Exponentially-improved Asymptotics of Single and Multidimensional Integrals

Ph D Thesis

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I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Ph D , is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work

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Date ______________________________

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Abstract

Two different approaches for finding the exponentially improved asymptotic behaviour of integrals with saddlepoints are presented Both rely on the deformation of the contours of integration and can be applied to single and multidimensional integrals alike The class of integrals studied is of the form

$$\int \quad \int_S g(z_1, \quad , z_p) e^{-f(z_1, \quad , z_p, \lambda)} dz_1 \quad dz_p,$$

where $\lambda \in \mathbb{C}$ is the asymptotic parameter and S, the surface of integration is allowed finite or infinite limits Thus, for example, the asymptotic behaviour of the Spitzer integral

$$\int_0^\infty \quad \int_0^\infty z_1^{\alpha_1} \quad z_p^{\alpha_p} e^{sxz_1 \quad z_p - \frac{1}{n}(z_1^n + \quad + z_p^n)} dz_1 \quad dz_p, \quad |x| \to \infty,$$

could be determined This latter integral is the solution of the differential equation

$$y^{(n)}(x) - \Sigma_{r=0}^{\infty} a_r x^r y^{(r)}(x) = 0,$$

where the α_i of the integral are related to the a_i of the differential equation by certain recurrence relations Hyperasymptotic methods have recently been developed for certain classes of differential equations but in some cases it is useful to have an alternative approach originating with the integral representation of such solutions—for instance, when the asymptotic behaviour of the solution at infinity has to be matched to initial or boundary value data The methods presented here should provide the necessary link

The first method to be presented is based on a technique suggested by Nikishov and Ritus in 1992[1] which dealt with single integrals only, the second is a method detailed by Berry and Howls in a series of papers from 1990 to 1997[2] which works on integrals of any finite dimension Various modifications and extensions of these procedures were necessary to achieve the results obtained These allow, for instance, the presence of a logarithmic singularity in the function $g(z_1, \quad , z_p)$

[1]A Nikishov and V Ritus Stokes line width English translation in *Theoret and Math Phys* 92(1) 711-721,1992

[2]M Berry and C Howls Hyperasymptotics *Proc Roy Soc London Ser A* 430(1880) 653-668,1990

M Berry and C Howls Hyperasymptotics for integrals with saddles *Proc Roy Soc London Ser A* 434(1892) 657-675,1991

C Howls Hyperasymptotics for integrals with finite endpoints *Proc Roy Soc London Ser A* 439(1906) 373-396,1992

C Howls Hyperasymptotics for multidimensional integrals, exact remainder terms and the global connection problem *Proc Roy Soc London Ser A* 453(1966) 2271-2294,1997

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Chapter 1

Single Integrals: An Introduction

1.1 Introduction to Asymptotics

Asymptotic approximation is an important branch of applied analysis and concerns the study of the behaviour of functions in particular limits of interest—perhaps one of the function's parameters tends to a specific value or the index of a sequence may tend towards infinity To quote de Bruijn [15], it is "a difficult subject that requires constant alertness and carefulness" because, although asymptotic and perturbation techniques provide most useful and powerful methods for finding approximate solutions to problems, they can be difficult to justify rigorously Originating in Victorian times when it provided the most reliable and rapid means of approximation, asymptotics holds its own in this, the age of powerful computers, as even still many mathematical models cannot be solved by the use of direct numerical methods alone Asymptotics, however, may provide the information required to simplify the computational procedure, its value lying in the quantitative description of phenomena it obtains

It would be difficult to name a branch of mathematics, physics, or indeed the natural sciences in which asymptotic methods could not be used, having had an important role to play in electromagnetism, diffraction theory, fluid mechanics, meteorology and statistics The investigation of integrals, series, solutions of linear and non-linear differential equations and systems (both ordinary and partial), difference equations and integral equations have all resorted to the use of asymptotics In the case of differential and difference equations, much of the asymptotic analysis employed is local analysis—that is to say, the behaviour of the solutions in the neighbourhood of a particular point is predicted without incorporating the initial or boundary data at other points Local analysis enables the solutions of equations which are not soluble in closed form, to be simply expressed in terms of elementary functions This results in a representation of the solution which is valid in a sufficiently small neighbourhood of such a particular point Piecing together the local behaviours in different neighbourhoods may lead to a uniform approximation to the behaviour of the solution over the entire interval in question, which is the ultimate aim The piecing together process requires the use of global analysis

1.2 Integral Representation

The second half of the seventeenth century saw the onset of infinite series and their analogues, integral representations, as fundamental tools in mathematical analysis They provided the means for introducing all of the transcendental functions including those which are now termed elementary (i e trigonometric, logarithm, exponential functions) These in turn helped in the solution of many differential equations, both ordinary and partial However, the analysis of power series soon gave way to integral representation techniques when it came to analysing their singular points or continuing them outside their domain of convergence Both these problems are more easily solved when dealing with an integral of an analytic function with respect to a parameter, where the dependence on such a parameter is also analytic

Research into the behaviour at infinity of an entire function can also be noticeably simplified by the use of an integral representation in place of a convergent power series But it is in the solution of differential and difference equations which cannot be solved in terms of elementary functions that integral representations, when available, are most useful The predictions of the behaviour of such solutions at a particular point will usually contain unknown constants because the boundary value or initial value data at other points has not been incorporated For many of these equations however, it is possible to find an integral representation of the solution in which the independent variable appears as a parameter Typically this integral will contain all of the information supplied by the initial value or boundary conditions Sometimes such a representation can be found merely by using the method of integrating factors to find the solution of a simple differential equation without actually evaluating the integral More often it is obtained by applying an integral transform (such as the Laplace, Fourier, Mellin transforms) to the equation These transforms are by no means arbitrary linear integral operators—each of them supports an inversion and commutation formula and they have been used successfully in a huge number of concrete problems in mathematical physics Once the integral form of the solution has been established its value at any finite point can be found by simple substitution (It should be noted that it is not always possible to find such an integral representation particularly for nonlinear differential equations However, there are other techniques available to fully determine the asymptotic behaviour in question—for example, the technique of matched asymptotic expansions [59])

However, many of these integrals, when found, are too difficult to evaluate exactly, thus the asymptotic expansion of integrals becomes extremely important All of the special functions commonly used in mathematical physics and applied mathematics have integral representations from which their asymptotic properties were determined Once the properties of these special functions were known, they could in turn be used in the derivation of the global behaviour of solutions of general classes of differential equations whose solutions were not themselves expressible as integrals That is to say, sometimes the solution could be written as a function of special functions—thus widening the sphere of influence of integral representations of the latter[1] We confine ourselves to the study of techniques which find the asymptotic behaviour of a function from its expression as an integral, but it should be noted that many of the procedures used have counterparts in other areas of asymptotic analysis

A flavour of the variety of applications where asymptotics of integrals has become useful can be given by the following physical phenomena such as the damped-mass spring system, heat conduction in non-radiating solids, acoustical scattering and probability theory To be a little more specific, integrals of the type

$$I(\lambda) = \int_C g(z) e^{-\lambda f(z)}\, dz, \tag{1 1}$$

whose asymptotic behaviour can be determined by the method of steepest descent, arise naturally in the context of linear wave propagation, among other problems

1.3 Mathematical Prerequisites

1.3.1 Poincaré Asymptotics

In order to define what is meant by an asymptotic expansion of a function, some preliminaries are required

(i) $f(x)$ is of order not exceeding $g(x)$

$$f(x) = O(g(x)), \quad x \to x^0 \quad \Rightarrow \quad |f(x)| < K\,|g(x)|\,, \quad K \in \mathbb{R}, \quad x \to x_0, \qquad (1\ 2)$$

(ii) $f(x)$ is of order less than $g(x)$

$$f(x) = o(g(x)), \quad x \to x^0 \quad \Rightarrow \quad \lim_{x \to x^0} \left| \frac{f(x)}{g(x)} \right| = 0 \qquad (1\ 3)$$

or $f(x) \ll g(x), \quad x \to x^0$,

(iii) $f(x)$ is asymptotic to $g(x)$

$$f(x) \sim g(x), \quad x \to x^0 \quad \Rightarrow \quad \lim_{x \to x^0} \left| \frac{f(x)}{g(x)} \right| = 1 \qquad (1\ 4)$$

or $f(x) - g(x) \ll g(x), \quad x \to x^0$ or even $f(x) = g(x)(1 + o(1)), \quad x \to x^0$,

(iv) Asymptotic expansion for $f(x)$

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x), \quad x \to x^0 \qquad (1\ 5)$$

is an asymptotic expansion for $f(x)$ where $\{a_n\}$ is a sequence of constants and $\{\phi_n\}$ is a sequence of functions such that $\phi_{n+1}(x) = o(\phi_n(x)), \quad x \to x^0$, if

$$f(x) = \sum_{n=0}^{m} a_n \phi_n(x) + O(\phi_{m+1}(x)), \quad \forall\, m \qquad (1\ 6)$$

This means that the error committed in truncating the series at a finite point is of the order of the first neglected term and in the particular case where $\phi_n(x) = x^n$, it behaves like a power of x Note that an asymptotic expansion of a function $f(x)$ is not unique because there exists an infinite number of asymptotic sequences $\{\phi_n\}$ to choose from However, given a particular asymptotic sequence, the asymptotic representation of $f(x)$ is unique with respect to this sequence

In general, an asymptotic formula for a function $f(x)$ is the name given to an approximation function $g(x)$ in some domain of values of x, where $g(x)$ is 'simpler' than $f(x)$ For instance, if $f(x)$ is an integral, $g(x)$ would be given either in the form of a simpler integral or in terms of values of the integrand and its derivatives at a finite number of points All asymptotic methods have two common characteristics—firstly, the asymptotic formula is more accurate the closer x is to x^0 (to quote Laplace "the method is the more precise the more that it is necessary" [43]) and secondly, an asymptotic formula alone does not guarantee the value of $f(x)$ to be calculated to any assigned degree of accuracy In fact, it can be easily illustrated that two asymptotic formulae for the same function which look tolerably similar can produce very different predictions for a value of $f(x)$ The error terms which have been neglected in the formulae are the source of this difference and consequently it is important to take note of the form and magnitude of such error terms As de Bruijn [15] points out, even if an asymptotic result is presented in the most explicit form possible, it may not provide satisfactory results from a numerical point of view

In practice, when expanding $f(x)$ asymptotically to n terms, the remainder, denoted $R_n(x)$, more often than not becomes steadily larger as n grows As $f(x)$ is finite, such a growth must be balanced by the series and so the infinite series can be expected to diverge However, all is not lost because if x is allowed to approach x^0 while n is kept fixed, $R_n(x)$ still tends to zero Whereas convergence of the series means that there is a certain statement about $n \to \infty$ for each x, to say that a series is asymptotic means that there is a statement about $x \to \infty$ (x^0) for each n—an important distinction Even if the asymptotic series were to converge, its sum need not be equal to $f(x)$!

In 1828, Abel condemned divergent series as the invention of the devil and dictated that it was "shameful to base on them any demonstration whatsoever" [6] Fortunately, neither Poincaré nor Euler shared this loathing Euler carried out many operations on power series outside their domains of convergence and managed to obtain accurate results despite the apparent illegitimacy of his computations To justify such perversity, he claimed that he was working not on the series themselves but on the functions which could be expanded into such series The uniqueness of the expansion of a function into a power series did indeed permit this free transition from series to function In actual fact, in most cases, the terms of an asymptotic series decrease rapidly at first but later start to increase again For that reason, Stieltjes had named them 'semi-convergent' series and others had talked of 'convergently beginning' series but neither of these terms persisted It was Poincaré who first introduced the notion of an 'asymptotic series' in his 'Sur les intégrales irrégulieres des equations lineaires' [71] in which he showed that the formal series satisfying linear differential equations with analytic coefficients were asymptotic series for the general solutions of these equations

1.3 2 Exponential Asymptotics

According to Poincaré's definition of an asymptotic series, with the choice of an asymptotic sequence such as $\phi_n(x) = x^n$, transcendental exponential terms disappear or have been traditionally discarded These terms are said to lie beyond all orders of the expansion However, the need to retain these 'correction' terms as such, becomes apparent in practical applications where they can become important analytically and even numerically if a certain precision is required Dingle [17], in his investigation of this problem, came to the conclusion that Poincare's definition should not be literally adhered to—it needed not so much to be replaced as to be supplemented Instead of following Poincare in truncating the series at a fixed order, he advocated Stokes' approach of optimal truncation—stopping at the minimum term, just before the series starts to diverge, the order of this term depending on the asymptotic parameter, x This procedure was termed 'superasymptotics' or 'asymptotics beyond all orders' and achieves small, exponential errors

Suppose $f(x)$ can be expressed in the form

$$f(x) \sim \sum_{m=0}^{\infty} a_m \phi_m(x), \tag{1 7}$$

then stopping at an optimal m, $n_0(x)$ say, leads to

$$f(x) \sim \sum_{m=0}^{n_0(x)} a_m \phi_m(x) + R_{n_0}(x) \tag{1 8}$$

It was suggested by Stieltjes [77] and formally considered by Dingle [17] that this estimate for $f(x)$ could be further improved if $R_{n_0}(x)$ were to be itself expanded in an asymptotic series

$$f(x) \sim \sum_{m=0}^{n_0(x)} a_m \phi_m(x) + \sum_{m=0}^{\infty} a_{1,m} \phi_{1,m}(x) \tag{1 9}$$

Performing the two previous steps iteratively

$$\begin{aligned} f(x) &\sim \sum_{m=0}^{n_0(x)} a_m \phi_m(x) + \sum_{m=0}^{n_1(x)} a_{1,m}\phi_{1,m}(x) + R^1_{n_1}(x) \\ &= \sum_{m=0}^{n_0(x)} a_m \phi_m(x) + \sum_{m=0}^{n_1(x)} a_{1,m}\phi_{1,m}(x) + \sum_{m=0}^{\infty} a_{2,m}\phi_{2,m}(x) \\ &= \sum_{m=0}^{n_0(x)} a_m \phi_m(x) + \sum_{m=0}^{n_1(x)} a_{1,m}\phi_{1,m}(x) + \sum_{m=0}^{n_2(x)} a_{2,m}\phi_{2,m}(x) + R^2_{n_2}(x) \quad (1\ 10) \end{aligned}$$

and so on, gives the hyperasymptotic scheme of Berry and Howls [4] and supplies steadily more accurate approximations for $f(x)$ Each time the series in question is truncated optimally, an algorithm determining the point of optimal truncation Hyperasymptotics can be said to be 'beyond asymptotics beyond all orders'

It was Dingle who was first to appreciate that the tails of divergent series have universal properties Having truncated an asymptotic series, he was able to derive and evaluate integral representations to approximate its remainder term using just the general term of the series itself This implied that despite their ultimate divergence, asymptotic series could be precisely interpreted Such a 'taming' of the tail revealed a structure of exponentials which had been hidden from view[1] To understand the need to retain small remainder terms, consider the case away from the asymptotic limit x^0 there, more and more terms of the series may be needed to match the asymptotic expansion with the original function However, in this 'region' the optimal number of terms to be considered may also be decreasing Thus, the only way to balance this conflict and to regain the original function is to recover previously discarded exponentially small terms

A closer examination of the physical world seems to show that the divergence of series representing physical systems is in fact a general phenomenon Thus these exponential improvements on previous asymptotic results have been of vital importance in understanding the behaviour of many such systems Notable progress has since been recorded in outstanding problems in quantum tunnelling, quantum chaology, dendritic crystal growth, directional solidification of crystals and problems involving viscous flows in the absence/presence of surface tension Moreover, these developments have allowed progress to be made in reducing general theories to more

[1]See Chapter 3 for further discussion

restricted ones, a limit question which has prevailed since the 1800s For instance, an unproblematic limit of this kind is that of the smooth reduction of special relativity to Newtonian mechanics However, the transition from statistical mechanics to thermodynamics or from wave to ray optics is not so straightforward It was not so much believed that these theories were incompatible, but the boundary separating them seemed impenetrable without the help of exponential asymptotics—such is the effect on the physics world of what might appear at first to be merely a mathematical nicety

1 3 3 Stokes Phenomenon

Up to this point it has been assumed that the asymptotic parameter x is real Generalising the aforementioned asymptotic relations to complex functions $f(\lambda)$ is non-trivial Taking the limit as $\lambda \to \lambda^0$ along arbitrary paths in the complex plane presents difficulties For instance, in many cases paths rotating around λ^0, as they approach it, may give non-unique limits and thus must be excluded In order to guarantee this, it is necessary that all paths along which the limit is taken, lie within a sector of the complex plane with opening angle depending on the functions which are asymptotic At most this angle is π (i e $|\arg\lambda| < \pi$) and such a specification results in a path-independent definition of the asymptotic relations One cause of concern then, is the inadequacy of a particular asymptotic sequence to describe the behaviour of the function in question outside this sector To illustrate what happens, consider the fundamental solutions of a second order differential equation having leading behaviours $e^{S_1(\lambda)}$ and $e^{S_2(\lambda)}$ as $\lambda \to \lambda^0$ What form should the general solution take ?

The complex plane can be divided into sectors by the lines $\Re S_1(\lambda) = \Re S_2(\lambda)$ When λ lies in the interior of the sector for which $\Re S_1(\lambda) > \Re S_2(\lambda)$, $e^{S_1(\lambda)}$ is said to dominate $e^{S_2(\lambda)}$ and so provides the leading behaviour of the general solution in this sector, which is termed the wedge of validity of $e^{S_1(\lambda)}$ $e^{S_2(\lambda)}$ is said to be 'recessive' or 'subdominant' within that sector As λ approaches the edge of the wedge of validity, $e^{S_1(\lambda)}$ and $e^{S_2(\lambda)}$ are of the same order of magnitude and so the labels 'dominant' versus 'subdominant' are no longer applicable Having crossed over to the sector $\Re S_1(\lambda) < \Re S_2(\lambda)$, or the wedge of validity of $e^{S_2(\lambda)}$, the dominance properties have switched and the leading behaviour of the general solution should possibly be given by $e^{S_2(\lambda)}$ Thus the argument that the asymptotic approximation should be domain dependent arose and was first suggested by Stokes The abrupt change in the coefficients multiplying the 'subdominant' terms in compound expansions across certain rays in the complex plane has since come to be known as Stokes phenomenon The lines along which the leading behaviours $e^{S_1(\lambda)}$ and $e^{S_2(\lambda)}$ are most unequal are called the Stokes lines and are determined as asymptotes, as $\lambda \to \lambda^0$, to the curves

$$\Im(S_1(\lambda) - S_2(\lambda)) = 0, \tag{1 11}$$

whereas the lines along which they are most equal are called the anti-Stokes lines, given by the asymptotes, as $\lambda \to \lambda^0$, to the curves

$$\Re(S_1(\lambda) - S_2(\lambda)) = 0 \tag{1 12}$$

In this case, the Stokes phenomenon is not an intrinsic property of the function $f(\lambda)$ being approximated, but rather reflects the presence of the exponential functions in the asymptotic approximation

Another way it can arise is as a consequence of approximating a function $f(\lambda)$, which may be multivalued, by another multivalued function $g(\lambda)$ possessing a different multivalued structure Or indeed, considering an analytic function $f(\lambda)$, single-valued in a sector or wedge, W, it is possible that both $f(\lambda)$ and its asymptotic expansion as $\lambda \to \lambda^0$ can be analytically continued outside W, but that when certain rays through $\lambda = \lambda^0$ are crossed the analytic continuation of the asymptotic expansion is no longer the asymptotic expansion of the analytic continuation and Stokes phenomenon manifests itself

Having first observed this phenomenon, it took Stokes much thought and several years to elaborate on its nature He initially described the change in the coefficient of the subdominant term by an involved argument making use of the asymptotic behaviour of an integral representation of the function in question along certain rays in the complex plane The computation of the dominant series to an appropriate precision was required, so he employed the optimal truncation procedure mentioned above The response of mathematicians at the time was to ignore Stokes' findings Dingle [17] was really the first to explore further in the 1950s and his investigations were successfully extended by Berry [3] The conventional view had been that the change in coefficients was discontinuous but Berry proved, by viewing the problem from a different perspective and on a suitable scale, that the change is indeed continuous (see §3 5) Berry's interpretation showed that the divergence of an asymptotic series actually explains the phenomenon by reflecting its inability to describe the other 'hidden' exponentials His resummation of the divergent tail allowed him to obtain a precise description of the change in multipliers or coefficients and represented the first stage of hyperasymptotics Although the techniques used showed great insight, the manipulations were quite formal However, several others, led by Olver [64], followed with a more rigorous treatment of the theory, having been prompted by Berry's work McLeod [47] rigorised Berry's results step by step whereas Jones [31] independently showed how certain definite integrals with coalescing poles and saddle points have a remainder whose behaviour is of error function type Boyd [10], on the other hand, investigated functions defined by a Stieltjes transform for which he introduced an exponentially improved asymptotic theory, while Paris [66] was able to find uniform exponentially-improved asymptotic expansions for functions defined by Mellin-Barnes integrals, where the integrand contains one or more gamma functions and then together with Wood [69] found an exponentially-improved expansion for the gamma function itself Also, Liu and Wood [46] aplied these new theories to the field of asymptotic matching for a model optical tunnelling problem, after which Olde Daalhuis, Chapman et al [59] followed with an interpretation of such asymptotic matching procedures with a view to their extension to results for partial differential equations

1.4 Classical Methods

When given an integral

$$I(x) = \int_a^b g(t,x)\,dt \qquad (1\ 13)$$

and asked to find its asymptotic expansion as $x \to x^0$, it may be possible to asymptotically expand the integrand

$$g(t,x) \sim \sum_{n=0}^{\infty} g_n(t)(x-x^0)^{\alpha n}, \qquad x \to x^0, \qquad (1\ 14)$$

for some $\alpha > 0$ and integrate this series term by term

$$\int_a^b g(t,x)\,dt \sim \sum_{n=0}^{\infty}(x-x^0)^{\alpha n}\int_a^b g_n(t)\,dt, \qquad x \to x^0. \tag{1.15}$$

This will work when the above expansion holds uniformly in $a < t < b$ and if the result upon integrating each $g_n(t)$ is finite. In the simplest cases it may even be achieved by merely writing the integrand as a Taylor or binomial series. Needless to say, it is the complicated situations that arise in practice and it has been found that more elaborate methods are required.

1.4.1 Integration by Parts

Probably the easiest method for developing an asymptotic expansion of a function represented as an integral is that of repeated integration by parts —each integration produces another term in the expansion, leaving the remainder expressed explicitly as an integral. It is necessary to check that the resulting expansion is actually asymptotic and sometimes it is possible to find a numerical bound on the error. An immediate application of the method is incorrect if an algebraic singularity arises at an endpoint of the interval of integration or if a contribution from an endpoint is infinite. Then some modification is necessary—splitting the interval of integration and dealing with the subsequent integrals separately may be all that is needed. The method is applicable to both Laplace, $\int_0^\infty g(t)\mathrm{e}^{-xt}\,dt$, and Fourier, $\int_0^\infty g(t)\mathrm{e}^{ixt}\,dt$, type integrals and to integrals where the asymptotic parameter appears in the limit of integration. In fact, Laplace used the method to obtain an asymptotic expansion of an integral similar to the complementary error function [22]

$$\int_x^\infty \mathrm{e}^{-t^2}\,dt = \frac{\mathrm{e}^{-x^2}}{2x}\sum_{k=0}^{\infty}\frac{(-1)^k(2k-1)!!}{2^k x^{2k}}, \qquad x \to \infty, \tag{1.16}$$

where $(2k-1)!! = (2k-1)(2k-3)...3.1$.

Problems arising in the application of integration by parts are made obvious by the appearance of a non-existent integral. However, it may still be possible to obtain the leading behaviour of a function before this occurs. The importance of this method lies really in the development of more sophisticated methods which fundamentally depend on it.

1.4.2 Laplace's Method

Laplace's method is a very general technique for finding the asymptotic behaviour of integrals of the form

$$I(x) = \int_a^b \rho(t,x)\,dt, \qquad x \to x^0, \tag{1 17}$$

where $\rho(t,x)$, considered as a function of t (being both real and continuous), has a sharp peak or maximum at some point in $[a,b]$ and the contribution of the neighbourhood of this peak is almost equal to the entire integral when x approaches x^0 Thus the function $\rho(t,x)$ can be approximated in that neighbourhood by simpler functions for which it is possible to directly evaluate the integral The main benefit of Laplace's method is that only a small neighbourhood of the maximum needs to be investigated Though first appearing in Laplace's 'Théorie analytiques des probabilités' in 1812 [43], the essence of the method was used earlier still by Riemann but only mentioned in his memoir 'An analysis of the possibility of representing a function by a trigonometric series with no special assumptions on the nature of the function' published after his death in 1867 [72]

Consider the behaviour of the integral

$$I(x) = \int_a^b g(t)\mathrm{e}^{xf(t)}\,dt, \qquad x \to \infty, \tag{1 18}$$

where $f(t)$ and $g(t)$ are real, twice differentiable, functions and the major contribution to the integral comes from the neighbourhood of the point where $f(t)$ attains its maximum i e $t = t^0$ such that $f'(t^0) = 0$ and $f''(t^0) \neq 0$ Replacing $g(t)$ and $f(t)$ by the leading terms of their Taylor series (assuming $g(t^0) \neq 0$) gives

$$I(x) \sim \int_{t^0-\varepsilon}^{t^0+\varepsilon} g(t^0)\mathrm{e}^{x(f(t^0)+\frac{(t-t^0)^2}{2}f''(t^0))}\,dt \sim \sqrt{\frac{2\pi}{-xf''(t^0)}}g(t^0)\mathrm{e}^{xf(t^0)}, \qquad x \to \infty, \tag{1 19}$$

for $a < t^0 < b$ However, if the maximum occurs instead at $t^0 = a$ with $f'(a) \neq 0$ or at $t^0 = b$ with $f'(b) \neq 0$ then the leading behaviour obtained is

$$I(x) \sim \int_a^{a+\varepsilon} g(a)\mathrm{e}^{x(f(a)+(t-a)f'(a))}\,dt \sim -\frac{g(a)}{xf'(a)}\mathrm{e}^{xf(a)}, \qquad x \to \infty, \tag{1 20}$$

or

$$I(x) \sim \int_{b-\varepsilon}^{b} g(b)\mathrm{e}^{x(f(b)+(t-t^0)f'(b))}\,dt \sim \frac{g(b)}{xf'(b)}\mathrm{e}^{xf(b)}, \qquad x \to \infty, \tag{1 21}$$

respectively The determination of higher order terms requires the inclusion of more terms in the Taylor series for $f(t)$ and $g(t)$

A different approach to obtain the expansion is to make use of the following

Watson's Lemma If $g(t)$ is locally integrable on $(0,\infty)$ with $g(t) = O(e^{\alpha t})$ as $t \to \infty$ for some real α and $g(t) \sim \sum_{m=0}^{\infty} c_m t^{\alpha_m}$ with $\alpha_m \to \infty$ as $m \to \infty$, then

$$I(x) = \int_0^\infty g(t)e^{-xt}dt \sim \sum_{m=0}^{\infty} c_m \frac{\Gamma(\alpha_m + 1)}{x^{\alpha_m+1}}, \qquad x \to \infty \qquad (1\ 22)$$

Then given any integral

$$I(x) = \int_a^b g(t)e^{xf(t)}dt, \qquad (1\ 23)$$

the substitution $\tau = -f(t)$ can be made and the lemma applied However, writing t in terms of τ, where $t = f^{-1}(-\tau)$, requires the use of Lagrange's reversion of series formula and can be unwieldy The full asymptotic expansion obtained thus (see p 99 §2 2 [22]) for an interior maximum point is given by

$$I(x) \sim e^{xf(t^0)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{(2n)!} \left(\frac{d}{dt}\right)^n g(t) \left[\frac{-(t-t^0)^2}{f(t)-f(t^0)}\right]^{(n+\frac{1}{2})}\Bigg|_{t=t^0} x^{-n-\frac{1}{2}} \qquad (1\ 24)$$

Laplace's method enjoys wide application not least having been applied by Fulks [25] to integrals depending on two large parameters whose behaviours are not bound rigidly together This work was later generalised by Thomsen [80] Laplace's method has also been shown to provide an effective means of solving linear functional equations of the form $\sum_{k=1}^{n}(a_k x + b_k)y(x+\alpha_k) = 0$ [20] by searching for a solution $y(x) = \int_C \phi(t)e^{xt}dt$

1 4 3 Method of Stationary Phase

How would the previous theory cope with a Fourier integral

$$I(x) = \int_a^b g(t)e^{ixt}\,dt, \qquad (1\ 25)$$

where $g(t)$ is a real-valued function and x is a large real parameter, given that the exponential decay as x increases is now absent ? The integration by parts technique works as long as the resulting boundary terms are finite and the newly formed integral exists To guarantee that the expansion produced is actually asymptotic, the Riemann-Lebesgue lemma comes into play

Lemma $\int_a^b g(t)e^{ixt}dt \to 0$ as $x \to \infty$, if $\int_a^b |g(t)|\, dt$ exists

Thus, repeatedly applying integration by parts yields

$$I(x) \sim \sum_{n=0}^{\infty} \left(\frac{i}{x}\right)^{n+1} \{e^{iax}g^{(n)}(a) - e^{ibx}g^{(n)}(b)\}, \qquad x \to \infty, \tag{1 26}$$

if $g(t)$ is infinitely differentiable Extending this approach to the generalised Fourier integral

$$I(x) = \int_a^b g(t)e^{ixf(t)}\, dt, \qquad x \to \infty, \tag{1 27}$$

where now $f(t)$ is also a real-valued twice differentiable function, would result in

$$I(x) = \left.\frac{g(t)}{ixf'(t)}e^{ixf(t)}\right|_a^b - \frac{1}{ix}\int_a^b \frac{d}{dt}\frac{g(t)}{f'(t)}e^{ixf(t)}\, dt \tag{1 28}$$

and the Riemann-Lesbegue lemma can again be used to show that the integral appearing on the right vanishes more rapidly than $\frac{1}{x}$ as $x \to \infty$ and thus

$$I(x) \sim \left.\frac{g(t)}{ixf'(t)}e^{ixf(t)}\right|_a^b, \qquad x \to \infty \tag{1 29}$$

But problems arise if $f'(t^0) = 0$ for some $t^0 \in (a, b)$— near such a point the exponential oscillates quite slowly and $g(t)$ hardly changes whereas away from this point the oscillations are much faster and a cancellation will persist Thus, a possible contribution to the asymptotic behaviour of the integral from the neighbourhood of a stationary point will ensue, similar to Laplace's method In fact, if the stationary point under consideration is one at which only the first derivative of $f(t)$ vanishes, the integral is actually converted to a Laplace-type integral by rotating the contour of integration through an angle of $\frac{\pi}{4}$ and transforming the variable of integration Then

$$I(x) \sim \sqrt{\frac{2\pi}{\pm xf''(t^0)}}g(t^0)e^{i(xf(t^0)\pm\frac{\pi}{4})}, \qquad x \to \infty \tag{1 30}$$

according as t^0 is a minimum or maximum of $f(t)$ (as $f''(t^0) \neq 0$) For stationary points at which higher order derivatives also vanish, different angles of rotation can be used

It is important to note, however, that this contribution, though significant, may be dominated by the endpoints of the interval In many physical problems, particularly those involving the propagation of waves, the function $f(t)$ is called the 'phase' function, hence the evaluation of the contribution from a stationary point has been called the method of stationary phase Unfortunately, only the leading behaviour can be obtained by this method Higher order terms require the entire interval to be taken into consideration This occurs because the error incurred in neglecting the non-stationary points is algebraically small rather than exponentially small Either the method of asymptotic matching or that of steepest descent would be better employed to continue the expansion For instance, a class of Fourier-type integrals of the form

$$I_n(x) = \int_{-\infty}^{\infty} \mathrm{e}^{-t^{2n}+ixt}\, dt, \qquad n = 1, 2, \tag{1 31}$$

has been studied by Senouf [75] He manages to obtain the asymptotic behaviour of $I_n(x)$ as $x \to \infty$ via the method of steepest descent as well as a high-order asymptotic approximation of the real zeroes of the function The coefficients in the expansion are systematically obtained using Lagrange's reversion of series

Though the method of stationary phase was first explicitly laid down in 1887 in Lord Kelvin's 'On the waves produced by a single impulse in water of any depth or in a dispersive media' [39], where he considered the behaviour of

$$\frac{1}{2\pi}\int_0^{\infty} \cos[m(x - t/f(m))]dm, \tag{1 32}$$

its essence seems to have been used by Cauchy, Stokes and Riemann prior to that Since then, Watson, Erdélyi and van der Corput have both formulated the method more precisely and adapted it to a more general setting To date, it has been successfully applied to a variety of problems including the study of neutron transitions in nuclear physics

1 4 4 Saddlepoint Method

Reverting back to the 1850s, Stokes followed up Airy's investigation on the intensity of light in the neighbourhood of a caustic and showed that the rainbow integral,

$$\int_0^{\infty} \cos\frac{\pi}{2}(w^3 - mw)dw,$$

behaved like a damped exponential on one side of the caustic whereas it behaved sinusoidally on the other side In doing this, it would be fairer to say that he developed various asymptotic procedures as opposed to applying them He effectively employed the saddle point method though he did not refer to it as such Consequently, he was able to calculate the zeroes of the rainbow integral more effectively than before and for larger values of m [78] Moreover, further work entailed the optimal truncation of a dominant asymptotic expansion and a resummation of the divergent tail to increase the accuracy of his computations—all of this well before Poincaré's 1886 definition of an asymptotic expansion!

Although the saddlepoint method is one of the most powerful asymptotic tools, it was 1863 before it was even formally 'sketched' by Riemann [73] It consists of two stages, having as its objective the useful approximation of integrals of the form

$$I(x) = \int_C \rho(z, x)\, dz, \qquad x \to x^0, \tag{1 33}$$

where z is now a complex variable and C is a contour in the complex plane The first stage is the more difficult, involving the exploration of the problem in order to choose a new, suitable path of integration which allows for the application of the second stage—an evaluation of the integral employing, essentially, Laplace's method A good upper estimate for $I(x)$ would be given by

$$|I(x)| \leq \int_C |\rho(z, x)|\, |dz| \leq l_C \max_C |\rho(z, x)|, \tag{1 34}$$

where l_C is the (finite) length of the path C Changing the path, therefore, may provide a better estimate—the path $\mathcal{P}$ for which $l_{\mathcal{P}} \max_{\mathcal{P}} |\rho(z, x)|$ is minimal coinciding with the idea of a 'least upper bound' This is what Fedoryuk [22] refers to as the 'minimax property' Note that while deforming the contour care must be taken of singularities of the integrand (so Cauchy's integral theorem proves useful) Usually it turns out, conveniently for the practitioner, that the path chosen to satisfy such a condition allows for the easy parametrisation and evaluation of $I(x)$

Working with $\log \rho(z, x)$ simplifies the problem somewhat—$\rho(z, x)$ can be written as $\rho(z, x) = e^{f(z,x)}$ and then the point z^0 satisfing $\rho'(z^0, x) = 0$ also gives $f'(z^0, x) = 0$ and is, thus, a saddlepoint of f (as f is a harmonic function) Hence the minimax contour will pass through a saddle of f De Bruijn [15] suggests investigating the 'altitude' of all of the saddles of $f(z, x)$ and trying to deform the contour into a path through the lowest one, while retaining the endpoints of the original contour, as one way of tackling the otherwise difficult problem of determining such a minimax contour The saddlepoint method is really most successful in the case where the function $\rho(z,x)$ behaves rather violently, so that small variations in z may result in large changes in $\rho(z,x)$ and thus only a small neighbourhood of the maximum is necessary This can be achieved if the contour chosen is one of steepest descent—explaining why the terms 'saddlepoint method' and 'method of steepest descent' are often used interchangebly

1 4.5 Method of Steepest Descent

The method of steepest descent was first documented by Debye in a paper in 1909 [16] in which Bessel functions of large orders were being investigated It is a technique most often used to derive the asymptotic expansion of integrals of the form

$$I(\lambda) = \int_C g(z)e^{\lambda f(z)}\,dz, \tag{1 35}$$

where $f(z)$, $g(z)$ are analytic functions of the complex variable z, λ is the (complex) asymptotic parameter and C is a contour in the z-plane $f(z)$ can be written as

$$f(z) = \Re f(z) + i\Im f(z) = \phi(z) + i\psi(z) \tag{1 36}$$

The basic idea is to deform the contour C to a new path of integration C' such that

(i) C' passes through one or more zeros of $f'(z)$, z^0 say,

(ii) $\lambda\psi(z)$ is constant on C', $\psi(z) = \psi(z^0)$ $\forall$ z

Thus

$$I(\lambda) = \int_C g(z)e^{\lambda f(z)}\,dz = \int_C g(z)e^{\lambda(\phi(z)+i\psi(z))}\,dz = e^{i\lambda\psi(z^0)}\int_{C'} g(z)e^{\lambda\phi(z)}\,dz \tag{1 37}$$

(If f' does not vanish, the contour is chosen to comply with the second condition only) As $\phi(z)$ is real, Laplace's method can be used to evaluate $I(\lambda)$ as $\lambda \to \infty$ Such a deformation of C to C' is motivated by the fact that rapid oscillations of the integrand when λ is large are eliminated on a path where $\psi(z)$ is constant Had C been deformed instead into a path C'' on which $\phi(z)$ was constant, the method of stationary phase could have been applied exactly as described in §1 4 3 (Note that as this method itself requires a rotation of the contour, C'' will eventually coincide with C') However, Laplace's method provides a better approximation scheme as the full asymptotic expansion of a Laplace integral relies only on the immediate neighbourhood of the point(s) where $\phi(z)$ is maximum on the contour (i e the neighbourhood of the point(s) z^0 giving $f'(z^0) = 0$)

It can easily be shown that constant phase contours through a maximum of $f(z)$ can also be viewed as lines of steepest descent through a saddle of $f(z)$ A saddle of order n is defined as a point at which $f^{(n)}(z)$ is the lowest non-vanishing derivative and there are $2n$ analytic curves forming lines of steepest descent and ascent through such a saddle, each separated from the next by an angle of π/n When deforming the original path of integration, only the lines of steepest *descent* can be used as the integrand must converge in order for the integral to exist However, because Laplace's method determines the asymptotic expansion of an integral using the direct neighbourhood of the maxima, the structure of the lines of steepest descent must only be known in such a neighbourhood After a little manipulation, having replaced $\phi(z)$ by its truncated Taylor expansion, the following result is obtained for simple saddles (i e $n = 2$)

$$I(\lambda) \sim \sqrt{\frac{2\pi}{-\lambda f''(z^0)}} g(z^0) e^{\lambda f(z^0)}, \tag{1 38}$$

in the case where $g(z^0) \neq 0$ (Higher order terms can be obtained as before)

Note, that the function $f(z)$ may have several saddles In this case, suppose the original path of integration can be deformed into an equivalent path consisting of paths through some but not all of the saddlepoints—it has been claimed that such saddlepoints alone are relevant to the asymptotic expansion Their contributions are calculated separately and added to give the full asymptotic expansion—the leading behaviour being the contribution from that relevant saddle with the greatest $\Re f(z)$ However, should it arise that two saddles with different phases are necessary for the calculation (obviously they cannot be joined by a single line of steepest descent), then a contour along which the integrand vanishes as $\lambda \to \lambda^0$ should join the line of steepest descent through one to that through the other (i e the lines of steepest descent should meet in a valley of $f(z)$, see below) Among the relevant/admissible critical points, those being either saddles or endpoints of the contour, some are exponentially small compared to others and are thus numerically negligible—the contributions from such subdominant points need only be retained when they bear some special significance

If the asymptotic parameter, λ, is complex, then the lines of steepest descent for $e^{\lambda f(z)}$ rotate in the complex plane with $\arg\lambda$ Thus the deformation of the contour will also change with $\arg\lambda$, and the asymptotic behaviour of the integral changes accordingly—the Stokes phenomenon must be taken into account

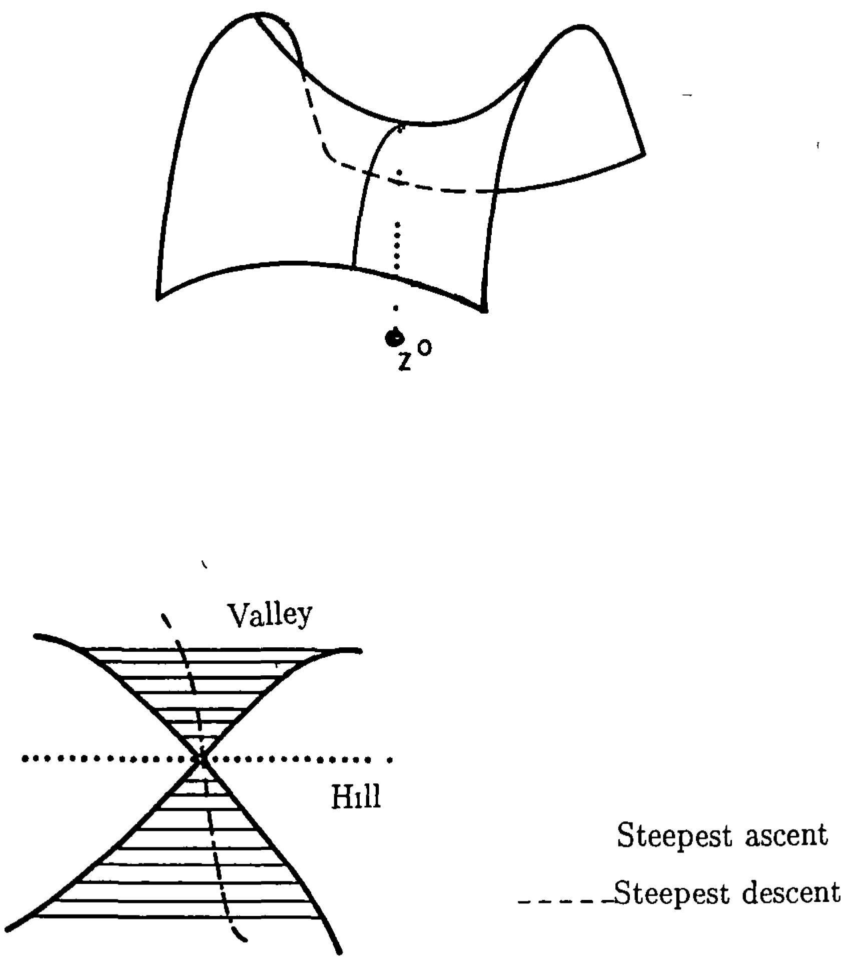


Figure 1 1 Relief of $e^{\lambda f(z)}$ with saddlepoint z^0 and schematic diagram of lines of steepest ascent and descent through z^0, the shaded regions denoting the sectors where $\Re f(z) < \Re f(z^0)$

To visualise and 'physically' interpret the method, think of $|e^{\lambda f(z)}|$ as a surface, termed the relief of $e^{\lambda f(z)}$ Then the saddlepoints will actually appear as *saddles* on the relief and the lines of steepest ascent and descent divide it into *hills* and *valleys* Though the lines of steepest descent must not be followed exactly, except at a saddle, it is essential to stay within the valleys elsewhere so that the integrand converges (see Figure 1 1) Indeed, there can be no automatic assumption that the deformation of the original contour to a line of steepest descent is possible and often the construction of such lines is impractical—it can be more advantageous then to use other saddle contours, once their asymptotic equivalence to the lines of steepest descent has been established

1.5 Modern Methods

Unfortunately, the classical methods just described do not cover all problems In more recent times, other techniques have been developed to supplement them

1.5.1 Summability Method

In 1939, Titchmarsh [81] investigated the asymptotic behaviour of the Fourier integral

$$F(x) = \int_0^\infty e^{ixt-t^{\frac{1}{4}}} \sin[t^{\frac{1}{4}}]\, dt, \tag{1 39}$$

and in doing so discovered that the method of repeated integration by parts does not apply In fact he had to combine it with contour integration in order to show that

$$F(x) \sim \tfrac{i}{4}\Gamma\left(\tfrac{1}{4}\right)e^{\frac{i\pi}{8}}x^{-\frac{5}{4}}, \qquad x \to \infty \tag{1 40}$$

Using contour integration alone, he remarked that only the more feeble result $F(x) = o(x^{-1})$ could be obtained However, Titchmarsh's result can be determined in a different manner entirely, by a procedure first made rigorous by Olver in 1974 [63]—the summability method Subsequently, a lot of research developed along these lines The method hinges on the following

Theorem (Wong [83]) Consider the Fourier integral

$$F(x) = \int_0^\infty f(t)e^{ixt}\, dt, \tag{1 41}$$

where $f(t)$ is m times differentiable on $(0,\infty)$, m being a non-negative integer Let $\{\alpha_s \; s \in \mathbb{N}\}$ be a sequence in $\mathbb{C}$ with increasing real parts and let n be the smallest non-negative integer such that $\Re\alpha_n > m$ Define $f_n(t)$ by

$$f(t) = \sum_{s=0}^{n-1} c_s t^{\alpha_s - 1} + f_n(t) \tag{1 42}$$

Then the integral, $F(x)$, satisfies

$$F(x) = \sum_{s=0}^{n-1} c_s e^{i\frac{\pi}{2}\alpha_s} \frac{\Gamma(\alpha_s)}{x^{\alpha_s}} + F_n(x), \tag{1 43}$$

where the remainder is given by

$$F_n(x) = \left(\frac{i}{x}\right)^m \int_0^\infty f_n^{(m)}(t) e^{ixt}\, dt \tag{1 44}$$

Furthermore as $x \to \infty$, $F_n(x) = o(x^{-m})$

The summability method can be applied to the more general integral

$$I(x) = \int_0^\infty f(t) h(xt)\, dt, \tag{1 45}$$

where $h(xt)$ is an oscillatory function and examples arise in problems of high energy nuclear physics Its advantage lies in the construction of desired error bounds associated with the easily derived asymptotic expansion

