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Estimation of *p*-adic sizes of Common Zeros of Partial Derivatives Associated with a Cubic Form

(Penganggaran Saiz *p*-Adic Pensifar Sepunya Polinomial-polinomial Terbitan Separa disekutukan dengan Bentuk Kubik)

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ABSTRACT

In this paper we determined the estimate of p*-adic sizes of common zeros of partial derivative polynomials associated with a cubic form whose indicator diagrams have one overlapping segment by using Newton polyhedron technique. We showed that the* p*-adic sizes of such common zeros can be found explicitly on the overlapping segment of the indicator diagrams associated with the polynomials.*

Keywords: Indicator diagram; newton polyhedron; overlapping segment; p*-adic sizes*

ABSTRAK

Dalam makalah ini kami menganggarkan saiz p*-adic pensifar sepunya polinomial-polinomial terbitan separa disekutukan dengan bentuk kubik dan gambar rajah penunjuknya mempunyai satu tembereng bertindih dengan menggunakan teknik polihedron Newton. Kami menunjukkan bahawa saiz* p*-adic bagi pensifar sepunya tersebut boleh didapati secara eksplisit pada tembereng bertindih dalam gambar rajah penunjuk disekutukan dengan polinomial terbabit.*

Kata kunci: Gambar rajah penunjuk; polihedron Newton; saiz p*-adic; tembereng bertindih*

INTRODUCTION

In our discussion, p will denote a prime. We use the notations Z_p to denote the ring of p-adic integers, Q_p the field of p-adic numbers, $\overline{Q}_p$ the closure of Q_p and Ω_p to denote the algebraically closed and complete extension of the field $\overline{Q}_p$ respectively. For a rational number x we denote by $ord_p x$ the p-adic size of x, by which we mean the highest power of p dividing x. It follows that for two rational numbers a and b, $ord_p b = ord_p a + ord_p b$, $ord_p \frac{a}{b} = ord_p a - ord_p b$ for $b \neq 0$ and $ord_p(a \pm b) \geq min\{ord_p a, ord_p b\}$. By convention $ord_p x = \infty$ if $x = 0$.

A Newton polyhedron associated with a polynomial $f(x,y) = \Sigma a_{ij} x^i y^j$ with coefficients in Ω_p is the lower convex hull of the set of points $(i,j,ord_p a_{ij})$. It consists of faces and edges on and above which lie the points $(i,j,ord_p a_{ij})$. The Newton polyhedron technique as developed by Mohd Atan and Loxton (1986) is an analogue of Newton polygon as defined by Koblitz (1977). Figure 1 shows an example of the Newton polyhedron associated with the given polynomial.

The indicator diagram is defined as the set of the line segments in real Euclidean plane joining pairs of vertices (x_i, y_j) and (x_i', y_i') that correspond to the upward pointing normals $(x_i, y_j, 1)$ and $(x_i', y_j', 1)$, respectively to adjacent faces in the associated Newton polyhedron. Figure 2 shows an example of the indicator diagram associated with the Newton polyhedron in Figure 1.

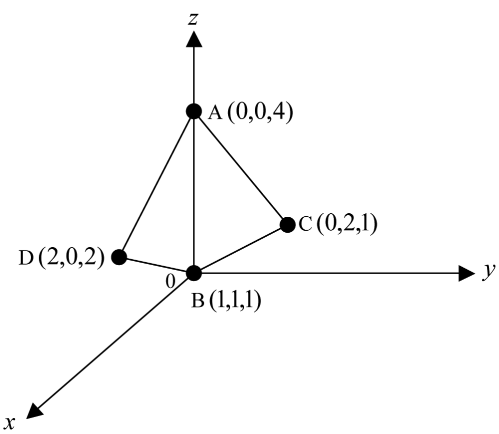

FIGURE 1. The Newton polyhedron of polynomial $f(x,y) = 4x^2 - 2xy + 2y^2 - 16$ with $p = 2$

Mohd Atan (1986b), Chan and Mohd Atan (1997), Heng and Mohd Atan (1999) found the estimate of p-adic sizes of common zeros for lower degree two-variable polynomials by using the Newton polyhedron technique. Mohd Atan and Loxton (1986) conjectured that the p-adic orders of common zeros occur at intersection points of the indicator diagrams associated with the polynomials. Sapar and Mohd Atan (2002) proved the conjecture for the cases in which intersection points occur at the vertices or they are simple points of intersection. Cases involving overlapping segments that occur in the combination of the indicator diagrams were not studied up till now.

