# On Intensional Aspects of Concepts Defined in Rough Set Theory

Lech Polkowski[1] and Maria Semeniuk–Polkowska[2]

[1] Polish–Japanese Institute of Information Technology
Koszykowa 86, 02008 Warsaw, Poland
Department of Mathematics and Computer Science
University of Warmia and Mazury
Zolnierska 14, 10560 Olsztyn, Poland
[2] Chair of Formal Linguistics, Warsaw University
Browarna 8/12, 00650 Warsaw,Poland
polkow@pjwstk.edu.pl; m.polkowska@uw.edu.pl

**Abstract.** Intensionality is bound to global mappings from sets of possible worlds to truth states relative to a set of predicates, witness schemes like Montague Grammar. We discuss this aspect in the frame of rough set theory where concepts arise as collections of objects constrained by bounds on values of chosen sets of attributes. We apply the idea of a rough inclusion as similarity measure for objects, and rough inclusions measure truth state values of predicates relative to possible worlds – granules of objects.
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## 1 Introductory Notions

We give in a nutshell basic notions relevant to our discussion, cf. [13], [4]. An information system is a pair $(U, A)$, of a set of *entities/objects* $U$ with a set $A$ of *attributes*. Each attribute is a mapping on $U$ with values in a set $V_a$; a decision system is a triple $(U, A, d)$, where the attribute $d$, the *decision*, represents the external knowledge about $U$ by an oracle/expert.

*Concepts* with respect to a given data are defined formally as subsets of the universe $U$. Concepts can be written down in the language of *descriptors*.

For an attribute $a$ and its value $v$, the *descriptor* defined by the pair $a, v$ is the atomic formula $(a = v)$. Descriptors can be made into formulas by means of sentential connectives: $\vee, \wedge, \neg, \Rightarrow$: formulas of descriptor logics are elements of the smallest set which contains all atomic descriptors and is closed under the mentioned above sentential connectives. Introducing for each object $u \in U$ its *information set* $Inf(u) = \{(a = a(u)) : a \in A\}$, we can define the basic *indiscernibility relation* $IND(A) = \{(u, v) : Inf(u) = Inf(v)\}$. Replacing $A$ with a subset $B$ of attribute set, we define the *B–indiscernibility relation* $IND(B)$.

A descriptor $(a = v)$ is interpreted semantically in the universe $U$; the meaning $[a = v]$ of this descriptor is the concept $\{u \in U; a(u) = v\}$. Meanings of atomic descriptors are extended to meanings of formulas of descriptor logic by recursive conditions,

$[p \vee q]$=$[p] \cup [q]$; $[p \wedge q]$=$[p] \cap [q]$; $[\neg p]$=$U \setminus [p]$ etc. etc.

Any relation IND(B) partitions the universe $U$ into blocks – equivalence classes $[u]_B$ of IND(B), regarded as *elementary $B$–exact concepts*. Unions of families of elementary $B$–exact concepts constitute *$B$–exact concepts*.

In terms of exact concepts, one can express dependencies among attributes [13]: in the simplest case, a set $D$ of attributes *depends functionally* on a set $C$ of attributes if and only if $IND(C) \subseteq IND(D)$; the meaning is that any class $[\bigwedge_{a \in C}(a = v_a)]$ is contained in a unique class $[\bigwedge_{a \in D}(a = w_a)]$ so there is a mapping $U/IND(C) \rightarrow U/IND(D)$. We write down this dependency as $C \mapsto D$.

Dependency need not be functional; in such case, the relation $IND(C) \subseteq IND(D)$ can be replaced [13] with a weaker notion of a *(C,D)–positive set* which is defined as the union $Pos_C(D) = \bigcup\{[u]_C : [u]_C \subseteq [u]_D\}$; clearly then, $IND(C)|Pos_C(D) \subseteq IND(D)$. In [13] a factor $\gamma(B, C) = \frac{|Pos_B(C)|}{|U|}$ was proposed as the measure of degree of dependency of $D$ on $C$, where $|X|$ is the number of elements in $X$. This form of dependency is denoted symbolically as $C \hookrightarrow_\gamma D$.

Dependencies have a logical form in logic of descriptors as sets of implications of the form

$$\bigwedge_{a \in C}(a = v_a) \Rightarrow \bigwedge_{a \in D}(a = w_a); \tag{1}$$

in a particular case of a decision system $(U, A, d)$, dependencies of the form $C \hookrightarrow_\gamma \{d\}$ are called decision algorithms and individual relations of the form $\bigwedge_{a \in C}(a = v_a) \Rightarrow (d = w_d)$ are said to be *decision rules*. There have been proposed various measures of the truth degree of a decision rule, under the name of a *rule quality*, see, e.g., [13].

