Singularly Perturbed Volterra Integral Equations

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I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy in Applied Mathematical Sciences is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work

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To Respicius

Contents

Abstract

This thesis studies singularly perturbed Volterra integral equations of the form

$$\varepsilon u(t) = f(t,\varepsilon) + \int_0^t g(t,s,u(s))\,ds, \quad 0 \le t \le T,$$

where $\varepsilon > 0$ is a small parameter The function $f(t,\varepsilon)$ is defined for $0 \le t \le T$ and $g(t,s,u)$ for $0 \le s \le t \le T$ There are many existence and uniqueness results known that ensure that a unique continuous solution $u(t,\epsilon)$ exists for all small $\epsilon > 0$ The aim is to find asymptotic approximations to these solutions This work is restricted to problems where there is an initial-layer, various hypotheses are placed on $g(t,s,u)$ to exclude other behaviour A major part of this work is that formal solutions of the nonlinear problem are determined and rigorously proved to be asymptotic approximations to the exact solutions Formal approximate solutions

$$U_N(t,\epsilon) = \sum_{n=0}^{N} \varepsilon^n u_n(t,\varepsilon), \quad u_n(t,\varepsilon) = O(1) \text{ as } \varepsilon \to 0,$$

are obtained using the additive decomposition method Algorithms which improve the method used in Angell and Olmstead (1987), are presented for obtaining these solutions Assuming a stability condition in the boundary layer, it shown that there is a constant c_N such that

$$|u(t,\epsilon) - U_N(t,\epsilon)| \le c_N \epsilon^{N+1} \quad \text{as } \epsilon \to 0,$$

uniformly for $t \in [0,T]$, thus establishing that $U_N(t,\varepsilon)$ is an asymptotic solution Skinner (1995) has proved similar results, but almost all the theorems here were discovered before Skinner's work was found and are largely independent of it Lange and Smith (1988) prove results for the case $g(t,s,u) = k(t,s)u$, where $k(t,s)$ is continuous and satisfies a stability condition in the boundary layer These results are carefully developed here and similar results for linear integrodifferential equations The problem of extending these to the class of weakly singular equations with

$$g(t,s,u) = \frac{k(t,s)}{(t-s)^{\beta}} u, \quad 0 \le \beta < 1,$$

is discussed An interesting aspect of this problem and others for which the boundary layer stability condition fails, is that the solutions decay algebraically rather exponentially within the boundary layer

Chapter 1

Introduction

1.1 Singular Perturbation Problems

In this work we study singularly perturbed Volterra integral and integrodifferential equations which depend on a small parameter in such a way that the solutions of the problem behave nonuniformly as the parameter tends to zero Such singular perturbation problems involving Volterra integral operators arise in applied mechanics, population dynamics and heat conduction The practical aim is to calculate a uniformly valid approximation to the exact solution, which can be used to understand and interpret the unknown exact solution Unlike regular perturbation, in singular perturbation theory there need be no solution to the reduced problem obtained by setting the small parameter to zero If a solution to the reduced problem does exist, its qualitative features can be distinctly different from those of the solution to the full singular perturbation problem

The nature of the nonuniformity of the solutions can vary Here we limit attention to problems in which such nonuniformity occurs in a narrow region called an initial or boundary layer In this region, the solution of the problem changes rapidly The width of the initial layer must approach zero as the parameter decreases to zero In problems with layers one approach is to seek (at least) two expansions, called the inner and outer expansions, neither of which is uniformly valid but whose domains of validity overlap and cover the whole domain This is the ***method of matched asymptotic expansions*** Its purpose is to replace the problem on the whole domain by a sequence of simpler tractable equations on the inner and outer regions For many problems the ***additive decomposition method*** (otherwise known as the O'Malley-Hoppensteadt or boundary function method) is simpler In this thesis we apply the method to several integral equations, and describe some standard, general techniques for mathematically justifying the results Estimates are provided using relatively simple differential inequalities

The additive decomposition method was first applied to singularly perturbated systems of ordinary differential equations of the form

$$\frac{dx}{dt} = f(x, y, t, \varepsilon), \quad x(0) = \alpha(\varepsilon) \tag{1 1 1a}$$

$$\varepsilon \frac{dy}{dt} = g(x, y, t, \varepsilon), \quad y(0) = \beta(\varepsilon) \tag{1 1 1b}$$

Here the data $f(x, y, t, \varepsilon)$, $g(x, y, t, \varepsilon)$, $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ are assumed to possess power series expansions in ε with smooth coefficients An asymptotic solution of (1 1 1) is sought in the form

$$x(t, \varepsilon) = X(t, \varepsilon) + \varepsilon\xi(t/\varepsilon, \varepsilon),$$

$$y(t, \varepsilon) = Y(t, \varepsilon) + \eta(t/\varepsilon, \varepsilon),$$

with an outer expansion

$$\begin{pmatrix} X(t, \varepsilon) \\ Y(t, \varepsilon) \end{pmatrix} \sim \sum_{j=0}^{\infty} \varepsilon^j \begin{pmatrix} X_j(t) \\ Y_j(t) \end{pmatrix}$$

and an initial layer correction

$$\begin{pmatrix} \xi(\tau, \varepsilon) \\ \eta(\tau, \varepsilon) \end{pmatrix} \sim \sum_{j=0}^{\infty} \varepsilon^j \begin{pmatrix} \xi_j(t) \\ \eta_j(t) \end{pmatrix},$$

whose terms tend to zero as $\tau \to \infty$ Related to (1 1 1) are two important problems The *reduced system* is

$$\frac{dx}{dt} = f(x, y, t, 0), \quad x(0) = \alpha(0)$$

$$0 = g(x, y, t, 0) \tag{1 1 2}$$

and the associated *boundary-layer equation*

$$\frac{dz}{d\tau} = g(x(0), z, 0, 0), \qquad z(0) = \beta(0) \tag{1 1 3}$$

Hoppensteadt investigated in [13] the behaviour of the solution of (1 1 1) on the interval $0 \leq t < \infty$ as $\varepsilon \to 0$ In order to treat this case of t being allowed to range over the entire positive real axis, [13] requires that both the reduced system (1 1 2) and the boundary layer equation (1 1 3) satisfy severe stability conditions Hoppensteadt's main result is that, under restrictive assumptions, the solutions of the system (1 1 1) exist for all $t \geq 0$ and converge as $\varepsilon \to 0$ to the solutions of the reduced system uniformly on closed but not necessarily bounded subsets of $(0, \infty)$ In particular, solutions converge on sets of the form $[t_1, \infty)$ with $t_1 > 0$ This result is significant in the sense that the hypotheses cannot be significantly weakened

Different results for (1 1 1) have been obtained on bounded intervals of the form $0 \leq t \leq T$ These include many by O'Malley, full references for which can be found in [20], [21] or Smith [25] In order to obtain these results, less severe stability conditions are imposed on the boundary-layer equation (1 1 3) and the reduced system (1 1 2) than in Hoppensteadt's theory Boundary value problems have also been extensively investigated, see for example the books of O'Malley [20] and Smith [25]

In some problems of the form (1 1) the additive decomposition method gives spurious results in cases for which the method of matched asymptotic expansions works Examples of this have been discussed in Fraenkel [8] and Lange [14]

1.2 Summary of Thesis

Chapter 2 considers the singularly perturbed linear Volterra equation

$$\varepsilon \mathbf{u}(t) = \mathbf{f}(t) + \int_0^t \mathbf{A}(t,s)\mathbf{u}(s)\,ds, \quad 0 \leq t \leq T, \tag{1 2 1}$$

where $0 < \varepsilon << 1$ The vector-valued function $\mathbf{f}(t)$ is continuous for $0 \leq t \leq T$ and the matrix-valued kernel $\mathbf{A}(t,s)$ is continuous for $0 \leq s \leq t \leq T$ The aim is to find asymptotic approximations to the continuous vector-valued solution $t \mapsto \mathbf{u}(t,\varepsilon)$ of (1 2 1) as $\varepsilon \to 0$ We impose the *boundary layer stability condition* that all eigenvalues of $\mathbf{A}(t,t)$ have negative real parts This not only forces an initial layer, but forces the solution $\mathbf{u}(t,\varepsilon)$ of (1 2 1) to decay exponentially in the boundary-layer

Angell and Olmstead in [1] and [2] used the additive decomposition method to find the first few terms in the formal solutions of linear and nonlinear singularly perturbed Volterra integral and differential equations However their approach has the shortcoming that general equations for the coefficients in the formal solution cannot be determined Also Lange and Smith [15] used the additive decomposition method in their study of singularly perturbed linear Fredholm equations They deduced general expansions for the formal solution and rigorous estimates to show its closeness to the exact solution Following the same approach, we derive in Section 2 4 equations for the terms in a formal solution

$$\mathbf{U}_N(t,\varepsilon) = \sum_{n=0}^{N+1} \mathbf{u}_n(t,\varepsilon)\varepsilon^{n-1}$$

Then in Section 2 5 it is shown that

$$\varepsilon \mathbf{U}_N(t,\varepsilon) = \mathbf{f}(t) + \int_0^t \mathbf{A}(t,s)\mathbf{U}_N(s,\varepsilon)\,ds + O(\varepsilon^{N+1}),$$

and in Section 2 6 we prove that

$$|\mathbf{u}(t,\varepsilon) - \mathbf{U}_N(t,\varepsilon)| = O(\varepsilon^{N+1}) \qquad (1\ 2\ 2)$$

uniformly for $0 \le t \le T$ as $\varepsilon \to 0$

Similar results are proved in Chapter 3 for the linear Volterra integrodifferential equation

$$\varepsilon \mathbf{u}'(t) = \mathbf{f}(t) + \mathbf{B}(t)\mathbf{u}(t) + \int_0^t \mathbf{A}(t,s)\mathbf{u}(s)\,ds, \quad \mathbf{u}(0) = \mathbf{a}$$

We construct in Section 3 2 a formal solution $\mathbf{U}_N(t,\varepsilon)$ for this problem using the additive decomposition method and prove the estimate (1 2 2) provided the above boundary layer stability condition holds In chapter 4 we consider the weakly singular linear scalar Volterra integral equation

A major part of this thesis is Chapter 5, where formal solutions of the nonlinear problem

$$\varepsilon u(t) = f(t,\varepsilon) + \int_0^t g(t,s,u(s))\,ds, \quad 0 \le t \le T, \qquad (1\ 2\ 3)$$

are determined and rigorously proved to be asymptotic approximations to the exact solutions Here we require that $\lim_{\varepsilon\to 0} f(0,\varepsilon) = 0$, and allow f to have the asymptotic expansion

$$f(t,\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j f_j(t) \quad \text{as } \varepsilon \to 0$$

Again the additive decomposition method is used The boundary layer stability assumption takes the form that there is a constant $\alpha > 0$ such that

$$\partial_3 g(t,t,y_0(t)) \le -\alpha < 0, \quad \text{for all } 0 \le t \le T,$$

$$\partial_3 g(0,0,v) \le -\alpha < 0, \quad \text{for all suitable } v$$

Skinner [24] has proved similar results, but almost all the work in Chapter 5 was done before Skinner's work was found and is largely independent of it However for the sake of clarity we have integrated some of Skinner's improvements into the exposition of Chapter 5 In particular Skinner's method of deriving the equations for the formal solution is adapted there Skinner's work builds on that of Smith [25], Ch 6, O'Malley [20], Ch 4 and O'Malley [21], Ch 2 on singularly perturbed initial value problems for nonlinear ordinary differential equations These were major sources for this thesis

We also investigate linear Volterra equations for which the boundary layer stability condition fails to hold In Section 2 8 we view the simple example

$$\varepsilon u(t) = f(t) - \int_0^t su(s)\,ds, \tag{1 2 4}$$

from the point of view of the additive decomposition method, looking for an expansion

$$u(t,\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j y_j(t) + \frac{1}{\varepsilon}\sum_{j=0}^{\infty} \varepsilon^{j/2} z_j(t/\varepsilon^{1/2})$$

Because not all the boundary layer correction terms $z_j(\tau) \to 0$ exponentially as $\tau \to \infty$ but only algebraically, greater care is required in applying the O'Malley-Hoppensteadt method Similarly in Chapter 4 the weakly singular scalar Volterra integral equation

$$\varepsilon u(t) = f(t) + \frac{1}{\Gamma(\beta)}\int_0^t \frac{k(t,s)}{(t-s)^{1-\beta}} u(s)\,ds,$$

is considered with $0 < \beta < 1$ and $k(t,t) = -1$ This problem exhibits an initial layer at $t = 0$ like the equations with continuous kernels considered in Chapter 2 The stability condition fails and there is only algebraic decay of solutions in the initial layer We construct a formal solution $U_0(t,\varepsilon)$ and can demonstrate in particular examples that $|u(t,\varepsilon) - U_0(t,\varepsilon)| = O(\varepsilon)$ A proof of this in general is not yet known

Chapter 2

Linear Integral Equations with Continuous Kernels

2.1 Introduction

This chapter considers the singularly perturbed linear Volterra equation

$$\varepsilon\mathbf{u}(t) = \mathbf{f}(t) + \int_0^t \mathbf{A}(t,s)\mathbf{u}(s)\,ds, \quad 0 \le t \le T, \tag{2 1 1}$$

where $0 < \varepsilon << 1$ The vector-valued function $\mathbf{f(t)}$ is continuous for $0 \le t \le T$ and the matrix-valued kernel $\mathbf{A}(t,s)$ is continuous for $0 \le s \le t \le T$ Our interest is in finding asymptotic approximations to the continuous vector-valued solution $t \mapsto \mathbf{u}(t,\varepsilon)$ of (2 1 1) as $\varepsilon \to 0$ The results here are not presented because they are new, but rather to explain in this simple context how the method of additive decomposition can be applied to integral equations In later chapters it is employed to find asymptotic approximations to the solutions of more complicated equations The results here are easily generalised to the case of $\mathbf{f}$ and $\mathbf{A}$ depending in a regular way on ε, though here it is assumed that they are independent of ε

The singular nature of (2 1 1) is easily seen For $\varepsilon > 0$, (2 1 1) is a Volterra equation of the second kind which has a continuous solution $\mathbf{u}(t,\varepsilon)$ satisfying $\varepsilon\mathbf{u}(0,\varepsilon) = \mathbf{f}(0)$ For $\varepsilon = 0$, (2 1 1) reduces to a Volterra equation the first kind

$$\mathbf{0} = \mathbf{f}(t) + \int_0^t \mathbf{A}(t,s)\mathbf{v}(s)\,ds, \quad 0 \le t \le T, \tag{2 1 2}$$

which does not have a continuous solution unless $\mathbf{f}(0) = \mathbf{0}$ Even in this case, (2 1 2) has a continuous solution only if $\mathbf{f}(t)$ is continuously differentiable So there is a loss of regularity for $\mathbf{v(t)}$ compared to the solution $\mathbf{u}(t,\varepsilon)$ of (2 1 1) for $\varepsilon > 0$ Indeed, if the solution of (2 1 2) is such that $\mathbf{v}(0) \neq \lim_{\varepsilon\to 0}\mathbf{u}(0,\varepsilon)$, then $\mathbf{v(t)}$ cannot provide a uniformly valid approximation of the solution $\mathbf{u}(t,\varepsilon)$ of (2 1 1) on $[0,T]$

The behaviour of the kernel plays an important role in determining the asymptotic character of the continuous solution $\mathbf{u}(t,\varepsilon)$ of (2 1 1) for small values of ε In this chapter we impose the condition that all of the eigenvalues of $\mathbf{A}(t,t)$ have negative real parts This not only forces an initial layer, but forces the solution $\mathbf{u}(t,\varepsilon)$ of (2 1 1) to decay exponentially in the initial-layer The solution $\mathbf{u}(t,\varepsilon)$ is slowly varying for $O(\varepsilon) \leq t \leq T$ as $\varepsilon \to 0$, but changes exponentially on a small interval $0 \leq t \leq O(\varepsilon)$ This small interval of rapid change is called the *inner region*, *initial layer* or *layer of rapid transition*, and the region of slow variation of $\mathbf{u}(t,\varepsilon)$ as the *outer region* The thickness ε of the initial layer approaches zero as $\varepsilon \to 0$

The aim of this chapter is to obtain asymptotic approximations to $\mathbf{u}(t,\varepsilon)$ which are uniformly valid for all $0 \leq t \leq T$ as $\varepsilon \to 0$ Our interest is in problems whose solutions have initial layers, solutions with rapid initial exponential growth will not be discussed here Exponential decay in the boundary layer of the solution $\mathbf{u}(t,\varepsilon)$ suggests the use of the additive decomposition method, as was employed by Lange and Smith [15] in their study of singularly perturbed linear Fredholm equations In Section 2 2, we introduce some notation and explain our basic assumptions Section 2 3 explains the fundamental ideas of the additive decomposition method, and how it regularizes the singular perturbation problem (2 1 1) We derive a formal solution $\sum_{n=-1}^{\infty} \mathbf{u}_n(t,\varepsilon)\varepsilon^n$ in Section 2 4 In Section 2 5 it is shown that this is an asymptotic series and that

$$\varepsilon \mathbf{U}_N(t,\varepsilon) = \mathbf{f}(t) + \int_0^t \mathbf{A}(t,s)\mathbf{U}_N(s,\varepsilon)\,ds + O(\varepsilon^{N+1}),$$

where

$$\mathbf{U}_N(t,\varepsilon) = \sum_{n=0}^{N+1} \mathbf{u}_n(t,\varepsilon)\varepsilon^{n-1}$$

In Section 2 6 we prove that

$$|\mathbf{u}(t,\varepsilon) - \mathbf{U}_N(t,\varepsilon)| = O(\varepsilon^{N+1}) \qquad (2\ 1\ 3)$$

uniformly for $0 \leq t \leq T$ as $\varepsilon \to 0$ This result is important because the method of additive decomposition can lead to spurious solutions (see for example Lange [14]) The method is illustrated in Section 2 7 by an example from Angell and Olmstead [2]

2 2 Notation and Assumptions

The n-dimensional space $\mathbb{R}^n$ is given the norm $|\mathbf{x}| = \max_{1\leq i\leq n} |x_i|$ for each $\mathbf{x}$ in $\mathbb{R}^n$, and the space $\mathbb{R}^{n\times n}$ of $n \times n$ matrices with real entries is given the norm $|\mathbf{M}| = \max_{1\leq i,j\leq n} |M_{ij}|$ for all $\mathbf{M}$ in $\mathbb{R}^{n\times n}$ The spectrum $\sigma(\mathbf{M})$ of $\mathbf{M}$ is the set of eigenvalues of $\mathbf{M}$ It is well-known (see, for example

Hirsch and Smale [12], Ch 7, Thm 1) that, if $\mathrm{Re}\,\lambda < \alpha \leq \alpha_1 < 0$ for all $\lambda \in \sigma(\mathbf{M})$, there is a constant $\kappa > 0$ such that

$$|e^{\mathbf{M}t}\mathbf{x}| \leq \kappa e^{-\alpha_1 t}|\mathbf{x}| \tag{2 2 1}$$

The kernel $\mathbf{A}\ \ \triangle_T \to \mathbb{R}^{n\times n}$ is defined on

$$\triangle_T = \{(t,s) \in \mathbb{R}^2 \quad 0 \leq s \leq t \leq T\} \tag{2 2 2}$$

It is convenient to use the notation

$$\mathbf{B}(t) = \mathbf{A}(t,t) \tag{2 2 3}$$

Partial derivatives are usually denoted by $\partial_1 \mathbf{A}$ and $\partial_2 \mathbf{A}$ instead of $\partial \mathbf{A}/\partial t$ and $\partial \mathbf{A}/\partial s$ respectively Similarly the derivative of $\mathbf{u}$ is usually denoted by $\mathbf{u}'(t)$ rather than $d\mathbf{u}/dt$

The following assumptions are used throughout this chapter The first is a regularity assumption on the data $\mathbf{f}$ and $\mathbf{A}$, the second is a stability condition for the solution within the boundary layer

($\mathbf{H}_1$) The functions $\mathbf{f}\ \ [0,T] \to \mathbb{R}^n$ and $\mathbf{A}\ \ \triangle_T \to \mathbb{R}^{n\times n}$ are both C^∞

