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Parameter Estimation of Stochastic Differential Equation

(Penganggaran Parameter Persamaan Pembeza Stokastik)

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ABSTRACT

Non-parametric modeling is a method which relies heavily on data and motivated by the smoothness properties in estimating a function which involves spline and non-spline approaches. Spline approach consists of regression spline and smoothing spline. Regression spline with Bayesian approach is considered in the first step of a two-step method in estimating the structural parameters for stochastic differential equation (SDE). The selection of knot and order of spline can be done heuristically based on the scatter plot. To overcome the subjective and tedious process of selecting the optimal knot and order of spline, an algorithm was proposed. A single optimal knot is selected out of all the points with exception of the first and the last data which gives the least value of Generalized Cross Validation (GCV) for each order of spline. The use is illustrated using observed data of opening share prices of Petronas Gas Bhd. The results showed that the Mean Square Errors (MSE) for stochastic model with parameters estimated using optimal knot for 1,000, 5,000 and 10,000 runs of Brownian motions are smaller than the SDE models with estimated parameters using knot selected heuristically. This verified the viability of the two-step method in the estimation of the drift and diffusion parameters of SDE with an improvement of a single knot selection.

Keywords: Bayesian approach; regression spline; stochastic differential equation; truncated power series basis

ABSTRAK

Permodelan tak-berparameter adalah satu kaedah yang sangat bergantung kepada data dan dimotivasi oleh ciri kelicinan dalam menganggar fungsi yang melibatkan pendekatan splin dan bukan-splin. Pendekatan splin terdiri daripada splin regresi dan splin pelicinan. Pendekatan pertama dengan kaedah Bayesian digunakan dalam langkah pertama untuk kaedah dua-langkah bagi menganggar parameter struktur persamaan pembeza stokastik (SDE). Pemilihan buku dan tertib splin boleh dilakukan secara heuristik berdasarkan rajah. Untuk mengatasi proses pemilihan bilangan buku dan tertib splin yang subjektif dan memakan masa, satu prosedur penyelesaian dikemukakan. Buku tunggal terbaik dengan nilai pengesahan silang teritlak (GCV) minimum dipilih daripada semua titik kecuali data pertama dan terakhir. Penggunaannya ditunjukkan menggunakan data cerapan saham pembukaan Petronas Gas Bhd. Hasil kajian menunjukkan nilai min ralat kuasadua bagi model stokastik yang menggunakan knot tunggal terbaik sebagai penganggaran parameter bagi larian gerakan Brown 1,000, 5,000 dan 10,000 adalah lebih kecil berbanding model stokastik dengan parameter dianggar menggunakan buku yang dipilih secara heuristik. Ini mengesahkan kebolehjayaan kaedah dua-langkah dalam menganggar parameter hanyut dan jerap SDE dengan menambahbaik kaedah pemilihan buku tunggal.

Kata kunci: Basis siri kuasa terpangkas; pendekatan Bayesan; persamaan pembeza stokastik; regresi splin

INTRODUCTION

Classical methods such as Maximum Likelihood Estimation (MLE), Methods of Moment, Least Squares Estimation (LSE) have been commonly employed in the estimation of ODE (Ordinary Differential Equation) parameters. Some drawbacks of classical methods such as MLE includes the decrease of the estimators efficiency due to computational problem which arise from huge exploration of local optima and computation time (Brunel 2008). An alternative approach to MLE is introduced such as The two-step methods the solution of ODE is obtained in the first step by the non-parametric method and the estimation of ODE parameter in the second step by minimizing a given distance. The problem with classical method such as MLE persists in SDE, is it involves the approximation of the transition density which includes intense computation, poor accuracy and difficulty for multivariate SDE. The motivation of this study was to extend the two-step method in SDE, utilizing a fully non-likelihood approach with the implemention of non-parametric criterion in both steps. This approach does not involve the approximation of the transition density, thus avoiding previously mentioned problems and providing simpler alternative procedure in estimating SDE parameters.

