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## Free Convection Boundary Layer Flow of a Nanofluid from a Convectively Heated Vertical Plate with Linear Momentum Slip Boundary Condition

(Aliran Lapisan Sempadan Olakan Bebas Nanobendalir terhadap Plat Menegak yang dipanaskan Secara Berolak dengan Syarat Sempadan Gelinciran Momentum Linear)

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### ABSTRACT

*Two dimensional steady laminar boundary layer flow of a nanofluid over a convectively heated vertical flat plate with linear momentum slip boundary condition has been studied numerically. The governing boundary layer equations are non-dimensionalized and transformed into a two point boundary value problem of coupled nonlinear ordinary differential equations in similarity variable before being solved numerically. The resulting equations with corresponding boundary conditions have been solved numerically by Maple 13 which uses Runge-Kutta-Fehlberg fourth- fifth order numerical algorithm for solving nonlinear ordinary boundary value problems. Our analysis reveals that the similarity solution is possible if the convective heat transfer coefficient is directly proportional to  $x^{-1/4}$ , where  $x$  is the axial distance from the leading edge of the plate. Solutions depend on the seven parameters: Prandtl number, buoyancy ratio, Brownian motion, thermophoresis, Lewis number, momentum slip and convective heat transfer. The effects of the governing parameters on the flow and heat transfer characteristics have been shown graphically and discussed. Comparisons of the present numerical solution with the existing results in the literature are made and our results are in very good agreement. Results for the skin friction factor, the reduced Nusselt and the Sherwood numbers are provided in tabular form for various values of the convective heat transfer parameter. It is found that the skin friction coefficient reduces with the momentum slip and the buoyancy ratio parameters whilst it enhances with the convective heat transfer parameter. It is also found that mass transfer rate enhances with the Lewis number and the convective heat transfer parameter whilst it falls with the thermophoresis parameter.*

*Keywords: Free convection; momentum slip boundary condition; nanofluids; thermal convective boundary condition*

### ABSTRAK

*Aliran lapisan sempadan berlaminar dua dimensi yang mantap bagi nanobendalir ke atas plat menegak yang dipanaskan secara berolak dengan syarat sempadan gelinciran momentum linear dikaji secara berangka. Persamaan menakluk lapisan sempadan dijadikan tanpa dimensi dan dijelmakan kepada masalah nilai sempadan dua titik yang terdiri daripada persamaan perbezaan biasa tidak linear terganding dalam bentuk pemboleh ubah keserupaan sebelum diselesaikan secara berangka. Persamaan yang terhasil bersama-sama syarat sempadan yang sepadan telah diselesaikan secara berangka melalui Maple 13 dengan menggunakan algoritma berangka Runge-Kutta-Fehlberg peringkat keempat-kelima untuk menyelesaikan masalah nilai sempadan tak linear. Analisis ini mendedahkan bahawa penyelesaian keserupaan adalah mungkin sekiranya pekali pemindahan haba berolak berkadar langsung dengan  $x^{-1/4}$ , dengan  $x$  ialah jarak paksi dari pinggir depan plat. Penyelesaian bergantung kepada parameter berikut: pergerakan Brownian, perpindahan haba, nombor Lewis, nombor Prandtl, gelinciran momentum dan pemindahan haba berolak. Kesan parameter menakluk ke atas aliran dan ciri-ciri pemindahan haba telah dipamerkan secara bergraf dan dibincangkan. Perbandingan keputusan berangka kajian ini dilakukan dan didapati menepati keputusan yang sedia ada dalam kajian lepas. Keputusan bagi faktor geseran kulit, nombor Nusselt terturun dan nombor Sherwood diberikan dalam bentuk berjadual untuk pelbagai nilai parameter pemindahan haba berolak. Didapati bahawa faktor geseran kulit mengurang dengan slip momentum dan parameter nisbah keapungan dan pada masa yang sama ia meningkat dengan parameter pemindahan haba perolakan. Kadar pemindahan jisim juga didapati meningkat dengan nombor Lewis dan parameter pemindahan haba berolak dan pada masa yang sama ia jatuh dengan parameter thermophoresis.*

*Kata kunci: Nanobendalir; olakan bebas; syarat sempadan gelinciran momentum; syarat sempadan haba berolak*