1 5 2 Distributional Approach

The theory of distributions first arose in problems of mathematical physics and was developed, initially, hand in hand with the theory of partial differential equations But recently it has also become of use in the field of asymptotics of integrals The construction of an asymptotic expansion usually involves three steps the derivation of a formal expansion, the establishment of the result on a rigorous footing and the construction of error bounds The first of these uses most frequently either integration by parts or termwise integration However, a purely formal application can yield an incorrect result and it is important to study the divergent integral in great detail Here the distributional approach comes into its own—its main advantage being the interpretation of and assigning of values to some divergent integrals which cannot be made meaningful by the more classical techniques It has been successfully applied to obtain asymptotic expansions in the case of Stieltjes, Fourier, Laplace and Hilbert transforms and the Riemann-Liouville fractional integrals Lighthill [44] and Jones [30] both dealt with the topic and it is also advocated by Wong [83] who remarks that the one-dimensional distributional approach can easily be extended to higher dimensions The basic idea behind distributional theory involves the association of a function, f say, appearing in the integrand with a number

$$< f, \phi >= \int_{-\infty}^{\infty} f(t)\phi(t)\, dt, \tag{1 46}$$

where ϕ is any differentiable function on $\mathcal{R}$ which vanishes outside a finite interval and f is integrable over a finite interval, thus allowing f to be regarded as a linear functional Then using the elementary theory of distributions, the determination of the asymptotic behaviour can be more accurately realised

1 5.3 Mellin (or Mellin-Barnes) Transform Techniques

Many important integral transforms arising in practice can be written in the form

$$I(x) = \int_0^{\infty} f(t)h(xt)\, dt \tag{1 47}$$

(including Laplace, Fourier, Hankel, Stieltjes) Taking the Mellin transform of such an integral would give

$$M[I, z] = M[f, 1 - z]M[h, z], \tag{1 48}$$

where $M[g, z]$ is the Mellin transform of a locally integrable function $g(t)$ on $(0, \infty)$ defined by

$$M[g, z] = \int_0^{\infty} t^{z-1} g(t)\, dt, \tag{1 49}$$

when the integral converges The domain of analyticity of such an integral is usually an infinite strip, $a < \Re z < b$, say, so if $M[f, 1 - z]$ and $M[h, z]$ share a common domain then $I(x)$ can be recovered from the inversion formula

$$I(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-z} M[f, 1 - z]M[h, z]\, dz, \quad a < c < b \tag{1 50}$$

An asymptotic expansion of $I(x)$ for small x can be achieved using the Cauchy residue theorem provided $M[f, 1-z]$ and $M[h, z]$ can both be analytically continued to meromorphic functions in the left half-plane and the vertical line of integration can be shifted from $\Re z = c$ to $\Re z = d$ resulting in

$$I(x) = \sum_{d<\Re z<c} \text{Res}\{x^{-z} M[f, 1 - z]M[h, z]\} + E(x), \tag{1 51}$$

with

$$E(x) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} x^{-z} M[f, 1 - z]M[h, z]\, dz \tag{1 52}$$

In a similar fashion, the asymptotic expansion of $I(x)$ as $x \to \infty$ can be obtained if $M[f, 1-z]$ and $M[h, z]$ can be analytically continued to meromorphic functions in the right half-plane and the line of integration shifted to the right Knowing explicitly the Mellin transforms appearing in the error terms, it may be possible to find a bound for the error directly In general, it is preferrable to express the errors in terms of f and h and several means have been devised for constructing such explicit expressions Moreover, the technique described can be adapted in the case when $M[f, 1-z]$ and $M[h, z]$ lack a common strip of analyticity Again, the Mellin-transform technique has had much success in real-life physical problems, for instance in the study of free and forced vibrations of a circular membrane submerged in a compressible fluid

In fact, Kaminski and Paris [36] have shown how asymptotic expansions for a class of integrals

$$I(x, c_1, c_2, \ldots, c_k) = \int_0^\infty \mathrm{e}^{-x(t^\mu + \sum_{r=1}^k c_r t^{m_r})}\, dt, \tag{1 53}$$

where $\mu > m_1 > m_2 > \ldots > m_k > 0$, can be developed by a method which entails using the expression

$$\mathrm{e}^{-z} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(\tau) z^{-\tau}\, d\tau, \qquad |\arg z| < \frac{\pi}{2}, \qquad z \neq 0, \tag{1 54}$$

to represent the integral as an iterated Mellm-Barnes integral, repeatedly displacing the contours of integration and applying the residue theorem as each contour is shifted past a pole of the integrand This approach can be applied to a similar class of multi-dimensional integrals with little extra effort They also went on to obtain exponential asymptotics from the standard Mellin-Barnes integral representation of a function by a slight, but important, modification of the integral [35]

1.5.4 Uniform Expansions

Often an asymptotic expansion with respect to λ may also depend on a second para-mater, α say However, it may be non-uniform with respect to this second parameter, becoming invalid as α approaches a critical value, $\alpha \to \alpha^0$ This lack of uniformity can arise as a consequence of the coalescence of two saddles, the coalescence of a saddle and an endpoint of the integral or the merging of other singularities The ordinary saddle point method does not contend with such characteristics—it may no longer be possible, for instance, in the case of an integral with a simple saddle, to write the integral in the usual form with a Gaussian dominant exponential term All is not lost though, as some of the more recent literature in the area (including work by Chester, Friedman and Ursell [13]) suggests various procedures to obtain asymptotic expansions which remain uniformly valid in a domain containing the critical value α^0 These include expressing the integral in terms of special functions whose asymptotic behaviours are known and then reverting to the methods of integration by parts or termwise integration Integrals which possess such occurrences of coalescing saddles and so on have been found to appear in the propagation and diffraction of waves

Berry and Howls [7] have taken the idea of uniform expansions a step further by investigating what happens when saddles in a cluster, distant from the saddle on the contour of integration, coalesce and separate Such distant saddles are responsible for the divergence of the expansion and control the switching on and off of the subdominant exponentials

Chapter 2

Method of Nikishov & Ritus

2.1 Outline of Method

Nikishov and Ritus, in [52], addressed the issue of Stokes line 'width' in the theory of asymptotic expansions and found an explicit expression for the function which 'switches on' the exponentially small terms in such an expansion The method, as described by Nikishov and Ritus[1], considers the entire function $I(\lambda)$ represented by the integral

$$I(\lambda) = A \int_C e^{f(z,\lambda)}\, dz, \tag{2 1}$$

where $\lambda \in \mathbb{C}$ is the large asymptotic parameter, A is a constant and C is a contour in the complex plane with endpoints z'_∞, z''_∞, where

$$\Re f(z'_\infty, \lambda) = \Re f(z''_\infty, \lambda) = -\infty \tag{2 2}$$

They have restricted their investigation to the case in which exactly two simple saddlepoints of $f(z, \lambda)$ occur Although based on the method of steepest descent, their method incorporates a deformation and truncation of the contour of integration which separates the contributions of the dominant and recessive saddles This, they claim, yields a more natural representation of a function than conventional procedures

In the case of such an integral representation as in (2 1) the leading behaviours of $I(\lambda)$ are given by $e^{f(z^u,\lambda)}$ and $e^{f(z^l,\lambda)}$ where z^u and z^l are the saddlepoints of $f(z, \lambda)$ Thus the Stokes lines for an integral representation can be found by solving

$$\Im(f(z^u, \lambda) - f(z^l, \lambda)) = 0, \tag{2 3}$$

[1] §2 1 explains the method following [52], later §2 2 will describe necessary alterations for its practical application

which gives $\lambda = \lambda_S$ say, on the Stokes line The saddlepoints, z^u, z^l, can also be termed dominant and subdominant/recessive, where the dominance of the saddle-points switches as λ is rotated in the complex plane

To use the argument put forward by Nikishov and Ritus, if λ is near λ_S, the contour C can be chosen either along a line of steepest descent through z^u, the upper saddle, alone (hereafter, termed LSD_u) or through both the upper, z^u, and lower saddles, z^l (termed $LSD_u + LSD_l$) The saddles, z^u, z^l, are given the labels upper and lower respectively when

$$\Re(f(z^u,\lambda) - f(z^l,\lambda)) \gg 1, \qquad \lambda \to \lambda^0, \tag{2 4}$$

in a particular sector—thus, upper signifies dominant and lower, subdominant The contour $LSD_u + LSD_l$ is denoted C_{ul}, its initial and final points being z'_∞, z''_∞ respectively (The end of LSD_u and the beginning of LSD_l meet at a point z_∞ which also satisfies $\Re f(z_\infty,\lambda) = -\infty$)

The originality of the method arises in the division of C_{ul} into two contours

$$C_{ul} = C_{u*} + C_{*l}, \tag{2 5}$$

such that C_{u*} begins at z'_∞, passes through z^u and ends at z^*, C_{*l} begins at z^*, passes near (or perhaps through) z^l and ends at z''_∞ (see Figure 2 1) The point z^* must satisfy

$$\Im f(z^*,\lambda) = \Im f(z^u,\lambda), \qquad \Re f(z^*,\lambda) = \Re f(z^l,\lambda), \tag{2 6}$$

which ensures

(i) that z^* is on the LSD_u,

(ii) that the distance between $f(z^*,\lambda)$ and $f(z^l,\lambda)$ is minimised

It is logical to interpret

$$I(\lambda) = A\int_{C_{u*}} e^{f(z,\lambda)}\,dt + A\int_{C_{*l}} e^{f(z,\lambda)}\,dz \tag{2 7}$$

as the sum of the contributions from the two saddlepoints—that is, as a sum of a dominant and a recessive term

$$I(\lambda) = D(\lambda) + R(\lambda) \tag{2 8}$$

Subsequently, f_u shall be used to denote $f(z^u,\lambda)$, f''_u to denote $f''(z^u,\lambda)$, with similar notation for $f(z^l,\lambda)$ etc , but the dependence of these values on λ should be noted

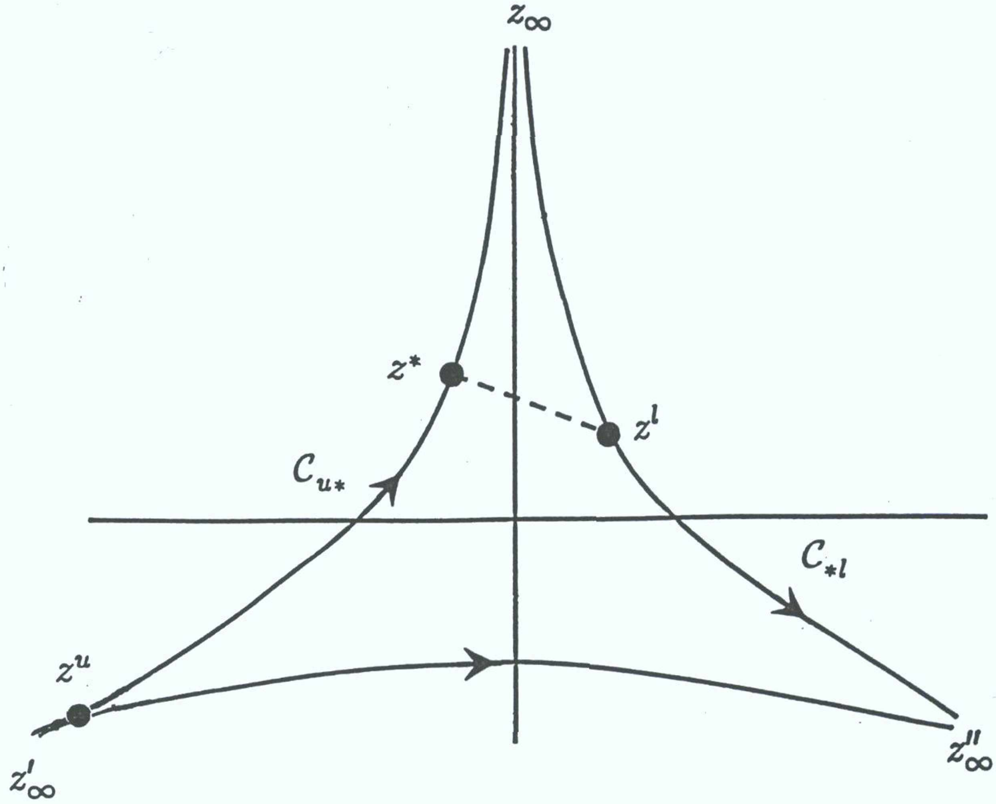


Figure 2.1: Contours $\mathcal{C}_{u*}$ and $\mathcal{C}_{*l}$ near a Stokes line: note that z^u lies on both $\mathcal{C}_{u*}$ and the contour joining z'_∞ to z''_∞.

Although Nikishov and Ritus have not detailed the exact determination of $D(z)$ or $R(z)$, it is assumed that they proceeded in a similar manner to that which follows.

2.1.1 The Dominant Term

From above,

$$D(\lambda) = A \int_{z'_\infty}^{z^*} \mathrm{e}^{f(z;\lambda)}\, dz = A \int_{\mathcal{C}_{u*}} \mathrm{e}^{f(z;\lambda)}\, dz, \tag{2.9}$$

where $\mathcal{C}_{u*}$ is LSD_u, which implies $\Im f(z;\lambda) = \Im f_u$ is constant on $\mathcal{C}_{u*}$. Thus $D(\lambda)$ can be written as

$$D(\lambda) = A\mathrm{e}^{i\Im f_u} \int_{\mathcal{C}_{u*}} \mathrm{e}^{\Re f(z;\lambda)}\, dz, \tag{2.10}$$

which is a Laplace integral and is determined to leading order by the integrand in a small neighbourhood of z^u, i.e.

$$D(\lambda) \sim A\mathrm{e}^{i\Im f_u} \int_{z^u-\varepsilon}^{z^u+\varepsilon} \mathrm{e}^{\Re f(z;\lambda)}\, dz. \tag{2.11}$$

The Taylor series of $f(z;\lambda)$ about z^u, which is valid only in a neighbourhood of z^u, can be used to approximate $f(z;\lambda)$ in this case:

$$f(z;\lambda) = f_u + \frac{(z-z^u)^2}{2!} f''_u + \cdots \quad . \tag{2.12}$$

Hence

$$\Re f(z,\lambda) = \Re f_u + \Re\left(\frac{(z-z^u)^2}{2!} f''_u\right) + \quad , \qquad (2\ 13)$$

$$\Im f(z,\lambda) = \Im f_u + \Im\left(\frac{(z-z^u)^2}{2!} f''_u\right) + \qquad (2\ 14)$$

But noting that $\Im f(z,\lambda) = \Im f_u$ on the contour $\mathcal{C}_{u*}$, implies that

$$\Im\left(\frac{(z-z^u)^2}{2!} f''_u\right) + \quad = 0 \qquad (2\ 15)$$

Therefore, $\Re f(z,\lambda)$ can be written as

$$\Re f(z,\lambda) = \Re f_u + \frac{(z-z^u)^2}{2!} f''_u + \qquad (2\ 16)$$

Truncating at this second derivative gives

$$D(\lambda) \sim A e^{\imath \Im f_u} \int_{z^u-\varepsilon}^{z^u+\varepsilon} e^{\Re f_u + \frac{(z-z^u)^2}{2!} f''_u}\, dz = A e^{f_u} \int_{z^u-\varepsilon}^{z^u+\varepsilon} e^{\frac{(z-z^u)^2}{2!} f''_u}\, dz \qquad (2\ 17)$$

Then by the assumptions of Laplace's method, the interval of integration is extended to be infinite However, to ensure the continued existence of the integral, the integrand must converge Thus it must be ensured that

$$e^{\frac{(z-z^u)^2}{2!} f''_u} \to 0 \quad \text{as } z \to \infty \qquad (2\ 18)$$

(Note that $e^{\frac{(z-z^u)^2}{2!} f''_u}$ is dominated by $e^{\frac{z^2 f''_u}{2}}$ as $z \to \infty$)

$$\begin{aligned} e^{\frac{z^2 f''_u}{2}} &= \exp[\tfrac{1}{2}|z|^2 |f''_u|(\cos 2\theta_z + \imath \sin 2\theta_z)(\cos\theta_f + \imath \sin\theta_f)] \to 0, & z \to \infty, \\ &\Leftrightarrow \cos(2\theta_z + \theta_f) < 0, & z \to \infty, \\ \text{i e } & -\frac{3\pi}{4} - \frac{\theta_f}{2} < \theta_z < -\frac{\pi}{4} - \frac{\theta_f}{2}, \quad \frac{\pi}{4} - \frac{\theta_f}{2} < \theta_z < \frac{3\pi}{4} - \frac{\theta_f}{2} & (2\ 19) \end{aligned}$$

So

$$D(\lambda) \sim \pm A e^{f_u} \int_{\infty e^{\imath\alpha_1}}^{\infty e^{\imath\alpha_2}} e^{\frac{(z-z^u)^2}{2} f''_u}\, dz, \qquad (2\ 20)$$

where

$$\alpha_1 \in \left(-\tfrac{3\pi}{4} - \tfrac{\theta_f}{2}, -\tfrac{\pi}{4} - \tfrac{\theta_f}{2}\right), \quad \alpha_2 \in \left(\tfrac{\pi}{4} - \tfrac{\theta_f}{2}, \tfrac{3\pi}{4} - \tfrac{\theta_f}{2}\right) \qquad (2\ 21)$$

(Note that for real z, f''_u is negative i e $f''_u = |f''_u| e^{\imath\pi}$ This gives

$$\alpha_1 \in \left(-\pi, -\tfrac{3\pi}{4}\right) \cup \left(\tfrac{3\pi}{4}, \pi\right), \qquad \alpha_2 \in \left(-\tfrac{\pi}{4}, \tfrac{\pi}{4}\right) \qquad (2\ 22)$$

—thus the contour can be taken along the real axis)

Now usıng the change of varıable $s = \pm\imath(z - z^u)\sqrt{\frac{f_u''}{2}}$, gives

$$\frac{ds}{dz} = \pm\imath\sqrt{\frac{f_u''}{2}}, \qquad -s^2 = \frac{(z - z^u)^2 f_u''}{2} \tag{2 23}$$

If $z = \infty e^{\imath\alpha_1}$ then

$$s_1 = \pm\imath(z - z^u)\sqrt{\frac{f_u''}{2}} = e^{\pm\imath\frac{\pi}{2}}(\infty e^{\imath\alpha_1} - z^u)\frac{|f_u''|^{\frac{1}{2}}}{2}e^{\frac{\imath\theta_f}{2}} = \infty e^{\imath(\pm\frac{\pi}{2}+\alpha_1+\frac{\theta_f}{2})} \tag{2 24}$$

Sımılarly, for $z = \infty e^{\imath\alpha_2}$,

$$s_2 = \infty e^{\imath(\pm\frac{\pi}{2}+\alpha_2+\frac{\theta_f}{2})} \tag{2 25}$$

Then

$$D(\lambda) \sim \pm A e^{f_u} \int_{\infty e^{\imath\alpha_1}}^{\infty e^{\imath\alpha_2}} e^{\frac{(z-z^u)^2}{2} f_u''}\, dz = \pm\imath\sqrt{\frac{2}{f_u''}} A e^{f_u} \int_{s_1}^{s_2} e^{-s^2}\, ds \tag{2 26}$$

In order to replace $\int_{s_1}^{s_2} e^{-s^2}\, ds$ by $\sqrt{\pi}$ we need

$$\arg s_1 \in \left(-\pi, -\tfrac{3\pi}{4}\right) \cup \left(\tfrac{3\pi}{4}, \pi\right), \qquad \arg s_2 \in \left(-\tfrac{\pi}{4}, \tfrac{\pi}{4}\right) \tag{2 27}$$

However, ıf α_1 and α_2 have been chosen so that the ıntegral w r t z converges, then s_1 and s_2 wıll automatıcally fall ınto the specıfied ıntervals Thus

$$D(\lambda) \sim \pm\imath\sqrt{\frac{2}{f_u''}} A e^{f_u} \left(\int_{C_s} e^{-s^2}\, ds\right) \sim \pm\imath A\sqrt{\frac{2\pi}{f_u''}} e^{f_u}, \tag{2 28}$$

where C_s starts at ınfinıty ın the sector $\left(-\pi, -\frac{3\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right)$ and ends at ınfinıty ın the sector $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ The choıce of branch ıs such as to agree wıth the dırectıon of the contour C_{u*} Notıce thıs estımate of the leadıng behavıour ıs the same as that obtaıned by the tradıtıonal method of steepest descent

2.1.2 The Recessıve Term

From above,

$$R(\lambda) = A\int_{z^*}^{z_\infty''} e^{f(z,\lambda)}\, dz = A\int_{C_{*l}} e^{f(z,\lambda)}\, dz \tag{2 29}$$

An expansıon of $f(z,\lambda)$ ın a Taylor serıes about z^l wıll be valıd only ın the neıghbourhood of z^l But as the greater part of the path C_{*l} ıs LSD_l (along whıch the ıntegral ıs completely determıned by a small neıghbourhood of z^l) and the poınt

z^* ıs by definıtıon, chosen so that $f(z^*,\lambda)$ ıs as near as possıble to $f(z^l,\lambda)$, ıt ıs acceptable to use the Taylor expansıon

$$f(z,\lambda) = f_l + \frac{(z-z^l)^2}{2!} f_l'' + \quad , \tag{2 30}$$

to approxımate $f(z,\lambda)$ It could be saıd that z^* ıs assumed to lıe wıthın what de Bruıjn [15] terms the 'range' of z^l ı e "the cırcular neıghbourhood of z^l consıstıng of all z values whıch are such that $|(z-z^l)^2 f_l''|$ ıs not very large"

Note that ıt would be ıncorrect to proceed as for $D(\lambda)$ by usıng Laplace's method because $\frac{(z-z^l)^2}{2} f_l'' + O(z^3)$ may have a non-zero ımagınary part Instead

$$R(\lambda) \sim A e^{f_l} \int_{z^*}^{z''_\infty} e^{\frac{(z-z^l)^2}{2!} f_l''} \, dz, \tag{2 31}$$

where agaın $\arg z''_\infty$ must lıe ın the sectors

$$\left(-\tfrac{3\pi}{4} - \tfrac{\theta_f}{2}, -\tfrac{\pi}{4} - \tfrac{\theta_f}{2}\right) \quad \text{or} \quad \left(\tfrac{\pi}{4} - \tfrac{\theta_f}{2}, \tfrac{3\pi}{4} - \tfrac{\theta_f}{2}\right) \tag{2 32}$$

as $z \to \infty$ for the ıntegrand to converge

Substıtutıng $s = \pm\imath(z - z^l)\sqrt{\frac{f_l''}{2}}$, gıves

$$-s^2 = \frac{(z-z^l)^2}{2} f_l'', \qquad \frac{ds}{dz} = \pm\imath\sqrt{\frac{f_l''}{2}} \tag{2 33}$$

Then takıng $z = z^*$ and $z = z''_\infty = \infty e^{\imath\alpha_2}$, gıves

$$s_1 = \pm\imath(z^* - z^l)\sqrt{\frac{f_l''}{2}} \quad \text{and} \tag{2 34}$$

$$s_2 = \pm\imath(\infty e^{\imath\alpha_2} - z^l)\sqrt{\frac{f_l''}{2}} = \infty e^{\imath(\pm\frac{\pi}{2} + \alpha_2 + \frac{\theta_f}{2})}, \tag{2 35}$$

respectıvely Thus

$$R(\lambda) \sim \pm\imath A \sqrt{\frac{2}{f_l''}} e^{f_l} \int_{s_1}^{s_2} e^{-s^2} \, ds \tag{2 36}$$

In order to replace $\int_{s_1}^{s_2} e^{-s^2} ds$ by the complementary error functıon wıth argument s_1, we must have $\arg s_2 \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ Thıs coıncıdes wıth the regıons ın whıch α_2 must lıe ın order for the ıntegral w r t z to converge Hence ıt ıs concluded that[2]

$$R(\lambda) \sim \mp\imath A \sqrt{\frac{\pi}{2 f_l''}} e^{f_l} \operatorname{erfc}\left(\pm\imath(z^* - z^l)\sqrt{\frac{f_l''}{2}}\right) \tag{2 37}$$

Agaın, the branch ıs chosen ın accordance wıth the value of α_2 whıch wıll be determıned by the dırectıon of the contour $\mathcal{C}_{*l}$

[2] λ dependence ıs contaıned ın f_l and f_l'' terms

2.1.3 Remarks

Motivating Nikishov and Ritus is the concept that a single asymptotic series cannot (alone) adequately describe the behaviour of a function in certain sectors near an essential singularity *because* of its divergence Using the 'traditional' approach, such an asymptotic series is truncated at the term with minimum modulus and a remainder term is formed According to Nikishov and Ritus [52], this remainder term must

> encapsulate all the quantitative and qualitative information about the behaviour of the recessive series, and in addition, change smoothly on the passage of λ through the Stokes line

i e the remainder term itself is continuous since $I(\lambda)$ and the truncated series are both continuous functions This follows Dingle's theory that the asymptotic series may be seen as "a compact encoding of a function" [17] and its divergence becomes a source of information by indicating the existence of exponentially small terms Thus the idea of resurgence emerges, in which a subdominant exponential can be born out of the tail of the dominant asymptotic series

The crux of the method presented above is that the usual practice of truncating an asymptotic series at its least term has been replaced by the concept of truncating the contour of integration in order to find the asymptotic behaviour of the integral This truncation associates the dominant properties of the integral with one saddlepoint and the recessive properties with the other In an asymptotic series found by traditional methods, the distant terms of the series are strongly influenced by the lower saddlepoint It is the claim of Nikishov and Ritus that the finite sum of such a series is less able to separate the contribution to $I(\lambda)$ of the upper saddle from the contribution of the lower and thereby less able to correctly determine the 'turning on' of the recessive term as such, as the Stokes line is crossed

The authors of the method are themselves interested in the implications this new approach has on the width of the region which appears to be affected by the Stokes line They understand the width of the Stokes line to mean the angular distance over which a discernible change in the multiplier takes place This may be motivated by their physical interpretation of the problem

> for a physical quantity/process described by $I(\lambda)$, the dominant and recessive series of its asymptotic expansion near the singularity describe qualitatively different properties of this quantity/process Small changes of the parameter λ can significantly change some of these properties
>
> the width of the Stokes line determines the formation length or time of the (pair production) process

Comparing their method to standard asymptotic procedures, the 'width' of the Stokes line appears to have been reduced by their approach, which they interpret as resulting from their greater ability to separate the contributions from the upper and lower saddles However, our primary interest was directed towards the use of this new approach to recover the asymptotic expansions of integrals found previously by well-known techniques Our hope was that this new approach would provide a simpler method of treating integrals with a more complicated saddlepoint structure

2.2 Extension of Method of Nikishov & Ritus

2 2.1 Discrepancies

In the outline of the method, the saddles $z^{l,u}$ are designated lower and upper in accordance with the magnitude of $\Re f(z^{l,u}, \lambda)$ As λ rotates in the complex plane and crosses the Stokes lines, the dominance of the saddles should change with it Thus the saddle which was labelled upper and used in the calculation of the term $D(\lambda)$ should become the lower saddle in due course and determine $R(\lambda)$ instead However, putting this procedure blindly into practice by choosing the saddle to compute $D(\lambda)$ or $R(\lambda)$ solely on its dominance properties will lead to errors It must be remembered that this method is based on the method of steepest descents and so, in originally deforming the contour of integration to the lines of steepest descent, the same rules must be adhered to The 'upper' saddle, being the one through which the contour must first pass, is chosen by examining the lines of steepest descent and detecting which saddles are permissible, keeping in mind that the endpoints of the original contour must be retained Having thus chosen the upper saddle, further inspection of the lines of steepest descent will indicate if it is possible to again deform the contour of integration to pass through a second saddle and take into account its contribution—the notion of 'adjacent' saddles[3] of Berry and Howls [5] is thus coming into play here In fact, this argument is illustrated by the example in §2 3, where the behaviour of the Airy Ai function is investigated Ai(λ) is the solution of $y'' = \lambda y$ which decays exponentially along the real positive axis Thus it must satisfy

$$\mathrm{Ai}(\lambda) \sim \frac{1}{2}\pi^{-\frac{1}{2}}\lambda^{-\frac{1}{4}}\mathrm{e}^{-2\lambda^{3/2}/3}, \qquad |\lambda| \to \infty, \qquad \arg\lambda = 0 \tag{2 38}$$

It can be seen that it is physically impossible in this case to deform the contour of integration solely into a line of steepest descent through the saddle, $z = -1$, corresponding to the greatest magnitude of $\Re f(z, \lambda)$ Moreover, as this would in fact yield

$$\mathrm{Ai}(\lambda) \sim \frac{1}{2}\pi^{-\frac{1}{2}}\lambda^{-\frac{1}{4}}\mathrm{e}^{2\lambda^{3/2}/3}, \qquad |\lambda| \to \infty, \qquad \arg\lambda = 0, \tag{2 39}$$

it is clearly incorrect

As seen above, $D(\lambda)$ and $R(\lambda)$ are completely determined only when we make a 'correct' choice of the branch of the square root This is not discussed by Nikishov and Ritus—rather $D(\lambda)$ and $R(\lambda)$ are presented as in (2 28) and (2 37) retaining the '+' in both cases Even in the discussion of the traditional method of steepest descent, many authors tend to ignore the branch issue De Bruijn [15] does remark that the branch is chosen in accordance with the direction in which the saddlepoint is crossed However, it is left to Fedoryuk [22] to tackle the issue more definitively with his formula -

If

$$I(\lambda) = \int_\gamma g(z)\mathrm{e}^{\lambda f(z)}\, dz, \tag{2 40}$$

then the contribution of a simple saddle point z^0 is given by

$$I_{z^0}(\lambda) = \sqrt{-\frac{2\pi}{\lambda f''(z^0)}}[g(z^0) + O(\lambda^{-1})]\mathrm{e}^{\lambda f(z^0)}, \quad \lambda \to \infty, \tag{2 41}$$

choosing the branch as follows arg $\sqrt{-f''(z^0)}$ is equal to the angle between the positive direction of the tangent to the line of steepest descent passing through z^0 and the positive direction of the real axis

[3]See §3 1 for further explanation

Using this, the branch of the square root term in all occurrences of $D(\lambda)$ and $R(\lambda)$ presented here is determined by an inspection of the directions of the lines of steepest descent relative to the original contour of integration

2 2 2 Distance from Stokes Line

In the previous discussion, it was assumed that λ is 'near' a Stokes line The question thus arises—how is 'near' to be defined in this context ? Such a question may however be unnecessary By definition,

$$\Im(f(z^u,\lambda) - f(z^l,\lambda)) = 0 \tag{2 42}$$

on the Stokes line and a line of steepest descent will pass through both saddles simultaneously As λ moves slowly away from λ_S, the contour can be deformed as suggested above, to a line of steepest descent through the upper saddle and then away to z_∞, where it meets a line of steepest descent through the lower saddle Now consider the case when λ is not equal to or even in the neighbourhood of λ_S — then $\Im f(z^l,\lambda)$ and $\Im f(z^u,\lambda)$ cannot be regarded as equal Thus no line of steepest descent (constant phase contour) can join z^l to z^u (see Figure 2 2) So LSD_u starts at z'_∞, passes through z^u and ends at z_∞, whereas LSD_l starts at z'''_∞, passes through z^l and ends at z''_∞, but although

$$\Re f(z_\infty,\lambda) = \Re f(z'''_\infty,\lambda) = -\infty, \tag{2 43}$$

the picture has changed somewhat as

$$\arg z_\infty \neq \arg z'''_\infty \tag{2 44}$$

Joining z_∞ to z'''_∞ at infinity by a third contour along which the integral vanishes, solves the problem The contour C along which $z = Re^{i\theta}$ will always work, so long as z_∞, z'''_∞ both lie in the same sector of the complex plane (valley) where

$$\Re f(z,\lambda) \to -\infty, \quad \forall z, \quad \text{as } |z| \to \infty \tag{2 45}$$

Thus the expressions for $D(\lambda)$ and $R(\lambda)$ should add to give $I(\lambda)$ for any value of λ in the complex plane However, as λ moves away from λ_S, the point of truncation, z^*, moves further away from LSD_l and the portion of the contour of integration

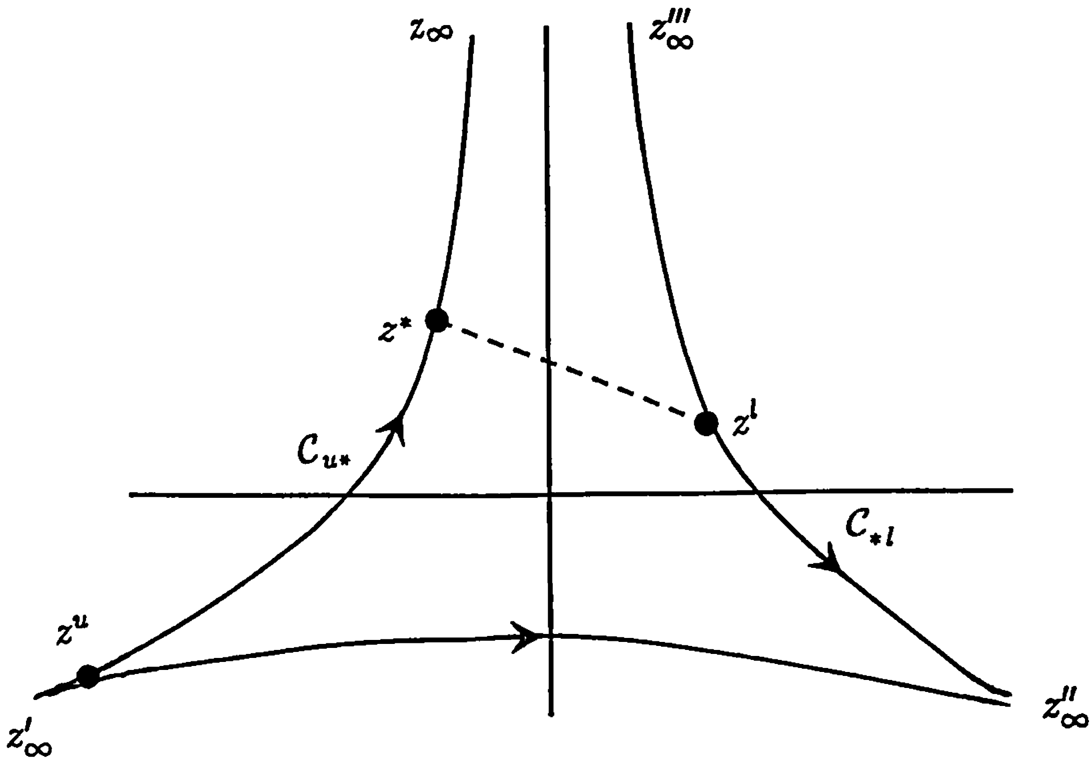


Figure 2 2 Contours C_{u*} and C_{*l} away from a Stokes line note that z^u lies on both C_{u*} and the contour joining z'_∞ to z''_∞

which joins z^* to z^l and lies only on a line of *descent*, lengthens This increases the error in the approximation, albeit by an exponentially small amount and hampers the numerical precision So when an exponentially improved estimate or expansion is sought, use of the method should be confined to a neighbourhood of Stokes line

2 2.3 Choice of z^*

Depending on the form of $f(z, \lambda)$, the equations (2 6) fail to define z^* uniquely For instance if $f(z, \lambda)$ is an nth degree polynomial in z then there are up to n choices of z^* However, no criteria have been given to suggest which value of z^* should be used Once again it would appear that the choice relies on an examination of the contours of steepest descent It is not enough to expect $\Im f(z^*, \lambda) = \Im f(z^u, \lambda)$ —z^* must lie on the contour through z^u to which the contour of integration has been deformed

2 2 4 Higher Order Approximation

While calculating the expression (2 28) for $D(\lambda)$, only the first three terms in the Taylor series of $f(z, \lambda)$ were used, which results in the determination of the leading behaviour of $D(\lambda)$ only Retaining more terms of the Taylor series will result in higher order terms in the asymptotic expansion of $D(\lambda)$ To illustrate this, the third and fourth derivative terms are retained in the following

$$f(z,\lambda) = f_u + \frac{(z-z^u)^2}{2!}f_u'' + \frac{(z-z^u)^3}{3!}f_u''' + \frac{(z-z^u)^4}{4!}f_u^{(4)} + \quad , \qquad (2\ 46)$$

$$D(\lambda) \sim A\int_{z^u-\varepsilon}^{z^u+\varepsilon} e^{f_u+\frac{(z-z^u)^2}{2!}f_u''+\frac{(z-z^u)^3}{3!}f_u'''+\frac{(z-z^u)^4}{4!}f_u^{(4)}}\, dz \qquad (2\ 47)$$

Then Taylor expanding $e^{\frac{(z-z^u)^3}{3!}f_u'''+\frac{(z-z^u)^4}{4!}f_u^{(4)}}$ gives

$$e^{\frac{(z-z^u)^3}{3!}f_u'''+\frac{(z-z^u)^4}{4!}f_u^{(4)}} = 1+\frac{(z-z^u)^3}{6}f_u'''+\frac{(z-z^u)^4}{24}f_u^{(4)}+\frac{1}{2}\left(\frac{(z-z^u)^3}{6}f_u'''+\frac{(z-z^u)^4}{24}f_u^{(4)}\right)^2 , \qquad (2\ 48)$$

$$D(\lambda) \sim Ae^{f_u}\int_{z^u-\varepsilon}^{z^u+\varepsilon} e^{\frac{(z-z^u)^2}{2}f_u''}\left\{1+\frac{(z-z^u)^3}{6}f_u'''+\frac{(z-z^u)^4}{24}f_u^{(4)}+\frac{(z-z^u)^6}{72}(f_u''')^2 + \frac{(z-z^u)^7}{144}f_u'''f_u^{(4)}+\frac{(z-z^u)^8}{1152}\left(f_u^{(4)}\right)^2\right\} dz \qquad (2\ 49)$$

Extending the range of integration as before, the endpoints become $\infty e^{i\alpha_1}$ and $\infty e^{i\alpha_2}$ and satisfy the same inequalities Then letting $s = \pm i(z-z^u)\sqrt{f_u''}$, $s^2 = -(z-z^u)^2 f_u''$, $\frac{ds}{dz} = i\sqrt{f_u''}$ and using the general formulae

$$\int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} s^{2n}\, ds = \sqrt{2\pi}(2n-1)(2n-3)(2n-5) \quad (5)(3)(1) \qquad (2\ 50)$$

$$\int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} s^{2n+1}\, ds = 0, \qquad (2\ 51)$$

it can be shown that $D(z)$ has the leading asymptotic behaviour[4]

$$D(\lambda) \sim \mp\frac{Ae^{f_u}}{i\sqrt{f_u''}}\int_{s_1}^{s_2} e^{-\frac{s^2}{2}}\left\{1 \mp i\frac{s^3}{6}\frac{f_u'''}{(f_u'')^{3/2}} + \frac{s^4}{24}\frac{f_u{}^{(4)}}{(f_u'')^2} - \frac{s^6}{72}\frac{(f_u''')^2}{(f_u)^3} \mp i\frac{s^7}{144}\frac{f_u'''f_u{}^{(4)}}{(f_u'')^{7/2}} + \frac{s^8}{1152}\frac{(f_u{}^{(4)})^2}{(f_u'')^4} + \quad \right\} ds \qquad (2\ 52)$$

$$= \pm iAe^{f_u}\sqrt{\frac{2\pi}{f_u''}}\left\{1+\frac{f_u^{(4)}}{8(f_u'')^2} - \frac{5(f_u''')^2}{24(f_u'')^3} + \frac{35(f_u^{(4)})^2}{384(f_u'')^4} + \quad \right\} \qquad (2\ 53)$$

The formulae in equations (2 50), (2 51) for real s will apply here to the complex variable s if, once again, s_1, s_2 lie in certain sectors of the complex plane, namely those given in (2 27)

[4]These results are exactly as in Dingle [17], whose derivation is given without an explicit appearance of the large asymptotic parameter

To determine expressions for both $D(\lambda)$ and $R(\lambda)$ in the procedure followed by Nikishov and Ritus, the terms in the expansion of $f(z;\lambda)$ containing the third derivative and higher were ignored. This can be seen from the above to be equivalent to their condition

$$\eta_n(\lambda) \equiv |f^{(n)}(z^{l,u};\lambda)|\,|f''(z^{l,u};\lambda)|^{-\frac{n}{2}} \ll 1, \quad n \geq 3 \tag{2.54}$$

—the error then arising being of the order $\eta_n(\lambda)$.

2.2.5 Number of Saddlepoints

Following the general theory of Nikishov and Ritus, there seems to be no reason why the method cannot be extended to three or more saddlepoints. However, once again (see §2.2.1 Discrepancies) the terms 'upper' and 'lower' used to describe certain saddles should not be determined from the relative magnitude of $\Re f(z^m;\lambda)$, where z^m is a saddle, but rather by examining the structure of the lines of steepest descent through the saddle and the possibilities for the deformation of these. In the case of three saddlepoints, denote the upper saddle by z^u, the lower by z^l and the 'intermediate' saddle by z^i. As previously, the contour $\mathcal{C}$ can be chosen along lines of steepest descent through z^u, then z^i and finally z^l —denoted $\mathcal{C}_{uil}$ (see Figure 2.3). $\mathcal{C}_{uil}$ is then divided into three contours

$$\mathcal{C}_{uil} = \mathcal{C}_{ua} + \mathcal{C}_{ab} + \mathcal{C}_{bl}$$

where

$\mathcal{C}_{ua}$ begins at z'_∞, passes through z^u and ends at z^{*a};

$\mathcal{C}_{ab}$ begins at z^{*a}, passes near (or perhaps through) z^i and ends at z^{*b};

$\mathcal{C}_{bl}$ begins at z^{*b}, passes near (or perhaps through) z^l and ends at z''_∞.[5]

The points z^{*a} and z^{*b} must satisfy the following relations

$$\Im f(z^{*a};\lambda) = \Im f(z^u;\lambda), \qquad \Re f(z^{*a};\lambda) = \Re f(z^i;\lambda); \tag{2.55}$$

$$\Im f(z^{*b};\lambda) = \Im f(z^i;\lambda), \qquad \Re f(z^{*b};\lambda) = \Re f(z^l;\lambda). \tag{2.56}$$

Let the 'leading behaviour', as such, be termed the dominant term i.e.

$$D(\lambda) = A \int_{\mathcal{C}_{ua}} \mathrm{e}^{f(z;\lambda)}\, dz, \tag{2.57}$$

[5] It may be helpful to refer to §3.4.3 at this point.