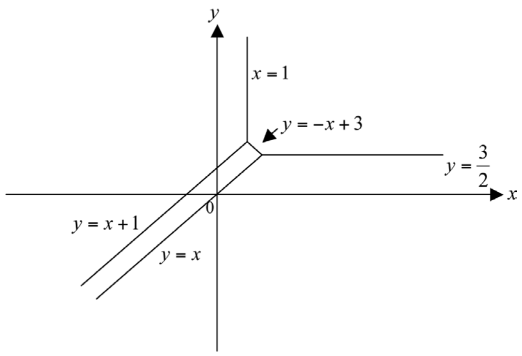

FIGURE 2. Indicator diagram associated with the Newton polyhedron of $f(x,y) = 4x^2 - 2xy + 2y^2 - 16$ with $p = 2$

Both Mohd Atan (1986b) and Sapar and Mohd Atan (2007) applied the Newton polyhedron technique at simple points of intersections in the combination of indicator diagrams associated with a pair of polynomials in $Z_p[x,y]$ where $Z_p[x,y]$ denote the set of polynomials in x and y with coefficients in Z_p. They used this technique to estimate p-adic sizes of common zeros of partial derivative polynomials associated with a polynomial $f(x,y)$ of degree at most six. Our work involves application of the same technique to determine p-adic sizes of common roots of partial derivative polynomials associated with a quadratic form and investigate cases involving single overlapping segment of the indicator diagrams associated with the polynomials. This is an early attempt to investigate cases in which overlapping segments occur in the combination of indicator diagrams associated with two polynomials. Our initial work which appears in this paper studies cases involving quadratic polynomials in two variables, and subsequently partial derivative polynomials associated with cubic forms.

p-ADIC ORDERS OF ZEROS OF A POLYNOMIAL

Mohd Atan (1986a) proved that the points on an indicator diagram associated with a polynomial correspond to the p-adic orders of roots of the polynomial.

Conversely Mohd Atan and Loxton (1986) conjectured that for every root of a polynomial equation there exists a point on the indicator diagram which gives the p-adic order of the root. If there exists common zero for two polynomials, then the p-adic order of the common zero is at an intersection point of the indicator diagrams associated with the polynomials.

Our investigation concentrates on the points on the overlapping segments of indicator diagrams associated with polynomials of the form $f(x,y) = ax^2 + bxy + cy^2 + d$ and $g(x,y) = rx^2 + sxy + ty^2 + q$. Our results are stated in Theorem 1 and Theorem 2. We prove the following lemma first.

LEMMA 1

Let Ω_p denote the algebraically closed and complete extension of the field $\overline{Q}_p$ the closure of Q_p the field of p-adic numbers. Suppose $f(x,y) = ax^2 + bxy + cy^2 + d$ and $g(x,y) = rx^2 + sxy + ty^2 + q$ are polynomials in $Z_p[x,y]$. Suppose that $ord_p dr > ord_p aq$, $ord_p dt = ord_p cq$ and $ord_p cr > ord_p at$. Let $A = \lfloor a(cs-bt)^2 + (at-cr)(cs-bt) + c(at-cr)^2$. Suppose that $A \neq 0$. Then there exists (ξ,η) in Ω_p^2 such that $f(\xi,\eta) = g(\xi,\eta) = 0$ and $(ord_p\xi, ord_p\eta)$ is a point on the overlapping segment of the indicator diagrams of f and g such that $ord_p\xi \geq \frac{1}{2} ord_p \frac{d}{a}$ and $ord_p\eta = \frac{1}{2} ord_p \frac{d}{c}$.

Proof:
With the given conditions, the combination of the indicator diagrams associated with f and g is as in Figure 3.

