Sains Malaysiana 42(3)(2013): 347–358

A Step Variational Iteration Method for Solving Non-Chaotic and Chaotic Systems

(Kaedah Lelaran Ubahan Langkah bagi Menyelesaikan Sistem Kalut dan Tak Kalut)

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ABSTRACT

In this paper, a new reliable method called the step variational iteration method (SVIM) based on an adaptation of the variational iteration method (VIM) is presented to solve non–chaotic and chaotic systems. The SVIM uses the general Lagrange multipliers for constructing the correction functional for the problems. The SVIM yields a step analytical solution of the form of a rapidly convergent infinite power series with easily computable terms and obtain a good approximate solution for larger intervals. The accuracy of the presented solution obtained is in an excellent agreement with the previously published solutions.

Keywords: Chaotic and non-chaotic systems; Lagrange multiplier; multistage variational iteration method; step variational iteration method; variational iteration method

ABSTRAK

Dalam kertas ini, kaedah baru dinamakan kaedah lelaran ubahan langkah (KLUL) berasaskan satu adaptasi kaedah lelaran ubahan digunakan untuk sistem tak-kalut dan kalut. KLUL menggunakan pendarab Lagrange umum untuk membina fungsian pembetulan bagi mengatasi masalah berkenaan. KLUL menghasilkan penyelesaian analisis dalam bentuk siri kuasa tak terhingga yang menumpu pantas dengan sebutan yang mudah dikira. Penyelesaian penghampiran diperoleh adalah baik untuk selang yang lebih besar. Ketepatan penyelesaian yang diperoleh adalah sangat baik bila dibandingkan dengan penyelesaian yang terdahulu.

Kata kunci: Kaedah lelaran ubahan; kaedah lelaran ubahan langkah; kaedah lelaran ubahan multitahap; sistem tak-kalut dan kalut; pendarab Lagrange

INTRODUCTION

Most scientific problems and phenomena are modelled by non-chaotic and chaotic systems which have been developed and analysed over the past years. A chaotic system is a nonlinear deterministic system having complex and unpredictable behaviour (Werndl 2009). Observations of chaotic behaviour in nature include changes in weather, the dynamics of satellites in the solar system, the time evolution of the magnetic field of celestial bodies, population growth in ecology and the dynamics of the action potentials in neurons and molecular vibrations (Ivancevic & Ivancevic 2008). Research is still active over the existence of chaotic dynamics in plate tectonics and economics (Serletis & Gogas 1997, 1999, 2000).

The exact solutions of most of the chaotic system cannot be found easily because chaotic systems are highly sensitive to initial conditions. Thus, semi-analytical and numerical methods are used as alternatives. Since chaotic systems are highly sensitive on initial conditions especially for a large time interval, most researches use multistage techniques to overcome this problem. In relation to this study, Noorani et al. (2007) used the multistage Adomian decomposition method (MADM) to obtain the solution of the non-chaotic and chaotic Chen systems. Furthermore, Chowdhury and Hashim (2009) applied the multistage homotopy perturbation method (MHPM) to solve the Chen systems. In the same year, Chowdhury et al. (2009) used MHPM to obtain an excellent approximate solution of the Lorenz system and Alomari et al. (2009) used multistage homotopy analysis method to gain a good approximate solution of the non-chaotic and chaotic Chen system until $t \in [0,10]$.

Another powerful method is the variational iteration method (VIM). It was proposed by He (1997a, 1997b) and other researchers had applied VIM to solve various problems (He 1997a, 1997b, 1998, 2007; He & Wu 2006, 2007). Some highlights of recent developments of VIM include the work by Yulita Molliq et al. (2009a, 2009b) that solved the fractional heat and wave-like equation, fractional Zakharov-Kuznetsov equations whereby good approximate solutions to the respective equations have been obtained. Faraz et al. (2010, 2011) used VIM to help solve the two-dimensional viscous flow with a shrinking sheet and differential-difference equation. Khan et al. (2011) used VIM with modified Riemann-Liouville's fractional derivatives approach for solving fractional initial-boundary value problems arising in the application of nonlinear science. Recently, Rangkuti and Noorani (2012) got the exact solution of delay differential equation using VIM by ignoring small terms and Yulita Molliq and

Batiha (2012) solved fractional Zakharov-Kuznetsov using VIM with fractional complex transform approach.

The solution of the chaotic system using VIM is still a hot topic for research, since the standard VIM cannot solve the problem for longer time. To overcome this shortcoming, Batiha et al. (2007b) introduced a new method, the so-called multistage variational iteration method (MVIM) which was very successful to solve a class of nonlinear system of ODEs. Afterward, Goh et al. (2009a, 2009b) used MVIM to solve non-chaotic and chaotic Rössler system and chaotic Genesio system, respectively. Nevertheless, MVIM has a particular disadvantage that it requires a longer computational time (Batiha et al. 2007b).

Alomari et al. (2010) introduced step homotopy analysis method (SHAM) which yields an approximate analytical solution in terms of a rapidly convergent infinite power series with easily computable terms for solving fractional Lorenz system. Inspired and motivated by Alomari et al. (2010), we modify the standard VIM by step sizes and adopt the SHAM technique in this paper and call it the step variational iteration method (SVIM). In this paper, SVIM was used to solve dynamical system with chaotic behaviour and non-chaotic behaviour. In particular, the Rössler and Genesio systems are considered to demonstrate the efficiency of the new method and the fourth–order Runge-Kutta method (RK4) and MVIM are used for comparison.

VARIATIONAL ITERATION METHOD

Consider the following general system of first-order ordinary differential equations (ODEs):

$$\frac{du_i(t)}{dt} = f_i\left(t, u_1(t), \ldots, u_m(t)\right), \tag{1}$$

where f_i are (linear or non–linear) real–valued functions, $u_i(t_0) = c_i$, $i = 1, 2, \ldots, m$ are the initial conditions and $c_i \in R$ are arbitrary number.

To illustrate the basic idea of VIM, the following nonlinear differential equation is presented below:

$$Lu_i + Nu_i = g_i, \tag{2}$$

where L is a linear operator, N is a nonlinear operator and g is an inhomogeneous term. According to VIM, one can construct a correction functional as follows:

$$u_{i,n+1}(t) = u_{i,n}(t) + \int_0^t \lambda_i(\xi)\left[Lu_{i,n}(\xi) + N\tilde{u}_{i,n}(\xi) - g_{i,n}(\xi)\right]d\xi, \tag{3}$$

where λ_i, $i = 1, 2, \ldots, m$ is the general Lagrange multiplier which can be identified optimally via the variational theory (Inokuti et al. 1978), $\tilde{u}_{i,n}$ is considered as restricted variations (Finlayson 1972), i.e. $\delta\tilde{u}_{i,n} = 0$. The subscript n denotes the n^{th} approximation. The approximate solution takes the form:

$$u_i(t) \approx u_{i,n}(t), \quad i = 1, 2, \ldots, m. \tag{4}$$

where n is the final iteration step.

It has been shown that the approximate solutions for a class of system of ODEs are not valid for large t (Batiha et al. 2007a). Goh et al. (2008, 2009a, 2009b) have shown that the standard VIM is not reliable for chaotic systems (related to some biological dynamical systems and Rössler and Genesio system) since their accuracy is only valid on a very short time span. Due to this unreliability, many modifications to VIM have been done by researchers in order to improve its efficiency such as MVIM. Even though MVIM has been proven to be an effective method in solving many systems (Goh et al. 2008, 2009a, 2009b), its weakness lies in long computational time and it still not valid for longer time interval t. Therefore, we shall introduce a modification of VIM based on step technique for better accuracy and efficiency purposes.