TWO-STEP METHOD IN ORDINARY DIFFERENTIAL EQUATION (ODE)

The estimation of structural parameters using the two-step method in ordinary differential equations (ODE) was initiated by Varah (1982). This method is computationally easier to perform compared with some classical parametric estimators such as maximum likelihood method (MLE) or derivative based methods. The parameter of ODE was estimated by employing least squares procedure by firstly fitting the given data with cubic B-spline function with knots chosen interactively. The parameters were estimated by finding the solution of least squares equation of the spline function and the ordinary differential equation.

Ramsay and Silverman (1997) had considered two-step approach in functional data analysis (FDA) framework. It is based on the transformation of data into functions with smoothing cubic splines. Ramsay (1996) proposed principal differential analysis (PDA) and using the basis function such as B-splines to estimate the parameters of ODE. The extension of PDA was done by applying it to nonlinear ODE and the iterated PDA (iPDA), thus repeating the two-steps method in Poyton et al. (2006).

The iPDA had been extended with the introduction of generalized smoothing approach by Ramsay et al. (2007) where the smoothing and estimation of ODE parameters were done simultaneously. They proposed a generalized profiling procedure which was a variant of the collocation method based on basis function expansion in the form of penalized log-likelihood criterion. The procedure was applied using noisy measurements on a subset of variables to estimate the parameters defining a system of non-linear differential equation. For simulated data from models in chemical engineering, they derived the point estimates and the confidence interval and had shown that these have low bias and good coverage properties. The method had also been applied to real data from chemistry and from the progress of the autoimmune disease lupus.

Referring to Ramsay et al. (2007), Brunel (2008) proposed a general method of estimating the parameters of ODE from time series data. Brunel (2008) used the nonparametric estimator of regression function as the first step of constructing the M-estimator minimizing:

$$R_w^q(\theta) = \left\| \dot{\hat{x}} - F(t, \hat{x}_n, \theta) \right\|_{q,w}, \quad (1)$$

where θ is the parameter of ODE, n is the number of observations, t is the observation time, w is the weight function, $\dot{\hat{x}}$ is the derivative of the non parametric estimator of solution of ODE and $F(t, \hat{x}_n, \theta)$ is the ODE. The method is able to alleviate computational difficulties encountered by the classical parametric method. The consistency of the derived estimator $\hat{\theta}$ with detail analysis when $q = 2$ was also shown. For the case of spline estimators, the asymptotic normality and the rate of convergence of the parametric estimators was also proven.

TWO-STEP METHOD IN STOCHASTIC DIFFERENTIAL EQUATION (SDE)

Parameter estimation of stochastic differential equation (SDE) is largely based on parametric methods; non-linear least squares, maximum likelihood, methods of moment and filtering such as the extended Kalman filter. Non-parametric approach in estimating the parameters of SDE has recently been introduced by Varziri et al. (2008) who developed Approximate Maximum Likelihood Estimation (AMLE). They proposed a new version of a two-step method via the minimization of the negative of natural logarithm of approximate probability density function to estimate the drift and the spline parameters of the SDE. The estimated disturbance intensity was then repeatedly improved by a noise estimator. This approach however causes computational burden since it involves the approximation of transitional probability. Furthermore, Bayesian approach with spline implementations have not been considered in parameter estimation of SDE. Wide literatures may be found in the implementation of regression spline, for example Budiantara (2001), Calderon et al. (2010), Hunt and Li (2006), Leathwick et al. (2005), Lee (2002) and Molinari et al. (2004) but few involving Bayesian approach. Works employing Bayesian regression spline include Li and Yu (2006) who estimated the term structure with Bayesian regression splines based on nonlinear least absolute deviation. The method was found to be robust to outliers in a chosen case study. Lang and Brezger (2004) proposed a Bayesian version for P-splines for generalized additive models. The approach has the advantages of allowing simultaneous estimation of smooth function and smoothing parameter and had been extended to more complex formulations. Wallstrom et al. (2008) implemented BARS (Bayesian Adaptive Regression Splines) in C by manipulating B-splines for normal and Poison cases. This has improved the original implementation of BARS in S.