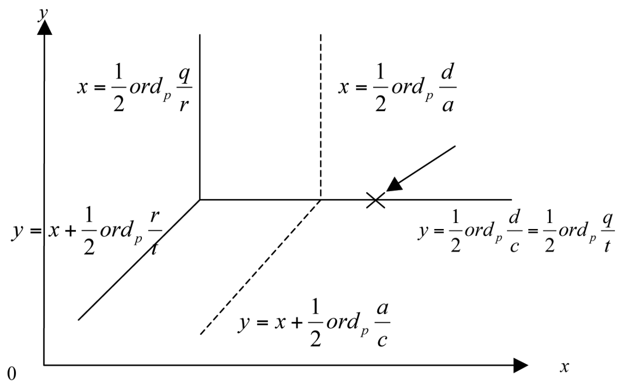

FIGURE 3. The combination of the indicator diagrams associated with f (in broken lines) and g (in bold) under the conditions : $ord_p dr > ord_p aq$, $ord_p dt = ord_p cq$ and $ord_p cr > ord_p at$

Let (μ, λ) be a point on the overlapping segment. From the figure,

$$\mu \geq \frac{1}{2}\ ord_p \frac{d}{a},\ \lambda = \frac{1}{2}\ ord_p \frac{q}{t}$$

We next show that common roots of f and g exist. We have two complete quadratic polynomial equations as follows.

$$f(x,y) = ax^2 + bxy + cy^2 + d = 0 \quad (1)$$

$$g(x,y) = rx^2 + sxy + ty^2 + q = 0. \quad (2)$$

By solving this two equations simultaneously, we get

$$x = \frac{(at-cr)y^2(aq-dr)}{(br-as)y} \quad (3)$$

and

$$y = \frac{(at-cr)x^2(dt-cq)}{(cs-bt)x}. \qquad (4)$$

We substitute (4) into (1), we will have

$$Ax^4 + Bx^2 + C = 0$$

with

$$A = \left\lfloor a(cs-bt)^2 + b(at-cr)(cs-bt) + c(at-cr)^2 \right\rfloor$$
$$B = \left\lfloor b(cs-bt)(dt-cq) + 2c(at-cr)(cs-bt) + d(cs-bt)^2 \right\rfloor$$
$$C = c(dt-cq)^2.$$

Since $ord_p dt = ord_p cq$, two possibilities occur for C. They are either $dt \neq cq$ or $dt = cq$.

i. If $dt \neq cq$, then $C = c(dt - cq)^2 \neq 0$. This means at least one of A and B is not zero. Therefore, there exists a root for this equation.
ii. If $dt = cq$, then the equation will be reduced to $x^2(Ax^2 + B) = 0$ and the root is of the form $x = 0$ or $x = \pm\sqrt{\frac{-B}{A}}$.

Since $A \neq 0$, then there exists a root for this equation. We let ξ be the root of this equation.

We substitute (3) into (1), we get

$$Dy^4 + Ey^2 + F = 0$$

with

$$D = \left\lfloor a(at-cr)^2 + b(at-cr)(br-as) + c(br-as)^2 \right\rfloor$$
$$E = \left\lfloor 2a(at-cr)(aq-dr) + b(aq-dr)(br-as) + d(br-as)^2 \right\rfloor$$
$$F = a(aq-dr)^2.$$

Since $ord_p dr > ord_p aq$, $F = a(aq - dr)^2 \neq 0$. This means at least one of D and E is not zero. Therefore, there exists a root for this equation. We let η be the root of this equation. Therefore, (ξ, η) is a common root of $f(x, y)$ and $g(x.y)$.

Since (ξ, η) is the common root of f and g, by a theorem of Mohd Atan and Loxton (1986), $(ord_p\xi, ord_p\eta)$ is a point on the common segment of the indicator diagrams of f and g. Since there is only one overlapping segment, the point $(ord_p\xi, ord_p\eta)$ lies on this segment.

Hence,

$$ord_p\xi \geq \frac{1}{2}\ ord_p\frac{d}{a},\ ord_p\eta = \frac{1}{2}\ ord_p\frac{q}{t} = \frac{1}{2}\ ord_p\frac{d}{c}.$$

LEMMA 2

Suppose $f(x, y) = ax^2 + bxy + cy^2 + d$ and $g(x, y) = rx^2 + sxy + ty^2 + q$ are polynomials in $Z_p[x, y]$. Suppose that $ord_p aq > ord_p dr$, $ord_p dt = ord_p cq$ and $ord_p at > ord_p cr$. Let $A = \lfloor a(cs - bt)^2\ b(at - cr)(cs - bt) + c(at - cr)^2$. Suppose that $A \neq 0$. Then there exists (ξ, η) in $\Omega_p^{\,2}$ such that $f(\xi, \eta) = g(\xi, \eta) = 0$ and $(ord_p\xi, ord_p\eta)$ is a point on the overlapping segment of the indicator diagrams of f and g such that $ord_p\xi \geq \frac{1}{2}\ ord_p\frac{q}{r}$ and $ord_p\eta = \frac{1}{2}\ ord_p\frac{q}{t}$.