STEP VARIATIONAL ITERATION METHOD

In this section, we shall now look at how this new modification of VIM works to find the approximate solution for longer time span t. Here, interval $[0, T]$ is regarded as an interval, then the simple idea is to divide the interval to subintervals with time step t and the solution at each subinterval of (1) will be obtained. It is necessary to satisfy the initial condition at each of the subinterval (Alomari et al. 2010). Accordingly, the initial values $u_{1,0}, u_{2,0}, \ldots, u_{m,0}$ will be changed for each subinterval, i.e. and it should be satisfied through the initial conditions $u_{i,n}(t^*) = 0$ for all $n \geq 1$. The new formulas will be calculated recursively. Thus, the formula can be written as:

$$u_{i,n+1}(t) = u_{i,n}(t) + \int_0^{t-t^*} \lambda_i(\xi)\,[Lu_{i,n}(\xi) + Nu_{i,n}(\xi) - g_i(\xi)]d\xi, \tag{5}$$

where L is linear operator, N is nonlinear operator and g_i is inhomogeneous term for $i = 0, 1, 2, \ldots, m$. Notice that this strategy gives a new construction of the correction functional (5) with variable $t - t^*$ as the upper limit of the integration instead of a fixed upper limit of t in (3). The fixed limits are the same as used in the classical VIM which can be seen in (He 1997a, 1997b, 1998, 2007; He & Wu 2006, 2007). The approximate solution takes the form:

$$u_i(t) \approx u_{i,n}(t - t^*), \quad i = 1, 2, \ldots, m, \tag{6}$$

where t^* starting from $t_0 = 0$ until $t_j = T$, J is the number of subinterval. To carry out the solution on every subinterval of equal length Δt, the values of the following initial conditions are shown below:

$$c_i^* = u_i(t^*), \quad i = 1, 2, \ldots, m. \tag{7}$$

In general, we do not have these information of our clearance except at the initial point $t^* = t_0 = 0$, but these values can be obtained by assuming that the new initial condition is the solution in previous interval (i.e. if the solution in interval $[t_j, t_{j+1}]$ is necessary, then the initial conditions of this interval will be as follows:

$$c_i = u_i(t) \approx u_{i,n}(t_j, t_{j-1}), \tag{8}$$

where c_i, $i = 0, 1, \ldots, m$ are the initial conditions in the interval $[t_j, t_{j+1}]$.

APPLICATIONS TO RÖSSLER SYSTEM

To demonstrate the accuracy of SVIM, two examples of non–chaotic and chaotic systems have been implemented. SVIM will be compared with MVIM and RK4, respectively. The first system can be written in the following form (Rössler 1976):

$$\frac{du}{dt} = -v - w, \tag{9}$$

$$\frac{dv}{dt} = u + \alpha v, \tag{10}$$

$$\frac{dw}{dt} = \beta + uw - \gamma w, \tag{11}$$

where u, v, w are the state variables and α, β, γ are positive constants.

To solve (9) to (11), SVIM will be applied through 5 steps as follows which are made lucid for the Rössler.

Step 1. First, the correction functional is constructed as used by VIM to find the Lagrange multiplier in the following forms:

$$u_{n+1} = u_n + \int_0^t \lambda_1(\xi)\left[\frac{du_n}{d\xi} + v_n + w_n\right]d\xi, \tag{12}$$

$$v = v_n + \int_0^t \lambda_2(\xi)\left[\frac{dv_n}{d\xi} - u_n + \alpha v_n\right]d\xi, \tag{13}$$

$$w_{n+1} = w_n + \int_0^t \lambda_3(\xi)\left[\frac{dw_n}{d\xi} - \beta - \tilde{u}_n\tilde{w}_n + \gamma v_n\right]d\xi, \tag{14}$$

where λ_1, λ_2 and λ_3 are the general Lagrange multipliers and $\tilde{u}_n, \tilde{v}_n, \tilde{w}_n, \tilde{u}_n, \tilde{w}_n$ considered as restricted variations, i.e. by taking variation with respect to the independent variables u_n, v_n and w_n the following forms can be obtained below:

$$\delta u_{n+1} = \delta u_n + \delta\int_0^t \lambda_1(\xi)\left[\frac{du_n}{d\xi} + \tilde{v}_n + \tilde{w}_n\right]d\xi, \tag{15}$$

$$\delta v_{n+1} = \delta v_n + \delta\int_0^t \lambda_2(\xi)\left[\frac{dv_n}{d\xi} - \tilde{u}_n - \alpha v_n\right]d\xi, \tag{16}$$

$$\delta w_{n+1} = \delta w_n + \delta\int_0^t \lambda_3(\xi)\left[\frac{dw_n}{d\xi} - \beta - \tilde{u}_n\tilde{w}_n + \gamma v_n\right]d\xi. \tag{17}$$

Making each of the correction functional in (15) to (17) stationary and also observe that $\delta\tilde{u}_n = \delta\tilde{v}_n = \delta\tilde{w}_n = \delta\tilde{u}_n\tilde{w}_n = 0$, therefore the three sets of stationary conditions can be obtained for each λ_i, $i = 1, 2, 3$. The general Lagrange multipliers therefore can be easily identified as $\lambda_1 = -1$, $\lambda_2 = -e^{\alpha(t-\xi)}$ and $\lambda_3 = -e^{\gamma(\xi-t)}$.

Step 2. Secondly, each general Lagrange multiplier is substituted from (11) to (13) then, the following iteration formulas will be obtained:

$$u_{n+1} = u_n - \int_0^t \left[\frac{du_n}{d\xi} + v_n + w_n\right]d\xi, \tag{18}$$

$$v_{n+1} = v_n - \int_0^t e^{\alpha(t-\xi)}\left[\frac{dv_n}{d\xi} - u_n - \alpha v_n\right]d\xi, \tag{19}$$

$$w_{n+1} = w_n - \int_0^t e^{\gamma(\xi-t)}\left[\frac{dw_n}{d\xi} - \beta - u_n w_n + \gamma v_n\right]d\xi, \tag{20}$$

Step 3. The interval [0,20] is divided to subintervals with the time step (Δt) to obtain the solution at each subinterval. In this case, the initial condition is satisfied at each of the subinterval (Alomari et al. 2010), i.e. $u(t^*) = c_1^* = u_0$, $v(t^*) = c_2^* = v_0$ and $w(t^*) = c_3^* = w_0$ and the initial conditions should be satisfied $u_n(t^*) = 0$, $v_n(t^*) = 0$ and $w_n(t^*) = 0$ for all $n \geq 1$. So, (18) to (20) can be written as:

$$u_{n+1} = u_n - \int_0^{t-t^*} \left[\frac{du_n}{d\xi} + v_n + w_n\right]d\xi, \tag{21}$$

$$v_{n+1} = v_n - \int_0^{t-t^*} e^{\alpha(t-\xi)}\left[\frac{dv_n}{d\xi} - u_n - \alpha v_n\right]d\xi, \tag{22}$$

$$w_{n+1} = w_n - \int_0^{t-t^*} e^{\gamma(\xi-t)}\left[\frac{dw_n}{d\xi} - \beta - u_n w_n + \gamma v_n\right]d\xi., \tag{23}$$

The above formula is calculated recursively.

