

**SOME NEW INTEGRAL OPERATORS: SUFFICIENT CONDITIONS  
FOR THEIR UNIVALENCE**

(Beberapa Pengoperasi Baharu Kamiran: Syarat Cukup untuk Keunivalenannya)

GEORGIA IRINA OROS, GHEORGHE OROS &amp; ALINA ALB LUPAS

*ABSTRACT*

In this paper we define the integral operators with the forms:

$$G_{\alpha,M}(z) = \left[ \frac{\alpha}{M} \int_0^z (g(u))^{\frac{\alpha}{M}-1} du \right]^{\frac{M}{\alpha}}$$

$$F_{\alpha,M}(z) = \left[ \frac{\alpha}{M} \int_0^z u^{\frac{\alpha}{M}-1} \left( \frac{g(u)}{u} \right) du \right]^{\frac{M}{\alpha}}$$

$$H_{\alpha,\beta,M}(z) = \left[ \frac{\beta}{M} \int_0^z u^{\frac{\beta}{M}-1} \left( \frac{g(u)}{u} \right)^{\frac{M}{\alpha}} du \right]^{\frac{M}{\beta}}.$$

The operators generalise some integral operators studied by Owa, Pascu and Pescar. The original results contained in the paper give sufficient conditions for the univalence of those integral operators.

*Keywords:* Analytic function; integral operator; univalent function

*ABSTRAK*

Dalam makalah ini ditakrifkan pengoperasi kamiran dalam bentuk:

$$G_{\alpha,M}(z) = \left[ \frac{\alpha}{M} \int_0^z (g(u))^{\frac{\alpha}{M}-1} du \right]^{\frac{M}{\alpha}}$$

$$F_{\alpha,M}(z) = \left[ \frac{\alpha}{M} \int_0^z u^{\frac{\alpha}{M}-1} \left( \frac{g(u)}{u} \right) du \right]^{\frac{M}{\alpha}}$$

$$H_{\alpha,\beta,M}(z) = \left[ \frac{\beta}{M} \int_0^z u^{\frac{\beta}{M}-1} \left( \frac{g(u)}{u} \right)^{\frac{M}{\alpha}} du \right]^{\frac{M}{\beta}}.$$

Pengoperasi ini mengitlakkan beberapa pengoperasi kamiran yang dikaji oleh Owa, Pascu dan Pescar. Keputusan asal dalam makalah ini memberi syarat cukup untuk keunivalenan pengoperasi kamiran tersebut.

*Kata kunci:* Fungsi analisis; pengoperasi kamiran; fungsi univalen

## 1. Introduction and Preliminaries

Let  $H$  be the class of analytic functions in the open unit disc

$$U = \{z \in \mathbb{C} : |z| < 1\},$$

$$A_n = \{f \in H : f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots, z \in U\},$$

with  $A_1 = A$ ,

$$S = \{f \in A : f \text{ is univalent in } U\}.$$

In order to prove our main results we shall make use of the following lemma:

**Lemma A.** (Pascu 1985) *Let  $\alpha \in \mathbb{C}$  with  $\operatorname{Re} \alpha > 0$ , and  $f \in A$ . If*

$$\frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad \forall z \in U,$$

then the function

$$F_\alpha(z) = \left[ \alpha \int_0^z t^{\alpha-1} f'(t) dt \right]^{\frac{1}{\alpha}}$$

is univalent.

**Lemma B.** (Pascu 1987) *Let  $\alpha \in \mathbb{C}$ ,  $\operatorname{Re} \alpha > 0$  and  $f \in A$ . If*

$$\frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad \forall z \in U,$$

then  $\forall \beta \in \mathbb{C}, \operatorname{Re} \beta \geq \operatorname{Re} \alpha$ , the function

$$F_\beta(z) = \left[ \beta \int_0^z t^{\beta-1} f'(t) dt \right]^{\frac{1}{\beta}}$$

is univalent.

## 2. Main Results

**Definition 1.** Let  $M \geq 1$ ,  $\alpha$  with  $\operatorname{Re} \alpha > 0$  be a complex number. For a function  $g \in A$  we define the integral operator

$$G_{\alpha, M}(z) = \left[ \frac{\alpha}{M} \int_0^z (g(u))^{\frac{\alpha}{M}-1} du \right]^{\frac{M}{\alpha}} = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots, \quad z \in U.$$

This operator generalises the integral operator defined by V. Pescar (Pescar 2006). For this operator we prove the next theorem which contains a sufficient condition for its univalence:

**Theorem 1.** Let  $M \geq 1$  and  $\alpha$  with  $\operatorname{Re} \alpha > 0$  be a complex number such that

$$|\alpha - M| \leq \frac{\operatorname{Re} \alpha}{M + 2}, \quad (1)$$

If  $g \in A$  satisfies the conditions

$$|g(z)| \leq M, \quad (2)$$

$$\left| \frac{z^2 g'(z)}{g^2(z)} - 1 \right| \leq \frac{1}{M}, \quad (3)$$

for all  $z \in U$ , then the function

$$G_{\alpha, M}(z) = \left[ \frac{\alpha}{M} \int_0^z (g(u))^{\frac{\alpha}{M}-1} du \right]^{\frac{M}{\alpha}} \quad (4)$$

is in the class  $S$ .