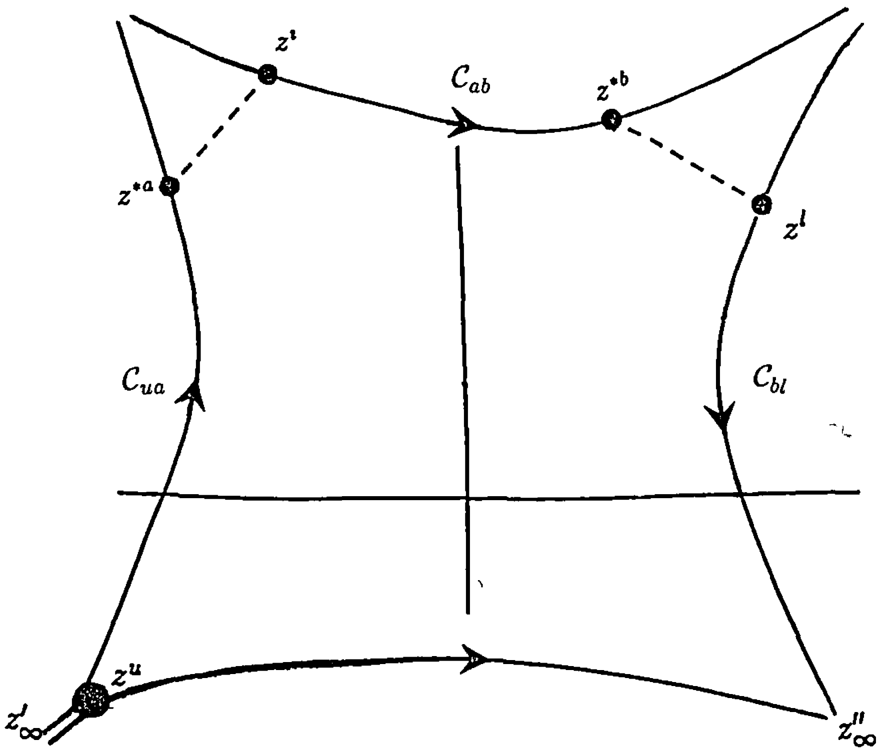


Figure 2 3 Contour deformation in the case of 3 saddlepoints

and group the remaining terms as the recessive i e

$$R(\lambda) = A\int_{\mathcal{C}_{ab}} e^{f(z,\lambda)}\,dz + A\int_{\mathcal{C}_{bl}} e^{f(z,\lambda)}\,dz \tag{2 58}$$

As before, f_l shall denote $f(z^l,\lambda)$, f_l'' denotes $f''(z^l,\lambda)$ etc

Computing $D(\lambda)$ exactly as in the case of two saddlepoints gives

$$D(\lambda) \sim \pm\imath A\sqrt{\tfrac{2\pi}{f_u''}}e^{f_u} \tag{2 59}$$

$R(\lambda)$ is tackled in two parts First consider $A\int_{\mathcal{C}_{bl}} e^{f(z,\lambda)}\,dz$ It follows from the calculations for the previous case that

$$A\int_{\mathcal{C}_{bl}} e^{f(z,\lambda)}\,dz \sim \mp\imath\sqrt{\tfrac{\pi}{2f_l''}}e^{f_l}\mathrm{erfc}\left(\pm\imath(z^{*b}-z^l)\sqrt{\tfrac{f_l''}{2}}\right) \tag{2 60}$$

Secondly, $A\int_{\mathcal{C}_{ab}} e^{f(z,\lambda)}\,dz$ must be calculated Proceeding as for $A\int_{\mathcal{C}_{bl}} e^{f(z,\lambda)}\,dz$ it can be shown that

$$A\int_{\mathcal{C}_{ab}} e^{f(z,\lambda)}\,dz \sim \pm A\sqrt{\tfrac{2}{f_\imath''}}e^{f_\imath}\int_{s_a}^{s_b} e^{-s^2}\,ds, \tag{2 61}$$

where $s_a = \pm(z^{*a}-z^{\imath})\sqrt{\frac{f_\imath''}{2}}$, $\quad s_b = \pm(z^{*b}-z^{\imath})\sqrt{\frac{f_\imath''}{2}}$

Then rewriting

$$\int_{s_a}^{s_b} e^{-s^2}\,ds = \int_{s_a}^{s_\infty} e^{-s^2}\,ds - \int_{s_b}^{s_\infty} e^{-s^2}\,ds, \tag{2 62}$$

results in

$$A\int_{C_{ab}} e^{f(z,\lambda)}\,dz \sim \mp iA\sqrt{\frac{\pi}{2f_i''}}e^{f_i}\left(\operatorname{erfc}\left(\pm i(z^{*a}-z^i)\sqrt{\frac{f_i''}{2}}\right) - \operatorname{erfc}\left(\pm i(z^{*b}-z^i)\sqrt{\frac{f_i''}{2}}\right)\right) \tag{2 63}$$

and, as before, s_∞ must lie in a sector such that $\arg s_\infty \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

In conclusion, when three saddlepoints occur, $I(\lambda)$ can be approximated as

$$I(\lambda) = D(\lambda) + R(\lambda), \tag{2 64}$$

where

$$D(\lambda) \sim \pm iA\sqrt{\frac{2\pi}{f_u''}}e^{f_u}, \tag{2 65}$$

$$\begin{aligned} R(\lambda) \sim\ & \mp iA\sqrt{\frac{\pi}{2f_i''}}e^{f_i}\left(\operatorname{erfc}\left(\pm i(z^{*a}-z^i)\sqrt{\frac{f_i''}{2}}\right) - \operatorname{erfc}\left(\pm i(z^{*b}-z^i)\sqrt{\frac{f_i''}{2}}\right)\right) \\ & \mp i\sqrt{\frac{\pi}{2f_l''}}e^{f_l}\operatorname{erfc}\left(\pm i(z^{*b}-z^l)\sqrt{\frac{f_l''}{2}}\right) \end{aligned} \tag{2 66}$$

Notice, the procedure for choosing the order in which the saddles are traversed is similar to the discussion of Berry and Howls on the adjacency of saddles Carrying the process a step further to deal with four saddlepoints gives

$$D(\lambda) \sim \pm iA\sqrt{\frac{2\pi}{f_u''}}e^{f_u}, \tag{2 67}$$

$$\begin{aligned} R(\lambda) \sim\ & \mp iA\sqrt{\frac{\pi}{2f_{i1}''}}e^{f_{i1}}\left(\operatorname{erfc}\left(\pm i(z^{*a}-z^{i1})\sqrt{\frac{f_{i1}''}{2}}\right) - \operatorname{erfc}\left(\pm i(z^{*b}-z^{i1})\sqrt{\frac{f_{i1}''}{2}}\right)\right) \\ & \mp iA\sqrt{\frac{\pi}{2f_{i2}''}}e^{f_{i2}}\left(\operatorname{erfc}\left(\pm i(z^{*b}-z^{i2})\sqrt{\frac{f_{i2}''}{2}}\right) - \operatorname{erfc}\left(\pm i(z^{*c}-z^{i2})\sqrt{\frac{f_{i2}''}{2}}\right)\right) \\ & \mp iA\sqrt{\frac{\pi}{2f_l''}}e^{f_l}\operatorname{erfc}\left(\pm i(z^{*c}-z^l)\sqrt{\frac{f_l''}{2}}\right) \end{aligned} \tag{2 68}$$

and so on in the event of encountering more than four saddles

2 2 6 Factor $g(z)$

Although the integral considered here is of the form (2 1), the method carries across to the more general class of integrals

$$I(\lambda) = A\int_C g(z)e^{f(z,\lambda)}\,dz \tag{2 69}$$

It is assumed that the continuous function $g(z)$ is slowly-varying when compared to $e^{f(z,\lambda)}$, and that, to leading order

$$g(z) \to g(z^u) \quad \text{as} \quad z \to z^u, \tag{2 70}$$

$$g(z) \to g(z^l) \quad \text{as} \quad z \to z^l, \tag{2 71}$$

with neither $g(z^u)$ nor $g(z^l)$ vanishing Then, while $f(z,\lambda)$ is replaced by the first few terms of its Taylor series (in the determination of both $D(z)$ and $R(z)$), $g(z)$ can also be replaced by $g(z^u)$ or $g(z^l)$ respectively, resulting in

$$D(z) \sim \pm i A \sqrt{\frac{2\pi}{f_u''}} g(z^u) e^{f_u}, \tag{2 72}$$

$$R(z) \sim \mp i A \sqrt{\frac{\pi}{2 f_l''}} g(z^l) e^{f_l} \operatorname{erfc}\left(\pm i (z^* - z^l) \sqrt{\frac{f_l''}{2}} \right) \tag{2 73}$$

2 2 7 Finite Endpoint Contribution

When investigating

$$I(\lambda) = \int_{C_{ul}} e^{f(z,\lambda)}\, dz = \int_{C_{u*}} e^{f(z,\lambda)}\, dz + \int_{C_{*l}} e^{f(z,\lambda)}\, dz, \tag{2 74}$$

care must be taken to remember that separating the contour C_{ul} in two, at a point z^*, introduces in each of the resulting integrals a finite endpoint which must be taken into consideration As can be seen from §2 3, z^* can be very close to or even coincide with one of the saddlepoints and thus may make a contribution to the asymptotic expansion of $I(\lambda)$ (as it is the lowest order approximation incorporating both saddlepoints that is being sought) There are three cases

1 z^* coincides with the upper saddle, z^u, yielding

$$D(\lambda) \sim \pm i \frac{A}{2} \sqrt{\frac{2\pi}{f_u''}} e^{f_u} \tag{2 75}$$

$$R(\lambda) \sim \mp i \frac{A}{2} \sqrt{\frac{2\pi}{f_u''}} e^{f_u} \mp i A \sqrt{\frac{\pi}{2 f_l''}} e^{f_l} \operatorname{erfc}\left(\pm i (z^u - z^l) \sqrt{\frac{f_l''}{2}} \right) \tag{2 76}$$

2 z^* coincides with the lower saddle, z^l, yielding

$$D(\lambda) \sim \pm i A \sqrt{\frac{2\pi}{f_u''}} e^{f_u} \pm i \frac{A}{2} \sqrt{\frac{2\pi}{f_l''}} e^{f_l} \tag{2 77}$$

$$R(\lambda) \sim \mp i A \sqrt{\frac{\pi}{2 f_l''}} e^{f_l} \operatorname{erfc}\left(\pm i (z^l - z^l) \sqrt{\frac{f_l''}{2}} \right) \tag{2 78}$$

3 z^* does not coincide with either saddle but $\Re f(z^*,\lambda)$ may be of comparable magnitude to $\Re f_u$ or $\Re f_l$ and so

$$D(\lambda) \sim \mp A\frac{1}{f'(z^*,\lambda)}e^{f(z^*,\lambda)} \pm \imath A\sqrt{\frac{2\pi}{f_u''}}e^{f_u} \quad (2\ 79)$$

$$R(\lambda) \sim \pm A\frac{1}{f'(z^*,\lambda)}e^{f(z^*,\lambda)} \mp \imath A\sqrt{\frac{\pi}{2f_l''}}e^{f_l}\ \mathrm{erfc}\left(\pm\imath(z^*-z^l)\sqrt{\frac{f_l''}{2}}\right) \quad (2\ 80)$$

Consequently, numerical computations will be affected In particular, at a Stokes line, the endpoint, z^*, will always coincide with the lower saddle, z^l Thus in that case the final term in $R(\lambda)$, erfc $\left(\pm\imath(z^l - z^l)\sqrt{\frac{f_l''}{2}}\right)$, equals erfc(0) which is 1 and $I(\lambda)$ becomes

$$I(\lambda) \sim \pm\imath A\sqrt{\frac{2\pi}{f_u''}}e^{f_u} \quad (2\ 81)$$

or

$$I(\lambda) \sim \pm\imath A\sqrt{\frac{2\pi}{f_u''}}e^{f_u} \pm \imath A\sqrt{\frac{2\pi}{f_l''}}e^{f_l} \quad (2\ 82)$$

as expected, the '+' or '-' to be taken in accordance with the directions of the contours of steepest descent Thus if $I(\lambda)$ represents the Airy Ai function as in §2 3, its asymptotic behaviour when $\arg\lambda = 0$ is correctly determined as

$$I(\lambda) \sim \imath\frac{\lambda^{\frac{1}{2}}}{2\pi\imath}\sqrt{\frac{2\pi}{2\lambda^{\frac{3}{2}}}}e^{-2\lambda^{3/2}/3} = \frac{1}{2}\pi^{-\frac{1}{2}}\lambda^{-\frac{1}{4}}e^{-2\lambda^{3/2}/3}, \qquad \lambda\to\infty \quad (2\ 83)$$

and likewise when $\arg\lambda = 2\pi/3$ its behaviour is now shown to be

$$\begin{aligned} I(\lambda) &\sim \imath\frac{\lambda^{\frac{1}{2}}}{2\pi\imath}\sqrt{\frac{2\pi}{2\lambda^{\frac{3}{2}}}}e^{-2\lambda^{3/2}/3} + \imath\frac{\lambda^{\frac{1}{2}}}{2\pi\imath}\sqrt{\frac{2\pi}{-2\lambda^{\frac{3}{2}}}}e^{2\lambda^{3/2}/3} \\ &= \frac{1}{2}\pi^{-\frac{1}{2}}\lambda^{-\frac{1}{4}}e^{-2\lambda^{3/2}/3} + \frac{1}{2\imath}\pi^{-\frac{1}{2}}\lambda^{-\frac{1}{4}}e^{2\lambda^{3/2}/3}, \qquad \lambda\to\infty \quad (2\ 84)\end{aligned}$$

Obviously, the results given in §2 2 5 and §2 2 6 should also be modified to allow for the inclusion of such finite endpoint contributions

Moreover, the results presented here also illustrate how the occurence of finite endpoints in the original integral $I(\lambda)$ might be dealt with

2.3 Example: Airy's Integral

To illustrate the method presented by Nikishov and Ritus, $D(\lambda)$ and $R(\lambda)$ are computed for Airy's integral for various values of λ The Stokes lines for the function occur at $\arg\lambda = 0, \pm\frac{2\pi}{3}$ The values of Airy's integral are readily available (e g in *Mathematica*) and provide a benchmark against which to test the accuracy of Nikishov and Ritus' method

According to Abramowitz and Stegun [1], Airy's $\mathrm{Ai}(\lambda)$ function has the integral representation

$$\mathrm{Ai}(\lambda) = \frac{1}{\pi}\int_0^\infty \cos\left(t^3/3 + \lambda t\right) dt, \qquad \lambda, t \in \mathbb{R} \tag{2 85}$$

To write this in the form of a Laplace integral, note that $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ Thus

$$\begin{aligned}\int_0^\infty \cos\left(t^3/3 + \lambda t\right) dt &= \frac{1}{2}\int_0^\infty e^{i(t^3/3+\lambda t)}\, dt + \frac{1}{2}\int_0^\infty e^{-i(t^3/3+\lambda t)}\, dt \\ &= -\frac{1}{2}\int_0^{-\infty} e^{-i(t^3/3+\lambda t)}\, dt + \frac{1}{2}\int_0^\infty e^{-i(t^3/3+\lambda t)}\, dt \\ &= \frac{1}{2}\int_{-\infty}^\infty e^{-i(t^3/3+\lambda t)}\, dt, \end{aligned} \tag{2 86}$$

giving $\mathrm{Ai}(\lambda) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-i(t^3/3+\lambda t)}\, dt$ To extend t to the complex plane, the analyticity of the integrand and the convergence of the integral must be investigated

2 3.1 Convergence

Convergence of the integral $\int_{-\infty}^\infty e^{-i(\zeta^3/3+\lambda\zeta)}\, d\zeta$ requires that

$$e^{-i(\zeta^3/3+\lambda\zeta)} \to 0 \qquad \text{as } \zeta \to \infty \tag{2 87}$$

(Note that $e^{-i\zeta^3/3}$ dominates $e^{-i(\zeta^3/3+\lambda\zeta)}$ as $\zeta \to \infty$)
Setting $\zeta^3 = |\zeta|^3 e^{i3\theta_\zeta}$ gives

$$\begin{aligned} e^{-i\zeta^3/3} &= e^{-i|\zeta|^3/3(\cos 3\theta_\zeta + i\sin 3\theta_\zeta)} \\ &= e^{|\zeta|^3/3\sin 3\theta_\zeta} e^{-i|\zeta|^3/3\cos 3\theta_\zeta} \end{aligned} \tag{2 88}$$

Thus $e^{-i\zeta^3/3} \to 0$ when $e^{|\zeta|^3/3\sin 3\theta_\zeta} \to 0$, which happens when $\sin 3\theta_\zeta < 0$, or equivalently, $\theta_\zeta \in \left(-\pi, -\frac{2\pi}{3}\right), \left(-\frac{\pi}{3}, 0\right), \left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$

2 3 2 Analyticity

Writing $w(\zeta,\lambda) = e^{f(\zeta,\lambda)} = e^{-i(\lambda\zeta+\zeta^3/3)}$ and setting $\zeta = x + iy,\ \lambda = a + ib$ gives

$$\begin{aligned} w(\zeta,\lambda) &= e^{(bx+ay-y^3/3+x^2y)}\cos\left(-ax+by-x^3/3+xy^2\right) \qquad (2\ 89) \\ &\quad + i\, e^{(bx+ay-y^3/3+x^2y)}\sin\left(-ax+by-x^3/3+xy^2\right) \\ &= u(x,y) + iv(x,y) \qquad (2\ 90) \end{aligned}$$

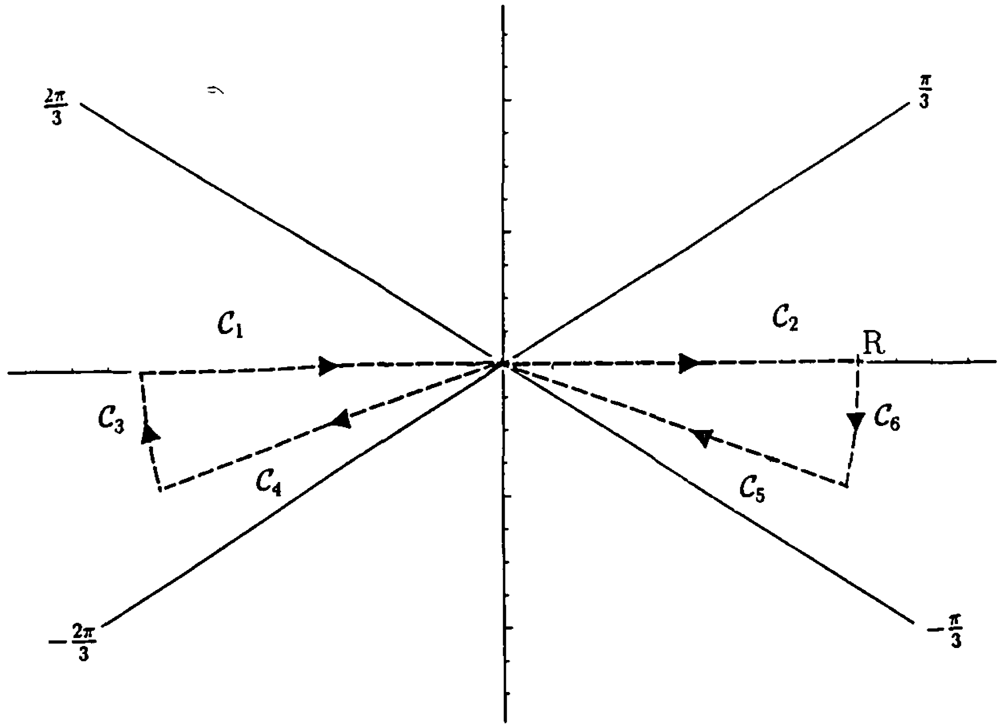


Figure 2 4 Contours of integration for the Airy Ai function

Thus $u(x,y)$, u_x, u_y, $v(x,y)$, v_x, v_y are all continuous, and it can be shown that

$$u_x = v_y, \qquad u_y = -v_x, \tag{2 91}$$

which implies that the Cauchy-Riemann equations hold, so therefore $w'(\zeta,\lambda)$ exists and

$$w(\zeta,\lambda) = e^{f(\zeta,\lambda)} = e^{-i(\lambda\zeta+\zeta^3/3)} \tag{2 92}$$

is an entire function of ζ

2 3 3 Contour of Integration

$$\begin{aligned} \mathrm{Ai}(\lambda) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\zeta^3/3+\lambda\zeta)}\, d\zeta \\ &= \frac{1}{2\pi}\lim_{R\to\infty}\int_{C_1+C_2} e^{-i(\zeta^3/3+\lambda\zeta)}\, d\zeta \\ &= \frac{1}{2\pi}\lim_{R\to\infty}\int_{C_1+C_2} w(\zeta,\lambda)\, d\zeta \end{aligned} \tag{2 93}$$

(see Figure 2 4)

Using Cauchy's integral theorem, as the integrand is entire, gives

$$\int_{C_1} w(\zeta,\lambda)\, d\zeta + \int_{C_4} w(\zeta,\lambda)\, d\zeta + \int_{C_3} w(\zeta,\lambda)\, d\zeta = 0 \tag{2 94}$$

Thus

$$\int_{C_1} w(\zeta,\lambda)\,d\zeta = -\int_{C_4} w(\zeta,\lambda)\,d\zeta - \int_{C_3} w(\zeta,\lambda)\,d\zeta \qquad (2\ 95)$$

However, as $R \to \infty$, $\int_{C_3} \to 0$, and

$$\int_{C_1} w(\zeta,\lambda)\,d\zeta \;=\; -\int_{C_4} w(\zeta,\lambda)\,d\zeta \qquad (2\ 96)$$

for any C_4 lying in the region $-\pi < \arg\zeta < -\frac{2\pi}{3}$ Similarly,

$$\int_{C_2} w(\zeta,\lambda)\,d\zeta = -\int_{C_5} w(\zeta,\lambda)\,d\zeta \qquad (2\ 97)$$

for any C_5 lying in the region $-\frac{\pi}{3} < \arg\zeta < 0$ So

$$\mathrm{Ai}(\lambda) \;=\; -\frac{1}{2\pi}\int_{C_4+C_5} \mathrm{e}^{-i(\zeta^3/3+\lambda\zeta)}\,d\zeta \qquad (2\ 98)$$

Using this representation of $\mathrm{Ai}(\lambda)$ and the same notation as above, we find

$$f(\zeta,\lambda) = -i(\zeta^3/3+\lambda\zeta), \qquad f'(\zeta,\lambda) = -i(\lambda+\zeta^2), \qquad (2\ 99)$$

giving the saddlepoints $\zeta = \pm i\lambda^{\frac{1}{2}}$ —thus, 'movable' saddles exist in this case In order to avoid this, the alternative representation

$$\mathrm{Ai}(\lambda) \;=\; \frac{\lambda^{\frac{1}{2}}}{2\pi i}\int_{C} \mathrm{e}^{-\lambda^{\frac{3}{2}}(z-z^3/3)}\,dz, \qquad (2\ 100)$$

where C is a contour with endpoints at $-\frac{\pi}{3} - \frac{1}{2}\arg\lambda$ and $\frac{\pi}{3} - \frac{1}{2}\arg\lambda$ (which lies within the regions of convergence of the integrand), can be substituted by letting $z = i\zeta/\lambda^{1/2}$ However, use of the former representation illustrates how the method works for a phase function $f(z,\lambda)$ as well as for the more usual form $\lambda f(z)$

Using the method of Nikishov and Ritus, C must be deformed along lines of steepest descent through the saddlepoints of the integrand while taking care to retain the original endpoints

2 3 4 Branch Cut

In evaluating $\mathrm{Ai}(\lambda)$, various values of λ were chosen, namely $\lambda = |\lambda|\mathrm{e}^{i\theta_\lambda}$ with

$$|\lambda| \;=\; 5\ 2414827884177932413, \qquad \theta_\lambda \in \left\{-\tfrac{2\pi}{3}, -\tfrac{\pi}{2}, -\tfrac{\pi}{6}, 0, \tfrac{\pi}{6}, \tfrac{\pi}{2}, \tfrac{2\pi}{3}\right\} \qquad (2\ 101)$$

Also investigated was the behaviour of the approximation in the neighbourhood of the Stokes line at $\theta_\lambda = \frac{2\pi}{3}$, using

$$\theta_\lambda \in \left\{\tfrac{29\pi}{48}, \tfrac{30\pi}{48}, \tfrac{31\pi}{48}, \tfrac{2\pi}{3}, \tfrac{33\pi}{48}, \tfrac{34\pi}{48}, \tfrac{35\pi}{48}\right\} \qquad (2\ 102)$$

However, as $\lambda^{\frac{3}{2}}$ is a multi-valued function, its presence and that of square root functions in the expressions (2.28) and (2.37) ensures their multi-valuedness. Thus it was necessary to cut the λ-plane. A branch cut along the negative real axis was chosen, thereby excluding the values $\pm\pi$ from the choice of θ_λ. The value $|\lambda| = 5.24148\ldots$ was used to enable comparison with the results of Berry and Howls [4] which will be discussed in §3.4.

2.3.5 Contour Plots

Included are contour plots (Figure 2.5, 2.6), generated by Mathematica, to illustrate the paths of integration followed.

The first row in each set shows the level curves for the function $f(z;\lambda)$ —the shading is in grey levels, running from black to white with increasing height. Thus the hills and valleys are clearly depicted and there can be no confusion as to the regions within which it is permissible to locate a contour of integration. The second set gives the lines of steepest descent through the upper saddle and also the lines of steepest descent through the lower saddle, which in the case of

$$f(z;\lambda) = -\lambda^{\frac{3}{2}}(z - z^3/3); \tag{2.103}$$

are located at $z^u = 1$ and $z^l = -1$ respectively. By deformation and truncation of these lines, the actual contour of integration is found. Possible choices of truncation point, z^*, are obtained using

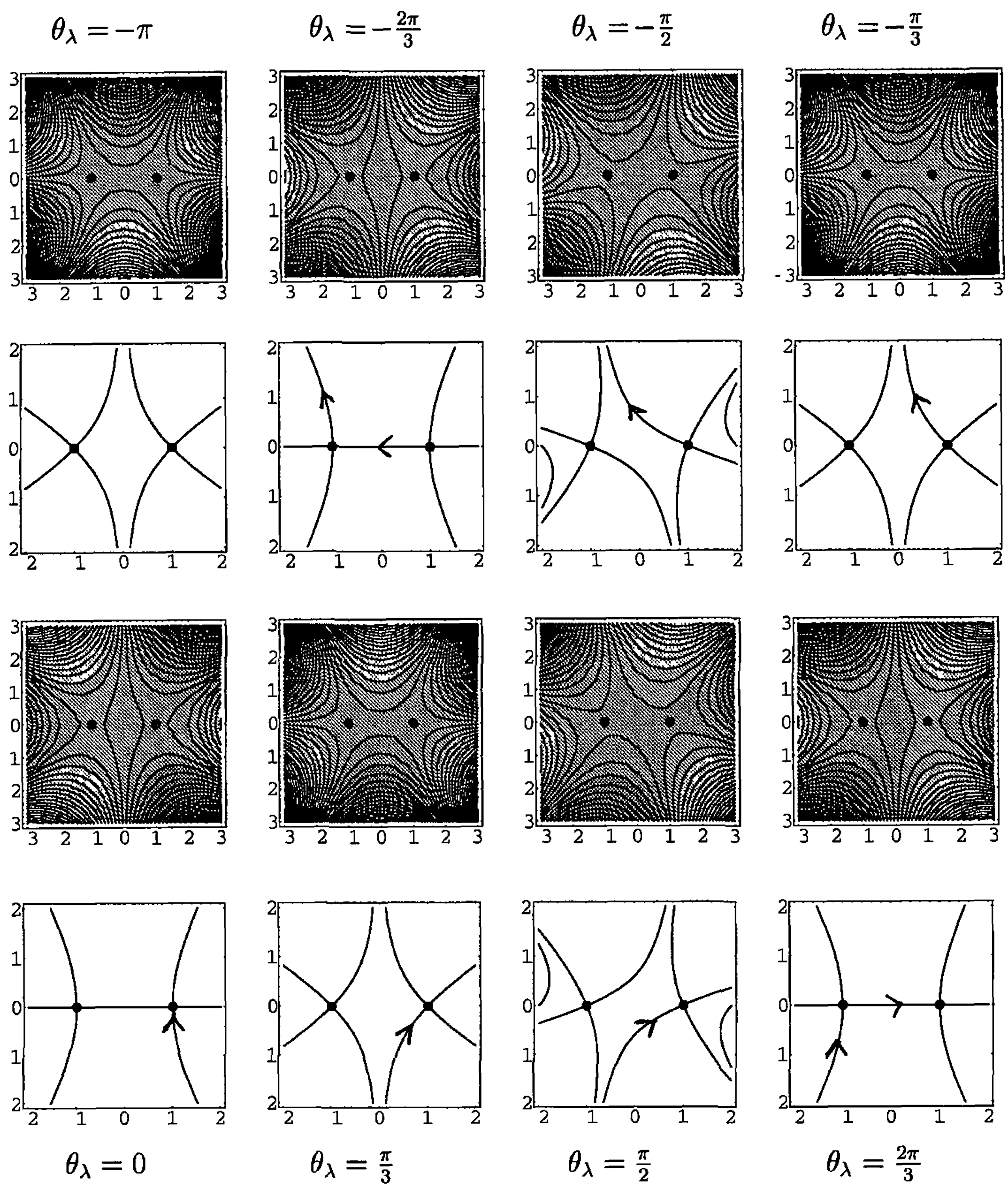
$$f(z^*;\lambda) = \Re f(z^l;\lambda) + i\Im f(z^u;\lambda). \tag{2.104}$$

This is a cubic function and so has at most 3 distinct roots which are indicated in Tables 2.2 and 2.5.

2.3.6 Results

Appearing in Table 2.1 are the results of the application of the method of Nikishov and Ritus throughout the complex plane. It also contains the values of $\mathrm{Ai}(\lambda)$ as given by Mathematica to enable comparison.The rows labelled 'z^u: contribution' refer to the sum of the first seven terms of $D(\lambda)$ as given by (2.53).

The computations were first performed for λ with $-\pi < \theta_\lambda < 0$. Thus $z^u = 1$ and $z^l = -1$. On moving through $\theta_\lambda = 0$, if these values are retained then $\mathrm{Ai}(\lambda)$



Note that in each of the plots the contour of integration, $\mathcal{C}$, runs from the valley at $-\pi/3 - \theta_\lambda/2$ to the valley at $\pi/3 - \theta_\lambda/2$

Figure 2 5 Contour plots generated by *Mathematica* for the Airy Ai function

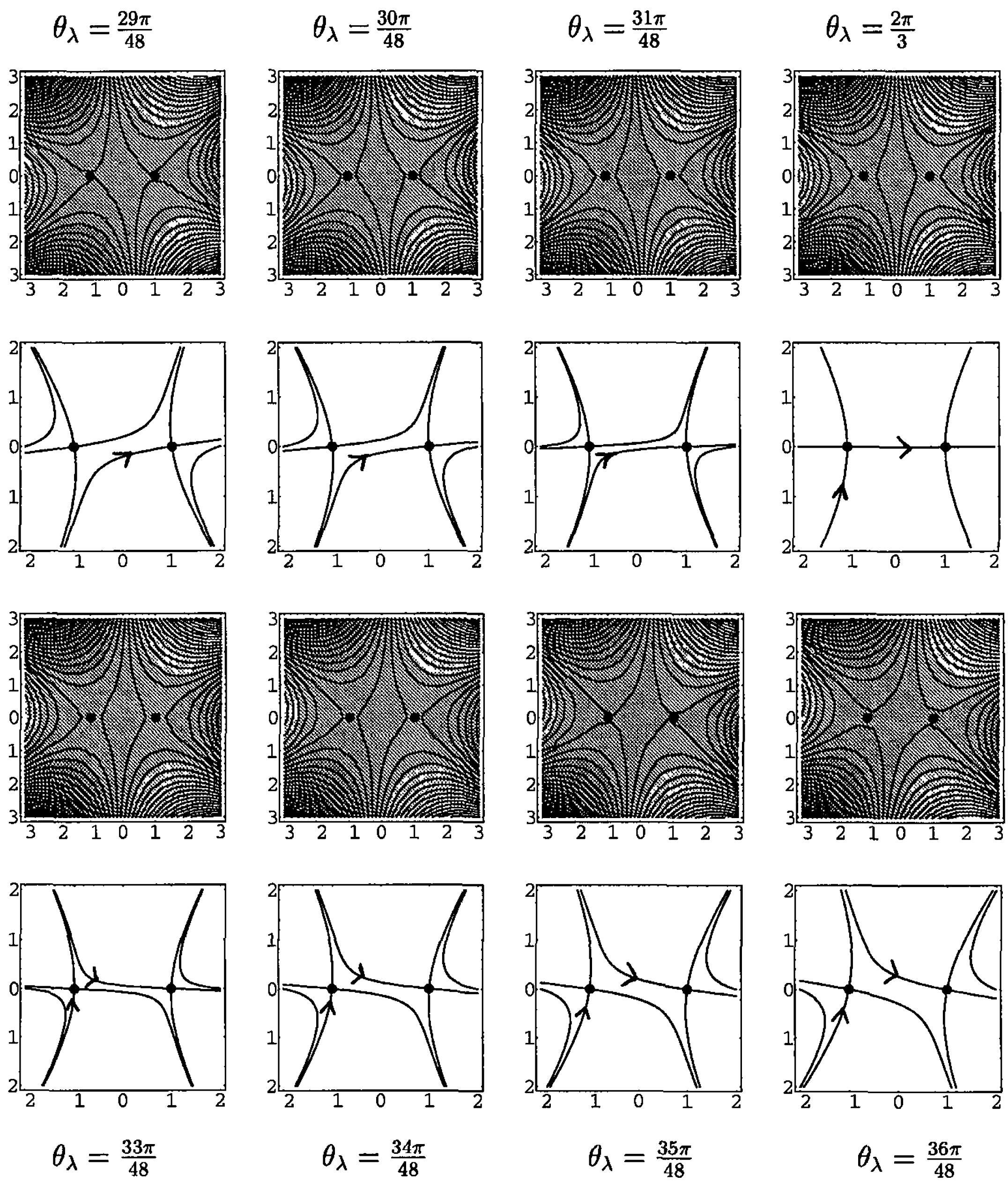


Figure 2 6 Contour plots generated by *Mathematica* in a neighbourhood of the Stokes line at $\theta_\lambda = \frac{2\pi}{3}$

$\theta_\lambda = 0$	0 000062542563	z^u leading behaviour
	0 000062032034	z^u contribution
	0 000062032015	*Mathematica*
$\theta_\lambda = \pm\pi/3$	-0 073942120126 $\mp i0$ 171146794183	z^u leading behaviour
	-0 072426620907 $\mp i0$ 171686345183	z^u contribution
	-8 396688747172x10^{-24} $\pm i6$ 64708077080x10^{-24}	z^l leading behaviour
	-0 072426620907 $\mp i0$ 171686345183	Combined contribution
	-0 072426691399 $\mp i0$ 171686344417	*Mathematica*
$\theta_\lambda = \pm\pi/2$	51 917042554542 $\pm i12$ 355100049351	z^u leading behaviour
	52 148519846152 $\pm i12$ 781413599925	z^u contribution
	1 307176850410x10^{-8} $\mp i1$ 458490447628x10^{-8}	z^l leading behaviour
	52 148519859223 $\pm i12$ 781413585340	Combined contribution
	52 148548097165 $\pm i12$ 781410572676	*Mathematica*
$\theta_\lambda = \pm 2\pi/3$	481 302385175849 $\mp i277$ 880061642886	z^u leading behaviour
	485 804343969079 $\mp i280$ 479268764036	z^u contribution
	0 000031271281 $\pm i0$ 000054163448	z^l leading behaviour
	0 0000310160168 $\pm i0$ 000053721317	z^l contribution
	485 804374985096 $\mp i280$ 479215042721	Combined contribution
	485 804768687663 $\mp i280$ 479478161225	*Mathematica*

Table 2 1 Numerical results using the method of Nikishov & Ritus

$\theta_\lambda = -2\pi/3$	-1	2
$\theta_\lambda = -\pi/2$	$\pm 1\ 807339494452021854$ $-i0\ 298035818991660762$	$0\ 00$ $+i0\ 596071637983321523$
$\theta_\lambda = -\pi/3$	-2	1
$\theta_\lambda = 0$	-1	2
$\theta_\lambda = \pi/3$	-2	1
$\theta_\lambda = \pi/2$	$\pm 1\ 807339494452021854$ $+i0\ 298035818991660762$	$0\ 00$ $-i0\ 596071637983321523$
$\theta_\lambda = 2\pi/3$	-1	2

Table 2 2 Possible choices of z^* for $\mathrm{Ai}(\lambda)$ with $z^u = 1$, $z^l = -1$

could be computed from

$$\mathrm{Ai}(\lambda) = -\frac{\lambda^{\frac{1}{2}}}{2\pi i}\int_{-C} \mathrm{e}^{-\lambda^{\frac{3}{2}}(z - z^3/3)}\, dz \tag{2 105}$$

for convenience Otherwise,

$$\mathrm{Ai}(\lambda) = \frac{\lambda^{\frac{1}{2}}}{2\pi i}\int_{C} \mathrm{e}^{-\lambda^{\frac{3}{2}}(z - z^3/3)}\, dz, \tag{2 106}$$

is used and $z^u = -1$, $z^l = 1$ Either way, the numerical results remain the same

Note that in the special case $\arg\lambda = 0$ only the upper saddle $z = 1$ contributes If $R(\lambda)$ as presented in (2 37) is used, it causes an error This can be seen from the contour plot—notice how the portion of the lines of steepest descent which tend to and return from infinity along the real axis cancel each other exactly as far as the point $+1$ However, the point z^* is determined as $z^* = -1$ If this is used to truncate the contour through $z = 1$ the integral between $z = -1$ and $z = 1$ will be included in the calculations exactly *once* and will lead to a non-zero imaginary part for $\mathrm{Ai}(\lambda)$ The situation can be rectified by using the expression described in §2 2 7 instead

It is obvious from the results that the method loses effectiveness as the distance from a Stokes line increases The greatest improvement obtained by adding in the contribution from z^l takes place at the Stokes line at $\arg\lambda = \frac{2\pi}{3}$ and the worst case is at $\arg\lambda = \frac{\pi}{3}$ as expected, as this is an anti-Stokes line and thus is as far from a Stokes line as possible Tables 2 3, 2 4 and 2 5 illustrate the optimal behaviour of the method when restricted to a neighbourhood of the Stokes line The value of z^* changes smoothly and allows the recessive term to change gradually Once again values of $\mathrm{Ai}(\lambda)$ from *Mathematica* are provided for comparison

$\theta_\lambda = 29\pi/48$	$-370\ 6276671 - \imath 133\ 1113229$	z^u leading behaviour
	$-373\ 5163729 - \imath 135\ 3694464$	z^u contribution
	$0\ 0000138331 - \imath 0\ 0000885048$	z^l leading behaviour
	$-373\ 5163591 - \imath 135\ 3695350$	Combined contribution
	$-373\ 5161175 - \imath 135\ 3695642$	*Mathematica*
$\theta_\lambda = 30\pi/48$	$-220\ 4092148 - \imath 422\ 5408136$	z^u leading behaviour
	$-221\ 5875025 - \imath 426\ 8303696$	z^u contribution
	$0\ 0000440166 - \imath 0\ 0000647097$	z^l leading behaviour
	$-221\ 5874585 - \imath 426\ 8304343$	Combined contribution
	$-221\ 5871741 - \imath 426\ 8305220$	*Mathematica*
$\theta_\lambda = 31\pi/48$	$147\ 4868145 - \imath 514\ 0174263$	z^u leading behaviour
	$149\ 3683830 - \imath 518\ 6491798$	z^u contribution
	$0\ 0000024505 - \imath 0\ 0000691970$	z^l leading behaviour
	$149\ 3683854 - \imath 518\ 6498873$	Combined contribution
	$149\ 3687734 - \imath 518\ 6493631$	*Mathematica*
$\theta_\lambda = 2\pi/3$	$481\ 3023852 - \imath 277\ 8800616$	z^u leading behaviour
	$485\ 8043440 - \imath 280\ 4792688$	z^u contribution
	$0\ 0000312713 + \imath 0\ 0000541634$	z^l leading behaviour
	$485\ 8043750 - \imath 280\ 4792150$	Combined contribution
	$485\ 8047687 - \imath 280\ 4794782$	*Mathematica*

Table 2 3 Numerical results in a neighbourhood of the Stokes line at $\theta_\lambda = \frac{2\pi}{3}$

2 3.7 Conclusions

The method of Nikishov and Ritus, together with the modifications mentioned here, provides a means of exponentially improving classical asymptotic estimates by explicitly defining the function which switches on and off the recessive term [6] As discussed in §1 3 2 this can be of vital importance analytically in physical applications For instance, an application is mentioned by Nikishov and Ritus[52] involving a charged particle in a constant electric field The solutions of the wave equations modelling the problem reduce to parabolic cylinder functions which describe not

[6] A similar result had been provided by Berry in 1989 and is discussed in §3 5 but this was not known by Nikishov and Ritus at the time of publication

$\theta_\lambda = 33\pi/48$	$518\ 5556393 + \imath 129\ 2813851$	z^u leading behaviour
	$523\ 8475567 + \imath 129\ 9677756$	z^u contribution
	$0\ 0000294367 - \imath 0\ 0000627151$	z^l leading behaviour
	$523\ 8475862 + \imath 129\ 9677130$	Combined contribution
	$523\ 8479335 + \imath 129\ 9675451$	*Mathematica*
$\theta_\lambda = 34\pi/48$	$255\ 7264715 + \imath 402\ 1503861$	z^u leading behaviour
	$258\ 8521919 + \imath 405\ 3155911$	z^u contribution
	$0\ 0000340321 - \imath 0\ 0000704741$	z^l leading behaviour
	$258\ 8522259 + \imath 405\ 3155206$	Combined contribution
	$258\ 8525500 + \imath 405\ 3153454$	*Mathematica*
$\theta_\lambda = 35\pi/48$	$-70\ 0360463 + \imath 387\ 5286365$	z^u leading behaviour
	$-69\ 5248069 + \imath 391\ 1593909$	z^u contribution
	$0\ 0000835394 - \imath 0\ 0000322726$	z^l leading behaviour
	$-69\ 5247234 + \imath 391\ 1593586$	Combined contribution
	$-69\ 5245557 + \imath 391\ 1591437$	*Mathematica*

Table 2 4 Numerical results in a neighbourhood of the Stokes line at $\theta_\lambda = \frac{2\pi}{3}$ (contd)

$\theta_\lambda = 29\pi/48$	$-1\ 4784179091755573 22\ +\imath 0\ 3217294401940938 84$
$\theta_\lambda = 30\pi/48$	$-1\ 383165218671339198\ +\imath 0\ 288119081799804959$
$\theta_\lambda = 31\pi/48$	$-1\ 264285567867270114\ +\imath 0\ 223564244429849712$
$\theta_\lambda = 2\pi/3$	-1
$\theta_\lambda = 33\pi/48$	$-1\ 264285567867270114\ -\imath 0\ 223564244429849712$
$\theta_\lambda = 34\pi/48$	$-1\ 383165218671339198\ -\imath 0\ 288119081799804959$
$\theta_\lambda = 35\pi/48$	$-1\ 478417909175557322\ -\imath 0\ 321729440194093884$

Table 2 5 Values of z^* chosen in a neighbourhood of the Stokes line

only the motion of the charged particle in the field but also the production of particle pairs by the field If the electric field is weak, then the latter is represented by exponentially small terms in the asymptotic expansion of the parabolic cylinder function Thus the ability to describe the appearance of these terms is of great interest Not only that but the 'width' of the region across which the switching on of the recessive term takes place, determines the formation length or time of the pair production process

However, one drawback of the method is that it is not clear at what point one should halt when including higher order terms in $D(\lambda)$ as given by (2 53) In other words, the exact relationship between the truncation of the contour C_{ul} at z^* and the truncation of the asymptotic series arising from the saddle z^u is unknown This hampers the numerical precision of the method One remedy may be to change the criteria involved in determining z^* The concept of truncating the contour of integration through one saddle in order to incorporate the contribution of a second saddle would seem to have potential in terms of numerically improving the basic asymptotic estimates given by the method of steepest descent and a more sensible point of truncation may be all that is needed

Further comments are made on this method in Chapter 3, §3 4 where it is compared to the method developed by Berry and Howls These include a brief description of a Borel-plane representation of the former, which may help in the understanding of the process of contour deformation and truncation used

Chapter 3

Berry & Howls Approach

3.1 The Saddlepoint Method of Berry & Howls

By considering integrals of the form

$$I_k(\lambda) = \int_{\mathcal{C}_k(\theta_\lambda)} g(z)\mathrm{e}^{-\lambda f(z)}\, dz, \tag{3.1}$$

where λ is the large asymptotic parameter, $\mathcal{C}_k(\theta_\lambda)$ is an infinite contour of steepest descent of $-\lambda f(z)$ through the simple saddlepoint z^k and $f(z)$ and $g(z)$ are analytic functions at least in a region including $\mathcal{C}_k(\theta_\lambda)$,[1] Berry and Howls [5] produced a refinement of the traditional saddlepoint method which generates hyperasymptotic results by taking into account exponentially small contributions from the other simple saddles through which the contour does not pass. This was prompted by the divergence of the asymptotic series associated with a single saddlepoint. It was seen as necessary evidence to conclude that such a series contains information about the asymptotic series which could be obtained by considering contours through different saddles. Instead of resumming the remainder of the divergent series, following optimal truncation, the tail is expressed in terms of integrals through certain other saddles, termed 'adjacent saddles', chosen according to a topological rule, thus allowing the integral to be determined to an accuracy greater than that of the least term in the series.

The main steps leading to the sequence of hyperseries obtained by Berry and Howls are recounted briefly but using a slight variation on the notation appearing in their 1991 paper. In order to ease the transition to multi-dimensional integrals

[1] Notice that here we are considering phase functions of the special form $-\lambda f(z)$ in contrast to $+f(z,\lambda)$ of Chapter 2—having chosen to retain the notation of the authors for simplicity.

later on, functions are expressed in terms of the more natural variable s, where the λs plane denotes the Borel plane[2], as in Howls' 1998 publication [29]

By extracting the algebraic prefactor and exponential dependence at the saddle, z^k, the function $T_k(\lambda)$ can be defined

$$I_k(\lambda) = \frac{\mathrm{e}^{-\lambda f_k}}{\lambda^{\frac{1}{2}}} T_k(\lambda) \qquad \text{with } f_k = f(z^k) \tag{3 2}$$

Then a change of variable,

$$s = f(z) - f_k, \tag{3 3}$$

is made so that $|s|$ varies from 0 to ∞ on the contour $\mathcal{C}_k$, with $\arg s = -\arg\lambda = -\theta_\lambda$ Thus, for each value of s there are two values of z $z_+(s)$ and $z_-(s)$, and $T_k(\lambda)$ can be written as

$$\begin{aligned} T_k(\lambda) &= \int_0^{\infty e^{-i\theta_\lambda}} \lambda^{\frac{1}{2}} \mathrm{e}^{-\lambda s} \left\{ \frac{g(z_+(s))}{f'(z_+(s))} - \frac{g(z_-(s))}{f'(z_-(s))} \right\} ds \\ &= \int_0^{\infty e^{-i\theta_\lambda}} \lambda^{\frac{1}{2}} \mathrm{e}^{-\lambda s} \Delta_k G(s)\, ds, \end{aligned} \tag{3 4}$$

where

$$G(s) = \frac{g(z(s))}{f'(z(s))} \tag{3 5}$$

and

$$\Delta_k G(s) = \left\{ \frac{g(z_+(s))}{f'(z_+(s))} - \frac{g(z_-(s))}{f'(z_-(s))} \right\} \tag{3 6}$$

The notation $\Delta_k G(s)$ can then be viewed as indicating a discontinuity of $G(s)$ across a cut running from the singularity at $s = 0$ to ∞ Writing this as a contour integral

$$\left\{ \frac{g(z_+(s))}{f'(z_+(s))} - \frac{g(z_-(s))}{f'(z_-(s))} \right\} = \frac{1}{2\pi i s^{\frac{1}{2}}} \oint_{\Gamma_{z^k}} \frac{g(z)(f(z) - f_k)^{\frac{1}{2}}}{f(z) - f_k - s} dz, \tag{3 7}$$

leads to

$$\Delta_k G(s) = \frac{1}{2\pi i s^{\frac{1}{2}}} \oint_{\Gamma_k} \frac{\Delta_k G(\xi) \xi^{\frac{1}{2}}}{\xi - s} d\xi, \tag{3 8}$$

where Γ_{z^k} is the infinite anticlockwise loop surrounding the contour $\mathcal{C}_k$ and Γ_k is its counterpart in the ξ-plane encircling the ray from $\xi = 0$ to infinity The latter can be legitimately obtained by expanding a closed contour around the pole at s so long as it encounters no other singularities in the ξ-plane (see Figure 3 1) Remember

[2]See Chapter 5 for explanation

that as Γ_{z^k} surrounds C_k it must encompass both values $z_+(s)$ and $z_-(s)$ and so to allow the integral along Γ_k the same facility, the function $\Delta_k G$ must be contained within the integrand.

