The opportunistic replacement and inspection problem for components with a stochastic life time

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Abstract

The problem of finding efficient maintenance and inspection schemes in the case of components with a stochastic life time is studied and a mixed integer programming solution is proposed. The problem is compared with the two simpler problems of which the studied problem is a generalisation: *The opportunistic replacement problem*, assuming components with a deterministic life time and *The opportunistic replacement problem for components with a stochastic life time*, for maintenance schemes without inspections.

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1 Introduction

This report concerns how to find efficient maintenance plans by mixed integer programming methods.

1.1 Participants and Contributions

Markus Bohlin: DUST project leader, discussions, experiments using AMPL and CPLEX

Jan Ekman: ideas, discussions, experiments using AMPL and CPLEX, producing the final report with figures and text, giving a presentation of the work

Anders Holst: discussions

1.2 On the maintenance problems studied in this report

Production and maintenance planning is often a problem which is not easily separated into production planning and a maintenance planning. Therefor the overall goal of maintenance planning is manifold such as finding desirable and legal maintenance personnel planning schemes, avoiding unnecessary production loss, avoiding maintenance and inspections with low gain, avoiding corrective maintenance in the cases this is expensive and making planning negotiations possible by presenting preliminary maintenance early in the planning process. In this report we will consider the general and simplified problem of maintenance planning by assuming that it is possible to estimate costs of making maintenance and estimate risks for failures. It may be that the maintenance costs is to be interpreted as the result of encoding all desirable and unwanted aspects of a plan. We assume that components of units to maintain have a stochastic life time such that there is always a small chance of failure regardless of how often service is made. By defining the maintenance problem this way it will be possible to compare any two plans and decide which one is the best, although the overall goal may be manifold and somewhat vague. In the mathematical model of the problem it may either be the case that risks of failures are represented as costs or that a part of the model is that the risks of failures are below given thresholds.

The problem studied in this report is a very general one but still not well-studied, although realistic and highly relevant. It is a generalisation of the much simpler *opportunistic replacement problem* (P1) which assumes deterministic life time of components. It is also a generalisation of *the opportunistic replacement problem for components with a stochastic life time* (P2) which does not take inspections into account. Thus, considering both stochastic life times of components and inspections we will call our problem *the opportunistic replacement and inspection problem for components with a stochastic life time* (P3).

This problem (P3) has a specific nature, which not so much accentuated in the first two problems, (P1) and (P2), in that the option to re-plan is central for the problem. The reason for this is of course that nothing is gained by the planned inspections unless we react on them by re-planning. If we among inspections include continuous surveillance by sensors then we need to react to information that may arrive at any time and especially during the execution of the plan itself. In order to be able to decide which of two plans is to prefer we need to estimate the gain of re-planning during the course of the plan. The problems (P1) and (P2) are generally solved without considering re-planning during the execution of the plan. In reality re-planning often cannot be avoided. Hence taking re-planning into account may be interpreted as making the mathematical model more realistic. It may also be considered as a way of finding less sub-optimal solutions to maintenance problems.

1.3 Related work

The opportutunistic replacement problem is studied in [1] and [2]. A survey of optimal maintenance for multi component systems is given in [5]. For an introduction to AMPL and integer programming see for instance [3] and [4].

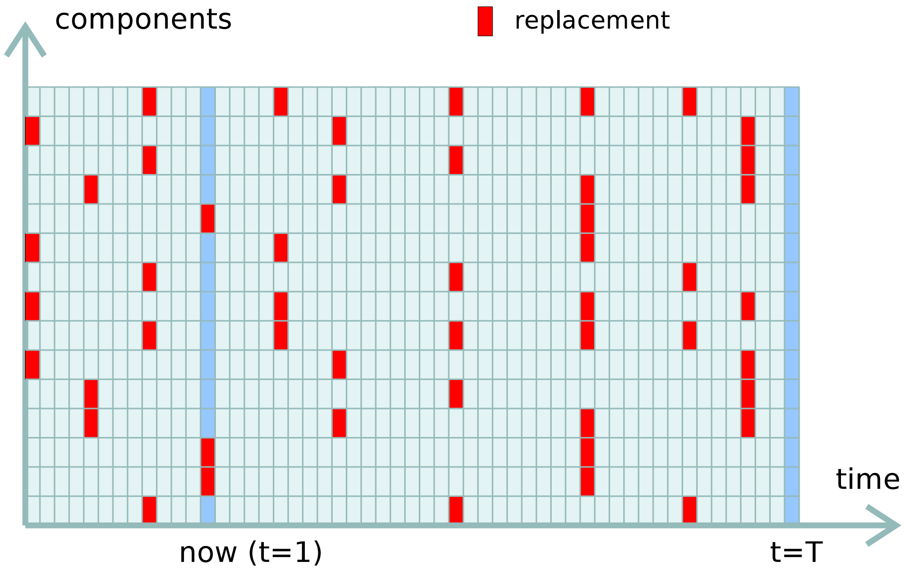

Figure 1: A maintenance scheme

2 The opportunistic replacement problem

2.1 Introduction

As mentioned above the opportunistic replacement problem refers to the case that the components life time are deterministic. In this case inspections has no value. The phrase opportunistic refers to that the model takes into account that it may be beneficial to make several replacements at the same time, thus making use of the *opportunity* offered by a replacement occasion to make another replacement at the same time.