The objective of this paper was to estimate SDE parameters, with Bayesian regression spline in the first step for estimating the spline parameters. An algorithm for selecting an optimal knot with the least GCV is proposed in this step. For the second step, a criterion introduced by Varah (1982) and our proposed criterion with a non-likelihood based with a spline approach are used to estimate the SDE parameters. The paper is organized as follows. The next three sections introduce the general form of SDE followed by the theoretical outline of regression spline and the derivation of non parametric criterion for SDE parameter estimation. The fourth section onwards are devoted to discussions on the outcome of research followed by some conclusions.

PROPOSED METHODS

Consider a one dimensional Itô SDE given by:

$$\frac{dx(t)}{dt} = f(x,t,\theta) + g(x,t,\phi)\frac{dW(t)}{dt}, \quad (2)$$

where $f(x,t,\theta)$ is the average drift term, $g(x,t,\phi)$ is the diffusion term and $dW(t)$ is the Brownian noise. A two-step method with a non-likelihood based approach will be used to estimate the structural parameter of SDE by firstly estimating the parameters of regression spline with Bayesian approach in the first step and next estimate the parameters of the drift and diffusion term in SDE in the second step.

TWO STEP METHOD: THE FIRST STEP

The general equation of regression splines with truncated power series basis is:

$$s(t) = \sum_{j=1}^{m} \alpha_j t^{j-1} + \sum_{j=1}^{k} \delta_j \left(t-\xi_j\right)_+^{m-1}, \tag{3}$$

where s is known as a spline of order m with knots ξ_1, …, ξ_k, t is the independent variable, α_1, …, α_m and δ_1, …, δ_k are some sets of coefficients. Given a choice of $\lambda = (\xi_1, \xi_2, \ldots, \xi_k)$, let $x_1(t) = 1$, $x_2(t) = t$, …, $x_m(t) = t^{m-1}$, $x_{m+1}(t) = \left(t-\xi_1\right)_+^{m-1}$ and set $\beta = (\alpha_1, \ldots, \alpha_m, \delta_1, \ldots, \delta_k)$. The least squares spline estimator can be rewritten as $s(t) = \sum_{j=1}^{m+k} \beta_{\lambda j} x_j(t)$. From Eubank (1988), adhoc rules for locating knots are as follows: for $m = 2$, linear splines, place knots at points where the data show a change in slope; for $m = 3$, quadratic splines, the knots are located near the local minimum, maximum or inflexion points of the data and for $m = 4$, cubic splines, the knots are arranged near the inflexion points in the data.

In the first step of the proposed procedure, the values of α and δ are estimated using the Bayesian approach with the assumption of normal error and diffuse prior for the parameters. The estimation is carried out in Winbugs with 10^6 MCMC simulations after selecting the suitable number and location of knots. The best number and location of knots of the fitted spline are determined by calculating the Generalized Cross Validation (GCV):

$$GVC(\lambda) = \frac{\sum_{i=1}^{n}\left(y_i - \sum_{j=1}^{m+k} \beta_{\lambda j} x_j\left(t_i\right)\right)^2}{\left(1-\frac{(m+k)}{n}\right)^2}, \tag{4}$$

where y_i is the observed data, $s(t) = \sum_{j=1}^{m+k} \beta_{\lambda j} x_j\left(t_i\right)$ is the spline equation, k is the number of knots, m is the degree of splin and n is the number of the observations. The least value of GCV indicates the best fit of $s(t)$.

TWO-STEP METHOD: THE SECOND STEP

In the second step, we first estimate the parameter of the average drift equation by:

$$\text{minimizing } \sum_{i=1}^{n}\left[\hat{x} - f\left(\hat{x}, t, \theta\right)\right]^2, \tag{5}$$

where $\hat{x}_i$ is the derivative of the spline approximation of the true solution of ordinary differential equation (ODE) which is used to represent the average drift term, $\hat{x}$ the consistent estimator of the true solution, t is the independent variable and θ the parameter of ordinary differential equation. $f(\hat{x}, t, \theta)$ is the ODE used to represent the average drift term in SDE. Criterion (5) was primarily introduced by Varah (1982) and used by many authors including Brunel (2008). By minimizing (5), it is expected to minimize the deviation between the differential of spline and the estimated ordinary differential equation.