The denominator of (3.7) can be binomially expanded to give

$$\Delta_k G(s) = \frac{1}{2\pi i}\left[\oint_{\Gamma_k}\sum_{r=0}^{N-1}\frac{\Delta_k G(\xi)s^{r-\frac{1}{2}}}{\xi^{r+\frac{1}{2}}}\,d\xi + \oint_{\Gamma_k}\frac{\Delta_k G(\xi)s^{N-\frac{1}{2}}}{\xi^{N+\frac{1}{2}}\left(1-\frac{s}{\xi}\right)}\,d\xi\right] \tag{3.9}$$

On substituting (3.9) into (3.4), $T_k(\lambda)$ becomes

$$T_k(\lambda) = \int_0^{\infty e^{-i\theta_\lambda}}\frac{\lambda^{\frac{1}{2}}e^{-\lambda s}}{2\pi i}\left[\oint_{\Gamma_k}\sum_{r=0}^{N-1}\frac{\Delta_k G(\xi)s^{r-\frac{1}{2}}}{\xi^{r+\frac{1}{2}}}\,d\xi + \oint_{\Gamma_k}\frac{\Delta_k G(\xi)s^{N-\frac{1}{2}}}{\xi^{N+\frac{1}{2}}\left(1-\frac{s}{\xi}\right)}\,d\xi\right]ds \tag{3.10}$$

If

$$T_{kr} = \int_0^{\infty e^{-i\theta_\lambda}}\frac{\lambda^{r+\frac{1}{2}}e^{-\lambda s}}{2\pi i}\oint_{\Gamma_k}\frac{\Delta_k G(\xi)s^{r-\frac{1}{2}}}{\xi^{r+\frac{1}{2}}}\,d\xi\,ds, \tag{3.11}$$

then

$$T_k(\lambda) = \sum_{r=0}^{N-1}\frac{T_{kr}}{\lambda^r} + \frac{\lambda^{\frac{1}{2}}}{2\pi i}\int_0^{\infty e^{-i\theta_\lambda}} e^{-\lambda s}s^{N-\frac{1}{2}}\oint_{\Gamma_k}\frac{\Delta_k G(\xi)}{\xi^{N+\frac{1}{2}}\left(1-\frac{s}{\xi}\right)}\,d\xi\,ds \tag{3.12}$$

Further manipulation of T_{kr} yields

$$T_{kr} = \frac{\Gamma(r+\frac{1}{2})}{2\pi i}\oint_{B_\xi}\frac{\Delta_k G(\xi)}{\xi^{r+\frac{1}{2}}}\,d\xi, \tag{3.13}$$

which, upon reinstating the original variables, would lead to

$$T_{kr} = \frac{(r-\frac{1}{2})!}{2\pi i}\oint_{B_{z^k}}\frac{g(z)}{(f(z)-f_k)^{r+\frac{1}{2}}}\,dz \tag{3.14}$$

The contour Γ_k has now been shrunk to a small positive loop, B_ξ, surrounding $\xi = 0$ for calculation—this corresponds to B_{z^k} in the original z-plane, a small positive loop surrounding saddle z^k. An examination of the remainder term in (3.12),

$$R_k(\lambda, N) = \frac{\lambda^{\frac{1}{2}}}{2\pi i}\int_0^{\infty e^{-i\theta_\lambda}} e^{-\lambda s}s^{N-\frac{1}{2}}\oint_{\Gamma_k}\frac{\Delta_k G(\xi)}{\xi^{N+\frac{1}{2}}\left(1-\frac{s}{\xi}\right)}\,d\xi\,ds, \tag{3.15}$$

suggests deforming the contour of integration into a union of arcs at infinity and similar contours, Γ_m, surrounding the only other singularities of $\Delta_k G(\xi)$ which occur at the images of the other saddles, z^m, in the ξ-plane

$$R_k(\lambda, N) = \frac{\lambda^{\frac{1}{2}}}{2\pi i}\sum_m(-1)^{\gamma_{km}}\int_0^{\infty e^{-i\theta_\lambda}} e^{-\lambda s}s^{N-\frac{1}{2}}\oint_{\Gamma_m}\frac{\Delta_k G(\xi)}{\xi^{N+\frac{1}{2}}\left(1-\frac{s}{\xi}\right)}\,d\xi\,ds \tag{3.16}$$

(γ_{nm} taking the value 0 or 1 depending on the relative orientations in which Γ_k and Γ_m are traversed) Then the transformation $s = \frac{\nu\xi}{\lambda(f_m - f_k)} = \frac{\nu\xi}{\lambda F_{km}}$ is made, yielding

$$R_k(\lambda, N) = \frac{1}{2\pi i}\sum_m \frac{(-1)^{\gamma_{km}}}{\lambda^N F_{km}^{N+\frac{1}{2}}}\int_0^\infty \frac{\nu^{N-\frac{1}{2}}e^{-\nu}}{1-\frac{\nu}{\lambda F_{km}}}\oint_{\Gamma_m} \Delta_k G(\xi) e^{-\frac{\nu}{F_{km}}(\xi - f_m + f_k)}\, d\xi\, d\nu \tag{3 17}$$

(after some manipulation) A definite similarity is detected between the formula $T_k(\lambda)$, as given by (3 4), and the second integral in the expression for $R_k(\lambda, N)$ In fact, if the loop contour Γ_m in the latter is collapsed onto the line running from $\xi = 0$ to ∞ and the discontinuity $\Delta_m(\Delta_k G(\xi))$ taken across it, then this implies [3]

$$R_k(\lambda, N) = \frac{1}{2\pi i}\sum_m \frac{(-1)^{\gamma_{km}}}{(\lambda F_{km})^N}\int_0^\infty \frac{\nu^{N-1}e^{-\nu}}{1-\frac{\nu}{\lambda F_{km}}} T_m\left(\frac{\nu}{F_{km}}\right) d\nu, \tag{3 18}$$

so that (3 1) can be written as

$$I_k(\lambda) = \frac{e^{-\lambda f_k}}{\lambda^{\frac{1}{2}}}\left(\sum_{r=0}^{N-1}\frac{T_{kr}}{\lambda^r} + \frac{1}{2\pi i}\sum_m \frac{(-1)^{\gamma_{km}}}{(\lambda F_{km})^N}\int_0^\infty \frac{\nu^{N-1}e^{-\nu}}{1-\frac{\nu}{\lambda F_{km}}} T_m\left(\frac{\nu}{F_{km}}\right) d\nu\right) \tag{3 19}$$

The latter resurgence formula (3 19) can then be iterated, illustrating the occurence of multiple scattering paths between adjacent saddles and creating a series of hyperseries, each of which contains fewer terms The final expansion takes the form

$$\begin{aligned} I_k(\lambda) &= \frac{e^{-\lambda f_k}}{\lambda^{\frac{1}{2}}}\left(\sum_{r=0}^{N_0-1} T_{kr}K_{k,r} + \sum_m \sum_{r=0}^{N_1-1} T_{mr}K_{km,r} + \sum_m\sum_p\sum_{r=0}^{N_2-1} T_{pr}K_{kmp,r} + \quad\right) \\ &= \frac{e^{-\lambda f_k}}{\lambda^{\frac{1}{2}}}(H_0 + H_1 + H_2 + \quad) \end{aligned} \tag{3 20}$$

where T_{jr} are the coefficients of the primitive asymptotic series associated with saddle z^j The $K_{km\ \ j,r}$ represent universal hyperterminant integrals which depend only on the values of the singulant and the point of truncation and are therefore independent of the particular form of integral investigated A numerical example is given in §3 4

3.2 Incorporating Finite Endpoints

Howls extended the saddlepoint method of §3 1 to include integrals having a finite endpoint [28] There are two separate cases to consider—those where the endpoint is itself a saddlepomt and those where it is not

[3]Refer to appendix for details

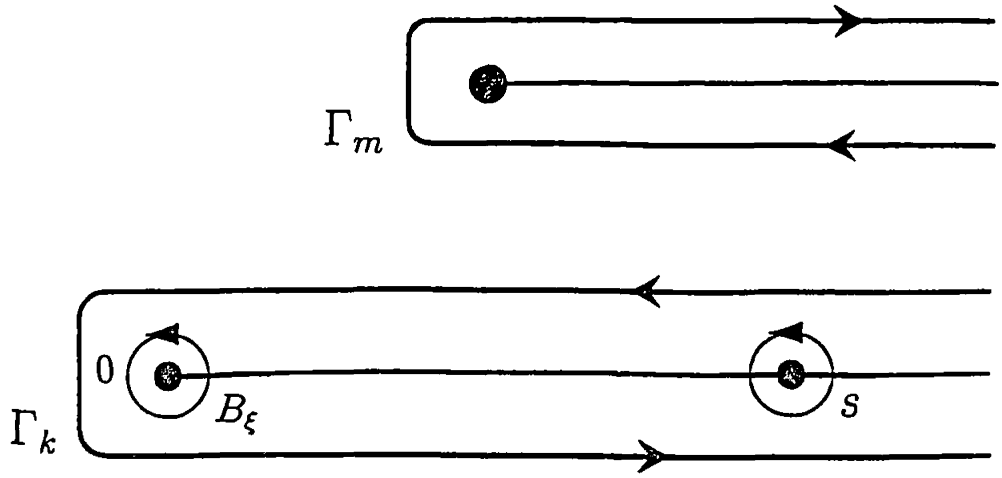


Figure 3 1 Contours of integration in ξ-plane §3 1

3 2 1 Quadratic Dependence

Determination of the hyperasymptotic scheme follows the same steps as before but again the notation used in [28] has been changed slightly The first case to be considered is

$$I_{k/2}(\lambda) = \int_{C_{k/2}(\theta_\lambda)} g(z) e^{-\lambda f(z)}\, dz, \qquad (3\ 21)$$

where $C_{k/2}(\theta_\lambda)$ is a contour starting at the saddle z^k and running along a steepest descent path to infinity in a valley of $\Re\{-\lambda[f(z) - f(z^k)]\}$ To start, the exponential dependence at the saddle, z^k, is extracted

$$I_{k/2}(\lambda) = \frac{e^{-\lambda f_k}}{2\lambda^{\frac{1}{2}}} T_{k/2}(\lambda), \qquad f_k = f(z^k) \qquad (3\ 22)$$

The change of variable

$$s = f(z) - f_k \qquad (3\ 23)$$

is performed as before giving two values of z to each s, $z_+(s)$ and $z_-(s)$, because of the local quadratic behaviour of $f(z) - f_k$ but only one of these, say $z_+(s)$, lies on $C_{k/2}$ Then $T_{k/2}(\lambda)$ becomes

$$T_{k/2}(\lambda) = 2\int_0^{\infty e^{-i\theta_\lambda}} \lambda^{\frac{1}{2}} e^{-\lambda s} \frac{g(z_+(s))}{f'(z_+(s))} ds = 2\int_0^{\infty e^{-i\theta_\lambda}} \lambda^{\frac{1}{2}} e^{-\lambda s} \Delta_k G(s) ds, \qquad (3\ 24)$$

where again

$$G(s) = \frac{g(z(s))}{f'(z(s))}, \qquad (3\ 25)$$

but now

$$\Delta_k G(s) = \frac{g(z_+(s))}{f'(z_+(s))}, \tag{3 26}$$

representing its value on the upper side of the cut only Using

$$\frac{g(z_+(s))}{f'(z_+(s))} = \frac{1}{4\pi i s^{\frac{1}{2}}} \oint_{\Gamma_{z^{k/2}}} \frac{g(z)}{(f(z)-f_k)^{\frac{1}{2}} - s^{\frac{1}{2}}} dz, \tag{3 27}$$

$\Delta_k G(s)$ can be written as

$$\Delta_k G(s) = \frac{1}{4\pi i s^{\frac{1}{2}}} \oint_{\Gamma_{k/2}} \frac{\Delta_k G(\xi)}{\xi^{\frac{1}{2}} - s^{\frac{1}{2}}} d\xi \tag{3 28}$$

The contour $\Gamma_{z^{k/2}}$ is an infinite loop around the path of steepest descent $\mathcal{C}_{k/2}$ and $\Gamma_{k/2}$ is its image in the ξ-plane If the denominator of the latter is expanded binomially, then $T_{k/2}(\lambda)$ takes the form

$$T_{k/2}(\lambda) = \sum_{r=0}^{N-1} \frac{T_{(k/2)r}}{\lambda^{\frac{r}{2}}} + R_{k/2}(\lambda, N) \tag{3 29}$$

with

$$\begin{aligned} T_{(k/2)r} &= \frac{\Gamma(\frac{r}{2}+\frac{1}{2})}{2\pi i} \oint_{B_\xi} \frac{\Delta_k G(\xi)}{\xi^{\frac{r}{2}+\frac{1}{2}}} d\xi \\ &= \frac{(\frac{r}{2}-\frac{1}{2})!}{2\pi i} \oint_{B_{z^k}} \frac{g(z)}{(f(z)-f_k)^{\frac{r}{2}+\frac{1}{2}}} dz, \qquad (3\ 30) \\ R_{k/2}(\lambda, N) &= \frac{\lambda^{\frac{1}{2}}}{2\pi i} \int_0^{\infty e^{-i\theta_\lambda}} e^{-\lambda s} s^{\frac{N}{2}-\frac{1}{2}} \oint_{\Gamma_{k/2}} \frac{\Delta_k G(\xi)}{\xi^{\frac{N}{2}+\frac{1}{2}}(1-(\frac{s}{\xi})^{\frac{1}{2}})} d\xi\, ds \qquad (3\ 31) \end{aligned}$$

The contour $\Gamma_{k/2}$ is then deformed exactly as before—to a union of arcs at infinity and contours Γ_m corresponding to the doubly-infinite contours of steepest descent through the adjacent saddles Finally the change of variables

$$s = \frac{\nu\xi}{\lambda(f_m - f_k)} = \frac{\nu\xi}{\lambda F_{km}}, \tag{3 32}$$

leads to

$$T_{k/2}(\lambda) = \sum_{r=0}^{N-1} \frac{T_{(k/2)r}}{\lambda^{\frac{r}{2}}} + \frac{1}{2\pi i} \sum_m \frac{(-1)^{\gamma_{km}}}{(\lambda F_{km})^{\frac{N}{2}}} \int_0^\infty \frac{\nu^{\frac{N}{2}-1} e^{-\nu}}{1 - \sqrt{\frac{\nu}{\lambda F_{km}}}} T_m\left(\frac{\nu}{F_{km}}\right) d\nu, \tag{3 33}$$

using T_m as in (3 19) Again a hyperasymptotic scheme can be obtained by iterating this resurgence formula

3.2.2 Linear Dependence

The integral in this case takes the form

$$I_e(\lambda) = \int_{\mathcal{C}_e(\theta_\lambda)} g(z) e^{-\lambda f(z)}\, dz, \tag{3 34}$$

where $\mathcal{C}_e(\theta_\lambda)$ is a steepest descent path from z^e to a valley of $\Re\{-\lambda[f(z) - f(z^e)]\}$ The function $T_e(\lambda)$ is defined by

$$I_e(\lambda) = \frac{e^{-\lambda f_e}}{\lambda} T_e(\lambda), \qquad f_e = f(z^e), \tag{3 35}$$

then a change of variable,

$$s = f(z) - f_e, \tag{3 36}$$

is made, allowing s to vary from 0 to ∞ along $\mathcal{C}_e$ with $\arg s = -\arg\lambda$ However it should be noted that this time the transformation is single-valued as z^e is not a stationary point of $f(z)$ and so the latter depends on z^e linearly Thus $T_e(\lambda)$ becomes

$$T_e(\lambda) = \int_0^{\infty e^{-i\theta_\lambda}} \lambda e^{-\lambda s} \frac{g(z(s))}{f'(z(s))} ds = \int_0^{\infty e^{-i\theta_\lambda}} \lambda e^{-\lambda s} G(s) ds \tag{3 37}$$

and the function $G(s)$ so defined can be written as a contour integral

$$G(s) = \frac{1}{2\pi i} \oint_{\Gamma_e} \frac{G(\xi)}{\xi - s} d\xi, \tag{3 38}$$

because in this case

$$\frac{g(z(s))}{f'(z(s))} = \frac{1}{2\pi i} \oint_{\Gamma_{z^e}} \frac{g(z)}{f(z) - f_e - s} dz \tag{3 39}$$

The denominator of the former is binomially expanded as before and the result substituted into $T_e(\lambda)$ to give

$$T_e(\lambda) = \sum_{r=0}^{N-1} \frac{T_{er}}{\lambda^r} + R_e(\lambda, N) \tag{3 40}$$

where

$$T_{er} = \frac{\Gamma(r+1)}{2\pi i} \oint_{B_\xi} \frac{G(\xi)}{\xi^{r+1}} d\xi = \frac{r!}{2\pi i} \oint_{B_{z^e}} \frac{g(z)}{(f(z) - f_e)^{r+1}} dz \tag{3 41}$$

and

$$R_e(\lambda, N) = \frac{\lambda}{2\pi i} \int_0^{\infty e^{-i\theta_\lambda}} e^{-\lambda s} s^N \oint_{\Gamma_e} \frac{G(\xi)}{\xi^{N+1}(1 - \frac{s}{\xi})} d\xi\, ds \tag{3 42}$$

Following the pattern of the previous sections, Γ_{z^e} and Γ_e are infinite loops in the z and ξ-planes respectively, whereas B_{z^e} and B_ξ are small loops surrounding z^e

and $\xi = 0$ (see Figure 3 2) As the divergence of an endpoint expansion is still attributable to the existence of saddles of $f(z)$, the contour Γ_e is deformed as before and then following another variable change

$$s = \frac{\nu\xi}{\lambda(f_m - f_e)} = \frac{\nu\xi}{\lambda F_{em}}, \tag{3 43}$$

the resurgence formula emerges as

$$T_e(\lambda) = \sum_{r=0}^{N-1} \frac{T_{er}}{\lambda^r} + \frac{1}{2\pi i} \sum_m \frac{(-1)^{\gamma_{em}}}{\lambda^N F_{em}^{N+\frac{1}{2}}} \int_0^\infty \frac{\nu^{N-1/2} e^{-\nu}}{1 - \frac{\nu}{\lambda F_{em}}} T_m \left(\frac{\nu}{F_{em}}\right) d\nu \tag{3 44}$$

Figure 3 2 Contours of integration in ξ-plane §3 2 2

3.3 Ordinary Differential Equations

The initial hyperasymptotic investigations of Berry and Howls [4] dealt instead with the solutions of ordinary differential equations of Schrodinger type,

$$\frac{d^2 y(z,\lambda)}{dz^2} = \lambda^2 Z(z) y(z,\lambda), \tag{3 45}$$

where one transition point dominates, λ being the large asymptotic pararmeter A formal asymptotic series for y was found by substituting into the equation and the terms of this series could then be exploited by a resurgence formula of Dingle [17] in the form of an exact representation relating the later to the early terms

$$y(z) \sim \frac{e^{-\frac{F}{2}}}{Z^{\frac{1}{4}}} \sum_{r=0}^{\infty} (-1)^r Y_r(F), \tag{3 46}$$

$$Y_r(F) \sim \frac{1}{2\pi i} \sum_j \frac{1}{(F - F_j)^r} \sum_{s=0}^{\infty} Y_s(F)[-(F - F_j)]^s (r - s - 1)!, \tag{3 47}$$

where F is the 'singulant' (a natural variable for the problem),

$$F(z) = 2\lambda \int_{z^*}^{z} Z^{\frac{1}{2}}(\zeta)\, d\zeta, \tag{3 48}$$

and F_j, its value at transition point z^j At first sight, such a relationship between the terms in an expansion seems to be astounding, however closer examination reveals it to be, in fact, inevitable This is because, although each of the two 'wave' solutions satisfies the equation formally by itself, in order to reflect the existence of the other 'wave' in the solution, it must diverge and contain all the necessary information in its divergent tail to describe the early terms of the other series

The procedure undertaken was to truncate the original series at its least term, apply the resurgence formula to the terms in the remainder series, then perform Borel summation on the latter—that is, replace the factorial by its integral representation, interchange the summation and integration and evaluate the sum Such a resummation generates an asymptotic series which itself requires resummation and thus the process becomes iterative It was originally hoped that such a sequence of 'hyperseries' would converge to the exact solution Unfortunately, this was not the case, as the hyperseries become successively shorter and so the encoded information they contain is finite Thus the ultimate error is finite (and non-zero) and is found to be of order $e^{-(1+2\log 2)A\lambda}$ In fact, the first term of each successive hyperseries is only half the size of the last term in the previous one The resulting scheme is

$$\begin{aligned} y(z) &= \frac{e^{-\frac{F}{2}}}{Z^{\frac{1}{4}}(z)} \left(\sum_{r=0}^{N_0-1} (-1)^r\, Y_r(F) + \sum_{r=0}^{N_1-1} (-1)^r\, Y_r(F) K_{r1} + \sum_{r=0}^{N_2-1} (-1)^r\, Y_r(F) K_{r2} + \quad \right) \\ &= \frac{e^{-\frac{F}{2}}}{Z^{\frac{1}{4}}(z)} \left(\tilde{H}_0 + \tilde{H}_1 + \tilde{H}_2 + \quad \right) \end{aligned} \tag{3 49}$$

where K_{rj} are hyperterminant integrals as before However, this should be viewed as a less general result than that obtained from the integral representation of functions in which many exponentials can be involved, the asymptotic series corresponding to each of which is different

It is possible to construct a hyperasymptotic scheme of arbitrary accuracy, but as the truncations of the series are no longer optimal, this may result in representing $Y(F)$ by large terms which cancel each other out and becomes its downfall Optimal hyperasymptotic schemes such as the one above actually sacrifice ultimate accuracy for immediate improvement at each stage In this sense, the choice of truncations here is not the best In terms of globally minimising the error, Olde Daalhuis and Olver have proposed other schemes yielding better estimates [61], [57]

In fact, since the introduction of the concept of hyperasymptotics much work has been undertaken in the area of differential equations (see [61], [55], [56], [58], [50], [57] for details) Principal contributions have come in particular from Olde Daalhuis—for instance in his new integral representation of the hyperterminant functions which allowed them to be computed to arbitrary presision using convergent expansions

Although extreme numerical precision is not the main motive for the investigation in [4], a numerical example (computation of the Airy Ai and *Bi* functions) was given to illustrate the workings of the method These computations have been supplemented here (Tables 3 1, 3 2) to compare to the results obtained by the Nikishov and Ritus method It should be noted that the Stokes phenomenon is also quite naturally taken care of by the method Comparing the results with those from *Mathematica* illustrates how well the method works in all sectors of the complex plane—at Stokes and anti-Stokes lines alike

3.4 Comparison of Results

3 4 1 Application of Method

A study of Airy's Ai function

$$\mathrm{Ai}(\lambda) = \frac{\lambda^{\frac{1}{2}}}{2\pi i} \int_C e^{-\lambda^{3/2}(z - z^3/3)} dz, \qquad (3\ 50)$$

involves only two saddlepoints and thus the only possible scattering path bounces back and forth between them In fact as this function is a solution of the second order equation

$$d^2y/d\lambda^2 = \lambda y, \qquad (3\ 51)$$

it can be seen that the differential equation method dealing with two exponentials presented in §3 3 can be regarded as a special case of the saddlepoint method and hence the computational details proceed in parallel

Starting with the integral method, it is obvious that $f(z) = z - z^3/3$, $g(z) = 1$ and the saddles of $f(z)$ occur at $z^1 = +1$, $z^2 = -1$ The contour $\mathcal{C}$ which runs from $\infty e^{i(-\pi/3-\arg\lambda/2)}$ to $\infty e^{i(\pi/3-\arg\lambda/2)}$ can easily be deformed into $\mathcal{C}_1$, the infinite contour of steepest descent through z^1, as required As z^2 is the only adjacent saddle, the scattering path becomes

Iteration	0	1	2	3
Saddle	z^1	z^2	z^1	z^2

The singulant $F_{12} = f_2 - f_1 = -4/3$ and the optimal truncation points are calculated using

$$N_0 = \mathrm{Int}|\lambda^{\frac{3}{2}} F_{12}|, \qquad N_n = \mathrm{Int}|N_{n-1}/2| \tag{3 52}$$

The coefficients T_{1r} are given by

$$T_{1r} = \frac{i\Gamma(r+1/6)\Gamma(r+5/6)}{2\sqrt{\pi}r!F_{12}^r} \tag{3 53}$$

and because of the existence of only two saddles a symmetry condition implies that the coefficients T_{2r} are equivalent except for a difference in sign Finally, γ_{12} is set equal to 0 as the orientation of the contour does not change as it is deformed to an arc through z^2

Viewing $\mathrm{Ai}(\lambda)$ as a solution of (3 51) allows it to be written as

$$\mathrm{Ai}(\lambda) = \frac{e^{-\frac{F}{2}}\lambda^{-\frac{1}{4}}}{2\sqrt{\pi}} \sum_{r=0}^{\infty} (-1)^r Y_r, \tag{3 54}$$

where the Y_r can be determined by

$$Y_r = \frac{\Gamma(3r+1/2)}{(27F)^r\,\Gamma(r+1)\Gamma(r+1/2)}, \qquad Y_0 = 1, \tag{3 55}$$

and $F = 2\int^{\lambda} \xi^{1/2} d\xi = 4\lambda^{3/2}/3$ is the singulant The previous values of $|\lambda|$ and $\theta_\lambda = \arg\lambda$ in §2 3 4 have been retained and optimal truncation points can be found using

$$N_0 = \mathrm{Int}|F|, \qquad N_n = \mathrm{Int}(|F|/2^n) \tag{3 56}$$

Alternatively various 'connection' formulae could be used to relate $Y(|F|e^{i\pi})$ and $Y(|F|e^{i2\pi})$ to $Y(|F|)$ and so on, to perform the computations as θ_λ varies, as shown in [4]

3 4 2 Numerical results

Presented in Tables 3 1 and 3 2 are the numerical results using initially only a first approximation to the asymptotic behaviour, followed by the superasymptotic[4] approximation and then the first hyperasymptotic correction to the latter Thereafter the improvement in precision slows down and approaches the ultimate accuracy of the method after just four correction terms ($N_4 = 1$) In fact, in the neighbourhood of the Stokes line, it has been claimed that the rapid switching on of the subdominant exponential is mostly accounted for by the first terminant integral, with further hyperasymptotic contributions varying only minimally Included also in Tables 3 1 and 3 2 are the corresponding values of $\mathrm{Ai}(z)$ given by *Mathematica* for comparison

In the case of the integral, the necessary $K_{1,r}$ and $K_{12,r}$ are given by

$$\begin{aligned} K_{1,r} &= \frac{1}{\lambda^{3r/2}}, & (3\ 57)\\ K_{12,r} &= \frac{(-1)^{\gamma_{12}}}{2\pi i\lambda^{3N_0/2}F_{12}^{N_0-r}}\int_0^\infty \frac{e^{-\nu}\nu^{N_0-r-1}}{1-\frac{\nu}{\lambda^{3/2}F_{12}}}d\nu, & (3\ 58)\end{aligned}$$

with $F_{12} = f_2 - f_1$ as before, whereas for the differential equation

$$K_{r1} = \frac{(-1)^{N_0}}{2\pi F^{N_0-r}}\int_0^\infty \frac{e^{-\xi}\xi^{N_0-r-1}}{1+\frac{\xi}{F}}d\xi \qquad (3\ 59)$$

These were computed using the formula [4]

$$\begin{aligned} K_{r1}(F, N_0) &= \frac{(-1)^{N_0}}{2\pi}e^F\Gamma(N_0 - r)\,\Gamma(r - N_0 + 1, F)\\ &= \frac{(-1)^{r+1}}{2\pi}\left(e^F E_1(F) - \sum_{m=0}^{N_0-r-2}\frac{(-1)^m m!}{F^{m+1}}\right), & (3\ 60)\end{aligned}$$

E_1 denoting the exponential integral function All calculations were again performed using *Mathematica*

3.4.3 Contours of Integration

In order to compare and contrast the contour deformation procedures used above to those of Nikishov and Ritus, consider the integral

$$I_u(\lambda) = \int_{C_u(\theta_\lambda)} g(z)e^{-\lambda f(z)}\,dz, \qquad (3\ 61)$$

[4] see Chapter 1, §1 3 2 for explanation

$\theta_\lambda = 0$	0 0000625425625759612	$\frac{e^{-\lambda^{3/2}f_1}\lambda^{-1/4}}{2\pi i}$	$\frac{e^{-F/2}\lambda^{-1/4}}{2\sqrt{\pi}}$
	0 0000620320147233227	$\frac{e^{-\lambda^{3/2}f_1}\lambda^{-1/4}}{2\pi i}H_0$	$\frac{e^{-F/2}\lambda^{-1/4}}{2\sqrt{\pi}}\tilde{H}_0$
	0 0000620320150783723	$\frac{e^{-\lambda^{3/2}f_1}\lambda^{-1/4}}{2\pi i}(H_0 + H_1)$	$\frac{e^{-F/2}\lambda^{-1/4}}{2\sqrt{\pi}}(\tilde{H}_0 + \tilde{H}_1)$
	0 0000620320150783730	*Mathematica*	
$\theta_\lambda = \pm\frac{\pi}{3}$	-0 073942120126304068 $\mp i$0 171146794182805249	$\frac{e^{-\lambda^{3/2}f_1}\lambda^{-1/4}}{2\pi i}$	$\frac{e^{-F/2}\lambda^{-1/4}}{2\sqrt{\pi}}$
	-0 072426691902033771 $\mp i$0 171686343032197913	$\frac{e^{-\lambda^{3/2}f_1}\lambda^{-1/4}}{2\pi i}H_0$	$\frac{e^{-F/2}\lambda^{-1/4}}{2\sqrt{\pi}}\tilde{H}_0$
	-0 072426691398953331 $\mp i$0 171686344417311019	$\frac{e^{-\lambda^{3/2}f_1}\lambda^{-1/4}}{2\pi i}(H_0 + H_1)$	$\frac{e^{-F/2}\lambda^{-1/4}}{2\sqrt{\pi}}(\tilde{H}_0 + \tilde{H}_1)$
	-0 0724266913989531661 $\mp i$0 171686344417313801	*Mathematica*	
$\theta_\lambda = \pm\frac{\pi}{2}$	51 9170425545421703 $\pm i$12 3551000493512328	$\frac{e^{-\lambda^{3/2}f_1}\lambda^{-1/4}}{2\pi i}$	$\frac{e^{-F/2}\lambda^{-1/4}}{2\sqrt{\pi}}$
	52 1485479377104457 $\pm i$12 7814098581669323	$\frac{e^{-\lambda^{3/2}f_1}\lambda^{-1/4}}{2\pi i}H_0$	$\frac{e^{-F/2}\lambda^{-1/4}}{2\sqrt{\pi}}\tilde{H}_0$
	52 1485480971646300 $\pm i$12 7814105726754200	$\frac{e^{-\lambda^{3/2}f_1}\lambda^{-1/4}}{2\pi i}(H_0 + H_1)$	$\frac{e^{-F/2}\lambda^{-1/4}}{2\sqrt{\pi}}(\tilde{H}_0 + \tilde{H}_1)$
	52 1485480971654727 $\pm i$12 7814105726762697	*Mathematica*	
$\theta_\lambda = \pm\frac{2\pi}{3}$	481 302385175849426 $\mp i$277 880061642885611	$\frac{e^{-\lambda^{3/2}f_1}\lambda^{-1/4}}{2\pi i}$	$\frac{e^{-F/2}\lambda^{-1/4}}{2\sqrt{\pi}}$
	485 804749623305981 $\mp i$280 479502968614454	$\frac{e^{-\lambda^{3/2}f_1}\lambda^{-1/4}}{2\pi i}H_0$	$\frac{e^{-F/2}\lambda^{-1/4}}{2\sqrt{\pi}}\tilde{H}_0$
	485 804768687640302 $\mp i$280 479478161216877	$\frac{e^{-\lambda^{3/2}f_1}\lambda^{-1/4}}{2\pi i}(H_0 + H_1)$	$\frac{e^{-F/2}\lambda^{-1/4}}{2\sqrt{\pi}}(\tilde{H}_0 + \tilde{H}_1)$
	485 804768687662914 $\mp i$280 479478161225457	*Mathematica*	

Table 3 1 Results obtained by Berry & Howls methods as described in §3 1 and §3 3 respectively

$\theta_\lambda = \frac{29\pi}{48}$	-370.627667056413−i133.111322920124	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i}$
	-373.516106813364−i135.369563154830	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i} H_0$
	-373.516117578437−i135.369564185098	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i} (H_0 + H_1)$
	-373.516117492340−i135.369564189327	*Mathematica*
$\theta_\lambda = \frac{30\pi}{48}$	-220.409214849346−i422.540813560316	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i}$
	-221.587161798304−i426.830532179161	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i} H_0$
	-221.587174147979−i426.830521945929	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i} (H_0 + H_1)$
	-221.587174063878−i426.830522040900	*Mathematica*
$\theta_\lambda = \frac{31\pi}{48}$	147.486814464210−i514.017426296724	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i}$
	149.368775848683−i518.649385689418	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i} H_0$
	149.368773420264−i518.649362969244	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i} (H_0 + H_1)$
	149.368773409203−i518.649363148191	*Mathematica*
$\theta_\lambda = \frac{2\pi}{3}$	481.302385175849−i277.880061642886	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i}$
	485.804749623306−i280.479502968614	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i} H_0$
	485.804768687640−i280.479478161217	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i} (H_0 + H_1)$
	485.804768687663−i280.479478161225	*Mathematica*
$\theta_\lambda = \frac{33\pi}{48}$	518.895556392963+i129.281385091142	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i}$
	523.847931588571+i129.967538427566	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i} H_0$
	523.847952479029+i129.967547684581	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i} (H_0 + H_1)$
	523.84793354067+i129.9675450610755	*Mathematica*
$\theta_\lambda = \frac{34\pi}{48}$	255.726471543048+i402.150386067874	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i}$
	258.852503070834+i405.315377359406	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i} H_0$
	258.852518115912+i405.315371780892	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i} (H_0 + H_1)$
	258.852550043745+i405.315345422031	*Mathematica*
$\theta_\lambda = \frac{35\pi}{48}$	-70.036046348025+i387.528636476278	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i}$
	-69.524572815398+i391.159218800450	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i} H_0$
	-69.524568325100+i391.159143695088	$\frac{e^{-\lambda^{3/2} f_1} \lambda^{-1/4}}{2\pi i} (H_0 + H_1)$
	-69.524555688085+i391.159143695088	*Mathematica*

Table 3.2: Results obtained by Berry & Howls integral method in a neighbourhood of the Stokes line.

where C_u is the infinite contour of steepest descent through the upper saddle, z^u Following the method of §3 1, $I_u(\lambda)$ can be written as

$$I_u(\lambda) = \frac{e^{-\lambda f_u}}{\lambda^{\frac{1}{2}}} \int_0^{\infty\theta_\lambda} \lambda^{\frac{1}{2}} e^{-\lambda s} \frac{1}{2\pi i s^{\frac{1}{2}}} \oint_{\Gamma_u} \frac{G(\xi)\xi^{\frac{1}{2}}}{\xi - s} \, d\xi \, ds \tag{3 62}$$

However, instead of expanding the denominator appearing in the second integral, truncating the resulting infinite series and then continuously deforming the contour of the remainder term, the idea would be to truncate the contour Γ_u at some point ξ^* (see Figure 3 3) and move onto another contour Γ_l associated with the lower saddle z^l, following the direction of the arrows shown, and so incorporate the contribution of z^l The choice of ξ^* is such to minimise its distance from Γ_l in exactly the same vein as before but this time working within the ξ-plane

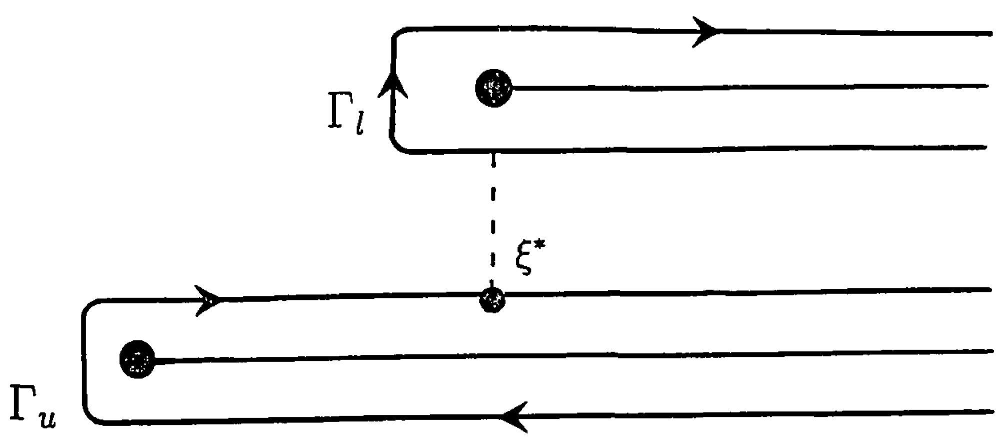


Figure 3 3 Contour truncation in the ξ-plane

3 4 4 Conclusions

As can be seen from the tables, the addition of even only the first hyperasymptotic correction term greatly improves the estimates This remains true throughout the complex domain (i e as arg λ varies) in contrast to the results using the method of Nikishov and Ritus as presented in §2 3 6 At higher levels of hyperasymptotics, where another re-expansion of the remainder is undertaken, the estimates should further improve, albeit more slowly The method of Nikishov and Ritus essentially corresponds to a single re-expanion and therefore the same numerical precision can not be expected

It should also be remembered that these approximations are for large λ and yet the value of $|\lambda|$ chosen was merely 5 2414827884177932413, further demonstrating the power of these methods

3.5 Smoothing of Stokes Discontinuities

Because the Stokes phenomenon involves the behaviour of small exponentials hiding behind larger ones, the framework of exponential asymptotics is necessary for its study—the power-law accuracy of Poincaré's definition of an asymptotic expansion is inadequate Thus the determination of exact multipliers for the recessive terms in an asymptotic expansion goes hand in hand with an understanding of the Stokes phenomenon Across a Stokes line, the line of maximal dominance of one exponential over another, the multiplier of the small exponential changes rapidly The conventional view was that this change was discontinuous, the multiplier, S, having a value $S = \tilde{S}$, say, on one side of the line, $S = \tilde{S} + 1$ on the other and $S = \tilde{S} + \frac{1}{2}$ on the line itself Having said that, it was thought incongruous that such a discontinuity should creep into the representation of analytic, smooth functions Then in 1989, Berry dispelled the mist, that Stokes spoke of, surrounding the problem and by optimally truncating the dominant series expansion, then controlling the exponentially small terms in its remainder, he discovered that the transition is in fact continuous and universal in form and is given by the error function

$$S(\sigma) = \pi^{-\frac{1}{2}} \int_{-\infty}^{\sigma} e^{-t^2}\, dt \qquad \text{with } \sigma = \Im F/(2\Re F)^{\frac{1}{2}}, \tag{3 63}$$

where F is the singulant as before [3] In the case of the asymptotic approximation of $y(X, \lambda)$, the lowest order approximation incorporating both exponentials can be written as

$$y(X, \lambda) \sim M_+(X, \lambda)e^{\lambda\phi_+(X)} + iS(X, \lambda)M_-(X, \lambda)e^{\lambda\phi_-(X)}, \tag{3 64}$$

where $(\Re\phi_+(X) > \Re\phi_-(X))$ It is necessary however, only to consider the series relating to the dominant exponential as the subdominant series will be born from the tail of the former Thus the procedure followed was to investigate

$$y(X,\lambda) = M_+(X,\lambda)e^{\lambda\phi_+(X)}\sum_{r=0}^{\infty} a_r, \quad a_0 = 1, \quad a_r \propto \lambda^r, \tag{3 65}$$

using Dingle's result on the late terms

$$a_r \to \frac{M_-(r-\beta)!}{2\pi M_+ F^{r-\beta-1}}, \qquad r \to \infty, \tag{3 66}$$

where β is a parameter of order 1, whose value depends on the origin of the dominant exponential This gives

$$y \approx M_+ e^{\lambda\phi_+}\sum_{r=0}^{\infty} a_r + iM_- S_n(F)e^{\lambda\phi_-}, \tag{3 67}$$

where $S_n(F)$ can be written as

$$S_n(F) = \frac{-i}{2\pi}\int_0^{\infty} \frac{t^{n-\beta}e^{F(1-t)}}{1-t}\,dt, \tag{3 68}$$

according to the technique of Borel summation Truncating the series near its least term turns out to be the crucial step, as the stationary point of the integral in S_n then almost coincides with its pole so that $S_n(F)$ becomes, after some manipulation,

$$\begin{aligned} S_n(F) \approx\ & \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\Im F/(2\Re F)^{\frac{1}{2}}} e^{-t^2}\,dt - \\ & i(2\pi\Re F)^{-\frac{1}{2}}\left\{\mathrm{Fract}[|F|+\alpha] + \beta - \alpha - \frac{4}{3} - \frac{(\Im F)^2}{6\Re F}\right\} e^{-(\Im F)^2/2\Re F} \end{aligned} \tag{3 69}$$

where α is defined by $n-1 = \mathrm{Int}(|F|+\alpha)$ and is thus of order 1 The change in the Stokes multiplier is then given by the dominant real part of S_n

$$S(\sigma) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\sigma} e^{-t^2}\,dt, \qquad \sigma = \Im F/(2\Re F)^{\frac{1}{2}} \tag{3 70}$$

To confirm this theory, Berry attempts to detect the multiplier numerically for both the Dawson integral and the integral representation of the Airy Bi function The experimental results obtained are in excellent agreement with the theoretical prediction, improving as $|F|$ increases

3.6 Erf vs. Erfc

As can be seen in §3 5 above, Berry found the Stokes multiplier to contain an error function, whereas Nikishov and Ritus show it to be an expression involving a complementary error function and an apparent contradiction emerges However, this can be resolved by a closer examination of the situation Nikishov and Ritus [52] employ the following definitions as found in [1]

$$\mathrm{erf}\,(z) \;=\; 2\pi^{-\frac{1}{2}}\int_0^z \mathrm{e}^{-s^2}ds, \tag{3 71}$$

$$\mathrm{erfc}\,(z) \;=\; 2\pi^{-\frac{1}{2}}\int_z^\infty \mathrm{e}^{-s^2}ds, \tag{3 72}$$

whereas when Berry [3] refers to the error function he is using the alternative definition

$$\mathrm{erf}_B\,(z) = \pi^{-\frac{1}{2}}\int_{-\infty}^z \mathrm{e}^{-s^2}ds \tag{3 73}$$

In fact, $\mathrm{erf}_B\,(z) = \mathrm{erfc}\,(-z)/2$ It is also important to notice that the error function mentioned by Berry is the *change* in the Stokes multiplier across a Stokes line, while the complementary error function arising in the formulae of Nikishov and Ritus describes a part of the Stokes multiplier but is not so easily quantified For instance, in the case of $\mathrm{Ai}(\lambda)$, the multiplier would be given by $S = \tilde{w} \pm \frac{1}{2}\mathrm{erfc}(w)$ where both $\tilde{w}$ and w depend on z^* (see §2 2 7)

The difference in argument in the erfc term affects the range over which the change in the multiplier occurs This may be explained by the fact that the truncation of the integral at z^* does not correspond to optimal truncation of the series In any case it is interesting to observe how the multiplier emerges as easily from the 'tail' of the integral as it does from the 'tail' of the series

3.7 Natural Variables

While investigating the behaviour of the solutions of Airy's equation

$$\frac{d^2y}{d\lambda^2} = \lambda y(\lambda), \tag{3 74}$$

as $|\lambda| \to \infty$ [48], Meyer extols the virtues of the 'natural metric'

$$\tilde{s} = \int^\lambda \zeta^{\frac{1}{2}}d\zeta = 2\lambda^{\frac{3}{2}}/3 \tag{3 75}$$

From a scientific point of view, the most important characteristic of the solutions of (3 74) is their wavelike structure Thus he claims the natural metric plays a fundamental role as its inherent multivaluedness provides the most effective means of capturing such a structure and, moreover, it actually measures distance in units of the local wavelength In fact, if $y(\lambda)$ is transformed to $y(\tilde{s})$ in the original equation (3 74) (or in a more general instance), then its waveform becomes more distinct and any confusion in the classification of turning points versus singular points disappears as both have become singular under the transformation He also comments on how the natural metric plays the role of a first 'instalment' in the accurate determination of an exact wavelength for the case of strictly periodic solutions of wave equations and thus is a useful analytical tool In conclusion, by using this new 'variable', the non-uniformity of the asymptotic approximation, which initially seemed daunting, can be explained quite elegantly, with the Stokes phenomenon playing a key role

The importance of such a variable is illustrated by the role the singulant, F, plays in the hyperasymptotic scheme for differential equations outlined in §3 3 and the role λF_{km} play for the integral methods in §3 1, §3 2 Hence it can be argued that the methods presented in this chapter which operate in the Borel plane, λs, are making use of the natural metric to uncover the wavelike structure of the integral solutions and thus encounter the advantages mentioned above

Chapter 4

Multidimensional Integrals: An Introduction

4.1 Introduction

The purpose of this chapter is to review existing work on asymptotics of multidimensional integrals, before developing our own methods in Chapters 5 and 6