In descriptor logic setting, a decision rule $r : \alpha_C \Rightarrow \beta_d$ is said to be *true* if and only if the meaning $[r] = U$ which is equivalent to the condition that $[\alpha_C] \subseteq [\beta_d]$.

The above introduced constituents: entities, indiscernibility relations, concepts, dependencies, form building blocks from which knowledge is discovered as a set of statements about those constituents.

It is our purpose to construct a formal logical system in which one would be able to define values of truth states of formulas of knowledge, decision rules in particular, in a formal manner, preserving the notion of truth as recalled above, but in a localized version, with respect to a particular exact concept of entities.

Rough set theory discerns between *exact concepts* which are unions of indiscernibility classes and *rough concepts* which are not any union of indiscernibility classes. Passing from rough to exact concepts is achieved by means of approximations: the *lower approximation* to a concept $W \subseteq U$ is defined as $\underline{W} = \{u \in U : [u]_A \subseteq W\}$, and, the *upper approximation* $\overline{W} = \{u \in U : [u]_A \cap W \neq \emptyset\}$.

Any attempt at assigning various degrees of truth to logical statements places one in the realm of many–valued logic. These logics describe formally logical functors as mappings on the set of truth values/states into itself hence they operate a fortiori with values of statements typically as fractions or reals in the unit interval $[0,1]$, see in this respect, e.g., [6], [7], [8], [9], and, as a survey, see [4].

In many of those logics, the functor of implication is interpreted as the *residual implication* induced by a continuous t–norm. We recall, cf., e.g., [4] or [**?**] that a *t–norm* is a mapping $t : [0,1]^2 \rightarrow [0,1]$ which satisfies the conditions,

(TN1)$t(x,y) = t(y,x)$ (symmetry);

(TN2) $t(x,t(y,z)) = t(t(x,y),z)$ (associativity);

(TN3) $t(x,1) = x; t(x,0) = 0$ (boundary conditions);

(TN4) $x > x'$ implies $t(x,y) \geq t(x'),y)$ (monotonicity coordinate–wise);

and additionally,

(TN5) $t$ can be continuous.

A continuous t–norm is *Archimedean* in case $t(x,x) = x$ for $x = 0,1$ only; for such t–norms, it was shown, see [5], that a formula holds,

$$t(x,y) = g(f(x) + f(y)), \tag{2}$$

with a continuous decreasing function $f : [0,1] \rightarrow [0,1]$ and $g$ – the pseudo–inverse to $f$.

Examples of Archimedean t–norms are,

**The Łukasiewicz t–norm** $L(x,y) = max\{0, x+y-1\}$;

**The product (Menger) t–norm** $P(x,y) = x \cdot y$.

The two are up to an automorphism on [0,1] the only Archimedean t–norms [12].

An example of a t–norm which is not any Archimedean is

**Minimum t–norm** $Min(x,y) = min\{x,y\}$. It is known, see [1], that for Min the representation (2) with $f$ continuous does not exist.

### 1.1 Residual implications

*Residual implication* $x \Rightarrow_t y$ induced by a continuous t–norm $t$ is defined as,

$$x \Rightarrow_t y = max\{z : t(x, z) \leq y\}. \tag{3}$$

As $t(x, 1) = x$ for each t–norm, it follows that $x \Rightarrow_t y = 1$ when $x \leq y$ for each t–norm $t$.

In case $x > y$, one obtains various semantic interpretations of implication depending on the choice of $t$, see, e.g., [4] for a review. Exemplary cases are,

**The Łukasiewicz implication** $x \Rightarrow_L y = min\{1, 1 - x + y\}$, see [7];

**The Goguen implication** $x \Rightarrow_P y = \frac{y}{x}$;

**The Goedel implication** $x \Rightarrow_{Min} y = y$.

### 1.2 Logics of residual implications vs. logical containment in decision rules

In logics based on implication given by residua of t–norms, negation is defined usually as $\neg x = x \Rightarrow_t 0$. Thus, the Łukasiewicz negation is $\neg_L x = 1 - x$ whereas Goguen as well as Goedel negation is $\neg_G x = 1$ for x=0 and is 0 for $x > 0$. Other connectives are defined with usage of the t–norm itself as semantics for the strong conjunction and ordinary conjunction and disjunction are interpreted semantically as, respectively, *min, max*. Resulting logics have been a subject of an intensive research, cf., a monograph [4].