A new criterion is proposed in estimating the diffusion term. Consider a one dimensional Itô SDE given by,

$$\frac{dx(t)}{dt} = f(x,t,\theta) + g(x,t,\phi)\frac{dW(t)}{dt}, \tag{6}$$

where $f(x, t, \theta)$ is the average drift term, $g(x, t, \phi)$ is the diffusion term, and $dW(t)$ is the Brownian noise. Rearranging (6), we have:

$$\frac{dx(t)}{dt} - f(x,t,\theta) = g(x,t,\theta)\frac{dW(t)}{dt}, \tag{7}$$

discrete approximation of true solution of SDE using numerical descretization such as Euler or Milstein may be considered. Here, Milstein numerical approximation is used to estimate the solution of (6) given as:

$$x_{i+1} = x_i + h_i f\left(x_i\right) + g\left(x_i\right)\Delta W_i + \frac{1}{2} g\left(x_i\right) g'\left(x_i\right)\left(\left(\Delta W_i\right) - h_i\right).$$

SDE can be rewritten in a form of difference quotient such as in Taylan et al. (2008) where:

$$\dot{\bar{x}}_i \approx f\left(x_i\right) + g\left(x_i\right)\frac{\Delta W_i}{h_i} + \frac{1}{2} g\left(x_i\right) g'\left(x_i\right)\left(\frac{\left(\Delta W_i\right)^2}{h_i} - 1\right), \tag{8}$$

(RHS is the Milstein scheme). In order to estimate θ and ϕ they minimize:

$$\sum_{i=1}^{N}\left(\dot{\bar{x}}_i - f\left(x_i\right) - g\left(x_i\right)\frac{\Delta W_i}{h_i} - \frac{1}{2} g\left(x_i\right) g'\left(x_i\right)\left(\frac{\left(\Delta W_i\right)^2}{h_i} - 1\right)\right)^2, \tag{9}$$

where x_s' s are the observed data, $i = 1, \ldots, N$, Therefore, there is an argument to support (6) can be approximated by a difference quotient $\dot{\bar{x}}_i = \frac{x_{i+1} - x_i}{h_i}$. The average drift term in (6) is of ODE form and approximated by regression spline (Brunel 2008; Varah 1982). Thus, $f(x,t,\theta)$ can be approximated by $f(\hat{x},t,\theta)$. Therefore, the approximation of (7) is obtained, that is $\dot{\bar{x}}_i - \hat{x} \approx g(x,t,\phi)\frac{dW(t)}{dt}$. In order for the values of ϕ to produce the least variation between both RHS and LHS of (7), the squared difference of both quantities are minimized. By rearranging the terms in the second step in order to estimate ϕ, a new criterion is introduced as follows:

$$\text{minimizing } \sum_{i=1}^{n}\left[\dot{\bar{x}}_i - \hat{x}_i - g(x,t,\phi)\frac{dW(t)}{dt}\right]^2. \tag{10}$$

The Brownian motion $dW(t)$ is approximated by $\Delta W_t \sim N(0, h_i)$, where $\Delta W_t = Z_i\sqrt{\Delta t_i} = Z_i\sqrt{h_i}$ and

$\frac{dW(t)}{dt} \approx \frac{Z_i}{\sqrt{h_i}}$. It can be seen that Z_i has a standard normal distribution. The approximated values of $\frac{dW(t)}{dt}$ can be generated by standard normal random numbers generator in MATLAB.

RESULTS AND DISCUSSION

DATA BACKGROUND

In this section, the proposed method will be applied to a commonly used financial model which is a form of linear SDE given as:

$$dX(t) = \theta X(t)dt + \phi X(t)dW(t) \quad t \in [0,T], \tag{11}$$

where $\theta X(t)$ is the average drift term, $\phi X(t)$ is the diffusion term and $dW(t)$ is the Brownian noise. A set of data collected from Yahoo Finance website namely share prices of Petronas Gas Bhd. was used to estimate the diffusion and drift parameter of stochastic differential equation (Figure 1).