We consider maintenance of a unit consisting of a fixed number n of components and we represent time by discrete time steps and consider a plan for a limited duration in time. This will make it possible to model the opportunistic replacement problem as a mixed integer program. A maintenance scheme, see figure 1, determines in each time step which components are replaced. The scheme must satisfy given conditions on how often components need to be replaced. We assume that these so called *life times* of each of the components are known. The life time may be different for different components. The opportunistic replacement problem aims at finding a maintenance scheme that satisfies the conditions and minimises the cost of maintenance.

2.2 Notation

Constants and indexes

T the total number time steps that the optimisation concerns

t a time step, $t \in [1, T]$

n number of components

i component index, $i \in [1, n]$

c_i cost of replacing component i

d replacement occasion cost

For the mixed integer formulation of the problem we use the following binary variables for defining the events of a service plan.

x_{it} component i is replaced in time step t

z_t some component is replaced in time step t

2.3 A mixed integer formulation of the problem

Without presenting all the details we can say that *the opportunistic replacement problem* is basically to minimise the following maintenance cost C_{fix} subject to *given* conditions on life times.

$$C_{\text{fix}} = \sum_{t=1}^{T} \sum_{i=1}^{n} c_i x_{it} + \sum_{t=1}^{T} d z_t$$

The purpose of the subscript *fix* here is to lead the mind to something that has nothing to do with randomness. C_{fix}, in this case is a value which in no part is composed of a random variable or an estimate, such as average value, of a random variable.

That the conditions on life times are *given* means just that the components life times are not variables as part of the mixed integer problem. The life times may for instance be estimated from historical data or the result of mechanical calculations.

This simple formulation of the problem, as minimising the cost C_{fix} above, needs to be modified by adding costs and constraints if we aim at an appropriate way to handle the period boundaries. At the plan period start the maintenance history, see figure 1, needs to be taken as an input to the problem. This means adding constraints to the problem. In most cases a plan with recently replaced components at the plan period end has a higher value than a plan were all components have to be replaced soon after that the plan ends. Hence, we also need an additional cost or gain related to the maintenance state at the plan period end. In addition to these modifications of the problem there may of course, in the specific cases, be a lot of other constraints, for instance concerning production loss and the availability of personnel and other resources for making maintenance.

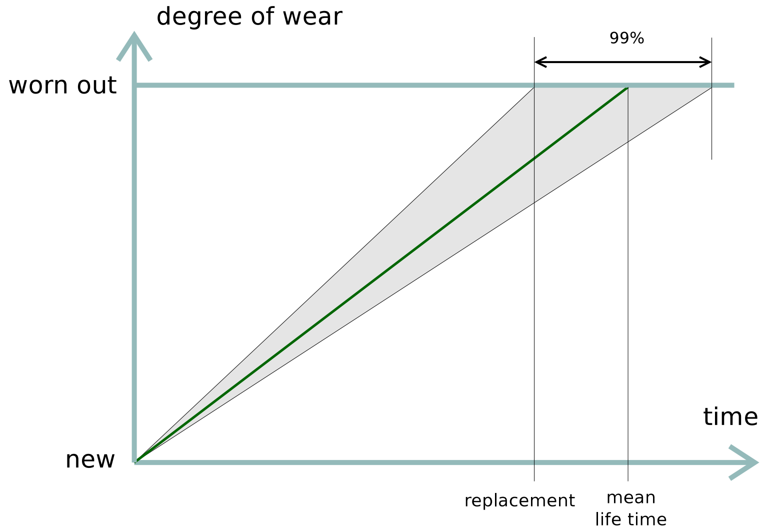

Figure 2: A model of component wear with a 99% certainty interval and a planned replacement with 1% risk of failure

3 The opportunistic replacement problem for components with a stochastic life time

3.1 Introduction

The problem in this section differs from the problem in the previous section only in that we consider components with a stochastic life time. For the problem in this section we will assume that we have some knowledge of the component wear which makes it possible to model the wear statistically, see as an example figure 2. The wear model, though, need not be linear in time, as is depicted in the figure. Such statistical models of wear may be estimated from historical data obtained at replacements, from inspections or from surveillance of components or it may be obtained otherwise as for example from mechanical properties of the components.

For the problem in the previous section there were only one kind of maintenance activity, that is replacements of components. For the problem in this section there are two: replacements and corrective maintenance, where corrective maintenance means maintenance of *failing* components. The plan will still only consist of replacements though. For the problem formulation we, in addition to the cost C_{fix} of the plan given in the previous section, need to consider another plan cost C_{risk}, the average cost for corrective maintenance.