In this approach a rule $\alpha \Rightarrow \beta$ is evaluated by evaluating the truth state $[\alpha]$ as well as the truth state $[\beta]$ and then computing the values of $[\alpha] \Rightarrow_t [\beta]$ for a chosen t–norm $t$. Similarly other connectives are evaluated.

In the rough set context, this approach would pose the problem of evaluating the truth state of a conjunct $\alpha$ of descriptors; to this end, one can invoke the idea of Łukasiewicz [6] and assign to $\alpha$ a value $[\alpha]_L = \frac{[\alpha]}{|U|}$, where $[\alpha]$ is the meaning already defined, i.e., the set $\{u \in U : u \models \alpha\}$. Clearly, this approach does not take into account the logical containment or its lack between $\alpha$ and $\beta$, and this fact makes the many–valued approach of a small use when data mining tasks are involved.

For this reason, we propose an approach to logic of decision rules which is based on the idea of measuring the state of truth of a formula against a concept constructed as a granule of knowledge; concepts can be regarded as "worlds" and our logic becomes intensional, cf., e.g., [2], [11]: logical evaluations at a given world are extensions of the intension which is the mapping on worlds valued in the set of logical values of truth.

To implement this program, we need to develop the following tools:

1 a tool to build worlds, i.e, a granulation methodology based on a formal mechanism of granule formation and analysis;

2 a methodology for evaluating states of truth of formulas against worlds.

In both cases 1, 2, our approach exploits tools provided by *rough mereology*, see, e.g., [14] Similarity measures – rough inclusions – provide means for all necessary definitions of relevant notions. Here we dispense with any account of granulation for space's sake.

## 2 Rough inclusions: The basic facts

We now recall some exemplary means for inducing rough inclusions along with some new results; for a general discussion, cf., e.g. [14].

### Rough inclusions on [0,1]

**Proposition 1.** *[14]. For each continuous t–norm t, the residual implication* $\Rightarrow_t$ *defines a rough inclusion by* $\mu_t(x, y, r) \Leftrightarrow x \Rightarrow_t y \geq r$.

There exist rough inclusions not definable in this way, e.g., the *drastic rough inclusion*,

$$\mu_0(x, y, r) \text{ if and only if either } x = y \text{ and } r = 1 \text{ or } r = 0. \tag{4}$$

Clearly, $\mu_0$ is associated with the ingredient relation $=$ and, a fortiori, the part relation is empty, whereas any rough inclusion $\mu$ induced by a residual implication in the sense of Prop. 1, is associated to the ingredient relation $\leq$ with the part relation $<$.

In case of Archimedean t–norms, it is well–known that a representation formula holds for them, see (2), which implies the residual implication in the form,

$$x \Rightarrow y = g(f(x) - f(y)). \tag{5}$$

This formula will be useful in case of information systems which is going to be discussed in the next section.

**Rough inclusions on sets** For our purpose it is essential to extend rough inclusions to sets; we use the t–norm $t_L$ of Łukasiewicz, along with the representation $t_L(r, s) = g(f(r) + f(s))$ already mentioned in (2), which in this case is $g(y) = 1 - y$, $f(x) = 1 - x$. We denote these kind of inclusions with the generic symbol $\nu$.

For sets $X, Y \subseteq U$, we let,

$$\nu_L(X, Y, r) \text{ if and only if } g(\frac{|X \setminus Y|}{|X|}) \geq r; \tag{6}$$

as $g(y) = 1 - y$, we have that $\nu_L(X, Y, r)$ holds if and only if $\frac{|X \cap Y|}{|X|} \geq r$. Let us observe that $\nu_L$ is *regular*, i.e., $\nu_L(X, Y, 1)$ if and only if $X \subseteq Y$ and $\nu_L(X, Y, r)$ only with $r = 0$ if and only if $X \cap Y = \emptyset$.

Thus, the ingredient relation associated with a regular rough inclusion is the improper containment $\subseteq$ whereas the underlying part relation is the strict containment $\subset$.

Other rough inclusion on sets which we exploit is the 3–valued rough inclusion $\nu_3$ defined via the formula, see [14],

$$\nu_3(X,Y,r)\text{if and only if}\begin{cases} X \subseteq Y \text{ and } r = 1 \\ X \cap Y = \emptyset \text{ and } r = 0 \\ r = \frac{1}{2} \text{ otherwise,} \end{cases} \tag{7}$$

The rough inclusion $\nu_3$ is also regular.