### INTRODUCTION

Boundary value problems involving the principles of transport phenomena where the results are directly influenced by the process of fluid motion known as

convection process. When fluid motion is determined by the external agent for examples fan, blower, the wind, or the motion of the heated object itself, the process is known as forced convection. On the other hand, when fluid

flows naturally for example due to the density difference which results from the temperature or the concentration difference in a body force field, such as the gravitational field, centrifugal force field, the Coriolis force field and the electromagnetic force field, the process is known as natural convection (Ram 1991). Natural convection phenomena can be observed in the atmosphere, in bodies of water, adjacent to domestic heating radiators, in nuclear reactor, in cooling system, in electronic power supplies and so forth (Incropera et al. 2007). In general, transport phenomena are governed by complicated system of partial differential equations and their exact solutions is quite complicated or even impossible. Similarity analysis technique which reduces the number of independent variables and transform the partial differential equations with the pertinent boundary conditions to ordinary differential equations, followed by numerical computations has been used and proved to be efficient to solve transport problem for the last few decades. Recently, nonsimilar solutions for the transport problems are readily available and the finite difference method and in some cases the finite element method, finite volume, are so far may be best tool to obtain such nonsimilar solutions. Interested readers are referred to the books by Patankar and Spalding (1970) and Jaluria (1980).

Nanoparticles are made from various materials, such as oxide ceramics ( $\text{Al}_2\text{O}_3$ , CuO), nitride ceramics (AlN, SiN), carbide ceramics (SiC, TiC), metals (Cu, Ag, Au), semiconductors, carbon nanotubes and composite materials such as alloyed nanoparticles or nanoparticle core–polymer shell composites. Nanofluids consist of a base fluid and ultrafine nanoparticles aim to achieve the maximum possible thermal properties at the minimum possible concentrations (preferably <1% by volume) by uniform dispersion and stable suspension of nanoparticles (preferably <10 nm) in host fluids (Kakac et al. 2010; Murshed et al. 2008). Nanofluids are able to enhance thermophysical properties such as thermal conductivity; thermal diffusivity, viscosity and convective heat transfer coefficients compared with those of base fluids like oil or water (Kaufui & Omar 2010). Many studies on nanofluids are being conducted by scientists and engineers due to their diverse technical and biomedical applications. Examples include nanofluid coolant: electronics cooling, vehicle cooling, transformer cooling, super powerful and small computers cooling and electronic devices cooling; medical applications: cancer therapy and safer surgery by cooling and process industries; materials and chemicals: detergency, food and drink, oil and gas, paper and printing and textiles. Ultra high-performance cooling is necessary for many industrial technologies. However, poor thermal conductivity is a drawback in developing energy-efficient heat transfer fluids necessary for ultra high–performance cooling. One possible mechanism for increasing thermal conductivity of nanofluids is the Brownian motions of the nanoparticles inside the base fluids (Kandasamy et al. 2011). A good number of research papers have been published on nanofluids to understand their diverse performance so that they can be used to elevate the heat transfer performance

in wide range of applications. A comprehensive study of convective transport in nanofluids was made by Buongiorno (2006). Nield and Kuznetsov (2009) have investigated the problem of natural convection flow near a vertical plate in a porous medium which is saturated by a nanofluid. The nanofluid model of Nield and Kuznetsov (2009) incorporates the effects of Brownian motion and thermophoresis. Kuznetsov and Nield (2010a) presented a similarity solution of natural convective boundary-layer flow of a nanofluid past a vertical plate. They have shown that the reduced Nusselt number is a decreasing function of each of buoyancy-ratio number  $Nr$ , a Brownian motion number  $Nb$  and a thermophoresis number  $Nt$ . Analytical study on the onset of convection in a horizontal layer of a porous medium with the Brinkman model and the Darcy model filled with a nanofluid were presented by Kuznetsov and Nield (2010b, 2010c). Godson et al. (2010) presented the recent experimental and theoretical studies on convective heat transfer in nanofluids, their thermo-physical properties and applications and clarifies the challenges and opportunities for future research. Recently, Arifin et al. (2011) studied forced convective boundary layer flow and heat transfers near the stagnation point on a permeable stretching/shrinking surface in a nanofluid. Gorla and Chamkha (2011) studied the natural convective flow past a horizontal plate in a porous medium filled with a nanofluid. Very recently, Uddin et al. (2012) studied the free convective boundary layer flow of a nanofluid over a horizontal plate with linear momentum slip model.

In case of fluid flows in micro electro mechanical systems, the no slip condition at the solid–fluid interface is no longer valid and must be replaced by slip condition (Aziz 2010). The slip flow model states a relationship between the tangential components of the velocity at the surface and the velocity gradient normal to the surface (Hak 2002). Many researchers studied the effect of linear momentum slip  $|u| = l \left| \frac{\partial u}{\partial y} \right|$ , and nonlinear slip  $|u| = l \left| \frac{\partial u}{\partial y} \right|^n$ , where  $l > 0$  is the slip length and  $n > 0$  is a certain power parameter, on the hydrodynamic / magnetohydrodynamic boundary layer flow with heat / mass transfer of free / forced / combined convection past different geometries. This includes Martin and Boyd (2010), Mathews and Hill (2007), Mukhopadhyay and Andersson (2009) and Wang (2009).