Step 4. The other components are obtain as follow:

$$u_1 = c_1 - (c_2 - c_3)(t - t^*), \tag{24}$$

$$v_1 = (5c_1 + c_2)\, e^{0.2(t-t^*)} - 5c_1, \tag{25}$$

$$w_1 = c_3 + \frac{0.2}{\gamma}\left[\left(-1 - 5c_3c_1 + 5\gamma c_3\right)e^{0.2(t-t^*)} + 1 + 5c_3c_1 - 5\gamma c_3\right], \tag{26}$$

$$\begin{aligned} u_2 &= c_1 - c_2(t-t^*) - c_3(t-t^*) - \frac{0.2}{\gamma}\,[-1-125c_1\gamma^2 - 25c_2\gamma^2 \\ &+ (125c_1 + 25c_2)e^{0.2(t-t^*)}\gamma_2 + (1 + 5c_3c_1 - 5\gamma c_3)e^{-\gamma(t-t^*)} \\ &+ (\gamma - 25c_1\gamma^2 + 5c_1c_3\gamma - 5c_3\gamma^2 - 5c_2\gamma^2)(t-t^*)] - 5c_3c_1 \\ &+ 5\gamma c_3, \end{aligned} \tag{27}$$

$$\begin{aligned} v_2 &= (5c_1 - 24c_2 - 25c_3)e^{0.2(t-t^*)} - 5c_1 + 25c_2 + 25c_3 \\ &- [4.167c_1 + 2.028c_2 - 0.8c_3 - 0.694c_1^{\,2}]e^{-12(t-t^*)} \\ &+ (5c_2 + 5c_3)(t-t^*), \end{aligned} \tag{28}$$

$$w_2 = \frac{0.2}{\gamma}[-e^{-\gamma(t-t^*)} - 5e^{-\gamma(t-t^*)}c_3c_1 + 5\gamma c_3 e^{-\gamma(t-t^*)} + 5c_1c_3]$$
$$-\frac{0.1}{\gamma^2}[2c_2+10c_1c_3^{\,2} + 2c_3 + 10c_1c_2c_3)\gamma(t-t^*)e^{\gamma(t-t^*)}$$
$$-(10c_1c_3\gamma^2 - 2c_1\gamma - 10c_1^{\,2}c_3\gamma - 3c_3 - 10c_1c_3^{\,2} - 10c_1c_2c_3,$$
$$-2c_1)e^{\gamma(t-t^*)} - (10c_1c_3\gamma^3 - 10c_3c_1^{\,2} - 2c_1\gamma^2)(t-t^*)$$
$$+(5c_3c_2\gamma^2 - 5c_2^{\,3}c_1\gamma^2 - 5c_3c_1^{\,2}\gamma^2 - c_3\gamma^2$$
$$+ 5c_2^{\,3}\gamma^3)(t-t^*)^2 - 10\gamma^2c_1c_3 + 10\gamma c_1^{\,2}c_3 + 2\gamma c_1 + 2c_1$$
$$+ 2c_3 + 10c_1c_3^{\,2} + 10c_1c_2c_3 + \frac{0.2}{\gamma}, \tag{29}$$

Step 5. The iteration of SVIM will be chosen until second iteration as in (Goh et al. 2009b) and written as:

$$u(t) \approx u_2(t - t^*), \tag{30}$$

$$v(t) \approx v_2(t - t^*), \tag{31}$$

$$w(t) \approx w_2(t - t^*), \tag{32}$$

where t^* start from $t_0 = 0$ until $t_J = T = 20$. To carry out the solution on every subinterval of equal length Δt, the values of the following initial conditions are presented below:

$$c_1 = u(t^*),\ c_2 = v(t^*),\ c_3 = w(t^*). \tag{33}$$

In general, these information at our disposal cannot be obtained except at the initial point $t^* = t_0 = 0$, but these values can also be achieved by assuming that the new initial condition is the solution in previous interval (i.e. if the solution in interval $[t_j, t_{j+1}]$ is required, then the initial conditions of this interval will be as the following:

$$c_1 = u(t) \approx u_2(t_j, t_{j-1}), \tag{34}$$

$$c_2 = v(t) \approx v_2(t_j, t_{j-1}), \tag{35}$$

$$c_3 = w(t) \approx w_2(t_j, t_{j-1}), \tag{36}$$

where c_1, c_2, c_3 are the initial conditions in the interval $[t_j, t_{j+1}]$).

APPLICATION TO GENESIO SYSTEM

Consider chaotic Genesio system which is written in the form (Genesio & Tesi 1992):

$$\frac{du}{dt} = v, \tag{37}$$

$$\frac{dv}{dt} = w, \tag{38}$$

$$\frac{dw}{dt} = cu + bv + aw - u^2, \tag{39}$$

where *a, b* and *c* are constants, satisfying $ab<c$.

To solve (36) to (38), again SVIM shall be applied through 5 steps as follow. First step, one constructs the correction functional as in VIM to find the general Lagrange multiplier as written in the following forms:

$$u_{n+1} = u_n + \int_0^t \lambda_1\left(\frac{du_n}{d\xi} - \tilde{v}_n\right)d\xi, \tag{40}$$

$$v_{n+1} = v_n + \int_0^t \lambda_2\left(\frac{dv_n}{d\xi} - \tilde{w}_n\right)d\xi, \tag{41}$$

$$w_{n+1} + w_n + \int_0^t \lambda_3\left(\frac{dw_n}{d\xi} - c\tilde{u}_n - b\tilde{v}_n - aw_n + \tilde{u}_n^2\right)d\xi, \tag{42}$$

where λ_1, λ_2, λ_3 are the general Lagrange multipliers and $\tilde{u}_n, \tilde{v}_n, \tilde{w}_n, \tilde{u}_n^2$ denoted as restricted variations, i.e. $\delta\tilde{u}_n = \delta\tilde{v}_n = \delta\tilde{w}_n = \delta\tilde{u}_n^2 = 0$. By taking variation with respect to the independent variables u_n, v_n and w_n the following form the results:

$$\delta u_{n+1} = \delta u_n + \delta\int_0^t \lambda_1\left(\frac{du_n}{d\xi} - \tilde{v}_n\right)d\xi, \tag{43}$$

$$\delta v_{n+1} = \delta v_n + \delta\int_0^t \lambda_2\left(\frac{dv_n}{d\xi} - \tilde{w}_n\right)d\xi, \tag{44}$$

$$\delta w_{n+1} = \xi w_n + \delta\int_0^t \lambda_2\left(\frac{dw_n}{d\xi} - c\tilde{u}_n - b\tilde{v}_n - a\tilde{w}_n + \tilde{u}_n^2\right)d\xi, \tag{45}$$

For making each of the correction functional in (43) to (45) stationary and we also observe that $\delta\tilde{u}_n = \delta\tilde{v}_n = \delta\tilde{w}_n = \delta\tilde{u}_n^2 = 0$. Therefore, the three sets of stationary conditions can be obtained for each λ_i, $i = 1, 2, 3$. The Lagrange multipliers can be easily identified as $\lambda_1 = -1$, $\lambda_2 = -1$ and $\lambda_3 = -e^{a(\xi-t)}$, respectively. Second step, each general Lagrange multiplier is substituted from (40) to (42) and the following iteration formulas are obtained:

$$u_{n+1} = u_n - \int_0^t\left(\frac{du_n}{d\xi} - v_n\right)d\xi, \tag{46}$$

$$v_{n+1} = v_n - \int_0^t\left(\frac{dv_n}{d\xi} - w_n\right)d\xi, \tag{47}$$

$$w_{n+1} = w_n - \int_0^t e^{a(\xi-t)}\left(\frac{dw_n}{d\xi} - cu_n - bv_n - aw_n + u_n^2\right)d\xi, \tag{48}$$

Third step, the interval [0,20] is divided to subintervals with time step (Δt) to obtain the solution at each subinterval. In this case, the initial condition is satisfied at each of the subinterval (Alomari et al. 2010), i.e. $u(t^*) = c_1^* = u_0$, $v(t^*) = c_2^* = v_0$, and $w(t^*) = c_3^* = w_0$, and the initial conditions should be satisfied $u_n(t^*) = 0$, $v_n(t^*) = 0$, and $w_n(t^*) = 0$ for all $n \geq 1$. So, (46) to (48) can be written as:

$$u_{n+1} = u_n - \int_0^{t-t^*} \left(\frac{du_n}{d\xi} - v_n \right) d\xi, \tag{49}$$

$$v_{n+1} = v_n - \int_0^{t-t^*} \left(\frac{dv_n}{d\xi} - w_n \right) d\xi, \tag{50}$$

$$w_{n+1} = w_n - \int_0^{t-t^*} e^{a(\xi - t)} \left(\frac{dw_n}{d\xi} - cu_n - bv_n - aw_n + u_n^2 \right) d\xi. \tag{51}$$