It is only because of that we are concerned with stochastic component life times that we need to take corrective maintenance into account. With deterministic life times we can make service or replace components before they fail. But also in this case, with deterministic life times, one service strategy is to make nothing but corrective maintenance, i.e. we wait till the components break down to make service. That kind of service is of course also one possibility for the case of statistically modelled component life times. The difference between a deterministic life time model and a stochastic is though that, in the latter case, such a strategy of no preventive maintenance may appear as the solution to the problem of finding the optimal maintenance scheme.

3.2 Notation

We use the notation of section 2.2 together with the following constants

a_i a fixed cost for failure of component i

r_{iu} the chance of failure of component i if not replaced in the last u steps in time

$C_{\mathbf{risk}}(i,u)$ an average cost for failure of component i if not replaced in the last u steps in time

α_{iu} a cost correction term (explained below)

and the binary variable $h_i(t_1,t_2)$

$h_i(t_1,t_2)$ component i is not replaced in the time interval $[t_1,\ t_2]$

Assuming that the risk of failure, according to a degeneration model, will increase with the time since last replacement we will for a component i have an increasing average failure cost

$$C_{\text{risk}}(i,u) = a_i\, r_{iu}$$

with increasing time u since last service.

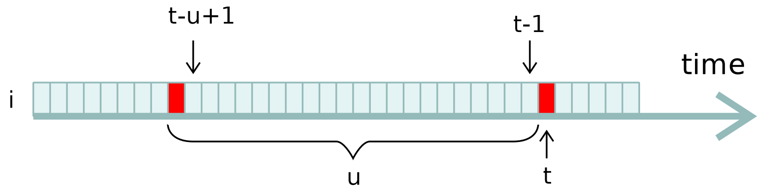

Figure 3: Taking failure risks into consideration

3.3 A mixed integer formulation of the problem

Without presenting all the details we can say that *the opportunistic replacement problem for components with a stochastic life time* is basically to minimise the maintenance cost $C_{\text{fix}} + C_{\text{risk}}$, see figure 3.

$$C_{\text{fix}} = \sum_{i=1}^{n} \sum_{t=1}^{T} c_i\, x_{it} + \sum_{t=1}^{T} d\, z_t$$

$$C_{\text{risk}} \;=\; \sum_{i=1}^{n} \sum_{u=2}^{T'} \sum_{t=1}^{T} (C_{\text{risk}}(i,u) - \alpha_{iu})\; h_i(t-u+1,\; t-1)$$

Here α_{iu} is a correction term for that we in C_{risk} incorrectly will include the risk costs for all the sub-intervals of each interval $[t-u+1,\; t-1]$ in which component i is not replaced. The reason for still using the variable $h_i(t_1, t_2)$ and not another formulation, where the correction could have be avoided, is that using the variable $h_i(t_1, t_2)$ gives high performance in solving the mixed integer program.

Concerning the conditions subject to the problem this problem and the deterministic component life time problem differs in that, for the problem in this section, we do not have to have any life times conditions. For specific cases we may of course still have conditions on maximum and minimum time of component usage. Similar to the deterministic problem an appropriate managing of the period boundaries requires some modifications of costs and condition and in specific cases there may be yet additional constraints.

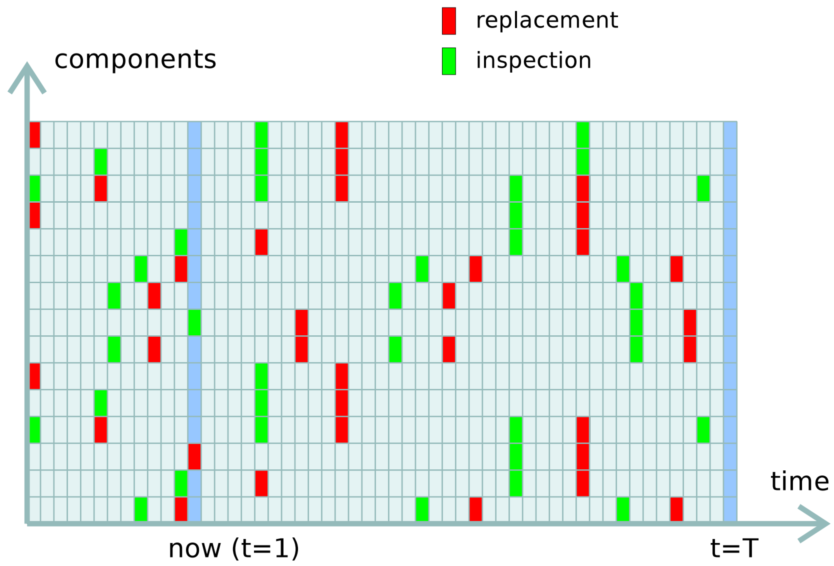

Figure 4: A maintenance scheme with replacements and inspections

4 The opportunistic replacement and inspection problem for components with a stochastic life time

4.1 Introduction

We now turn to the case that the maintenance scheme consists of both replacements and inspections, see figure 4. In this case the maintenance in consists of three types of actions:

- component inspections
- component replacements
- corrective maintenance

The inspections and replacements are called events and may be either *planned* or *pre-historic*. Just as for the previously studied problems we aim at finding a maintenance scheme that minimises the total plan cost. The big difference between this problem and the previous ones is that the total plan cost need to include the gain of re-scheduling as a response to the outcomes of the inspections. Re-scheduling is the only way to gain anything from making inspections.