**Proof.** From (4), we have

$$G_{\alpha, M}(z) = \left[ \frac{\alpha}{M} \int_0^z u^{\left(\frac{\alpha}{M}-1\right)} \left[ \frac{g(u)}{u} \right]^{\left(\frac{\alpha}{M}-1\right)} du \right]^{\frac{M}{\alpha}}, \quad z \in U. \quad (5)$$

We consider

$$f(z) = \int_0^z \left[ \frac{g(u)}{u} \right]^{\left(\frac{\alpha}{M}-1\right)} du. \quad (6)$$

The function  $f$  is regular in  $U$ .

Differentiating (6), we obtain

$$f'(z) = \left[ \frac{g(z)}{z} \right]^{M-1},$$

$$f''(z) = \frac{\alpha - M}{M} \left[ \frac{g(z)}{z} \right]^{\alpha-M-1} \cdot \frac{g'(z)z - g(z)}{z^2}, \quad z \in U,$$

and

$$\frac{zf''(z)}{f'(z)} = \frac{\alpha - M}{M} \left[ \frac{zg'(z)}{g(z)} - 1 \right], \quad z \in U. \quad (7)$$

Using (1), (2), (3) and (7), we calculate

$$\begin{aligned} \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \left| \frac{zf''(z)}{f'(z)} \right| &= \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \frac{|\alpha - M|}{M} \cdot \left| \frac{zg'(z)}{g(z)} - 1 \right| \\ &\leq \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \frac{|\alpha - M|}{M} \left[ \left| \frac{zg'(z)}{g(z)} \right| + 1 \right] \\ &\leq \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \frac{|\alpha - M|}{M} \left[ \left| \frac{z^2 g'(z)}{g^2(z)} \right| \cdot \left| \frac{g(z)}{z} \right| + 1 \right] \\ &\leq \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \frac{|\alpha - M|}{M} \left[ \left| \frac{z^2 g'(z)}{g^2(z)} \right| \cdot M + 1 \right] \\ &\leq \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \frac{|\alpha - M|}{M} \left[ \left| \frac{z^2 g'(z)}{g^2(z)} - 1 \right| \cdot M + M + 1 \right] \\ &\leq \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \frac{|\alpha - M|}{M} (M + 2) \end{aligned} \quad (8)$$

$$\leq \frac{|\alpha - M|}{\operatorname{Re} \alpha} (M + 2) \leq 1, \text{ for all } z \in U .$$

From (8), using Lemma A, we have that  $G_{\alpha, M}$  is in the class  $S$ .

**Remark 1.** For  $M = 1$ , the result was obtained in Pescar (2006).

**Remark 2.** For  $M = 1$ , the condition (3) expresses a sufficient condition for univalence of function  $g$  and this result can be found in Ozaki and Nunokawa (1972, Lemma C).

**Definition 2.** Let  $1 \leq M \leq 2$  and  $\alpha$  be a complex number such that  $\operatorname{Re} \alpha \geq 3M$ . For a function  $g \in A$  we define the integral operator

$$F_{\alpha, M}(z) = \left[ \frac{\alpha}{M} \int_0^z u^{\frac{\alpha}{M}-1} \left( \frac{g(u)}{u} \right) du \right]^{\frac{M}{\alpha}} = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots, z \in U .$$

This operator generalises the integral operator defined by Owa *et al.* (1999). For this operator we prove the next theorem which contains a sufficient condition for its univalence:

**Theorem 2.** Let  $1 \leq M \leq 2$  and  $\alpha$  be a complex number such that

$$\operatorname{Re} \alpha \geq 3M. \tag{9}$$

If  $g \in A$  satisfies the conditions

$$|g(z)| \leq M, \tag{10}$$

$$\left| \frac{z^2 g'(z)}{g^2(z)} - 1 \right| \leq \frac{2 - M}{M}, \tag{11}$$

for all  $z \in U$ , then the function

$$F_{\alpha, M}(z) = \left[ \frac{\alpha}{M} \int_0^z u^{\frac{\alpha}{M}-1} \left( \frac{g(u)}{u} \right) du \right]^{\frac{M}{\alpha}} \tag{12}$$

is in the class  $S$ .