Proof:
With the given conditions, the arguments in the proof is similar to that for Lemma 1, by exchanging the positions of the indicator diagrams associated with f and g.

LEMMA 3

Suppose $f(x, y) = ax^2 + bxy + cy^2 + d$ and $g(x,y) = rx^2 + sxy + ty^2 + q$ are polynomials in $Z_p[x,y]$. Suppose that $ord_p aq = ord_p dr$, $ord_p cq > ord_p dt$ and $ord_p cr > ord_p at$. Let $D = \lfloor a(at - cr)^2 + b(at - cr)(br - as) + c(br - as)^2$. Suppose that $D \neq 0$. Then there exists (ξ, η) in $\Omega_p^{\,2}$ such that $f(\xi, \eta) = g(\xi, \eta) = 0$ and $(ord_p\xi, ord_p\eta)$ is a point on the overlapping segment of the indicator diagrams of f and g such that $ord_p\xi\frac{1}{2} = ord_p\frac{q}{r}$ and $ord_p\eta \geq \frac{1}{2}\ ord_p\frac{q}{t}$.

Proof:
With the given conditions and Figure 4, the arguments in the proof is similar to that for Lemma 1.

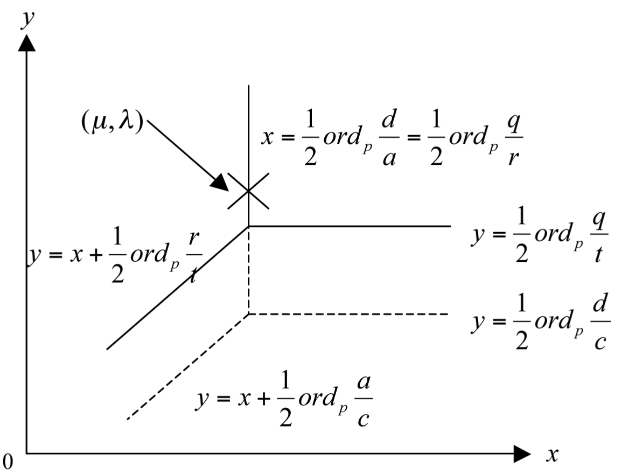

FIGURE 4. The combination of the indicator diagrams associated with f (in broken lines) and g (in bold) under the conditions : $ord_p aq = ord_p dr$, $ord_p cq > ord_p dt$ and $ord_p cr > ord_p at$

LEMMA 4

Suppose $f(x,y) = ax^2 + bxy + cy^2 + d$ and $g(x,y) = rx^2 + ssy + ty^2\ \ q$ are polynomials in $Z_p[x,y]$. Suppose that $ord_p aq = ord_p dr$, $ord_p dt > ord_p cq$ and $ord_p at > ord_p cr$. Let $D = \lfloor a(at - cr)^2 + b(at - cr)(br - as) + c(br - as)^2$. Suppose that $D \neq 0$. Then there exists (ξ,η) in $\Omega_p^{\,2}$ such that $f(\xi,$

$\eta) = g(\xi, \eta) = 0$ and $(ord_p\xi, ord_p\eta)$ is a point on the overlapping segment of the indicator diagrams of f and g such that $ord_p\xi \frac{1}{2} = ord_p \frac{d}{a}$ and $ord_p\eta \geq \frac{1}{2} ord_p \frac{d}{c}$.

Proof:
With the given conditions, the arguments in the proof is similar to that for Lemma 3, by exchanging the positions of the indicator diagrams associated with f and g.