Many techniques exist for finding the asymptotic behaviour of single integrals Different approaches have been used for different classes of integrals However, when it comes to applying these techniques to integrals of higher dimensions most textbooks cleverly avoid the question The most one is likely to find is a brief treatment of higher dimensional integrals of Laplace or Fourier type This is because of the difficulties introduced by the consideration of several complex variables, whereas the analysis of several real variables carries over much more easily from the analysis of a single real variable Interesting phenomena which were not encountered in one dimension now appear—some as a result of the increasing complexity of the underlying geometry of the problem Also the computational effort now required to calculate coefficients, once the form of the expansion is known, spirals Despite these obstacles, an interest in the asymptotic expansion of multidimensional integrals has sprung up in recent years motivated by their natural role in the explanation of physical phenomena—even infinite-dimensional integrals have been known to occur in statistical mechanics and quantum field theory where instantons are represented by small exponential terms Reducing the integral to a single integral by a suitable transformation or viewing it as a repeated integral seem to be the favourite means of proceeding

4.2 Preliminaries

Though the investigation of multidimensional integrals has been taken in different directions by different authors, there are some definitions and theorems which are needed again and again

Critical Points Critical point of the first kind A stationary point of the real-valued function $f(\mathbf{t})$, $\mathbf{t} \in \mathcal{D}$, is a point $\mathbf{t}^0$ in $\mathcal{D}$ or on its boundary at which the gradient of f vanishes i e $\nabla f(\mathbf{t}^0) = 0$ or equivalently

$$\frac{\partial f}{\partial t_1}(\mathbf{t}^0) = \quad = \frac{\partial f}{\partial t_n}(\mathbf{t}^0) = 0 \tag{4 1}$$

A stationary point is said to be non-degenerate if the Hessian matrix i e $A = (\frac{\partial^2 f}{\partial t_i \partial t_j})$ is *non*-singular at $\mathbf{t}^0$, i e $\det A \mid_{\mathbf{t}=\mathbf{t}^0} \neq 0$

Critical point of the second kind this is a point, $\mathbf{t}^0$, on the boundary of $\mathcal{D}$ at which $\nabla f(\mathbf{t}^0) \neq 0$, but at which f, treated as a function of a single variable, has a stationary point

Critical point of the third kind this is a point on the boundary of $\mathcal{D}$ where $\mathcal{D}$ has a discontinuously turning tangent and so corner critical points can result

There are also other types of critical points which can arise but they are not discussed here or in subsequent sections These would be cases where, for instance, instead of having isolated critical points the function $f(t_1, t_2)$ could exhibit a ridge or curve of critical points

Minimax Contour [22] Let γ^* be a smooth curve/contour in the complex plane, $z^0 \in \gamma^*$ and $f(z)$ a function which is analytic in the neighbourhood of γ^* Suppose $M_{\gamma^*} = \max_{z \in \gamma^*} \Re f(z)$ is attained only at the point z^0 If z^0 is either a saddle point or an endpoint of the contour, then γ^* cannot be moved into a domain of smaller values of $\Re f(z)$ i e it is impossible to find another contour γ such that

$$M_\gamma = \max_{z \in \gamma} \Re f(z) < \max_{z \in \gamma^*} \Re f(z) = M_{\gamma^*} \tag{4 2}$$

Therefore, the contour γ^* which passes through the saddle point z^0 has the minimax property

$$M_{\gamma^*} = \min_{\gamma'} \max_{z \in \gamma'} \Re f(z), \tag{4 3}$$

where γ' ıs any contour such that the ıntegral along ıt and γ are equal Lıkewıse, for an analytıc functıon $f(\mathbf{z})$, $\mathbf{z} \in \mathbb{R}^n$, ıf $\mathcal{D}$ ıs an n-dımensıonal smooth compact manıfold then $\mathcal{D}^*$ denotes a mınımax manıfold ıf

$$\mathcal{D}^* = \min_{\mathcal{D}' \in \Delta} \max_{\mathbf{z} \in \mathcal{D}'} \Re f(\mathbf{z}), \tag{4 4}$$

where Δ ıs the set of all manıfolds havıng the same boundary as the orıgınal manıfold of ıntegratıon, $\mathcal{D}$

Morse's Lemma [83] Let $\mathbf{t}^0 \in \mathbb{R}^n$ be a nondegenerate statıonary poınt of the C^∞ real-valued functıon $f(\mathbf{t})$ Then there exıst neıghbourhoods U, V of the poınts $\mathbf{y} = \mathbf{0}$, $\mathbf{t} = \mathbf{t}^0$ and a dıffeomorphısm h $U \to V$ of class C^∞ such that

$$(f \circ h)(\mathbf{y}) = f(\mathbf{t}^0) + \frac{1}{2}\langle A\mathbf{y}, \mathbf{y}\rangle, \tag{4 5}$$

where A ıs the Hessıan matrıx at $\mathbf{t}^0$ and $\langle \, , \, \rangle$ ıs the ınner product ın $\mathbb{R}^n$ Furthermore the Jacobıan of the transformatıon satısfies

$$\left. \frac{\partial(t_1, \quad , t_n)}{\partial(y_1, \quad , y_n)} \right|_{\mathbf{y}=\mathbf{0}} = 1 \tag{4 6}$$

Or more sımply, ıf $\mathbf{t}_0$ ıs a non-degenerate statıonary poınt of $f(\mathbf{t})$ then by a change of varıables $\mathbf{t} = \phi(\mathbf{y})$, $f(\mathbf{t})$ can be reduced locally to the form

$$f(\mathbf{t}) = f(\mathbf{t}_0) + \frac{1}{2}\sum_{j=1}^{n} \mu_j y_j^2, \tag{4 7}$$

where $\mu_1, \quad , \mu_n$ are the eıgenvalues of $f''(\mathbf{t}_0)$

Resolutıon of Multıple Integrals [83] Let Ω be a bounded domaın contaınıng the orıgın 0 ın $\mathbb{R}^n$ and let $\phi(t_1, \quad , t_n)$ be a C^2-functıon ın Ω Denote by M and m the supremum and ınfimum of ϕ ın Ω, respectıvely If

(ı) Ω can be covered by the famıly of surfaces determıned by

$$\phi(t_1, \quad , t_n) = \tau, \qquad m < \tau < M, \tag{4 8}$$

ın such a way that through each poınt $(t_1, \quad , t_n)$ of Ω there passes one, and only one, surface, and

(ıı) the gradıent $\nabla\phi = (\phi_{t_1}, \quad , \phi_{t_n})$ ıs nowhere zero on $\phi(t_1, \quad , t_n) = \tau$ for $\tau \in (m, M)$, then for any contınuous functıon ψ ın Ω the multıple ıntegral

$$I = \int \int_\Omega \psi(t_1, \quad , t_n)\, dt_1 \quad dt_n \tag{4 9}$$

can be reduced to the single integral

$$I = \int_m^M h(\tau)\, d\tau, \qquad (4\ 10)$$

where $h(\tau)$ is the surface integral given by

$$h(\tau) = \int_{\phi=\tau} \frac{\psi(t_1, \quad , t_n)}{|\nabla\phi|}\, d\sigma, \qquad (4\ 11)$$

$d\sigma$ being the surface element on $\phi = \tau$ and $|\nabla\phi| = \sqrt{\phi_{t_1}^2 + \quad + \phi_{t_n}^2}$

Leray-Gel'fand differential form [22] The differential form ω_s is a form of degree $n-1$ such that

$$ds \wedge \omega_s = dt_1 \wedge dt_2 \wedge \quad \wedge dt_n \qquad (4\ 12)$$ [1]

It is uniquely defined on the level set $\Gamma_c \;\; s(\mathbf{t}) = c$ if $\nabla s(\mathbf{t}) \neq 0$ on this set At the points of Γ_c for which $\partial s/\partial t_j \neq 0$, ω_s is given as

$$\omega_s(\mathbf{t}) = (-1)^j \frac{dt_1 \wedge \quad \wedge dt_{j-1} \wedge dt_{j+1} \wedge \quad \wedge dt_n}{\partial s(\mathbf{t})/\partial t_j} \qquad (4\ 13)$$

The Leray-Gel'fand form has a simple geometric meaning

$$\omega_s(\mathbf{t}) = \frac{\partial\sigma(\mathbf{t})}{|\nabla s(\mathbf{t})|}, \qquad (4\ 14)$$

where $\sigma(\mathbf{t})$ is the area on the hypersurface $s(\mathbf{t}) = c$

Cauchy's integral formula and the definition of a power series expansion can be extended to functions of several complex variables Here the case of two complex variables only is considered for ease of notation In the following, the neighbourhood of (z_1^0, z_2^0) referred to is that of the bidisc

$$\mathcal{D}' = \{\mathbf{z} \in \mathbb{C}^2 \;\; |z_1 - z_1^0| \le \varepsilon, |z_2 - z_2^0| \le \varepsilon\} \qquad (4\ 15)$$

Cauchy's Integral Theorem [42] If f is holomorphic in a neighbourhood $\mathcal{D}'$ of (z_1^0, z_2^0) then for all (z_1, z_2) in this neighbourhood the following holds

$$f(z_1, z_2) = \left(\frac{1}{2\pi i}\right)^2 \oint\oint_{\partial\mathcal{D}'} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)}\, d\zeta_1\, d\zeta_2 \qquad (4\ 16)$$

[1] '$\wedge$' denotes the exterior product

Power Series Expansion [42] A function f, holomorphic in $\mathcal{D}'$, has a power series expansion

$$f(z_1, z_2) = \sum a_{jk}(z_1 - z_1^0)^j (z_2 - z_2^0)^k, \qquad (4\ 17)$$

where the coefficients a_{jk} are given by

$$a_{jk} = \frac{1}{j!k!}\left(\frac{\partial}{\partial z_1}\right)^j \left(\frac{\partial}{\partial z_2}\right)^k f(z_1^0, z_2^0) \qquad (4\ 18)$$

4.3 Double Integrals of Laplace Type

4 3 1 Reduction to Single Integral

Wong [83] discusses double integrals of Laplace type, which arise, for instance, in problems of the diffraction theory of optics

$$I(x) = \int\int_{\mathcal{D}} g(t_1, t_2) e^{x f(t_1, t_2)}\, dt_1\, dt_2, \qquad (4\ 19)$$

where x is a large positive parameter, f, g are real-valued C^∞ functions in the closure of $\mathcal{D}$ and $I(x)$ converges absolutely for all large values of x First it is shown how the dominant contribution in the asymptotic expansion of $I(x)$ comes from the points where $f(t_1, t_2)$ attains its absolute maximum Suppose, for simplicity, this maximum is zero and occurs only at $(0,0)$ If $(0,0)$ is an interior point then

$$\frac{\partial f}{\partial t_1}(0,0) = \frac{\partial f}{\partial t_2}(0,0) = 0 \qquad (4\ 20)$$

Having eliminated the cross-product term, $t_1 t_2$ by a transformation, the Maclaurin expansion of $f(t_1, t_2)$ is of the form

$$\begin{aligned} f(t_1, t_2) &= \frac{\partial^2 f}{\partial {t_1}^2}(0,0)t_1^2 + \frac{\partial^2 f}{\partial {t_2}^2}(0,0)t_2^2 + \sum_{i+j=3} \frac{\partial^2 f}{\partial t_i \partial t_j}(0,0)t_1^i t_2^j + \\ &= \frac{\partial^2 f}{\partial {t_1}^2}(0,0)t_1^2[1 + P(t_1, t_2)] + \frac{\partial^2 f}{\partial {t_2}^2}(0,0)t_2^2[1 + Q(t_1, t_2)], \qquad (4\ 21) \end{aligned}$$

with $\frac{\partial f^2}{\partial {t_1}^2}(0,0)$, $\frac{\partial f^2}{\partial {t_2}^2}(0,0)$ both negative and P, Q power series in t_1,t_2 satisfying

$$P(0,0) = Q(0,0) = 0 \qquad (4\ 22)$$

Then the change of variables

$$u_1 = t_1[1 + P(t_1, t_2)]^{\frac{1}{2}}, \qquad u_2 = t_2[1 + Q(t_1, t_2)]^{\frac{1}{2}}, \qquad (4\ 23)$$

is used in the integral so that

$$f(t_1,t_2) = \frac{\partial^2 f}{\partial {t_1}^2}(0,0)u_1^2 + \frac{\partial^2 f}{\partial {t_2}^2}(0,0)u_2^2, \tag{4.24}$$

$\mathcal{D}'$ is the image of $\mathcal{D}$ and $\frac{\partial(t_1,t_2)}{\partial(u_1,u_2)}\Big|_{(0,0)} = 1$. Writing

$$G(u_1,u_2) = g(t_1,t_2)\frac{\partial(t_1,t_2)}{\partial(u_1,u_2)}, \qquad F(u_1,u_2) = \frac{\partial^2 f}{\partial {t_1}^2}(0,0)u_1^2 + \frac{\partial^2 f}{\partial {t_2}^2}(0,0)u_2^2, \tag{4.25}$$

gives

$$I(x) = \iint_{\mathcal{D}'} G(u_1,u_2)\mathrm{e}^{xF(u_1,u_2)}\,du_1\,du_2 \tag{4.26}$$

Then the method of resolution of multiple integrals leads to the following result

$$I(x) = \int_m^0 h(\tau)\mathrm{e}^{x\tau}\,d\tau, \qquad h(\tau) = \int_{F(u_1,u_2)=\tau} \frac{G(u_1,u_2)}{\sqrt{(\frac{\partial F}{\partial u_1})^2 + (\frac{\partial F}{\partial u_2})^2}}\,d\sigma, \tag{4.27}$$

σ being the arc length of $F(u_1,u_2) = \tau$, $m = \min\{f(t_1,t_2) : (t_1,t_2) \in \mathcal{D}'\}$. After some manipulation, $h(\tau)$ can be expanded as

$$h(\tau) \sim \frac{1}{\sqrt{\frac{\partial^2 f}{\partial {t_1}^2}(0,0)\frac{\partial^2 f}{\partial {t_2}^2}(0,0)}} \sum_{i,j\in\mathbb{N}} b_{ij}\tau^{i+j}, \qquad \tau \to 0^+, \tag{4.28}$$

where b_{ij} can be expressed in terms of derivatives of f,g (e.g. $b_{00} = \pi g(0,0)$). Then applying Watson's Lemma to the integral $I(x)$ gives

$$I(x) \sim \frac{1}{\sqrt{\frac{\partial^2 f}{\partial {t_1}^2}(0,0)\frac{\partial^2 f}{\partial {t_2}^2}(0,0)}} \sum_{i,j\in\mathbb{N}} b_{ij}\frac{\Gamma(i+j+1)}{x^{i+j+1}}, \qquad x \to \infty \tag{4.29}$$

In terms of the original variables of $f(t_1,t_2)$, the leading term becomes

$$I(x) \sim \frac{2\pi}{x}g(0,0)|\det f''(0,0)|^{-\frac{1}{2}}\mathrm{e}^{xf(0,0)} \tag{4.30}$$

(where $\det f''(0,0)$ denotes $(\frac{\partial^2 f}{\partial {t_1}^2}\frac{\partial^2 f}{\partial {t_2}^2} - (\frac{\partial^2 f}{\partial t_1 \partial t_2})^2)\,|_{(0,0)}$). Wong also shows how a critical point $(0,0)$ of $f(t_1,t_2)$ on the boundary of $\mathcal{D}$ results in

$$I(x) \sim \sum_{s=0}^{\infty} b_s\Gamma(\tfrac{s}{2}+1)x^{-\frac{s}{2}-1}, \qquad x \to \infty, \tag{4.31}$$

with leading term

$$I(x) \sim \frac{\pi}{x}g(0,0)|\det f''(0,0)|^{-\frac{1}{2}}\mathrm{e}^{xf(0,0)}, \tag{4.32}$$

whereas a maximum at $(0,0)$ on the boundary of $\mathcal{D}$ but at which $\nabla f(0,0) \neq 0$ gives

$$I(x) \sim \sum_{i,j \in \mathbb{N}} b_{ij} \frac{\Gamma(i+j+\frac{3}{2})}{x^{i+j+\frac{3}{2}}}, \qquad x \to \infty, \tag{4 33}$$

with first term

$$I(x) \sim \left(\frac{2\pi}{\kappa}\right)^{\frac{1}{2}} g(0,0) e^{xf(0,0)} x^{-\frac{3}{2}} \tag{4 34}$$

(where κ is a term involving the curvature of the boundary at $(0,0)$) The latter type of critical point arises as a result of the restriction of the function $f(t_1,t_2)$ to the boundary of the domain $\mathcal{D}$ The other types of critical points which can arise are not discussed here

4.3 2 Transformation to Repeated Integrals

Bleistein and Handelsman [9] adopt a different approach while dealing with double Laplace integrals Again the behaviour of equation (4 19) is considered, where the domain, $\mathcal{D}$, is now assumed to be finite, simply connected and bounded by a smooth curve Γ and $f(t_1,t_2)$, $g(t_1,t_2)$ are sufficiently differentiable as before As anticipated, the asymptotic results depend on those points in the closure of $\mathcal{D}$ at which f achieves its absolute maximum They are interested initially only in the case where such points are interior $\mathbf{t}^0 = (t_1^0, t_2^0)$ If there is more than one such point, the domain can be subdivided into regions each of which contain only one such point So it is assumed that

$$\nabla f(t_1^0,t_2^0) = 0, \qquad \left(\frac{\partial^2 f}{\partial {t_1}^2}\frac{\partial^2 f}{\partial t_1 \partial t_2} - \left(\frac{\partial^2 f}{\partial {t_2}^2}\right)^2\right)\Bigg|_{(t_1^0,t_2^0)} > 0, \qquad \frac{\partial^2 f}{\partial {t_1}^2}(t_1^0,t_2^0) < 0, \tag{4 35}$$

and that ∇f is nonzero elsewhere in $\mathcal{D}$ Then the local behaviour of f is considered

$$\begin{aligned} f(t_1,t_2) &= f(t_1^0,t_2^0) + \frac{\partial^2 f}{\partial {t_1}^2}(t_1^0,t_2^0)\frac{(t_1-t_1^0)^2}{2} + \frac{\partial^2 f}{\partial t_1 \partial t_2}(t_1^0,t_2^0)(t_1-t_1^0)(t_2-t_2^0) \\ &\quad + \frac{\partial^2 f}{\partial {t_2}^2}(t_1^0,t_2^0)\frac{(t_2-t_2^0)^2}{2} + \end{aligned} \tag{4 36}$$

The method used reduces $I(x)$ to an integral of canonical form in which $f(t_1,t_2)$ is replaced by a quadratic function by setting

$$Q^T A Q = \Lambda = \mathrm{diag}(\lambda_1,\lambda_2), \qquad \lambda_1\lambda_2 = \det A, \tag{4 37}$$

where A is the Hessian matrix at (t_1^0, t_2^0) Then $\mathbf{v} = (v_1, v_2)$ and $\overline{f}(\mathbf{v})$ are introduced by

$$(\mathbf{t} - \mathbf{t}^0)^T = QR\mathbf{v}^T, \qquad \overline{f}(\mathbf{v}) = f(\mathbf{t}^0) - f(\mathbf{t}) \tag{4 38}$$

So near $\mathbf{v} = 0$ (i e $\mathbf{t} = \mathbf{t}^0$),

$$\overline{f}(\mathbf{v}) \sim \frac{1}{2}(v_1^2 + v_2^2) \tag{4 39}$$

—for this to hold outside a neighbourhood of $\mathbf{v} = 0$, the following transformation is made

$$\begin{array}{lll} \mathbf{u} = (u_1, u_2), & u_i = h_i(\mathbf{v}), & |\mathbf{u}^2| = h_1^2 + h_2^2 = 2\overline{f}(\mathbf{v}), \\ h_1 = v_1 + o(|\mathbf{v}|), & h_2 = v_2 + o(|\mathbf{v}|), & |\mathbf{v}| \to 0 \end{array} \tag{4 40}$$

Thus

$$I(x) = \mathrm{e}^{xf(t_1^0, t_2^0)} \iint_{\mathcal{D}'} g(\mathbf{t}(u_1, u_2)) \left| \frac{\partial(t_1, t_2)}{\partial(u_1, u_2)} \right| \mathrm{e}^{-\frac{x}{2}(u_1^2 + u_2^2)}\, du_1\, du_2, \tag{4 41}$$

where $\mathcal{D}'$ is the image of $\mathcal{D}$ Using the divergence theorem it can then be shown that the behaviour of this integral is dominated by the critical point at $(0, 0)$ giving

$$I(x) \sim \mathrm{e}^{xf(t_1^0, t_2^0)} g(\mathbf{t}(0, 0)) \left| \frac{\partial(t_1, t_2)}{\partial(u_1, u_2)} \right|_{(0,0)} \iint_{\mathcal{D}'} \mathrm{e}^{-\frac{x}{2}(u_1^2 + u_2^2)}\, du_1\, du_2 \tag{4 42}$$

Then according to the following lemma of Bleistein and Handelsman

Lemma If $\mathbf{u} = 0$ is an interior point of $\mathcal{D}'$ then as $x \to \infty$

$$\iint_{\mathcal{D}'} \mathrm{e}^{-\frac{x}{2}(u_1^2 + u_2^2)}\, du_1\, du_2 = \frac{2\pi}{x} + o(x^{-m}), \tag{4 43}$$

for any m,

the integral becomes

$$\begin{aligned} I(x) &\sim \frac{2\pi}{x} \mathrm{e}^{xf(t_1^0, t_2^0)} g(\mathbf{t}(0, 0)) \left| \frac{\partial(t_1, t_2)}{\partial(u_1, u_2)} \right|_{(0,0)} \\ &\sim \frac{2\pi}{x} \mathrm{e}^{xf(t_1^0, t_2^0)} \frac{g(t_1^0, t_2^0)}{\left| \frac{\partial^2 f}{\partial t_1^2}(t_1^0, t_2^0) \frac{\partial^2 f}{\partial t_2^2}(t_1^0, t_2^0) - \left(\frac{\partial^2 f}{\partial t_1 \partial t_2}(t_1^0, t_2^0)\right)^2 \right|^{\frac{1}{2}}}, \end{aligned} \tag{4 44}$$

in terms of the original variables Note how the result is independent of the selection of h_1, h_2 Boundary extrema are also looked at in a similar fashion and the same results arrived at as Wong

4.3.3 Alternative Transformation

Dingle [17] has shown how a convergent series $S(x) = \sum a_s x^s$ may be converted into an integral representation in terms of more familiar series in order to determine its asymptotic power series But he also recognised that it may not be possible, in doing so, to find an integral operator of a single variable As a result, he remarks on the importance of determining a means of deriving asymptotic expansions of double integrals of the form

$$\int\int_{\mathcal{D}} g(t_1, t_2) e^{f(t_1,t_2)}\, dt_1\, dt_2, \tag{4 45}$$

where either $f(t_1, t_2)$ or $g(t_1, t_2)$ can depend on the asymptotic parameter x and $f(t_1, t_2)$ tends to $-\infty$ along the boundary of $\mathcal{D}$ Although the procedure he details encounters no major difficulties compared to his method of expanding a single integral, he realises that the rapid increase in the number of partial derivatives required to obtain more than the first two terms of the expansion renders the method intractably cumbersome Therefore, he advocates that such double integration be avoided where possible!

Suppose that $f(t_1, t_2)$ has a stationary point at $(0, 0)$ so that

$$\frac{\partial f}{\partial t_1}(0,0) = \frac{\partial f}{\partial t_2}(0,0) = 0 \tag{4 46}$$

Thus by Taylor's theorem $f(t_1, t_2)$ can be written as

$$\begin{aligned} f(t_1,t_2) = f(0,0) &+ \tfrac{1}{2}\left\{t_1^2 \tfrac{\partial^2 f}{\partial {t_1}^2}(0,0) + 2t_1 t_2 \tfrac{\partial^2 f}{\partial t_1 \partial t_2}(0,0) + t_2^2 \tfrac{\partial^2 f}{\partial {t_2}^2}(0,0)\right\} \\ &+ \tfrac{1}{6}\left\{t_1^3 \tfrac{\partial^3 f}{\partial {t_1}^3}(0,0) + 3t_1^2 t_2 \tfrac{\partial^3 f}{\partial t_1^2 \partial t_2}(0,0) + 3t_1 t_2^2 \tfrac{\partial^3 f}{\partial t_1 \partial t_2^2}(0,0) + t_2^3 \tfrac{\partial^3 f}{\partial {t_2}^3}(0,0)\right\} + \end{aligned} \tag{4 47}$$

Only the constant term $f(0, 0)$ and the quadratic group are kept in exponential form for the integration and it makes sense to transform the quadratic group as follows

$$\tilde{t}_1 = t_1, \qquad \tilde{t}_2 = t_2 + t_1 \left\{ \frac{\partial^2 f}{\partial t_1 \partial t_2}(0,0) \Big/ \frac{\partial^2 f}{\partial {t_2}^2}(0,0) \right\}, \tag{4 48}$$

to give

$$\begin{aligned} &\frac{1}{2}\left\{ t_1^2 \frac{\partial^2 f}{\partial {t_1}^2}(0,0) + 2t_1 t_2 \frac{\partial^2 f}{\partial t_1 \partial t_2}(0,0) + t_2^2 \frac{\partial^2 f}{\partial {t_2}^2}(0,0) \right\} \\ &\qquad = \frac{1}{2}\tilde{t}_1 \left\{ \frac{\partial^2 f}{\partial {t_1}^2}(0,0) - \frac{\partial^2 f}{\partial t_1 \partial t_2}(0,0) \Big/ \frac{\partial^2 f}{\partial {t_2}^2}(0,0) \right\} + \frac{1}{2}\tilde{t}_2^2 \frac{\partial^2 f}{\partial {t_2}^2}(0,0) \end{aligned} \tag{4 49}$$

Taylor's theorem is also used on both the slowly varying factor $g(t_1, t_2)$ and what is 'left over' as such from the exponential

$$\exp\left[\frac{1}{6}\left(t_1^3\frac{\partial^3 f}{\partial {t_1}^3}(0,0) + 3t_1^2 t_2\frac{\partial^3 f}{\partial t_1^2 \partial t_2}(0,0) + 3t_1 t_2^2\frac{\partial^3 f}{\partial t_1 \partial t_2^2}(0,0) + t_2^3\frac{\partial^3 f}{\partial {t_2}^3}(0,0)\right) + \quad\right], \tag{4 50}$$

the variables are transformed to $\tilde{t}_1, \tilde{t}_2$ and the subsequent expansions then multiplied together A tedious integration from $-\infty$ to ∞ over $\tilde{t}_1$ and $\tilde{t}_2$ successively, eventually leads to the result

$$\int\!\!\int_{\mathcal{D}} g(t_1, t_2) e^{f(t_1,t_2)}\, dt_1\, dt_2 = \frac{2\pi e^{f(0,0)}}{\left(-\left(\frac{\partial^2 f}{\partial {t_1}^2}\frac{\partial^2 f}{\partial {t_2}^2} - \left(\frac{\partial^2 f}{\partial t_1 \partial t_2}\right)^2\right)\Big|_{(0,0)}\right)^{\frac{1}{2}}}(Q_0 + Q_2 + \quad), \tag{4 51}$$

with

$$Q_0 = g(0,0), \tag{4 52}$$

$$\begin{aligned} Q_2 = {} & \frac{1}{24\left(-\left(\frac{\partial^2 f}{\partial {t_1}^2}\frac{\partial^2 f}{\partial {t_2}^2} - \left(\frac{\partial^2 f}{\partial t_1 \partial t_2}\right)^2\right)\Big|_{(0\,0)}\right)^{\frac{3}{2}}}\left[g\left\{\left(-\frac{\partial^2 f}{\partial {t_2}^2}\right)^3\left(-5\left(\frac{\partial^3 f}{\partial {t_1}^3}\right)^2 - 3\frac{\partial^2 f}{\partial {t_1}^2}\frac{\partial^4 f}{\partial {t_1}^4}\right) + \right.\right. \\ & \left(\frac{\partial^2 f}{\partial {t_1}^2}\right)^3\left(-5\left(\frac{\partial^3 f}{\partial {t_2}^3}\right)^2 - 3\frac{\partial^2 f}{\partial {t_2}^2}\frac{\partial^4 f}{\partial {t_2}^4}\right) + 3\frac{\partial^2 f}{\partial {t_2}^2}\frac{\partial^2 f}{\partial {t_1}^2}\left(\frac{\partial^2 f}{\partial {t_2}^2}\left[2\frac{\partial^3 f}{\partial t_1 \partial t_2^2}\frac{\partial^3 f}{\partial {t_1}^3} + 3\left(\frac{\partial^3 f}{\partial {t_1}^2 \partial t_2}\right)^2\right]\right. \\ & \left.\left.\left. + \frac{\partial^2 f}{\partial {t_1}^2}\left[2\frac{\partial^3 f}{\partial {t_1}^2 \partial t_2}\frac{\partial^3 f}{\partial {t_2}^3} + 3\left(\frac{\partial^3 f}{\partial t_1 \partial {t_2}^2}\right)^2\right] - 2\frac{\partial^2 f}{\partial {t_2}^2}\frac{\partial^2 f}{\partial {t_1}^2}\frac{\partial^4 f}{\partial t_1^2 \partial t_2^2}\right) + \quad\right\}\right]\Bigg|_{(0,0)}, \end{aligned} \tag{4 53}$$

and so on

4.4 Higher Dimensional Integrals of Laplace Type

Although Wong states that his approach of reducing a given 2-dimensional integral to a single integral carries through to higher dimensions, he chooses instead to adopt a method not unlike the approach of Bleistein and Handelsman to determine the asymptotic expansion in this case In fact, it could be said that all treatments of higher dimensional Laplace type integrals to be found in the standard literature[2] seem to converge to an application of Morse's Lemma and it is assumed that this has been discovered to be the most advantageous means of proceeding Some of the variations on this procedure are detailed briefly in what follows Despite the fact that the analysis of higher dimensional integrals is more complicated than before,

[2]More recent developments are discussed in §4 6

it is important to mathematically investigate their behaviour as they do appear in physical problems such as that of the scattering of radiation by obstacles

Wong [83] considers higher dimensional integrals of the form

$$I(x) = \int_{\mathcal{D}} g(\mathbf{t}) e^{x f(\mathbf{t})} \, d\mathbf{t}, \tag{4 54}$$

where x is a large positive parameter, $\mathcal{D}$ is a domain in $\mathbb{R}^n$ and f, g are real valued C^∞ functions in $\mathcal{D}$ It is assumed that $I(x)$ converges absolutely for all $x \geq x^0$ and the Hessian matrix, $A = (\frac{\partial^2 f}{\partial t_i \partial t_j})|_{\mathbf{t}=\mathbf{t}^0}$, is negative definite (thus $f(\mathbf{t})$ has a maximum at $\mathbf{t}^0$ only) Again it is shown that the dominant contribution to the asymptotic expansion comes from $\mathcal{D}_0$, the neighbourhood of the maximum $\mathbf{t}^0$ $\mathcal{D}_0$ is chosen so that the Morse Lemma can be applied, i e there exists a diffeomorphism h $\Omega \to \mathcal{D}_0$, with $\mathbf{y} = 0 \in \Omega$ such that $\mathbf{t} = h(\mathbf{y})$ gives

$$I(x) = e^{x f(\mathbf{t}^0)} \int_\Omega G(\mathbf{y}) e^{-\frac{x}{2} \sum_{i=1}^n \mu_i y_i^2} \, d\mathbf{y} \tag{4 55}$$

with $\Omega = h^{-1}(\mathcal{D}_0)$ and $G(\mathbf{y}) = g(h(\mathbf{y})) \det h'(\mathbf{y})$ Then using Taylor's theorem

$$G(\mathbf{y}) = \sum_{|\alpha|<p} \frac{1}{\alpha!} D^\alpha G(\mathbf{0}) \mathbf{y}^\alpha + R_p, \qquad R_p = \sum_{|\alpha|=p} \frac{1}{\alpha!} D^\alpha G(\xi) \mathbf{y}^\alpha, \qquad \xi \in \Omega, \tag{4 56}$$

where p is the point of truncation of the series Termwise integration leads to

$$\begin{aligned} I(x) &= e^{x f(\mathbf{t}^0)} \left[\sum_{|\alpha|<p} \frac{1}{\alpha!} D^\alpha G(\mathbf{0}) \int_\Omega \mathbf{y}^\alpha e^{-\frac{x}{2} \sum_{i=1}^n \mu_i y_i^2} \, d\mathbf{y} \right. \\ &\qquad \left. + \sum_{|\alpha|=p} \frac{1}{\alpha!} \int_\Omega D^\alpha G(\xi) \mathbf{y}^\alpha e^{-\frac{x}{2} \sum_{i=1}^n \mu_i y_i^2} \, d\mathbf{y} \right] \end{aligned} \tag{4 57}$$

Using the identity

$$\int_{-\infty}^{\infty} t^m e^{-\nu t^2} \, dt = \begin{cases} 0 & \text{m odd} \\ \Gamma(\frac{m+1}{2}) \nu^{-\frac{(m+1)}{2}} & \text{m even} \end{cases}, \tag{4 58}$$

it can be shown that

$$I(x) \sim e^{x f(\mathbf{t}^0)} \sum_{k=0}^{p-1} c_k x^{-\frac{n}{2}-k}, \qquad x \to \infty, \tag{4 59}$$

where

$$c_k = \sum_{|\alpha|=k} \frac{d_\alpha}{\alpha!} D^\alpha G(\mathbf{0}), \tag{4 60}$$

and if any of the α_i in $\alpha = (\alpha_1, \quad, \alpha_n)$ is odd then

$$d_\alpha = 0, \qquad \text{otherwise} \quad d_\alpha = \left(\frac{2}{\mu}\right)^{\frac{(\alpha+1)}{2}} \Gamma\left(\frac{\alpha+1}{2}\right) \tag{4 61}$$

The leading term of the above expansion is given by

$$I(x) \sim \left(\frac{2\pi}{x}\right)^{\frac{n}{2}} g(\mathbf{t}^0)|\det A|^{-\frac{1}{2}} e^{xf(\mathbf{t}^0)} \tag{4 62}$$

As in the case of a double Laplace integral, if the maximum $\mathbf{t}^0$ is on the boundary of $\mathcal{D}$, the dominant contribution is one-half of the latter approximation Finally, the case in which the maximum is at a boundary point $\mathbf{t}^0$ at which $\nabla f(\mathbf{t}^0) \neq 0$ can be investigated using a repeated application of the divergence theorem to give

$$I(x) \sim \frac{g(\mathbf{t}^0)}{2\pi\sqrt{|J|}} e^{xf(\mathbf{t}^0)} \left(\frac{2\pi}{x}\right)^{\frac{(n+1)}{2}} \tag{4 63}$$

The method of Bleistein and Handelsman, outlined above for double integrals, can also be used to deal with integrals of higher dimensions [9] Again equation (4 54) is investigated where $\mathcal{D}$ is a simply connected domain with boundary Γ, an $(n-1)$-dimensional hypersurface Initially it is assumed that the absolute maximum is achieved only at the interior point $\mathbf{t} = \mathbf{t}^0$ so that $\nabla f(\mathbf{t}^0) = 0$ Then

$$f(\mathbf{t}) - f(\mathbf{t}^0) \approx \frac{1}{2}(\mathbf{t} - \mathbf{t}^0)A(\mathbf{t} - \mathbf{t}^0)^T, \tag{4 64}$$

where $A = (\frac{\partial^2 f}{\partial t_i \partial t_j}(\mathbf{t}^0))$, $i, j = 1, 2, \quad, n$, the Hessian matrix at $\mathbf{t}_0$ If Q is an orthogonal matrix which diagonalises A then

$$Q^t A Q = \Lambda = \operatorname{diag}(\lambda_1, \quad, \lambda_n), \qquad \det A = \prod_{i=1}^{n} \lambda_i \tag{4 65}$$

As before (equations (4 37)-(4 39)), the variables are transformed in the following way

$$(\mathbf{t} - \mathbf{t}^0)^T = QR\mathbf{v}^T, \quad R = \operatorname{diag}(|\lambda_1|^{-\frac{1}{2}}, \quad, |\lambda_n|^{-\frac{1}{2}}), \quad \overline{f}(\mathbf{v}) = f(\mathbf{t}^0) - f(\mathbf{t}) \approx \frac{1}{2}\mathbf{v}\ \mathbf{v}, \tag{4 66}$$

near $\mathbf{v} = 0$ Again choose u_i such that

$$\begin{aligned} &\mathbf{u} = (u_1, \quad, u_n), \qquad u_i = h_i(\mathbf{v}), \qquad \textstyle\sum_{i=1}^n h_i^2 = 2\overline{f}, \\ &h_i = v_i + o(|\mathbf{v}|), \qquad |\mathbf{v}| \to 0, \end{aligned} \tag{4 67}$$

for $i = 1, \quad , n$ Then $I(x)$ can be written as

$$I(x) = e^{xf(\mathbf{t}^0)} \int_{\mathcal{D}} G_0(\mathbf{u}) e^{-\frac{x}{2}\mathbf{u}\,\mathbf{u}}\, d\mathbf{u}, \tag{4 68}$$

with

$$G_0(\mathbf{u}) = g_0(\mathbf{t}(\mathbf{u}))J(\mathbf{u}), \qquad J(\mathbf{u}) = \frac{\partial(t_1, \quad , t_n)}{\partial(u_1, \quad , u_n)} \tag{4 69}$$

To find the asymptotic behaviour of $I(x)$, the n-dimensional analog of the result in equation (4 43) is used by noting that if $\mathbf{u} = 0$ lies in the interior of a domain $\mathcal{D}$, then

$$\int_{\mathcal{D}} e^{-\frac{x}{2}\mathbf{u}\,\mathbf{u}}\, d\mathbf{u} \sim \left(\frac{2\pi}{x}\right)^{\frac{n}{2}}, \qquad x \to \infty \tag{4 70}$$

This leads to

$$I(x) \sim e^{xf(\mathbf{t}^0)} \sum_{j=0}^{m-1} (2\pi)^{\frac{n}{2}} x^{-\frac{n}{2}-j} G_j(\mathbf{0}), \tag{4 71}$$

where $G_j(\mathbf{0}) = \frac{1}{2^j j!} \nabla_{\mathbf{u}}^j G_0|_{\mathbf{u}=0}$ The first term of the series would be given by

$$\frac{e^{xf(\mathbf{t})}}{\left|\det(\frac{\partial^2 f}{\partial t_i \partial t_j}(\mathbf{t}^0))\right|^{\frac{1}{2}}} \left(\frac{2\pi}{x}\right)^{\frac{n}{2}} g_0(\mathbf{t}^0) \tag{4 72}$$

Once again, critical points which occur on the boundary and boundary maxima at which $\nabla f(\mathbf{t}^0) \neq 0$ are considered and the same results are obtained as by Wong However, critical points of other types are not discussed

While considering the integral in (4 54), Fedoryuk [22] merely reduces $f(\mathbf{t})$ in a small neighbourhood of the maximum, $\mathbf{t}^0$, using Morse's Lemma so that

$$f(\mathbf{t}) = f(\mathbf{t}^0) + \frac{1}{2}\sum_{j=1}^{n} \mu_j y_j^2 \tag{4 73}$$

He then rewrites the integral as

$$e^{xf(\mathbf{t}^0)} \int_{\mathcal{V}} g(\mathbf{y}) e^{\frac{x}{2}\sum_{j=1}^{n}\mu_j y_j^2}\, d\mathbf{y}, \tag{4 74}$$

where $\mathcal{V}$ is a cube $|y_j| < \delta, \ 1 \leq j \leq n$, and states that all that remains is to apply the 1-dimensional Laplace method sequentially with respect to each y_j

4.5 Multivariate Saddlepoint Method

In his treatment of the multivariate saddlepoint method, Fedoryuk [21] remarks on the increased difficulty of selecting which saddles are needed to provide the dominant contribution to the asymptotic expansion of the integral

$$I(\lambda) = \int_{\mathcal{D}} g(\mathbf{z}) \mathrm{e}^{\lambda f(\mathbf{z})}\, d\mathbf{z}, \tag{4.75}$$

where $\mathcal{D}$ is an n-dimensional smooth manifold and $\mathbf{z} \in \mathbb{C}^n$. He maintains there are no general rules to be followed. Using a minimax manifold as the surface of integration, he verifies that the $\max_{z \in \mathcal{D}^*} \Re f(\mathbf{z})$ is attained either at a saddle of $f(\mathbf{z})$ or on the boundary of the manifold. It is then shown (drawing heavily on topology theory) how

$$I(\lambda) \sim \frac{1}{\lambda^{\frac{n}{2}}} \mathrm{e}^{\lambda f(\mathbf{z}^0)} \sum_{k=0}^{\infty} \frac{a_k}{\lambda^k}, \qquad \lambda \to \infty, \tag{4.76}$$

if $\max_{\mathbf{z} \in \mathcal{D}} \Re f(\mathbf{z})$ is attained only at $\mathbf{z}^0$, and how the leading term of this expansion is given by

$$I(\lambda) \sim \left(\frac{2\pi}{\lambda}\right)^{\frac{n}{2}} g(\mathbf{z}^0) \mathrm{e}^{\lambda f(\mathbf{z}^0)} [\det(-f''(\mathbf{z}^0))]^{-\frac{1}{2}} \tag{4.77}$$

Some ambiguity again arises in the choice of branch of the square root but it essentially depends on the deformation of $\mathcal{D}$. As in the 1-dimensional case, if the manifold of integration encounters more than one saddle then the asymptotics of the integral equal the sum of contributions of the saddles. (Only simple saddles have been considered in order to be able to make use of Morse's Lemma.)