Since re-scheduling is part of the planning process we may consider maintenance planning as consisting of two phases:

1. making an initial plan consisting of inspections and component service
2. the operative re-planning phase resulting in the carried out maintenance

The total plan cost we aim at minimising is

$$C = C_{\text{fix}} + C_{\text{risk}} + C_{\text{insp}}$$

The fixed costs C_{fix} are the sum of the costs for the planned replacements and inspections. The risk cost C_{risk} is an estimated average cost obtained from the risk for a failure and an estimated cost for corrective maintenance of a failed component. $-C_{\text{insp}}$ is an estimation of the inspection information gain. If $C_{\text{insp}} > 0$ we would get a better plan by removing all the inspections. Hence $C_{\text{insp}} < 0$ for any reasonable plan and therefor we consider $-C_{\text{insp}}$ as a gain. Since C_{insp} occurs as a part of the plan cost, we will nevertheless use *the inspection information cost* to refer to C_{insp}.

As for the previously studied problems, we assume that there is a plan pre-history. The inspections and replacements that occurs at or after the plan start we call *planned* and the ones at or before the plan start we call pre-historic. We assume that the pre-history, for each component, contains information on at least one event, that is an inspection or a replacement, for each component, see figure 4.

We will restrict the problem by, for each component, not allowing a planned inspection to be immediately succeeded by another inspection. That an event, that is an inspection or a replacement, is immediately preceded or succeeded by another event here means that there is no third event in between these two events. For any component, the pre-historic inspections may be succeeded by any number of inspections. For any component, an inspection may also be preceded by any number of pre-historic inspections. The restriction is non-essential since we count on re-planning as a response to the inspections whenever inspection information arrives. That is, for the resulting maintenance the inspection of any component may be succeeded by any number of inspections.

We will begin by presenting a mixed integer formulation of the problem under the assumption that we know how to estimate the information gain of a single inspection. After that we study the information gains, or costs, of single inspections.

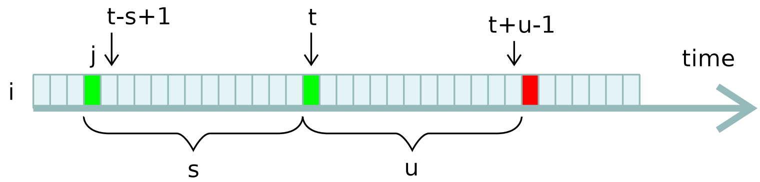

Figure 5: A planned inspection immediately preceded by a pre-historic inspection with outcome j and immediately succeeded by a replacement

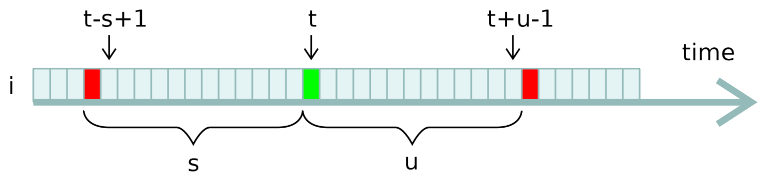

Figure 6: A planned inspection immediately preceded by a replacement and immediately succeeded by another replacement

4.2 Notation

An inspection who's outcome is dependent on the pre-history, we will be call a *initial* inspection. The other inspections are simply called *non-initial*. A non-initial inspection of a component occurs after a planned replacement of the same component, that is a replacement not belonging to the plan pre-history.

Constants and indexes:

T the total number time steps that the optimisation concerns

t, u time steps, $t, u \in [1, T]$

n the number of components

i a component index, $i \in [1, n]$

b_i the cost of inspecting component i

c_i the cost of replacing component i

d a replacement occasion cost

a_i a fixed cost for failure of component i

r_{iu} the chance of failure of component i if not replaced in the last u steps in time

Derived costs:

α_{iu}, β_{ijsu} cost correction terms, explained in the text

C_{fix} the total costs for the planned replacements and inspections

$C_{\text{risk}}(i,u)$ an average cost for failure of component i if not replaced in the last u steps in time

C_{risk} the total costs for the planned replacements and inspections

$C^{(1)}(i,s,u)$ the average cost of re-scheduling as a response to the outcomes of an inspection of component i time s after a replacement and time u before a another replacement of the same component, see figure 5.

$C^{(2)}(i,j,s,u)$ the average cost of re-scheduling as a response to the outcomes of an inspection of component i time s after an inspection with outcome j and and time u before a replacement of the same component, see figure 6.

$C^{(1)}_{\text{insp}}$ the average total cost of re-scheduling as a response to the inspections outcomes for the initial inspections, that is the inspections for which the gain is dependent on the plan pre-history, see figure 5.

$C^{(2)}_{\text{insp}}$ the average total cost of re-scheduling as a response to the inspections outcomes for the non-initial inspections, that is the inspection for which the gain is not dependent on the plan pre-history, see figure 6.