THEOREM 1

Le $f(x, y) = ax^2 + bxy + cy^2 + d$ and $g(x,y) = rx^2 + sxy + ty^2 + q$ be polynomials in $Z_p[x,y]$. Suppose there exists one overlapping segment in the construction of indicator diagrams of f and g. Then there exists (ξ, η) in $\Omega_p^{\ 2}$ such that $f(\xi, \eta) = g(\xi, \eta) = 0$ and $(ord_p\xi, ord_p\eta)$ is a point on the overlapping segment.

Proof:
From Lemma 1, Lemma 2, Lemma 3 and Lemma 4 respectively, there exists (ξ, η) in $\Omega_p^{\ 2}$ such that $f(\xi, \eta) = g(\xi, \eta) = 0$ and $(ord_p\xi, ord_p\eta)$ is a point on the overlapping segment of the indicator diagrams of f and g.

We now apply Theorem 1 to prove Lemma 5 in the following section.

p-ADIC SIZES OF COMMON ROOTS IN THE NEIGHBOURHOOD OF A POINT IN $\Omega_p^{\ 2}$

LEMMA 5

Let $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 + ex + my + n$ be a polynomial in $Z_p[x, y]$ and (x_0, y_0) in $\Omega_p^{\ 2}$. Suppose that $ord_p bf_x(x_0, y_0) > ord_p af_y(x_0, y_0)$, $ord_p df_x(x_0, y_0) = ord_p cf_y(x_0, y_0)$, and $ord_p bc > ord_p ad$. Let $\alpha > 0$, with $ord_p f_x(x_0, y_0)$, $ord_p f_y(x_0, y_0) \geq \alpha$ and $\delta = \max\{ord_p a, ord_p b, ord_p c, ord_p d\}$. Then there exists (ξ, η) in $\Omega_p^{\ 2}$ such that $f_x(\xi, \eta) = 0$, $f_y(\xi, \eta) = 0$ and $ord_p(\xi - x_0), ord_p(\eta - y_0) \geq \frac{1}{2}(\alpha - \delta)$.

Proof:

$$f(x, y) = ax^3 + bx^2 y + cxy^2 + dy^3 + ex + my + n$$

$$f_x(x, y) = 3ax^2 + 2bxy + cy^2 + e$$

$$f_y(x, y) = bx^2 + 2cxy + 3dy^2 + m.$$

Let $X = x - x_0$, $Y = y - y_0$, and $g = f_x$, $h = f_y$.
Then,

$$g(X + x_0, Y + y_0) = 3aX^2 + 2bXY + cY^2 + (6ax_0 + 2by_0)X + (2bx_0 + 2cy_0)Y + (3ax_0^{\ 2} + 2bx_0y_0 + cy_0^{\ 2} + e)$$

and

$$h(X + x_0, Y + y_0) = bX^2 + 2cXY + 3dY^2 + (2bx_0 + 2cy_0)X + (2cx_0 + 6dy_0)Y + (bx_0^{\ 2} + 2cy_0y_0 + 3dy_0^{\ 2} + m)$$

Let $G(X,Y) = g(X + x_0, Y + y_0)$ and $H(X, Y) = h(X + x_0, Y + y_0)$. Then,

$$G(X,Y) = 3aX^2 + 2bXY + cY^2 + (6ax_0 + 2by_0)X + (2bx_0 + 2cy_0)Y + f_x(x_0,y_0) \tag{5}$$

and

$$H(X,Y) = bX^2 + 2cXY + 3dY^2 + (2bx_0 + 2cy_0)X + (2cx_0 + 6dy_0)Y + f_y(x_0,y_0). \tag{6}$$

With the given conditions, the combination of the indicator diagrams associated with p-adic Newton polyhedra of $G(X,Y)$ and $H(X,Y)$ is as in Figure 5.

By considering the p-adic orders of coefficients of $G(X,Y)$ and $H(X,Y)$, there are two possibilities that will occur. They are either p-adic orders of coefficients of dominant terms are less than or equal to p-adic orders of coefficients of X and Y or p-adic orders of coefficients of dominant terms are greater than p-adic orders of coefficients of X and Y. The combination of indicator diagrams of the first case is as shown in Figure 5 and its result is stated as below. For the second case, a similar diagram is obtained in which the far-right vertical line is of the equation $x = ord_p \frac{f_y(x_0,y_0)}{(6ax_0+2by_0)}$. The p-adic orders of their common roots will occur on the overlapping horizontal segment $y = ord_p \frac{f_x(x_0,y_0)}{(2bx_0+2cy_0)} = ord_p \frac{f_y(x_0,y_0)}{(2cx_0+6dy_0)}$ to the right of this vertical line. Since $x = ord_p \frac{f_y(x_0,y_0)}{(6ax_0+2by_0)} \geq \frac{1}{2} ord_p \frac{f_x(x_0,y_0)}{a}$, the p-adic orders of the common roots will be greater than $\frac{1}{2} ord_p \frac{f_x(x_0,y_0)}{a}$. The rest of the proof is similar to the proof below.