**Proof.** Let us consider the function

$$f(z) = \int_0^z \frac{g(u)}{u} du. \tag{13}$$

Differentiating (13), we obtain

$$f'(z) = \frac{g(z)}{z}, \quad f''(z) = \frac{zg'(z) - g(z)}{z^2}, \quad z \in U,$$

and

$$\frac{zf''(z)}{f'(z)} = \frac{zg'(z)}{g(z)} - 1. \quad (14)$$

Using (9), (10), (11), (14), we calculate

$$\begin{aligned} \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \left| \frac{zf''(z)}{f'(z)} \right| &= \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \left| \frac{zg'(z)}{g(z)} - 1 \right| \\ &\leq \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \left[ \left| \frac{zg'(z)}{g(z)} \right| + 1 \right] \\ &= \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \left[ \left| \frac{z^2g'(z)}{g^2(z)} \right| \cdot \left| \frac{g(z)}{z} \right| + 1 \right] \\ &\leq \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \left[ \left| \frac{z^2g'(z)}{g^2(z)} \right| \cdot M + 1 \right] \\ &\leq \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \left[ \left| \frac{z^2g'(z)}{g^2(z)} - 1 \right| \cdot M + M + 1 \right] \\ &\leq \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \left[ \frac{2 - M}{M} \cdot M + M + 1 \right] \\ &\leq \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot 3M \leq \frac{3M}{\operatorname{Re}\alpha} \leq 1. \end{aligned} \quad (15)$$

From (14) and using Lemma A we have that  $F_{\alpha, M}$  is in the class  $S$ .

**Remark 3.** For  $M = 1$ , the result was obtained in Owa *et al.* (1999).

**Remark 4.** For  $M = 1$ , the condition (11) expresses a sufficient condition for univalence of function  $g$  (Ozaki & Nunokawa 1972, Lemma C).

**Definition 3.** Let  $M \geq 1$  and let  $\alpha$  be a complex number such that

$$\operatorname{Re} \alpha \geq \frac{M^5 + M^4 + M^3}{|\alpha|}. \text{ For a function } g \in A \text{ we define the integral operator}$$

$$H_{\alpha, \beta, M}(z) = \left[ \frac{\beta}{M} \int_0^z u^{\frac{\beta}{M}-1} \left( \frac{g(u)}{u} \right)^{\frac{M}{\alpha}} du \right]^{\frac{M}{\beta}} = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots, z \in U.$$

This operator generalises the integral operator defined by V. Pescar (Pescar 2006). For this operator we prove the next theorem which contains a sufficient condition for its univalence:

**Theorem 3.** Let  $M \geq 1$ ,  $\alpha$  a complex number such that

$$\operatorname{Re} \alpha \geq \frac{M^5 + M^4 + M^3}{|\alpha|}. \quad (16)$$

If  $g \in A$  satisfies

$$|g(z)| \leq M, \quad (17)$$

$$\left| \frac{z^2 g'(z)}{g^2(z)} - 1 \right| \leq M^2, \quad (18)$$

for all  $z \in U$ , then for every complex number  $\beta$ , with  $\operatorname{Re} \beta \geq \operatorname{Re} \alpha$ , the function

$$H_{\alpha, \beta, M}(z) = \left[ \frac{\beta}{M} \int_0^z u^{\frac{\beta}{M}-1} \left( \frac{g(u)}{u} \right)^{\frac{M}{\alpha}} du \right]^{\frac{M}{\beta}} \quad (19)$$

is in the class  $S$ .

**Proof.** Let us consider the function

$$h(z) = \int_0^z \left[ \frac{g(u)}{u} \right]^{\frac{M}{\alpha}} du, \quad z \in U. \quad (20)$$

The function  $h$  is regular in  $U$ . From (20), we have

$$h'(z) = \left[ \frac{g(z)}{z} \right]^{\frac{M}{\alpha}}, \quad h''(z) = \frac{M}{\alpha} \left[ \frac{g(z)}{z} \right]^{\frac{M}{\alpha}-1} \frac{zg'(z) - g(z)}{z^2},$$

and

$$\frac{zh''(z)}{h'(z)} = \frac{M}{\alpha} \left( \frac{zg'(z)}{g(z)} - 1 \right), \text{ for all } z \in U. \quad (21)$$