C_{insp} the average total cost of re-scheduling as a response to the inspections outcomes

C the total cost of the maintenance scheme

Binary variables:

x_{it} component i is replaced in time step t

y_{it} component i is inspected in time step t

z_t some component is replaced in time step t

$h_i(t_1,t_2)$ component i is not replaced in the time interval $[t_1,\ t_2]$

$w^{(1)}_{istu}$ at time t there is an inspection of component i which is at least time s after a replacement and at least time u before a another replacement of the same component, see figure 6.

$w^{(2)}_{ijstu}$ at time t there is an inspection of component i which is at least time s after another inspection with outcome j and at least time u before a replacement of the same component, see figure 5.

4.3 A mixed integer formulation of the problem

In this section we formulate the opportunistic replacement and inspection problem for components with a stochastic life time as a mixed integer problem. This formulation is made under the assumption that we know how to estimate the gain of an inspection. In the coming sections we will show how to estimate inspection gains.

Without presenting all the details we can say that *the opportunistic replacement and inspection problem for components with a stochastic life time* is basically to minimise the maintenance cost C defined as

$$C \ = = C_{\text{fix}} + C_{\text{risk}} + C_{\text{insp}} \qquad C_{\text{insp}} \ = \ C^{(1)}_{\text{insp}} + C^{(2)}_{\text{insp}}$$

where

$$C_{\text{fix}} = \sum_{i=1}^{n}\sum_{t=1}^{T} b_i\, y_{it} + \sum_{i=1}^{n}\sum_{t=1}^{T} c_i\, x_{it} + \sum_{t=1}^{T} d\, z_t$$

$$C_{\text{risk}} \ = \ \sum_{i=1}^{n}\sum_{u=2}^{T'}\sum_{t=1}^{T} (C_{\text{risk}}(i,u) - \alpha_{iu})\ h_i(t-u+1,\ t-1)$$

$$C^{(1)}_{\text{insp}} \ = \ \sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T'}\sum_{u=1}^{T''} \left(C^{(1)}(i,s,u) - \beta_{i1su}\right) w^{(1)}_{istu}$$

$$C^{(2)}_{\text{insp}} \ = \ \sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T'}\sum_{u=1}^{T''} \left(C^{(2)}(i,j,s,u) - \beta_{ijsu}\right) w^{(2)}_{ijstu}$$

and where the variables $w^{(1)}_{istu}$ and $w^{(2)}_{ijstu}$ may be expressed in terms of $h_i(t_1, t_2)$ as follows

$$w^{(1)}_{istu} = y_{it} \times\ h_i(t-s+1,\ t-1)\ \times\ h_i(t+1,\ t+u-1)$$

$$w^{(2)}_{ijstu} = y'_{it}\ \times\ \text{init}(i,j,s-t) \times\ h_i(t-s+1,\ t-1)\ \times\ h_i(t+1,\ t+u-1)$$

where y'_{it} denotes that an initial inspection of component i takes place at time t and $\text{init}(i,j,v)$ is an input constant meaning that at v steps in time before the plan start there is pre-historic inspection with outcome j of component i.

For similar reasons as the formulations of the cost C_{risk} uses the correction terms α_{iu} also the inspection costs $C^{(1)}_{\text{insp}}$ and $C^{(2)}_{\text{insp}}$ have correction terms β_{ijsu}. As for the previous problem without in sections the problem in this section may be subject to some conditions and additional costs or gains

Concerning the conditions subject to the problem this problem and the deterministic component life time problem differs in that, for the problem in this section, we do not have to have any life times conditions. For specific cases we may of course still have conditions on maximum and minimum time of component usage. Similar to the deterministic problem an appropriate managing of the period boundaries requires some modifications of costs and condition and in specific cases there may be yet additional constraints.

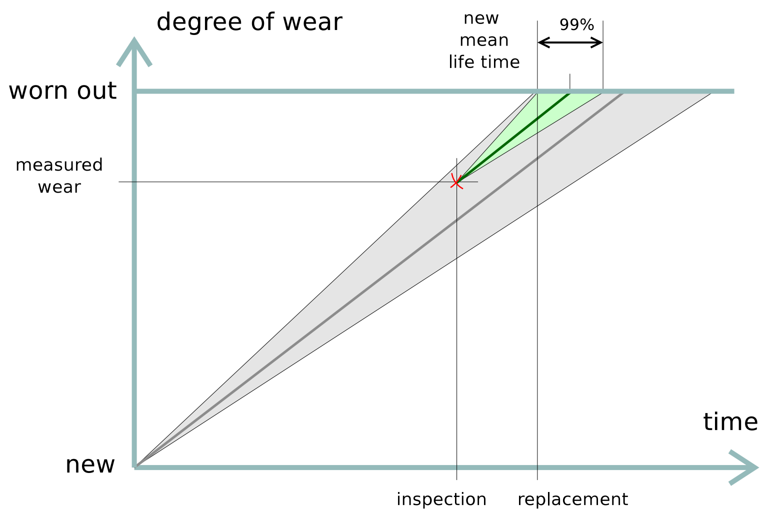

Figure 7: An updated model of component wear as the result of an inspection result that shows that the wear is above the average

5 Estimation of information gain of inspections

5.1 Introduction

Figures 7 and 8 aim at showing what we gain by an inspection. By an inspection the remaining life time in general will be clearer. In statistical terms the life time variance will decrease. This is indicated in the figures by the green triangles showing an updated statistical model as a result of the inspection outcome.