From Figure 5 and Lemma 1, there exists $(\hat{X},\hat{Y})$ in $\Omega_p^{\ 2}$ such that $G(\hat{X},\hat{Y}) = 0$, $H(\hat{X},\hat{Y}) = 0$ and $ord_p\hat{X} = \mu$, $ord_p\hat{Y} = \lambda$ with

$$\mu \geq \frac{1}{2} ord_p \frac{f_x(x_0,y_0)}{a}, \ \lambda = \frac{1}{2} ord_p \frac{f_y(x_0,y_0)}{d}.$$

Hence, $ord_p\hat{X} \geq \frac{1}{2} ord_p \frac{f_x(x_0,y_0)}{a}$,
$ord_p\hat{Y} = \frac{1}{2} ord_p \frac{f_y(x_0,y_0)}{d}$.

Since $ord_p f_x(x_0, y_0) \geq \alpha$, $ord_p f_y(x_0, y_0) \geq \alpha$ and $\delta = \max\{ord_p a, ord_p b, ord_p c, ord_p d\}$, we have

$$ord_p\hat{X} \geq \frac{1}{2}(\alpha - \delta), \ ord_p\hat{Y} \geq \frac{1}{2}(\alpha - \delta)$$

Let $\xi = \hat{X} + x_0$ and $\eta = \hat{Y} + y_0$. Then $\hat{X} = \xi - x_0$ and $\hat{Y} = \eta - y_0$.

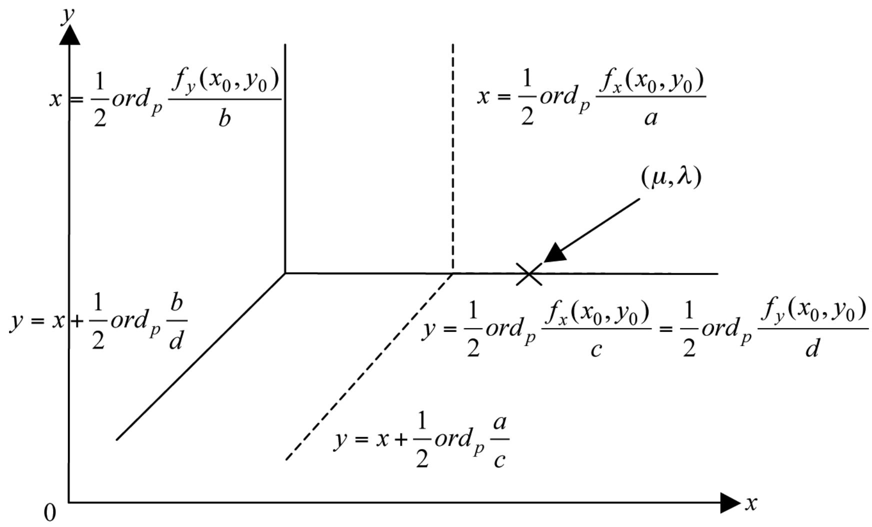

FIGURE 5. The combination of the indicator diagrams associated with $G(X,Y)$ (in broken lines) and $H(X,Y)$ (in bold) under the conditions: $ord_p bf_x(x_0,y_0) > ord_p af_y(x_0,y_0)$, $ord_p df_x(x_0,y_0) = ord_p cf_y(x_0,y_0)$ and $ord_p bc > ord_p ad$