($\mathbf{H}_2$) There exists a number $\alpha > 0$ such that

$$\max_{\lambda \in \sigma(\mathbf{B}(t))} \{\mathrm{Re}(\lambda)\} \leq -\alpha,$$

for all $0 \leq t \leq T$

2 3 Heuristic Analysis

In this section, we describe how the additive decomposition technique can be applied to integral equations of the type (2 1 1) The method of additive decomposition, also called the O'Malley and Hoppensteadt method, was initially applied by O'Malley [20], [21] and Hoppensteadt [13] to investigate the behaviour of solutions of singularly perturbed systems of ordinary differential equations The book Smith [25] contains a clear account of its application to singularly perturbed ordinary differential equations This method was later employed by Angell and Olmstead in [2] and [1] to get formal solutions of singularly perturbed Volterra integral equations, linear and nonlinear Lange and Smith in [15] in a very careful study of singularly perturbed linear Fredholm equations applied the method systematically to get a complete formal solution and proved estimates of the type (2 1 3) The singularly perturbed Fredholm equations investigated in [15] have the additional complication of two boundary layers It is also indicated there how internal layers can be analysed The additive decomposition has also been employed by Lange and Smith [16] and Skinner [24] The presentation is similar to §3 and §6 of Lange and Smith [15]

The analysis in this and the next section is formal The forcing function $\mathbf{f}(t)$ and kernel $\mathbf{A}(t,s)$ are assumed to be C^{∞} The solution $\mathbf{u}(t,\varepsilon)$ of (2 1 1) can be represented as

$$\mathbf{u}(t,\varepsilon) = \frac{1}{\varepsilon}\mathbf{f}(t) + \frac{1}{\varepsilon}\int_0^t \mathbf{\Gamma}(t,s,\varepsilon)\mathbf{f}(s)\,ds, \quad 0 \le t \le T, \tag{2 3 1}$$

where $\mathbf{\Gamma}(t,s,\varepsilon)$ is the resolvent kernel of $\mathbf{A}(t,s)/\varepsilon$, which by definition is the solution of

$$\mathbf{\Gamma}(t,s,\varepsilon) = \frac{1}{\varepsilon}\mathbf{A}(t,s) + \frac{1}{\varepsilon}\int_s^t \mathbf{A}(t,v)\mathbf{\Gamma}(v,s,\varepsilon)\,dv, \quad 0 \le s \le t \le T$$

$\mathbf{\Gamma}(t,s\ \varepsilon)$ is also C^{∞} Detailed accounts of the theory of linear nonconvolution Volterra equations can be found in Miller [19] ch IV and Gripenberg, Londen and Staffans [10] Ch 9

To model an initial layer for $\mathbf{u}(t,\varepsilon)$ we introduce a new scaled time scale $\tau = \frac{t}{\mu(\varepsilon)}$ The idea is that if the initial layer region is described with respect to the new time scale no rapid variation in the solution should be exhibited A solution $\mathbf{u}(t,\varepsilon)$ is sought in the form

$$\mathbf{u}(t,\varepsilon) = \mathbf{y}(t,\varepsilon) + \varphi(\varepsilon)\mathbf{z}(t/\mu(\varepsilon),\varepsilon), \tag{2 3 2}$$

where $\mathbf{y}(t,\varepsilon)$ represents the outer approximation and $\mathbf{z}(\tau,\varepsilon)$ an initial layer correction function The function $\mu(\varepsilon)$ describes the width of the layer and $\varphi(\varepsilon)$ describes the magnitude of $\mathbf{u}(t,\varepsilon)$ in the layer Therefore we require that[1]

$$\mathbf{y}(t,\varepsilon) = \mathrm{ord}(1), \quad \mathbf{z}(\tau,\varepsilon) = \mathrm{ord}(1) \quad \text{as } \varepsilon \to 0$$

At any fixed $t > 0$, the outer approximation, $\mathbf{y}(t,\varepsilon)$ should give a good approximation to $\mathbf{u}(t,\varepsilon)$ as $\varepsilon \to 0$, we impose the condition

$$\mathbf{z}(\tau,\varepsilon) \to \mathbf{0}, \quad \text{as } \tau \to \infty \tag{2 3 3}$$

[1]Two functions $\theta(\varepsilon)$ and $\psi(\varepsilon)$ defined in a neighbourhood $(0,\varepsilon_0)$ satisfy $\theta(\varepsilon) = \mathrm{ord}(\psi(\varepsilon))$ if $\theta(\varepsilon) = O(\psi(\varepsilon))$ but $\theta(\varepsilon) \neq o(\psi(\varepsilon))$ as $\varepsilon \to 0$

The substitution of (2 3 2) into (2 1 1) gives

$$\varepsilon \mathbf{y}(t,\varepsilon) + \varepsilon\varphi(\varepsilon)\mathbf{z}(t/\mu(\varepsilon),\varepsilon) = \int_0^t \mathbf{A}(t,s)\mathbf{y}(s,\varepsilon)\,ds + \varphi(\varepsilon)\mu(\varepsilon)\int_0^{t/\mu(\varepsilon)} \mathbf{A}(t,\mu(\varepsilon)\sigma)\mathbf{z}(\sigma,\varepsilon)\,d\sigma + \mathbf{f}(t) \tag{2 3 4}$$

This is equivalent to

$$\varepsilon \mathbf{y}(\mu(\varepsilon)\tau,\varepsilon) + \varepsilon\varphi(\varepsilon)\mathbf{z}(\tau,\varepsilon) = \int_0^{\mu(\varepsilon)\tau} \mathbf{A}(\mu(\varepsilon)\tau,s)\mathbf{y}(s,\varepsilon)\,ds + \varphi(\varepsilon)\mu(\varepsilon)\int_0^{\tau} \mathbf{A}(\mu(\varepsilon)\tau,\mu(\varepsilon)\sigma)\mathbf{z}(\sigma,\varepsilon)\,d\sigma + \mathbf{f}(\mu(\varepsilon)\tau) \tag{2 3 5}$$

The width $\mu(\varepsilon)$ and amplitude $\varphi(\varepsilon)$ in the boundary layer can be found by examining the dominate balance Of course $\mu(\varepsilon) = o(1)$ as $\varepsilon \to 0$ We shall only consider the leading order terms in $\mathbf{y}(t,\varepsilon)$ and $\mathbf{z}(\tau,\varepsilon)$, and therefore write

$$\mathbf{y}(t,\varepsilon) = \mathbf{y}_0(t) + o(1), \quad \mathbf{z}(\tau,\varepsilon) = \mathbf{z}_0(\tau) + o(1) \quad \text{as } \varepsilon \to 0$$

Of course $\mathbf{z}_0(\tau) \to 0$ as $\tau \to \infty$ Also we assume that there is a real number γ and nontrivial kernels $\mathbf{B}(\tau,\sigma)$ and $\mathbf{C}(t,\sigma)$ such that

$$\mathbf{A}(\varepsilon\tau,\varepsilon\sigma) \sim \varepsilon^{\gamma}\mathbf{B}(\tau,\sigma)$$

$$\mathbf{A}(t,\varepsilon\sigma) \sim \varepsilon^{\gamma}\mathbf{C}(t,\sigma)$$

uniformly as $\varepsilon \to 0$ For simplicity we suppose that $\mathbf{f}(0) \neq \mathbf{0}$ Equations (2 3 4) and (2 3 5) imply that as $\varepsilon \to 0$

$$\varepsilon \mathbf{y}_0(t) + \varepsilon\varphi(\varepsilon)\mathbf{z}_0(t/\mu(\varepsilon)) \sim \int_0^t \mathbf{A}(t,s)\mathbf{y}_0(s)\,ds + \varphi(\varepsilon)\mu(\varepsilon)\int_0^{t/\mu(\varepsilon)} \mathbf{A}(t,\mu(\varepsilon)\sigma)\mathbf{z}(\sigma,\varepsilon)\,d\sigma + \mathbf{f}(t) \tag{2 3 6}$$

$$\varepsilon \mathbf{y}_0(\mu(\varepsilon)\tau) + \varepsilon\varphi(\varepsilon)\mathbf{z}_0(\tau) \sim \int_0^{\mu(\varepsilon)\tau} \mathbf{A}(\mu(\varepsilon)\tau,s)\mathbf{y}_0(s)\,ds + \varphi(\varepsilon)\mu(\varepsilon)^{\gamma+1}\int_0^{\tau} \mathbf{B}(\tau,\sigma)\mathbf{z}_0(\sigma)\,d\sigma + \mathbf{f}(0) \tag{2 3 7}$$

Examining the dominant balance in the second relation, we see that

$$\operatorname{ord}(\varepsilon\varphi(\varepsilon)) = \operatorname{ord}(\mu(\varepsilon)^{\gamma+1}\varphi(\varepsilon)) = \operatorname{ord}(1) \quad \text{as } \varepsilon \to 0$$

Hence we choose

$$\mu(\varepsilon) = \varepsilon^{\frac{1}{1+\gamma}}, \quad \varphi(\varepsilon) = \frac{1}{\varepsilon}$$

It then follows by letting $\varepsilon \to 0$ with $\tau \geq 0$ fixed in (2 3 5), that $\mathbf{z}_0$ obeys the equation

$$\mathbf{z}_0(\tau) = \int_0^{\tau} \mathbf{B}(\tau,\sigma)\mathbf{z}_0(\sigma)\,d\sigma + \mathbf{f}(0)$$

To get an equation for $\mathbf{y}_0$ the order as $\varepsilon \to 0$ of the term

$$\varphi(\varepsilon)\mu(\varepsilon)\int_0^{t/\mu(\varepsilon)} \mathbf{A}(t,\mu(\varepsilon)\sigma)\mathbf{z}(\sigma,\varepsilon)\,d\sigma \tag{2 3 8}$$

in (2 3 4) must be calculated In the standard case of exponential decay in the boundary layer, each of the integrals

$$\int_0^{t/\mu(\varepsilon)} \mathbf{A}(t,\mu(\varepsilon)\sigma)\{\mathbf{z}(\sigma,\varepsilon)-\mathbf{z}_0(\sigma)\}\,d\sigma,$$
$$\int_0^{t/\mu(\varepsilon)} \{\mathbf{A}(t,\mu(\varepsilon)\sigma)-\mu(\varepsilon)^\gamma\mathbf{C}(t,\sigma)\}\mathbf{z}_0(\sigma)\,d\sigma,$$
$$\mu(\varepsilon)^\gamma\int_{t/\mu(\varepsilon)}^{\infty} \mathbf{C}(t,\sigma)\mathbf{z}_0(\sigma)\,d\sigma,$$

can be formally shown to vanish, and hence (2 3 8) has leading order

$$\int_0^{\infty} \mathbf{C}(t,\sigma)\mathbf{z}_0(\sigma)\,d\sigma \tag{2 3 9}$$

in this case However finding the order of (2 3 8) as $\varepsilon \to 0$ in the case of algebraic decay of the solution in the boundary layer is not so straightforward Indeed in Section 2 8 an example is discussed for which the evaluation of the layer limit in (2 3 8) requires knowledge of the asymptotic behaviour of higher order terms in $\mathbf{z}(\tau,\varepsilon)$ not just the leading order term $\mathbf{z}_0(\tau)$ For the standard case of exponentially decaying boundary layers, we find by letting $\varepsilon \to 0$ with $0 < t \le T$ fixed in (2 3 4) that $\mathbf{y}_0$ obeys

$$0 = \int_0^t \mathbf{A}(t,s)\mathbf{y}_0(s)\,ds + \int_0^{\infty} \mathbf{C}(t,\sigma)\mathbf{z}_0(\sigma)\,d\sigma + \mathbf{f}(t)$$

It is easy to see that if $(\mathbf{H_2})$ holds then

$$\mathbf{A}(\varepsilon\tau,\varepsilon\sigma) \sim \mathbf{A}(0,0) \tag{2 3 10}$$

$$\mathbf{A}(t,\varepsilon\sigma) \sim \mathbf{A}(t,0) \tag{2 3 11}$$

as $\varepsilon \to 0$, where $\mathbf{A}(0,0)$ and $\mathbf{A}(t,0)$ are non-zero Then the width and amplitude of the boundary become

$$\mu(\varepsilon) = \varepsilon, \quad \varphi(\varepsilon) = \frac{1}{\varepsilon} \tag{2 3 12}$$

In the standard case $\mathbf{y}_0$ and $\mathbf{z}_0$ then satisfy

$$0 = \int_0^t \mathbf{A}(t,s)\mathbf{y}_0(s)\,ds + \mathbf{A}(t,0)\int_0^{\infty} \mathbf{z}_0(\sigma)\,d\sigma + \mathbf{f}(t) \tag{2 3 13}$$

$$\mathbf{z}_0(\tau) = \int_0^{\tau} \mathbf{A}(0,0)\mathbf{z}_0(\sigma)\,d\sigma + \mathbf{f}(0) \tag{2 3 14}$$

A consequence of the magnitude $O(\varepsilon^{-1})$ of the boundary layer is that the term $\varepsilon\mathbf{u}(t)$ on the right of (2 1 1) contributes to equation (2 3 14) for the inner correction term It also follows from (2 3 6) that (2 3 9) is the contribution to the integral in (2 1 1) from narrow initial layer $0 \leq t \leq O(\varepsilon)$ is $O(1)$ as $\varepsilon \to 0$ with $t > 0$ fixed Also note that the integral equation (2 3 13) is not the reduced equation (2 1 2), unless the second integral on the right side is zero In the special case where $\mathbf{f}(0) = \mathbf{0}$, the boundary layer has $O(1)$ magnitude and the leading order term $\mathbf{z}_0$ obeys a different equation

The solution of (2 3 14) is $\mathbf{z}_0(\tau) = e^{A(0,0)\tau}\mathbf{f}(0)$ If $(\mathbf{H_2})$ holds,

$$\int_0^\infty \mathbf{z}_0(\tau)\,d\tau = -\mathbf{A}(0,0)^{-1}\mathbf{f}(0),$$

and (2 3 13) becomes

$$0 = \int_0^t \mathbf{A}(t,s)\mathbf{y}_0(s)\,ds + \mathbf{f}(t) - \mathbf{A}(t,0)\mathbf{A}(0,0)^{-1}\mathbf{f}(0),$$

which has a smooth solution

2.4 Derivation of the Formal Solution

In this section we assume that (2 3 10), (2 3 11) and (2 3 12) hold, so that we seek a formal solution in the form

$$\mathbf{u}(t,\varepsilon) = \mathbf{y}(t,\varepsilon) + \frac{1}{\varepsilon}\mathbf{z}(t/\varepsilon,\varepsilon) \tag{2 4 1}$$

The vector functions $\mathbf{y}(t,\varepsilon)$ and $\mathbf{z}(\tau,\varepsilon)$ are given asymptotically by

$$\mathbf{y}(t,\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j \mathbf{y}_j(t), \tag{2 4 2}$$

$$\mathbf{z}(\tau,\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j \mathbf{z}_j(\tau), \tag{2 4 3}$$

as $\varepsilon \to 0$ To ensure that (2 3 3) holds we assume that

$$\lim_{\tau\to\infty} \mathbf{z}_j(\tau) = 0, \quad j = 0,1,2,$$

Putting $\mathbf{y}_{-1}(t) = \mathbf{0}$, it follows from (2 3 4) that

$$\sum_{j=0}^{\infty} \varepsilon^j \mathbf{y}_{j-1}(t) + \sum_{j=0}^{\infty} \varepsilon^j \mathbf{z}_j(t/\varepsilon) \sim \mathbf{f}(t) + \sum_{j=0}^{\infty} \varepsilon^j \int_0^t \mathbf{A}(t,s)\mathbf{y}_j(s)\,ds + \sum_{j=0}^{\infty} \varepsilon^j \int_0^{t/\varepsilon} \mathbf{A}(t,\varepsilon\sigma)\mathbf{z}_j(\sigma)\,d\sigma \tag{2 4 4}$$

The orders of the terms in

$$\sum_{j=0}^{\infty} \varepsilon^j \int_0^{t/\varepsilon} \mathbf{A}(t, \varepsilon\sigma)\mathbf{z}_j(\sigma)\, d\sigma \tag{2 4 5}$$

in (2 4 4) as $\varepsilon \to 0$ depend on the decay rate of the layer term $\mathbf{z}_j(\tau)$ We assume that $\mathbf{z}_j(\tau)$ decays exponentially so that, for each integer $j \geq 0$, there are positive constants β_j and c_j such that

$$|\mathbf{z}_j(\tau)| \leq c_j e^{-\beta_j \tau}, \quad \tau \geq 0 \tag{2 4 6}$$

By writing out the Taylor expansion of $\mathbf{A}(t, \varepsilon\sigma)$ we find that

$$\mathbf{A}(t, \varepsilon\sigma) \sim \sum_{i=0}^{\infty} \varepsilon^i \mathbf{E}_i(t, \sigma) \quad \text{as } \varepsilon \to 0,$$

where

$$\mathbf{E}_i(t, \sigma) = \frac{1}{i!}\sigma^i \left[\partial_2^i \mathbf{A}\right](t, 0) \tag{2 4 7}$$

Hence, noting that $\mathbf{E}_i(t, \sigma)$ is defined for all (t, σ) in $\mathbb{R}^+ \times \mathbb{R}^+$, (2 4 5) has the asymptotic expansion

$$\begin{aligned}
\sum_{j=0}^{\infty} \varepsilon^j \sum_{i=0}^{\infty} \varepsilon^i \int_0^{t/\varepsilon} \mathbf{E}_i(t, \sigma)\mathbf{z}_j(\sigma)\, d\sigma &= \sum_{j=0}^{\infty} \varepsilon^j \sum_{i=0}^{\infty} \varepsilon^i \int_0^{\infty} \mathbf{E}_i(t, \sigma)\mathbf{z}_j(\sigma)\, d\sigma \\
&\quad - \sum_{j=0}^{\infty} \varepsilon^j \sum_{i=0}^{\infty} \varepsilon^i \int_{t/\varepsilon}^{\infty} \mathbf{E}_i(t, \sigma)\mathbf{z}_j(\sigma)\, d\sigma \\
&\sim \sum_{j=0}^{\infty} \varepsilon^j \sum_{i=0}^{j} \int_0^{\infty} \mathbf{E}_i(t, \sigma)\mathbf{z}_{j-i}(\sigma)\, d\sigma - \mathbf{J}(t/\varepsilon, \varepsilon)
\end{aligned}$$

as $\varepsilon \to 0$, where

$$\mathbf{J}(\tau, \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j \sum_{i=0}^{\infty} \varepsilon^i \int_{\tau}^{\infty} \mathbf{E}_i(\varepsilon\tau, \sigma)\mathbf{z}_j(\sigma)\, d\sigma$$

We introduce the homogeneous polynomial of degree i

$$\mathbf{F}_i(\tau, \sigma) = \frac{1}{i!}\left[(\tau\partial_1 + \sigma\partial_2)^i \mathbf{A}\right](0, 0), \tag{2 4 8}$$

which has the property that

$$\sum_{i=0}^{\infty} \varepsilon^i \mathbf{E}(\varepsilon\tau, \sigma) \sim \sum_{i=0}^{\infty} \varepsilon^i \mathbf{F}(\tau, \sigma), \quad \text{as } \varepsilon \to 0$$

It follows that

$$\mathbf{J}(\tau, \varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j \mathbf{J}_j(\tau) \quad \text{as } \varepsilon \to 0,$$

where

$$\mathbf{J}_j(\tau) = \sum_{i=0}^{j} \int_\tau^\infty \mathbf{F}_i(\tau,\sigma)\mathbf{z}_{j-i}(\sigma)\,d\sigma$$

However it follows from (2 4 6) that for any $0 \le l \le i$

$$\tau^l \left| \int_\tau^\infty \sigma^{i-l}\mathbf{z}_{j-i}(\sigma)\,d\sigma \right| \le \tau^l c_{i-j} \int_\tau^\infty \sigma^{i-l} e^{-\tau\beta} {}_{-j}\, d\sigma \to 0 \quad \text{as } \tau \to 0,$$

and hence from (2 4 8) that

$$\mathbf{J}_j(\tau) \to 0 \quad \text{as } \tau \to 0$$

Equation (2 4 4) can be decomposed into functions of t and functions of t/ε which decay to zero The following Lemma is used to derive the coefficients $\mathbf{y}_j(t)$ and $\mathbf{z}_j(\tau)$ of (2 4 2) and (2 4 3)