All the above works assumed either isothermal or isoflux thermal boundary conditions. However, the idea of using the thermal convective heating boundary condition was first introduced by Aziz (2009) to analyze the Blasius flow. After his pioneering works, several authors used convective boundary condition to study various problems of transport phenomena. Representative examples are the papers by Aziz et al. (2012), Bachok et al. (2011), Ishak (2010), Magyari (2011), Makinde and Aziz (2011), Uddin et al. (2012), Yacob et al. (2011) and Yao et al. (2011). So far as we are concern, the problem of natural convective flow of a nanofluid over a flat solid heated plate with

thermal convective and linear momentum slip boundary conditions has yet to be reported in the literature in a thorough manner.

In this paper, our main objective was to investigate the effect of linear momentum slip and convective heat transfer parameters on the dimensionless axial velocity, temperature, nanoparticle volume fraction, the skin friction factor, the heat transfer rate and the rate of nanoparticle volume fraction transfer within the respective boundary layers. The governing boundary layer equations with the associated boundary conditions have been transformed into a two-point boundary value problem in similarity variable  $\eta$ . The transformed equations have been solved numerically. The effects of the emerging flow controlling parameters on the dimensionless velocity, temperature, nanoparticle volume fraction, the skin friction factor, the rate of heat transfer and the rate of nanoparticle volume fraction have been shown graphically. The possible application area of the present nanofluids model is in advanced cooling systems, micro/nano electromechanical systems and many others. These applications include manufacturing processes of a continuous casting, glass fiber production, metal extrusion, geothermal reservoirs, processes involving polymer composites, hot rolling, manufacturing of plastics, textiles and paper productions.

#### FORMULATION OF THE PROBLEM

We consider a two-dimensional free convective flow of a nanofluid over a convectively heated solid vertical flat plate with linear momentum slip boundary condition. Left surface of the plate is heated by convection from a hot fluid at temperature  $T_f(\bar{x})$  which gives a heat transfer coefficient  $h_f(\bar{x})$ . Temperature of the right surface of the plate is  $T_w$ . The nanoparticle fraction at the wall is  $C_w$ . The field variables are the velocity vector  $\bar{v}$ , the temperature  $T$  and the nanoparticle volume fraction  $C$ . The ambient values of the temperature and nanoparticle volume fraction are denoted by  $T_\infty$  and  $C_\infty$ , respectively. It is assumed that  $T_f > T_w > T_\infty$  and all the physical properties are constant except density in the body force term in the momentum equation. The Oberbeck–Boussinesq approximation is utilized and the four field equations are the equation of conservation of mass, momentum, thermal energy and the nanoparticles volume fraction. These equations can be written in terms of dimensional forms as (Kuznetsov & Nield 2010).

$$\bar{\nabla} \cdot \bar{v} = 0, \quad (1)$$

$$\rho_f \left( \frac{\partial \bar{v}}{\partial t} + \bar{v} \cdot \bar{\nabla} \bar{v} \right) = -\bar{\nabla} p + \mu \bar{\nabla}^2 \bar{v} + [C \rho_p + (1-C) \{ \rho_f (1 - \beta (T - T_\infty)) \}] \mathbf{g}, \quad (2)$$

$$(\rho c)_f \left( \frac{\partial T}{\partial t} + \bar{v} \cdot \bar{\nabla} T \right) = k \bar{\nabla}^2 T + (\rho c)_p \left[ D_B \bar{\nabla} C \cdot \bar{\nabla} T + \left( \frac{D_T}{T_\infty} \right) \bar{\nabla} T \cdot \bar{\nabla} T \right], \quad (3)$$

$$\frac{\partial C}{\partial t} + \bar{v} \cdot \bar{\nabla} C = D_B \bar{\nabla}^2 C + \frac{D_T}{T_\infty} \bar{\nabla}^2 T. \quad (4)$$