Fourth step, the other component are obtain and can be written as:

$$u_1 = c_1 + c_2(t - t^*), \tag{52}$$

$$v_1 = c_2 + c_3(t - t^*), \tag{53}$$

$$w_1 = (5c_1 + 2.43c_2 + c_3 - 0.83c_1^2) \tag{54}$$

Finally, as above we obtain the solutions:

$$u(t) \approx u_5(t - t^*), \tag{58}$$

$$v(t) \approx v_5(t - t^*), \tag{59}$$

$$w(t) \approx w_5(t - t^*), \tag{60}$$

where t^* start from $t_0 = 0$ until $t_J = T = 20$. To carry out the solution on every subinterval of equal length Δt, we need to know the values of the following initial conditions:

$$c_1 = u(t^*), \quad c_2 = v(t^*), \quad c_3 = w(t^*). \tag{61}$$

In general, these information were not possessed at but can be obtained disposal except at the initial point $t^* = t_0 = 0$, but these values can be obtained by assuming that the new initial condition is the solution in previous interval (i.e. if the solution is required in interval $[t_j, t_{j+1}]$, then the initial conditions of this interval will be as follows:

$$c_1 = u(t) \approx u_5(t_j, t_{j-1}), \tag{62}$$

$$c_2 = v(t) \approx v_5(t_j, t_{j-1}), \tag{63}$$

$$c_3 = w(t) \approx w_5(t_j, t_{j-1}), \tag{64}$$

where c_1, c_2, c_3 are the initial conditions in the interval $[t_j, t_{j+1}]$).

Results and Discussion

The accuracy of SVIM for the solution of both non-chaotic and chaotic systems i.e. non-chaotic and chaotic Rössler system and chaotic Genesio system were presented in this paper. The SVIM algorithm was coded in the computer algebra package Maple. The simulations were done in this paper for the time span $t \in [0, 20]$ and comparison was done by the fourth-order Runge Kutta (RK4) method.

RÖSSLER SYSTEM

Firstly, in the Rössler system, it is necessary to fix the value of the parameters α and β at 0.2 with $\gamma = 2.3$ (for non–chaotic) and $\gamma = 5.7$ (for chaotic). The initial conditions used were $u(0) = 2.0$, $v(0) = 3.0$ and $w(0) = 2.0$. Based on the previous calculations (Goh et al. 2009b), author has decided to use 2^{nd} iteration in the step variational iteration series solutions.

NON-CHAOTIC CASE

The value $\Delta t = 0.001$ was chosen as our benchmark for comparison as mentioned in (Goh et al. 2009b). Table 1 shows that the SVIM at 2^{nd} iteration with $\Delta t = 0.01$, which has maximum tolerance of $|10^{-2}|$ and when $\Delta t = 0.001$, SVIM has maximum tolerance of $|10^{-4}|$. The result which is shown in Table 1 is the same result with MVIM of Goh et al. (2009b). Since MVIM uses more partitions to get better results, this shows the disadvantage of MVIM. The plots of *uv, uw, vw* of the Rössler attractor in the 2-*d* space are presented in Figures 1, 2 and 3. Figures 4, 5 and 6 present the plots of various differences between 2^{nd} iteration of SVIM with time step ($\Delta t = 0.01$ and $\Delta t = 0.001$) versus RK4 with $\Delta t = 0.001$, respectively.

TABLE 1. Absolute errors between solution by SVIM and RK4 solutions when Δt = 0.001 (non–chaotic case)

	SVIM $\Delta t = 0.01$			SVIM $\Delta t = 0.001$		
t	Δu	Δv	Δw	Δu	Δv	Δw
2	7.438×10^{-5}	2.629×10^{-4}	3.396×10^{-7}	7.615×10^{-7}	2.633×10^{-6}	3.441×10^{-9}
4	4.125×10^{-4}	2.449×10^{-4}	1.259×10^{-5}	4.115×10^{-6}	2.482×10^{-6}	1.257×10^{-7}
6	9.530×10^{-5}	1.972×10^{-3}	3.603×10^{-4}	1.258×10^{-6}	2.020×10^{-5}	4.009×10^{-6}
8	2.063×10^{-3}	6.791×10^{-4}	2.374×10^{-5}	2.132×10^{-5}	6.769×10^{-6}	2.448×10^{-7}
10	2.052×10^{-3}	1.851×10^{-3}	2.331×10^{-4}	2.095×10^{-5}	1.925×10^{-5}	2.299×10^{-6}
12	6.479×10^{-4}	2.193×10^{-4}	1.968×10^{-4}	6.279×10^{-6}	2.376×10^{-6}	2.055×10^{-6}
14	2.806×10^{-4}	9.312×10^{-4}	1.048×10^{-6}	2.491×10^{-6}	9.173×10^{-6}	8.275×10^{-9}
16	7.939×10^{-4}	8.062×10^{-4}	4.906×10^{-4}	8.060×10^{-6}	7.699×10^{-6}	4.644×10^{-6}
18	1.338×10^{-3}	6.981×10^{-4}	1.340×10^{-4}	1.334×10^{-5}	6.732×10^{-6}	1.307×10^{-6}
20	1.709×10^{-3}	1.230×10^{-3}	1.941×10^{-5}	1.679×10^{-5}	1.234×10^{-5}	1.902×10^{-7}

(a)

(b)

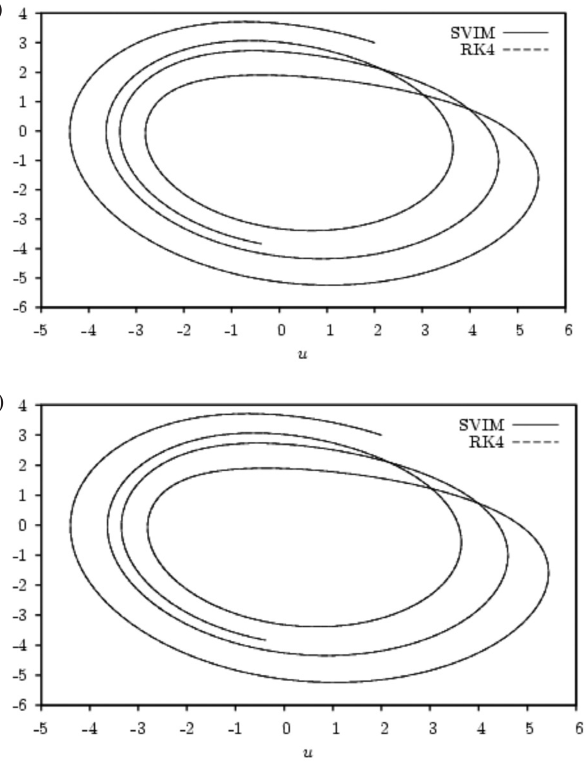

FIGURE 1. The Rössler attractor of the 2-*d* space of non-chaotic case for *u* versus *v*: (a) 2-iterate of SVIM (Δt = 0.01) and (b) 2-iterate of SVIM (Δt = 0.001)

(a)

(b)

