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A Fourth-Order Compact Finite Difference Scheme for the Goursat Problem

(Skema Beza Terhingga Padat Peringkat Empat untuk Masalah Goursat)

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ABSTRACT

A high-order uniform Cartesian grid compact finite difference scheme for the Goursat problem is developed. The basic idea of high-order compact schemes is to find the compact approximations to the derivatives terms by differentiating centrally the governing equations. Our compact scheme will approximate the derivative terms by involving the higher terms and reducing the number of grid points. The compact finite difference scheme is given for general form of the Goursat problem in uniform domain and illustrates the performance by applying a linear problem. Numerical experiments have been conducted with the new scheme and encouraging results have been obtained. In this paper we present the compact finite difference scheme for the Goursat problem. With the aid of computational software the scheme was programmed for determining the relative errors of linear Goursat problem.

Keywords: Compact finite difference; consistency; convergence; Goursat problem; stability

ABSTRAK

Skema beza terhingga padat bagi grid Kartesan seragam peringkat tinggi untuk masalah Goursat dibincangkan. Idea asas bagi skema padat peringkat tinggi ialah untuk mendapatkan penghampiran padat sebutan-sebutan terbitan dengan membezakan secara memusat persamaan yang bersekutu. Skema padat kami akan membuat penghampiran sebutan-sebutan terbitan dengan melibatkan sebutan-sebutan peringkat lebih tinggi dan mengurangkan bilangan titik-titik grid. Skema beza terhingga padat diberikan dalam bentuk am untuk masalah Goursat di dalam domain seragam dan prestasinya digambarkan dengan mengaplikasi satu masalah linear. Uji kaji berangka dengan skema baru telah dijalankan dan keputusan memberangsangkan telah diperoleh. Dalam kertas ini kami berikan skema beza terhingga padat untuk masalah Goursat. Dengan bantuan perisian pengkomputeran, skema telah diatur cara untuk menentukan pelbagai ralat relatif bagi masalah Goursat linear.

Kata kunci: Beza terhingga padat; ketekalan; kestabilan; masalah Goursat; penumpuan

INTRODUCTION

The Goursat problems have been studied and the solutions of this problems using the aid computational devices have been widely introduced by many researchers such as Evans and Sanugi (1988), Day (1966), Nasir and Ismail (2004) and Wazwaz (1993, 1995). Applications of this problem can be obtained in many area disciplines such as inverse acoustic scattering problem (McLaughlin et al. 1994), wave equations in nonhomogeneous media (Hillion 1992), sine-Gordon equation (Kaup & Newell 1978) and electric field problem (Frisch & Cheo 1972).

Traditional numerical discretization schemes for approximating the Goursat problem usually employ forwarded differentiating for the mix derivatives term (Evans & Sanugi 1988; Wazwaz 1993). The existing scheme (AM scheme-standard scheme) has used arithmetic mean averaging of functional values in finite difference approximation. Evans and Sanugi (1988) and Wazwaz (1993) have used geometric and harmonic mean averaging of functional values, the so-called GM and HM schemes, respectively, for the Goursat problem. However, we note that for linear problems the GM and HM schemes results in non linear difference scheme and thus would suffer the possible disadvantages of iteration problems.

In recent years, high-order compact finite difference approximations have been applied to solve several differential equations: convergence for second-order elliptic equations (Zhao et al. 2006), nonlinear dispersive waves problem (Li & Visbal 2006), compressible Navier-Stokes equations (Boersma 2005), financial applications of symbolically (Zhao et al. 2005), dispersive media problem (Li & Chen 2004), elliptic equations on irregular domains and interface problems and their applications (Kyei 2004), unsteady viscous incompressible flows (Li & Tang 2001), numerical solution of partial differential equation (Ahmed 1997) and initial boundary problems for mixed systems (Bodenmann & Schroll 1996).

A class of high order compact finite difference schemes exhibits higher accuracy at the grid points and in that applies a compact stencil. The governing differential equations will be used to approximate leading truncation error terms in the central difference scheme (Spotz 1995). The objective of this research was to study the theoretical aspects of a numerical scheme widely used to

solve the problem by considering its application to model linear problem. We obtain results relating to the stability (using Von Neumann stability analysis), consistency and convergence of the scheme. We verify these theoretical results with data from computational experiments.