Finally, we consider the drastic rough inclusion on sets, $\nu_1$,

$$\nu_1(X,Y,r) \text{ if and only if} \begin{cases} X = Y \text{ and } r = 1 \\ X \neq Y \text{and } r = 0. \end{cases} \tag{8}$$

Clearly, $\nu_1$ is not regular.

## 3 Rough Mereological Logics

Given an information system $(U, A)$, along with a rough inclusion $\nu$ on the subsets of the universe $U$, for a collection of predicates (unary) $Pr$, interpreted in the universe $U$ (meaning that for each predicate $\phi \in Pr$ the meaning $[\phi]$ is a subset of $U$), we define the intensional logic $grm_\nu$ on $Pr$ by assigning to each predicate $\phi$ in $Pr$ its intension $I_\nu(\phi)$ defined by the family of extensions $I_\nu^\vee(g)$ at particular granules $g$, as,

$$I_\nu^\vee(g)(\phi) \geq r \text{ if and only if } \nu(g, [\phi], r). \tag{9}$$

With respect to the rough inclusion $\nu_L$, the formula (9) becomes,

$$I_{\nu_L}^\vee(g)(\phi) \geq r \text{ iff } \frac{|g \cap [\phi]|}{|g|} \geq r. \tag{10}$$

The counterpart for $\nu_3$ is specified by definition (7).

We say that a formula $\phi$ interpreted in the universe $U$ of an information system $(U, A)$ is *true* at a granule $g$ with respect to a rough inclusion $\nu$ if and only if $I_\nu^\vee(g)(\phi) = 1$.

**Proposition 2.** *For every regular rough inclusion $\nu$, a formula $\phi$ interpreted in the universe $U$, with meaning $[\phi]$, is true at a granule $g$ with respect to $\nu$ if and only if $g \subseteq [\phi]$. In particular, for a decision rule $r : p \Rightarrow q$ in the descriptor logic, the rule $r$ is true at a granule $g$ with respect to a regular rough inclusion $\nu$ if and only if $g \cap [p] \subseteq [q]$.*

Indeed, truth of $\phi$ at $g$ means that $\nu(g, [\phi], 1)$ which in turn, by regularity of $\nu$ is equivalent to the inclusion $g \subseteq [\phi]$.

We will say that a formula $\phi$ is a *theorem* of our intensional logic if and only if $\phi$ is true at every world $g$.

The preceding proposition implies that

**Proposition 3.** *For every regular rough inclusion $\nu$, a formula $\phi$ is a theorem if and only if $Cls$(all granules $g$ considered) $\subseteq [\phi]$; in the case when granules considered cover the universe $U$ this condition simplifies to $[\phi] = U$. This means for a decision rule $p \Rightarrow q$ that it is a theorem if and only if $[p] \subseteq [q]$.*

### 3.1 Relations to many–valued logics

Here we examine some axiomatic schemes for many–valued logics with respect to their meanings under the stated in introductory section proviso that $[p \Rightarrow q] = (U \setminus [p]) \cup [q]$, $[\neg p] = U \setminus [p]$.

We examine first axiom schemes for 3–valued Łukasiewicz logic investigated in [15] (Wajsberg schemes).
(W1) $q \Rightarrow (p \Rightarrow q)$;
(W2) $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$;
(W3) $((p \Rightarrow \neg p) \Rightarrow p) \Rightarrow p$;
(W4) $(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$.

We have as meanings of those formulas,

$[W1] = (U \setminus [q]) \cup (U \setminus [p]) \cup [q] = U$;
$[W2] = ([p] \setminus [q]) \cup ([q] \setminus [r]) \cup (U \setminus [p]) \cup [r] = U$;
$[W3] = (U \setminus [p]) \cup [p] = U$;
$[W4] = ([p] \setminus [q]) \cup [q] = U$.

Thus, *all instances of Wajsberg axiom schemes for 3–valued Łukasiewicz logic are theorems of our intensional logic in case of regular rough inclusions on sets.*

The deduction rule in 3–valued Łukasiewicz logic is Modus Ponens: $\frac{p, p \Rightarrow q}{q}$.

In our setting this is a valid deduction rule: if $p, p \Rightarrow q$ are theorems than $q$ is a theorem. Indeed, if $[p] = U = [p \Rightarrow q]$ then $[q] = U$.