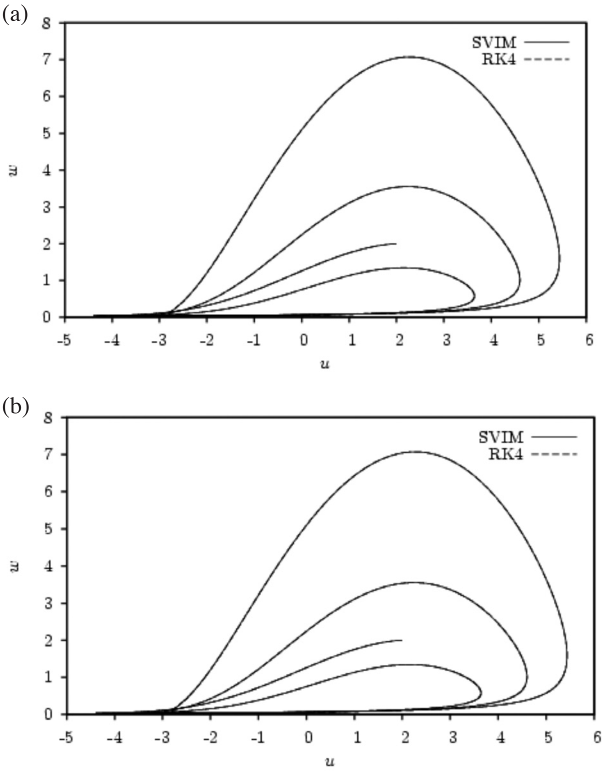

FIGURE 2. The Rössler attractor of the 2-*d* space of non-chaotic case for *u* versus *w*: (a) 2-iterate SVIM (Δt = 0.01) and (b) 2-iterate SVIM (Δt = 0.001)

(a)

(b)

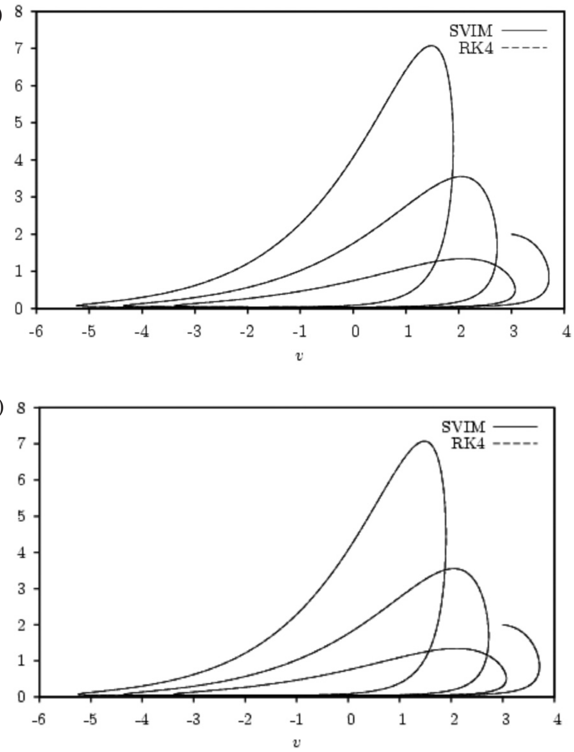

FIGURE 3. The Rössler attractor of the 2-*d* space of non-chaotic case for *v* versus *w*: (a) 2-iterate SVIM (Δt 0.01) and (b) 2-iterate SVIM (Δt = 0.001)

(a)

(b)

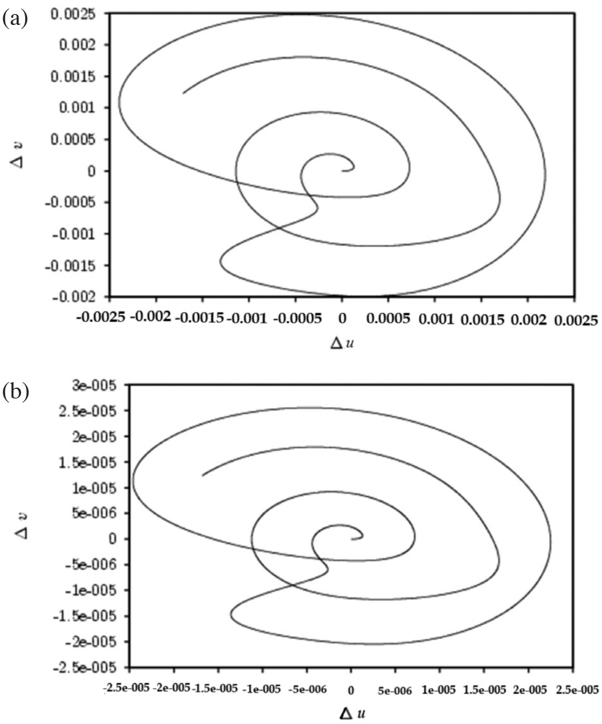

FIGURE 4. The error curves of Rössler system between *u* and *v* for non–chaotic case: (a) 2-iterate SVIM (Δt = 0.01) vs RK4 (Δt = 0.001) and (b) 2-iterate SVIM (Δt = 0.001) vs RK4 (Δt = 0.001)

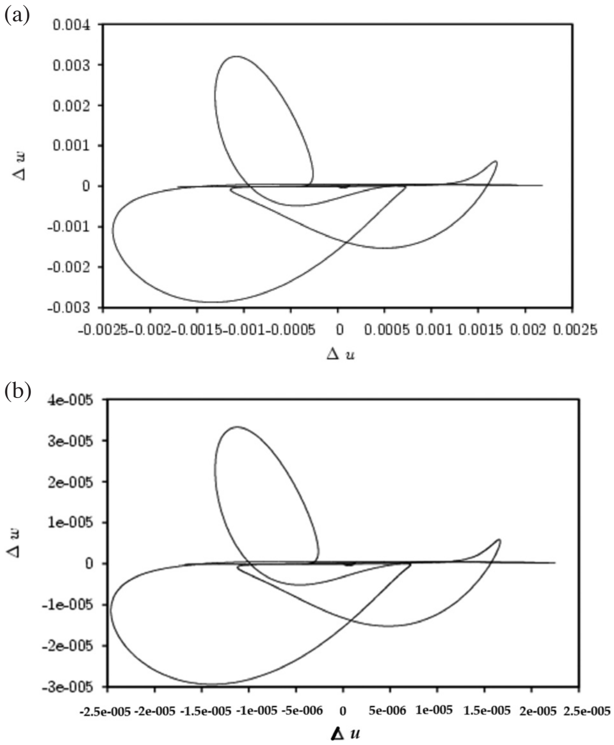

FIGURE 5. The error curves of Rössler system between u and w for non–chaotic case: (a) 2-iterate SVIM ($\Delta t = 0.01$) vs RK4 ($\Delta t = 0.001$) and (b) 2-iterate SVIM ($\Delta t = 0.001$) vs RK4 ($\Delta t = 0.001$)

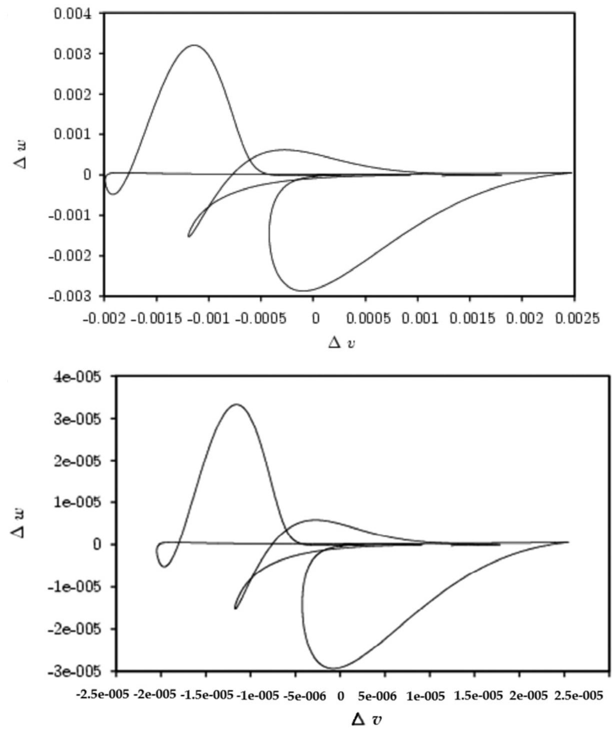

FIGURE 6. The error curves of Rössler system between v and w for non–chaotic case: (a) 2-iterate SVIM ($\Delta t = 0.01$) vs RK4 ($\Delta t = 0.001$) and (b) 2-iterate SVIM ($\Delta t = 0.001$) vs RK4 ($\Delta t = 0.001$)