Lemma 2 1 *For each integer $j \ge 0$, let $\mathbf{p}_j(t)$ be a continuous function on $[0,T]$ and $\mathbf{q}_j(\tau)$ a continuous function on $[0,\infty)$ such that $\mathbf{q}_j(\tau) \to 0$ as $\tau \to \infty$ Suppose that for every integer $N \ge 1$,*

$$\sum_{j=0}^{N-1} \{\mathbf{p}_j(t) + \mathbf{q}_j(t/\varepsilon)\}\,\varepsilon^j = O(\varepsilon^N), \tag{2 4 9}$$

uniformly as $\varepsilon \to 0$ Then $\mathbf{p}_j = \mathbf{0}$ and $\mathbf{q}_j = \mathbf{0}$ for every $j \ge 0$

Proof There is a uniformly bounded function $\mathbf{r}_0$, defined for all $0 \le t \le T, \tau \ge 0$ and $0 < \varepsilon \le \varepsilon_0$, such that

$$\mathbf{p}_0(t) + \mathbf{q}_0(t/\varepsilon) = \varepsilon\mathbf{r}_0(t, t/\varepsilon, \varepsilon)$$

By letting $\varepsilon \to 0$ for each fixed $t \in (0,T]$, it follows that $\mathbf{p}_0(t) = \mathbf{0}$ The continuity of $\mathbf{p}_0$ then implies $\mathbf{p}_0 = \mathbf{0}$ on $[0,T]$ Therefore substituting $t = \varepsilon\tau$, we have

$$\mathbf{q}_0(\tau) = \varepsilon\mathbf{r}_0(\varepsilon\tau, \tau, \varepsilon)$$

Hence, on taking the limit as $\varepsilon \to 0$ for each fixed $\tau > 0$, we deduce that $\mathbf{q}_0 = \mathbf{0}$ An obvious induction argument completes the proof □

It has been shown that (2 4 4) can be expressed in the form (2 4 9) with $\mathbf{p}_j$ and $\mathbf{q}_j$ given by

$$\mathbf{p}_j(t) = \mathbf{y}_{j-1}(t) - \delta_{j0}\mathbf{f}(t) - \int_0^t \mathbf{A}(t,s)\mathbf{y}_j(s)\,ds - \sum_{i=0}^{j} \int_0^\infty \mathbf{E}_{j-i}(t,\sigma)\mathbf{z}_j(\sigma)\,d\sigma,$$

$$\mathbf{q}_j(\tau) = \mathbf{z}_j(\tau) + \mathbf{J}_j(\tau)$$

It is convenient to introduce

$$\psi_j(\tau) = \begin{cases} \mathbf{0}, & j = 0, \\ \sum_{i=0}^{j-1} \int_\tau^\infty \mathbf{F}_{j-i}(\tau,\sigma)\mathbf{z}_i(\sigma)\,d\sigma, & j \geq 1, \end{cases} \tag{2 4 10}$$

$$\phi_j(t) = \begin{cases} \mathbf{f}(t) + \int_0^\infty \mathbf{A}_0(t,0)\mathbf{z}_0(\sigma)\,d\sigma, & j = 0, \\ \sum_{i=0}^{j} \int_0^\infty \mathbf{E}_{j-i}(t,\sigma)\mathbf{z}_i(\sigma)\,d\sigma, & j \geq 1 \end{cases} \tag{2 4 11}$$

It is important to note that ψ_j and ϕ_{j-1} are determined by $\mathbf{z}_0, \quad , \mathbf{z}_{j-1}$ Later we use the identity

$$\phi_j(0) = \psi_j(0) + \int_0^\infty \mathbf{A}(0,0)\mathbf{z}_j(\sigma)\,d\sigma \tag{2 4 12}$$

From (2 4 10) and (2 4 11)

$$\mathbf{p}_j(t) = \mathbf{y}_{j-1}(t) - \int_0^t \mathbf{A}(t,s)\mathbf{y}_j(s)\,ds - \phi_j(t),$$
$$\mathbf{q}_j(\tau) = \mathbf{z}_j(\tau) + \int_\tau^\infty \mathbf{A}(0,0)\mathbf{z}_j(\sigma)\,d\sigma + \psi_j(t)$$

By applying Lemma 2 1 we obtain the following equations for $\mathbf{y}_j(t)$ and $\mathbf{z}_j(\tau)$

$$\mathbf{y}_{j-1}(t) = \int_0^t \mathbf{A}(t,s)\mathbf{y}_j(s)\,ds + \phi_j(t), \tag{2 4 13}$$

$$\mathbf{z}_j(\tau) = -\int_\tau^\infty \mathbf{A}(0,0)\mathbf{z}_j(\sigma)\,d\sigma - \psi_j(\tau), \tag{2 4 14}$$

The integral equations are augmented by initial conditions Since

$$\mathbf{u}(0,\varepsilon) = \frac{\mathbf{f}(0)}{\varepsilon} \sim \sum_{j=0}^{\infty} \varepsilon^j \left(\mathbf{y}_j(0) + \frac{1}{\varepsilon}\mathbf{z}_j(0)\right),$$

we impose the conditions

$$\mathbf{z}_j(0) = \begin{cases} \mathbf{f}(0), & j = 0, \\ -\mathbf{y}_{j-1}(0), & j \geq 1 \end{cases} \tag{2 4 15}$$

2.5 Properties of the Formal solution

In this section, we first show in Proposition 2 2 that there exists solutions $\mathbf{y}_j$ and $\mathbf{z}_j$ to equations (2 4 13) and (2 4 14) satisfying the initial condition (2 4 15) Moreover $\mathbf{z}_j(\tau) \to \mathbf{0}$ exponentially as $\tau \to \infty$ Therefore

$$\mathbf{u}_n(t,\varepsilon) = \mathbf{y}_{n-1}(t) + \mathbf{z}_n(t/\varepsilon),$$

can be defined for $n \geq 0$ Then

$$\mathbf{U}(t,\varepsilon) = \sum_{n=0}^{\infty} \mathbf{u}_n(t,\varepsilon)\varepsilon^{n-1} \tag{2 5 1}$$

is an asymptotic series as $\varepsilon \to 0$ If we define the truncated sum

$$\mathbf{U}_N(t,\varepsilon) = \sum_{n=0}^{N+1} \mathbf{u}_n(t,\varepsilon)\varepsilon^{n-1}, \tag{2 5 2}$$

then we can consider the residual $\boldsymbol{\rho}_N(t,\varepsilon)$ given by

$$\varepsilon\mathbf{U}_N(t,\varepsilon) = \mathbf{f}(t) + \int_0^t \mathbf{A}(t,s)\mathbf{U}_N(s,\varepsilon)\,ds - \boldsymbol{\rho}_N(t,\varepsilon) \tag{2 5 3}$$

Thus $\mathbf{U}_N(t,\varepsilon)$ satisfies the original equation (2 1 1) approximately with a residual $\boldsymbol{\rho}_N(t,\varepsilon)$ We express $\boldsymbol{\rho}'_N(t,\varepsilon)$ as the sum of a function of t and a function of t/ε In the same manner as in the construction of the formal solution, functions of t/ε contribute only in the initial layer region, away from the layer, functions of t dominate In Proposition 2 4 various results are given which demonstrate that $\boldsymbol{\rho}_N(t,\varepsilon)$ is small for $0 \le t \le T$ as $\varepsilon \to 0$ Similar results are given in Chapter 5 of [25] for a linear overdamped initial-value problems The estimates in Lemma 2 4 are stronger than those of Section 7 of Smith and Lange [16]

Proposition 2 2 *Suppose that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold, and let* $0 < \beta < \alpha$ *Then for every integer* $j \ge 0$ *there exist solutions* $\mathbf{y}_j \in C^\infty([0,T],\mathbb{R}^n)$ *of* (2 4 13) *and solutions* $\mathbf{z}_j(\tau) \in C^\infty([0,\infty),\mathbb{R}^n)$ *of* (2 4 14) *and* (2 4 15) *Moreover there are positive constants* c_j *such that*

$$|\mathbf{z}_j(\tau)| \le c_j e^{-\beta\tau}, \quad \tau \ge 0 \tag{2 5 4}$$

Proof We choose α_1 such that $\beta < \alpha_1 < \alpha$ Consider the hypothesis that for some integer $N \ge 0$ there are solutions $\mathbf{y}_j(t)$ of (2 4 13) for all $0 \le j \le N-1$ and solutions $\mathbf{z}_j(\tau)$ (2 4 14) for all $0 \le j \le N$ satisfying

$$|\mathbf{z}_j(\tau)| \le e^{-\alpha_1\tau}p_j(\tau), \quad \tau \ge 0, \tag{2 5 5}$$

where $p_j(\tau)$ is a polynomial of degree j with positive coefficients Once this hypothesis has been established for all $N \ge 0$, Proposition 2 2 follows immediately

The solution of

$$\mathbf{z}_0(\tau) = -\int_\tau^\infty \mathbf{A}(0,0)\mathbf{z}_0(\sigma)\,d\sigma, \quad \mathbf{z}_0(0) = \mathbf{f}(0),$$

is $\mathbf{z}_0(\tau) = e^{\mathbf{A}(0,0)\tau}\mathbf{f}(0)$ Hence by (2 2 1),

$$|\mathbf{z}_0(\tau)| \le \kappa e^{-\alpha_1\tau}|\mathbf{f}(0)|, \quad \tau \ge 0$$

Also $\mathbf{y}_{-1}(t) = \mathbf{0}$ Hence the induction hypothesis is true for $M = 0$

Suppose now that it holds for some $M \geq 0$ Then $\phi_M(t)$ is well-defined and smooth The equation

$$\mathbf{A}(t,t)^{-1}[\mathbf{y}'_{M-1}(t) - \phi'_M(t)] = \mathbf{y}_M(t) + \int_0^t \mathbf{A}(t,t)^{-1}\partial_1\mathbf{A}(t,s)\mathbf{y}_M(s)\,ds, \tag{2 5 6}$$

which is obtained by differentiating (2 4 13), is a Volterra equation of the second kind Since the kernel and forcing function are C^∞, so is the unique solution $\mathbf{y}_M(t)$ It follows that

$$\mathbf{y}_{M-1}(t) = \int_0^t \mathbf{A}(t,s)\mathbf{y}_M(s)\,ds + \phi_M(t) + \text{constant}$$

However the constant is zero because (2 4 12) and (2 4 14) give $-\mathbf{z}_M(0) = \phi_M(0)$, and the induction hypothesis implies that the initial condition $\mathbf{z}_M(0) = -\mathbf{y}_{M-1}(0)$ holds

The induction hypothesis also implies that $\psi_{M+1}(\tau)$ is well-defined Moreover a tedious calculation using (2 2 1) and (2 5 4) establishes that

$$|\psi_{M+1}(\tau)| \leq e^{-\alpha_1\tau}P_{M+1}(\tau),$$

where $P_{M+1}(\tau)$ is a polynomial of degree M $\mathbf{z}_{M+1}$ satisfies the ordinary differential equation

$$\mathbf{z}'_{M+1}(\tau) = \mathbf{A}(0,0)\mathbf{z}_{M+1}(\tau) - \psi'_{M+1}(\tau), \quad \mathbf{z}_{M+1}(0) = \mathbf{y}_M(0)$$

The solution of this can be found using variation of parameters and written as

$$\mathbf{z}_{M+1}(\tau) = e^{\mathbf{A}(0,0)\tau}[\mathbf{y}_M(0) - \psi_{M+1}(0)] + \psi_{M+1}(\tau) + \mathbf{A}(0,0)\int_0^\tau e^{\mathbf{A}(\tau-\sigma)}\psi_{M+1}(\sigma)\,d\sigma \tag{2 5 7}$$

The norm of the last integral is easily bounded by

$$\begin{aligned}\left|\mathbf{A}(0,0)\int_0^\tau e^{\mathbf{A}(\tau-\sigma)}\psi_{M+1}(\sigma)\,d\sigma\right| &\leq |\mathbf{A}(0,0)|\int_0^\tau |e^{\mathbf{A}(\tau-\sigma)}\psi_{M+1}(\sigma)|\,d\sigma \\ &\leq \kappa|\mathbf{A}(0,0)|\int_0^\tau e^{-\alpha_1\tau}P_{M+1}(\sigma)\,d\sigma,\end{aligned}$$

and it can be shown from (2 5 7) that $\mathbf{z}_{M+1}(\tau)$ satisfies an estimate of the form

$$|\mathbf{z}_{M+1}(\tau)| \leq e^{-\alpha_1\tau}p_{M+1}(\tau), \quad \tau \geq 0,$$

where $p_{M+1}(\tau)$ is a polynomial of degree $M+1$ This proves the induction hypothesis □

Remark 2 3 The formal series (2 5 1) is a uniform asymptotic series, because

$$\frac{|\mathbf{u}_{n+1}(t,\varepsilon)|}{|\mathbf{u}_n(t,\varepsilon)|} \to \frac{|\mathbf{y}_{n+1}(t)|}{|\mathbf{y}_n(t)|} \quad \text{as } \varepsilon \to 0,$$

implying that $\mathbf{u}_{n+1}(t,\varepsilon)\varepsilon^{n+1} = o(\mathbf{u}_n(t,\varepsilon)\varepsilon^n)$ uniformly for $0 \le t \le T$ as $\varepsilon \to 0$

Proposition 2 4 *Suppose that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold Then for each* $N \ge 0$,

$$|\rho_N(t,\varepsilon)| = O(\varepsilon^{N+1})$$

uniformly for $0 \le t \le T$ *as* $\varepsilon \to 0$, *and there are positive constants* d_N *and* e_N *such that*

$$|\rho_N'(t,\varepsilon)| \le e_N\varepsilon^{N+1}, \qquad \int_0^t |\rho_N'(s,\varepsilon)|\,ds \le d_N\varepsilon^{N+1}, \tag{2 5 8}$$

for all $0 < \varepsilon \le \varepsilon_0$ *and for all* t *in* $[0,T]$, *for some* $\varepsilon_0 > 0$

Proof Later we shall use the estimates in (2 5 8), and therefore only prove these in detail To demonstrate the other result an almost identical argument is used

Since $\rho_N(0,\varepsilon) = \mathbf{0}$, differentiation of (2 5 3) gives

$$\rho_N'(t,\varepsilon) = -\varepsilon\mathbf{U}_N'(t,\varepsilon) + \mathbf{f}'(t) + \mathbf{A}(t,t)\mathbf{U}_N(t,\varepsilon) + \int_0^t \partial_1\mathbf{A}(t,s)\mathbf{U}_N(s,\varepsilon)\,ds \tag{2 5 9}$$

The substitution of (2 5 2) and the differentiated version of (2 4 13) into this yields

$$\begin{aligned}\rho_N'(t,\varepsilon) = &-\varepsilon^{N+1}\mathbf{y}_N'(t) - \sum_{i=0}^{N+1}\varepsilon^{i-1}\mathbf{z}_i'(t/\varepsilon) + \sum_{i=0}^{N+1}\varepsilon^{i-1}\mathbf{A}(t,t)\mathbf{z}_i(t/\varepsilon)\\ &-\sum_{i=0}^{N}\varepsilon^i\sum_{k=0}^{i}\int_0^\infty \partial_1\mathbf{E}_{i-k}(t,\sigma)\mathbf{z}_k(\sigma)\,d\sigma + \sum_{i=0}^{N}\varepsilon^i\int_0^{t/\varepsilon}\partial_1\mathbf{A}(t,\varepsilon\sigma)\mathbf{z}_i(\sigma)\,d\sigma\end{aligned} \tag{2 5 10}$$

Using the Taylor expansion of $\mathbf{A}(t,\varepsilon\sigma)$ we can derive

$$\begin{aligned}\sum_{i=0}^{N}\varepsilon^i\sum_{k=0}^{\infty}\varepsilon^k\int_0^{t/\varepsilon}\partial_1\mathbf{E}_k(t,\sigma)\mathbf{z}_i(\sigma)\,d\sigma = &\sum_{i=0}^{N}\varepsilon^i\sum_{k=0}^{\infty}\varepsilon^k\int_0^{\infty}\partial_1\mathbf{E}_k(t,\sigma)\mathbf{z}_i(\sigma)\,d\sigma\\ &-\sum_{i=0}^{N}\varepsilon^i\sum_{k=0}^{\infty}\varepsilon^k\int_{t/\varepsilon}^{\infty}\partial_1\mathbf{E}_k(t,\sigma)\mathbf{z}_i(\sigma)\,d\sigma\end{aligned}$$

By substituting this into (2 5 10), we get

$$\begin{aligned}\rho_N'(t,\varepsilon) = &-\varepsilon^{N+1}\mathbf{y}_N'(t) + \sum_{i=0}^{N+1}\varepsilon^{i-1}\mathbf{z}_i'(t/\varepsilon) + \sum_{i=0}^{N+1}\varepsilon^{i-1}\sum_{k=0}^{\infty}\varepsilon^k\mathbf{F}_k(t/\varepsilon,t/\varepsilon)\mathbf{z}_i(t/\varepsilon)\\ &+\sum_{i=N+1}^{\infty}\varepsilon^i\sum_{k=0}^{i}\int_0^\infty \partial_1\mathbf{E}_{i-k}(t,\sigma)\mathbf{z}_k(\sigma)\,d\sigma - \sum_{i=0}^{\infty}\varepsilon^i\sum_{k=0}^{i}\int_{t/\varepsilon}^{\infty}\mathbf{F}_{i-k}'(t/\varepsilon,\sigma)\mathbf{z}_k(\sigma)\,d\sigma,\end{aligned} \tag{2 5 11}$$

where

$$\mathbf{F}'_i(\tau,\sigma) = \frac{1}{i!}[(\tau\partial_1 + \sigma\partial_2)^i \partial_1 \mathbf{A}](0,0)$$

By putting the differentiated version of (2 4 14) into (2 5 11), we obtain

$$\rho'_N(t,\varepsilon) = -\varepsilon^{N+1}\mathbf{y}'_N(t) + \sum_{i=N+1}^{\infty} \varepsilon^i \sum_{k=0}^{i} \int_0^\infty \partial_1 \mathbf{E}_{i-k}(t,\sigma)\mathbf{z}_k(\sigma)\,d\sigma$$
$$+ \sum_{i=N+1}^{\infty} \varepsilon^i \sum_{k=0}^{i} \mathbf{F}_{i-k+1}(t/\varepsilon, t/\varepsilon)\mathbf{z}_k(t/\varepsilon) + \sum_{i=N+1}^{\infty} \varepsilon^i \sum_{k=0}^{i} \int_{t/\varepsilon}^\infty \mathbf{F}'_{i-k}(t/\varepsilon,\sigma)\mathbf{z}_k(\sigma)\,d\sigma,$$

where the following relation has been used

$$\partial_1 \mathbf{F}_i(\tau,\sigma) = \mathbf{F}'_{i-1}(\tau,\sigma)$$

To summarise it has been shown that

$$\rho^1_N(t,\varepsilon) = \rho^1_N(t,\varepsilon) + \rho^2_N(t/\varepsilon,\varepsilon) + O(\varepsilon^{N+2}),$$

where

$$\rho^1_N(t,\varepsilon) = \varepsilon^{N+1}\left\{-\mathbf{y}_N(t) + \sum_{k=0}^{N+1}\int_0^\infty \partial_1 \mathbf{E}_{N+1-k}(t,\sigma)\mathbf{z}_k(\sigma)\,d\sigma\right\}, \tag{2 5 12}$$

$$\rho^2_N(\tau,\varepsilon) = \sum_{i=N+1}^{\infty} \varepsilon^i \sum_{k=0}^{i}\left\{\mathbf{F}_{i-k+1}(\tau,\tau)\mathbf{z}_k(\tau) - \int_\tau^\infty \mathbf{F}'_{i-k}(\tau,\sigma)\mathbf{z}_k(\sigma)\,d\sigma\right\} \tag{2 5 13}$$