Given $Wt \sim N(0,1)$, $\Delta W_t = W_{t_{i+1}} - W_{t_i}$ and $\Delta t = t_{i+1} - t_i$, then $\Delta W_t \sim N(0, \Delta t)$ and $Z = \frac{\Delta W_t}{\sqrt{\Delta t}} \sim N(0,1)$. We fixed $h = 1$ for every interval, thus, the distribution of ΔW_t is a standard normal distribution.

REGRESSION SPLINE

Heuristic Knot Selection Firstly, a suitable nonparametric representation of the data will be determined using Bayesian regression spline. The knots are placed heuristically through visual inspection of the scatter plot with only linear splines, $m = 2$ will be considered. The spline parameters will also be estimated by Bayesian approach with Winbugs software at 10^6 Markov Chain Monte Carlo (MCMC) simulations.

Table 1 lists the corresponding GCV values of Bayesian regression spline for opening share prices of Petronas Gas Bhd. The values of θ were estimated using (5). The estimated values of θ at respective knot locations shown with the least GCV at $t = 150$ parameter of the regression spline is -0.0001. Figure 2 shows the plot of the regression spline model.

In order to estimate the value of ϕ, (10) is minimized. Table 2 summarizes estimated values of ϕ at three different runs 1,000, 5,000 and 10,000 for Bayesian regression spline at knot location with the least GCV.

The MSE from stochastic model is the least at 1000 run of Brownian noise simulation that is 0.2808 with corresponding mean value of ϕ is 0.0001665, standard

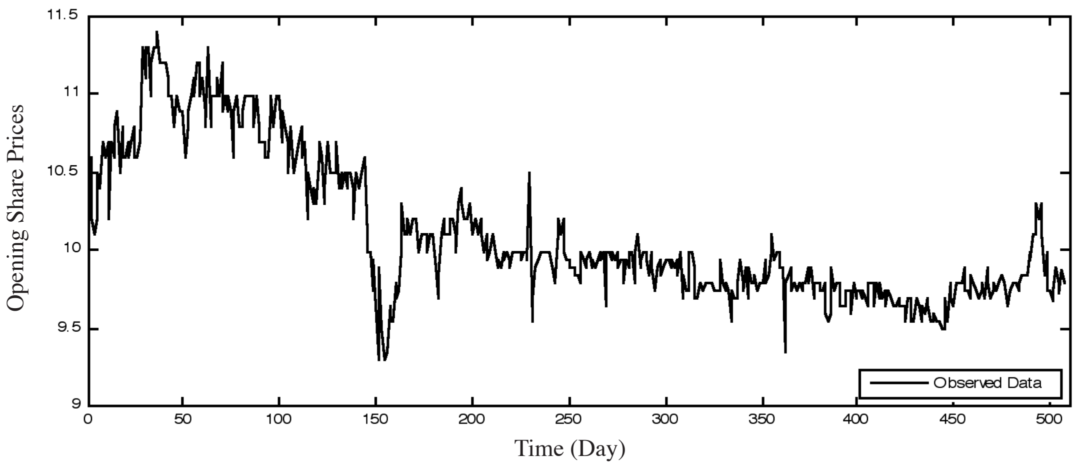

FIGURE 1. Real opening share prices of Petronas Gas Bhd

TABLE 1. GCV values of Bayesian regression spline of Opening share prices

No. of knots (k)	Knots	GCV
1	70	31
1	150	24
1	300	29
2	70,150	1512
2	70,300	89
2	150,300	12658
3	70,150,300	1518

