega^* \cong T_A C \otimes \omega^*.$$

In any case, $T_A C \otimes \omega^*$ is a divisible torsion $H^*B\mathbb{T}$-module, isomorphic to a finite sum of copies of $H_*B\mathbb{T}$. Applying Proposition A.8, we have the natural isomorphism

$$H^*_{\mathbb{T}}(X; \mathcal{O}^{\wedge}_A \otimes \omega^*) \cong \mathrm{Hom}_{H^*B\mathbb{T}}(H^{\mathbb{T}}_*(X), T_A C \otimes \omega^*). \quad \square$$

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