Using (16), (17), (18), (21), we calculate

$$\begin{aligned} \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \left| \frac{zh''(z)}{h'(z)} \right| &= \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \frac{M}{|\alpha|} \cdot \left| \frac{zg'(z)}{g(z)} - 1 \right| \\ &\leq \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \frac{M^2}{|\alpha|} \cdot \left[ \left| \frac{zg'(z)}{g(z)} \right| + 1 \right] \\ &= \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \frac{M^2}{|\alpha|} \cdot \left[ \left| \frac{z^2 g'(z)}{g^2(z)} \right| \cdot \left| \frac{g(z)}{z} \right| + 1 \right] \\ &\leq \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \frac{M^2}{|\alpha|} \cdot \left[ \left| \frac{z^2 g'(z)}{g^2(z)} \right| \cdot M + 1 \right] \\ &\leq \frac{1 - |z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}} \cdot \frac{M^2}{|\alpha|} \cdot \left[ \left| \frac{z^2 g'(z)}{g^2(z)} - 1 \right| M + M + 1 \right] \\ &\leq \frac{M^2}{\operatorname{Re}\alpha|\alpha|} [M^3 + M^2 + M] \leq 1. \end{aligned} \quad (22)$$

Using (22), we apply Lemma A. Combining this result with the condition  $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$ , we apply Lemma B and we have that  $H_{\alpha,\beta,M}$  is in the class  $S$ .

**Remark 5.** For  $M = 1$ , the result was obtained in Pescar (2006).



**Remark 6.** For  $M = 1$ , the condition (18) expresses a sufficient condition for univalence of function  $g$  (Ozaki & Nunokawa 1972, Lemma C).

**Example 1.** For  $M = 1$ ,  $\frac{2-M}{M} = 1$ ,  $\alpha = 4 + 3i$  and  $g \in A$ ,

$$g(z) = z + \frac{1}{2}z^2, \quad z \in U,$$

with  $\operatorname{Re} \alpha = 4 \geq 3$  we obtain

$$|g(z)| = \left| z + \frac{1}{2}z^2 \right| \leq |z| + \frac{1}{2}|z|^2 \leq 1 + \frac{1}{2} = \frac{3}{2} < \frac{8}{5},$$

$$\left| \frac{z^2(1+z)}{z^2\left(1+\frac{1}{2}z\right)^2} - 1 \right| = \left| \frac{1+z-1-z-\frac{1}{4}z^2}{\left(1+\frac{1}{2}z\right)^2} \right| = \frac{\frac{1}{4}|z^2|}{\left|\left(1+\frac{1}{2}z\right)^2\right|} \leq \frac{\frac{1}{4}|z^2|}{\frac{1}{4}} \leq |z|^2 \leq 1.$$

Using Theorem 2, we have

$$F_{(4+3i),1}(z) = \left[ (4+3i) \int_0^z u^{3+3i} \left(1 + \frac{1}{2}u\right) du \right]^{\frac{4-3i}{25}}$$

$$= \left[ z^{4+3i} + \frac{29+3i}{68} z^{5+3i} \right]^{\frac{4-3i}{25}}, \quad z \in U,$$

is in the class  $S$ .

**Example 2.** For  $M = 1$ ,  $\alpha = 4 + 3i$ ,  $\beta = 5 - i$  and  $g \in A$ ,

$$g(z) = z + \frac{1}{2}z^2, \quad z \in U,$$

with  $\operatorname{Re} \alpha = 4$ ,  $\operatorname{Re} \beta = 5$ ,  $\operatorname{Re} \beta \geq \operatorname{Re} \alpha$ , we obtain

$$|g(z)| = \left| z + \frac{1}{2}z^2 \right| \leq |z| + \frac{1}{2}|z|^2 \leq 1 + \frac{1}{2} = \frac{3}{2} < \frac{8}{5},$$

$$\left| \frac{z^2(1+z)}{z^2\left(1+\frac{1}{2}z\right)^2} - 1 \right| = \left| \frac{1+z-1-z-\frac{1}{4}z^2}{\left(1+\frac{1}{2}z\right)^2} \right| = \frac{\frac{1}{4}|z^2|}{\left|\left(1+\frac{1}{2}z\right)^2\right|} \leq \frac{\frac{1}{4}|z^2|}{\frac{1}{4}} \leq |z|^2 \leq 1.$$

Using Theorem 3, we have

$$H_{(4+3i), (5-i), 1}(z) = \left[ (5-i) \int_0^z u^{4-i} \left(1 + \frac{1}{2}u\right) du \right]^{\frac{1}{5-i}}$$

$$= \left[ z^{5-i} + \frac{31-i}{74} z^{6-i} \right]^{\frac{5+i}{26}}, \quad z \in U,$$

is in the class  $S$ .

**Remark 7.** Similar results for other univalent integral operators can also be seen in Oros (2009), Oros *et al.* (2008), and Pescar and Breaz (2008).

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*Department of Mathematics*  
*University of Oradea*  
*Str. Universității No.1*  
*410087 Oradea, ROMÂNIA*  
*E-mail: georgia\_oros\_ro@yahoo.co.uk\*, gh\_oros@yahoo.com, dalb@uoradea.ro*

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\* Corresponding author