By (2 5 12)

$$|\rho^1_N(t,\varepsilon)| \le \gamma^1_N \varepsilon^{N+1}, \quad \int_0^t |\rho^1_N(s,\varepsilon)|\,ds \le \gamma^2_N \varepsilon^{N+1}, \tag{2 5 14}$$

uniformly for all $0 \le t \le T$, where γ^1_N and γ^2_N are positive constants Using (2 5 5) the function $\rho^2_N(\tau,\varepsilon)$ satisfies

$$|\rho^2_N(\tau,\varepsilon)| \le \varepsilon^{N+1} Q_N(\tau) e^{-\alpha_1 \tau} \le \varepsilon^{N+1}\gamma^3_N e^{-\beta\tau},$$

where Q_N is a polynomial with positive coefficients, and $\beta < \alpha_1 < \alpha$ Hence there is a positive γ^4_N such that

$$\frac{1}{\varepsilon}\int_0^t |\rho_{N^2}(s/\varepsilon,\varepsilon)|\,ds \le \gamma^4_N \varepsilon^{N+1},$$

uniformly for $0 \le t \le T$ as $\varepsilon \to 0$ The conclusions of the proposition now follow □

2.6 Asymptotic Solution

In this section we state and prove our main result It says that for $N \geq 0$,

$$|\mathbf{u}(t,\varepsilon) - \mathbf{U}_N(t,\varepsilon)| = O(\varepsilon^{N+1}) \quad \text{as } \varepsilon \to 0,$$

and hence that $\mathbf{U}(t,\varepsilon)$ given by (2 5 1) is an *asymptotic solution*

Theorem 2 5 *Suppose* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *in Section 2 2 are satisfied Let* $\mathbf{u}(t,\varepsilon)$ *be the solution of (2 1 1) and* $\mathbf{U}_N(t,\varepsilon)$ *the partial sum given in (2 5 2) Then for each integer* $N \geq 0$, *there are positive constants* C_{N+1} *and* ε_0, *independent of* ε, *such that*

$$|\mathbf{u}(t,\varepsilon) - \mathbf{U}_N(t,\varepsilon)| \leq C_{N+1}\varepsilon^{N+1}, \tag{2 6 1}$$

uniformly for $0 \leq t \leq T$ *and* $0 < \varepsilon \leq \varepsilon_0$

Proof It is convenient to fix $N \geq 0$ and define

$$\mathbf{r}_N(t,\varepsilon) = \mathbf{u}(t,\varepsilon) - \mathbf{U}_N(t,\varepsilon)$$

By subtracting (2 5 3) from (2 1 1) we get

$$\varepsilon\mathbf{r}_N(t,\varepsilon) = \boldsymbol{\rho}_N(t,\varepsilon) + \int_0^t \mathbf{A}(t,s)\mathbf{r}_N(s,\varepsilon)\,ds$$

Differentiation yields

$$\mathbf{r}'_N(t,\varepsilon) = \frac{1}{\varepsilon}\mathbf{B}(t)\mathbf{r}_N(t,\varepsilon) + \frac{1}{\varepsilon}\boldsymbol{\rho}'_N(t,\varepsilon) + \frac{1}{\varepsilon}\int_0^t \partial_1\mathbf{A}(t,s)\mathbf{r}_N(s,\varepsilon)\,ds, \quad \mathbf{r}_N(0,\varepsilon) = \mathbf{0}, \tag{2 6 2}$$

where $\mathbf{B}(t)$ is given by (2 2 3)

The solution of the ordinary differential equation

$$\mathbf{r}'_N(t,\varepsilon) = \frac{1}{\varepsilon}\mathbf{B}(t)\mathbf{r}_N(t,\varepsilon) + \mathbf{g}(t),$$

can be represented using variation of parameters as

$$\mathbf{r}_N(t,\varepsilon) = \boldsymbol{\Phi}(t,0,\varepsilon)\mathbf{r}_N(0,\varepsilon) + \int_0^t \boldsymbol{\Phi}(t,s,\varepsilon)\mathbf{g}(s)\,ds \tag{2 6 3}$$

where

$$\boldsymbol{\Phi}(t,s,\varepsilon) = \mathbf{R}(t,\varepsilon)\mathbf{R}(s,\varepsilon)^{-1}, \tag{2 6 4}$$

and $\mathbf{R}(t,\varepsilon)$ is the fundamental matrix solution satisfying

$$\mathbf{R}'(t,\varepsilon) = \frac{1}{\varepsilon}\mathbf{B}(t)\mathbf{R}(t,\varepsilon)$$

It is a result of Flatto and Levinson [7] that there are constants $\kappa_1 > 0$ and $0 < \alpha_2 < \alpha$ such that

$$|\mathbf{\Phi}(t,s,\varepsilon)| \le \kappa_1 e^{-\alpha_2(t-s)/\varepsilon}, \tag{2 6 5}$$

since $(\mathbf{H_2})$ holds

Application of the representation (2 6 3) to (2 6 2) yields

$$\mathbf{r}_N(t,\varepsilon) = \frac{1}{\varepsilon}\int_0^t \mathbf{\Phi}(t,s,\varepsilon)\rho_N'(s,\varepsilon)\,ds + \frac{1}{\varepsilon}\int_0^t \left(\int_v^t \mathbf{\Phi}(t,s,\varepsilon)\partial_1\mathbf{A}(s,v)\,ds\right)\mathbf{r}_N(v,\varepsilon)\,dv \tag{2 6 6}$$

However it follows from (2 6 5) that

$$\begin{aligned}\frac{1}{\varepsilon}\left|\int_v^t \mathbf{\Phi}(t,s,\varepsilon)\partial_1\mathbf{A}(s,v)\,ds\right| &\le \frac{\kappa_1}{\varepsilon}\int_v^t \mathrm{e}^{-\alpha_2(t-s)/\varepsilon}|\partial_1\mathbf{A}(s,v)|\,ds \\ &\le \frac{\kappa_1}{\alpha_2}\max_{(t,s)\in\Delta_T}|\partial_1\mathbf{A}(t,s)| = \kappa_2\end{aligned}$$

Similarly we see from (2 5 8) and (2 6 5) that

$$\frac{1}{\varepsilon}\left|\int_0^t \mathbf{\Phi}(t,s,\varepsilon)\rho_N'(s,\varepsilon)\,ds\right| \le e_N\varepsilon^N\int_s^t \mathrm{e}^{-\alpha_2(t-s)/\varepsilon}\,ds \le \frac{e_N}{\alpha_2}\varepsilon^{N+1}$$

Hence (2 6 6) implies that

$$|\mathbf{r}_N(t,\varepsilon)| \le \frac{e_N}{\alpha_2}\varepsilon^{N+1} + \kappa_2\int_0^t |\mathbf{r}_N(v,\varepsilon)|\,dv$$

By Gronwall's inequality,

$$|\mathbf{r}_N(t,\varepsilon)| \le \frac{e_N}{\alpha_2}\varepsilon^{N+1}e^{\kappa_2 t},$$

and the theorem is proved □

2.7 Example

To illustrate the method, let us consider the following example from [1] and [2]

$$\varepsilon u(t) = f(t) - \int_0^t \{(t-s)\omega(s) + \theta(s)\}u(s)\,ds, \quad t \ge 0 \tag{2 7 1}$$

where $\theta(t) > 0$ Equation (2 7 1) is equivalent to "over-damped" initial value second-order ordinary differential equation

$$\varepsilon u''(t) + \theta(t)u'(t) + \{\omega(t) + \theta'(t)\}u(t) = f''(t), \; t > 0, \tag{2 7 2}$$

with initial conditions

$$u(0) = \frac{1}{\varepsilon}f(0), \quad u'(0) = -\frac{1}{\varepsilon^2}\theta(0)f(0) + \frac{1}{\varepsilon}f'(0)$$

For simplicity we take

$$\omega(t) = 1, \quad \theta(t) = 1, \; f(t) = t + t^2 + \frac{1}{6}t^3$$

because the exact solution of (2 7 1) can be obtained using Laplace transforms as

$$u(t,\varepsilon) = t + 1 + \frac{1}{\gamma_1 - \gamma_2}\left[\left(\gamma_2 - 1 + \frac{1}{\varepsilon}\right)e^{\gamma_1 t} - \left(\gamma_1 - 1 + \frac{1}{\varepsilon}\right)e^{\gamma_2 t}\right], \tag{2 7 3}$$

where

$$\gamma_1, \gamma_2 = \frac{1}{2\varepsilon}(-1 \pm \sqrt{1 - 4\varepsilon})$$

In this example $f(0) = 0$ and we should use an asymptotic representation other than (2 4 1) However we find that $z_0(\tau) = 0$ and our representation agrees with the correct one Note that in this example $a(t,s) = -t + s - 1$ and the boundary layer stability condition holds For $j \geq 0$, the inner correction solution $z_j(\tau)$ is given by

$$z_j(\tau) = e^{-\tau}z_j(0) - \int_0^\tau e^{-(\tau-\sigma)}\psi_j'(\sigma)\,d\sigma, \tag{2 7 4}$$

where

$$\psi_j(\tau) = \sum_{i=0}^{j-1}\int_\tau^\infty F_{j-i}(\tau,\sigma)z_i(\sigma)\,d\sigma$$

Since in this example

$$F_i(\tau,\sigma) = \begin{cases} -1, & i = 0, \\ -(\tau - \sigma), & i = 1, \\ 0, & i \geq 2 \end{cases}$$

it follows that

$$\psi_j(\tau) = -\int_\tau^\infty (\tau - \sigma)z_{j-1}(\sigma)\,d\sigma$$

Therefore we get

$$z_j(\tau) = e^{-\tau}z_j(0) - \int_0^\tau e^{-(\tau-\sigma)}\int_\sigma^\infty z_{j-1}(v)\,dv\,d\sigma,$$

where

$$z_0(0) = f(0), \quad z_j(0) = y_{j-1}(0), \quad j \geq 1$$

By (2 4 13) the outer solution $y_j(t)$ satisfies

$$y_{j-1}(t) = -\int_0^t (t - s + 1) y_j(s)\, ds + \phi_j(t),$$

or

$$y''_{j-1}(t) - \phi''_j(t) = -y'_j(t) - y_j(t),$$

Since

$$E_i(t, \sigma) = \begin{cases} -(1+t), & i = 0, \\ \sigma, & i = 1, \\ 0, & i \geq 2, \end{cases}$$

we find that

$$\phi_j(t) = \sum_{i=j-1}^{j} \int_0^\infty E_{j-1}(t, \sigma) z_i(\sigma)\, d\sigma$$

Therefore

$$\phi''_j(t) = \begin{cases} 2 + t, & j = 0, \\ 0, & j \geq 1, \end{cases}$$

and

$$y_j(t) = -z_{j+1}(0)\mathrm{e}^{-t} - \int_0^t \mathrm{e}^{-(t-s)} (y''_{j-1}(s) - \phi''_j(s))\, ds$$

From the above equations we see that

$$z_0(\tau) = 0, \quad z_1(\tau) = -\mathrm{e}^{-\tau}, \quad z_2(\tau) = -\tau \mathrm{e}^{-\tau},$$

$$y_0(t) = 1 + t, \quad y_1(t) = 0,$$

from which we calculate the first two partial sums of the asymptotic solution to be

$$U_0(t, \varepsilon) = 1 + t - \mathrm{e}^{-t/\varepsilon}, \quad U_1(t, \varepsilon) = 1 + t - \mathrm{e}^{-t/\varepsilon} - t\mathrm{e}^{-t/\varepsilon}$$

To verify that $U_0(t, \varepsilon)$ is a uniformly valid asymptotic approximation, note that

$$u(t, \varepsilon) - U_0(t, \varepsilon) = \mathrm{e}^{-t/\varepsilon} - \mathrm{e}^{(1-1/\varepsilon)t} + O(\varepsilon^2) = -t\mathrm{e}^{-t/\varepsilon} + O(\varepsilon^2),$$

implying that $|u(t,\varepsilon) - U_0(t,\varepsilon)| \le C_0\varepsilon$ Similarly

$$u(t,\varepsilon) - U_1(t,\varepsilon) = -\frac{t^2}{2}\mathrm{e}^{-t/\varepsilon} + \varepsilon^2(\mathrm{e}^{-t} - \mathrm{e}^{-t/\varepsilon}) + O(\varepsilon^3),$$

so that $|u(t,\varepsilon) - U_1(t,\varepsilon)| \le C_1\varepsilon^2$ Therefore the terms $U_0(t,\varepsilon)$ and $U_1(t,\varepsilon)$ found by additive decomposition method are uniformly valid asymptotic approximations to $u(t,\varepsilon)$ for all $0 \le t \le T$ as $\varepsilon \to 0$

Having established a uniformly valid asymptotic expansion using the method of additive decomposition we developed, we now form a composite expansion from the exact solution (2 7 3) The outer expansion is found by fixing $t > 0$ and letting $\varepsilon \to 0$ in (2 7 3), obtaining

$$V(t,\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j v_j(t),$$

where

$$v_0(t) = 1 + t, \quad v_1(t) = 0, \quad v_2(t) = \mathrm{e}^{-t},$$

Similarly expressing (2 7 3) in terms of the inner variable, τ and then taking the inner limit by fixing $\tau > 0$ and letting $\varepsilon \to 0$, the inner expansion takes the form

$$u(\varepsilon\tau,\varepsilon) = W(\tau,\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j w_j(\tau),$$

where

$$w_0(\tau) = 1 - \mathrm{e}^{-\tau}, \quad w_1(\tau) = \tau(1 - \mathrm{e}^{-\tau}), \quad w_2(\tau) = 1 - (1 + \tau^2/2)\mathrm{e}^{-\tau},$$

To obtain these expansion we have used

$$\gamma_1 = -1 + O(\varepsilon), \quad \gamma_2 = -\frac{1}{\varepsilon} + 1 + O(\varepsilon), \quad \varepsilon \to 0,$$
$$\frac{1}{\gamma_1 - \gamma_2}\left(\gamma_2 - 1 + \frac{1}{\varepsilon}\right) = \varepsilon^2 + 4\varepsilon^3 + O(\varepsilon^4),$$
$$\frac{1}{\gamma_1 - \gamma_2}\left(\gamma_1 - 1 + \frac{1}{\varepsilon}\right) = 1 + \varepsilon^2 + 4\varepsilon^3 + O(\varepsilon^4),$$

all as $\varepsilon \to 0$ Using a standard procedure we can obtain a uniform approximation to $u(t,\varepsilon)$ by forming a composite expansions from the inner and outer expansions In fact, we find that $U_0(t,\varepsilon)$ and $U_1(t,\varepsilon)$ are first two composite expansions

2 8 Example of Boundary Layer Stability Condition Failing

Both Lange and Smith [15] and Angell and Olmstead [3] study the integral equation

$$\varepsilon^2 u(t) = f(t) - \int_0^t s u(s)\, ds \tag{2 8 1}$$

To avoid fractional powers, ε^2 replaces ε The exact solution of equation is found, after differentiating, to be

$$u(t,\varepsilon) = \frac{e^{-t^2/(2\varepsilon^2)}}{\varepsilon^2}\left\{ f(0) + \int_0^t e^{s^2/(2\varepsilon^2)} f'(s)\, ds \right\} \tag{2 8 2}$$

For this example $a(t,s) = -s$ and $a(t,t) < 0$ only for $t > 0$ Hence the analysis in Sections 2 5 and 2 6 is no longer applicable

Smith and Lange [15] observe that (2 8 1) has a number of interesting features Firstly expansions for the inner and outer solutions can be calculated from (2 8 2) We see that

$$u(t,\varepsilon) \sim V(t,\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^{2j} v_j(t), \tag{2 8 3}$$

where

$$v_0(t) = -\frac{f'(t)}{t}, \quad v_1(t) = \frac{1}{t}\left(\frac{f'(t)}{t}\right)' \tag{2 8 4}$$

Notice that the integrals

$$\int_0^t s v_j(s)\, ds$$

do not exist for $j \geq 1$ Similarly

$$u(\varepsilon\tau,\varepsilon) \sim W(\tau,\varepsilon) = \frac{1}{\varepsilon^2}\sum_{j=0}^{\infty} \varepsilon^j w_j(\tau), \tag{2 8 5}$$

where

$$w_0(t) = f(0)e^{-\tau^2/2}, \quad w_1(\tau) = f'(0)\int_0^\tau e^{-(\tau^2-\sigma^2)/2}\, d\sigma, \quad w_2(\tau) = f''(0)(1 - e^{-\tau^2/2}) \tag{2 8 6}$$

From (2 8 3), (2 8 4), (2 8 5) and (2 8 6) the composite expansion can be computed such that

$$u(t,\varepsilon) = \frac{f(0)}{\varepsilon^2}e^{-t^2/(2\varepsilon^2)} + \frac{f'(0)}{\varepsilon}\int_0^{t/\varepsilon} e^{-(t^2/\varepsilon^2-\sigma^2)/2}\, d\sigma - f''(0)e^{-t^2/(2\varepsilon^2)} + \frac{f'(t) - f'(0)}{t} + O(\varepsilon) \tag{2 8 7}$$

as $\varepsilon \to 0$ uniformly for $0 \le t \le T$

The analysis of Section 2 3 holds for (2 8 1) even though ($\mathbf{H}_2$) does not In fact it shows that the initial layer should have magnitude $O(\varepsilon^{-2})$ and width $O(\varepsilon)$ However Smith and Lange point out that the ansatz

$$u(t,\varepsilon) = y_0(t) + \frac{1}{\varepsilon^2} z_0(t/\varepsilon) + o(1)$$

and exponential decay for all the inner correction terms produces a false leading order approximate solution

$$-\frac{f(0)}{\varepsilon^2} e^{-t^2/(2\varepsilon^2)} + \frac{f'(t)}{t}, \tag{2 8 8}$$

which is not uniformly valid for all $0 \le t \le T$

We look for an asymptotic solution of the form

$$u(t,\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^{2j} y_j(t) + \frac{1}{\varepsilon^2} \sum_{j=0}^{\infty} \varepsilon^j z_j(t/\varepsilon) \tag{2 8 9}$$

Since $\varepsilon^2 u(0,\varepsilon) = f(0)$, $y_j(t)$ and $z_j(\tau)$ satisfy the initial conditions

$$z_0(0) = f(0), \quad z_1(0) = 0, \quad z_{2j}(0) = -y_{j-1}(0), \quad z_{2j+1}(0) = 0$$

for $j \ge 1$ Substituting (2 8 9) into (2 8 1) gives

$$\begin{aligned} \varepsilon^2 y_0(t) + z_0(t/\varepsilon) + \varepsilon z_1(t/\varepsilon) + \varepsilon^2 z_2(t/\varepsilon) &= f(t) - \int_0^t s y_0(s)\, ds \\ &- \frac{1}{\varepsilon^2}\int_0^t s z_0(s/\varepsilon)\, ds - \frac{1}{\varepsilon^1}\int_0^t s z_1(s/\varepsilon)\, ds - \int_0^t s z_2(s/\varepsilon)\, ds + O(\varepsilon^2) \end{aligned} \tag{2 8 10}$$

This is equivalent to

$$\begin{aligned} \varepsilon^2 y_0(\varepsilon\tau) + z_0(\tau) + \varepsilon z_1(\tau) + \varepsilon^2 z_2(\tau) &= f(\varepsilon\tau) - \int_0^{\varepsilon\tau} s y_0(s)\, ds \\ &- \int_0^\tau \sigma z_0(\sigma)\, d\sigma - \varepsilon \int_0^\tau \sigma z_1(\sigma)\, d\sigma - \varepsilon^2 \int_0^\tau \sigma z_2(\sigma)\, d\sigma + O(\varepsilon^2) \end{aligned}$$

It follows that

$$z_j(\tau) = \psi_j(\tau) - \int_0^\tau \sigma z_j(\sigma)\, d\sigma \tag{2 8 11}$$

where

$$\psi_0(\tau) = f(0), \quad \psi_1(\tau) = f'(0)\tau, \quad \psi_2(\tau) = \frac{1}{2}(f''(0) - y_0(0))\tau^2 - y_0(0)$$

Therefore,

$$z_0(\tau) = f_0(0)e^{-\tau^2/2}, \quad z_1(\tau) = f'(0)\int_0^\tau e^{-(\tau^2-\sigma^2)/2}\,d\sigma,$$
$$z_2(\tau) = f''(0)(1 - e^{-\tau^2/2}) - y_0(0)$$

In order to calculate the outer solution, we express all terms in (2 8 11) in terms of the outer variable t and substitute them into (2 8 10), giving

$$\varepsilon^2 y_0(t) = f(t) - f(0) - f'(0)t - \frac{1}{2}(f''(0) - y_0(0))t^2 - \int_0^t s y_0(s)\,ds + O(\varepsilon^2)$$