We write  $\bar{v} = (\bar{u}, \bar{v})$ . Here  $\rho_f$  is the density of the base fluid,  $\mu$ ,  $k$  and  $\beta$  are dynamic viscosity, thermal conductivity and volumetric volume expansion coefficient of the nanofluid and  $\rho_p$  is the density of the particles.  $\mathbf{g}$  is the acceleration due to gravity. Here  $D_B$  stand for the Brownian diffusion coefficient and  $D_T$  stands for the thermophoretic diffusion coefficient. We consider a steady state flow. In keeping with the Oberbeck–Boussinesq approximation and an assumption that the nanoparticle concentration is sufficiently dilute and with a suitable choice for the reference pressure, we can linearize the momentum equation and write (2) as (Kandasamy et al. 2011)

$$\rho_f \left( \frac{\partial \bar{v}}{\partial t} + \bar{v} \cdot \bar{\nabla} \bar{v} \right) = -\bar{\nabla} p + \mu \bar{\nabla}^2 \bar{v} + [(\rho_p - \rho_{f\infty})(C - C_\infty) + (1 - C_\infty) \rho_{f\infty} \beta (T - T_\infty)] \mathbf{g}, \quad (5)$$

An order of magnitude analysis of the continuity, momentum, energy and nanoparticle volume fraction equations using boundary layer approximation is done to neglect the small order terms and hence we obtain

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0, \quad (6)$$

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 \bar{u}^2}{\partial y^2} - \rho_f \left( \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} \right) \quad (7)$$

$$+ [1 - C_\infty] \rho_{f\infty} \beta g \beta (T - T_\infty) - (\rho_p - \rho_{f\infty}) g (C - C_\infty), \quad (8)$$

$$\frac{\partial p}{\partial y} = 0,$$

$$\bar{u} \frac{\partial T}{\partial x} + \bar{v} \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \tau \left[ D_B \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} + \frac{D_T}{T_\infty} \left( \frac{\partial T}{\partial y} \right)^2 \right], \quad (9)$$

$$\bar{u} \frac{\partial C}{\partial x} + \bar{v} \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty} \frac{\partial^2 T}{\partial y^2}, \quad (10)$$

where  $\tau = \frac{(\rho c)_p}{(\rho c)_f}$  is the ratio of nanoparticle heat capacity and  $\alpha = \frac{k}{(\rho c)_f}$  the base fluid heat capacity and is the thermal diffusivity of the fluid.

The boundary conditions are taken to be

$$\bar{u} = N_1 \bar{v} \frac{\partial \bar{u}}{\partial y}, \bar{v} = 0, -k \frac{\partial T}{\partial y} = h_f(\bar{x})(T_f - T_w),$$

$$C = C_w, \text{ at } \bar{y} = 0,$$

$$\bar{u} \rightarrow 0, T \rightarrow T_\infty, C \rightarrow C_\infty \text{ as } \bar{y} \rightarrow \infty. \quad (11)$$

Here  $(\bar{u}, \bar{v})$ : the velocity components along and perpendicular to the plate ( $\bar{x}$  and  $\bar{y}$  - axis),  $N_1$  is the velocity slip factor with dimension (velocity)<sup>-1</sup>. One can eliminate  $p$  from (7) and (8) by cross-differentiation. Following dimensionless variables are introduced to non-dimensionalise (6)-(11):

$$\begin{aligned} x &= \frac{\bar{x}}{L}, \quad y = \frac{\bar{y} Ra^{1/4}}{L}, \quad u = \frac{\bar{u} L}{\alpha Ra^{1/2}}, \\ v &= \frac{\bar{v} L}{\alpha Ra^{1/4}}, \quad \theta = \frac{T - T_\infty}{T_f - T_\infty}, \quad \phi = \frac{C - C_\infty}{C_w - C_\infty}. \end{aligned} \tag{12}$$

Here  $L$  is the plate characteristic length and  $Ra = \frac{(1 - C_\infty) g \beta \rho_{f\infty} \Delta T L^3}{\alpha \nu}$  is the Rayleigh number.

A stream function  $\psi$  defined by  $u = \frac{\partial \psi}{\partial y}$ ,  $v = -\frac{\partial \psi}{\partial x}$  is introduced into (6)-(11) to reduce the number of dependent variables and the number of equations. Note that (6) is satisfied identically. We are then left with the following three dimensionless equations:

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} - \text{Pr} \frac{\partial^3 \psi}{\partial y^3} - \text{Pr}[\theta - Nr\phi] = 0, \tag{13}$$

$$\frac{\partial \psi}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} - \frac{\partial^2 \theta}{\partial y^2} - Nb \frac{\partial \theta}{\partial y} \frac{\partial \phi}{\partial y} - Nt \left( \frac{\partial \theta}{\partial y} \right)^2 = 0, \tag{14}$$

$$\frac{\partial \psi}{\partial y} \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial y} - \frac{1}{Le} \frac{\partial^2 \phi}{\partial y^2} - \frac{Nt}{Nb} \frac{1}{Le} \frac{\partial^2 \theta}{\partial y^2} = 0. \tag{15}$$

subject to the boundary conditions:

$$\begin{aligned} \frac{\partial \psi}{\partial y} &= N_1 v \frac{Ra^{1/4}}{L} \frac{\partial^2 \psi}{\partial y^2}, \quad \frac{\partial \psi}{\partial x} = 0, \\ \frac{\partial \theta}{\partial y} &= -\frac{Lh_f(x)}{\kappa Ra^{1/4}} (1 - \theta), \quad \phi = 1 \text{ at } y = 0, \\ \frac{\partial \psi}{\partial y} &\rightarrow 0, \quad \theta \rightarrow 0, \quad \phi \rightarrow 0 \text{ as } y \rightarrow \infty. \end{aligned} \tag{16}$$

We adopt the following similarity transformations (Kuznetsov & Nield 2010):

$$\eta = \frac{y}{\sqrt[4]{x}}, \quad \psi = x^{3/4} f(\eta), \quad \theta = \theta(\eta), \quad \phi = \phi(\eta). \tag{17}$$

On substituting the transformations in (17) into the governing (13)-(15), we obtain the following similarity equations:

$$f''' + \frac{1}{4\text{Pr}} (3ff'' - 2f'^2) + \theta - Nr\phi = 0, \tag{18}$$

$$\theta'' + \frac{3}{4} f\theta' + Nb \theta'\phi' + Nt \theta'^2 = 0, \tag{19}$$

$$\phi'' + \frac{3}{4} Le f\phi' + \frac{Nt}{Nb} \theta'' = 0. \tag{20}$$

subject to the boundary conditions:

$$\begin{aligned} f(0) &= 0, \quad f'(0) = a f''(0), \quad \theta'(0) = -\gamma[1 - \theta(0)], \\ \phi(0) &= 1, \quad f'(\infty) = \theta(\infty) = \phi(\infty) = 0, \end{aligned} \tag{21}$$

where the parameters are defined by  $\text{Pr} = \nu / \alpha$  is Prandtl number,  $Nt = \tau D_T \Delta T / \alpha T_\infty$  is the thermophoresis parameter,  $t = \tau D_B \Delta C / \alpha$  is the Brownian motion parameter,  $Nr = (\rho_p - \rho_{f\infty}) \Delta C / \rho_{f\infty} \beta \Delta T (1 - C_\infty)$  is the buoyancy ratio parameter,  $Le = \alpha / D_B$  is the Lewis number,  $a = Ra^{1/4} / LN_1(x) x^{-1/4}$  is the slip parameter,  $\gamma = h(x) x^{1/4} k / Ra^{1/4} k$  is the thermal convective parameter. For true similarity solutions slip factor  $N_1(x)$  must be proportional to  $x^{1/4}$  whilst the convective heat transfer coefficient is proportional to  $x^{-1/4}$ . Primes denote derivative with respect to  $\eta$ .

It is interesting to note that if we put  $a = Nt = 0$ ,  $Nb \rightarrow 0$  and  $\gamma \rightarrow \infty$ , we have the same equations as derived by Khair and Bejan (1985). It is also note that when  $a = 0$  and  $\gamma \rightarrow \infty$ , we have the same equations as derived recently by Kuznetsov and Nield (2010).

Quantities of practical interest are the skin friction factor  $C_f$ , the reduced Nusselt number  $Nur$  and the reduced Sherwood number  $Shr$  which are proportional to  $f''(0)$ ,  $-\theta'(0)$  and  $-\phi'(0)$  respectively.

#### NUMERICAL SOLUTION

The set of coupled non-linear ordinary differential equations (18)- (20) with boundary conditions in (21) formed a two point boundary value problem and have been solved numerically by using an efficient Runge-Kutta-Fehlberg fourth-fifth order numerical method under Maple 13 proposed by Aziz (2009). The accuracy of the method has been tested in various transport problems. As a further confirmation, very recently the method was applied by Aziz et al. (2011), Makinde and Aziz (2011) and Khan and Aziz (2011) and found to reproduce their results well. The asymptotic boundary conditions given in (21) were replaced by finite values of 8 to 15 for similarity variable  $\eta_{\max}$ . The choice of  $\eta_{\max}$  ensured that all numerical solutions approached the far field asymptotic values in the correct manner. This is sometimes overlooked. It was noticed by Pantokratotars (2009) that in the convective heat transfer problems some results are erroneous as the graphs for the velocity, temperature and the nanoparticle volume fraction (concentration) distributions in the boundary layers do not approach the correct values in as asymptotic manner due to small value of  $\eta_{\max}$ .