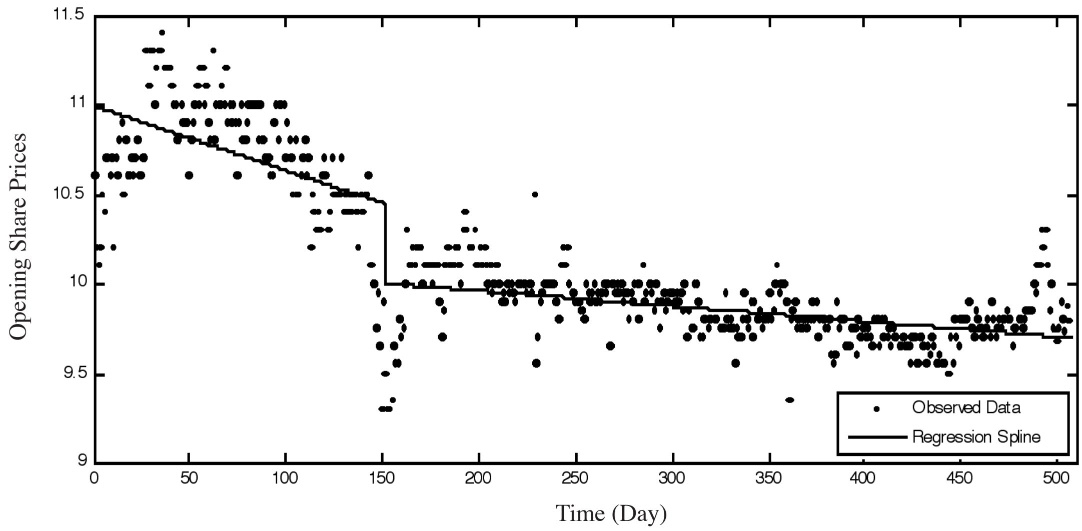

FIGURE 2. Observed values and regression spline model at $\xi = 150$

TABLE 2. Estimated values of ϕ at several runs for $\xi = 150$

Run	Mean	Standard deviation	Lowerbound 95% C.I	Upperbound 95% C.I
1,000	0.0001665	0.0006211	0.0001123	0.0002207
5,000	0.0015000	0	0.0015000	0.0015000
10,000	0.0004245	0	0.0004245	0.0004245

deviation 0.0006211 and 95% confidence interval (0.0001123, 0.0002207). Figure 3 depicts the plot of predicted and observed values of SDE model (10) for the opening share prices of Petronas Gas Bhd.

The prediction quality of stochastic model is measured by the values of mean square error given as $\text{MSE} = \sum_{i=1}^{N} \frac{(x_i - f_i)^2}{N}$, where x_i is the observed values, f_i is the predicted values and N is total observations. The value of the MSE and RMSE (root mean square error) of the approximated true solution with parameters estimated by the two-step method is given in Table 3.

Optimal Knot Selection Selection of number of knots, location of knots and order of spline are subjective issues

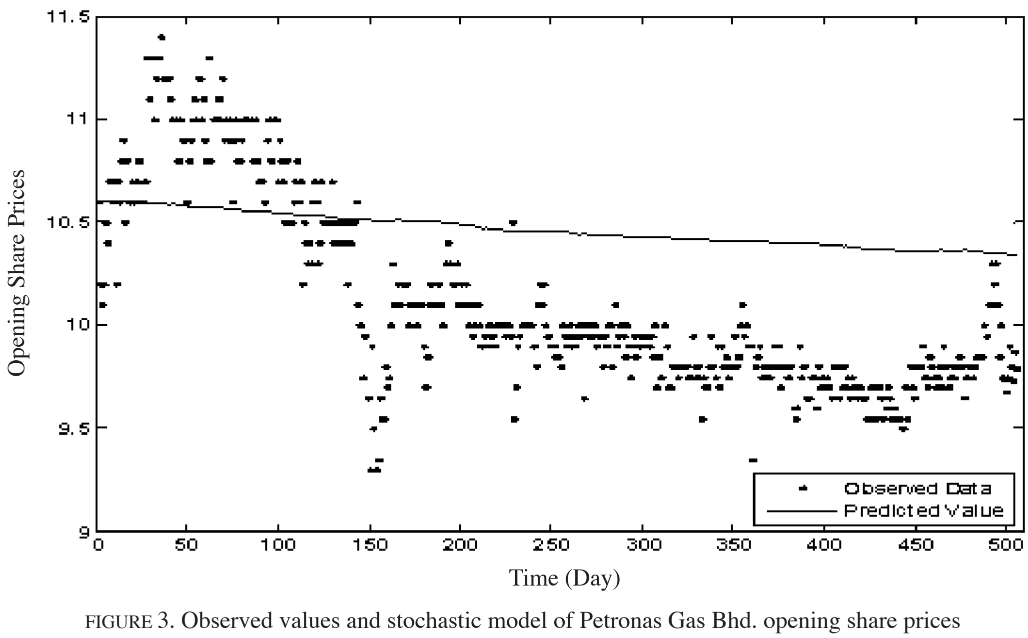

FIGURE 3. Observed values and stochastic model of Petronas Gas Bhd. opening share prices