We have obtained

**Proposition 4.** *Each theorem of 3–valued Łukasiewicz logic is a theorem of rough mereological granular logic in case of a regular rough inclusion on sets.*

In an analogous manner, we examine axiom schemes for infinite valued Łukasiewicz logic, proposed by Łukasiewicz [9], with some refinements showing redundancy of a scheme due to Meredith [10] and Chang [3], cf., in this respect [**?**] for an account of the reasoning.

(L1) $q \Rightarrow (p \Rightarrow q)$;
(L2) $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$;
(L3) $((q \Rightarrow p) \Rightarrow p) \Rightarrow ((p \Rightarrow q) \Rightarrow q)$;
(L4) $(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$.

As (L1) is (W1), (L2) is (W2) and (L4) is (W4), it remains to examine (L3). In this case, we have $[(q \Rightarrow p) \Rightarrow p]$=$(U \setminus [q \Rightarrow p]) \cup [p]$)=$(U \setminus ((U \setminus [q]) \cup [p])) \cup$

$[p]$=$([q] \setminus [p]) \cup [p]$= $[q] \cup [p]$. Similarly, $[(p \Rightarrow q) \Rightarrow q]$ is $[p] \cup [q]$ by symmetry, and finally the meaning $[L3]$ is $(U \setminus ([q] \cup [p])) \cup [p] \cup [q] = U$.

It follows that,

*all instances of axiom schemes for infinite–valued Łukasiewicz logic are theorems of rough mereological granular logic.*

As Modus Ponens remains a valid deduction rule in infinite–valued case, we obtain, analogous to Prop. 4,

**Proposition 5.** *Each theorem of infinite–valued Łukasiewicz logic is a theorem of rough mereological granular logic in case of a regular rough inclusion on sets.*

It follows from Prop.5 that all theorems of *Basic logic*, see [4], i.e. logic which is intersection of all many–valued logics with implications evaluated semantically by residual implications of continuous t–norms are theorems of rough mereological granular logic.

The assumption of regularity of a rough inclusion $\nu$ is essential: considering the drastic rough inclusion $\nu_1$, we find that an implication $p \Rightarrow q$ is true only at the world $(U \setminus [p]) \cup [q]$, so it is not any theorem – this concerns all schemes (W) and (L) above as they are true only at the global world $U$.

### 3.2 Graded notion of truth

The graded relaxation of truth is given obviously by the condition, a formula $\phi$ is *true to a degree at least* $r$ *at* $g, \nu$ if and only if $I_\nu^\vee(g)(\phi) \geq r$, i.e., $\nu(g, [\phi], r)$ holds. In particular, $\phi$ is *false* at $g, \nu$ if and only if $I_\nu^\vee(g)(\phi) \geq r$ implies $r = 0$, i.e. $\nu(g, [\phi], r)$ implies $r = 0$.

The following properties hold.

1. For each regular $\nu$, a formula $\alpha$ is true at $g, \nu$ if and only if $\neg\alpha$ is false at $g, \nu$.
2. For $\nu = \nu_L, \nu_3$, $I_\nu^\vee(g)(\neg\alpha) \geq r$ if and only if $I_\nu^\vee(g)(\alpha) \geq s$ implies $s \leq 1 - r$.
3. For $\nu = \nu_L, \nu_3$, the implication $\alpha \Rightarrow \beta$ is true at $g$ if and only if $g \cap [\alpha] \subseteq [\beta]$ and $\alpha \Rightarrow \beta$ is false at $g$ if and only if $g \subseteq [\alpha] \setminus [\beta]$.
4. For $\nu = \nu_L$, if $I_\nu^\vee(g)(\alpha \Rightarrow \beta) \geq r$ then $\Rightarrow_L (t, s) \geq r$ where $I_\nu^\vee(g)(\alpha) \geq t$ and $I_\nu^\vee(g)(\beta) \geq s$.

The functor $\Rightarrow$ in 4. is the Łukasiewicz implication of many–valued logic: $\Rightarrow_{t_L} (t, s) = min\{1, 1 - t + s\}$.

Further analysis should be split into the case of $\nu_L$ and the case of $\nu_3$ as the two differ essentially with respect to the form of reasoning they imply.

## 4 Reasoning with $\nu_L$

The last property 4. shows in principle that the value of $I_\nu^\vee(g)(\alpha \Rightarrow \beta)$ is bounded from above by the value of $I_\nu^\vee(g)(\alpha) \Rightarrow_{t_L} I_\nu^\vee(g)(\beta))$.