CHAOTIC CASE

In this section, the value $\Delta t = 0.001$ from Goh et al. (2009b) is taken as the benchmark for comparing with various SVIM. Table 2, the SVIM at 2nd iteration with $\Delta t = 0.01$, has maximum tolerance of $|10^{-7}|$ and $\Delta t = 0.001$ has maximum tolerance of $|10^{-11}|$ when it compares to RK4 with $\Delta t = 0.001$. The plot *uv, uw, vw* attractor of the chaotic Rössler system using the 2nd iteration of SVIM solutions as in Figures 7, 8 and 9. Figures 10, 11 and 12 present the plots of various differences between 2nd iteration of SVIM with time step ($\Delta t = 0.01$ on left and $\Delta t = 0.001$ on right) versus RK4 with $\Delta t = 0.001$. The results of non–chaotic and chaotic Rössler system also have the same results with MVIM (Goh et al. (2009b). Even so, the computation is faster by SVIM.

GENESIO SYSTEM

In the Genesio system, the value of parameters $a = 1.2$, $b = 2.92$ and $c = 6$ (for chaotic) were fixed. The initial conditions $u(0) = 0.2$, $v(0) = -0.3$ and $w(0) = 0{,}1$. We

TABLE 2. Absolute errors between various SVIM and RK4 solutions when $\Delta t = 0.001$ (chaotic case)

	SVIM Δ t = 0.01			SVIM $\Delta t = 0.001$		
t	Δu	Δv	Δw	Δu	Δv	Δw
2	5.375×10^{-5}	2.388×10^{-4}	1.321×10^{-7}	5.497×10^{-7}	2.393×10^{-6}	1.184×10^{-9}
4	4.088×10^{-4}	1.861×10^{-4}	2.272×10^{-6}	4.083×10^{-6}	1.889×10^{-6}	2.295×10^{-8}
6	5.107×10^{-4}	6.374×10^{-4}	1.043×10^{-4}	5.124×10^{-6}	6.373×10^{-6}	1.057×10^{-6}
8	5.070×10^{-4}	1.018×10^{-3}	1.074×10^{-6}	5.042×10^{-6}	1.021×10^{-5}	1.055×10^{-8}
10	1.760×10^{-3}	2.102×10^{-4}	7.793×10^{-6}	1.761×10^{-5}	2.038×10^{-6}	7.835×10^{-8}
12	1.168×10^{-2}	7.052×10^{-3}	7.123×10^{-4}	1.241×10^{-3}	7.341×10^{-5}	7.321×10^{-5}
14	1.313×10^{-2}	2.744×10^{-3}	2.142×10^{-5}	1.379×10^{-4}	2.986×10^{-5}	2.249×10^{-7}
16	5.138×10^{-3}	1.568×10^{-2}	2.173×10^{-5}	5.269×10^{-5}	1.651×10^{-4}	2.158×10^{-7}
18	5.778×10^{-3}	8.125×10^{-3}	1.362×10^{-2}	6.258×10^{-5}	8.422×10^{-5}	1.423×10^{-4}
20	1.112×10^{-2}	2.344×10^{-3}	1.176×10^{-5}	1.167×10^{-4}	2.172×10^{-5}	1.231×10^{-7}

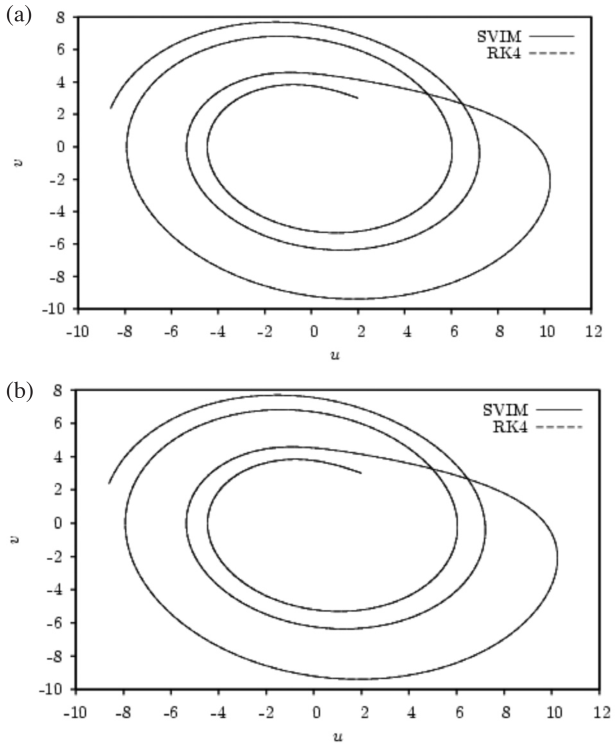

FIGURE 7. The Rössler attractor in the 2-d space between u and v for chaotic case: (a) 2-iterate SVIM ($\Delta t = 0.01$) vs RK4 ($\Delta t = 0.001$) and (b) 2-iterate SVIM ($\Delta t = 0.001$) vs RK4 ($\Delta t = 0.001$)

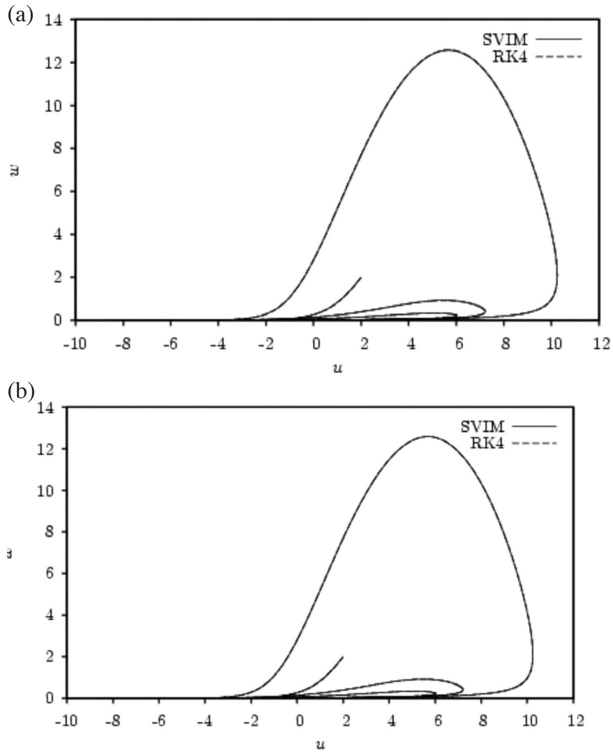

FIGURE 8. The Rössler attractor in the 2-d space between u and w for chaotic case: (a) 2-iterate SVIM ($\Delta t = 0.01$) vs RK4 ($\Delta t = 0.001$) and (b) 2-iterate SVIM ($\Delta t = 0.001$) vs RK4 ($\Delta t = 0.001$)