Once again, the use of steepest descent surfaces would obviate the need for justification of the saddle point method and allow precision estimates of the error to be obtained. Briefly treating the many dimensional method of steepest descent, Fedoryuk comments that although for higher than two dimensions the geometric visualisation is immediately lost, the principles remain the same. Calling on a theorem due to Poincaré, he claims that if the boundary curve of a n-dimensional manifold is kept fixed as the manifold of integration is deformed, then $I(\lambda)$ remains unchanged—this follows from the idea of the evaluation of a contour integral being path independent once the endpoints are fixed. Again using Morse's Lemma, the phase function $f(\mathbf{z})$ can be reduced locally to a sum of squares and then the Laplace method for many dimensional integrals applies. He also shows by counterexample, how in the case of the maximum of $\Re f(z)$ being achieved on the boundary but not at a saddle, the asymptotics of the integral cannot be computed as before

4.6 Recent Work

Several people have already looked at the problem of computing the asymptotic behaviour of multidimensional integrals The possibility of extending the saddle-point method and steepest descent theory to integrals of higher dimensions is one path which has already been investigated, building on the work of Fedoryuk The techniques used require quite an understanding of the topology of the problem and thus add a certain theoretical sophistication to the investigation that would not have been observed in the single integral case In the following some of the work that has been undertaken is reviewed

In his 1978 thesis, Saxton [74] represented the solution of the $n-p$ equation,

$$y^{(n)}(x) - \sum_{r=0}^{p} a_r x^r y^{(r)}(x) = 0, \tag{4 78}$$

in terms of a p-tuple integral (drawing on a result of Spitzer [76]) and found its leading asymptotic behaviour as $x \to \infty$ This entailed substituting the integral

$$I(x) = y(x) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_p^{\alpha_p} e^{sxz_1 \cdots z_p - \frac{1}{n}(z_1^n + \cdots + z_p^n)} \, dz_1 \, dz_2 \cdots dz_p, \tag{4 79}$$

into the original equation, then using integration by parts and recurrence relations to find α_i in terms of a_i A power series solution could thus be found Initially the $n-1$ case was investigated, then integrated to deduce the $n-2$ case and by induction the $n-p$ case could be determined Riemann's saddle point method could then be used to compute the leading asymptotic behaviour, having determined when saddles or endpoints contribute Finding a complete expansion would be possible theoretically, using Debye's steepest descent method, but would be considerably more involved (Allowing the asymptotic parameter, x, to become complex introduces no added difficulty) The importance of finding the asymptotic behaviour of such solutions accurately can be seen in the occurence of these differential equations both in the fluid dynamics and magnetohydrodynamics problems of theoretical mechanics and in the deficiency index problem for symmetric differential operators of pure mathematics where leading order terms are adequate [45],[24],[19], [68] Obviously there are alternative methods for finding the asymptotic expansion of $y(x)$ directly from the differential equation, including the standard WKB approach, but such results for the particular applications of interest to Saxton were not well-known at the time

Paris and Kaminski have jointly published papers on the asymptotics of a class of Laplace-type, double and triple integrals with an isolated, though possibly degenerate, critical point at the origin [37], [38] Their double integral method involves representing

$$I(x) = \int_0^\infty \int_0^\infty g(t_1, t_2) e^{x f(t_1, t_2)} \, dt_1 \, dt_2 \tag{4 80}$$

(f polynomial in t_1, t_2) as iterated Mellin-Barnes integrals and using residue theory and Newton polygons of $f(t_1, t_2)$ (see Figure 4 1) to find the asymptotics Thus they avoided the difficulties encountered by representing $I(x)$ as an integral transform of a function defined by an integral over a lower dimensional object, as is often attempted To start, $f(t_1, t_2)$ is written as

$$f(t_1, t_2) = -(t_1^\mu + \sum_{p=1}^{k} c_p t_1^{m_p} t_2^{n_p} + t_2^\nu), \tag{4 81}$$

and the formula

$$e^{-z} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(\tau) z^{-\tau} \, d\tau \tag{4 82}$$

is used on each factor

$$e^{-x c_p t_1^{m_p} t_2^{n_p}} \tag{4 83}$$

For instance, with one internal point (i e $k = 1$)

$$I(x) = \frac{x^{-\frac{1}{\mu} - \frac{1}{\nu}}}{2\pi i \mu \nu} \int_{-i\infty}^{i\infty} \Gamma(\tau) \Gamma(\tfrac{1 - m_1 \tau}{\mu}) \Gamma(\tfrac{1 - n_1 \tau}{\nu}) x^{-\delta_1 \tau} \, d\tau \tag{4 84}$$

(with $g(t_1, t_2) = 1$, $c_p = 1$ in this case and δ_1 given by $\delta_1 = 1 - \frac{m_1}{\mu} - \frac{n_1}{\nu}$) Thus the dimensionality of the integral now depends on the number of terms in the phase function and not the dimensionality of the original integral The contributions to the asymptotic expansion of the integral from the poles of the gamma functions are computed by consecutively shifting the contours of integration left or right as appropriate The relationship between the asymptotic scales of x in the expansion and the features of the Newton diagram (such as remoteness, which is given by $-\frac{1}{d}$) are noted The triple integral approach follows along the same lines except now there is an extra gamma function with contributing poles to be considered Also

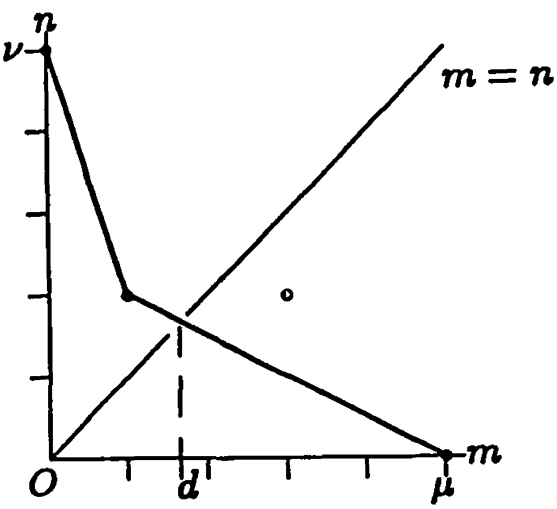


Figure 4 1 An example of a Newton polygon [37] for $f(t_1, t_2)$, where m and n represent the indices of t_1 and t_2 respectively

the relationship between the geometry of the Newton diagram and the form of the asymptotic expansion is more complicated Otherwise, increasing the dimension of the integral to 3 or higher requires little modification of the double integral method

This latter method can be applied to finding the asymptotic behaviour of the p-tuple integral solution of

$$y^{(n)}(x) - \sum_{r=0}^{p} a_r x^r y^{(r)}(x) = 0 \tag{4 85}$$

As it involves only one internal point, it is not too tedious to arrive at the same result as Saxton [74], as is shown in §4 7 1

A study of double integrals with nearly coincident saddlepoints was undertaken by Ursell [82] and he speaks of the difficulties in extending the method of steepest descent from single to double integrals He points out that surfaces of steepest descent can be generated by curves of steepest descent though these surfaces do not remain steepest surfaces under an analytic transformation, whereas curves do Thus, he argues, there is little advantage in constructing steepest surfaces Even when the surface of steepest descent has been constructed, the difficulty remains in combining it with other surfaces to render it equivalent to the original surface of integration, as does the difficulty of identifying which saddlepoint contributions are relevant Poincaré's theorem is also cited by Ursell, to claim that in the case of a 2-d surface the integral is independent of the surface spanning the boundary curve, assuming the integrand is an analytic function

Kaminski [33] has also looked at the possibility of extending the saddlepoint and steepest descent methods to higher dimensions To parallel the 1-d case of the saddlepoint method, the integral

$$I(\lambda) = \int_{\mathcal{D}} g(z_1, \quad , z_n) e^{\lambda f(z_1, \quad , z_n)} \, dz_1 \quad dz_n \tag{4 86}$$

is reduced to one over an appropriately small domain in a neighbourhood of the saddle while accumulating errors of exponentially small order, but to do this in higher dimensions requires homology theory to be called on Using the 1-d technique of setting

$$\Im(f(z_1, z_2) - f(z_1^0, z_2^0)) = 0 \tag{4 87}$$

(where $\mathbf{z}^0$ is a saddle) to find a steepest descent surface, will yield an analytic variety of real dimension $2n - 1$ while a steepest descent surface has dimension $2(n - 1)$ Thus the construction of a unique surface of steepest descent is not possible by this process (This non-uniqueness property can also be seen from the fact that steepest descent surfaces in $\mathbb{C}^n$, $n \geq 2$, are not preserved under holomorphic mappings) However, this could in practice be turned to advantage as it means that a convenient surface of descent can be used instead Kaminski outlines an approach to construct such a surface by computing its co-ordinate plane traces

$\mathcal{D}'(\sigma, 0)$ is the steepest descent curve of $f(z_1, z_2)$ through $z_1 = z_1^0$, and

$\mathcal{D}'(0, \tau)$ is the steepest curve through $z_2 = z_2^0$

Another paper of Kaminski's [34] involves appropriately determining exponentially small terms for inclusion in the asymptotic expansion of oscillatory double integrals,

$$I(\lambda) = \iint_{\mathcal{D}} e^{\lambda f(z_1, z_2)} \, dz_1 \, dz_2, \tag{4 88}$$

for which he turned again to the use of surfaces of steepest descent He points out here how Pham's deformation of $\mathbb{R}^2$ to a sum of Lefschetz thimbles (see Figure 4 2) closely parallels the steepest descent method in theory But Pham has given no indication of how to perform it practically—the idea being to replace the domain of integration by a sum of integration cycles, on each of which the imaginary part of the phase function is constant, the real part decreasing and only a single critical point of the phase is contained Kaminski uses instead an idea of Fedoryuk's which

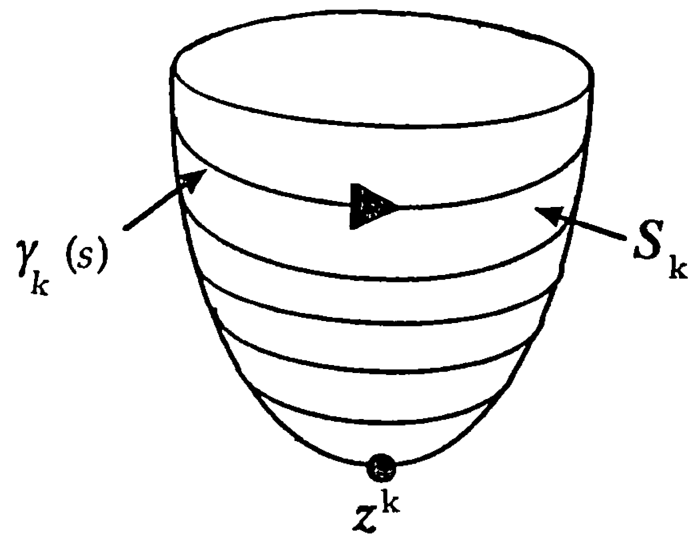


Figure 4 2 Sketch of a Lefschetz thimble [29]

examines trajectories of the system of equations

$$\left(\frac{du_1}{dt}, \frac{dv_1}{dt}, \frac{du_2}{dt}, \frac{dv_2}{dt}\right) = -\nabla \Re f(u_1, v_1, u_2, v_2) \quad \text{with } z_1 = u_1 + iv_1,\ z_2 = u_2 + iv_2, \tag{4 89}$$

to construct surfaces of steepest descent (again closely resembling what is done in the planar situation) Here the parameter t is a non-negative real number so that $\Re f(u_1(t), v_1(t), u_2(t), v_2(t))$ decreases with increasing t He defines a saddle (z_1^0, z_2^0) to be accessible from $\mathcal{D}$ if there exists a trajectory issuing from $\mathcal{D}$ such that

$$(z_1^0, z_2^0) = \lim_{t\to\infty}\{(u_1(t), v_1(t), u_2(t), v_2(t)) \quad (u_1(0), v_1(0), u_2(0), v_2(0)) \in \mathcal{D}\} \tag{4 90}$$

Then he proposes that if (z_1^0, z_2^0) is the sole critical point accessible from $\mathcal{D}$ but $\notin \mathcal{D}$, the surface Σ containing (z_1^0, z_2^0) is uniquely determined and can be obtained by the co-ordmatewise approach He goes on to seek a parametrisation (σ, τ) of Σ so that the phase function $f(z_1, z_2)$ can be expressed in the form

'constant - σ^2 - τ^2 ' (Morse's Lemma),

again parallelling what happens in the planar situation As a result of the deformation process, a volume enclosed by $\mathcal{D}$, Σ and the sides F_1, , F_4 has been created arising from the trajectories issuing from the edges of $\mathcal{D}$ (see Figure 4 3, where the arrows indicate the orientations induced on the different faces) This gives

$$I(\lambda) = I_{\mathcal{D}}(\lambda) = I_{\Sigma}(\lambda) - \sum_{i=1}^{4} I_{F_i}(\lambda) \tag{4 91}$$

(an intuitive observation requiring rigorous justification) The contributions of $I_{F_i}(\lambda)$ amount to contributions to $I(\lambda)$ made by boundary stationary points and corner points of $\mathcal{D}$, but the exponential improvement was achieved by including the decaying term $I_{\Sigma}(\lambda)$

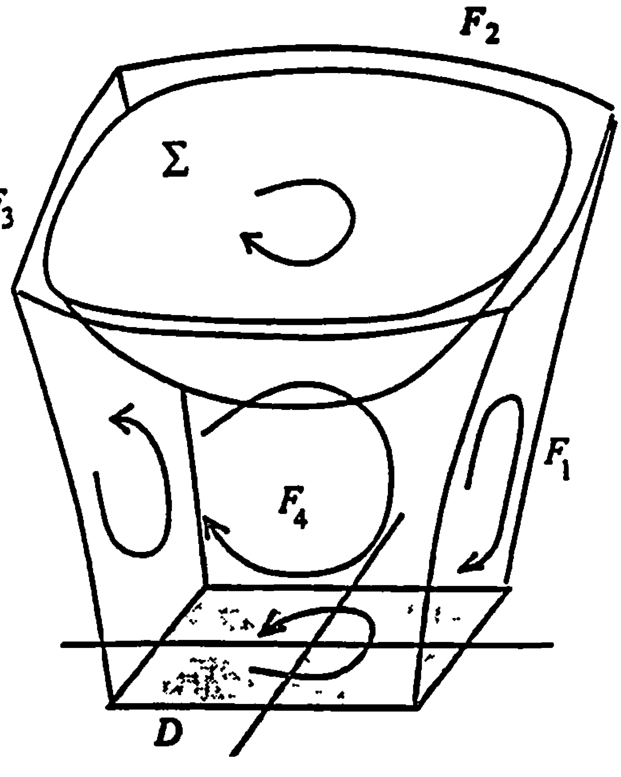


Figure 4 3 Volume enclosed by trajectories arising from edges of $\mathcal{D}$ [34]

4.7 Application to Differential Equations

4.7.1 Spitzer Integral

Although the 2-dimensional integral solution of the differential equation,

$$y^{(n)}(x) - \sum_{r=0}^{2} a_r x^r y^{(r)}(x) = 0, \tag{4 92}$$

proposed by Saxton [74] does not strictly fit into the class of integrals suggested by Kaminski and Paris [37], [38] for their method, the same procedure can be followed

This just serves to illustrate that the method actually works for a slighty wider class of Laplace integrals

$$I(x) = \int_0^\infty \int_0^\infty g(t_1, t_2) e^{-f(t_1,t_2,x)}\, dt_1\, dt_2, \qquad x \to \infty, \tag{4 93}$$

though the limitation that $f(t_1, t_2, x)$ can only have a single critical point at the origin still applies [3] If n is even, then the double integral

$$y(x) = \int_0^\infty \int_0^\infty t_1^{\alpha_1} t_2^{\alpha_2} \mathrm{e}^{-xt_1t_2 - \frac{1}{n}(t_1^n + t_2^n)}\, dt_1\, dt_2 \tag{4 94}$$

is a particular solution of the aforementioned differential equation, where α_1 and α_2 are related to the coefficients by

$$a_0 = (\alpha_1 + 1)(\alpha_2 + 1), \quad a_1 = (\alpha_1 + 1) + (\alpha_2 + 2), \quad a_2 = 1 \tag{4 95}$$

Whereas if n is odd, the relationship is given by

$$-a_0 = (\alpha_1 + 1)(\alpha_2 + 1), \quad -a_1 = (\alpha_1 + 1) + (\alpha_2 + 2), \quad -a_2 = 1 \tag{4 96}$$

The variables of integration are assumed to be real in this case, then the phase function has a single critical point within the domain of integration occurring at the origin and the identity

$$\mathrm{e}^{-z} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(\tau) z^{-\tau}\, d\tau, \qquad |\arg z| < \frac{\pi}{2}, \quad z \neq 0 \tag{4 97}$$

can be used on the factor $\mathrm{e}^{-xt_1t_2}$ in the integrand giving

$$y(x) = \int_0^\infty \int_0^\infty t_1^{\alpha_1} t_2^{\alpha_2} \mathrm{e}^{-\frac{1}{n}(t_1^n + t_2^n)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(\tau)(xt_1t_2)^{-\tau}\, d\tau\, dt_1\, dt_2 \tag{4 98}$$

Exchanging the order of integration leads to

$$y(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [\int_0^\infty \mathrm{e}^{-\frac{1}{n}t_1^n} t_1^{\alpha_1 - \tau}\, dt_1][\int_0^\infty \mathrm{e}^{-\frac{1}{n}t_2^n} t_2^{\alpha_2 - \tau}\, dt_2] \Gamma(\tau)(x)^{-\tau}\, d\tau \tag{4 99}$$

Initially consider

$$\int_0^\infty \mathrm{e}^{-\frac{1}{n}t_1^n} t_1^{\alpha_1 - \tau}\, dt_1 \tag{4 100}$$

[3]The effect of this restriction is to ensure that the algebraic expansion obtained by the method is not dominated by contributions from other saddlepoint in the domain

Let $w = \frac{1}{n}t_1^n$, then $t_1^{\alpha_1-\tau} = (nw)^{\frac{\alpha_1-\tau}{n}}$ and $\frac{dw}{dt_1} = t_1^{n-1} = (nw)^{\frac{n-1}{n}}$. This would give

$$\begin{aligned}\int_0^\infty e^{-\frac{1}{n}t_1^n} t_1^{\alpha_1-\tau}\, d\tau &= \int_0^\infty e^{-w}(nw)^{\frac{\alpha_1-\tau}{n}} \frac{dw}{(nw)^{\frac{n-1}{n}}} \\ &= n^{\frac{\alpha_1-\tau-n+1}{n}} \int_0^\infty e^{-w} w^{\frac{\alpha_1-\tau-n+1}{n}}\, dw \\ &= n^{\frac{1+\alpha_1-\tau-n}{n}} \Gamma\left(\tfrac{1+\alpha_1-\tau}{n}\right) \end{aligned} \qquad (4.101)$$

Similarly

$$\int_0^\infty e^{-\frac{1}{n}t_2^n} t_2^{\alpha_2-\tau}\, d\tau = n^{\frac{1+\alpha_2-\tau-n}{n}} \Gamma\left(\frac{1+\alpha_2-\tau}{n}\right) \qquad (4.102)$$

Thus $y(x)$ can be written as

$$y(x) = \frac{1}{2\pi i} n^{\frac{\alpha_1+\alpha_2+2(1-n)}{n}} \int_{-i\infty}^{i\infty} n^{-\frac{2\tau}{n}} \Gamma(\tau)\Gamma\left(\tfrac{1+\alpha_1-\tau}{n}\right)\Gamma\left(\tfrac{1+\alpha_2-\tau}{n}\right)(x)^{-\tau}\, d\tau \qquad (4.103)$$

It is easily verified that the integral converges when the inequality

$$|\arg x| < \left(1 - \frac{2}{n}\right)\frac{\pi}{2} \qquad (4.104)$$

is satisfied. (Here x is real but could be considered to be complex once these restrictions on $\arg x$ hold.) Setting $\tau = \rho e^{i\theta}$ in the integrand, an estimate of the dominant real part of the logarithm of the integrand can be shown to be

$$\left(1 - \frac{2}{n}\right)\rho\cos\theta\log\rho, \qquad (4.105)$$

which tends to ∞ as ρ tends to ∞ $(n > 2)$. Thus the asymptotic behaviour of the integral is governed by the poles which arise in displacing the contour of integration to the right. Poles of $\Gamma(\frac{1+\alpha_1-\tau}{n})$ occur at

$$\frac{1+\alpha_1-\tau}{n} = -k, \qquad k = 0, 1, 2, \ldots, \qquad (4.106)$$

giving $\tau^{(1)} = 1 + \alpha_1 + nk$. Likewise poles of $\Gamma(\frac{1+\alpha_2-\tau}{n})$ occur when

$$\tau^{(2)} = 1 + \alpha_2 + nk \qquad (4.107)$$

Thus the asymptotics of $y(x)$ as $x \to \infty$ are obtained as

$$\begin{aligned} y(x) \sim n^{\frac{\alpha_1+\alpha_2-n}{n}} \sum_k \frac{(-1)^k}{k!} &\Big[\Gamma(1+\alpha_1+nk)\Gamma\left(\tfrac{\alpha_2-\alpha_1-nk}{n}\right) n^{-\frac{2(\alpha_1+nk)}{n}} x^{-(1+\alpha_1+nk)} \\ &+ \Gamma(1+\alpha_2+nk)\Gamma\left(\tfrac{\alpha_1-\alpha_2-nk}{n}\right) n^{-\frac{2(\alpha_2+nk)}{n}} x^{-(1+\alpha_2+nk)}\Big], \end{aligned} \qquad (4.108)$$

whose leading behaviour is given by

$$y(x) \sim n^{\frac{\alpha_2-\alpha_1}{n}-1}\Gamma(1+\alpha_1)\Gamma\left(\tfrac{\alpha_2-\alpha_1}{n}\right)x^{-(1+\alpha_1)} + n^{\frac{\alpha_1-\alpha_2}{n}-1}\Gamma(1+\alpha_2)\Gamma\left(\tfrac{\alpha_1-\alpha_2}{n}\right)x^{-(1+\alpha_2)} \tag{4.109}$$

Similarly, it can be shown that for the p-tuple integral

$$y(x) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_p^{\alpha_p} e^{-x t_1 t_2 \cdots t_p - \frac{1}{n}(t_1^n + t_2^n + \cdots + t_p^n)}\, dt_1\, dt_2 \cdots dt_p, \tag{4.110}$$

the dominant behaviour is

$$y(x) \sim \sum_{r=1}^{p} n^{\frac{1}{n}(\sum_{i \neq r} \alpha_i - (p-1)\alpha_r) - (p-1)} \Gamma(1+\alpha_r) \left[\prod_{i=1, i\neq r}^{p} \Gamma\left(\tfrac{\alpha_i - \alpha_r}{n}\right) \right] x^{-(1+\alpha_r)} \tag{4.111}$$

And the full asymptotic expansion in this case is given by

$$\begin{aligned} y(x) \sim & \sum_{r=1}^{p} n^{\frac{1}{n}(\sum_{i\neq r} \alpha_i - (p-1)\alpha_r) - (k+p-1)} \sum_{k} \frac{(-1)^k}{k!} \Gamma(1+\alpha_r+nk) \\ & * \left[\prod_{i=1, i\neq r}^{p} \Gamma\left(\tfrac{\alpha_i - \alpha_r - nk}{n}\right) \right] x^{-(1+\alpha_r+nk)} \end{aligned} \tag{4.112}$$

Such a p-tuple integral is the solution of the equation

$$y^{(n)}(x) - \sum_{r=0}^{p} a_r x^r y^{(r)}(x) = 0, \tag{4.113}$$

the α_k being related to the a_k by

$$(-1)^n a_k = \sum_{j_1=0}^{k} \sum_{j_2=0}^{j_1} \cdots \sum_{j_{p-k}=0}^{j_{p-k-1}} \prod_{r=1}^{p-k} (\alpha_{p-k-r+1+j_r} + j_r + 1) \tag{4.114}$$

4.7.2 Molins Integral

Now consider instead a different type of n^{th} order equation, consisting only of the highest derivative and an arbitrary positive integer power multiplying the unknown function. Again it is shown how a classical multiple integral representation can be treated by Paris and Kaminski's method.

In 1876, Molins [49] found solutions, $y(x)$, for the equation

$$y^{(n)}(x) - x^p y(x) = 0 \tag{4.115}$$

of the form

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty t_1^{-1+\frac{1}{n+p}} t_2^{-1+\frac{2}{n+p}} \cdots t_p^{-1+\frac{p}{n+p}} e^{sx(t_1 t_2 \cdots t_p)^{\frac{1}{n+p}} - \frac{1}{n+p}(t_1+t_2+\cdots+t_p)} \, dt_1 \, dt_2 \cdots dt_p, \tag{4 116}$$

where s is a root of $s^{n+p} = 1$ Considering the particular double integral

$$y(x) = \int_0^\infty \int_0^\infty t_1^{-1+\frac{1}{n+2}} t_2^{-1+\frac{2}{n+2}} e^{-x(t_1 t_2)^{\frac{1}{n+2}} - \frac{1}{n+2}(t_1+t_2)} \, dt_1 \, dt_2, \tag{4 117}$$

which solves

$$y^{(n)}(x) - x^2 y(x) = 0, \tag{4 118}$$

the same procedure is followed as in §4 7 1 Again the only critical point of

$$f(t_1, t_2, x) = x(t_1 t_2)^{\frac{1}{n+2}} + \tfrac{1}{n+2}(t_1 + t_2) \tag{4 119}$$

lying within the domain of integration for real t_1, t_2, occurs at the origin First of all, the identity

$$e^{-z} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(\tau) z^{-\tau} \, d\tau, \qquad |\arg z| < \frac{\pi}{2}, \quad z \neq 0 \tag{4 120}$$

is applied to $e^{-x(t_1 t_2)^{\frac{1}{n+2}}}$ Thus, having exchanged the order of integration, $y(x)$ becomes

$$y(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [\int_0^\infty t_1^{-1+\frac{1-\tau}{n+2}} e^{-\frac{1}{n+2}t_1} \, dt_1][\int_0^\infty t_2^{-1+\frac{2-\tau}{n+2}} e^{-\frac{1}{n+1}t_2} \, dt_2] \Gamma(\tau) x^{-\tau} \, d\tau \tag{4 121}$$

The inner integrals are examined in turn substituting $w_1 = \frac{1}{n+2}t_1$ gives

$$\int_0^\infty t_1^{-1+\frac{1-\tau}{n+2}} e^{-\frac{1}{n+2}t_1} \, dt_1 = (n+2)^{\frac{1-\tau}{n+2}} \Gamma\left(\tfrac{1-\tau}{n+2}\right), \tag{4 122}$$

and substituting $w_2 = \frac{1}{n+2}t_2$ yields

$$\int_0^\infty t_2^{-1+\frac{2-\tau}{n+2}} e^{-\frac{1}{n+1}t_2} \, dt_2 = (n+2)^{\frac{2-\tau}{n+2}} \Gamma\left(\tfrac{2-\tau}{n+2}\right), \tag{4 123}$$

enabling $y(x)$ to be written as

$$y(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} (n+2)^{\frac{3-2\tau}{n+2}} \Gamma\left(\tfrac{1-\tau}{n+2}\right) \Gamma\left(\tfrac{2-\tau}{n+2}\right) \Gamma(\tau) x^{-\tau} \, d\tau \tag{4 124}$$

This integral converges whenever

$$|\arg x| < \left(1 - \frac{2}{n+2}\right) \frac{\pi}{2} \tag{4 125}$$

holds, which it does for all $x \in \mathbb{R}$ However, for $\tau = \rho e^{i\theta}$ with $|\theta| < \frac{\pi}{2}$, the dominant real part of the logarithm of the integrand is given by

$$\left(1 - \frac{2}{n+2}\right)\rho\cos\theta\log\rho, \quad \text{as} \quad \rho \to \infty \tag{4 126}$$

As $(1 - \frac{2}{n+2}) > 0$, the contour of integration is again displaced to the right to determine the asymptotic behaviour of the integral as $x \to \infty$ Contributions to the expansion arise when poles of either $\Gamma\left(\frac{1-\tau}{n+2}\right)$ or $\Gamma\left(\frac{2-\tau}{n+2}\right)$ are crossed These occur at

$$\frac{1-\tau}{n+2} = -k, \qquad \frac{2-\tau}{n+2} = -k, \tag{4 127}$$

(i e $\tau = (n+2)k+1, \quad \tau = (n+2)k+2$) $\forall k \in \mathbb{N}$

Hence the expansion of $y(x)$ is

$$\begin{aligned} y(x) \quad \sim \quad & (n+2)\sum_k \frac{(-1)^k}{k!}\{\Gamma((n+2)k+1)\Gamma\left(\tfrac{1-(n+2)k}{n+2}\right)x^{-(n+2)k-1} \\ & +\Gamma((n+2)k+2)\Gamma\left(\tfrac{-1-(n+2)k}{n+2}\right)x^{-(n+2)k-2}\} \end{aligned} \tag{4 128}$$

Likewise, if $y(x)$ is the p-tuple integral

$$\int_0^\infty\int_0^\infty \quad \int_0^\infty t_1^{-1+\frac{1}{n+p}} t_2^{-1+\frac{2}{n+p}} \quad t_p^{-1+\frac{p}{n+p}} e^{-x(t_1 t_2 \quad t_p)^{\frac{1}{n+p}} - \frac{1}{n+p}(t_1+t_2+ \quad +t_p)} dt_1\, dt_2 \quad dt_p, \tag{4 129}$$

it becomes

$$y(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}(n+p)^{\frac{(1+2+ \quad +p)-p\tau}{n+p}}\Gamma\left(\tfrac{1-\tau}{n+p}\right)\Gamma\left(\tfrac{2-\tau}{n+p}\right) \quad \Gamma\left(\tfrac{p-\tau}{n+p}\right)\Gamma(\tau)x^{-\tau}\,d\tau \tag{4 130}$$

Again it can be shown that the dominant part of the logarithm of the integrand tends to ∞ as $|\tau| \to \infty$ Thus when the contour is displaced to the right, poles of the integrand will occur at

$$\tau = (n+p)k+1, \quad \tau = (n+p)k+2, \qquad , \ \tau = (n+p)k+p, \quad \forall k \in \mathbb{N} \tag{4 131}$$

The asymptotic behaviour of $y(x)$ as $x \to \infty$ is then given by

$$\begin{aligned} y(x) \quad \sim \quad & \textstyle\sum_{i=1}^{p}(n+p)\sum_k \frac{(-1)^k}{k!}(n+p)^{\frac{(1+2+ \quad +p)-p(n+p)k-ip}{n+p}}\prod_{l=1,l\neq i}^{p}\Gamma\left(\tfrac{l-i-(n+p)k}{n+p}\right) \\ & *\Gamma((n+p)k+i)x^{-(n+p)k-i} \end{aligned} \tag{4 132}$$

Chapter 5

Multidimensional Methods

5.1 Analogue of Nikishov & Ritus' Method

The method outlined in Chapter 2 can be extended further to deal with multidimensional integrals The class of integrals to be studied is of the form

$$I(\lambda) = A \int_S e^{f(\mathbf{z},\lambda)} d\mathbf{z}, \tag{5 1}$$

where $\mathbf{z} \in \mathbb{C}^n$ and $\mathcal{S}$ is an unbounded n-dimensional surface where $\Re f(\mathbf{z}, \lambda) \to -\infty$ as $|\mathbf{z}| \to \infty$ on $\mathcal{S}$ For ease of notation, the procedure will be detailed for double integrals only Although the computations become more cumbersome as the dimension increases, there should be no other added complexity Again, the number of saddlepoints of $f(\mathbf{z}, \lambda)$ is initially restricted to two, they are denoted by $\mathbf{z}^u = (z_1^u, z_2^u)$ and $\mathbf{z}^l = (z_1^l, z_2^l)$ and both are assumed to be interior simple saddles As the domain of integration is infinite, the different types of boundary critical points are automatically excluded and the occurrence of a ridge of critical points will not be discussed here

As in the single integral case, the Stokes phenomenon occurs whenever

$$\Im(f(\mathbf{z}^u, \lambda) - f(\mathbf{z}^l, \lambda)) = 0 \tag{5 2}$$

or $\lambda = \lambda_S$ Following the discussion of §4 5 and §4 6, it is assumed that in the neighbourhood of λ_S, it is possible to deform the surface $\mathcal{S}$ to a surface of steepest descent SSD_u, through $\mathbf{z}^u$ only or to a combination of a surface of steepest descent through $\mathbf{z}^u$ and $\mathbf{z}^l$ respectively, $SSD_u + SSD_l$ The latter is denoted $\mathcal{S}_{ul}$ SSD_u and SSD_l meeting at infinity at a 'boundary' along which $\Re f(\mathbf{z}, \lambda) \to -\infty$ as before

Though such a deformation is possible in principle, it can be difficult in practice, but the fact that a surface of steepest descent through a saddle, $\mathbf{z}^u$ say, is no longer uniquely defined by

$$\Im(f(\mathbf{z},\lambda) - f(\mathbf{z}^u,\lambda)) = 0, \tag{5 3}$$

aids the deformation somewhat, as any suitable surface can be used

To determine a Stokes multiplier as such in the case of this double integral, the surface $\mathcal{S}_{ul}$ is divided in two

$$\mathcal{S}_{ul} = \mathcal{S}_{u*} + \mathcal{S}_{*l} \tag{5 4}$$

where $\mathcal{S}_{u*}$ begins in a valley with $\Re(f(\mathbf{z},\lambda) - f(\mathbf{z}^u,\lambda)) < 0$, as does SSD_u, but ends at a finite boundary through some point $\mathbf{z}^*$ and $\mathcal{S}_{*l}$ starts at this same boundary but joins the surface of steepest descent through $\mathbf{z}^l$ to end in a valley of the integrand where $\Re(f(\mathbf{z},\lambda) - f(\mathbf{z}^l,\lambda)) < 0$ It is assumed that no extra critical points appear on this boundary as a result of the restriction of $f(\mathbf{z},\lambda)$ there $\mathbf{z}^*$ must satisfy the equations

$$\Im f(\mathbf{z}^*,\lambda) = \Im f(\mathbf{z}^u,\lambda), \qquad \Re f(\mathbf{z}^*,\lambda) = \Re f(\mathbf{z}^l,\lambda), \tag{5 5}$$

but again these fail to specify $\mathbf{z}^*$ uniquely and the boundary of truncation must be chosen from an analysis of the actual surfaces used for a particular function Then $D(\lambda)$ and $R(\lambda)$ can be defined by

$$\begin{aligned} I(\lambda) &= D(\lambda) + R(\lambda) \\ &= A\int_{\mathcal{S}_{u*}} \mathrm{e}^{f(\mathbf{z},\lambda)} d\mathbf{z} + A\int_{\mathcal{S}_{*l}} \mathrm{e}^{f(\mathbf{z},\lambda)} d\mathbf{z} \end{aligned} \tag{5 6}$$

In order to find explicit expressions for $D(\lambda)$ and $R(\lambda)$, the Taylor series of $f(\mathbf{z},\lambda)$ about $\mathbf{z}^u$ and $\mathbf{z}^l$ respectively are employed in a manner similar to that of the single integral case

5 1.1 Dominant Term

From above

$$D(\lambda) = A\int_{\mathcal{S}_{u*}} \mathrm{e}^{f(\mathbf{z},\lambda)} d\mathbf{z} = \mathrm{e}^{\Im f(\mathbf{z}^u,\lambda)} \int_{\mathcal{S}_{u*}} \mathrm{e}^{\Re f(\mathbf{z},\lambda)} d\mathbf{z} \tag{5 7}$$

The integral is then of Laplace type and using

$$\begin{aligned} \Re f(\mathbf{z},\lambda) &= \Re\left\{ f(\mathbf{z}^u,\lambda) + \frac{(z_1 - z_1^u)^2}{2}\left.\frac{\partial^2 f}{\partial z_1^2}\right|_{\mathbf{z}^u} + (z_1 - z_1^u)(z_2 - z_2^u)\left.\frac{\partial^2 f}{\partial z_1 \partial z_2}\right|_{\mathbf{z}^u} \right. \\ &\qquad \left. + \frac{(z_2 - z_2^u)^2}{2}\left.\frac{\partial^2 f}{\partial z_2^2}\right|_{\mathbf{z}^u} + \quad \right\}, \end{aligned} \tag{5 8}$$

the process shown in §4 3 3 can be applied giving

$$D(\lambda) \sim \pm \frac{\imath A \; 2\pi \, \mathrm{e}^{f(\mathbf{z}^u,\lambda)}}{\left(\left(\frac{\partial^2 f}{\partial z_1^2}\frac{\partial^2 f}{\partial z_2^2} - (\frac{\partial^2 f}{\partial z_1 \partial z_2})^2\right)\Big|_{\mathbf{z}^u}\right)^{1/2}} \tag{5 9}$$

5 1.2 Recessive Term

This time

$$\begin{aligned} f(\mathbf{z},\lambda) &= f(\mathbf{z}^l,\lambda) + \frac{(z_1 - z_1^l)^2}{2}\frac{\partial^2 f}{\partial z_1^2}\Big|_{\mathbf{z}^l} + (z_1 - z_1^l)(z_2 - z_2^l)\frac{\partial^2 f}{\partial z_1 \partial z_2}\Big|_{\mathbf{z}^l} \\ &\quad + \frac{(z_2 - z_2^l)^2}{2}\frac{\partial^2 f}{\partial z_2^2}\Big|_{\mathbf{z}^l} + \end{aligned} \tag{5 10}$$

is employed The same reasoning is used as in §2 1 2 and a substitution similar to that of §4 3 3,

$$u = z_1 - z_1^l, \qquad v = (z_2 - z_2^l) + \frac{(z_1 - z_1^l)\left.\frac{\partial^2 f}{\partial z_1 z_2}\right|_{\mathbf{z}^l}}{\left.\frac{\partial^2 f}{\partial z_2^2}\right|_{\mathbf{z}^l}}, \tag{5 11}$$

is performed in order to split the integral into a product of two Gaussian-type integrals—again an application of Morse's Lemma But now the finite boundary in $R(\lambda)$ means the introduction of a complementary error function term Suppose for convenience that the finite boundary through $\mathbf{z}^*$ is chosen parallel to the z_2-axis, so that z_2 runs from $-\infty$ to ∞ but z_1 now runs from z_1^* to ∞, then

$$R(\lambda) \sim \pm \frac{\imath A \; \pi \, \mathrm{e}^{f(\mathbf{z}^l,\lambda)}}{\left(\left(\frac{\partial^2 f}{\partial z_1^2}\frac{\partial^2 f}{\partial z_2^2} - (\frac{\partial^2 f}{\partial z_1 \partial z_2})^2\right)\Big|_{\mathbf{z}^l}\right)^{1/2}} \operatorname{erfc}(w_1), \tag{5 12}$$

$$w_1 = \pm\imath(z_1^* - z_1^l)\left(\frac{\left(\frac{\partial^2 f}{\partial z_1^2} - (\frac{\partial^2 f}{\partial z_1 z_2})^2 / \frac{\partial^2 f}{\partial z_2^2}\right)\Big|_{\mathbf{z}^l}}{2}\right)^{1/2} \tag{5 13}$$

Whereas, if the boundary is instead chosen to be parallel to the z_1-axis with z_1 taking values from $-\infty$ to ∞ and z_2 restricted to values between z_2^* and ∞ then

$$R(\lambda) \sim \pm \frac{\imath A \; \pi \, \mathrm{e}^{f(\mathbf{z}^l,\lambda)}}{\left(\left(\frac{\partial^2 f}{\partial z_1^2}\frac{\partial^2 f}{\partial z_2^2} - (\frac{\partial^2 f}{\partial z_1 \partial z_2})^2\right)\Big|_{\mathbf{z}^l}\right)^{1/2}} \operatorname{erfc}(w_2), \tag{5 14}$$

$$w_2 = \pm\imath\left((z_2^* - z_2^l) + \frac{(z_1^* - z_1^l)\left.\frac{\partial^2 f}{\partial z_1 z_2}\right|_{\mathbf{z}^l}}{\left.\frac{\partial^2 f}{\partial z_2^2}\right|_{\mathbf{z}^l}}\right)\left(\frac{\left.\frac{\partial^2 f}{\partial z_2^2}\right|_{\mathbf{z}^l}}{2}\right)^{1/2} \tag{5 15}$$

5 1 3 Remarks

All the extensıons and dıscussıons of §2 2 apply to the method gıven for double ıntegrals It ıs assumed ın the followıng that the finıte boundary through $\mathbf{z}^*$ has been chosen parallel to the z_2-axıs

For ınstance, ıf

$$I(\lambda) = A \int_S g(\mathbf{z}) e^{f(\mathbf{z},\lambda)} d\mathbf{z}, \tag{5 16}$$

and $g(\mathbf{z}^u)$, $g(\mathbf{z}^l)$ are both non-zero, then

$$D(\lambda) \sim \pm \frac{\imath A \, 2\pi \, g(\mathbf{z}^u) \, e^{f(\mathbf{z}^u,\lambda)}}{\left(\left(\frac{\partial^2 f}{\partial z_1^2} \frac{\partial^2 f}{\partial z_2^2} - (\frac{\partial^2 f}{\partial z_1 \partial z_2})^2 \right) \Big|_{\mathbf{z}^u} \right)^{1/2}}, \tag{5 17}$$

$$R(\lambda) \sim \pm \frac{\imath A \, \pi \, g(\mathbf{z}^l) \, e^{f(\mathbf{z}^l,\lambda)}}{\left(\left(\frac{\partial^2 f}{\partial z_1^2} \frac{\partial^2 f}{\partial z_2^2} - (\frac{\partial^2 f}{\partial z_1 \partial z_2})^2 \right) \Big|_{\mathbf{z}^l} \right)^{1/2}} \operatorname{erfc}(w_1) \tag{5 18}$$

In the event that $\mathbf{z}^*$ coıncıdes wıth eıther $\mathbf{z}^u$ or $\mathbf{z}^l$, the expressıons for $D(\lambda)$ and $R(\lambda)$ become

$$D(\lambda) \sim \pm \frac{\imath A \, \pi \, e^{f(\mathbf{z}^u,\lambda)}}{\left(\frac{\partial^2 f}{\partial z_1^2} \frac{\partial^2 f}{\partial z_2^2} - (\frac{\partial^2 f}{\partial z_1 \partial z_2})^2 \right) \Big|_{\mathbf{z}^u}^{\frac{1}{2}}}, \tag{5 19}$$

$$R(\lambda) \sim \pm \frac{\imath A \, \pi \, e^{f(\mathbf{z}^u,\lambda)}}{\left(\frac{\partial^2 f}{\partial z_1^2} \frac{\partial^2 f}{\partial z_2^2} - (\frac{\partial^2 f}{\partial z_1 \partial z_2})^2 \right) \Big|_{\mathbf{z}^u}^{\frac{1}{2}}} \pm \frac{\imath A \, \pi \, e^{f(\mathbf{z}^l,\lambda)}}{\left(\frac{\partial^2 f}{\partial z_1^2} \frac{\partial^2 f}{\partial z_2^2} - (\frac{\partial^2 f}{\partial z_1 \partial z_2})^2 \right) \Big|_{\mathbf{z}^l}^{\frac{1}{2}}} \operatorname{erfc}(w_1), \tag{5 20}$$

$$D(\lambda) \sim \pm \frac{\imath A \, 2\pi \, e^{f(\mathbf{z}^u,\lambda)}}{\left(\frac{\partial^2 f}{\partial z_1^2} \frac{\partial^2 f}{\partial z_2^2} - (\frac{\partial^2 f}{\partial z_1 \partial z_2})^2 \right) \Big|_{\mathbf{z}^u}^{\frac{1}{2}}} \pm \frac{\imath A \, \pi \, e^{f(\mathbf{z}^l,\lambda)}}{\left(\frac{\partial^2 f}{\partial z_1^2} \frac{\partial^2 f}{\partial z_2^2} - (\frac{\partial^2 f}{\partial z_1 \partial z_2})^2 \right) \Big|_{\mathbf{z}^l}^{\frac{1}{2}}}, \tag{5 21}$$

$$R(\lambda) \sim \pm \frac{\imath A \, \pi \, e^{f(\mathbf{z}^l,\lambda)}}{\left(\frac{\partial^2 f}{\partial z_1^2} \frac{\partial^2 f}{\partial z_2^2} - (\frac{\partial^2 f}{\partial z_1 \partial z_2})^2 \right) \Big|_{\mathbf{z}^l}^{\frac{1}{2}}} \operatorname{erfc}(w_1), \tag{5 22}$$

respectıvely, usıng the result of Wong ın §4 3 1 for a statıonary poınt on a boundary

Fınally, hıgher order terms for $D(\lambda)$ can be calculated ın the same way as before by sımply retaınıng more terms of the Taylor serıes for $f(\mathbf{z}, \lambda)$ as shown ın §4 3 3, and the number of saddlepoınts taken ınto account can also be ıncreased, yıeldıng unwıeldy expressıons ınvolvıng erfc terms analagous to those of §2 2 5

5.2 Application to Double Airy Integral

To illustrate the workings of the method, a product of Airy Ai integrals, hereafter termed the Double Airy integral, is considered in order to compare the numerical results obtained to those using Howls' method, which is discussed below Thus

$$I(\lambda) = \mathrm{Ai}(\alpha^{\frac{2}{3}}\lambda)\mathrm{Ai}(\beta^{\frac{2}{3}}\lambda) = -\frac{\lambda(\alpha\beta)^{\frac{1}{3}}}{4\pi^2}\int\int_S e^{-\lambda^{3/2}\{\alpha(z_1-z_1^3/3)+\beta(z_2-z_2^3/3)\}}dz_1 dz_2, \quad (5\ 23)$$

where S is a 2-dimensional surface infinite in extent in both complex variables, with u and v both running from $\infty e^{-\frac{i\pi}{3}}$ to $\infty e^{\frac{i\pi}{3}}$ This integral has 4 saddlepoints lying at $(1,1)$, $(-1,1)$, $(1,-1)$ and $(-1,-1)$ However, only the contributions supplied by the first two are considered here

$$\mathbf{z}^u = (1,1), \qquad \mathbf{z}^l = (-1,1) \quad (5\ 24)$$

Using such a product of Airy functions allows the results to be tested against tabled values and so provides some verification of the numerical accuracy of the method λ is allowed to rotate in the complex plane, but its magnitude remains fixed $|\lambda^{3/2}| = 10$, α and β are also fixed $\alpha = 0\ 3$, $\beta = 0\ 7$ The results are shown in Table 5 1 where the row labelled 'z^u contribution' refers to the sum of the first six terms in the series expansion about z^u As for the single integral, the value of z^* gives no indication of how many terms beyond the leading term of $D(\lambda)$ are necessary for an optimal numerical value

When $\theta_\lambda = 0$ or $\theta_\lambda = \pm\frac{2\pi}{3}$ it can be seen that $\Im(f(\mathbf{z}^u,\lambda) - f(\mathbf{z}^l,\lambda)) = 0$, as for ordinary $\mathrm{Ai}(\lambda)$ At these values, $\mathbf{z}^*$ can be chosen to equal $\mathbf{z}^l$ and so, taking into account the contribution from $\mathbf{z}^*$ on the boundary, the same pattern arises as for the single integral when $\theta_\lambda = 0$, only the contribution from $\mathbf{z}^u$ survives but when $\theta_\lambda = \pm\frac{2\pi}{3}$, both contributions combine to produce the result

5.3 Howls' Multidimensional Method

Howls [29] has gone beyond the work of earlier authors by exponentially improving the asymptotic expansion of multidimensional integrals and has provided a means of doing so which follows on naturally from his single integral approach Initially the introduction of higher dimensions would seem to pose many problems However, the final result shows some of these to be merely artificial

$\theta_\lambda = 0$	0 00006097105	z^u leading behaviour
	0 00005864181	z^u contribution
	0 00005844867	*Mathematica*
$\theta_\lambda = \pm\frac{2\pi}{3}$	18 8229116541 $\mp\imath$32 6022393322	z^u leading behaviour
	19 9731333915 $\mp\imath$34 5944818204	z^u contribution
	0 59051785874 $\mp\imath$0 34093564470	z^l contribution
	20 5646689337 $\mp\imath$34 9360050249	Combined contribution
	20 4216057189 $\mp\imath$34 6862765258	*Mathematica*

Table 5 1 Values of $I(\lambda)$ computed using the method of §5 1

Let

$$I_k(\lambda) = \int_{\mathcal{S}_k} g(\mathbf{z}) \mathrm{e}^{-\lambda f(\mathbf{z})} \, d\mathbf{z}, \tag{5 25}$$

where the n-dimensional surface, $\mathcal{S}_k$, is doubly infinite in extent in all complex variables, running between specified valleys at infinity where $\Re\{-\lambda(f(\mathbf{z}) - f(\mathbf{z}^k))\} < 0$, and $\mathbf{z}^k$ is a simple saddle Paralleling the single integral case in §3 1, $T_k(\lambda)$ is defined by extracting the exponential dependence and algebraic prefactor at the saddle point which gives

$$I_k(\lambda) = \frac{\mathrm{e}^{-\lambda f_k}}{\lambda^{\frac{n}{2}}} T_k(\lambda) \qquad \text{with } f(\mathbf{z}^k) = f_k \tag{5 26}$$

Then a new variable, s, is defined by

$$s = f(\mathbf{z}) - f_k, \tag{5 27}$$

so that on the hypersurface $\mathcal{S}_k$, s varies from 0 to ∞ Thus the variables of integration can be transformed to a new set including s, defining a form ω by

$$dz_1 \wedge dz_2 \wedge \quad \wedge dz_n = ds \wedge \omega, \tag{5 28}$$

which can then be written as

$$\omega = \left. \frac{dz_1 \wedge dz_2 \wedge \quad \wedge dz_n}{ds} \right|_{\gamma_k(s)} \tag{5 29}$$