By letting $\varepsilon \to 0$ an equation for $y_0(t)$ is obtained with solution

$$y_0(t) = \frac{f'(t) - f'(0)}{t} - f''(0) + y_0(0)$$

Since $\lim_{t\to 0} y_0(t) = y_0(0)$, $z_2(\tau) \to f''(0) - y_0(0)$ as $\tau \to \infty$ and we choose $y_0(0) = f''(0)$ so that $z_2(\tau) \to 0$ as $\tau \to \infty$ as required Also by integrating by parts it can be shown that

$$z_1(\tau) = \frac{f'(0)}{\tau}\left[1 + \sum_{n=0}^{\infty} \frac{(1)(3)(5) \quad (2n-1)}{\tau^{2n}}\right] \quad \text{as } \tau \to \infty,$$

so there is only algebraic decay

The candidate leading order solution is given by

$$u_0(t,\varepsilon) = y_0(t) + \frac{1}{\varepsilon^2}z_0(t/\varepsilon) + \frac{1}{\varepsilon}z_1(t/\varepsilon) + z_2(t/\varepsilon),$$

which agrees with (2 8 7) It is not hard to directly show this is a uniformly valid asymptotic solution Also there is nontrivial contribution to the outer solution from $\lim_{\varepsilon\to 0} \varepsilon^{-1} z_1(t/\varepsilon) = f'(0)t^{-1}$ with $t > 0$ fixed, which would not be the case if $z_1(\tau)$ decayed exponentially

Our calculations suggest that the method of additive decomposition can also be applied to problems where there is no exponential decay in the boundary layer

Chapter 3

Integrodifferential Equations with Continuous Kernels

3.1 Introduction

This chapter considers the singularly perturbed linear Volterra integrodifferential equation

$$\varepsilon \mathbf{u}'(t) = \mathbf{f}(t) + \mathbf{B}(t)\mathbf{u}(t) + \int_0^t \mathbf{A}(t,s)\mathbf{u}(s)\,ds, \quad 0 < t \le T, \tag{3 1 1}$$

$$\mathbf{u}(0) = \mathbf{a}, \tag{3 1 2}$$

where $0 < \varepsilon << 1$ The vector-valued function $\mathbf{f}(t)$ is continuous for $0 \le t \le T$, the matrix-valued function $\mathbf{B}(t)$ is continuous for $0 \le t \le T$ and the matrix-valued kernel $\mathbf{A}(t,s)$ is continuous for $0 \le s \le t \le T$

For $\varepsilon > 0$, problem (3 1 1) is a Volterra integrodifferential equation which has a unique solution $\mathbf{u}(t,\varepsilon) \in C^1[0,T]$ It is given by

$$\mathbf{u}(t,\varepsilon) = \mathbf{\Gamma}(t,0,\varepsilon)\mathbf{a} + \int_0^t \mathbf{\Gamma}(t,s,\varepsilon)\mathbf{f}(s)\,ds, \quad 0 \le t \le T, \tag{3 1 3}$$

where $\mathbf{\Gamma}(t,s,\varepsilon)$, defined for $0 \le s \le t \le T$, is the resolvent matrix given by

$$\partial_2\mathbf{\Gamma}(t,s,\varepsilon) = -\mathbf{\Gamma}(t,s,\varepsilon)\mathbf{B}(s) - \int_s^t \mathbf{\Gamma}(t,v,\varepsilon)\mathbf{A}(v,s)\,dv, \tag{3 1 4}$$

and $\mathbf{\Gamma}(t,t,\varepsilon) = I$ For $\varepsilon = 0$, problem (3 1 1) reduces to

$$\mathbf{0} = \mathbf{B}(t)\mathbf{v}(t) + \mathbf{f}(t) + \int_0^t \mathbf{A}(t,s)\mathbf{v}(s)\,ds, \quad 0 \le t \le T \tag{3 1 5}$$

Problem (3 1 5) is a Volterra integral equation of the second kind which does have a continuous solution $\mathbf{v}\ [0,T] \to \mathbb{R}^n$ if either $\mathbf{B}(t)$ or $\mathbf{A}(t,t)$ is invertible and the data is C^1 If (3 1 5) has a continuous solution $\mathbf{v}(t)$ such that $\mathbf{v}(0) \ne \mathbf{a}$, then $\mathbf{v}(t)$ cannot approximate $\mathbf{u}(t,\varepsilon)$ uniformly on $[0,T]$ Thus, problem (3 1 1) is singularly perturbed We are interested in obtaining asymptotic approximations which are uniformly valid in $[0,T]$ as $\varepsilon \to 0$ of (3 1 1)

We construct in Section 3 2 a formal solution $\mathbf{U}(t,\varepsilon)$ using the additive decomposition method introduced in Chapter 2 The main result of this chapter is presented in Section 3 3 where it is proved that $\mathbf{U}_N(t,\varepsilon)$ is an *asymptotic solution* of (3 1 1) in the sense that

$$|\mathbf{u}(t,\varepsilon) - \mathbf{U}_N(t,\varepsilon)| = O(\varepsilon^{N+1}) \quad \text{as } \varepsilon \to 0$$

It is not surprising that results for (3 1 1) can be found using the techniques of Chapter 2, because there we had to first differentiate integral equations in Sections 2 4 and 2 5 to prove our results Formal expansions for the asymptotic solution of this problem have been obtained Angell and Olmstead [1] for the Volterra equations However their approach has a shortcoming in that general equations for the coefficients in the formal solution could not be determined Smith and Lange [15] deduced a general expansion and rigorous estimates for Fredholm integrodifferential equations from their theory of Fredholm integral equations The expansion procedure developed here modifies that of Smith and Lange [15] Both the papers cited use the additive decomposition method Lomov [18] gets rigorous results by employing a different multiple time scale method He introduces n new time scales, not just the one $\tau = t/\varepsilon$

3.2 Heuristic Analysis and Formal Solution

We seek a formal solution $\mathbf{u}(t,\varepsilon)$ of the form

$$\mathbf{u}(t,\varepsilon) = \mathbf{y}(t,\varepsilon) + \mathbf{z}(t/\varepsilon,\varepsilon), \tag{3 2 1}$$

where $\mathbf{y}(t,\varepsilon)$ and $\mathbf{z}(t/\varepsilon,\varepsilon)$ are represented by the asymptotic series (2 4 2) and (2 4 3) with

$$|\mathbf{z}_j(\tau)| = O(e^{-\beta_j \tau}), \quad \tau \to \infty, \quad j = 0, 1, \quad , \tag{3 2 2}$$

for some $\beta_j > 0$

We form the partial sum

$$\mathbf{U}_N(t,\varepsilon) = \sum_{n=0}^{N} \mathbf{u}_n(t,\varepsilon)\varepsilon^n, \tag{3 2 3}$$

and the formal sum

$$\mathbf{U}(t,\varepsilon) = \sum_{n=0}^{\infty} \mathbf{u}_n(t,\varepsilon)\varepsilon^n,$$

where

$$\mathbf{u}_j(t,\varepsilon) = \mathbf{y}_j(t) + \mathbf{z}_j(t/\varepsilon) \tag{3 2 4}$$

In this section we assume that $\mathbf{f}(t)$ and $\mathbf{A}(t,s)$ are C^∞ Clearly

$$\int_0^t \mathbf{A}(t,s)\mathbf{u}_j(s,\varepsilon)\,ds = \int_0^t \mathbf{A}(t,s)\mathbf{y}_j(s)\,ds + \int_0^t \mathbf{A}(t,s)\mathbf{z}_j(s/\varepsilon)\,ds$$

Decomposing this equation into functions of t and functions of t/ε as in Section 2 4, we get for all $m \geq 0$

$$\begin{aligned}\int_0^t \mathbf{A}(t,s)\mathbf{u}_j(s,\varepsilon)\,ds = \int_0^t \mathbf{A}(t,s)\mathbf{y}_j(s)\,ds + \sum_{i=1}^{m}\varepsilon^i \int_0^\infty \mathbf{E}_{i-1}(t,\sigma)\mathbf{z}_j(\sigma)\,d\sigma \\ - \sum_{i=1}^{m}\varepsilon^i \int_{t/\varepsilon}^\infty \mathbf{F}_{i-1}(t/\varepsilon,\sigma)\mathbf{z}_j(\sigma)\,d\sigma + O(\varepsilon^{m+1})\end{aligned} \tag{3 2 5}$$

where $\mathbf{E}_i(t,\sigma)$ and $\mathbf{F}_i(\tau,\sigma)$ are defined by (2 4 7) and (2 4 8) respectively The last integral above represents a boundary layer function

The residual $\rho_N(t,\varepsilon)$ is defined by the relation

$$\varepsilon\mathbf{U}'_N(t,\varepsilon) = \mathbf{f}(t) + \mathbf{B}(t)\mathbf{U}_N(t,\varepsilon) + \int_0^t \mathbf{A}(t,s)\mathbf{U}_N(s,\varepsilon)\,ds - \rho_N(t,\varepsilon), \tag{3 2 6}$$

By substituting (3 2 3) into this equation and using (3 2 5) to replace the integrals, we obtain

$$\begin{aligned}\rho_N(t,\varepsilon) = \mathbf{f}(t) + \sum_{j=0}^{N}\varepsilon^j\left(\mathbf{B}(t)\mathbf{y}_j(t) + \int_0^t \mathbf{A}(t,s)\mathbf{y}_j(s)\,ds + \sum_{i=0}^{j-1}\int_0^\infty \mathbf{E}_{j-i-1}(t,\sigma)\mathbf{z}_i(\sigma)\,d\sigma\right) \\ + \sum_{j=0}^{N}\varepsilon^j\left(\sum_{i=0}^{j}\mathbf{G}_{j-i}(\tau)\mathbf{z}_i(\tau) + \sum_{i=0}^{j-1}\int_{t/\varepsilon}^\infty \mathbf{F}_{j-i-1}(t/\varepsilon,\sigma)\mathbf{z}_i(\sigma)\,d\sigma\right) \\ - \sum_{j=0}^{N-1}\varepsilon^{j+1}\mathbf{y}'_j(t) - \sum_{j=0}^{N}\varepsilon^j\mathbf{z}'_j(t/\varepsilon) + O(\varepsilon^{N+1}),\end{aligned} \tag{3 2 7}$$

uniformly for $0 \leq t \leq T$ where

$$\mathbf{G}_i(\tau) = \frac{1}{i!}\tau^i \frac{d^i\mathbf{B}}{dt^i}(0)$$

Equation (3 2 7) is equivalent to

$$\rho_N(t,\varepsilon) = \sum_{j=0}^{N}\varepsilon^j(\mathbf{p}_j(t) + \mathbf{q}_j(t/\varepsilon)) + O(\varepsilon^{N+1}), \tag{3 2 8}$$

uniformly for $0 \leq t \leq T$, where

$$\mathbf{p}_j(t) = \mathbf{B}(t)\mathbf{y}_j(t) + \int_0^t \mathbf{A}(t,s)\mathbf{y}_j(s) + \phi_j(t) - \mathbf{y}'_{j-1}(t),$$

$$\mathbf{q}_j(\tau) = \mathbf{B}(0)\mathbf{z}_j(\tau) + \psi_j(\tau) - \mathbf{z}'_j(\tau),$$

with

$$\phi_j(t) = \begin{cases} \mathbf{f}(t), & j = 0, \\ \sum_{i=0}^{j-1} \int_0^{\infty} \mathbf{E}_{j-i-1}(t,\sigma)\mathbf{z}_i(\sigma)\,d\sigma, & j \geq 1, \end{cases} \tag{3 2 9}$$

$$\psi_j(\tau) = \begin{cases} \mathbf{0}, & j = 0, \\ \sum_{i=0}^{j-1} \mathbf{G}_{j-i}(\tau)\mathbf{z}_i(\tau) - \sum_{i=0}^{j-1} \int_\tau^{\infty} \mathbf{F}_{j-i-1}(\tau,\sigma)\mathbf{z}_i(\tau)\,d\sigma, & j \geq 1 \end{cases} \tag{3 2 10}$$

We observe that (3 2 9) and (3 2 10) imply that $\phi_j(t)$ and $\psi_j(\tau)$ are determined by $\mathbf{z}_i(\tau)$ for $i = 0, 1, \quad, j-1$

A calculation similar to that in Section 2 4 shows that $\mathbf{q}_j(\tau) \to \mathbf{0}$ as $\tau \to \infty$ if (3 2 2) holds If $\mathbf{U}(t,\varepsilon)$ is a formal solution, $\rho_N(t,\varepsilon) = O(\varepsilon^{N+1})$ for all $N \geq 0$, in which case Lemma 2 1 in Chapter 2 implies that, for each $j \geq 0$, $\mathbf{y}_j(t)$ satisfies

$$\mathbf{y}'_{j-1}(t) = \mathbf{B}(t)\mathbf{y}_j(t) + \int_0^t \mathbf{A}(t,s)\mathbf{y}_j(s)\,ds + \phi_j(t), \quad 0 \leq t \leq T, \tag{3 2 11}$$

and $\mathbf{z}_j(\tau)$ satisfies

$$\mathbf{z}'_j(\tau) = \mathbf{B}(0)\mathbf{z}_j(\tau) + \psi_j(\tau), \quad \tau > 0 \tag{3 2 12}$$

Also each $\mathbf{z}_j(\tau)$ obeys the initial condition

$$\mathbf{z}_j(0) = \begin{cases} \mathbf{a} - \mathbf{y}_0(0), & j = 0, \\ -\mathbf{y}_j(0), & j \geq 1 \end{cases} \tag{3 2 13}$$

Remark 3 1 It follows from (3 2 8) that if each $\mathbf{y}_j(t)$ satisfies (3 2 11) and each $\mathbf{z}_j(\tau)$ satisfies (3 2 12), then $|\rho_N(t,\varepsilon)| = O(\varepsilon^{N+1})$ as $\varepsilon \to 0$ uniformly for $0 \leq t \leq T$

3.3 Properties of Formal Solution

In this section, we show that the equations for $\mathbf{y}_j(t)$ and $\mathbf{z}_j(\tau)$ derived in Section 3 2 have the properties required in their derivation, and then prove that

$$|\mathbf{u}(t,\varepsilon) - \mathbf{U}_N(t,\varepsilon)| = O(\varepsilon^{N+1})$$

uniformly for $0 \leq t \leq T$ as $\varepsilon \to 0$

The following assumption will be used

($\mathbf{H}'_1$) The functions $\mathbf{f} \quad [0,T] \to \mathbb{R}^n$, $\mathbf{B} \quad [0,T] \to \mathbb{R}^{n\times n}$ and $\mathbf{A} \quad \Delta_T \to \mathbb{R}^{n\times n}$ are all C^∞, where Δ_T is defined as in (2 2 2)

Proposition 3 2 *Suppose that* $(\mathbf{H}_1')$ *and* $(\mathbf{H}_2)$ *hold Then for each* $j \geq 0$ *there is a* C^∞ *solution* $\mathbf{y}_j(t)$ *on* $[0,T]$ *of (3 2 11) and a* C^∞ *solution* $\mathbf{z}_j(\tau)$ *on* $[0,\infty)$ *of (3 2 12) and* (3 2 13), *moreover there are positive constants* $\beta < \alpha$ *and* c_j

$$|\mathbf{z}_j(\tau)| \leq c_j \, e^{-\beta\tau}, \quad \tau \geq 0 \tag{3 3 1}$$

The proof is similar to that of Proposition 2 2 in Chapter 2 and therefore is omitted

Lemma 3 3 *Suppose that* $(\mathbf{H}_1')$ *and* $(\mathbf{H}_2)$ *hold Then for each* $j \geq 0$ *the residual* $\boldsymbol{\rho}_j(t,\varepsilon)$ *defined in* (3 2 6) *satisfies*

$$|\boldsymbol{\rho}_j(t,\varepsilon)| \leq e_j \varepsilon^{j+1}, \tag{3 3 2}$$

as $\varepsilon \to 0$ *uniformly for all* $0 \leq t \leq T$, *for some fixed positive constant* e_j *independent of* ε

As pointed out in Remark 3 1 the result follows from what has already been done in Section 3 2 It can also be proved that there are positive constants d_j such that

$$\int_0^t |\boldsymbol{\rho}_j(s,\varepsilon)| \, ds \leq d_j \varepsilon^{j+1}$$

uniformly for all $0 \leq t \leq T$

Theorem 3 4 *Suppose that* $(\mathbf{H}_1')$ *and* $(\mathbf{H}_2)$ *hold Then there are constants* $C_N > 0$ *such that*

$$|\mathbf{u}(t,\varepsilon) - \mathbf{U}_N(t,\varepsilon)| \leq C_N \varepsilon^{N+1} \tag{3 3 3}$$

uniformly on $[0,T]$ *as* $\varepsilon \to 0$ *where* C_N *is independent of* ε

Proof We introduce the the remainder term

$$\mathbf{r}_N(t,\varepsilon) = \mathbf{u}(t,\varepsilon) - \mathbf{U}_N(t,\varepsilon),$$

as in Chapter 2 It satisfies the following problem

$$\varepsilon \mathbf{r}_N'(t,\varepsilon) = \boldsymbol{\rho}_N(t,\varepsilon) + \mathbf{B}(t)\mathbf{r}_N(t,\varepsilon) + \int_0^t \mathbf{A}(t,s)\mathbf{r}_N(s,\varepsilon) \, ds, \ t > 0,$$

with $\mathbf{r}_N(0,\varepsilon) = \mathbf{0}$ The variation of parameters formula enables us to see that its solution $\mathbf{r}_N(t,\varepsilon)$ satisfies

$$\mathbf{r}_N(t,\varepsilon) = \frac{1}{\varepsilon}\int_0^t \boldsymbol{\Phi}(t,s,\varepsilon)\boldsymbol{\rho}_N(s,\varepsilon) \, ds + \frac{1}{\varepsilon}\int_0^t \left(\int_v^t \boldsymbol{\Phi}(t,s,\varepsilon)\mathbf{A}(s,v) \, ds\right) \mathbf{r}_N(v,\varepsilon) \, dv, \tag{3 3 4}$$

where $\Phi(t,s,\varepsilon)$ is defined as in equation (2 6 4) The bound (3 3 3) follows from (3 3 4) using (H_2) and the estimates given in (3 3 2) The details are almost identical to those in the proof of Theorem 2 5, and are omitted □

Remark 3 5 The initial condition for $\mathbf{u}(0,\varepsilon)$ can depend on ε More precisely (3 1 2) can be replaced by

$$\mathbf{u}(0,\varepsilon) = \frac{1}{\varepsilon}\sum_{j=0}^{\infty}\varepsilon^j \mathbf{a}_j,$$

where each $\mathbf{a}_j$ is constant The case $\mathbf{a}_0 \neq 0$, leads to an analysis similar to that in Chapter 2 The analysis in this Chapter corresponds to the case where $\mathbf{a}_0 = 0$ The differences between the two cases are twofold Not only is the form of the asymptotic expansion different, but the outer solution can be constructed first in the case $\mathbf{a}_0 = 0$ whereas the initial layer correction solution must be found first in the case $\mathbf{a}_0 \neq 0$

Chapter 4

Volterra Equations with Weakly Singular Kernels

4.1 Introduction

This chapter considers the weakly singular scalar Volterra integral equation of the second kind

$$\varepsilon u(t) = f(t) + \frac{1}{\Gamma(\beta)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\beta}} u(s)\, ds, \quad 0 \le t \le T, \tag{4 1 1}$$

where $0 < \varepsilon << 1$ and $0 < \beta < 1$ The functions $f(t)$ and $k(t,s)$ are continuous and $k(t,t) = -1$ This problem (4 1 1) exhibits an initial layer at $t = 0$ like the equations with continuous kernels considered in Chapter 2, but with a narrower initial layer width of order $O(\varepsilon^{1/\beta})$ as $\varepsilon \to 0$

The weakly singular equation (4 1 1) has a solution $u(t,\varepsilon)$ in $C[0,T]$ for all $\varepsilon > 0$ For $\varepsilon = 0$ (4 1 1) reduces to the Abel integral equation

$$0 = f(t) + \frac{1}{\Gamma(\beta)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\beta}} v(s)\, ds, \quad 0 \le t \le T, \tag{4 1 2}$$