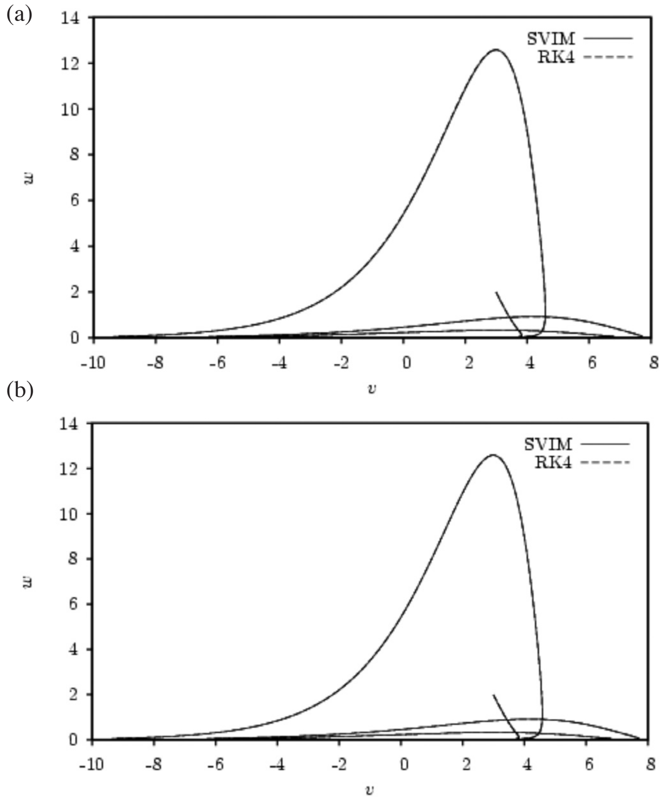

FIGURE 9. The Rössler attractor in the 2-d space between v and w for chaotic case: (a) 2-iterate SVIM ($\Delta t = 0.01$) vs RK4 ($\Delta t = 0.001$) and (b) 2-iterate SVIM ($\Delta t = 0.001$) vs RK4 ($\Delta t = 0.001$)

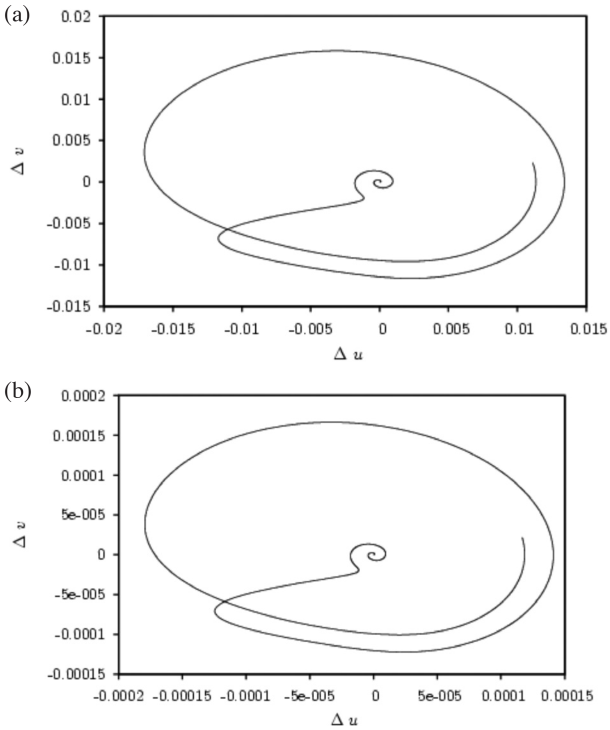

FIGURE 10. The error curves of Rössler system between u and v for chaotic case: (a) 2-iterate SVIM ($\Delta t = 0.01$) vs RK4 ($\Delta t = 0.001$) and (b) 2-iterate SVIM ($\Delta t = 0.001$) vs RK4 ($\Delta t = 0.001$)

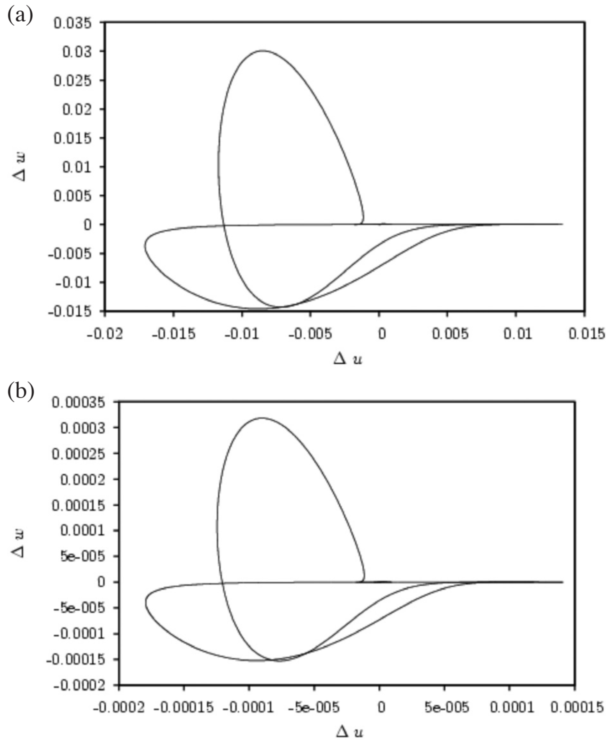

FIGURE 11. The error curves of Rössler system between u and w for chaotic case: (a) 2-iterate SVIM ($\Delta t = 0.01$) vs RK4 ($\Delta t = 0.001$) and (b) 2-iterate SVIM ($\Delta t = 0.001$) vs RK4 ($\Delta t = 0.001$)

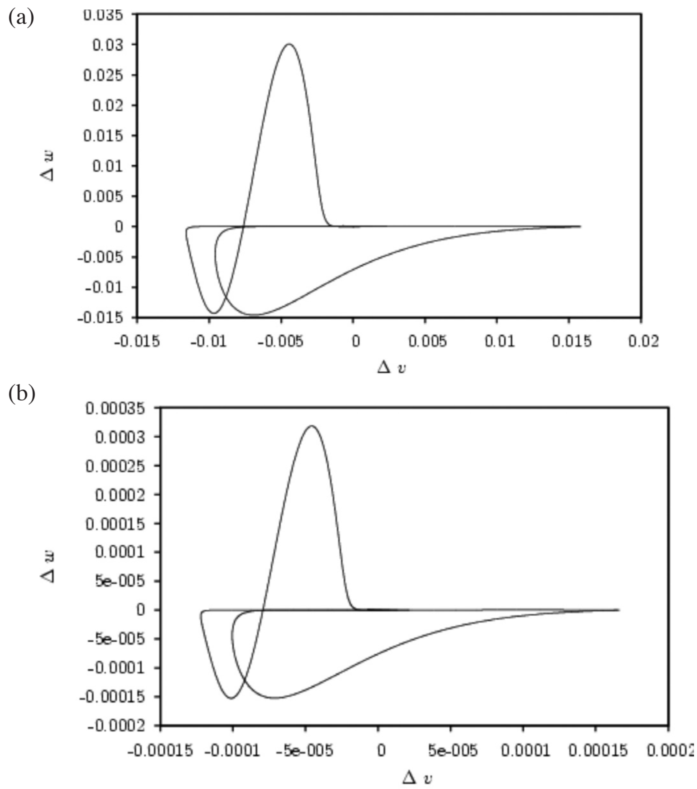

FIGURE 12. The error curves of Rössler system between v and w for chaotic case: (a) 2-iterate SVIM ($\Delta t = 0.01$) vs RK4 ($\Delta t = 0.001$) and (b) 2-iterate SVIM ($\Delta t = 0.001$) vs RK4 ($\Delta t = 0.001$)

used 5^{th} iteration of SVIM at time–step 5^{th} iteration of MVIM versus RK4 at $\Delta t = 0.01$, respectively. Because the efficiency in the calculation, $\Delta t = 0.01$ was chosen as the benchmark for comparing the various SVIM, (Table 3). Table 3 presents the comparisons between SVIM, MVIM and RK4 for $\Delta t = 0.01$, $t \in [0,20]$, $a = 1.2$, $b = 2.92$ and $c = 6$ where the system exhibits chaotic behaviour. Figure 13 shows that SVIM is in good agreement with RK4 with step size $\Delta t = 0.01$ compared with MVIM (Goh et al. 2009a).