THE GOURSAT PROBLEM AND FOURTH-ORDER COMPACT FINITE DIFFERENCE SCHEME

The Goursat problem is of the form (Wazwaz 1993):

$$\begin{aligned} &u_{xy} = f(x, y, u, u_x, u_y)\\ &u(x, 0) = \phi(x),\ u(0, y) = \psi(y),\ \phi(0) = \psi(0)\\ &0 \le x \le a,\ 0 \le y \le b. \end{aligned} \tag{1}$$

Here the unknown function u, is assumed to be continuously differentiable and has the required partial derivatives on the rectangular domain.

By manipulating the Taylor series expansion in two independent variables (Twizell 1984), it can be found that:

$$\begin{aligned} &u(x + \Delta x,y + \Delta y) - u(x + \Delta x,y - \Delta y) - u(x - \Delta x,y + \Delta y)\\ &+ u(x - \Delta x,y - \Delta y) = 4(\Delta x)(\Delta y)u_{xy} + \frac{2}{3}(\Delta x)^3(\Delta y)u_{xxxy}\\ &+ \frac{2}{3}(\Delta x)(\Delta y)^3 u_{xyyy} + O(\Delta x + \Delta y)^6. \end{aligned} \tag{2}$$

Thus, for the mixed derivatives u_{xy}, the expression is as follows:

$$\begin{aligned} u_{xy} = {} & \frac{\begin{array}{l} u\left(x + \Delta x, y + \Delta y\right) - u\left(x + \Delta x, y - \Delta y\right)\\ -u\left(x - \Delta x, y + \Delta y\right) + u\left(x - \Delta x, y - \Delta y\right) \end{array}}{4\left(\Delta x\right)\left(\Delta y\right)}\\ & - \frac{2}{12}\left(\Delta x\right)^2 u_{xxxy} - \frac{2}{12}\left(\Delta y\right)^2 u_{xyyy} + O\left(\frac{\left(\Delta x + \Delta y\right)^6}{4\left(\Delta x\right)\left(\Delta y\right)}\right). \end{aligned} \tag{3}$$

We shall focus our concern on the numerical solution of the Goursat problem (1). Hence, equating (1) and (3):

$$\begin{aligned} & \frac{\begin{array}{l} u\left(x + \Delta x, y + \Delta y\right) - u\left(x + \Delta x, y - \Delta y\right)\\ -u\left(x - \Delta x, y + \Delta y\right) + u\left(x - \Delta x, y - \Delta y\right) \end{array}}{4\left(\Delta x\right)\left(\Delta y\right)}\\ & - \frac{2}{12}\left(\Delta x\right)^2 u_{xxxy} - \frac{2}{12}\left(\Delta y\right)^2 u_{xyyy} + O\left(\frac{\left(\Delta x + \Delta y\right)^6}{4\left(\Delta x\right)\left(\Delta y\right)}\right)\\ & = f(x, y, u, u_x, u_y), \end{aligned} \tag{4}$$

where the initial conditions on the rectangle domain remains.

We rewrite (4) in indexing form as follows:

$$\frac{u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}}{4h^2} - \phi_i = f_i, \tag{5}$$

where $\Delta x = \Delta y = h$ and ϕ_i is the truncation error at node i given by:

$$\phi_i = \frac{2}{12}(\Delta x)^2 u_{xxxy} + \frac{2}{12}(\Delta y)^2 u_{xyyy} + O\left(\frac{(\Delta x + \Delta y)^6}{4(\Delta x)(\Delta y)}\right). \tag{6}$$

We use the initial conditions in (1) and the established finite difference algorithm (Nasir & Ismail 2005) to compute the initial values in (5) as follows:

$$u_{i+1,j+1} = u_{i+1,j} + u_{i,j+1} - u_{i,j} + \frac{h^2}{4}(f_{i+1,j+1} + f_{i,j} + f_{i+1j} + f_{i,j+1}). \tag{7}$$

The central finite difference scheme (5) is accurate up to $O(h^2)$ when ϕ_i is dropped from the formulation in (5) and clearly, if the leading term in (6) vanished the scheme would be at least $O(h^4)$. To obtain a fourth-order compact finite difference scheme, the discretization on the right hand side of (6) should be considered. Relations for u_{xxxy} and u_{xyyy} can be constructed by differentiating (1) to get,

$$u_{xxxy} = f_{xx}, \tag{8}$$

and

$$u_{xyyy} = f_{yy}. \tag{9}$$

It is of interest to note that source term f plays an important role in approximating u_{xxxy} and u_{xyyy}. If the derivatives of function f are known in an analitical approaches, it can be substituted in (8) and (9). However, if only a discrete approximation to function f is available, the central differences can be used in approximating the discretization (Spotz 1995).