#### RESULTS AND DISCUSSION

The effects of the emerging flow controlling parameters on the dimensionless axial velocity, temperature, the nanoparticle volume fraction, the skin friction factor, the rate of heat and nanoparticle volume fraction transfer are investigated and presented graphically in Figures 1-7. The results of the reduced Nusselt number  $-\theta'(0)$  is



compared to the available results for a special case which is shown in Table 1 and found to be in good agreement. This shows the validity of our numerical results for other cases. Variations of the skin friction factor, the rate of heat and nanoparticle volume fraction transfer with the convective heat transfer parameter ( $\gamma$ ) are depicted in Table 2. It is found that the skin friction factor which is proportional to  $f''(0)$  enhances with the rising of  $\gamma$  whilst it falls with the rising of  $a$ .

Figure 1 shows the effects of the convective heat transfer parameter on the dimensionless axial velocity. Note that the fluid velocity is zero at the plate surface and increases gradually away from the plate and then approaches toward the free stream value (here zero) satisfying the boundary conditions. It is apparent that a rise in the intensity of convective surface heat transfer  $\gamma$  produces a slight rises in the fluid velocity within the boundary layer. Physically, rising convective heat transfer parameter  $\gamma$  implies rising the heat transfer coefficient and consequently the cool fluid on right side of the plate is heated up by the hot fluid on the left hand side of plate, making it become lighter and flow faster. A cross flow occur at  $\eta = 3.038$  (approx.). The behavior of the velocity after cross flow is opposite of that before cross flow. This cross flow may be because of combined effects of the all parameters.

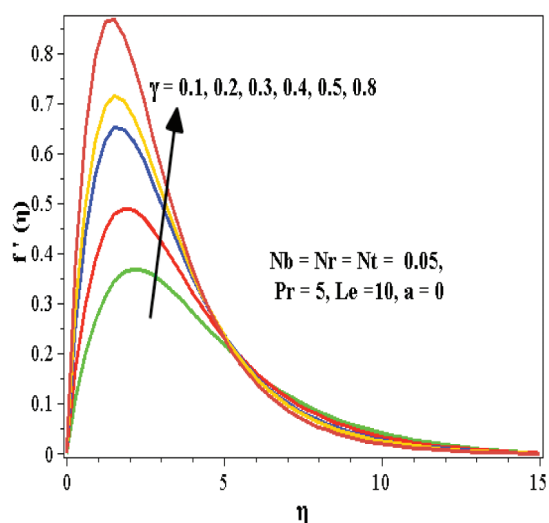


FIGURE 1. Effects of convective heat transfer parameter on the dimensionless velocity

Figure 2 depicts linear momentum slip effects on the dimensionless horizontal velocity profiles. It is found that velocity slip causes to increase the velocity near the plate and decrease far away the plate. A cross flow occur at  $\eta = 1.859$  (approx). It is noticed that the fluid dimensionless horizontal velocity at the plate surface is 0 when  $a = 0$  which represents the conventional no-slip case. As the linear momentum slip parameter  $a$  rises, the slip velocity at the wall increases. Again, it is clear

that as slip parameter enhances, the penetration of the stagnant surface through the fluid domain decreases leading to a reduction in the hydrodynamics boundary layer measured by the displacement thickness. Increasing slip factor may be looked at as a miscommunication between the stationary plate and the moving fluid. Note that hydrodynamics behaviour of the problem under consideration is more sensitive to the variations in small values of  $a$  as compared with the variations in large values of it.

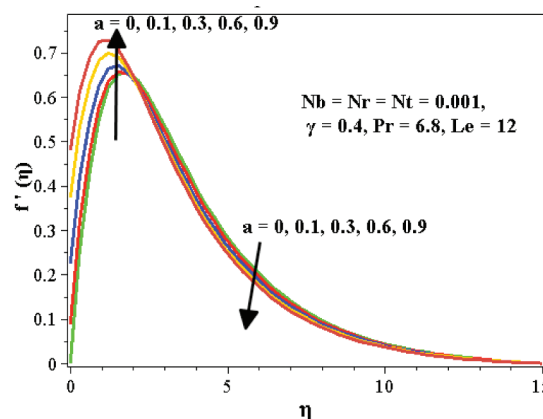


FIGURE 2. Effects of momentum slip parameter on the dimensionless velocity

Figure 3 shows the variation of the dimensionless temperature with the similarity independent variable  $\eta$  at various values of the linear hydrodynamic slip parameter and for the thermal convective boundary condition  $\gamma = 1$ . It is found that temperature is decreased with the increasing value of the linear momentum slip parameter. As the slip parameter  $a$  enhances, more flow will penetrate through the boundary layer due to slipping effect. As a result hot plate heats more amounts of fluid and this leads to decrease in the temperature.