It follows that $ord_p(\xi - x_0) \geq \frac{1}{2}(\alpha - \delta)$, $ord_p(\eta - y_0) \geq \frac{1}{2}(\alpha - \delta)$ and by back substitution in equations (5) and (6), we will have

$$f_x(\xi, \eta) = 0, f_y(\xi, \eta) = 0.$$

LEMMA 6

Let $f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 + ex + my + n$ be a polynomial in $Z_p[x, y]$ and (x_0, y_0) in Ω_p^2. Suppose that $ord_p af_y(x_0, y_0) > ord_p bf_x(x_0, y_0)$, $ord_p df_x(x_0, y_0) = ord_p cf_y(x_0, y_0)$ and $ord_p ad > ord_p bc$. Let $\alpha > 0$, with $ord_p f_x(x_0, y_0)$, $ord_p c_y(x_0, y_0) \geq \alpha$ and $\delta = \max\{ord_p a, ord_p b, ord_p c, ord_p d\}$. Then there exists (ξ, η) in Ω_p^2 such that $f_x(\xi, \eta) = 0, f_y(\xi, \eta) = 0$ and $ord_p(\xi - x_0)$, $ord_p(\eta - y_0) \geq \frac{1}{2}(\alpha - \delta)$.

Proof:
With the given conditions and Lemma 2, the arguments in the proof is similar to that for Lemma 5, by exchanging the positions of the indicator diagrams associated with $G(X, Y)$ and $H(X, Y)$.

LEMMA 7

Let $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 + ex + my + n$ be a polynomial in $Z_p[x, y]$ and (x_0, y_0) in Ω_p^2 . Suppose that $ord_p bf_x(x_0, y_0) = ord_p af_y(x_0, y_0)$, $ord_p cf_y(x_0, y_0) > ord_p df_x(x_0, y_0)$ and $ord_p bc > ord_p ad$. Let $\alpha > 0$, with $ord_p fx(x_0, y_0)$, $ord_p f_y(x_0, y_0) \geq \alpha$ and $\delta = \max\{ord_p a, ord_p b, ord_p c, ord_p d\}$. Then there exists (ξ, η) in Ω_p^2 such that $f_x(\xi, \eta) = 0, f_y(\xi, \eta) = 0$ and $ord_p(\xi - x_0)$, $ord_p(\eta - y_0) \geq \frac{1}{2}(\alpha - \delta)$.

Proof:
With the given conditions, Figure 6 and Lemma 3, the arguments in the proof is similar to that for Lemma 5.

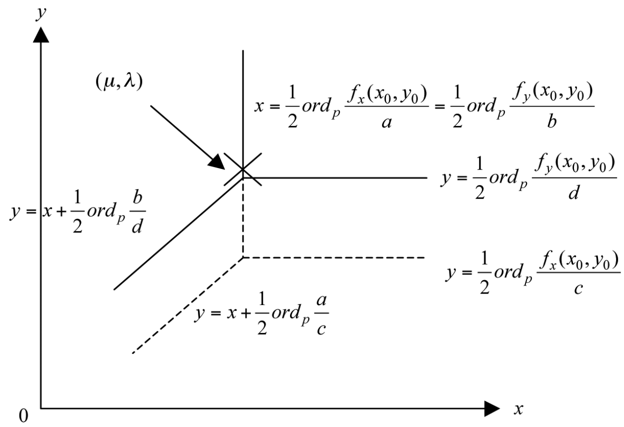

FIGURE 6. The combination of the indicator diagrams associated with $G(X, Y)$ (in broken lines) and $H(X, Y)$ (in bold) under the conditions: $ord\ pbfx(x_0, y_0) = ord_p af_y(x_0, y_0)$, $ord_p cf_y(x_0, y_0) > ord_p df_x(x_0, y_0)$ and $ord_p bc > _p rd_p ad$

LEMMA 8

Let $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 + ex + my + n$ be a polynomial in $Z_p[x, y]$ and (x_0, y_0) in Ω_p^2. Suppose that $ord_p af_y(x_0, y_0) = ord_p bf_x(x_0, y_0)$, $ord_p df_x(x_0, y_0) > ord_p cf_y(x_0, y_0)$ and $ord_p ad > ord_p bc$. Let $\alpha > 0$, with $ord_p f_x(x_0, y_0)$, $ord_p f_y(x_0, y_0) \geq \alpha$ and $\delta = \max\{ord_p a, ord_p b, ord_p c, ord_p d\}$. Then there exists (ξ, η) in Ω_p^2 such that $f_x(\xi, \eta) = 0, f_y(\xi, \eta) = 0$ and $ord_p(\xi - x_0)$, $ord_p(\eta - y_0) \geq \frac{1}{2}(\alpha - \delta)$.