In figure 7 the wear is above the average wear. However, as the figure shows, even if we do not re-schedule the replacement to a earlier point in time the risk of failure before the replacement is not higher than it was from start. This is the consequence of that the the variance is lower for the updated life time model. In figure 8 the wear is below the average and in this case we have the opportunity to re-schedule the replacement to a later point in time and still keep the failure risk at the initial level.

The gain of inspection information stems the from option to re-plan, based on the information. Let us for simplicity think of gain as a negative cost, such that we do not need to refer to both “cost” and “gain.” In order to estimate the plan cost we need to convert the information gain into something of the same sort as the other costs. Inspection takes two re-planning costs into consideration

- cost related to change of time for the next replacement
- cost related to the altered risk of failure

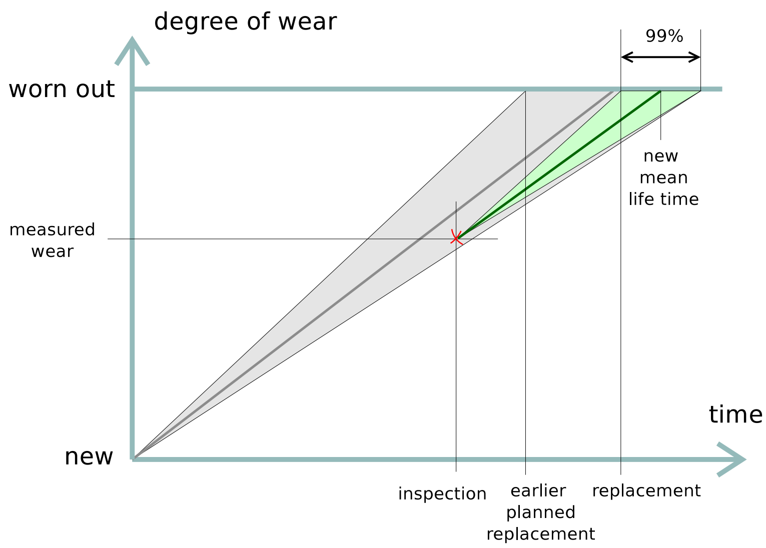

Figure 8: An updated model of component wear as the result of an inspection result that shows that the wear is below the average

Even if we do not re-plan, the inspection information will lead to a decrease in the risk of failure, for some of the outcomes of the inspection. For other outcomes, inspection information leads to that the risk of failure will increase, even if no re-planning takes place. For each point in time there is a risk of failure of each of the components. We do however assume that we do not discover the failures until inspections. Moreover we assume that the cost for a failure is constant and independent of how long it was since it appeared.

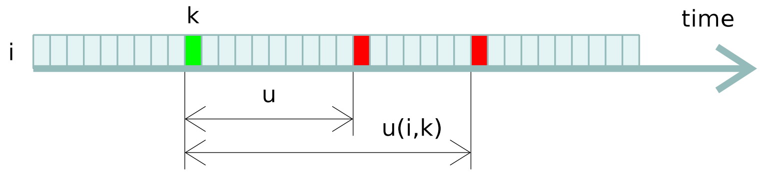

Figure 9: Re-scheduling a replacement from u time steps after inspection to $u(i,k)$ time steps after inspection

5.2 Notation

For the estimation of gain of inspections we use the following notation in addition to one the given in section 4.2 .

ω the number of inspection outcomes for each of the components

k, j inspection outcomes , $k, j \in [1, \omega]$

$u(i,k)$ a re-scheduling strategy for component i as the number of time steps from an inspection with outcome k to the immediately succeeding replacement

δ_i the gain of delaying replacement of component i one time step

r_{iu} the chance of failure of component i if not replaced in the last u steps in time

$g_{is}(j,k)$ the probability that an inspection of component i will have have k as outcome $s \geq 0$ time steps after a previous inspection with j as outcome.

b_{iuk} the gain of re-scheduling replacement from time u to $u(i,k)$ given the inspection result k

5.3 The information gain of a single inspection

To find a formula for the information gain of an inspection we will estimate the gain difference with and without a given inspection. We start by considering the information gain for a given inspection outcome.

We let the outcome k of an inspection of a component be a natural number in $[1, \omega]$, assumed to encode the component degree of wear. We will use the convention that 1 is the lowest degree of wear, for which the component is to be regarded as new. The highest degree of wear is the maximum outcome ω of an inspection and this outcome denotes *failure*. We assume that we have a model of degradation from which we have constructed an algorithm to derive the probability $g_{is}(j,k)$ that an inspection of component i will have have k as outcome at time $s \geq 0$ after a previous inspection with j as outcome. We have previously used r_{is} to denote the chance of failure of component i if not replaced in the last u steps in. Hence we have

$$r_{is} = g_{is}(1, \omega)$$

Given an outcome k as the result of an inspection, see figure 9, we assume that have a re-scheduling strategy $u(i,k)$ for component i, where $u(i,k)$ is the number of time steps from an inspection with outcome k to the planned replacement that immediately succeeds the inspection. This strategy may itself be the result of a separate optimisation. In this report we simply assume that the re-scheduling strategy $u(i,k)$ is given as input to the problem.