TABLE 3. MSE and RMSE of stochastic logistic model with the two-step method of parameter estimation for ξ = 150 using Milstein approximation

Runs	1000	5000	10000
MSE	0.2375	0.2808	0.2714
RMSE	0.4874	0.5296	0.5210

and quite tedious if heuristic approach is made where the location of the knots are chosen through visual inspection of the scatter plot. To reduce the time and improve efficiency, an algorithm is introduced for selecting the best single knot and order of spline giving the least GCV. Algorithm for optimal single knot selection: data entry with knots are iteratively selected with the exception of the first and the last data; data splitting based on knots to first interval and second interval; parameter estimation for each interval and each order of spline $m = 2$, $m = 3$ and $m = 4$; combine predicted value for each interval of each order of spline; new matrix formation for the combined estimation; Evaluation of the models on the GCV values and find optimum ξ and GCV values corresponding to the least GCV for each order.

This algorithm is applied to the previous data set where the computation is done through MATLAB. The parameters of the regression spline are calculated by the least squares method. Table 4 shows that since the GCV of quadratic and cubic spline is approximately equivalent, quadratic spline is chosen since it is more convenient in terms of computation. The results show the best knot is at $\xi = 161$ with $m = 3$.

Bayesian regression spline models with optimal knot selection has a smaller GCV value (for all order of spline) compared with model with heuristic knot selection as shown in Tables 1 and 4. This supports the argument that optimal knot selection does improve the GCV values resulting in models with better fit. Using Winbugs at 10^6 MCMC simulations with the assumption of normal errors and diffusion priors the spline parameters at $\xi = 161$ are calculated and the value is given in Table 5. The plot is shown in Figure 4.

The plot of regression spline and the observed data are given in Figure 4.

Structural Parameter Estimation of SDE By employing criteria (5) and (11), the drift and diffussion parameters of the stochastic model in (10) are calculated at 1,000, 5,000 and 10,000 runs of the Brownian motion. The results are shown in Table 6.

The corresponding MSE's and RMSE's at each run is illustrated in Table 7. Table 7 shows that, the least MSE was obtained at 1,000 runs of the Brownian motions, thus the value of estimated ϕ at 1,000 equals 0.00120 is chosen for the stochastic model. The plot of the stochastic model is given in Figure 5.

CONCLUSION

In this paper, a new criterion to estimate the diffusion parameters of SDE was introduced. This method was simpler and could be an alternative to the classical likelihood approach. This will avoid computational difficulties encountered by such method. An algorithm was also introduced to select the optimal single knot and the order of knot via Bayesian method which overcame the tedious process of heuristic selection. Using option price data set of Petronas Gas Bhd., the improvement was illustrated by comparing the models with heuristic knot selection and optimal knot selection based on the GCV values and the MSE values. Both quantities have been reduced significantly. We use the Bayesian approach since this method is considered more efficient.

TABLE 4. Optimal single knots and GCV values for linear, quadratic and cubic splines

Data	Optimal knot (ξ)	GCV
Opening Share	Linear splines (m=2) : 143	20.46832
Price	Quadratic splines (m=3) : 161	14.03534
	Cubic splines (m=4) : 161	14.05705

TABLE 5. Regression spline parameters at optimal knot location

Data	m	Location of knot	Interval	Regression spline parameters $\beta = (\alpha_1, \ldots, \alpha_m, \delta_1, \ldots, \delta_k)$
Option Price	3	161	$t < 161$	11.08,-0.004982,-0.000003327
			$t \geq 161$	10.94,-0.005723,0.000006911, 0.000000008771