Proof:
With the given conditions and Lemma 4, the arguments in the proof is similar to that for Lemma 7, by exchanging the positions of the indicator diagrams associated with $G(X, Y)$ and $H(X,Y)$.

THEOREM 2

Let $f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 + ex + my + n$ be a polynomial in $Z_p[x, y]$ and (x_0, y_0) in $\Omega_p^{\,2}$. Let $\alpha > 0$, with $ord_p f_x(x_0, y_0)$, $ord_p f_y(x_0, y_0) \geq \alpha$ and $\delta = \max\{ord_p a, ord_p b, ord_p c, ord_p d\}$. Then there exists (ξ, η) in $\Omega_p^{\,2}$ such that $f_x(\xi, \eta) = 0$, $f_y(\xi, \eta) = 0$ and $ord_p(\xi - x_0)$, $ord_p(\eta - y_0) \geq \frac{1}{2}(\alpha - \delta)$.

Proof:
From Lemma 5, Lemma 6, Lemma 7 and Lemma 8 respectively, there exists (ξ, η) in $\Omega_p^{\,2}$ with $f_x(\xi, \eta) = 0$, $f_y(\xi, \eta) = 0$ such that $ord_p(\xi - x_0)$, $ord_p(\eta - y_0) \geq \frac{1}{2}(\alpha - \delta)$.

CONCLUSION

Suppose $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 + ex + my + n$ is a polynomial in $Z_p[x, y]$ and (x_0, y_0) in $\Omega_p^{\,2}$, $\alpha > 0$ with $ord_p f_x(x_0, y_0) \geq \alpha$ and $\delta = \max\{ord_p a, ord_p b, ord_p c, ord_p d\}$. Our investigation on cases of indicator diagrams associated with f_x and f_y having one overlapping segment shows that there exist points on the overlapping segment that give estimates of the p-adic sizes in the neighbourhood of (x_0, y_0) of common zeros (ξ, η) of these partial derivative polynomials. These estimates are $ord_p(\xi - x_0) \geq \frac{1}{2}(\alpha - \delta)$ and $ord_p(\eta - y_0) \geq \frac{1}{2}(\alpha - \delta)$. The p-adic sizes of the common zeros in these cases are explicitly in terms of the p-adic orders of the coefficients of the polynomial. Our result improves the result of Sapar and Mohd Atan (2002) for cubic polynomials.

Our future research will be extended to cases involving occurrences of more than one overlapping segment in the combinations of indicator diagrams associated with the Newton polyhedra of higher degree polynomials.

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REFERENCES

Chan K.L. & Mohd. Atan K.A. Mohd. 1997. On the estimate to solutions of congruence equations associated with a quartic form. *Journal of Physical Science* 8: 21-34.

Heng S.H. & Mohd. Atan K.A. 1999. An estimation of exponential sums associated with a cubic form, *Journal of Physical Science* 10: 1-21.

Koblitz, N. 1977. *p-Adic Numbers, p-Adic Analysis and Zeta-Function*. New York: Springer-Verlag.

Mohd. Atan K.A. 1986a. Newton polyhedral method of determining p-adic orders of zeros common to two polynomials in $Q_p[x,y]$. *Pertanika* 9(3): 375-380.

Mohd. Atan K. A. 1986b. Newton polyhedra and p-Adic estimates of zeros of polynomials in $\Omega_p[x,y]$. *Pertanika* 9(1): 51-56.

Mohd. Atan K.A. & Loxton J.H. 1986. Newton polyhedra and solutions of congruences. In *Diophantine Analysis*. Loxton J.H. and A. Van der Poorten (ed.). Cambridge: Cambridge University Press.

Sapar S.H. & Mohd Atan K.A. 2002. Estimate for the cardinality of the set of solution to congruence equations. *Journal of Technology* 36(C): 13-40.

Sapar S.H. & Mohd Atan K.A. 2007. Penganggaran saiz p-Adic pensifar sepunya terbitan separa polinomial berdarjah enam, *Sains Malaysiana* 36(1): 77-82.

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