Assume that the planned replacement immediately succeeding an inspection is u time steps after the inspection as shown in figure 9. Assume that the inspection results in the outcome k and that after that the planned replacement is re-scheduled to take place $u(i,k)$ time steps after the inspection instead. Hence, as a consequence we have the following difference in risk of failure

$$g_{i\,u(i,k)}(k, \omega) - g_{iu}(k, \omega)$$

As mentioned in section 4.1, although the inspection information cost C_{insp} in general is negative we still refer to it as a cost. We will analogously, in the continuation, let the notation be given in terms of *costs*, although these costs in general are negative.

Remember that a_i is the fixed cost for failure of component i. We thus have the following cost difference with respect to failure

$$a_i \left(g_{i\,u(k)}(k, \omega) - g_{iu}(k, \omega) \right)$$

We In addition to the re-scheduling strategy $u(i,k)$ we also assume that we have an estimate of the gain δ_i of moving a replacement one time step ahead. Since moving a replacement one time step ahead means a negative cost, we have the following cost difference with respect to change of time for the next replacement

$$\delta_i \left(u - u(i,k) \right)$$

Hence the cost b_{iuk} of re-scheduling a planned replacement immediately succeeding an inspection from time u to $u(i,k)$ given the inspection result k is

$$b_{iuk} = \delta_i \left(u - u(i,k) \right) + a_i \left(g_{u(k)}(k, \omega) - g_u(k, \omega) \right)$$

Recall that $C^{(2)}(i, j, s, u)$ is the average cost of re-scheduling as a response to the outcome of an inspection at some time step, say t, of component i time s after an inspection with outcome j and and time u before a replacement of the same component, see figure 5. Hence to calculate this average cost we have to consider all possible outcomes k of the inspection at time t and the probability $g_{is}(j,k)$ of each of the outcomes. That is

$$C^{(2)}(i,j,s,u) = \sum_{k=1}^{\omega} g_{is}(j,k) b_{iuk}$$

Similarly for $C^{(1)}(i,s,u)$ the average cost of re-scheduling as a response to the outcomes of an inspection at some time step, say t, of component i time s after a replacement and time u before a another replacement of the same component, see figure 6

$$C^{(1)}(i,s,u) = \sum_{k=1}^{\omega} g_{is}(1,k) b_{iuk}$$

These two formulae above are the single inspection information cost formulae.

5.4 On the distribution g_{is} of inspection outcomes

To make it simple to express the properties of the inspection distribution we do the following:

1. consider service as a kind of inspection, with the special feature of making the component become new
2. let g_{is} be the probability matrix such that $g_{is}(j,k)$ is the probability that an inspection of component i will have have k as outcome at $s \geq 0$ units in time after a previous inspection with j as outcome.
3. let failure be one of the outcomes at inspection and let ω denote that outcome

With this notation $g_{is}(1,\omega)$ is the probability of failure of component i at s units in time after a replacement and $g_{is}(j,\omega)$ is the probability of failure of component i at s units in time after an inspection with j as outcome. We notice that for $k \neq \omega$ and for all $s \geq 0$

$$g_{is}(\omega,\omega) = 1 \qquad g_{is}(\omega,k) = 0 \qquad g_{i0}(k,k) = 1$$

The interpretation of $g_{is}(\omega,\omega) = 1$ is that: if a failure occurs, then it will persist. Hence g_t is upper triangular. We assume that a higher inspection outcome means a more degraded state, that is

$$j > k \;\rightarrow\; g_{is}(j,k) = 0$$

where $\omega > k$ for all $k \neq \omega$. Let m be the number of outcomes, we notice that each of the row sums is the probability of the outcome of the final inspection under the condition that a previous inspection has a given outcome j.

$$\sum_{i=1}^{m} g_{is}(j,i) = 1$$

The column sums does however not add up to one. We have the matrix composition rules

$$g_{(i,s+u)} = g_{is}g_{iu} \qquad g_{iu}g_{is} = g_{is}g_{iu} \qquad g_{is}g_{i0} = g_s$$

and we see that g_t is recursively given in terms of g_1.