This suggests that the idea of collapse attributed to S. Lesniewski can be applied to formulas of rough mereological logic in the following form: for a formula $q(x)$ we denote by the symbol $q^*$ the formula $q$ regarded as a sentential formula (i.e., with variable symbols removed) subject to relations:

$(\neg q(x))^*$ is $\neg(q(x)^*)$ and $(p(x) \Rightarrow q(x))^*$ is $p(x)^* \Rightarrow q(x)^*$. As the value $[q^*]_g$ of the formula $q(x)^*$ at a granule $g$, we take the value of $\frac{|g \cap [q(x)]|}{|g|}$, i.e, $argmax_r\{\nu_L(g, [q^*]_g, r)\}$. Thus, item 4 above can be rewritten in the form.

$$I_\nu^\vee(g)(\alpha \Rightarrow \beta) \leq [\alpha^*]_g \Rightarrow_{t_L} [\beta^*]_g. \tag{11}$$

The following statement is then obvious:

*if $\alpha \Rightarrow \beta$ is true at $g$ then the collapsed formula has the value 1 of truth at the granule $g$ in the Łukasiewicz logic.*

This gives a necessity condition for verification of implications of rough mereological logics:

*if $\Rightarrow_L ([\alpha^*]_g, [\beta^*]_g) < 1$ then the implication $\alpha \Rightarrow \beta$ is not true at $g$.*

This concerns in particular decision rules:

*for a decision rule $p(v) \Rightarrow q(v)$, the decision rule is true on a granule $g$ if and only if $[p^*]_g \leq [q^*]_g$.*

## 5 Reasoning with $\nu_3$

In case of $\nu_3$, one can check on the basis of definitions that $I_\nu^\vee(g)(\neg\alpha) \geq r$ if and only if $I_\nu^\vee(g)(\alpha) \leq 1 - r$; thus the negation functor in rough mereological logic based on $\nu_3$ is the same as the negation functor in the 3–valued Łukasiewicz logic. For implication, the relations between granular rough mereological logic and 3–valued logic of Łukasiewicz follow from truth tables for respective functors of negation and implication.

Table 1 shows truth values for implication in 3–valued logic of Łukasiewicz. We recall that these values obey the implication $x \Rightarrow_L y = min\{1, 1 - x + y\}$. Values of $x$ correspond to rows and values of $y$ correspond to columns in Table 1.

**Table 1.** Truth values for implication in $L_3$

| $\Rightarrow$ | 0 | 1 | $\frac{1}{2}$ |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | $\frac{1}{2}$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 |

Table 2 shows values of implication for rough mereological logic based on $\nu_3$. Values are shown for the extension $I_\nu^\vee(g)(p \Rightarrow q)$ of the implication $p \Rightarrow q$. Rows correspond to $p$, columns correspond to $q$.

**Table 2.** Truth values for implication $p \Rightarrow q$ in logic based on $\nu_3$

| $\Rightarrow$ | $I^{\vee}_{\nu_3}(g)(q) = 0$ | $I^{\vee}_{\nu_3}(g)(q) = 1$ | $I^{\vee}_{\nu_3}(g)(q) = \frac{1}{2}$ |
|---|---|---|---|
| $I^{\vee}_{\nu_3}(g)(p) = 0$ | 1 | 1 | 1 |
| $I^{\vee}_{\nu_3}(g)(p) = 1$ | 0 | 1 | $\frac{1}{2}$ |
| $I^{\vee}_{\nu_3}(g)(p) = \frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 when $g \cap [\alpha] \subseteq [\beta]$; $\frac{1}{2}$ otherwise |

We verify values shown in Table 2. First, we consider the case when $I^{\vee}_{nu_3}(g)(p) = 0$, i.e., the case when $g \cap [p] = \emptyset$. As $g \subseteq (U \setminus [p]) \cup [q]$ for every value of $[q]$, we have only values of 1 in the first row of Table 2.

Assume now that $I^{\vee}_{nu_3}(g)(p) = 1$, i.e., $g \subseteq [p]$. As $g \cap (U \setminus [p]) = \emptyset$, the value of $I^{\vee}_{\nu}(g)(p \Rightarrow q)$ depends only on a relation between $g$ and $[q]$. In case $g \cap [q] = \emptyset$, the value in Table 2 is 0, in case $g \subseteq [q]$ the value in Table 2 is 1, and in case $I^{\vee}_{\nu_3}(g)(q) = \frac{1}{2}$, the value in Table 2 is $\frac{1}{2}$.