We investigated the performance of a fourth-order compact scheme by applying a linear Goursat problem below (Wazwaz 1995):

$$\begin{aligned} &u_{xy} = u\\ &u(x, 0) = e^x\\ &u(0, y) = e^y\\ &0 \le x \le 2,\ 0 \le y \le 2. \end{aligned} \tag{10}$$

The analytical solution of (10) is e^{x+y}.

Differentiating (10) with respect to x, gives as follows:

$$\begin{aligned} &u_{xxy} = u_x\\ &u_{xxxy} = u_{xx}. \end{aligned} \tag{11}$$

In like manner,

$$\begin{aligned} &u_{xyy} = u_y\\ &u_{xyyy} = u_{yy}. \end{aligned} \tag{12}$$

Substituting terms (11) and (12) into (3), gives the discretization of $u_{xy} = u$ i.e.

$$\frac{u(x+\Delta x,y+\Delta y)-u(x+\Delta x,y-\Delta y)}{-u(x-\Delta x,y+\Delta y)+u(x-\Delta x,y-\Delta y)}{4(\Delta x)(\Delta y)}$$
$$-\frac{2}{12}(\Delta x)(\Delta y)(u_{xx}+u_{yy})+O\left(\frac{(\Delta x+\Delta y)^6}{4(\Delta x)(\Delta y)}\right)=u(x,y). \quad (13)$$

Summing of the Taylor series expansions, we get:

$$2(\Delta x)(\Delta y)(u_{xx}+u_{yy}) = u(x+\Delta x,y+\Delta y) + u(x+\Delta x,y-\Delta y) + u(x-\Delta x,y+\Delta y) + u(x-\Delta x,y-\Delta y) - 4u(x,y) + O(\Delta x+\Delta y)^4. \quad (14)$$

Substituting terms in (14) into (13), the fourth-order compact finite difference scheme for the partial differential equation in (10) in indexing form can be written as:

$$\frac{u_{i+1,j+1}-u_{i+1,j-1}-u_{i-1,j+1}+u_{i-1,j-1}}{4h^2} - \frac{u_{i+1,j+1}+u_{i+1,j-1}+u_{i-1,j+1}+u_{i-1,j-1}}{12} - \frac{2u_{i,j}}{3}=0. \quad (15)$$

Scheme (15) can be written as:

$$u_{i+1,j+1} = A(u_{i+1,j-1}+u_{i-1,j+1}) + \frac{8h^2}{3(1-r)}u_{i,j} - u_{i-1,j-1}. \quad (16)$$

By shifting the index: $i \to i+1$ and $j \to j+1$, the approximate value of all interior node points can be computed by these algorithm (Figure 1),

$$u_{i+2,j+2} = A(u_{i+2,j}+u_{i,j+2}) + \frac{8h^2}{3(1-r)}u_{i+1,j+1} - u_{i,j}, \quad (17)$$

where $A=\left(\frac{1+r}{1-r}\right)$ with $r=\frac{h^2}{3}>0$.

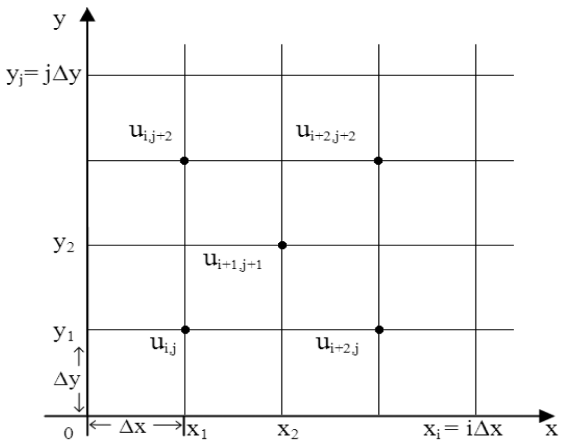

FIGURE 1. Finite grid network based on problem (10)

The initial values for algorithm (17) can be determined as disccussed in earlier section.