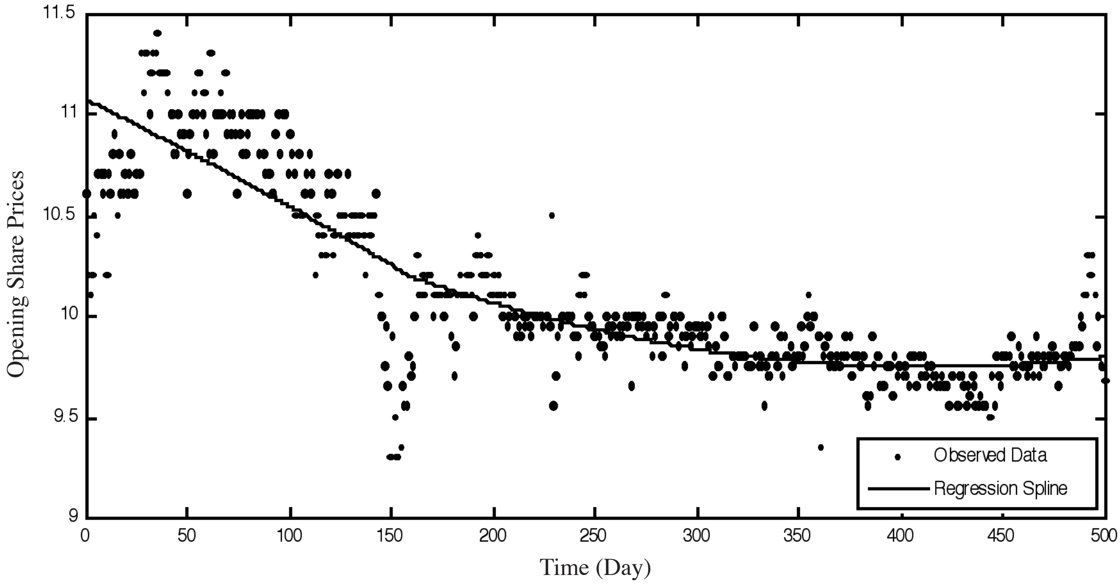

FIGURE 4. Observed values and regression spline model at $\xi = 161$

TABLE 6. The estimated parameters of the drift and diffusion parameters of the stochastic model

Knot	Estimated θ	Runs	Estimated ϕ
161	-0.000254	1,000	0.00120
		5,000	0.00039
		10,000	0.00015

TABLE 7. MSE of stochastic logistic model with the two-step method of parameter estimation for $\xi = 161$

	Runs	1,000	5,000	10,000
Milstein approximation	MSE	0.1077	0.1233	0.1286
	RMSE	0.3281	0.3511	0.3587

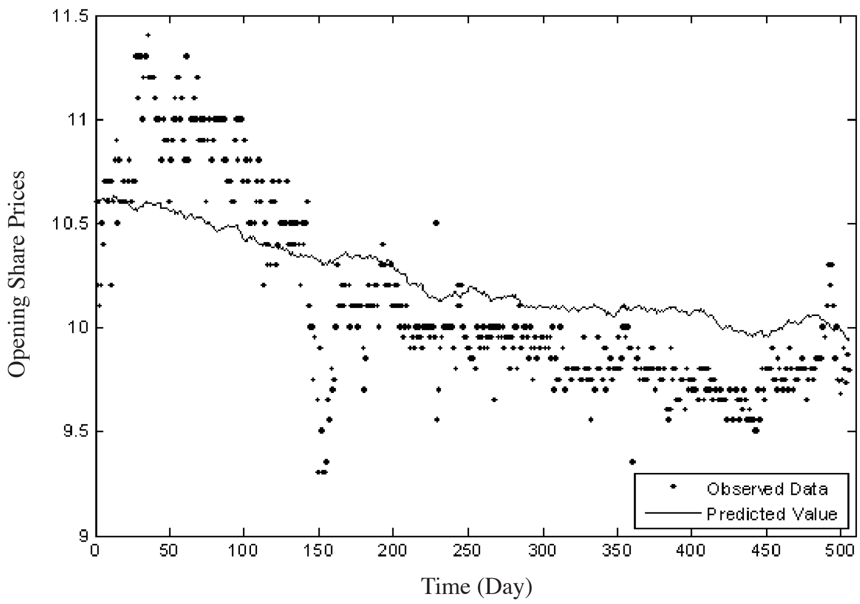

FIGURE 5. Observed values and stochastic model of opening share prices

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