It certainly does not have a continuous solution if $f(0) \neq 0$ The forcing function $f(t)$ must be smoother than the desired solution Even if (4 1 2) has a solution $v(t)$ in $C^0[0,T]$ it may not approximate $u(t,\varepsilon)$ uniformly for t in $[0,T]$ as $\varepsilon \to 0$

The kernel $a(t,s)$ in (4 1 1) given by

$$a(t,s) = \frac{k(t,s)}{\Gamma(\beta)(t-s)^{1-\beta}},$$

obviously does not satisfy the boundary layer stability condition $(\mathbf{H}_2)$ of section 2 2, though $\lim_{s\uparrow t} a(t,s) = -\infty$ because $k(t,t) = -1$ If an equation like (4 1 1) is encountered with $k(0,0) < 0$, a simple rescaling of ε leads to $k(0,0) = -1$ If $k(t,t) < 0$ the equation for $t \mapsto k(t,t)u(t)$ has the form of (4 1 1)

Our aim is to find asymptotic approximations $U_N(t,\varepsilon)$ which are uniformly close on $[0,T]$ to $u(t,\varepsilon)$ as $\varepsilon \to 0$ Problems of the type (4 1 1) do not exhibit an exponential decay in the initial layer and therefore the methodology developed in Chapter 2 must be modified To emphasise the fundamental ideas and illustrate the technical difficulties, we only attempt here to find the leading order term $U_0(t,\varepsilon)$ of the asymptotic solution It is proved that the residual $|\rho_0(t,\varepsilon)| = O(\varepsilon)$

uniformly as $\varepsilon \to 0$ It is not demonstrated that $|u(t,\varepsilon) - U_0(t,\varepsilon)| = O(\varepsilon)$ For an example with a known exact solution though, we do establish this estimate

4 2 Mathematical Preliminaries

In this section we review some of the results which are applied later in the chapter Firstly though we state some hypotheses which are used

(H_6) $0 < \beta < 1$

(H_7) $k(t,s)$ is a C^2 function on Δ_T with $k(t,t) = -1$, where

$$\Delta_T = \{(t,s), 0 \le t \le T\}$$

(H_8) The function $f(t)$ is C^2 on $[0,T]$ with $f(0) \neq 0$

4 2 1 Solution of Abel Equations

It is a classical result of Abel's that for $0 < \beta < 1$ the equation

$$\frac{1}{\Gamma(\beta)} \int_0^t \frac{1}{(t-s)^{1-\beta}} y(s)\, ds = \phi(t), \tag{4 2 1}$$

has the solution

$$y(t) = (D^\beta \phi)(t) \ = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{1}{(t-s)^\beta} \phi(s)\, ds$$

This relies on the useful formula

$$\int_0^t \frac{1}{(t-s)^{1-\beta} s^\beta} ds = \Gamma(\beta)\Gamma(1-\beta)$$

Tonelli proved that (4 2 1) has a solution in $L^1[0,T]$ if ϕ is absolutely continuous on $[0,T]$ In this section we consider the more general Abel equation

$$\frac{1}{\Gamma(\beta)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\beta}} y(s)\, ds = \phi(t) \tag{4 2 2}$$

Gorenflo and Vessella [9] give several existence and uniqueness for (4 2 2) We state here a special case of Theorem 5 1 4 of [9]

Theorem 4 1 *Suppose that* (H_6) *and* (H_7) *hold Let* $\mathrm{D}^{\beta}\phi$ *be continuous on* $[0,T]$ *Then* (4 2 2) *has a unique solution* y *in* $\mathrm{C}[0,T]$ *and*

$$||y||_{\mathrm{C}^1} \leq C||\mathrm{D}^{\beta}\phi||$$

for some constant $C > 0$ *depending on* T *and* $||k||_{\mathrm{C}^1(\Delta_T)}$

Later in this chapter we require knowledge of the asymptotic behaviour of solutions $y(t)$ of (4 2 2) The following result is Theorem 5 1 5 of [9] and comes from Atkinson [4]

Theorem 4 2 *Suppose that* (H_6) *and* (H_7) *hold Suppose that there is a function* $\tilde{\phi}(t)$ *in* C^1 *such that* $\phi(t) = t^{\mu}\tilde{\phi}(t)$, *with* $1 - \beta + \mu > 0$ *Then* (4 2 2) *has a unique solution* $y(t)$, *and this solution can be expressed as*

$$y(t) = t^{\mu-\beta}\tilde{y}(t),$$

where $\tilde{y}(t) = \nu + ty^*(t)$ *with* ν *constant and* y^* *continuous Moreover* $\nu = 0$ *if and only if* $\tilde{\phi}(0) = 0$, *and there is a constant* $\mathrm{c} > 0$ *such that*

$$||\tilde{y}||_{\mathrm{C}} \leq c||\tilde{\phi}||_{\mathrm{C}^1}$$

4 2 2 The Mittag-Leffler Function and its Asymptotic Expansion

In this section we present some of the properties of the Mittag-Leffler function, $\mathrm{E}_{\mu} \quad \mathbb{C} \to \mathbb{C}$ In particular we state formulae for $\mathrm{E}_{\mu}(z)$ for large $z \in \mathbb{C}$ For each $\mu > 0$ the Mittag-Leffler function is defined by

$$\mathrm{E}_{\mu}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\mu n + 1)} z^n \tag{4 2 3}$$

E_{μ} is *entire*, and

$$\mathrm{E}_1(z) = \mathrm{e}^z, \quad \mathrm{E}_2(z) = \cosh z, \quad \mathrm{E}_{1/2}(z^{1/2}) = 2\pi^{-1/2}\mathrm{e}^{-z}\,\mathrm{erfc}(-z^{1/2}) \tag{4 2 4}$$

An interesting property proved by Pollard [23] is $t \mapsto \mathrm{E}_{\mu}(-t)$ is completely monotonic on $[0,\infty)$ if $0 \leq \mu \leq 1$ Thus for μ in this parameter range $(-1)^n \mathrm{E}_{\mu}^{(n)}(-t) \geq 0$ for $t \geq 0$, where

$$\mathrm{E}_{\mu}^{n}(z) = \frac{d^n \mathrm{E}_{\mu}}{dz^n}(z)$$

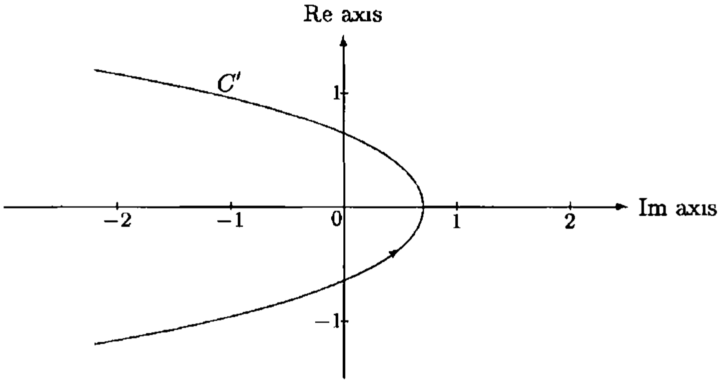

Figure 4 1 The contour of integration for the Mittag-Leffler function $E_\mu(z)$

A detailed discussion on the properties of the Mittag-Leffler function can be found in Chapter 18 of Erdelyi, Magnus, Oberhettinger and Tricomi [6] or Chapter 5 of Paris and Kaminski [22]

We are interested in the asymptotic expansion of $\mathrm{E}_\mu(z)$ only in case where $0 < \mu < 1$ However the asymptotic expansions formulae below are for all $0 < \mu < 2$ These expansions are derived from the representation

$$\mathrm{E}_\mu(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e(s) z^{-s}\, ds, \tag{4 2 5}$$

for some $0 < c < 1$ where

$$e(s) = \frac{\pi \cos \pi s}{\Gamma(1-\mu s)\sin \pi s}, \tag{4 2 6}$$

(4 2 5) comes from the formula for inverting a Mellin transform We decompose the path in (4 2 5) into a contour C' which is closed to the left It is shown in Figure 4 1 Now $e(s)z^{-s}$ has simple poles at $s = 0, -1, -2,$ Let a_n be the residue of $s \mapsto e(s)z^{-s}$ at $-n$ Then

$$a_n = \frac{z^n}{\Gamma(1+\mu n)}$$

To check that $e(s)$ above is the proper choice in (4 2 5)

$$\frac{1}{2\pi i}\int_{C'} e(s)z^{-s}\, ds = \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + 1)} = \mathrm{E}_\mu(z)$$

Using the integral representation in (4 2 5), it is shown in Erdelyi et al [6] and Paris and Kaminski [22] that for $0 < \mu < 2$, the controlling factor of the leading behaviour of $\mathrm{E}_\mu(z)$ is $e^{z^{1/\mu}}$ as $z \to \infty$ Stokes lines occur at $\mathrm{Re}\, z^{1/\mu} = 0$ or $\arg z = \pm\pi\mu$ and anti-Stokes lines occur at $\mathrm{Im}\, z^{1/\mu} = 0$ or $\arg z = \pm\frac{\pi}{2}\mu$

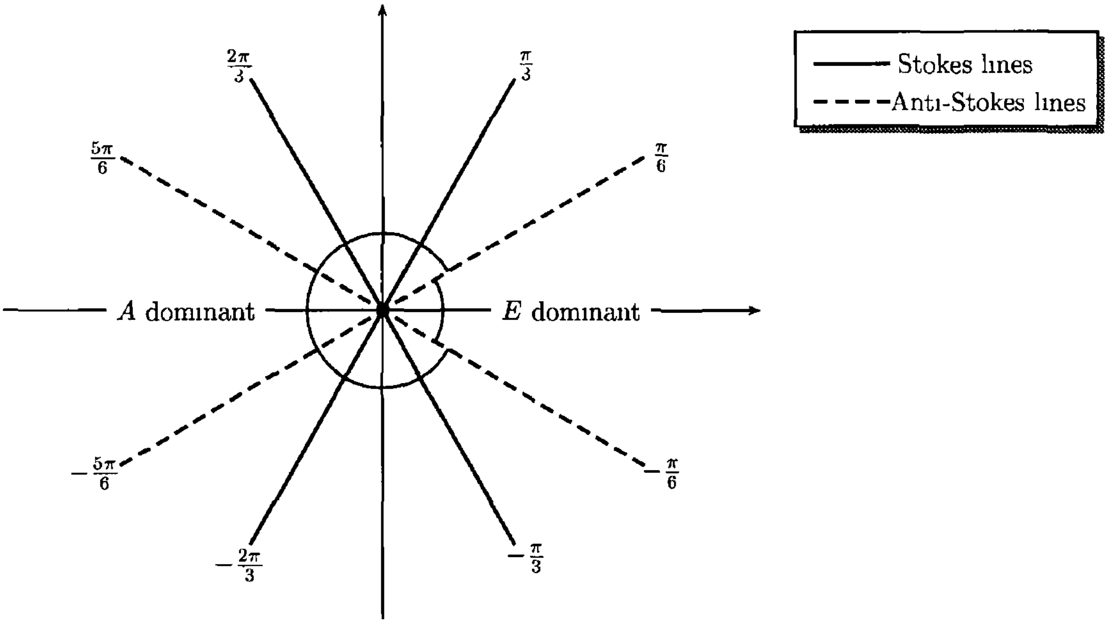

Figure 4 2 The Stokes lines are shown for the exponential term in (4 2 8a) corresponding to $\mu = 1/3$ Also shown is the sector E where the exponential term in (4 2 8a) dominates and the sector A where the algebraic term in (4 2 8a) and (4 2 8b) dominates

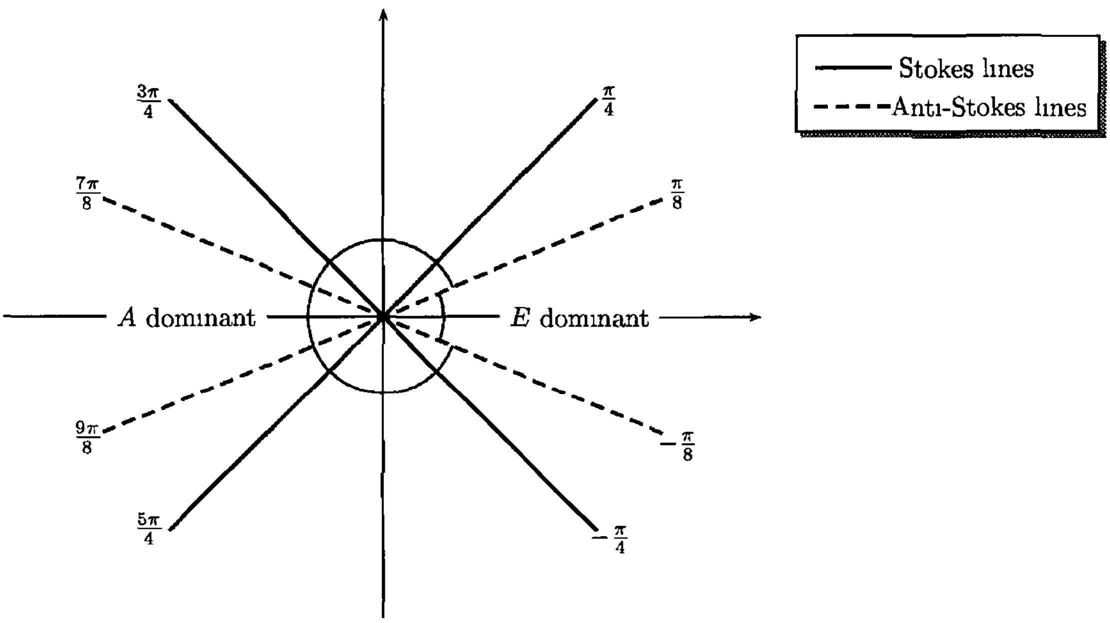

Figure 4 3 The Stokes lines are shown for the exponential term in (4 2 8a) corresponding to $\mu = 1/4$ Also shown is the sector E where the exponential term in (4 2 8a) dominates and the sector A where the algebraic term in (4 2 8a) and (4 2 8b) dominates

It is shown in Erdelyi et al [6] and Paris and Kaminski [22] that the expansion of $E_\mu(z)$ when $\mu < 2$ is given by

$$E_\mu(z) \sim \frac{1}{\mu} e^{z^{1/\mu}} - \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\mu k)}, \quad |\arg z| < \frac{3\pi\mu}{2}, \tag{4 2 7a}$$

$$E_\mu(z) \sim -\sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\mu k)}, \quad |\arg(-z)| < \frac{\pi}{2}(2-\mu) \tag{4 2 7b}$$

It should be noted that (4 2 7) is a valid asymptotic expansion in the Poincare sense The discussion in § 5 1 of [22] elucidates what is happening for $0 < \mu < 1$ The expansions have a common sectors $\pi\mu/2 < |\arg z| < 3\pi\mu/2$ In the sector $|\arg z| < \pi\mu$, expansion (4 2 7a) is valid However the exponential term is decaying for $\pi\mu/2 < |\arg z| < \pi\mu$ since the anti-Stokes lines at $\arg z = \pm\pi\mu/2$ have been crossed $E_\mu(z)$ is exponentially large as $|z| \to \infty$ for $|\arg z| < \pi\mu/2$ As $\arg z$ crosses the Stokes lines $\arg z = \pm\pi\mu$, the exponential term disappears from the leading order term and becomes subdominant It remerges as $\arg z$ crosses $\pm 2\pi\mu$, but it is exponentially decaying At $\arg z = 3\pi\mu/2$, expansion (4 2 7a) is no longer valid Expansion (4 2 7b) holds for $|\arg(-z)| < \pi\mu/2$ Since we are interested in the asymptotic expansion on the negative real axis, this sector particularly concerns us The conclusion is that we obtain the composite expansion

$$E_\mu(z) \sim \frac{1}{\mu} e^{z^{1/\mu}} - \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\mu k)}, \quad |\arg z| < \pi\mu, \tag{4 2 8a}$$

$$E_\mu(z) \sim -\sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\mu k)}, \quad |\arg(-z)| < \pi(1-\mu) \tag{4 2 8b}$$

We illustrate this in the Figures 4 2 and 4 3

4 2 3 Solution of a Simple Class of Abel -Volterra Equations

The Abel -Volterra equation

$$z(\tau) = \psi(\tau) - \frac{1}{\Gamma(\beta)} \int_0^\tau \frac{1}{(\tau-\sigma)^{1-\beta}} z(\sigma)\, d\sigma, \quad \tau \geq 0, \tag{4 2 9}$$

has an explicit solution in terms of the Mittag - Leffler function E_β

The following existence and uniqueness result, which is attributed to Hille and Tamarkin [11], is given in Geronflo and Vessella [9]

Theorem 4 3 *Let $0 < \beta < 1$ and $\psi(\tau)$ be continuous on $[0, \infty)$ Then equation (4 2 9) has the* *continuous solution $z(\tau)$ given by*

$$z(\tau) = \frac{d}{d\tau}\int_0^\tau \mathrm{E}_\beta(-(\tau-\sigma)^\beta)\psi(\sigma)\,d\sigma, \quad \tau \geq 0 \tag{4 2 10}$$

z is unique in the class $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+)$

4.3 Heuristic Analysis and Formal Solution

The analysis of Section 2 3 shows that we should introduce the new time scale $\tau = t/\varepsilon^\gamma$ where $\gamma = \beta^{-1}$ We call this the inner variable It is easily found that if $f(0) \neq 0$ then the magnitude of the boundary layer is ε^{-1} and the width ε^γ We seek an asymptotic solution $u(t,\varepsilon)$ in the form

$$u(t,\varepsilon) = y(t,\varepsilon) + \frac{1}{\varepsilon}z(t/\varepsilon^\gamma,\varepsilon), \tag{4 3 1}$$

and require that

$$\lim_{\tau\to\infty} z(\tau,\varepsilon) = 0$$

$z(t/\varepsilon^\gamma,\varepsilon)$ corrects the nonuniformity in the initial layer Substituting (4 3 1) into (4 1 1) gives

$$\varepsilon y(t,\varepsilon) + z(t/\varepsilon^\gamma,\varepsilon) = f(t) + \frac{1}{\Gamma(\beta)}\int_0^t \frac{k(t,s)}{(t-s)^{1-\beta}}y(s,\varepsilon)\,ds + \frac{1}{\Gamma(\beta)\varepsilon}\int_0^t \frac{k(t,s)}{(t-s)^{1-\beta}}z(s/\varepsilon^\gamma,\varepsilon)\,ds \tag{4 3 2}$$

It is assumed that $y(t,\varepsilon)$ and $z(\tau,\varepsilon)$ have asymptotic expansions of the form

$$y(t,\varepsilon) \sim \sum_{n=0}^\infty \varepsilon^n y_n(t), \quad z(\tau,\varepsilon) \sim \sum_{n=0}^\infty \varepsilon^{n\gamma} z_n(\tau),$$

as $\varepsilon \to 0$, so that

$$u(t,\varepsilon) \sim \sum_{n=0}^\infty \varepsilon^n y_n(t) + \sum_{n=0}^\infty \varepsilon^{n\gamma-1} z_n(t/\varepsilon^\gamma) \tag{4 3 3}$$

Moreover we require that for all $n \geq 0$,

$$\lim_{\tau\to\infty} z_n(\tau) = 0 \tag{4 3 4}$$

Firstly we restrict attention to

$$U_0(t,\varepsilon) = y_0(t) + \frac{1}{\varepsilon}z_0(t/\varepsilon^\gamma),$$

assuming that

$$z_0(\tau) \to \frac{f(0)}{\Gamma(1-\beta)\tau^\beta} \quad \text{as } \tau \to \infty \tag{4 3 5}$$

Defining the residual $\rho_0(t,\varepsilon)$ in the usual way, we see

$$\rho_0(t,\varepsilon)+\varepsilon y_0(t)+z_0(t/\varepsilon^\gamma)=f(t)+\frac{1}{\Gamma(\beta)}\int_0^t\frac{k(t,s)}{(t-s)^{1-\beta}}y_0(s)\,ds + \frac{1}{\varepsilon\Gamma(\beta)}\int_0^t\frac{k(t,s)}{(t-s)^{1-\beta}}z_0(s/\varepsilon^\gamma)\,ds \tag{4.3.6}$$