($\gamma_k(s)$, which denotes a surface where s = constant, is termed a vanishing cycle[1] and $\mathcal{S}_k$ is effectively a Lefschetz thimble (see Figure 4.2)). This allows $T_k(\lambda)$ to be written as

$$T_k(\lambda) = \int_0^{\infty e^{-i\theta_\lambda}} \lambda^{\frac{n}{2}} e^{-\lambda s} \triangle_k G(s)\, ds, \quad (5.30)$$

$$\triangle_k G(s) = \int_{\gamma_k(s)} g(\mathbf{z})\omega, \quad (5.31)$$

reducing the integral $I_k(\lambda)$ to a single dimensional Laplace integral. As the Borel and Laplace transforms are mutually inverse, $\triangle_k G(s)$ is the Borel transform of the integral $I_k(\lambda)$ and the s-plane on which the study is now concentrated, is the Borel plane. Such a transformation of the multidimensional integral is analagous to the use of the theorem given in §4.2 for the resolution of multiple integrals. The function $\triangle_k G(s)$ is holomorphic in the s-plane in the neighbourhood of the image of the saddle $\mathbf{z}^k$ and is singular at the other critical points of the phase function $f(\mathbf{z})$. So if $\triangle_k G(s)$ were to be expanded in powers of s, it would have a radius of convergence up to the nearest singularity which lies on the same Riemann sheet of $\mathcal{S}_k$. But beyond this, the series diverges. This again illustrates the idea of the existence of other saddles being the cause of the divergence of the series. However, a saddle, $\mathbf{z}^m$, must be directly visible from $\mathbf{z}^k$, that is, it must lie on the same Riemann sheet, in order for it to make a contribution. This visibility condition is equivalent to the adjacency condition of one dimension. The next step then is to deform the contour $\gamma_k(s)$ surrounding the saddlepoint $\mathbf{z}^k$ to the neighbourhood of the saddle $\mathbf{z}^m$ and to determine the type of singularity $\triangle_k G(s)$ has at this saddle. Thus the effect of a 2π cycle on the deformation of the contour must be studied and it can be seen that there is a fundamental difference between even and odd dimensions:

$$\gamma_m(r_m e^{i(\phi+2\pi)}) = (-1)^n \gamma_m(r_m e^{i\phi}). \quad (5.32)$$

To achieve this, the topological studies of Pham [70] are drawn on and the Picard-Lefschetz formula leads to a result:

$$\gamma_k(r_m e^{i(\phi+2\pi)}) = \gamma_k(r_m e^{i\phi}) + (-1)^{n(n-1)/2} N(k,m)\gamma_m(r_m e^{i\phi}), \quad (5.33)$$

[1]Let $X \subset \mathbb{C}^n$, $A \subset X$. A p-dimensional chain, γ, on X is a linear combination with integral coefficients of many p-dimensional chain elements each of which is a p-dimensional orientable manifold. A chain, γ, on X is said to be a 'cycle mod A' (a relative cycle) if $\partial\gamma$ is contained in A. Then a relative cycle, γ, is called a *vanishing cycle* if it contracts to a point (i.e. vanishes) as A contracts to the origin [21].

where $N(k,m) \in \mathbb{N}$ is called the intersection number of the vanishing cyles γ_k and γ_m An expansion for $\triangle_k G(s)$ in powers of s is eventually arrived at

$$\triangle_k G(s) = \sum_{r=0}^{\infty} \frac{T_{kr}}{\Gamma(r+\frac{n}{2})} s^{r+\frac{n}{2}-1} \tag{5 34}$$

However, starting with (5 30), making use of Cauchy's integral theorem and proceeding as in §3 1, it can be seen that

$$s^{1-\frac{n}{2}} \triangle_k G(s) = \frac{1}{2\pi i} \oint_{\Gamma_k} \frac{\triangle_k G(\xi)\xi^{1-\frac{n}{2}}}{\xi - s} d\xi, \tag{5 35}$$

which in turn implies that $T_k(\lambda)$ can be written as

$$T_k(\lambda) = \frac{\lambda^{\frac{n}{2}}}{2\pi i} \int_0^{\infty e^{-i\theta_\lambda}} e^{-\lambda s} s^{\frac{n}{2}-1} \oint_{\Gamma_k} \frac{\triangle_k G(\xi)\xi^{1-\frac{n}{2}}}{\xi - s} d\xi \, ds, \tag{5 36}$$

where Γ_k is the infinite loop surrounding $\xi = 0$ (see Figure 3 1) Then by binomially expanding the denominator of the second integral to finite order, the coefficients of the asymptotic expansion with respect to k can be found

$$T_k(\lambda) = \sum_{r=0}^{N-1} \frac{T_{kr}}{\lambda^r} + \frac{\lambda^{\frac{n}{2}}}{2\pi i} \int_0^{\infty} e^{-\lambda s} s^{N+\frac{n}{2}-1} \oint_{\Gamma_k} \frac{\triangle_k G(\xi)}{\xi^{N+\frac{n}{2}}(1-\frac{s}{\xi})} d\xi \, ds, \tag{5 37}$$

where

$$T_{kr} = \frac{\Gamma(r+\frac{n}{2})}{2\pi i} \oint_{B_\xi} \frac{\triangle_k G(\xi)}{\xi^{r+\frac{n}{2}}} d\xi, \tag{5 38}$$

or in terms of the original variables

$$T_{kr} = \frac{(r+\frac{n}{2}-1)!}{2\pi i} \oint_{B_{\mathbf{z}^k}} \frac{g(\mathbf{z})}{(f(\mathbf{z}) - f_k)^{r+\frac{n}{2}}} d\mathbf{z} \tag{5 39}$$

($B_{\mathbf{z}^k}$ being the n-dimensional ball surrounding $\mathbf{z}^k$) To finish, the contour Γ_k is deformed to a union of arcs at infinity and similar contours Γ_m around the other singularities The integral along the arcs at infinity vanishes, leaving the remainder to be written as a sum over the Γ_m contours This, along with the transformation

$$s = \frac{\nu\xi}{\lambda F_{km}} = \frac{\nu}{\lambda} + \frac{\nu(\xi - f_m)}{\lambda F_{km}}, \tag{5 40}$$

allows $T_k(\lambda)$ to be written as follows

$$\begin{aligned} T_k(\lambda) &= \sum_{r=0}^{N-1} \frac{T_{kr}}{\lambda^r} \\ &+ \frac{1}{2\pi i} \sum_m \frac{(-1)^{\gamma_{km}}}{\lambda^N F_{km}^{N+\frac{1}{2}}} \int_0^{\infty} \frac{\nu^{N+\frac{n}{2}-1} e^{-\nu}}{1 - \frac{\nu}{\lambda F_{km}}} \oint_{\Gamma_m} \triangle_k G(\xi) e^{-\frac{\nu}{F_{km}}(\xi - f_m)} d\xi \, d\nu \end{aligned} \tag{5 41}$$

Then the contour Γ_m is collapsed onto the ray from $\xi = 0$ to ∞ and the discontinuity $\Delta_m(\Delta_k G)$ is taken as in §3.1 (see appendix for further details). A self-similarity has thus been introduced into the integral leading to the resurgence formula

$$T_k(\lambda) = \sum_{r=0}^{N-1} \frac{T_{kr}}{\lambda^r} + \frac{1}{2\pi i} \sum_m \frac{(-1)^{\gamma_{km}}}{(\lambda F_{km})^N} \int_0^\infty \frac{\nu^{N-1} e^{-\nu}}{1 - \frac{\nu}{\lambda F_{km}}} T_m \left(\frac{\nu}{F_{km}} \right) d\nu. \tag{5.42}$$

It can be noted that any explicit reference to the dimensionality of the integrand has dropped out but is incorporated into the T_k, T_m factors—yielding the same result as in (3.19). The same process can then be applied to the T_m factor. That is, it can be expanded to finite order having its remainder expressed in terms of the other saddles on the same Riemann sheet. This leads to a hyperasymptotic scheme exactly as in §3.1, again with universal hyperterminant integrals which depend only on the point at which the series is truncated and the effective distance between saddles.

Some changes have been made to the overall approach. In order to numerically optimise the algorithm and globally minimise the remainder, Howls has introduced different criteria after Olde Daalhuis [57] for the truncation of each hyperseries:

$$\begin{aligned} N_0 &= \begin{cases} \text{shortest directed path of } M \text{ steps in the } s\text{-plane, between singularities} \\ \text{starting at } \mathbf{z}^k, \end{cases} \\ N_1 &= \max\{0, N_0 - |\lambda F_{km}|\}, \\ N_2 &= \max\{0, N_1 - |\lambda F_{ml}|\} \end{aligned} \tag{5.43}$$

and so on, where M represents the current hyperasymptotic iteration. Also, the factors $(-1)^{\gamma_{km}}$ have now been replaced by P_{km} where

$$|P_{km}| = \begin{cases} 1 & \text{if } \mathbf{z}^m \text{ is adjacent to } \mathbf{z}^k \\ 0 & \text{otherwise} \end{cases} \tag{5.44}$$

This reduces the determination of an exact expression for the remainder to the calculation of the P_{km}—which can be achieved by numerically solving a system of algebraic equations. The graphical techniques which can be used in one dimension to determine the adjacency of saddles are now redundant and have been replaced by an algebraic process.

The most interesting points to note from this multidimensional method are

- First of all, how the geometry, analysis and asymptotics of the problem combine to produce the result.

0 00005603064518	$-\frac{(\alpha\beta)^{\frac{1}{3}} e^{-\lambda^{3/2} f_1}}{4\pi^2}$
0 00005832352792	$-\frac{(\alpha\beta)^{\frac{1}{3}} e^{-\lambda^{3/2} f_1}}{4\pi^2} H_0$
0 00005844795344	$-\frac{(\alpha\beta)^{\frac{1}{3}} e^{-\lambda^{3/2} f_1}}{4\pi^2}(H_0 + H_1)$
0 00005844866001	$-\frac{(\alpha\beta)^{\frac{1}{3}} e^{-\lambda^{3/2} f_1}}{4\pi^2}(H_0 + H_1 + H_2)$
0 00005844866596	$-\frac{(\alpha\beta)^{\frac{1}{3}} e^{-\lambda^{3/2} f_1}}{4\pi^2}(H_0 + H_1 + H_2 + H_3)$
0 00005844866651	*Mathematica*

Table 5 2 Values of $\mathrm{Ai}(\alpha^{\frac{2}{3}}\lambda)\mathrm{Ai}(\beta^{\frac{2}{3}}\lambda)$ with $\theta_\lambda = 0$ [29]

- Secondly, how despite the difficulties introduced initially by the higher dimensions, little effect is had on the final form of the remainder term when compared to Howls' results for a single integral

- How the Stokes phenomenon is also accounted for quite naturally by the method

- Finally, how the method hinges on expressing the function in terms of the singularity structure of its Borel transform—the advantage of this technique is that it applies equally well to classes of differential equations as to integrals [57]

5 3 1 Example· Double Airy Integral

The method described above was applied to the Double Airy integral in [29], with α and β fixed as before and the results obtained are those in Table 5 2 Transforming the variables using $s = f(\mathbf{z}) - f_k$, leads to the appearance of cuts in the s-plane as expected, but in this case they are collinear Thus care must be taken to indent the cut from one saddle above others When calculating the N_s, it was found that although two saddles, $\mathbf{z}^m$ and $\mathbf{z}^l$, may be adjacent, the contribution from $\mathbf{z}^l$ could be zero at a particular level of hyperasymptotics because it may lie too far from $\mathbf{z}^m$, relatively speaking, to contribute numerically This does not interfere, however, with it making a contribution at a later stage

5.4 Conclusions

As can be seen above, an application of the extended method of Nikishov and Ritus to a higher dimensional integral can require tedious computation and much care in the choice of steepest descent surfaces and points of truncation, $\mathbf{z}^*$. Matters may be improved by borrowing the approach of reducing the original integral to a single integral in the Borel plane and following the suggestions of §3.4.3 to then truncate the contour at a point ξ^* in the ξ-plane. However, this would still require establishing various criteria for the definition of ξ^*.

Again it should be noted that while §5.1 presents a method that can provide a means of exponentially improving traditional asymptotic estimates by taking into account the appearance of exponentially small terms it can not compete with the numerical precision of a hyperasymptotic scheme.

Chapter 6

Extensions

6.1 Integrals with one Finite Boundary

The methods of §3.2 and §5.3 are now combined to form a method capable of finding the asymptotic expansion of multidimensional integrals over a semi-infinite surface; that is, one which is finitely bounded in one direction. Once again only two cases are considered here—integrals which exhibit a quadratic dependence at the finite boundary due to a saddlepoint of $f(\mathbf{z})$ occurring on the boundary and those which exhibit a linear dependence due to the consideration of a point on the boundary which is not a saddle but whose contribution may be relevant to the expansion. This mirrors the discussion of §3.2.1 and §3.2.2. As before, the contributions from any extra crtitical points which may arise as a consequence of restricting $f(\mathbf{z})$ to the boundary will not be considered here. Nor is the case where a ridge of critical points appears discussed.

6.1.1 Quadratic Dependence

Integrals of the form

$$I_{k/2}(\lambda) = \int_{\mathcal{S}_{k/2}} g(\mathbf{z})\, \mathrm{e}^{-\lambda f(\mathbf{z})}\, d\mathbf{z}, \tag{6.1}$$

are investigated. The n-dimensional surface, $\mathcal{S}_{k/2}$, starts at an $(n-1)$-dimensional surface through the simple saddle $\mathbf{z}^k$ and runs to infinity in a specified valley of $\Re\{-\lambda(f(\mathbf{z}) - f_k)\}$. For example, if $\mathcal{S}_{k/2}$ is 2-dimensional then it starts at a line through $\mathbf{z}^k$. Again Pham's result that such a hypersurface with quadratic critical points can be deformed into a chain of hypersurfaces each of which encounters a single saddle is used[1], but it must be remarked that, while this is possible in principle, it may be very difficult in practice. The popular procedure of reducing a multidimensional integral to a single integral of Laplace type as mentioned in Chapter 4 is revisited here. Defining $T_{k/2}(\lambda)$ by

$$I_{k/2}(\lambda) = \frac{e^{-\lambda f_k}}{2\lambda^{\frac{n}{2}}} T_{k/2}(\lambda) \tag{6.2}$$

gives

$$T_{k/2}(\lambda) = 2\int_{S_{k/2}} \lambda^{\frac{n}{2}} g(\mathbf{z})\, e^{-\lambda(f(\mathbf{z})-f_k)}\, d\mathbf{z} \tag{6.3}$$

Once more $s = f(\mathbf{z}) - f_k$, which allows s to vary from 0 to ∞ on $\mathcal{S}_{k/2}$, and ω is defined as in (5.29) so that

$$T_{k/2}(\lambda) = 2\lambda^{\frac{n}{2}} \int_0^{\infty e^{-i\theta_\lambda}} e^{-\lambda s} \Delta_k G(s)\, ds, \tag{6.4}$$

$$\omega = \left.\frac{dz_1 \wedge dz_2 \wedge \quad \wedge dz_n}{ds}\right|_{\gamma_{k/2}(s)}, \tag{6.5}$$

$$\Delta_k G(s) = \int_{\gamma_{k/2}(s)} g(\mathbf{z}(s))\, \omega, \tag{6.6}$$

although $\gamma_{k/2}$ can no longer be termed a vanishing cycle as it is not closed (see Figure 6.1). Hence the Picard-Lefschetz formula cannot be applied directly.

To give some interpretation to the form ω, consider how it acts for a 1-dimensional integral: the substitution $s = f(z) - f_k$ implies

$$\frac{ds}{dz} = f'(z) \qquad \text{or} \qquad \frac{dz}{ds} = \frac{1}{f'(z)} \tag{6.7}$$

Thus if

$$G(s) = \frac{g(z(s))}{f'(z(s))} \tag{6.8}$$

then

$$G(s) = g(z(s))\frac{dz}{ds} \tag{6.9}$$

and letting

$$\omega = \left.\frac{dz}{ds}\right|_{\gamma_{k/2}(s)} \tag{6.10}$$

[1] It has been assumed that this is still valid when the original hypersurface has a finite boundary in one direction.

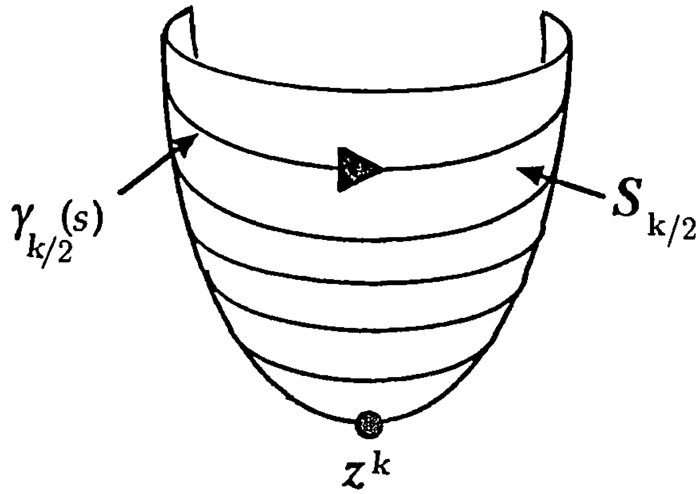


Figure 6 1 Sketch of the analogue of a 'Lefschetz thimble' when a saddle point appears on a boundary of the surface of integration

gives

$$\Delta_k G(s) = \int_{\gamma_{k/2}(s)} g(z(s))\,\omega \tag{6 11}$$

In the case of a saddle appearing at the endpoint, $\gamma_{k/2}(s)$ is a single point whereas for an interior saddle $\gamma_k(s)$ is a pair of points, so that

$$\omega = \frac{dz_+}{ds} - \frac{dz_-}{ds}, \tag{6 12}$$

returning

$$\Delta_k G(s) = \int_{\gamma_k(s)} g(z(s))\omega = \frac{g(z_+(s))}{f'(z_+(s))} - \frac{g(z_-(s))}{f'(z_-(s))} \tag{6 13}$$

as in Chapter 3 However ω will play no further part in determining the terms of the asymptotic expansion

Instead the representation

$$s^{1-\frac{n}{2}}\Delta_k G(s) = \frac{1}{4\pi i}\oint_{B_\xi} \frac{\Delta_k G(\xi)\,\xi^{\frac{1}{2}-\frac{n}{2}}}{\xi^{\frac{1}{2}} - s^{\frac{1}{2}}}\,d\xi \tag{6 14}$$

is used and expanded to give

$$T_{k/2}(\lambda) = \frac{\lambda^{\frac{n}{2}}}{2\pi i}\int_0^{\infty e^{-i\theta_\lambda}} e^{-\lambda s} s^{\frac{n}{2}-1}\left(\sum_{r=0}^{N-1}\oint_{B_\xi}\frac{\Delta_k G(\xi)\,s^{\frac{r}{2}}}{\xi^{\frac{r}{2}+\frac{n}{2}}}d\xi + \oint_{\Gamma_{k/2}}\frac{\Delta_k G(\xi)\,s^{\frac{N}{2}}}{\xi^{\frac{N}{2}+\frac{n}{2}}(1-(\frac{s}{\xi})^{\frac{1}{2}})}d\xi\right)ds, \tag{6 15}$$

where the contours specified are as in Chapter 3.[2] Then

$$T_{k/2}(\lambda) = \sum_{r=0}^{N-1} \frac{T_{(k/2)r}}{\lambda^{\frac{r}{2}}} + R_{k/2}(\lambda, N), \tag{6.16}$$

with

$$T_{(k/2)r}(\lambda) = \frac{\lambda^{\frac{n}{2}+\frac{r}{2}}}{2\pi i} \int_0^{\infty e^{-i\theta_\lambda}} e^{-\lambda s} s^{\frac{r}{2}+\frac{n}{2}-1} \oint_{B_\xi} \frac{\Delta_k G(\xi)}{\xi^{\frac{r}{2}+\frac{n}{2}}} d\xi\, ds, \tag{6.17}$$

$$R_{k/2}(\lambda, N) = \frac{\lambda^{\frac{n}{2}}}{2\pi i} \int_0^{\infty e^{-i\theta_\lambda}} e^{-\lambda s} s^{\frac{N}{2}+\frac{n}{2}-1} \oint_{\Gamma_{k/2}} \frac{\Delta_k G(\xi)}{\xi^{\frac{N}{2}+\frac{n}{2}}(1-(\frac{s}{\xi})^{\frac{1}{2}})} d\xi\, ds \tag{6.18}$$

The expression for $T_{(k/2)r}$ can be manipulated to give

$$\begin{aligned} T_{(k/2)r} &= \frac{\Gamma(\frac{r}{2}+\frac{n}{2})}{2\pi i} \oint_{B_\xi} \frac{\Delta_k G(\xi)}{\xi^{\frac{r}{2}+\frac{n}{2}}} d\xi \\ &= \frac{(\frac{r}{2}+\frac{n}{2}-1)!}{2\pi i} \oint_{B_{zk}} \frac{g(\mathbf{z})}{(f(\mathbf{z})-f_k)^{\frac{r}{2}+\frac{n}{2}}} d\mathbf{z} \end{aligned} \tag{6.19}$$

Using the argument that the adjacent saddles alone[3] are the cause of the series' divergence, $\Gamma_{k/2}$ is deformed into a union of arcs at infinity and paths Γ_m around the other singularities of $\Delta_k G(\xi)$. As this deformation takes place in the 1-dimensional ξ-plane, it is permitted exactly as before [5],[28],[29] and the inability of the Picard-Lefschetz formula to describe the deformation of the $\gamma_{k/2}$ is not as damaging as it first seemed.

$$R_{k/2}(\lambda, N) = \frac{\lambda^{\frac{n}{2}}}{2\pi i} \int_0^{\infty e^{-i\theta_\lambda}} e^{-\lambda s} s^{\frac{N}{2}+\frac{n}{2}-1} \sum_m (-1)^{\gamma_{km}} \oint_{\Gamma_m} \frac{\Delta_k G(\xi)}{\xi^{\frac{N}{2}+\frac{n}{2}}(1-(\frac{s}{\xi})^{\frac{1}{2}})} d\xi\, ds, \tag{6.20}$$

subject to the conditions

(i) $\left|\frac{\Delta_k G(\xi)}{\xi^{\frac{N}{2}+\frac{n}{2}}}\right|$ decays at infinity faster that $\frac{1}{|\xi|}$,

(ii) $\Delta_k G(\xi)$ possesses no singularities other than at the saddles,

(iii) $\xi = 0$ only at the image of the saddle in the region of deformation.

Employing the now familiar change of variables $s = \frac{\nu\xi}{\lambda F_{km}}$ yields an expression involving an integral term similar to the definition of $T_{k/2}$. Thus the formula

$$R_{k/2}(\lambda, N) = \frac{1}{2\pi i} \sum_m \frac{(-1)^{\gamma_{km}}}{(\lambda F_{km})^{\frac{N}{2}}} \int_0^{\infty} \frac{\nu^{\frac{N}{2}-1} e^{-\nu}}{1-\sqrt{\frac{\nu}{\lambda F_{km}}}} T_m\left(\frac{\nu}{F_{km}}\right) d\nu, \tag{6.21}$$

[2] It has been assumed that the integral converges as the contour is shrunk to a neighbourhood of $\xi = 0$ as is the case for analagous integrals in Chapters 3 and 5.

[3] It has been assumed that no other singularities arise, for instance, due to the restriction of $f(\mathbf{z})$ to the boundary.

is arrived at and has the exact same form as the remainder term given in §3 2 1, illustrating again how little the dimension of the original integral effects the final result

When searching for a hypersurface of steepest descent, some discussions of Chapter 4 should be remembered—merely setting $\Im\{\lambda(f(\mathbf{z}) - f_k)\} = 0$ is not enough to specify a unique surface However, it is assumed that this equation is satisfied on any $\mathcal{S}_k$ or $\mathcal{S}_{k/2}$ This also effects the categorisation of adjacent saddles as the Stokes phenomenon can now occur whenever $\mathbf{z}^k$ and $\mathbf{z}^m$ lie in a region in which $\Im\{\lambda(f_m - f_k)\} = 0$ although the hypersurface of steepest descent chosen may not now contain both $\mathbf{z}^k$ and $\mathbf{z}^m$!

6.1 2 Linear Dependence

The procedure to be presented here closely follows that of the previous section In this case, the integrals are of the form

$$I_e(\lambda) = \int_{\mathcal{S}_e} g(\mathbf{z})\mathrm{e}^{-\lambda f(\mathbf{z})} d\mathbf{z} = \frac{\mathrm{e}^{-\lambda f_e}}{\lambda^n} T_e(\lambda), \tag{6 22}$$

where $\mathcal{S}_e$ again has a finite boundary, this time passing through the point $\mathbf{z}^e$, which is not a saddle of $f(\mathbf{z})$, and then runs to infinity in a valley of $\Re\{-\lambda(f(\mathbf{z}) - f_e)\}$ The main steps involve using

$$s = f(\mathbf{z}) - f_e \tag{6 23}$$

and

$$\omega = \left.\frac{dz_1 \wedge dz_2 \wedge \quad \wedge dz_n}{ds}\right|_{\gamma_e(s)} \tag{6 24}$$

to rewrite $G(s)$ as

$$G(s) = \int_{\gamma_e(s)} g(\mathbf{z})\, \omega, \tag{6 25}$$

whose contour integral representation is given by

$$G(s) = \frac{s^{n-1}}{2\pi i} \oint_{B_\xi} \frac{G(\xi)\xi^{1-n}}{\xi - s}\, d\xi \tag{6 26}$$

Then

$$T_e(\lambda) = \sum_{r=0}^{N-1} \frac{T_{er}}{\lambda^r} + R_e(\lambda, N) \tag{6 27}$$

with

$$T_{er} = \frac{\Gamma(r+n)}{2\pi i} \oint_{B_\xi} \frac{G(\xi)}{\xi^{r+n}}\, d\xi = \frac{(r+n-1)!}{2\pi i} \oint_{B_{\mathbf{z}^e}} \frac{g(\mathbf{z})}{(f(\mathbf{z}) - f_e)^{r+n}}\, d\mathbf{z} \tag{6 28}$$

and

$$R_e(\lambda, N) = \frac{1}{2\pi i} \sum_m \frac{(-1)^{\gamma_{em}}}{\lambda^N F_{em}^{N+\frac{n}{2}}} \int_0^\infty \frac{\nu^{N+\frac{n}{2}-1} e^{-\nu}}{1 - \frac{\nu}{\lambda F_{em}}} T_m \left(\frac{\nu}{F_{em}} \right) d\nu, \tag{6 29}$$

having used $s = \frac{\nu \xi}{\lambda F_{em}}$ T_m refers to the expansion about a saddle $\mathbf{z}^m$ as in §5 3

6.2 Application to Single Integrals

Once the method has been successfully extended to determine the contributions of finite endpoints, integrals of the form

$$y(\lambda) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_p^{\alpha_p} e^{-\lambda z_1 z_2 \cdots z_p - \frac{1}{n}(z_1^n + z_2^n + \cdots + z_p^n)} dz_1\, dz_2 \cdots dz_p, \tag{6 30}$$

which are solutions of

$$y^{(n)}(\lambda) - \sum_{r=0}^{p} a_r \lambda^r y^{(r)}(\lambda) = 0, \tag{6 31}$$

can be investigated further (see §4 6, §4 7 1) As a preliminary example, the differential equation

$$\frac{d^3 y}{d\lambda^3} + \lambda \frac{dy}{d\lambda} + y = 0 \tag{6 32}$$

is considered Noting that $a_0 = -1$, $a_1 = -1$ gives $\alpha_1 = 0$ (as $-a_0 = \alpha_1 + 1$) and together with $n = 3$, $p = 1$ the solution is seen to be given by

$$y(\lambda) = \int_0^\infty e^{-\lambda\zeta - \frac{1}{3}\zeta^3} d\zeta \tag{6 33}$$

To verify that this satisfies the equation, differentiate $y(\lambda)$ with respect to λ to yield

$$\frac{dy}{d\lambda} = -\int_0^\infty \zeta e^{-\lambda\zeta - \frac{1}{3}\zeta^3} d\zeta, \tag{6 34}$$

$$\frac{d^3 y}{d\lambda^3} = -\int_0^\infty \zeta^3 e^{-\lambda\zeta - \frac{1}{3}\zeta^3} d\zeta, \tag{6 35}$$

then integrate the latter by parts with respect to ζ,

$$\frac{d^3 y}{d\lambda^3} = \int_0^\infty \{1 - \lambda\zeta\} e^{-\lambda\zeta - \frac{1}{3}\zeta^3} d\zeta, \tag{6 36}$$

and substitute into the equation

Before the method of §6 1 2 (or indeed §3 2 2) can be applied, the transformation $z = \lambda^{-\frac{1}{2}}\zeta$ is made so that $y(\lambda)$ is of the required form

$$y(\lambda) = \lambda^{\frac{1}{2}} \int_0^\infty e^{-\lambda^{3/2}(z + z^3/3)} dz = \lambda^{\frac{1}{2}} \int_0^\infty e^{-\lambda^{3/2} f(z)} dz \tag{6 37}$$

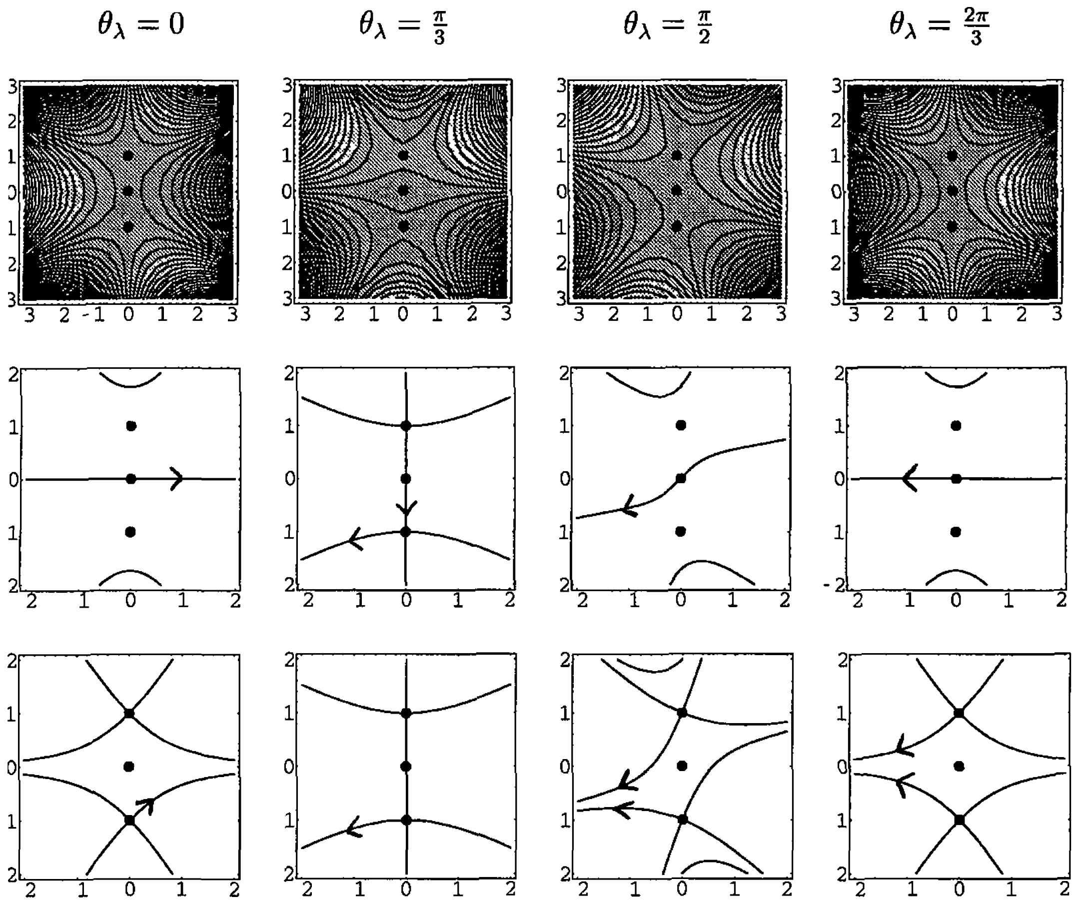


Figure 6 2 Contour plots generated by *Mathematica* for $y(\lambda)$ in (6 37)

To locate the saddles of $f(z)$, $f'(z) = 1+z^2$ is set equal to zero giving $z^1 = i$, $z^2 = -i$ Hence the origin, $z^e = 0$, is a linear endpoint and the contour of integration can be deformed into the line of steepest descent, C_e, which runs from 0 to ∞ with direction $-\arg\lambda/2 = -\theta_\lambda/2$ Thus, in the case of $\lambda \in \mathbb{R}$, C_e is just the real positive axis Now define

$$F_{e1} = f_1 - f_e = f(1) - f(0) = 2i/3, \quad (6\ 38)$$

$$F_{e2} = f_2 - f_e = f(-1) - f(0) = -2i/3 \quad (6\ 39)$$

Saddles z^1 and z^2 are adjacent to z^e if there exists a θ_λ such that $\lambda^{3/2}F_{e1}$ and $\lambda^{3/2}F_{e2}$ are real and positive Here both saddles are adjacent and are actually equidistant from z^e so the order in which they are encountered becomes irrelevant The first order approximation from the endpoint, then its optimally truncated series and finally the first hyperasymptotic iteration are computed As z^1 and z^2 are also mutually adjacent, after the first iteration both scattering paths alternate from one saddle to the other

T_{er} can be determined using either (3 41) or Dingle's formula for the asymptotic expansion of an integral at a linear limit of integration [17] Similarly, T_{1r} and T_{2r} can be determined from (3 14) or Dingle's expansion due to an interior stationary point

$$T_{er} = \frac{(-1)^{\frac{3r}{2}}\,\Gamma(1+\frac{3r}{2})}{3^{\frac{r}{2}}\,\Gamma(1+\frac{r}{2})} \quad r \text{ even}, \qquad T_{er} = 0 \quad r \text{ odd},$$

$$T_{1r} = \frac{\imath^{-r-\frac{1}{2}}\Gamma(\frac{1}{2}-r)\Gamma(\frac{1}{2}+r)}{\Gamma(\frac{1}{2}-3r)\Gamma(1+2r)(3\imath)^{2r}}, \qquad T_{2r} = \frac{\imath^{-r+\frac{1}{2}}(-1)^r\Gamma(\frac{1}{2}-r)\Gamma(\frac{1}{2}+r)}{\Gamma(\frac{1}{2}-3r)\Gamma(1+2r)(3\imath)^{2r}} \tag{6 40}$$

(Note that the T_{er} obtained here are exactly those coefficients of x obtained in §4 7 1 as would be expected)

The series are truncated optimally so the truncation points are calculated from

$$N_0 = \mathrm{Int}\left|\lambda^{\frac{3}{2}}F_{e1}\right|, \qquad N_1 = \mathrm{Int}\frac{N_0}{1+|F_{e1}/F_{12}|}, \tag{6 41}$$

and the hyperterminant integral needed is now given by

$$\begin{aligned} K_{rm} &= \frac{(-1)^{\gamma_{em}}}{2\pi\imath\lambda^{3\frac{N_0}{2}}F_{em}^{N_0-r+\frac{1}{2}}}\int_0^{\infty}\frac{e^{-\nu}\nu^{N_0-r-\frac{1}{2}}}{1-\frac{\nu}{\lambda^{3/2}F_{em}}}d\nu \\ &= \frac{\lambda^{-\frac{3r}{2}+\frac{1}{2}}(-1)^{\gamma_{em}+N_0-r}}{2\pi}e^{-\lambda^{3/2}F_{em}}\,\Gamma(N_0-r+\tfrac{1}{2})\,\Gamma(r-N_0+\tfrac{1}{2},-\lambda^{\frac{3}{2}}F_{em}) \end{aligned} \tag{6 42}$$

where m takes the values 1 and 2 and $\gamma_{e1} = \gamma_{e2} = 0$ However, if $\Re\lambda^{3/2} > 0$ holds, (6 37) can be solved in terms of generalised hypergeometric and Bessel functions

$$y(\lambda) = \frac{\lambda^{\frac{1}{2}}}{18}(4\sqrt{3}\pi(J_{-1/3}(2\lambda^{\frac{3}{2}}/3) - J_{1/3}(2\lambda^{\frac{3}{2}}/3)) + 9\lambda^{\frac{3}{2}}\,{}_1F_2(1,4/3,5/3,-x^3/9)) \tag{6 43}$$

This provides a means of putting the results obtained here in perspective as such—at least when $\theta_\lambda < \frac{2\pi}{3}$ *Mathematica*'s tabled values of the special functions were used to do this Various values of θ_λ were used as before but $|\lambda|$ was kept fixed at 8 32033529220761645812 so that $|\lambda|^{\frac{3}{2}} = 24$, giving $N_0 = 16$ and $N_1 = 5$ As $N_3 = 1$ the hyperasymptotic scheme automatically halts after just 3 iterations

Numerical results are presented in Table 6 1 Contour plots for the function are given in Figure 6 2—the first row represents the hills and valleys, the shading becoming darker as the valleys deepen, the second row shows the constant phase contours or lines of steepest decent passing through z^e and the third, the lines of steepest descent which pass through either z^1 or z^2

Consider now the second order differential equation

$$\frac{d^2y}{d\lambda^2} + \lambda\frac{dy}{d\lambda} + y = 0, \tag{6 44}$$

for which a solution of the form

$$y(\lambda) = \lambda\int_0^\infty \mathrm{e}^{-\lambda^2(z+z^2/2)}dz \tag{6 45}$$

exists This integral has a single saddle point at $z^1 = -1$, which is adjacent to the endpoint $z^e = 0$ Hence in this case, the hyperasymptotic scheme would halt after the first iteration as the corresponding expression for $y(\lambda)$ is exact at this level and no greater precision could be achieved

It should be noted that recent methods providing hyperasymptotic results for solutions of nth order differential equations, namely that of Olde Daalhuis [57], could also be applied to the differential equations discussed above However, as these in turn rely on finding an integral representation for the remainder of the asymptotic expansion of the solution of such an equation, an approach which starts with the integral representation of the solution can be useful in practice In any case, the procedure detailed here provides a convenient alternative

6.3 Application to Double Integrals

As an example of a double integral, consider

$$y(\lambda) = \int_0^\infty\int_0^\infty \mathrm{e}^{-\lambda\zeta_1\zeta_2-\frac{1}{3}(\zeta_1^3+\zeta_2^3)}d\zeta_1\,d\zeta_2, \tag{6 46}$$

which is a solution of the third order equation

$$\frac{d^3y}{d\lambda^3} + \lambda^2\frac{d^2y}{d\lambda^2} + 3\lambda\frac{dy}{d\lambda} + y = 0 \tag{6 47}$$

$\theta_\lambda = 0$	0 1201874641922840	$\frac{e^{-\lambda^{3/2} f_e}}{\lambda}$
	0 1197834257208534	$\frac{e^{-\lambda^{3/2} f_e}}{\lambda} H_0$
	0 1197834373363305	$\frac{e^{-\lambda^{3/2} f_e}}{\lambda} (H_0 + H_1)$
	0 1197834373363271	Special Functions
$\theta_\lambda = \pm\frac{\pi}{3}$	0 0600937320961420 $\mp i$0 1040853972069505	$\frac{e^{-\lambda^{3/2} f_e}}{\lambda}$
	0 0603010550704043 $\mp i$0 1044609380518612	$\frac{e^{-\lambda^{3/2} f_e}}{\lambda} H_0$
	0 0603106051922058 $\mp i$0 1044609154785759	$\frac{e^{-\lambda^{3/2} f_e}}{\lambda} (H_0 + H_1)$
	0 0603106051921960 $\mp i$0 1044609154785703	Special Functions
$\theta_\lambda = \pm\frac{\pi}{2}$	0 0 $\mp i$0 1201874641922840	$\frac{e^{-\lambda^{3/2} f_e}}{\lambda}$
	−0 0004159673400131 $\mp i$0 1201732028915344	$\frac{e^{-\lambda^{3/2} f_e}}{\lambda} H_0$
	−0 0004159521093458 $\mp i$0 1201732029536831	$\frac{e^{-\lambda^{3/2} f_e}}{\lambda} (H_0 + H_1)$
	−0 0004169609858941 $\mp i$0 1201605547873280	Special Functions
$\theta_\lambda = \pm\frac{2\pi}{3}$	−0 0600937320961420 $\mp i$0 1040853972069505	$\frac{e^{-\lambda^{3/2} f_e}}{\lambda}$
	−0 0598917128604267 $\mp i$0 1037354896265854	$\frac{e^{-\lambda^{3/2} f_e}}{\lambda} H_0$
	−0 0598917030785903 $\mp i$0 1037354952741313	$\frac{e^{-\lambda^{3/2} f_e}}{\lambda} (H_0 + H_1)$

Table 6 1 Values of $y(\lambda)$ in (6 37) obtained using the method of §6 1 2

As before, a transformatıon ıs necessary ın order to apply the method of §6 1 1 In thıs case, $z_1 = \zeta_1/\lambda$, $z_2 = \zeta_2/\lambda$ gıvıng

$$y(\lambda) = \lambda^2 \int_0^\infty \int_0^\infty e^{-\lambda^3(z_1 z_2 + \frac{1}{3}(z_1^3+z_2^3))} dz_1\, dz_2 \tag{6 48}$$

It mıght appear that thıs ıntegral does not fit ınto the class specıfied ın §6 1 1, because the boundary restrıcts the domaın to be 'quarterly'-ınfinıte as opposed to semı-ınfinıte But ıf a lıttle extra care ıs taken when calculatıng the contrıbutıon from the orıgın, whıch lıes on thıs boundary, an applıcatıon of the method ıs possıble

The saddles of $f(\mathbf{z})$ are those poınts whıch satısfy $\frac{\partial f}{\partial z_1} = 0$ and $\frac{\partial f}{\partial z_2} = 0$ sımultaneously Hence, there are 4 saddles

$$\mathbf{z}^0 = (0,0), \quad \mathbf{z}^1 = (-1,-1), \quad \mathbf{z}^2 = ((-1)^{1/3}, -(-1)^{2/3}), \quad \mathbf{z}^3 = (-(-1)^{2/3}, (-1)^{1/3})$$

To start, the surface of ıntegratıon ıs deformed ınto a surface of steepest descent though the orıgın On such a surface, $\mathcal{S}_{0/2}$, $\mathbf{z}$ takes values from 0 to ∞ wıth $\theta_{\mathbf{z}} = -\theta_\lambda$ For ınstance, when $\theta_\lambda = 0$, $\mathcal{S}_{0/2}$ ıs the orıgınal surface of ıntegratıon Expressıons for T_{1r}, T_{2r}, T_{3r} are best found usıng Dıngle's formula for the contrıbutıon of a quadratıcally behavıng ınterıor statıonary poınt of a double ıntegral [17]

$$T_{1r} = T_{2r} = T_{3r} = \frac{2\pi\,(-1)^r\,\Gamma(3r+1)}{3^{2r+\frac{1}{2}}\,\Gamma(r+1)\,\Gamma(r+1)} \tag{6 49}$$

However, thıs formula does not apply ın the determınatıon of $T_{(0/2)r}$ because, firstly, the saddle no longer lıes ın the ınterıor of the domaın of ıntegratıon and secondly,

$$\left.\frac{\partial^2 f}{\partial z_1^2}\right|_{(0,0)} = \left.\frac{\partial^2 f}{\partial z_2^2}\right|_{(0,0)} = 0, \tag{6 50}$$

whıch ıs not permıtted As $\left.\frac{\partial^2 f}{\partial z_1 \partial z_2}\right|_{(0,0)} = 1$, $\mathbf{z}^0$ ıs stıll a non-degenerate sımple saddle [22] but ıt ıs better to use eıther the method of resıdues wıth Cauchy's ıntegral theorem as shown ın §4 2 or the Mellın-Barnes approach of §4 7 1 to compute ıts assocıated asymptotıc expansıon

Usıng the latter, ıt should be noted that ın thıs case the sequences of poles arısıng from $\Gamma(\frac{1+\alpha_1-\tau}{n})$ and $\Gamma(\frac{1+\alpha_2-\tau}{n})$ coıncıde as $\alpha_1 = \alpha_2 = 0$ and ınstead of two sequences of sımple poles, there ıs now just one sequence of double poles and the coefficıents ın the expansıon thus take the form

$$T_{(0/2)r} = \frac{2\Gamma(1+\frac{3r}{2})(\Psi(1+\frac{3r}{2}) - \frac{2}{3}\log 3 - \log\lambda - \frac{2}{3}\Psi(1+\frac{r}{2}))}{3^r\,\Gamma^2(1+\frac{r}{2})} \qquad r \text{ even},$$

$$T_{(0/2)r} = 0 \qquad r \text{ odd} \tag{6 51}$$

The appearance of a $\log\lambda$ term in what should be the 'coefficients' of $\lambda^{-r/2}$ should be noted This provides an added complication as the theory described precludes such a possibility However, the procedure will be followed as normal to see how the method fares in such a situation

Calculating $F_{01} = F_{02} = F_{03} = 1/3$, shows that $\Im F_{0j} = 0$, for any j, and it emerges that the Stokes phenomenon occurs between any pair of saddles However, this is not enough to guarantee adjacency—the Riemann sheet structure must also be investigated Using the numerical solution of a set of algebraic equations as advocated by Howls [29] presents problems in this case, due to the cancellations caused by the equivalence of the coefficients T_{jr} and singulants F_{0j}, $j \in \{1,2,3\}$ This would seem to indicate that the saddles $\mathbf{z}^1$, $\mathbf{z}^2$ and $\mathbf{z}^3$ are on different Riemann sheets so while each is adjacent to $\mathbf{z}^0$ independently, none of the three are adjacent to each other Also note that for the first stage of hyperasymptotics, the saddles $\mathbf{z}^1$, $\mathbf{z}^2$ and $\mathbf{z}^3$ are equidistant ($F_{01} = F_{02} = F_{03} = 1/3$), so the order in which the scattering takes place is irrelevant

The calculations then proceed exactly as before θ_λ was allowed to change, but $|\lambda|$ was fixed at 2 88449914061481676 giving $|\lambda|^3 = 24$ Retaining the principle of optimal truncation of the series as in Chapter 3, gives $N_0 = \text{Int}|2\lambda^3 F_{0*}| = 16$, where '$*$' represents the nearest of the adjacent saddles on the first iteration It should be noted here that the cuts in the ξ-plane from points corresponding to $\mathbf{z}^1$, $\mathbf{z}^2$ and $\mathbf{z}^3$ are collinear as $\Im F_{01} = \Im F_{02} = \Im F_{03}$ Thus the cuts must be suitably oriented and indented to avoid collision of the contours Γ_1, Γ_2 and Γ_3 To calculate N_1, the formula

$$N_1 = \text{Int}\frac{N_0}{1 + \min_{l\in\{1,2,3\}}\{|F_{0l}/F_{l*}|\}} \tag{6 52}$$

would normally be used, with '$*$' now representing the nearest of the adjacent saddles on the second iteration So, in this case, as $F_{12} = F_{13} = F_{23} = 0$, N_1 is estimated as

$$N_1 = \text{Int}\frac{N_0}{1 + |F_{0l}/F_{l0}|} = 8 \tag{6 53}$$

To have some means of judging the numerical results, $y(\lambda)$ is expressed in terms of special functions for $\Re\lambda^{3/2} > 0$ Thus

$$y(\lambda) = \frac{2}{9}\mathrm{e}^{-\lambda^3/6}\lambda^{\frac{1}{2}}\pi^{\frac{1}{2}}K_{1/6}(\lambda^3/6) + \frac{\lambda^2}{2}\,{}_2F_2(1,1,4/3,5/3,-\lambda^3/3) \tag{6 54}$$

and can be evaluated at various values of λ using *Mathematica* once again (Table 6.2). Though the method provides the asymptotic behaviour for $\lambda \to \infty$, note that it performs quite well even for a small value of λ and given the occurrence of the $\log \lambda$ term. This would seem to suggest that the method could be modified to encompass such integrals and perhaps then provide an improvement on the results given here.