Finally, we consider the case when $I^{\vee}_{\nu_3}(g)(p) = \frac{1}{2}$, i.e., $g \cap [p] \neq \emptyset \neq g \setminus [p]$. In case $g \cap [q] = \emptyset$, we have $g \cap ((U \setminus [p]) \cup [q]) \neq \emptyset$ and it is not true that $g \subseteq ((U \setminus [p]) \cup [q])$ so the value in table is $\frac{1}{2}$. In case $g \subseteq [q]$, the value in Table is clearly 1. The case when $I^{\vee}_{\nu_3}(g)(q) = \frac{1}{2}$ remains. Clearly, when $g \cap [p] \subseteq [q]$, we have $g \subseteq (U \setminus [p]) \cup [q]$ so the value in Table is 1; otherwise, the value is $\frac{1}{2}$.

Thus, negation in both logic is semantically treated in the same way, whereas treatment of implication differs only in case of implication $p \Rightarrow q$ from the value $\frac{1}{2}$ to $\frac{1}{2}$, when $g \cap [p]$ is not any subset of $[q]$.

It follows from these facts that given a formula $\alpha$ and its collapse $\alpha^*$, we have,

$$I^{\vee}_{\nu_3}(g)(\neg\alpha) = [(\neg\alpha)^*]_{L_3}, I^{\vee}_{\nu_3}(g)(\alpha \Rightarrow \beta) \leq [(\alpha \Rightarrow \beta)^*]_{L_3}. \tag{12}$$

A more exact description of implication in both logics is as follows.

**Proposition 6.** *1. If $I^{\vee}_{nu_3}(g)(\alpha \Rightarrow \beta) = 1$ then $[(\alpha \Rightarrow \beta)^*]_{L_3} = 1$;*

*2. If $I^{\vee}_{nu_3}(g)(\alpha \Rightarrow \beta) = 0$ then $[(\alpha \Rightarrow \beta)^*]_{L_3} = 0$;*

*3. If $I^{\vee}_{nu_3}(g)(\alpha \Rightarrow \beta) = \frac{1}{2}$ then $[(\alpha \Rightarrow \beta)^*]_{L_3} \geq \frac{1}{2}$ and this last value may be 1.*

We offer a simple check–up on Proposition 6. In case 1, we have $g \subseteq ((U \setminus [\alpha]) \cup [\beta])$. For the value of $[(\alpha \Rightarrow \beta)^*]$, consider some subcases. Subcase 1.1: $g \subseteq U \setminus [\alpha]$. Then $[\alpha^*] = 0$ and $[(\alpha \Rightarrow \beta)^*]$ =$[\alpha^*] \Rightarrow [\beta^*]$ is always 1 regardless of a value of $[\beta^*]$. Subcase 1.2: $g \cap [\alpha] \neq \emptyset \neq g \setminus [\alpha]$ so $[\alpha^*] = \frac{1}{2}$. Then $g \cap [\beta] = \emptyset$ is impossible, i.e., $[\beta^*]$is at least $\frac{1}{2}$ and $[(\alpha \Rightarrow \beta)^*]$=1. Subcase 1.3: $g \subseteq [\alpha]$ so $[\alpha^*] = 1$; then $g \subseteq [\beta]$ must hold, i.e., $[\beta^*] = 1$ which means that $[(\alpha \Rightarrow \beta)^*]$=1.

For case 2, we have $g \cap ((U \setminus [\alpha]) \cup [\beta]) = \emptyset$ hence $g \cap [\beta] = \emptyset$ and $g \subseteq [\alpha]$, i.e., $[\alpha^*] = 1, [\beta^*] = 0$ so $[\alpha^*] \Rightarrow [\beta^*]$=0.

In case 3, we have $g \cap ((U \setminus [\alpha]) \cup [\beta]) \neq \emptyset$ and $g \cap [\alpha] \setminus [\beta] \neq \emptyset$. Can $[\alpha^*] \Rightarrow [\beta^*]$ be necessarily 0? This would mean that $[\alpha^*] = 1$ and $[\beta^*] = 0$, i.e., $g \subseteq [\alpha]$ and $g \cap [\beta] = \emptyset$ but then $g \cap ((U \setminus [\alpha]) \cup [\beta]) = \emptyset$, a contradiction. Thus the value

$[\alpha^*] \Rightarrow [\beta^*]$ is at least $\frac{1}{2}$. In the subcase: $g \subseteq [\alpha]$, $g \cap [\beta] \neq \emptyset \neq g \setminus [\beta]$, the value of $[\alpha^*] \Rightarrow [\beta^*]$ is $0 \Rightarrow_L \frac{1}{2} = 1$, and the subcase is consistent with case 3.