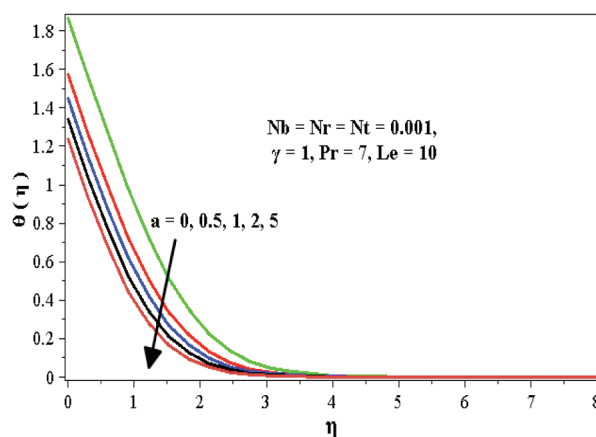


FIGURE 3. Effects of momentum slip parameter on the dimensionless temperature

Variation of the dimensionless nanoparticle volume fraction with the linear momentum slip parameter is depicted in Figure 4. It is found that nanoparticle volume fraction is decreased with the increasing value of the linear momentum slip parameter. It is further found that nanoparticle volume fraction is increased with the increasing value of convective heat transfer parameter. Physically, higher  $\gamma$  increases the nanoparticle volume fraction (concentration) as concentration distribution is driven by temperature distribution. The fluid on the right surface of the plate is heated up by the hot fluid on the left surface of the plate, making it become lighter and flow faster.

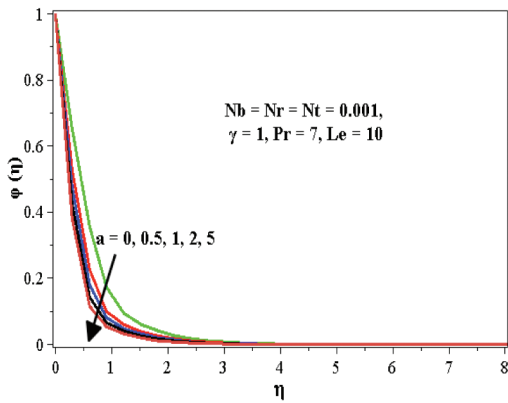


FIGURE 4. Effects of momentum slip parameter on the dimensionless nanoparticle volume fraction

Variation of the dimensionless nanoparticle volume fraction with the Lewis number is displayed in Figure 5. It is found that like regular fluid, nanoparticle volume fraction is decreased with the increasing value of Lewis number, as expected. Dimensionless concentration is found overshoot for small values of Lewis number. In nanofluid system, the Lewis number is assumed to be large because of small values of Brownian motion coefficient  $D_B$ . This overshoot may be because of taking large value of Brownian motion coefficient  $D_B$ . Physically, from the definition of Lewis number a higher value of Lewis number implies a lower Brownian motion coefficient  $D_B$  for a base fluid having kinematic viscosity  $\nu$ . Hence, higher Lewis number reduces nanoparticle volume fraction and its boundary layer thickness (Figure 5), consequently, increases the rate of nanoparticle volume fraction transfer (Figure 7). The dimensionless axial velocity is found to be increased whilst the dimensionless temperature decreased with increasing values of  $Le$ .

The skin friction factor variation vs convective heat transfer parameter for various values of the slip and buoyancy ratio parameters is shown in Figure 6. It is found that the skin friction factor increases with the increasing of the convective heat transfer parameter whilst it decreases with the rising of the momentum slip and the buoyancy ratio parameters.