STABILITY

The stability of a finite difference scheme can be investigated using the Von Neumann method (Fletcher 1990). In this method, the errors distributed along grid lines at one time level are expanded as a finite Fourier series. If the separate Fourier components of the error distribution amplify in progressing to the next time level, then the scheme is unstable.

The error equation for (16) is:

$$\xi_{i+1,j+1} = A(\xi_{i+1,j-1}+\xi_{i-1,j+1}) + \frac{8h^2}{3(1-r)}\xi_{i,j} - \xi_{i-1,j-1}, \quad (18)$$

where $\xi_{i,j}$ is the error at the (i, j) grid point. We write $\xi_{i,j}$ as $\lambda^j e^{\sqrt{-1}\theta_m i}$ where λ is the amplification factor for the m^{th} Fourier mode of the error distribution as it propagates one step forward in time and $\theta_m = m\pi h$. For linear schemes, it is sufficient to consider the propagation of the error due to just a single term of the Fourier series representation i.e. the subscript m can be dropped.

Substituting $\xi_{i,j} = \lambda^j e^{\sqrt{-1}\theta_m i}$ into (18) gives:

$$\lambda^{j+1}e^{\sqrt{-1}\theta(i+1)} = A(\lambda^{j-1}e^{\sqrt{-1}\theta(i+1)} + \lambda^{j+1}e^{\sqrt{-1}\theta(i-1)}) + \frac{8h^2}{3(1-r)}\lambda^j e^{\sqrt{-1}\theta i} - \lambda^{j-1}e^{\sqrt{-1}\theta(i-1)}, \quad (19)$$

i.e.

$$\lambda = A(\lambda^{-1}+\lambda e^{-2\sqrt{-1}\theta}) + \frac{8h^2}{3(1-r)}e^{-\sqrt{-1}\theta} - \lambda^{-1}e^{-2\sqrt{-1}\theta}. \quad (20)$$

For stability, it is required that $|\lambda| \le 1, \forall\theta$.

Due to the complexity of the algebraic manipulation, the Maple 8 code programming is used to illustrate the area of stability and given in Figure 2.

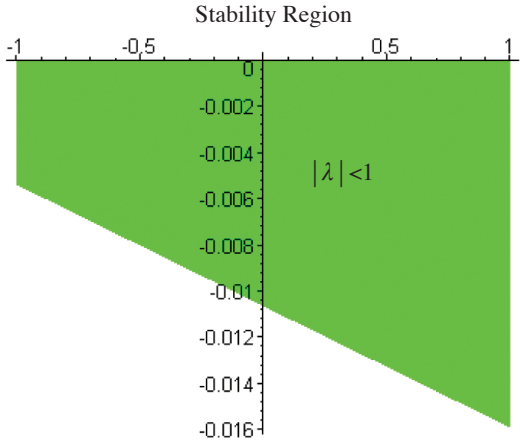

FIGURE 2. Stability in graphic interpretation with h = 0.05

CONSISTENCY

To test for consistency, the exact solution of the partial differential equation is substituted into the finite difference scheme and values at grid points expanded as a Taylor series. For consistency, the expression obtained should tend to the partial differential equation as the grid size tends to zero (Twizell 1984).

Equation (16) can be rewritten as:

$$u(x_{i+1}, y_{j+1}) = \left(\frac{3+h^2}{3-h^2}\right)(u(x_{i+1}, y_{j-1}) + u(x_{i-1}, y_{j+1})) + \frac{8h^2}{3-h^2}\, u(x_i, y_j) - u(x_{i-1}, y_{j-1}). \quad (21)$$