### 5.1 Dependencies and decision rules

It is an important feature of rough set theory that it allows for an elegant formulation of the problem of dependency between two sets of attributes, cf., [13], in terms of indiscernibility relations.

We recall, see sect.1 that for two sets $C, D \subseteq A$ of attributes, one says that $D$ *depends functionally on* $C$ when $IND(C) \subseteq IND(D)$, symbolically denoted $C \mapsto D$. Functional dependence can be represented locally by means of functional dependency rules of the form

$$\phi_C(\{v_a : a \in C\}) \Rightarrow \phi_D(\{w_a : a \in D\}), \tag{13}$$

where $\phi_C(\{v_a : a \in C\})$ is the formula $\bigwedge_{a \in C}(a = v_a)$, and $[\phi_C] \subseteq [\phi_D]$.

Clearly, *if* $\alpha : \phi_C \Rightarrow \phi_D$ *is a functional dependency rule as in* (13), *then* $\alpha$ *is a theorem of logic induced by* $\nu_3$.

Indeed, for each granule $g$, we have $g \cap [\phi_C] \subseteq [\phi_D]$. Let us observe that the converse statement is also true: if a formula $\alpha : \phi_C \Rightarrow \phi_D$ is a theorem of logic induced by $\nu_3$ then this formula is a functional dependency rule in the sense of (13). Indeed, assume that $\alpha$ is not any functional dependency rule, i.e., $[\phi_C] \setminus [\phi_D] \neq \emptyset$. Taking $[\phi_C]$ as the witness granule $g$, we have that $g$ is not any subset of $[\alpha]$, i.e, $I^{\vee}_{\nu_3}(g)(\alpha) \leq \frac{1}{2}$, so $\alpha$ is not true at $g$, a fortiori it is no theorem.

Let us observe that these characterizations are valid for each regular rough inclusion on sets $\nu$.

A more general and also important notion is that of a local proper dependency: a formula $\phi_C \Rightarrow \phi_D$ where $\phi_C(\{v_a : a \in C\})$ is the formula $\bigwedge_{a \in C}(a = v_a)$, similarly for $\phi_D$, is a local proper dependency when $[\phi_C] \cap [\phi_D] \neq \emptyset$.

We will say that a formula $\alpha$ is *acceptable with respect to a collection* $M$ *of worlds* when $I^{\vee}_{\nu_3}(g)(\alpha) \geq \frac{1}{2}$ for each world $g \in M$, i.e, when $\alpha$ is false at no world $g \in M$. Then,

*if a formula* $\alpha : \phi_C \Rightarrow \phi_D$ *is a local proper dependency rule, then it is acceptable with respect to all C-exact worlds.*

Indeed, for a C–exact granule $g$, the case that $I^{\vee}_{\nu_3}(g)(\alpha) = 0$ means that $g \subseteq [\phi_C]$ and $g \cap [\phi_D] = \emptyset$; as $g$ is C–exact and $[\phi_C]$ is a C–indiscernibility class, either $[\phi_C] \subseteq g$ or $[\phi_C] \cap g = \emptyset$. When $[\phi_C] \subseteq g$ then $[\phi_C] = g$ which makes $g \cap [\phi_D] = \emptyset$ impossible. When $[\phi_C] \cap g = \emptyset$, then $g \cap [\phi_D] = \emptyset$ is impossible. In either case, $I^{\vee}_{\nu_3}(g)(\alpha) = 0$ cannot be satisfied with any C–exact granule $g$.

Again, the converse is true: when $\alpha$ is not local proper, i.e., $[\phi_C] \cap [\phi_D] = \emptyset$, then $g = [\phi_C]$ does satisfy $I^{\vee}_{\nu_3}(g)(\alpha) = 0$.

A corollary of the same forms follows for *decision rules* in a given decision system $(U, A, d)$, i.e., dependencies of the form $\phi_C \Rightarrow (d = w)$.

## 6 Conclusions

Intensional logics $grm_\nu$ capture the basic aspects of reasoning in rough set theory as the construction of such logic is oriented toward logical dependency between premises and conclusions of an implicative rule.

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