By expressing this in terms of $\tau=t/\varepsilon^\gamma$,

$$\rho_0(\varepsilon^\gamma\tau,\varepsilon)+z_0(\tau)+\varepsilon y_0(\varepsilon^\gamma\tau)=f(\varepsilon^\gamma\tau)+\frac{1}{\Gamma(\beta)}\int_0^\tau\frac{k(\varepsilon^\gamma\tau,\varepsilon^\gamma\sigma)}{(\tau-\sigma)^{1-\beta}}z_0(\sigma)\,d\sigma + \varepsilon\int_0^\tau\frac{k(\varepsilon^\gamma\tau,\varepsilon^\gamma\sigma)}{(\tau-\sigma)^{1-\beta}}y_0(\varepsilon^\gamma\sigma)\,d\sigma$$

This can be rearranged as

$$\begin{aligned}\rho_0(\varepsilon^\gamma\tau,\varepsilon)+\varepsilon y_0(\varepsilon^\gamma\tau)=&\left(f(0)+\frac{1}{\Gamma(\beta)}\int_0^\tau\frac{k(0,0)}{(\tau-\sigma)^{1-\beta}}z_0(\sigma)\,d\sigma-z_0(\tau)\right)+f(\varepsilon^\gamma\tau)-f(0)\\&+\frac{1}{\Gamma(\beta)}\int_0^\tau\frac{k(\varepsilon^\gamma\tau,\varepsilon^\gamma\sigma)-k(0,0)}{(\tau-\sigma)^{1-\beta}}z_0(\sigma)\,d\sigma\\&+\frac{\varepsilon}{\Gamma(\beta)}\int_0^\tau\frac{k(\varepsilon^\gamma\tau,\varepsilon^\gamma\sigma)}{(\tau-\sigma)^{1-\beta}}y_0(\varepsilon^\gamma\sigma)\,d\sigma,\end{aligned}$$

and hence

$$\rho_0(\varepsilon^\gamma\tau,\varepsilon)=\left(f(0)+\frac{1}{\Gamma(\beta)}\int_0^\tau\frac{k(0,0)}{(\tau-\sigma)^{1-\beta}}z_0(\sigma)\,d\sigma-z_0(\tau)\right)+O(\varepsilon)+O(\varepsilon^\gamma)$$

We see that if $\rho_0(\varepsilon^\gamma\tau,\varepsilon)=o(1)$ as $\varepsilon\to 0$ for fixed $\tau>0$, then

$$z_0(\tau)=f(0)-\frac{1}{\Gamma(\beta)}\int_0^\tau\frac{1}{(\tau-\sigma)^{1-\beta}}z_0(\sigma)\,d\sigma,\quad \tau\ge 0 \tag{4.3.7}$$

To derive the leading order outer solution, we express (4.3.7) in terms of $t=\varepsilon^\gamma\tau$ and substitute into (4.3.6), giving

$$\rho_0(t,\varepsilon)+\varepsilon y_0(t)=f(t)-f(0)+\frac{1}{\Gamma(\beta)}\int_0^t\frac{k(t,s)}{(t-s)^{1-\beta}}y_0(s)\,ds + \frac{1}{\varepsilon\Gamma(\beta)}\int_0^t\frac{k(t,s)-k(0,0)}{(t-s)^{1-\beta}}z_0(s/\varepsilon^\gamma)\,ds \tag{4.3.8}$$

It follows from (4.3.5) and the Dominated Convergence Theorem that

$$\frac{1}{\varepsilon}\int_0^t\frac{\{k(t,s)-k(0,0)\}}{(t-s)^{1-\beta}}z_0(s/\varepsilon^\gamma)\,ds\to\frac{f(0)}{\Gamma(1-\beta)}\int_0^t\frac{\{k(t,s)-k(0,0)\}}{(t-s)^{1-\beta}s^\beta}\,ds,$$

as $\varepsilon\to 0$. If $\rho(t,\varepsilon)=o(1)$ as $\varepsilon\to 0$, we deduce from (4.3.6) that the leading order outer solution $y_0(t)$ satisfies

$$0=f(t)-f(0)+\frac{f(0)}{\Gamma(\beta)\Gamma(1-\beta)}\int_0^t\frac{k(t,s)-k(0,0)}{(t-s)^{1-\beta}s^\beta}\,ds+\frac{1}{\Gamma(\beta)}\int_0^t\frac{k(t,s)}{(t-s)^{1-\beta}}y_0(s)\,ds \tag{4.3.9}$$

If $y_0(t)$ satisfies (4 3 9) and $z_0(\tau)$ obeys (4 3 7), it follows from (4 3 8) that

$$\rho_0(t,\varepsilon) = -\varepsilon y_0(t) + \frac{1}{\varepsilon\Gamma(\beta)} \int_0^t \frac{k(t,s)-k(0,0)}{(t-s)^{1-\beta}} \left(z_0(s/\varepsilon^\gamma) - \frac{f(0)\varepsilon}{\Gamma(1-\beta)s^\beta} \right) ds \qquad (4\ 3\ 10)$$

4 4 Properties of the Formal Solution

In this section, we show that the solutions of the equations for $y_0(t)$ and $z_0(\tau)$ exist and elucidate some of their properties

Equation (4 3 9) for the outer solution can be rewritten as

$$0 = \phi(t) + \int_0^t \frac{k(t,s)}{(t-s)^{1-\beta}} y_0(s)\, ds, \quad 0 \le t \le T, \qquad (4\ 4\ 1)$$

where

$$\phi(t) = f(t) - f(0) + \frac{f(0)}{\Gamma(\beta)\Gamma(1-\beta)} \int_0^t \frac{k(t,s)-k(0,0)}{(t-s)^{1-\beta}s^\beta}\, ds$$

Note that

$$\left| \frac{f(0)}{\Gamma(\beta)\Gamma(1-\beta)} \int_0^t \frac{\{k(t,s)-k(0,0)\}}{(t-s)^{1-\beta}s^\beta}\, ds \right| \le f(0) \sup_{0\le s\le t} |k(t,s)-k(0,0)| \to 0$$

as $t \to 0$ This and $(\mathbf{H}_8)$ imply that $\phi(0) = 0$ Also

$$\begin{aligned} \frac{1}{t}\int_0^t \frac{k(t,s)-k(0,0)}{(t-s)^{1-\beta}s^\beta}\, ds &= \frac{1}{t}\int_0^1 \frac{\{k(t,t\theta)-k(0,0)\}}{(1-\theta)^{1-\beta}\theta^\beta}\, d\theta \\ &\to \partial_1 k(0,0) \int_0^1 \frac{1}{(1-\theta)^{1-\beta}\theta^\beta}\, d\theta + \partial_2 k(0,0) \int_0^1 \frac{\theta^{(1-\beta)}}{(1-\theta)^{1-\beta}}\, d\theta \end{aligned}$$

as $t \to 0$ Hence we can write

$$\phi(t) = t\tilde{\phi}(t) \qquad (4\ 4\ 2)$$

and show that $\tilde{\phi}(t)$ is C^1 Using Theorem 4 2 we can establish from (4 4 1) and (4 4 2) the following

Proposition 4 4 *Suppose that* $(\mathbf{H}_6)$, $(\mathbf{H}_7)$ *and* $(\mathbf{H}_8)$ *hold Then (4 3 9) has a unique continuous solution* $y_0(t)$ *which satisfies*

$$y_0(t) = t^{1-\beta}\tilde{y}_0(t), \qquad (4\ 4\ 3)$$

where $\tilde{y}_0$ *is continuous on* $[0,T]$

It is a simple corollary of Theorem 4 3 and (4 2 8b) that the following result is true

Proposition 4 5 *Suppose that* (H_6), (H_7) *and* (H_8) *hold Then (4 3 7) has the continuous solution*

$$z_0(\tau) = f(0)\,\mathrm{E}_\beta(-\tau^\beta), \tag{4 4 4}$$

for $\tau \geq 0$, *which satisfies*

$$z_0(\tau) \sim f(0)\sum_{j=1}^{\infty}(-1)^{j+1}\frac{\tau^{-\beta j}}{\Gamma(1-\beta j)} \quad \textit{as } \tau \to \infty \tag{4 4 5}$$

Remark 4 6 This results vindicates assumption (4 3 5) made in the derivation of (4 3 9) and (4 3 7) or $y_0(t)$ and $z_0(\tau)$

It is important to establish the asymptotic behaviour of $y_0(t)$ as $t \downarrow 0$ and $z_0(\tau)$ as $\tau \to \infty$ If we define $w(\tau, \varepsilon) = \varepsilon u(\varepsilon^\gamma \tau, \varepsilon)$, then

$$w(\tau, \varepsilon) = f(\varepsilon^\gamma \tau) + \int_0^\tau \frac{k(\varepsilon^\gamma \tau, \varepsilon^\gamma \sigma)}{(\tau - \sigma)^{1-\beta}} w(\sigma, \varepsilon)\, d\sigma$$

Therefore we expect the inner expansion to be

$$w(t, \varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^{j\gamma} w_j(\tau) \quad \text{as } \varepsilon \to 0$$

Comparing this to (4 3 3) we see that

$$w_0(\tau) + \varepsilon^\gamma w_1(\tau) + \quad \sim z_0(\tau) + \varepsilon y_0(\varepsilon^\gamma \tau) +$$

Since (4 4 3) implies that $\varepsilon y_0(\varepsilon^\gamma \tau) = \varepsilon^\gamma \tilde{y_0}(\varepsilon^\gamma \tau)$, the apparent anomaly of a $O(\varepsilon)$ term balancing with a $O(\varepsilon^\gamma)$ term does not arise

4.5 Example

Angell and Olmstead in [2] consider the following weakly singular linear singularly perturbed Volterra equation

$$\varepsilon u(t) = f(t) - \frac{1}{\pi^{1/2}} \int_0^t \frac{u(s)}{(t-s)^{1/2}}\, ds, \tag{4 5 1}$$

where

$$f(t) = \frac{1}{\pi^{1/2}} \int_0^t \frac{h(s)}{(t-s)^{1/2}}\, ds,$$

and $h(t)$ is C^2 with $h(0) \neq 0$ Since $\Gamma(1/2) = \pi^{1/2}$, this corresponds to (4 1 1) with $k(t,s) = -1$ and $\beta = 1/2$ Therefore $\gamma = 2$ The exact solution of (4 5 1) can be obtained by Laplace transforms or read directly from (4 2 4) and (4 2 10) It is given by

$$u(t,\varepsilon) = \frac{f(t)}{\varepsilon} - \frac{1}{\varepsilon^2}\int_0^t e^{(t-s)/\varepsilon^2} \operatorname{erfc}\left(\frac{(t-s)^{1/2}}{\varepsilon}\right) h(s)\, ds \tag{4 5 2}$$

Since $f(\varepsilon^2\tau) = 2\varepsilon h(0)\tau^{1/2}/\pi^{1/2} + O(\varepsilon^2)$, we look for an asymptotic solution of the form

$$u(t,\varepsilon) = \sum_{j=0}^{\infty} \left(\varepsilon^j y_j(t) + \varepsilon^{2j} z_j(t/\varepsilon^2)\right)$$

Following the formal method of Section 4 3, it is found that the leading order outer solution $y_0(t)$ obeys

$$0 = \int_0^t \frac{h(s) - y_0(s)}{(t-s)^{1/2}}\, ds, \quad t \geq 0,$$

and hence $y_0(t) = h(t)$

The inner correction term $z_0(\tau)$ is a solution of

$$\begin{aligned} z_0(\tau) &= -y_0(0) + \frac{2\tau^{1/2}}{\pi^{1/2}}(h(0) - y_0(0)) - \frac{1}{\pi^{1/2}}\int_0^\tau \frac{1}{(\tau-\sigma)^{1/2}} z_0(\sigma)\, d\sigma \\ &= -h(0) - \frac{1}{\pi^{1/2}}\int_0^\tau \frac{1}{(\tau-\sigma)^{1/2}} z_0(\sigma)\, d\sigma \end{aligned}$$

By (4 2 4) and (4 2 10)

$$z_0(\tau) = -h(0)e^{\tau} \operatorname{erfc}(\tau^{1/2}), \quad \tau \geq 0 \tag{4 5 3}$$

The asymptotic expansion of the integral

$$\operatorname{erfc}\sqrt{\tau} = \frac{2}{\sqrt{\pi}}\int_{\sqrt{\tau}}^{\infty} e^{-t^2}\, dt \sim \frac{e^{-\tau}}{\sqrt{\pi\tau}}\left\{1 - \frac{1}{2\tau} + \frac{3}{4\tau^2} + \quad \right\} \quad \text{as } \tau \to \infty \tag{4 5 4}$$

implies that

$$z_0(\tau) \sim -\frac{h(0)}{\sqrt{\pi\tau}}\left\{1 - \frac{1}{2\tau} + \frac{3}{4\tau^2} + \quad \right\}$$

so that $z_0(\tau) \to 0$ as $\tau \to \infty$, but only algebraically

Therefore up to the leading order, the formal solution of (4 5 1) is given by

$$U_0(t,\varepsilon) = h(t) - h(0)e^{t/\varepsilon^2} \operatorname{erfc}(t/\varepsilon^2) \tag{4 5 5}$$

To show directly that $U_0(t,\varepsilon)$ approximates the solution of (4 5 2) to within $O(\varepsilon)$ consider the difference

$$u(t,\varepsilon) - U_0(t,\varepsilon) = \frac{1}{\varepsilon}\int_0^t \frac{h(s)}{\pi^{1/2}(t-s)^{1/2}}\,ds - \frac{1}{\varepsilon^2}\int_0^t e^{(t-s)/\varepsilon^2}\operatorname{erfc}\left(\frac{(t-s)^{1/2}}{\varepsilon}\right)h(s)\,ds$$
$$- h(t) + h(0)e^{t/\varepsilon^2}\operatorname{erfc}(\frac{t^{1/2}}{\varepsilon}) \qquad (4\ 5\ 6)$$

Integrating by parts

$$\frac{1}{\varepsilon^2}\int_0^t e^{(t-s)/\varepsilon^2}\operatorname{erfc}\left(\frac{\sqrt{t-s}}{\varepsilon}\right)h(s)\,ds = h(0)e^{t/\varepsilon^2}\operatorname{erfc}\frac{\sqrt{t}}{\varepsilon} - h(t) + \frac{1}{\varepsilon}\int_0^t \frac{h(s)}{\sqrt{\pi(t-s)}}\,ds$$
$$+ \varepsilon^2 h'(0)e^{t/\varepsilon^2}\operatorname{erfc}\frac{\sqrt{t}}{\varepsilon} - \varepsilon^2 h'(t) - \varepsilon\int_0^t [\pi(t-s)]^{-1/2}h'(s)\,ds$$
$$- \varepsilon^2\int_0^t \left\{e^{(t-s)/\varepsilon^2}\operatorname{erfc}\frac{\sqrt{t-s}}{\varepsilon} - \frac{2\varepsilon\sqrt{t-s}}{\sqrt{\pi}}\right\}h''(s)\,ds \qquad (4\ 5\ 7)$$

Substituting this into (4 5 6), we get

$$u(t,\varepsilon) - U_0(t,\varepsilon) = -\varepsilon\int_0^t [\pi(t-s)]^{-1/2}h'(s)\,ds - \varepsilon^2 h'(t) + \varepsilon^2 e^{t/\varepsilon^2}\operatorname{erfc}\frac{\sqrt{t}}{\varepsilon}h'(0)$$
$$- \varepsilon^2\int_0^t \left\{e^{(t-s)/\varepsilon^2}\operatorname{erfc}\frac{\sqrt{(t-s)}}{\varepsilon} - \frac{2\varepsilon\sqrt{(t-s)}}{\sqrt{\pi}}\right\}h''(s)\,ds$$

This implies that

$$|u(t,\varepsilon) - U_0(t,\varepsilon)| = O(\varepsilon)$$

as $\varepsilon \to 0$ uniformly on $0 \le t \le T$

We now examine the exact solution (4 5 2) with the view of directly determining a valid asymptotic solution for $u(t,\varepsilon)$ Suppose now that $h(t)$ is C^∞ For the outer expansion, we fix $t > 0$ in (4 5 2) and let $\varepsilon \to 0$ Then

$$u(t,\varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n v_n(t) \quad \text{as } \varepsilon \to 0$$

The integration by parts in (4 5 7) gives

$$v_0(t) = h(t), \quad v_1(t) = \frac{h(0)}{(\pi t)^{1/2}} - \int_0^t \frac{1}{\pi^{1/2}(t-s)^{1/2}}h'(s)\,ds, \quad \text{etc} \qquad (4\ 5\ 8)$$

where the first term in v_1 follows from the first term in (4 5 7) and the asymptotic expansion (4 5 4) To get the inner expansion, we express (4 5 2) in terms of the inner variable $\tau = t/\varepsilon^2$ to get

$$u(\varepsilon^2\tau,\varepsilon) = w(\tau,\varepsilon) = \int_0^\tau \left\{\frac{1}{\pi^{1/2}(\tau-\sigma)^{1/2}} - e^{\tau-\sigma}\operatorname{erfc}(\tau-\sigma)^{1/2}\right\}h(\varepsilon^2\sigma)\,d\sigma$$

This suggests that the inner expansion has the form

$$w(\tau,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^{2n} w_n(\tau) \quad \text{as } \varepsilon \to 0$$

Equating the coefficients of like powers of ε, we get

$$w_n(\tau) = \frac{h^{(n)}(0)}{n!} \int_0^\tau \left\{ \frac{1}{\pi^{1/2}(\tau-\sigma)^{1/2}} - e^{\tau-\sigma} \operatorname{erfc}(\tau-\sigma)^{1/2} \right\} \sigma^n \, d\sigma$$

The leading order term in (4 5) is given by

$$w_0(\tau) = h(0) \int_0^\tau \left\{ \frac{1}{\pi^{1/2}(\tau-\sigma)^{1/2}} - e^{\tau-\sigma} \operatorname{erfc}(\tau-\sigma)^{1/2} \right\} d\sigma,$$

equivalently

$$w_0(\tau) = h(0) - h(0) e^{\tau} \operatorname{erfc}\sqrt{\tau} \tag{4 5 9}$$

The first order term $w_1(\tau)$ is given by

$$w_1(\tau) = h'(0) \int_0^\tau \left\{ \frac{1}{\pi^{1/2}(\tau-\sigma)^{1/2}} - e^{\tau-\sigma} \operatorname{erfc}(\tau-\sigma)^{1/2} \right\} \sigma \, d\sigma$$

which on integration by parts is

$$w_1(\tau) = h'(0)\tau - h'(0) \int_0^\tau e^{\sigma} \operatorname{erfc}\sqrt{\sigma}\, d\sigma$$

We then see that the leading order term in the outer expansion and the leading order term in the inner expansion form a composite expansion which is the uniformly valid asymptotic solution $U_0(t,\varepsilon)$ obtained by the methodology developed

Chapter 5

Nonlinear Scalar Volterra Integral Equations

5.1 Introduction

This chapter considers the nonlinear singularly perturbed Volterra integral equation,

$$\varepsilon u(t) = f(t,\varepsilon) + \int_0^t g(t,s,u(s))\,ds, \quad 0 \le t \le T, \tag{5 1 1}$$

where $0 < \varepsilon << 1$ The function $f(t,\varepsilon)$ is C^∞ and defined for $0 \le t \le T$ and $0 \le \varepsilon \le 1$, $g(t,s,u)$ is also C^∞ and defined for $0 \le s \le t \le T$ and $-\infty < u < \infty$ Also we require that $\lim_{\varepsilon\to 0} f(0,\varepsilon) = 0$ f has an asymptotic power series expansion,

$$f(t,\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j f_j(t) \quad \text{as } \varepsilon \to 0,$$

where each $f_j(t)$ is C^∞ Furthermore, we require that $f_0(0) = 0$ and $f_1(0)$ is nontrivial

Problem (5 1 1) depends on the parameter ε in such a way that the reduced equation

$$0 = f_0(t) + \int_0^t g(t,s,v(s))\,ds, \quad 0 \le t \le T,$$

is a Volterra equation of the first kind For this to have a continuous solution, $f_0(t)$ cannot be merely continuous Assuming that a stability condition for the boundary layer holds, we show that $u(t,\varepsilon)$ converges uniformly to $v(t)$ as $\varepsilon \to 0$