Substituting the exact solution into (21) leads to:

$$u + hu_x + hu_y + \frac{1}{2}(h^2u_{xx} + 2h^2u_{xy} + h^2u_{yy}) + \ldots = \left(\frac{3+h^2}{3-h^2}\right)(u + hu_x - hu_y + \frac{1}{2}(h^2u_{xx} - 2h^2u_{xy} + h^2u_{yy}) + \ldots + u - hu_x + hu_y + \frac{1}{2}(h^2u_{xx} - 2h^2u_{xy} + h^2u_{yy}) + \ldots) + \frac{8h^2}{3-h^2}\, u - (u - hu_x - hu_y + \frac{1}{2}(h^2u_{xx} + 2h^2u_{xy} + h^2u_{yy}) + \ldots), \quad (22)$$

Equation (22) can be simplified as:

$$u_{xy} = u + \frac{h^2\left(u_{xx} + u_{yy}\right)}{6} + \ldots \quad (23)$$

[note: all terms involving u in (22) are evaluated at (x_j, y_j)]

As $h \to 0$, (23) becomes $u_{xy} = u$. Thus, the condition for consistency is satisfied.

CONVERGENCE

A solution of the algebraic equations which approximate a given partial differential equation is said to be convergent if the approximate solution approaches the exact solution for each value of the independent variables as the grid spacing tends to zero (Nasir & Ismail 2004). The Lax equivalence theorem (Richtmyer & Morton 1967) states that given a properly (well) posed linear initial value problem and a finite difference equation that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence.

The Goursat problem is well posed and was established by Garabedian (1964) by transforming it into an integro differential equation and then solving it by the technique of successive approximations. We have established that the fourth-order compact scheme for the linear Goursat problem (10) is both stable and consistent. Thus from the Lax equivalence theorem we can conclude it is convergent.

COMPUTATIONAL EXPERIMENTS

By the assistance of computer software to implement scheme (17), the numerical solutions in calculating absolute errors at particular grid points are given in Tables 1 and 2.

For the grid sizes h = 0.025, 0.05, 0.1 and 0.25 we obtained the following average error results (Table 3).

TABLE 1. h values and absolute errors by using the fourth-order compact scheme at various grid points

	Absolute errors at grid points			
h	(0.5,0.5)	(1.0,1.0)	(1.5,1.5)	(2.0,2.0)
0.025	9.8791527e-08	3.8531802e-07	6.7314451e-07	8.8171770e-07
0.050	7.9763931e-07	3.1606038e-06	5.5686068e-06	7.3476793e-06
0.100	8.0916557e-05	2.6421645e-05	9.2698770e-05	6.3361226e-05
0.250	9.4307416e-05	4.5023080e-04	8.5375522e-04	1.1867231e-03
0.500	6.9233875e-03	3.5989691e-03	1.4775527e-02	1.1570979e-02

TABLE 2. h values and absolute errors by using the second-order compact scheme at various grid points

	Absolute errors at grid points			
h	(0.5,0.5)	(1.0,1.0)	(1.5,1.5)	(2.0,2.0)
0.025	3.3571265e-005	9.7655834e-005	1.7254422e-004	2.5289173e-004
0.050	1.3266089e-004	3.8425312e-004	6.7961862e-004	9.9845300e-004
0.100	3.0848996e-004	1.4848477e-003	2.5559089e-003	3.8783998e-003
0.250	3.1068239e-003	8.3086911e-003	1.4538331e-002	2.1651893e-002
0.500	6.9233875e-003	2.9149988e-002	3.2355098e-002	6.6818475e-002

TABLE 3. A comparatives study between fourth-order vs second-order scheme

Grid sizes	0.025	0.05	0.1	0.25
Average error of fourth-order scheme	9.3298241e-07	7.7107136e-06	6.5645706e-05	1.2121591e-03
Average error of second-order scheme	8.4279145e-05	3.3334195e-04	1.3020743e-03	7.5335660e-03

The numerical experiments showed that the absolute error becomes smaller as h is decreased for all grid points tested. It can be seen that the fourth-order compact scheme gives better accuracy for the linear Goursat problem.

CONCLUSION

Several studies of the finite difference solution of the Goursat problem have focused on the accuracy and implementation aspects. In this paper we have studied the accuracy as well as the theoretical aspects of the finite difference solution of a linear Goursat problem using the fourth-order compact finite difference scheme. Using the Von Neumann method we have shown that it is unconditionally stable and consistent. Invoking the Lax equivalence theorem, we deduced that the scheme is convergent.

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