CONCLUSIONS

In this paper, the algorithm for solving chaotic systems via step variational iteration method (SVIM) was developed. For computations and plots, the Maple package was used. The conclusions of SVIM are as follows; it was found that for Rössler systems, the SVIM was a suitable technique to solve the chaotic and non-chaotic systems. This modified method yields a step analytical solution in

TABLE 3. Absolute errors between the 5^{th} iteration of SVIM when $\Delta t = 0.01$, $\Delta t = 0.001$ and MVIM solutions when $\Delta t = 0.01$ for chaotic Genesio system, respectively

	SVIM $\Delta t = 0.01$			SVIM $\Delta t = 0.001$			MVIM $\Delta t = 0.01$		
t	Δu	Δv	Δw	Δu	Δv	Δw	Δu	Δv	Δw
2	1.378×10^{-7}	9.946×10^{-8}	5.021×10^{-7}	6.962×10^{-7}	1.142×10^{-6}	3.070×10^{-6}	3.998×10^{-7}	8.096×10^{-8}	5.508×10^{-6}
4	9.626×10^{-7}	1.606×10^{-6}	8.791×10^{-7}	6.473×10^{-7}	2.191×10^{-6}	4.171×10^{-6}	3.405×10^{-5}	8.884×10^{-5}	3.145×10^{-5}
6	7.179×10^{-6}	7.645×10^{-6}	1.360×10^{-5}	1.729×10^{-6}	1.765×10^{-6}	4.497×10^{-6}	2.861×10^{-4}	5.312×10^{-6}	8.111×10^{-4}
8	9.331×10^{-9}	7.064×10^{-6}	1.556×10^{-5}	3.480×10^{-6}	2.239×10^{-6}	2.853×10^{-5}	3.138×10^{-3}	5.548×10^{-3}	1.212×10^{-3}
10	9.967×10^{-6}	6.796×10^{-6}	2.868×10^{-5}	8.897×10^{-6}	8.217×10^{-6}	2.422×10^{-5}	0.1624	0.1726	0.4622
12	1.981×10^{-5}	1.843×10^{-4}	1.381×10^{-3}	5.954×10^{-5}	1.167×10^{-4}	1.072×10^{-4}	0.2455	1.288	3.716
14	2.142×10^{-4}	2.571×10^{-4}	5.008×10^{-4}	1.118×10^{-4}	6.377×10^{-6}	9.426×10^{-4}	0.3811	3.461	1.941
16	7.319×10^{-4}	2.460×10^{-4}	4.944×10^{-3}	1.534×10^{-3}	2.479×10^{-3}	2.109×10^{-3}	2.726	2.898	7.225
18	1.537×10^{-3}	3.063×10^{-4}	4.653×10^{-3}	4.751×10^{-3}	1.063×10^{-3}	1.227×10^{-2}	2.62	1.417	10.88
20	1.305×10^{-2}	2.170×10^{-2}	7.648×10^{-3}	1.876×10^{-2}	3.297×10^{-2}	1.098×10^{-2}	5.613	0.6713	8.679

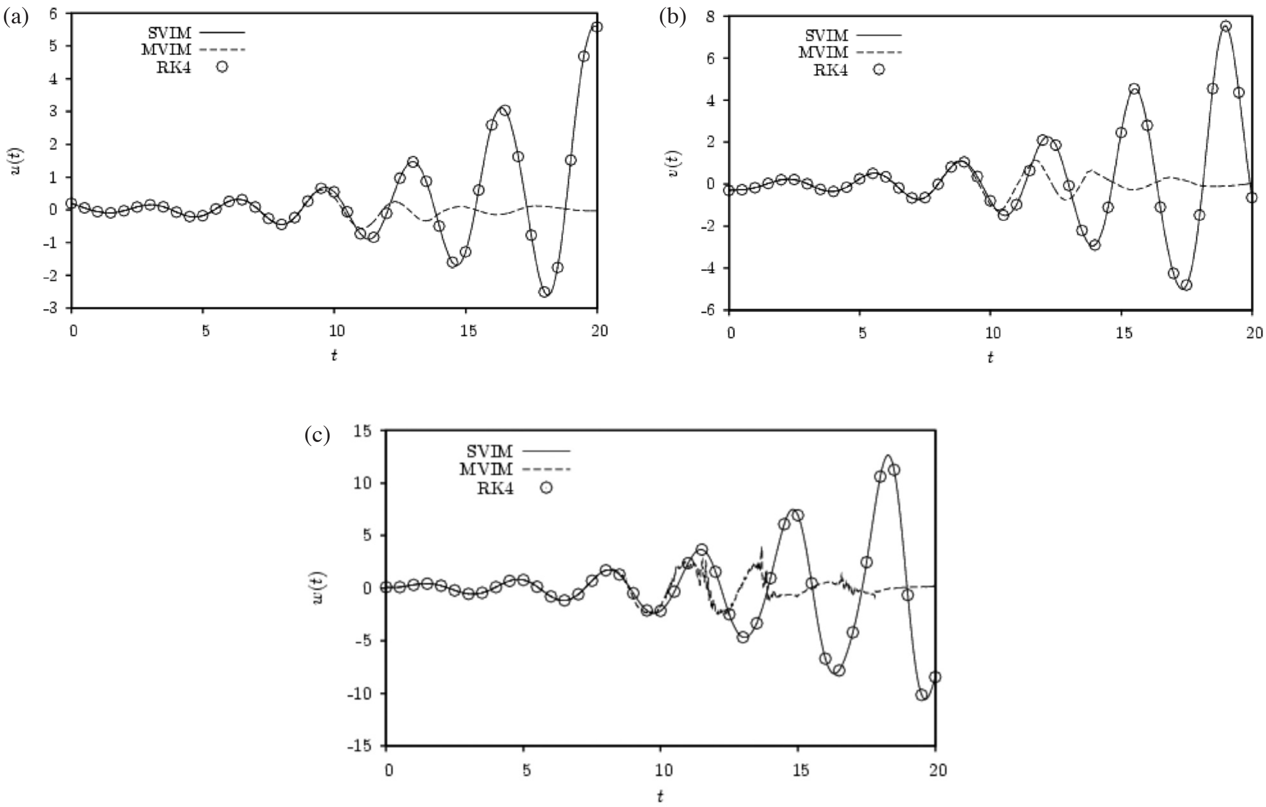

FIGURE 13. The approximate solution curves of 5^{th} iteration of SVIM, 5^{th} iteration of MVIM and RK4 for chaotic Genesio System with ($\Delta t = 0.01$), respectively; (a) $u(t)$, (b) $v(t)$, (c) $w(t)$

iterations of a rapid convergent infinite power series with easily computable terms and the interesting point for the Genesio system case, a good approximate solution was obtained for enlarged intervals when the SVIM solution was compared with MVIM solution (as proposed by Batiha et al. 2007b). Comparison between SVIM, MVIM and RK4 was made; the SVIM was found to be more accurate than the other two methods. SVIM was simple in its calculations and readily seen to be a more powerful method. It has potential for solving more complex systems which may arise in various fields of pure and applied sciences.

ACKNOWLEDGEMENT

The financial support received from UKM-DIP-2012-31 is gratefully acknowledged.

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Received: 25 May 2010
Accepted: 17 September 2012