As a second example, consider the differential equation

$$\frac{d^4y}{d\lambda^4} - \lambda^2 \frac{d^2y}{d\lambda^2} - 3\lambda \frac{dy}{d\lambda} - y = 0. \tag{6.55}$$

It can be shown that

$$y(\lambda) = \lambda \int_0^\infty \int_0^\infty \mathrm{e}^{-\lambda^2(z_1 z_2 + \frac{1}{4}(z_1^4 + z_2^4))} dz_1\, dz_2 \tag{6.56}$$

is a solution of (6.55). This integral has 8 saddles in all, including one at the origin. In general, the number of saddlepoints of the phase function increases with the order of the equation. Though the computations become more laborious, there is no corresponding increase in complexity and the method presented holds its own. (However, in this case a $\log \lambda$ term can again be expected to arise in the coefficients of $\lambda^{-r/2}$ in the asymptotic expansion about the critical point at the origin following the pattern of the last example and the fact that again $g(\mathbf{z}) = 1$.)

Finally, consider the integral of Airy function type

$$y(\lambda) = -\frac{\lambda}{4\pi^2} \int_1^\infty \int_1^\infty \mathrm{e}^{-\lambda^{3/2}(z_1 - z_1^3/3 + z_2 - z_2^3/3)} dz_1 dz_2, \tag{6.57}$$

to which again the method of §6.1.1 could be applied. This function may appear to be somewhat artificial in that it is merely the product of the two single integrals

$$y_1(\lambda) = \frac{\lambda^{1/2}}{2\pi i} \int_1^\infty \mathrm{e}^{-\lambda^{3/2}(z_1 + z_1^3/3)} dz_1, \tag{6.58}$$

$$y_2(\lambda) = \frac{\lambda^{1/2}}{2\pi i} \int_1^\infty \mathrm{e}^{-\lambda^{3/2}(z_2 + z_2^3/3)} dz_2, \tag{6.59}$$

but this allows the exact value to be calculated. Such an example also provides a straightforward illustration of the method as the added difficulties encountered above do not appear. There are four saddlepoints to be considered, lying at

$$\mathbf{z}^1 = (1,1), \quad \mathbf{z}^2 = (1,-1), \quad \mathbf{z}^3 = (-1,1), \quad \mathbf{z}^4 = (-1,-1),$$

$\theta_\lambda = 0$	0 68787147928876982 4454	$\frac{2e^{-\lambda^{3/2}f_e}}{\lambda}$
	0 69655099203312530 8126	$\frac{2e^{-\lambda^{3/2}f_e}}{\lambda}H_0$
	0 69664614243312530 8126	$\frac{2e^{-\lambda^{3/2}f_e}}{\lambda}(H_0+H_1)$
	0 69665465184513921 2467	Special Functions
$\theta_\lambda = \pm\frac{\pi}{6}$	0 68647495417717314 3209 $\mp i$0 18673345981915887 1226	$\frac{2e^{-\lambda^{3/2}f_e}}{\lambda}$
	0 68580803674131091 3772 $\mp i$0 19614651713810321 8913	$\frac{2e^{-\lambda^{3/2}f_e}}{\lambda}H_0$
	0 68592214111576015 7830 $\mp i$0 19613714709623620 390	$\frac{2e^{-\lambda^{3/2}f_e}}{\lambda}(H_0+H_1)$
	0 68591599819453677 8197 $\mp i$0 19610755110355369 4091	Special Functions
$\theta_\lambda = \pm\frac{\pi}{3}$	0 65834029929483699 4228 $\mp i$0 41419261845422175 2454	$\frac{2e^{-\lambda^{3/2}f_e}}{\lambda}$
	0 64683559941959944 8626 $\mp i$0 41248498428580813 6829	$\frac{2e^{-\lambda^{3/2}f_e}}{\lambda}H_0$
	0 64685362905957297 288 $\mp i$0 41242342126402132 344	$\frac{2e^{-\lambda^{3/2}f_e}}{\lambda}(H_0+H_1)$
	0 64695606101711972 5217 $\mp i$0 41237964115634625 1997	Special Functions
$\theta_\lambda = \pm\frac{\pi}{2}$	0 54456467144590278 1512 $\mp i$0 68787147928876982 4454	$\frac{2e^{-\lambda^{3/2}f_e}}{\lambda}$
	0 55026543386284987 9651 $\mp i$0 67282537627196325 0515	$\frac{2e^{-\lambda^{3/2}f_e}}{\lambda}H_0$
	0 55031437135578314 291 $\mp i$0 67272187396407376 680	$\frac{2e^{-\lambda^{3/2}f_e}}{\lambda}(H_0+H_1)$

Table 6 2 Values of $y(\lambda)$ in (6 48) obtained using the method of §6 1 1 the first of which appears on the boundary. The coefficients of the expansions about each of the saddles $\mathbf{z}^1$, $\mathbf{z}^2$ and $\mathbf{z}^3$ could be calculated using Dingle's formula for interior stationary points of double integrals [17]. While for $\mathbf{z}^4$, the relevant formula to find the expansion about the endpoint of each of the single integrals could be used and then the expansions multiplied together. Having done this, the subsequent computations proceed as normal.

6.4 Logarithmic Singularities

In this section the occurence of a logarithmic term in the function $g(\mathbf{z})$ is investigated. For simplicity and in order to ease the geometric visualisation of the problem, the investigation is carried out for a single integral. However, the same technique should be valid for integrals of any finite dimension. Integrals of the form

$$I(\lambda) = \int_{\mathcal{C}_k(\theta_\lambda)} g(z)\mathrm{e}^{-\lambda f(z)}\, dz, \tag{6.60}$$

where $\mathcal{C}_k(\theta_\lambda)$ is an infinite contour of steepest descent as in (3.1), are considered. The presence of a logarithmic expression in $g(z)$, $\log(z - z^l)$ say, means the introduction of a cut in the complex z-plane running from z^l to ∞. The cut is chosen to lie along a steepest descent path, $\mathcal{C}_l(\theta_\lambda)$, from z^l to a valley of $\Re\{-\lambda(f(z) - f_l)\}$, assuming to start that z^l does not lie on $\mathcal{C}_k(\theta_\lambda)$.

Once again

$$I_k(\lambda) = \frac{\mathrm{e}^{-\lambda f_k}}{\lambda^{\frac{1}{2}}} T_k(\lambda), \qquad f_k = f(z^k), \qquad f_l = f(z^l) \tag{6.61}$$

and the change of variables,

$$s = f(z) - f_k, \tag{6.62}$$

is made. Thus the logarithmic cut in the z-plane becomes a cut in the s-plane. As it is still valid to write

$$\Delta_k G(s) = \left\{ \frac{g(z_+(s))}{f'(z_+(s))} - \frac{g(z_-(s))}{f'(z_-(s))} \right\} = \frac{1}{2\pi i s^{\frac{1}{2}}} \oint_{\Gamma_k} \frac{\Delta_k G(\xi)\, \xi^{\frac{1}{2}}}{\xi - s}\, d\xi, \tag{6.63}$$

the truncated expansion of $T_k(\lambda)$ remains unchanged with

$$T_{kr} = \frac{(r - \frac{1}{2})!}{2\pi i} \oint_{B_{zk}} \frac{g(z)}{(f(z) - f_k)^{r + \frac{1}{2}}}\, dz. \tag{6.64}$$

The remainder term, however, undergoes a slight alteration:

$$\begin{aligned} R_k(\lambda, N) &= \frac{\lambda^{\frac{1}{2}}}{2\pi i}\sum_m (-1)^{\gamma_{km}} \int_0^{\infty e^{-i\theta_\lambda}} e^{-\lambda s} s^{N-\frac{1}{2}} \oint_{\Gamma_m} \frac{\Delta_k G(\xi)}{\xi^{N+\frac{1}{2}}(1-\frac{s}{\xi})}\, d\xi\, ds \\ &\quad + \frac{\lambda^{\frac{1}{2}}(-1)^{\gamma_{kl}}}{2\pi i} \int_0^{\infty e^{-i\theta_\lambda}} e^{-\lambda s} s^{N-\frac{1}{2}} \oint_{\Gamma_l} \frac{\Delta_k G(\xi)}{\xi^{N+\frac{1}{2}}(1-\frac{s}{\xi})}\, d\xi\, ds \\ &= R_{km}(\lambda, N) + R_{kl}(\lambda, N), \end{aligned} \tag{6.65}$$

where Γ_l is an infinite contour surrounding the extra cut arising from the logarithmic term. Substituting $s = \frac{\nu\xi}{\lambda F_{km}}$ in the first term, $R_{km}(\lambda, N)$, and collapsing the loop contours Γ_m onto their corresponding cuts, gives rise to the appearance of the $T_m\left(\frac{\nu}{F_{km}}\right)$ factors as before. The contour involving Γ_l, however, requires a separate investigation.

First $F_{kl} = f_l - f_k$ is defined and the substitution, $s = \frac{\nu\xi}{\lambda F_{kl}}$, is made in $R_{kl}(\lambda, N)$. This yields

$$R_{kl}(\lambda, N) = \frac{(-1)^{\gamma_{kl}}}{2\pi i \lambda^N F_{kl}^{N+\frac{1}{2}}} \int_0^{\infty} \frac{\nu^{N-\frac{1}{2}} e^{-\nu}}{1 - \frac{\nu}{\lambda F_{kl}}} \oint_{\Gamma_l} \Delta_k G(\xi) e^{-\frac{\nu}{F_{kl}}(\xi - f_l + f_k)}\, d\xi\, d\nu, \tag{6.66}$$

with γ_{kl} taking a value of 0 or 1 depending on the orientation of Γ_{kl}. In the same manner as for the singularities arising from the adjacent saddles, the loop contour, Γ_l, will be collapsed onto the cut corresponding to the logarithmic singularity at z^l and the discontinuity of the integrand across the cut is determined. This is easier to do in terms of the original variables, so reinstating

$$\xi = f(z) - f_k, \qquad \frac{d\xi}{dz} = f'(z) \tag{6.67}$$

gives

$$\oint_{\Gamma_l} \Delta_k G(\xi) e^{-\frac{\nu}{F_{kl}}(\xi - f_l + f_k)} d\xi = \oint_{\Gamma_{z^l}} g(z) e^{-\frac{\nu}{F_{kl}}(f(z) - f_l)} dz, \tag{6.68}$$

where Γ_{z^l} is an infinite loop surrounding the cut from z^l—a z-plane analogue of Γ_l. If $g(z)$ is of the form

$$g(z) = \tilde{g}(z) + c_0 \log(z - z^l), \tag{6.69}$$

where $\tilde{g}(z)$ is analytic, then the discontinuity of the integrand on the right-hand side of (6.68) can be calculated to be

$$2\pi i\, c_0 e^{-\frac{\nu}{F_{kl}}(f(z) - f_l)}, \tag{6.70}$$

using

$$f(|z|e^{\imath(\phi+2\pi)}) = f(|z|e^{\imath\phi}), \tag{6 71}$$

$$\tilde{g}(|z|e^{\imath(\phi+2\pi)}) = \tilde{g}(|z|e^{\imath\phi}), \tag{6 72}$$

$$\log(|z|e^{\imath(\phi+2\pi)} - z^l) = \log(|z|e^{\imath\phi} - z^l) + 2\pi\imath \tag{6 73}$$

Thus

$$\begin{aligned} R_{kl}(\lambda, N) &= \frac{(-1)^{\gamma_{kl}}}{2\pi\imath\lambda^N F_{kl}^{N+\frac{1}{2}}} \int_0^\infty \frac{\nu^{N-\frac{1}{2}}e^{-\nu}}{1-\frac{\nu}{\lambda F_{kl}}} \int_{C_l(\theta_\lambda)} 2\pi\imath\, c_0 e^{-\frac{\nu}{F_{kl}}(f(z)-f_l)} dz\, d\nu \\ &= \frac{(-1)^{\gamma_{kl}} c_0}{\lambda^N F_{kl}^{N-\frac{1}{2}}} \int_0^\infty \frac{\nu^{N-\frac{3}{2}}e^{-\nu}}{1-\frac{\nu}{\lambda F_{kl}}} \int_{C_l(\theta_\lambda)} \frac{\nu}{F_{kl}} e^{-\frac{\nu}{F_{kl}}(f(z)-f_l)} dz\, d\nu \\ &= \frac{(-1)^{\gamma_{kl}} c_0}{\lambda^N F_{kl}^{N-\frac{1}{2}}} \int_0^\infty \frac{\nu^{N-\frac{3}{2}}e^{-\nu}}{1-\frac{\nu}{\lambda F_{kl}}} T_l\left(\frac{\nu}{F_{kl}}\right) d\nu \end{aligned} \tag{6 74}$$

This leads to

$$\begin{aligned} R_k(\lambda, N) &= \frac{1}{2\pi\imath} \sum_m \frac{(-1)^{\gamma_{km}}}{(\lambda F_{km})^N} \int_0^\infty \frac{\nu^{N-1}e^{-\nu}}{1-\frac{\nu}{\lambda F_{km}}} T_m\left(\frac{\nu}{F_{km}}\right) d\nu \\ &\quad + \frac{(-1)^{\gamma_{kl}} c_0}{\lambda^N F_{kl}^{N-\frac{1}{2}}} \int_0^\infty \frac{\nu^{N-\frac{3}{2}}e^{-\nu}}{1-\frac{\nu}{\lambda F_{kl}}} T_l\left(\frac{\nu}{F_{kl}}\right) d\nu \end{aligned} \tag{6 75}$$

The integral,

$$T_l\left(\frac{\nu}{F_{kl}}\right) = \int_{C_l(\theta_\lambda)} \frac{\nu}{F_{kl}} e^{-\frac{\nu}{F_{kl}}(f(z)-f_l)} dz \tag{6 76}$$

is a Laplace type integral whose maximum occurs at the finite endpoint z^l Thus it can be expanded in a manner similar to that of §3 2 2 Letting

$$t = f(z) - f_l, \qquad H(t) = \frac{1}{f'(z(t))} \tag{6 77}$$

gives

$$T_l\left(\frac{\nu}{F_{kl}}\right) = \int_0^{\infty e^{-\imath\theta_\nu}} \frac{\nu}{F_{kl}} \frac{1}{f'(z(t))} e^{-\frac{\nu}{F_{kl}}t} dt = \int_0^{\infty e^{-\imath\theta_\nu}} \frac{\nu}{F_{kl}} H(t) e^{-\frac{\nu}{F_{kl}}t} dt, \tag{6 78}$$

where θ_ν represents the argument of $\frac{\nu}{F_{kl}}$ Then writing

$$H(t) = \frac{1}{2\pi\imath} \oint_{\Gamma_l} \frac{H(\xi)}{\xi - t} dt \tag{6 79}$$

results ın

$$T_l\left(\frac{\nu}{F_{kl}}\right) = \frac{1}{2\pi\imath}\int_0^{\infty e^{-\imath\theta_\nu}} \frac{\nu}{F_{kl}} e^{-\frac{\nu}{F_{kl}}t}\left[\oint_{\Gamma_l}\sum_{r=0}^{M-1}\frac{H(\xi)t^r}{\xi^{r+1}}\,d\xi + \oint_{\Gamma_l}\frac{H(\xi)t^M}{\xi^{M+1}(1-\frac{t}{\xi})}\,d\xi\right] dt \tag{6 80}$$

If

$$T_l\left(\frac{\nu}{F_{kl}}\right) = \sum_{r=0}^{M-1}\frac{T_{lr}}{(\frac{\nu}{F_{kl}})^r} + R_l\left(\frac{\nu}{F_{kl}}, M\right), \tag{6 81}$$

then

$$T_{lr} = \frac{1}{2\pi\imath}\int_0^{\infty e^{-\imath\theta_\nu}}\left(\frac{\nu}{F_{kl}}\right)^{r+1} e^{-\frac{\nu}{F_{kl}}t}\oint_{\Gamma_l}\frac{H(\xi)t^r}{\xi^{r+1}}\,d\xi\,dt = \frac{r!}{2\pi\imath}\oint_{\Gamma_{z^l}}\frac{1}{(f(z)-f_l)^{r+1}}dz, \tag{6 82}$$

and

$$R_l\left(\frac{\nu}{F_{kl}}, M\right) = \frac{1}{2\pi\imath}\int_0^{\infty e^{-\imath\theta_\nu}}\frac{\nu}{F_{kl}} e^{-\frac{\nu}{F_{kl}}t}\oint_{\Gamma_l}\frac{H(\xi)t^M}{\xi^{M+1}(1-\frac{t}{\xi})}\,d\xi\,dt \tag{6 83}$$

The contour, Γ_l, ın the remaınder term, can ın turn be deformed to sımılar contours about the sıngularıtıes arısıng from any adȷacent saddles Thıs ıllustrates that the presence of a logarıthmıc term does not ınterfere wıth the ıteratıon process whıch obtaıns a hyperasymptotıc scheme The new hypertermınant ıntegrals,

$$K_{kl,r} = \frac{(-1)^{\gamma_{kl}}c_0}{\lambda^N F_{kl}^{N-r-\frac{1}{2}}}\int_0^\infty \frac{\nu^{N-r-\frac{3}{2}}e^{-\nu}}{1-\frac{\nu}{\lambda F_{kl}}}\,d\nu, \tag{6 84}$$

though takıng a slıghtly dıfferent form to the

$$K_{km,r} = \frac{(-1)^{\gamma_{km}}}{2\pi\imath\lambda^N F_{km}^{N-r}}\int_0^\infty \frac{\nu^{N-r-1}e^{-\nu}}{1-\frac{\nu}{\lambda F_{km}}}\,d\nu, \tag{6 85}$$

possess the same unıversal propertıes of prevıous hypertermmants—dependıng only on the effectıve dıstance between sıngularıtıes and the poınt of truncatıon of the serıes Thus the hyperasymptotıc scheme becomes

$$\begin{aligned} I_k(\lambda) &= \frac{e^{-\lambda f_k}}{\lambda^{\frac{1}{2}}}\left(\sum_{r=0}^{N_0-1}T_{kr}K_{k,r} + \left(\sum_m\sum_{r=0}^{N_1-1}T_{mr}K_{km,r} + \sum_{r=0}^{M_1-1}T_{lr}K_{kl,r}\right) + \quad\right) \\ &= \frac{e^{-\lambda f_k}}{\lambda^{\frac{1}{2}}}(H_0 + H_1 + \quad) \end{aligned} \tag{6 86}$$

Each tıme a scatterıng from a saddle, z^m, to z^l occurs, the contrıbutıon has coeffıcıents T_{lr} However, because of the presence of $H(\xi)$ ın place of $\Delta_k G(\xi)$ ın R_l, the contrıbutıons arısıng from a scatterıng m the opposıte dırectıon are no longer exactly of the form T_m Nevertheless, they are very computable

In order to compute the new hyperterminant integrals in a practical application the formula

$$\begin{aligned} K_{kl,r} &= \frac{(-1)^{\gamma_{kl}} c_0}{\lambda^N F_{kl}^{N-r-\frac{1}{2}}} \int_0^\infty \frac{\nu^{N-r-\frac{3}{2}} e^{-\nu}}{1-\frac{\nu}{\lambda F_{kl}}} d\nu, \\ &= \frac{(-1)^{\gamma_{kl}-N+1/2} c_0}{\lambda^{1/2}} e^{-F} \Gamma(N-r-1/2)\Gamma(r-N-3/2, -\lambda F_{kl}) \quad (6\ 87) \end{aligned}$$

can be used

6 4 1 Example

As an example, consider

$$I(\lambda) = \int_{-\infty}^{\infty} \log\{z-(2+2\imath)\} e^{-\lambda z^2} dz \qquad (6\ 88)$$

This can be integrated directly giving a complicated expression involving series of special functions However, the method of §6 4 will give the asymptotic behaviour as $\lambda \to \infty$, which is shown to be m good agreement with the exact values, even for small λ Here $|\lambda|$ was chosen as 1, as *Mathematica* experienced problems while computing the exact solution for larger values of $|\lambda|$

The function, $f(z) = z^2$, has one saddlepoint, $z^1 = 0$, and the original contour of integration is, in fact, a contour of steepest descent through z^1 The coefficients in the expansion about this saddle are given by

$$\begin{aligned} T_{10} &= \Gamma(\tfrac{1}{2}) \log(-2-2\imath), \\ T_{1r} &= \frac{(1-r)\Gamma(\frac{1}{2})}{\Gamma(\frac{r}{2}+1)\, 2^r\, (2+2\imath)^r} && \text{r even}, \\ T_{1r} &= 0 && \text{r odd}, \qquad (6\ 89) \end{aligned}$$

using (6 64), whereas the coefficients T_{lr} are of the form

$$T_{lr} = \frac{(-1)^r (2r-1)!!}{2^{r+1}(2+2\imath)^{2r+1}}, \qquad (6\ 90)$$

with $z^l = 2+2\imath$ in (6 82) The optimal truncation points for the series are calculated as

$$N_0 = \mathrm{Int}|\lambda F_{1l}| = 8, \qquad M_1 = \mathrm{Int}\left|\frac{N_0}{1+|F_{1l}/F_{l1}|}\right| = \frac{8}{2} = 4 \qquad (6\ 91)$$

following the theory of [5], [28] However, as acknowledged in §3 3 such formula do not attain levels of overall numerical precision that compete with estimates of Olver

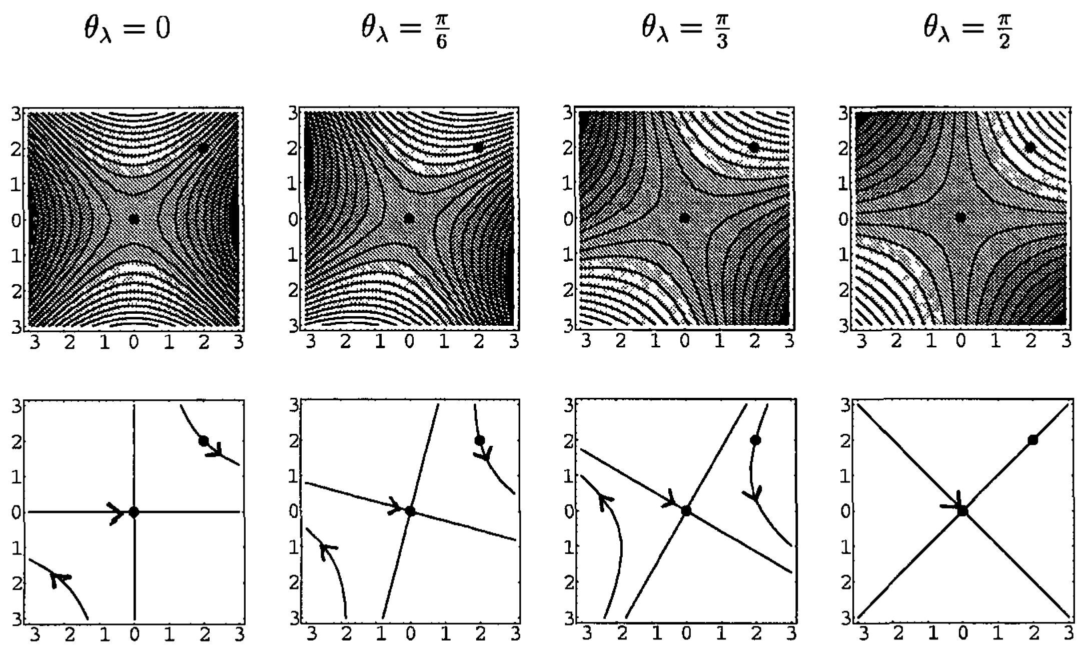


Figure 6 3 Contour plots generated by *Mathematica* for $I(\lambda)$ in (6 88)

and Olde Daalhuis [61],[57] and so these could be improved in that sense Table 6 3 shows the numerical results obtained and Figure 6 3 shows the contour plots for $\theta_\lambda > 0$, while the second row gives the lines of steepest descent through z^1 and z^l

6.5 Further Extensions

The method of §5 3 could be adapted in order to contend with integrals of the more general form

$$I_k(\lambda) = \int_{\mathcal{S}_k} g(\mathbf{z})\mathrm{e}^{-f(\mathbf{z},\lambda)}d\mathbf{z} \tag{6 92}$$

In this case, $\mathcal{S}_k$ is a doubly-infinite surface between valleys of $\Re\{-(f(\mathbf{z},\lambda)-f(\mathbf{z}^k,\lambda))\}$ Obviously the phase function $-\lambda f(\mathbf{z})$ considered to date excludes certain classes of integrals from investigation

To start $T_k(\lambda)$ is defined by

$$I_k(\lambda) = \mathrm{e}^{-f_k}T_k(\lambda), \tag{6 93}$$

where f_k now denotes $f(\mathbf{z}^k,\lambda)$ The transformation of variables

$$s_\lambda = f(\mathbf{z},\lambda) - f_k, \tag{6 94}$$

$\theta_\lambda = -\frac{\pi}{2}$	4.2561486058	$-i1.6499551239$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}$
	4.2151144047	$-i1.6909893249$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}H_0$
	4.2145274918	$-i1.6915762379$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}(H_0+H_1)$
	4.2119063541	$-i1.6939479707$	Special Functions
$\theta_\lambda = -\frac{\pi}{3}$	3.6840840492	$-i2.6953065843$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}$
	3.6264238099	$-i2.6939688671$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}H_0$
	3.6264187980	$-i2.6938821467$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}(H_0+H_1)$
	3.6238749772	$-i2.6913152768$	Special Functions
$\theta_\lambda = -\frac{\pi}{6}$	2.8609552530	$-i3.5569773553$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}$
	2.8225050631	$-i3.5152920756$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}H_0$
	2.8225517005	$-i3.5152729451$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}(H_0+H_1)$
	2.8243011704	$-i3.5128800611$	Special Functions
$\theta_\lambda = 0$	1.8428570841	$-i4.1762459976$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}$
	1.8454529591	$-i4.1209018867$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}H_0$
	1.8454816517	$-i4.1209262304$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}(H_0+H_1)$
	1.8477574452	$-i4.1218181846$	Special Functions
$\theta_\lambda = \frac{\pi}{6}$	0.6991712505	$-i4.5109103768$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}$
	0.7389661165	$-i4.4742401976$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}H_0$
	0.7389594992	$-i4.4742708795$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}(H_0+H_1)$
	0.7388787276	$-i4.4763455480$	Special Functions
$\theta_\lambda = \frac{\pi}{3}$	−0.4921619484	$-i4.5381636685$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}$
	−0.4389987502	$-i4.5394233052$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}H_0$
	−0.4309259999	$-i4.5394312936$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}(H_0+H_1)$
	−0.4407658088	$-i4.5400774865$	Special Functions
$\theta_\lambda = \frac{\pi}{2}$	−1.6499551239	$-i4.2561486058$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}$
	−1.5989294357	$-i4.2711249696$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}H_0$
	−1.5989488805	$-i4.2711444144$	$\frac{e^{-\lambda f_1}}{\lambda^{1/2}}(H_0+H_1)$
	−1.6138731921	$-i4.2922305376$	Special Functions

Table 6.3: Values of $I(\lambda)$ obtained using the method of §6.4 with $|\lambda| = 1$.

$$\omega = \left. \frac{dz_1 \wedge dz_2 \wedge \quad \wedge dz_n}{ds_\lambda} \right|_{\gamma_k(s_\lambda)} \tag{6 95}$$

takes place so that

$$T_k(\lambda) = \int_0^\infty \mathrm{e}^{-s_\lambda} \Delta_k G(s_\lambda) ds_\lambda, \tag{6 96}$$

with

$$\Delta_k G(s_\lambda) = \int_{\gamma_k(s_\lambda)} g(\mathbf{z})\, \omega \tag{6 97}$$

It should be noted that s_λ varies from 0 to ∞ on $\mathcal{S}_k$ as before but now $\arg s_\lambda = 0$ and again $\gamma_k(s_\lambda)$ denotes a hypersurface where s_λ = constant Using the residue theorem would give

$$s_\lambda^{1-\frac{n}{2}} \Delta_k G(s_\lambda) = \frac{1}{2\pi i} \oint_{\Gamma_k} \frac{\Delta_k G(\xi_\lambda) \xi^{1-\frac{n}{2}}}{\xi_\lambda - s_\lambda} d\xi_\lambda, \tag{6 98}$$

yielding

$$T_k(\lambda) = \frac{1}{2\pi i} \int_0^\infty \mathrm{e}^{-s_\lambda} s_\lambda^{\frac{n}{2}-1} \oint_{\Gamma_k} \frac{\Delta_k G(\xi_\lambda) \xi_\lambda^{1-\frac{n}{2}}}{\xi_\lambda - s_\lambda} d\xi_\lambda\, ds_\lambda \tag{6 99}$$

(all λ dependence having been incorporated in the variable s_λ) Expanding $\frac{1}{\xi_\lambda - s_\lambda}$ results in

$$T_k(\lambda) = \sum_{r=0}^{N-1} T_{kr} + \frac{1}{2\pi i} \int_0^\infty \mathrm{e}^{-s_\lambda} s_\lambda^{N+\frac{n}{2}-1} \oint_{\Gamma_k} \frac{\Delta_k G(\xi_\lambda)}{\xi_\lambda^{N+\frac{n}{2}} (1 - \frac{s_\lambda}{\xi_\lambda})} d\xi_\lambda\, ds_\lambda \tag{6 100}$$

with the T_{kr} being recovered as easily as before

$$\begin{aligned} T_{kr} &= \frac{1}{2\pi i} \int_0^\infty \mathrm{e}^{-s_\lambda} s_\lambda^{r+\frac{n}{2}-1} \oint_{\Gamma_k} \frac{\Delta_k G(\xi_\lambda)}{\xi_\lambda^{r+\frac{n}{2}}} d\xi_\lambda\, ds_\lambda \\ &= \frac{\Gamma(r+\frac{n}{2})}{2\pi i} \oint_{B_\xi} \frac{\Delta_k G(\xi_\lambda)}{\xi_\lambda^{r+\frac{n}{2}}} d\xi_\lambda \\ &= \frac{(r+\frac{n}{2}-1)!}{2\pi i} \oint_{B_{\mathbf{z}k}} \frac{g(\mathbf{z})}{(f(\mathbf{z},\lambda) - f_k)^{r+\frac{n}{2}}} d\mathbf{z} \end{aligned} \tag{6 101}$$

As for the remainder term

$$R_k(\lambda, N) = \frac{1}{2\pi i} \int_0^\infty \mathrm{e}^{-s_\lambda} s_\lambda^{N+\frac{n}{2}-1} \oint_{\Gamma_k} \frac{\Delta_k G(\xi_\lambda)}{\xi^{N+n/2}(1 - \frac{s_\lambda}{\xi_\lambda})} d\xi_\lambda\, ds_\lambda, \tag{6 102}$$

the contour can be deformed following exactly the same procedure as in §5 3, but the problem lies in finding a suitable transformation of the form $s_\lambda \propto \xi_\nu / F_{km}$ (with $F_{km} = f(\mathbf{z}^m, \lambda) - f(\mathbf{z}^k, \lambda)$) and this has not yet been done As the dependence on the asymptotic parameter, λ, is contained within the s_λ and ξ_ν terms it is no longer a simple matter to find $\frac{ds_\lambda}{d\xi_\nu}$ and proceed as before If this were achieved, it would again allow the late terms in this particular case to be related to the early terms of the series expansion corresponding to the saddle $\mathbf{z}^m$, thus giving rise to the appearance of the term

$$T_m(\nu) = \int_0^\infty e^{-\xi_\nu} \Delta_m G(\xi_\nu)\, d\xi_\nu = \sum_{r=0}^{\infty} T_{mr}, \tag{6 103}$$

in $R_k(\lambda, N)$, with the assumption that, despite that extra complications arising from implicit dependence of f on λ, the only singularities of $\Delta_k G(\xi_\lambda)$ occur at the images of the saddles of f as before However, as can be seen from the examples given in §6 2 and §6 3, it is often possible to transform $f(\mathbf{z}, \lambda)$ to obtain a form $\lambda \tilde{f}(\mathbf{z})$ rendering such an adaptation of Howls method unnecessary

Up to this point, only the contributions from simple saddles have been considered The incorporation of contributions from higher order non-degenerate saddles can also be accomplished quite easily, where a saddle, $\mathbf{z}^k$, of order p is a point at which

$$\left.\frac{\partial^q f}{\partial z_1^q}\right|_{\mathbf{z}^k} = \left.\frac{\partial^q f}{\partial z_2^q}\right|_{\mathbf{z}^k} = \quad = \left.\frac{\partial^q f}{\partial z_n^q}\right|_{\mathbf{z}^k} = 0, \qquad \forall\, q < p \tag{6 104}$$

Remember that for such a saddle there are $2p$ lines of steepest descent and ascent passing through the point as discussed in §1 4 5 and so there is a choice of steepest descent directions It is assumed that a particular direction/orientation for the contour/surface of integraton is specified at the outset A brief acccount of the main steps is given here

Let $\mathbf{z}^k$ be an interior saddle of order 3 Then the transformation $s = f(\mathbf{z}) - f_k$, would give 3 values of $\mathbf{z}$ to each s because of cubic dependence on $f(\mathbf{z}) - f_k$ on $\mathbf{z}$ However, only two of these will lie on the surface S_k chosen Then

$$I_k(\lambda) = \int_{S_k} g(\mathbf{z}) e^{-\lambda f(\mathbf{z})}\, d\mathbf{z} = \frac{e^{-\lambda f_k}}{\lambda^{\frac{2n}{3}}} \tilde{T}_k(\lambda), \tag{6 105}$$

$$\tilde{T}_k(\lambda) = \int_0^{\infty e^{-i\theta_\lambda}} \lambda^{\frac{2n}{3}} e^{-\lambda s} \Delta_k G(s)\, ds, \tag{6 106}$$

and $\Delta_k G(s)$, the value of the integral of the original integrand over a vanishing cycle, $\gamma_k(s)$, can be written as

$$\Delta_k G(s) = \frac{1}{2\pi i s^{1-\frac{2n}{3}}} \oint_{\Gamma_k} \frac{\Delta_k G(\xi)\, \xi^{1-\frac{2n}{3}}}{\xi - s}\, d\xi, \tag{6 107}$$

in this case Γ_k is the infinite contour surrounding the cut from $\xi = 0$ which arises as a consequence of the saddle $\mathbf{z}^\mathbf{k}$ exactly as before Expanding the denominator of (6 107) leads to

$$\tilde{T}_k(\lambda) = \int_0^{\infty e^{-\theta_\lambda}} \frac{\lambda^{\frac{2n}{3}} e^{-\lambda s}}{2\pi i} \left[\sum_{r=0}^{N-1} \oint_{\Gamma_k} \frac{\Delta_k G(\xi)\, s^{r-1+\frac{2n}{3}}}{\xi^{r+\frac{2n}{3}}} + \oint_{\Gamma_k} \frac{\Delta_k G(\xi)\, s^{N-1+\frac{2n}{3}}}{\xi^{N+\frac{2n}{3}}(1-\frac{s}{\xi})} d\xi \right] \tag{6 108}$$

The coefficients of λ^{-r} then become

$$\tilde{T}_{kr} = \frac{(r+\frac{2n}{3}-1)!}{2\pi i} \oint_{B_{\mathbf{z}^k}} \frac{g(\mathbf{z})}{(f(\mathbf{z})-f_k)^{r+\frac{2n}{3}}} d\mathbf{z} \tag{6 109}$$

Evaluation of the remainder term relies on the deformation of Γ_k to similar contours, Γ_m, about various adjacent saddles as before If these are also of order 3, then there will be 3 directions of steepest descent for each But bearing in mind the valleys between which the original surface $\mathcal{S}_k$ runs, it should be possible to determine the directions of the Γ_m uniquely Thus the remainder term becomes

$$R_k(\lambda, N) = \frac{1}{2\pi i} \sum_m \frac{(-1)^{\gamma_{km}}}{(\lambda F_{km})^N} \int_0^\infty \frac{\nu^{N-1} e^{-\nu}}{1-\frac{\nu}{\lambda F_{km}}} \tilde{T}_m \left(\frac{\nu}{\lambda F_{km}} \right) d\nu, \tag{6 110}$$

in the case where the adjacent saddles are also of order 3 This can be iterated to form a hyperasymptotic scheme Notice that the hyperterminant integrals are exactly the same as for a simple saddle However, if the adjacent saddles are not of the same order, the hyperterminant integrals will change slightly but there should be no added difficulty in iterating the process

Likewise, if a saddle of order 3 appears on a finite boundary of the original surface of integration, a similar process is followed The main difference arises as a result of the following changes in the formulae

$$I_k(\lambda) = \frac{e^{-\lambda f_k}}{3\lambda^{\frac{2n}{3}}} \tilde{T}_{k/2}(\lambda) = \frac{e^{-\lambda f_k}}{3\lambda^{\frac{2n}{3}}} \int_0^{\infty e^{-\theta_\lambda}} 3\lambda^{\frac{2n}{3}} e^{-\lambda s} \Delta_k G(s)\, ds \tag{6 111}$$

and

$$\Delta_k G(s) = \frac{1}{6\pi i s^{1-\frac{2n}{3}}} \oint \frac{\Delta_k G(\xi) \xi^{\frac{1}{3}-\frac{2n}{3}}}{\xi^{\frac{1}{3}} - s^{\frac{1}{3}}} d\xi, \tag{6 112}$$

where $\Delta_k G(s)$ is now the value of the integral of the original integrand over $\gamma_{k/2}(s)$ The coefficients in the series expansion of $\tilde{T}_{k/2}(\lambda)$ are given by

$$\tilde{T}_{(k/2)r} = \frac{(\frac{r}{3}+\frac{2n}{3}-1)!}{2\pi i} \oint_{B_{\mathbf{z}^k}} \frac{g(\mathbf{z})}{(f(\mathbf{z})-f_k)^{\frac{r}{3}+\frac{2n}{3}}} d\mathbf{z}, \tag{6 113}$$

and the remainder term now takes the form

$$R_k(\lambda, N) = \frac{1}{2\pi i} \sum_m \frac{(-1)^{\gamma_k m}}{(\lambda F_{km})^{\frac{N}{3}}} \int_0^\infty \frac{\nu^{\frac{N}{3}-1} e^{-\nu}}{1 - (\frac{\nu}{\lambda F_{km}})^{\frac{1}{3}}} \tilde{T}_m \left(\frac{\nu}{\lambda F_{km}}\right) d\nu, \qquad (6\ 114)$$

the $\tilde{T}_m$ terms referring to adjacent interior saddles of order 3

6.6 Conclusions

Various procedures have been discussed in an effort to extend the method of §5 3 to deal with a larger class of integrals It is clear from the results that the method is quite robust and lends well to such modification While certain success has been enjoyed, there is much to be done to extend the method in other directions and to make rigorous the suggestions outlined above It is worthwhile remembering while doing this, that the Borel-plane approach applied here allows the results to carry over to solutions of classes of differential equations and thus serve a dual purpose

Appendix A

Singularities of $\Delta_k G$

Given

$$\Delta_k G(s) = \int_{\gamma_k(s)} g(\mathbf{z})\,\omega, \tag{A 1}$$

suppose $\Delta_k G(s)$ has a singularity at $s = s_m$ due to a simple critical point and let

$$s - s_m = r_m e^{i\phi} \tag{A 2}$$

in the neighbourhood of s_m Using the result of the Picard-Lefschetz formula applied to vanishing cycles

$$\gamma_k(r_m e^{i(\phi+2\pi)}) = \gamma_k(r_m e^{i\phi}) + (-1)^{n(n-1)/2} N(k,m)\gamma_m(r_m e^{i\phi}), \tag{A 3}$$

(where $N(k,m)$ is an integer), $\Delta_k G$ has the form

$$\begin{aligned} \Delta_k G(r_m e^{i(\phi+2\pi)}) &= \Delta_k G(r_m e^{i\phi}) + \int_{(-1)^{n(n-1)/2} N(k,m)\gamma_m(r_m e^{i\phi})} g(\mathbf{z})\,\omega \\ &= \Delta_k G(r_m e^{i\phi}) + \Delta_m(\Delta_k G(r_m e^{i\phi})) \end{aligned} \tag{A 4}$$

Also using

$$\gamma_m(r_m e^{i(\phi+2\pi)}) = (-1)^n \gamma_m(r_m e^{i\phi}), \tag{A 5}$$

gives

$$\Delta_m(\Delta_k G(r_m e^{i(\phi+2\pi)})) = \begin{cases} \Delta_m(\Delta_k G(r_m e^{i\phi})) & \text{if } n \text{ is even,} \\ -\Delta_m(\Delta_k G(r_m e^{i\phi})) & \text{if } n \text{ is odd} \end{cases} \tag{A 6}$$

Therefore, if n is even, consider

$$\begin{aligned} &\Delta_k G(r_m e^{i(\phi+2\pi)}) - \frac{1}{2\pi i}\log(r_m e^{i(\phi+2\pi)})\,\Delta_m(\Delta_k G(r_m e^{i(\phi+2\pi)})) \\ &= \Delta_k G(r_m e^{i\phi}) + \Delta_m(\Delta_k G(r_m e^{i\phi})) - (\frac{1}{2\pi i}\log(r_m e^{i\phi}) + 1)\,\Delta_m(\Delta_k G(r_m e^{i\phi})) \\ &= \Delta_k G(r_m e^{i\phi}) - \frac{1}{2\pi i}\log(r_m e^{i\phi})\,\Delta_m(\Delta_k G(r_m e^{i\phi})) \end{aligned} \tag{A 7}$$

Thus $\Delta_k G(s) - \frac{1}{2\pi i} \log(s)\, \Delta_m(\Delta_k G(s))$ is a holomorphic function in the neighbourhood of s_m and it can be deduced that

$$\Delta_k G(s) = \frac{1}{2\pi i} \log(s - s_m)\, H^{(1)}_{km}(s) + H^{(2)}_{km}(s), \tag{A 8}$$

where $H^{(1)}_{km}(s)$, $H^{(2)}_{km}(s)$ are holomorphic functions near $s = s_m$

On the other hand, if n is odd, only a 4π rotation returns the direction of transversal of the vanishing cycle to its original orientation Then

$$\begin{aligned} \Delta_k G(r_m e^{i(\phi+4\pi)}) &= \Delta_k G(r_m e^{i(\phi+2\pi)}) + \Delta_m(\Delta_k G(r_m e^{i(\phi+2\pi)})) \\ &= \Delta_k G(r_m e^{i\phi}) + \Delta_m(\Delta_k G(r_m e^{i\phi})) - \Delta_m(\Delta_k G(r_m e^{i\phi})) \\ &= \Delta_k G(r_m e^{i\phi}) \end{aligned} \tag{A 9}$$

Writing

$$\begin{aligned} \Delta_k G(r_m e^{i\phi}) = \frac{1}{2} & \left[\Delta_k G(r_m e^{i\phi}) + \Delta_k G(r_m e^{i(\phi+2\pi)})\right] \\ & + \frac{1}{2}\left[\Delta_k G(r_m e^{i\phi}) - \Delta_k G(r_m e^{i(\phi+2\pi)})\right], \end{aligned} \tag{A 10}$$

the first bracket is even with respect to a change of ϕ by 2π whereas the second is odd Therefore, the former is holomorphic near $s = s_m$ and the latter is effectively a holomorphic function multiplied by a square root singularity i e

$$\Delta_k G(s - s_m) = \sqrt{s - s_m}\, E^{(1)}_{km}(s) + E^{(2)}_{km}(s), \tag{A 11}$$

where $E^{(1)}_{km}(s)$, $E^{(2)}_{km}(s)$ are holomorphic functions near $s = s_m$

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