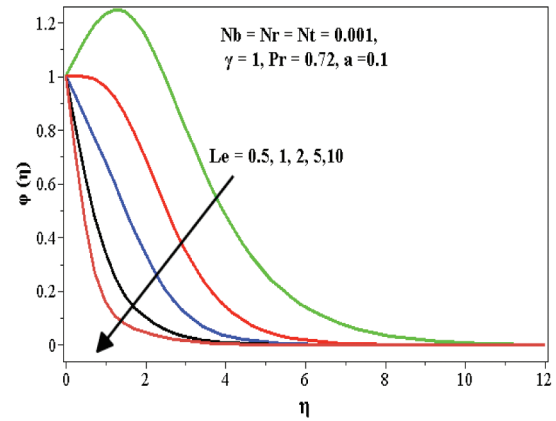


FIGURE 5. Effects of Lewis number on the dimensionless nanoparticle volume fraction

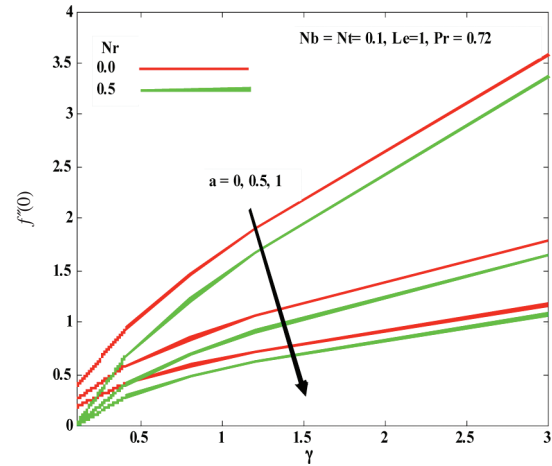


FIGURE 6. Variations of local skin friction factor with convective heat transfer parameter, slip parameter and buoyancy ratio parameter

Finally, variations of the dimensionless rate of nanoparticle volume fraction transfer vs the Lewis number for various values of the convective heat transfer and thermophoresis parameters is displayed in Figure 7. Observed that rate of nanoparticle volume fraction transfer increases with the increasing of the convective heat transfer parameter and the Lewis number whilst it decreases with the rising of thermophoresis parameter.

### CONCLUSION

We studied numerically a 2-D steady laminar incompressible free convective boundary layer flow from a heated solid (impermeable) vertical flat plate embedded in media which is filled with nanofluid taking into account the thermal convective and momentum slip boundary conditions. The governing boundary layer equations were transformed into highly nonlinear coupled ordinary differential equations using similarity transformations before being solved numerically. Following conclusions are drawn:

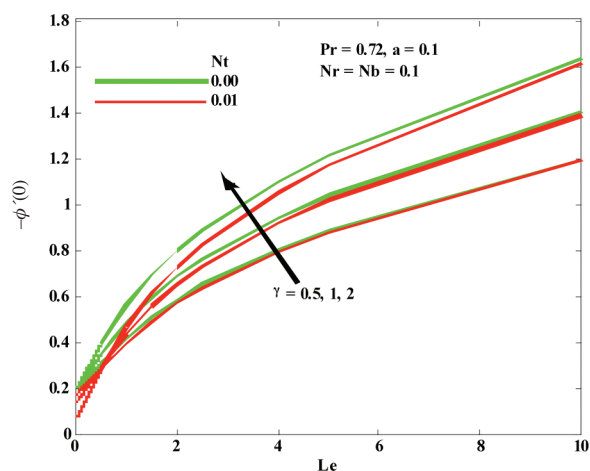


FIGURE 7. Variations of local mass transfer rate with convective heat transfer parameter, thermophoresis parameter and Lewis number

TABLE 1. Values of  $-\theta'(0)$  in the limiting case of a regular fluid. The present results are with  $Pr = 1$ ,  $Nr = Nb = Nt = 10^{-5}$ ,  $\gamma \rightarrow \infty$

Nur	$a = 0$	$a = 0.5$	$a = 3$
Bejan (1984)	0.401	-	-
Kuznetsov and Nield (2010)	0.401	-	-
Presents	0.40135	0.46823	0.49989

TABLE 2. Values of  $f''(0)$  for  $Pr = 0.72$ ,  $Le = 10$ ,  $Nr = Nb = Nt = 0.001$ ,  $\eta_{max} = 8$

$a$	$\gamma = 0.001$	$\gamma = 0.5$	$\gamma = 10$
0	0.01539	1.02662	6.22419
0.5	0.01356	0.62897	2.85913
1	0.01206	0.44753	1.82928

the dimensionless nano particle volume fraction decreases with rising of Lewis number; the dimensionless axial velocity, the temperature as well as the nanoparticle volume fraction increases with rising of the convective heat transfer parameter; the dimensionless axial velocity, the temperature as well as the nanoparticle volume fraction decreases with rising of the momentum slip parameter; the skin friction factor, the reduced Nusselt and the Sherwood number increase with the convective heat transfer parameter and the skin friction factor reduces with the slip parameter and the reduced Sherwood number reduces with the thermophoresis parameter.

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