We have for instance, using $j > k \ \rightarrow \ g_{is}(j,k) = 0$

$$g_{(i,s+u)}(j,k) = \sum_{i=j}^{k} g_{is}(j,i)g_{iu}(i,k)$$

5.5 Reformulation of the inspection information cost

Recall the formulae for the information cost of an inspection

$$C^{(1)}(i,s,u) = \sum_{k=1}^{\omega} g_{is}(1,k)b_{iuk} \qquad C^{(2)}(i,j,s,u) = \sum_{k=1}^{\omega} g_{is}(j,k)b_{iuk}$$

$$b_{iuk} = \delta_i\left(u - u(i,k)\right) + a_i\left(g_{u(k)}(k,\omega) - g_u(k,\omega)\right)$$

By using the results of the previous section and that $r_{is} = g_{is}(1,\omega)$ these formulae may reformulated as follows

$$C^{(1)}(i,s,u) = \delta_i A_1 + a_i A_2$$

$$C^{(2)}(i,j,s,u) = \delta_i B_1 + a_i B_2$$

$$A_1 = u - \sum_{k=1}^{\omega} g_{is}(1,k)\,u(i,k) \qquad A_2 = \left(\sum_{k=1}^{\omega} g_s(j,k)g_{u(k)}(k,\omega)\right) - r_{(i,s+u)}$$

$$B_1 = u - \sum_{k=1}^{\omega} g_{is}(j,k)\,u(i,k) \qquad B_2 = \left(\sum_{k=1}^{\omega} g_{is}(j,k)g_{u(k)}(k,\omega)\right) - g_{s+u}(j,\omega)$$

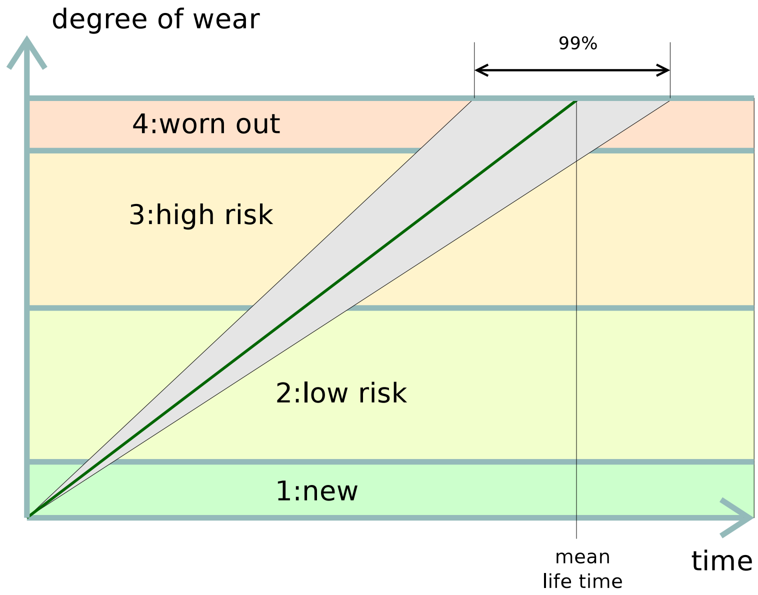

Figure 10: An example with four different outcomes of an inspection of a component

6 An example

6.1 A distribution

Let us as an example consider a unit consisting of just one component and the number inspection outcomes be just four, where 1 means “counts as new”, 2 means “low degradation”, 3 means “high degradation” and $\omega = 4$ means that the component has failed, see figure 10. Let $g_s(j,k)$ be the probability that an inspection of the component will have have k as outcome at $s \geq 0$ units in time after a previous inspection with j as outcome. Let g_s be defined as below.

$$g_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$g_1 = \begin{pmatrix} 0.9 & 0.09 & 0.009 & 0.001 \\ 0 & 0.9 & 0.09 & 0.01 \\ 0 & 0 & 0.9 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$g_2 = g_1 g_1 = \begin{pmatrix} 0.81 & 0.162 & 0.0243 & 0.0037 \\ 0 & 0.81 & 0.0162 & 0.028 \\ 0 & 0 & 0.81 & 0.19 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$g_3 = g_1 g_2 = \begin{pmatrix} 0.729 & 0.2187 & 0.04374 & 0.00856 \\ 0 & 0.729 & 0.2187 & 0.0523 \\ 0 & 0 & 0.729 & 0.271 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$g_4 = g_1 g_3 = \begin{pmatrix} 0.6561 & 0.26244 & 0.06561 & 0.01585 \\ 0 & 0.6561 & 0.26244 & 0.08146 \\ 0 & 0 & 0.6561 & 0.3439 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Consider figure 5 and assume that we make an inspection with outcome $j = 2$ in a time step $t - 3$, that is $s = 3$. Then the likelihood of having outcome $k = 3$ in time step t is $g_3(2, 3) = 0.2187$ and the probability of having a failure, $\omega = 4$, at time step $t + 4$, that is $u = 4$, is $g_4(2, 4) = 0.08146$.

6.2 Strategy

Let the strategy $u(k)$ be

$$u(k) = \begin{cases} 5 & k = 1 \\ 2 & k = 2 \\ 1 & k = 3 \\ 0 & k = \omega \end{cases}$$

With $j = 1$ the risk of failure at $t + u(k)$ is

$$\sum_{k=1}^{\omega} g_3(1, k) \times g_{u(k)}(k, \omega)$$

$$= g_3(1, 1) \times g_5(1, \omega) + g_3(1, 2) \times g_2(2, \omega) + g_3(1, 3) \times g_3(3, \omega) + g_3(1, \omega)$$

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