Angell and Olmsteadt [2] used the additive decomposition method to obtain the first few terms in a formal solution of (5 1 1) However Skinner [24] developed a method of generating all the terms of the formal solution and showed that the formal solution is an asymptotic solution His work builds on that of Smith [25], Ch 6, O'Malley [20], Ch 4 and O'Malley [21], Ch 2 on singularly perturbed initial value problems for nonlinear ordinary differential equations The study of the nonlinear integral equation (5 1 1) in this chapter was mostly done before the work of Skinner [24] was found, and therefore most of it is independent work However, an adaptation of Skinner's method of deriving the equations for the formal solution is included here

In Section 5 2, we construct a formal solution for (5 1 1) of the form

$$U_N(t,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j [y_j(t) + z_j(t/\varepsilon)], \tag{5 1 2}$$

using the O'Malley/Hoppensteadt method The analysis in this section is more complicated than that of Section 2 3 In Section 5 3 we prove that $y_j(t)$ and $z_j(\tau)$ have the properties assumed in their derivation Then in Section 5 3, we prove using the Banach fixed point theorem that

$$|u(t,\varepsilon) - U_N(t,\varepsilon)| = O(\varepsilon^{N+1}) \quad \text{as } \varepsilon \to 0,$$

uniformly for $0 \le t \le T$ An example from Angell and Olmstead [2] is discussed in Section 5 5 and one from Skinner [24] in Section 5 6

5 2 Derivation of the Formal Solution

We derive in this section a formal solution for the integral equation (5 1 1) using the additive decomposition method We suppose that the solution of (5 1 1) can be represented in the form

$$u(t,\varepsilon) = y(t,\varepsilon) + \phi(\varepsilon)z(t/\mu(\varepsilon),\varepsilon), \tag{5 2 1}$$

where

$$y(t,\varepsilon) = y_0(t) + o(1), \quad z(\tau,\varepsilon) = z_0(\tau) + o(1) \quad \text{as } \varepsilon \to 0$$

Firstly, we determine formally the width $\mu(\varepsilon)$ and the magnitude $\phi(\varepsilon)$ of the initial boundary layer, supposing that $\mu(\varepsilon) \to 0$ For this argument we assume that $g(0,0,u)$ is nontrivial We follow the analysis in Section 2 3 Substituting (5 2 1) into (5 1 1) gives

$$\varepsilon y(t,\varepsilon) + \varepsilon\phi(\varepsilon)z(t/\mu(\varepsilon),\varepsilon) = f(t,\varepsilon) + \int_0^t g(t,s,y(s,\varepsilon) + \phi(\varepsilon)z(s/\mu(\varepsilon),\varepsilon))\,ds, \tag{5 2 2}$$

which, letting $\tau = t/\mu(\varepsilon)$, is equivalent to

$$\varepsilon y(\mu(\varepsilon)\tau,\varepsilon) + \varepsilon\phi(\varepsilon)z(\tau,\varepsilon) = f(\mu(\varepsilon)\tau,\varepsilon) + \mu(\varepsilon)\int_0^\tau g(\mu(\varepsilon)\tau,\mu(\varepsilon)\sigma,y(\mu(\varepsilon)\sigma,\varepsilon) + \phi(\varepsilon)z(\sigma,\varepsilon))\,d\sigma$$

Hence, fixing $\tau > 0$ and letting $\varepsilon \to 0$,

$$\varepsilon y_0(0) + \varepsilon\phi(\varepsilon)z_0(\tau) = \varepsilon f_1(0) + \mu(\varepsilon)\int_0^\tau g(0,0,y_0(0) + \phi(\varepsilon)z_0(\sigma))\,d\sigma + o(\varepsilon) + o(\mu(\varepsilon))$$

Dominant terms can be balanced if we take

$$\mu(\varepsilon) = \varepsilon, \quad \phi(\varepsilon) = 1$$

To obtain a formal solution we now suppose that $y(t,\varepsilon)$ and $z(\tau,\varepsilon)$ have the asymptotic expansions

$$y(t,\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j y_j(t), \quad z(\tau,\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j z_j(\tau)$$

as $\varepsilon \to 0$ $y(t,\varepsilon)$ represents the outer solution, which approximates the solution outside the initial layer, while $z(t/\varepsilon,\varepsilon)$ represents the inner correction term which is required for uniform approximation of the solution of (5 1 1) inside the initial layer but is negligible outside the initial layer We require for each $j \geq 0$ that

$$z_j(\tau) = o(\tau^{-r}) \quad \text{as } \tau \to \infty \tag{5 2 3}$$

for all $r \geq 0$ The rapid decay in the initial layer is crucial for the application of the method of additive decomposition because then transcendentally small terms can be omitted from the asymptotic expansions

Since Theorem 2 1 from Skinner [24] is used later in this section, it is stated here

Lemma 5 1 *Suppose that $\eta(t,\tau,\varepsilon)$ is a C^∞ function on $[0,T] \times [0,\infty) \times [0,1]$ and $\eta(t,\tau,\varepsilon) = o(\tau^{-r})$ as $\tau \to \infty$ for all $r \geq 0$ Then*

$$\eta(t,t/\varepsilon,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j \eta_j(t/\varepsilon) + O(\varepsilon^{N+1}),$$

where $\eta_j(\tau)$ is a C^∞ function on $[0,\infty)$ and is the coefficient of ε^j in the Taylor expansion of $\varepsilon \mapsto \eta(\varepsilon\tau,\tau,\varepsilon)$ Also $\eta_j(\tau) = o(\tau^{-r})$ as $\tau \to \infty$ for all $r \geq 0$

We shall substitute (5 1 2) into (5 1 1) Therefore for a fixed integer $N \geq 0$ we first consider the term

$$\int_0^t g(t,s,U_N(s,\varepsilon))\,ds$$

We introduce

$$H(t,s,\varepsilon) = g(t,s,\sum_{j=0}^{N} \varepsilon^j y_j(s)),$$

$$K(t,s,\sigma,\varepsilon) = g(t,s,\sum_{j=0}^{N} \varepsilon^j (y_j(s) + z_j(\sigma))) - g(t,s,\sum_{j=0}^{N} \varepsilon^j y_j(s)),$$

so that

$$\int_0^t g(t,s,U_N(s,\varepsilon))\,ds = \int_0^t H(t,s,\varepsilon)\,ds + \varepsilon \int_0^{t/\varepsilon} K(t,\varepsilon\sigma,\sigma,\varepsilon)\,d\sigma \qquad (5\,2\,4)$$

By (5 2 3) and the Mean Value Theorem, $K(t,s,\sigma,\varepsilon) = o(\sigma^{-r})$ as $\sigma \to \infty$ for all $r \geq 0$ By applying Lemma 5 1 to $(s,\sigma,\varepsilon) \mapsto K(t,s,\sigma,\varepsilon)$, we deduce that

$$K(t,\varepsilon\sigma,\sigma,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j k_j(t,\sigma) + O(\varepsilon^{N+1}), \qquad (5\,2\,5)$$

with $k_j(t,\sigma) = o(\sigma^{-r})$ for all $r \geq 0$ Also, straightforward Taylor expansions yields

$$H(t,s,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j h_j(t,s) + O(\varepsilon^{N+1}), \qquad (5\,2\,6)$$

$$K(\varepsilon\tau,\varepsilon\sigma,\sigma,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j l_j(\tau,\sigma) + O(\varepsilon^{N+1}) \qquad (5\,2\,7)$$

The coefficients $h_j(t,s)$ in (5 2 6) are given by

$$h_0(t,s) = g(t,s,y_0(s)), \quad h_1(t,s) = \partial_3 g(t,s,y_0(s))y_1(s),$$

and in general for $j \geq 1$,

$$h_j(t,s) = \partial_3 g(t,s,y_0(s))y_j(s) + \Phi_j(t,s),$$

where $\Phi_j(t,s)$ is determined by $y_i(s)$, for $0 \leq i \leq j-1$ The first two terms of Φ_j are given by

$$\Phi_1(t,s) = 0, \quad \Phi_2(t,s) = \frac{1}{2}\partial_3^2 g(t,s,y_0(s))y_1^2(s)$$

The coefficients $k_j(t,\sigma)$ in (5 2 5) are given by

$$k_0(t,\sigma) = g(t,0,y_0(0)+z_0(\sigma)) - g(t,0,y_0(0)),$$

$$k_1(t,\sigma) = \partial_3 g(t,0,y_0(0)+z_0(\sigma))z_1(\sigma) + \Psi_1(t,\sigma),$$

and in general for $j \geq 1$,

$$k_j(t,\sigma) = \partial_3 g(t,0,y_0(0)+z_0(\sigma))z_j(\sigma) + \Psi_j(t,\sigma)$$

Here the function $\Psi_j(t,\sigma)$ is determined by $y_i(s)$ for $0 \leq i \leq j$ and $z_i(\sigma)$ for $0 \leq i \leq j-1$ The first two Ψ_j are given by

$$\Psi_1(t,\sigma) = \{\partial_2 g(t,0,y_0(0)+z_0(\sigma) - \partial_2 g(t,0,y_0(0))\}\sigma$$

$$+ \{\partial_3 g(t,0,y_0(0)+z_0(\sigma)) - \partial_3 g(t,0,y_0(0))\}(y_0'(0)\sigma + y_1(0)),$$

$$\Psi_2(t,\sigma) = \{\partial_3 g(t,0,y_0(0)+z_0(\sigma)) - \partial_3 g(t,0,y_0(0))\}(y_2(0)+y_1(0)\sigma + \frac{1}{2}y_0''(0)\sigma^2)$$

$$+ \{\partial_2\partial_3 g(t,0,y_0(0)+z_0(\sigma)) - \partial_2\partial_3 g(t,0,y_0(0))\}(y_0'(0)\sigma^2 + y_1(0)\sigma)$$

$$+ \partial_2\partial_3 g(t,0,y_0(0)+z_0(\sigma))z_1(\sigma)\sigma + \partial_3^2 g(t,0,y_0(0)+z_0(\sigma))z_1(\sigma)y_1(0)$$

$$+ \frac{1}{2}\{\partial_3^2 g(t,0,y_0(0)+z_0(\sigma)) - \partial_3^2 g(t,0,y_0(0))\}(y_0'(0)^2\sigma^2 + y_1^2(0)$$

$$+ 2y_0'(0)y_1(0)\sigma) + \frac{1}{2}\partial_3^2 g(t,0,y_0(0)+z_0(\sigma))\{z_1^2(\sigma) + z_1(\sigma)y_0'(0)\sigma\}$$

$$+ \frac{1}{2}\{\partial_2^2 g(t,0,y_0(0)+z_0(\sigma)) - \partial_2^2 g(t,0,y_0(0))\}\sigma^2$$

The coefficients $l_j(\tau,\sigma)$ in (5 2 7) are given by

$$l_0(\tau,\sigma) = g(0,0,y_0(0)+z_0(\sigma)) - g(0,0,y_0(0)),$$

$$l_1(\tau,\sigma) = \partial_3 g(0,0,y_0(0)+z_0(\sigma))z_1(\sigma) + \Xi_1(\tau,\sigma),$$

and in general for $j \geq 1$,

$$l_j(\tau,\sigma) = \partial_3 g(0,0,y_0(0)+z_0(\sigma))z_j(\sigma) + \Xi_j(\tau,\sigma),$$

where $\Xi_j(\tau,\sigma)$ is determined by y_i for $i \leq j$ and z_i for $i \leq j-1$ In particular,

$$\Xi_1(\tau,\sigma) = \{\partial_1 g(0,0,y_0(0)+z_0(\sigma)) - \partial_1 g(0,0,y_0(0))\}\tau$$

$$+ \{\partial_2 g(0,0,y_0(0)+z_0(\sigma)) - \partial_2 g(0,0,y_0(0))\}\sigma$$

$$+ \{\partial_3 g(0,0,y_0(0)+z_0(\sigma)) - \partial_3 g(0,0,y_0(0))\}(y_0'(0)\sigma + y_1(0))$$

It follows from (5 2 4) that

$$\int_0^t g(t,s,U_N(s,\varepsilon))\,ds = \sum_{j=0}^N \varepsilon^j \left(\int_0^t h_j(t,s)\,ds + \varepsilon\int_0^\infty k_j(t,\sigma)\,d\sigma\right)$$
$$- \sum_{j=0}^N \varepsilon^{j+1}\int_{t/\varepsilon}^\infty k_j(t,\sigma)\,d\sigma + O(\varepsilon^{N+1}) \qquad (5\ 2\ 8)$$

Since $k_j(t,\sigma) = o(\sigma^{-r})$ for all $r \geq 0$,

$$\int_\tau^\infty k_j(t,\sigma)\,d\sigma = o(\tau^{-r}),$$

for all $r \geq 0$, and Lemma 5 1 implies that

$$\int_{t/\varepsilon}^{\infty} k_j(t,\sigma)\,d\sigma = \int_{t/\varepsilon}^{\infty} \sum_{i=0}^{j} \varepsilon^i \tilde{k}_{j,i}(t/\varepsilon,\sigma)\,d\sigma + O(\varepsilon^{N+1}),$$

where $\tilde{k}_{j,i}(\tau,\sigma)$ is the coefficient of ε^i in the Taylor expansion of $\varepsilon \mapsto k_j(\varepsilon\tau,\sigma)$ Of course Lemma 5 1 also assures us that

$$\int_{\tau}^{\infty} \tilde{k}_{j,i}(\tau,\sigma)\,d\sigma = o(\tau^{-r}) \quad \text{as } \tau \to \infty$$

for all $r \geq 0$ Note also that if

$$K(\varepsilon\tau,\varepsilon\sigma,\sigma,\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j l_j(\tau,\sigma),$$

then

$$\sum_{i=0}^{j} \tilde{k}_{j-i,i}(\tau,\sigma) = l_j(\tau,\sigma) \tag{5 2 9}$$

It follows that (5 2 8) becomes

$$\begin{aligned}\int_0^t g(t,s,U_N(s,\varepsilon))\,ds = &\sum_{j=0}^{N} \varepsilon^j \left(\int_0^t h_j(t,s)\,ds + \varepsilon \int_0^{\infty} k_j(t,\sigma)\,d\sigma \right) \\ &- \sum_{j=0}^{N-1} \varepsilon^{j+1} \int_{\tau}^{\infty} l_j(t/\varepsilon,\sigma)\,d\sigma + O(\varepsilon^{N+1})\end{aligned}$$

Next we define the *residual* $\rho_N(t,\varepsilon)$ by

$$\varepsilon U_N(t,\varepsilon) = f(t,\varepsilon) + \int_0^t g(t,s,U_N(s,\varepsilon))\,ds - \rho_N(t,\varepsilon) \tag{5 2 10}$$

Then, putting $y_{-1}(t) = 0$ and $k_{-1}(t,\sigma) = 0$, we see that

$$\begin{aligned}\rho_N(t,\varepsilon) = &\sum_{j=0}^{N} \varepsilon^j \left(\int_0^t h_j(t,s)\,ds + \int_0^{\infty} k_{j-1}(t,\sigma)\,d\sigma + f_j(t) - y_{j-1}(t) \right) \\ &- \sum_{j=0}^{N-1} \varepsilon^{j+1} \left(z_j(t/\varepsilon) + \int_{t/\varepsilon}^{\infty} l_j(t/\varepsilon,\sigma) \right) + O(\varepsilon^{N+1})\end{aligned} \tag{5 2 11}$$

If $U_N(t,\varepsilon)$ is a formal solution for all $N \geq 0$, then $\rho_N(t,\varepsilon) = O(\varepsilon^{N+1})$ as $\varepsilon \to 0$ for all $N \geq 0$, in which case the argument of Lemma 2 1 shows that for every $j \geq 0$, $y_j(t)$ and $z_j(\tau)$ satisfy

$$y_{j-1}(t) = f_j(t) + \int_0^t h_j(t,s)\,ds + \int_0^{\infty} k_{j-1}(t,\sigma)\,d\sigma, \tag{5 2 12}$$

$$z_j(\tau) = -\int_{\tau}^{\infty} l_j(\tau,\sigma)\,d\sigma \tag{5 2 13}$$

There is also an initial condition for solutions of (5 2 13), obtained from $\varepsilon u(0,\varepsilon) = f(0,\varepsilon)$, namely that for all $j \geq 0$

$$z_j(0) = f_{j+1}(0) - y_j(0) \tag{5 2 14}$$

Remark 5 2 There is considerable simplication in the case $g(t,s,u) = a(t,s)u$ for which (5 1 1) is a linear equation It is found that

$$h_j(t,s) = a(t,s)y_j(s), \quad k_j(t,\sigma) = \sum_{i=0}^{j} e_i(t,\sigma)z_{j-i}(\sigma),$$

where

$$e_i(t,\sigma) = \frac{1}{i!}\partial_2^i a(t,0)\sigma^i$$

Remark 5 3 Equation (5 2 11) for the residual has been derived only assuming that (5 2 3) is true It follows that if (5 2 3) holds and (5 2 12) and (5 2 13) hold for $0 \leq j \leq N$, then $|\rho_N(t,\varepsilon)| = O(\varepsilon^{N+1})$ as $\varepsilon \to 0$

5.3 Properties of the Formal Solution

In this section it is shown that there are unique solutions $y_j(t)$ and $z_j(\tau)$ of (5 2 12) and (5 2 13), and that they have the important properties assumed in their derivation It is convenient to rewrite these equations as

$$0 = f_0(t) + \int_0^t g(t,s,y_0(s))\,ds, \tag{5 3 1}$$

$$z_0(\tau) = -\int_\tau^\infty \left(g(0,0,y_0(0) + z_0(\sigma)) - g(0,0,y_0(0))\right)\,d\sigma, \tag{5 3 2}$$

and $j \geq 1$,

$$0 = \phi_j(t) + \int_0^t \partial_3 g(t,s,y_0(s))y_j(s)\,ds, \tag{5 3 3}$$

$$z_j(\tau) = -\int_\tau^\infty \partial_3 g(0,0,y_0(0) + z_0(\sigma))z_j(\sigma)\,d\sigma + \psi_j(\tau) \tag{5 3 4}$$

Here we used the definitions

$$\phi_j(t) = f_j(t) + \int_0^t \Phi_j(t,s)\,ds + \int_0^\infty k_{j-1}(t,\sigma)\,d\sigma - y_{j-1}(t), \tag{5 3 5}$$

$$\psi_j(\tau) = -\int_\tau^\infty \Xi_j(\tau,\sigma)\,d\sigma \tag{5 3 6}$$

We see that the leading order solutions (outer and inner correction) are given by nonlinear equations while the higher order terms are given by linear equations

We use the following hypotheses on the functions $f(t,\varepsilon)$ and the kernel $g(t,s,u)$ They are based on the assumptions used in O'Malley [21], Ch 4

($\mathbf{H}_3$) The function f $[0,T]\times[0,1]\to\mathbb{R}$ is $C^\infty$and $f(0,0)=0$ Also g $\Delta_T\times\mathbb{R}\to\mathbb{R}$ is a C^∞ function where

$$\Delta_T=\{(t,s),0\le s\le t\le T\}$$

($\mathbf{H}_4$) There exists a C^∞ solution y_0 $[0,T]\to\mathbb{R}$ to (5 **3** 1) which is unique in the class of continuous functions on $[0,T]$

($\mathbf{H}_5$) There is a positive constant α such that

$$\partial_3 g(t,t,y_0(t))\le-\alpha<0,\quad \text{for all } 0\le t\le T,$$

$$\partial_3 g(0,0,v)\le-\alpha<0,$$

for all v between $y_0(0)$ and $y_0(0)+f_1(0)$

Remark 5 4 If ($\mathbf{H_3}$) holds, $f(t,\varepsilon)$ has the asymptotic expansion

$$f(t,\varepsilon)\sim\sum_{j=0}^{\infty}\varepsilon^j f_j(t),\quad \text{as } \varepsilon\to 0,$$

where each $f_j(t)$ is C^∞ on $[0,T]$

Remark 5 5 (5 3 1) is a Volterra integral equation of the first kind for $y_0(t)$ An existence and uniqueness theorem for this equation is given in Linz [17], Ch 5, Th 5 2 It is obtained by applying the method of successive approximations to the differentiated version of (5 3 1)

Remark 5 6 Skinner [24] proves similar results to those presented in this chapter, except that he replaces $g(t,s,u)$ by $g(t,s,u\ \ \varepsilon)$, where

$$g(t,s,u,\varepsilon)\sim\sum_{j=0}^{\infty}\varepsilon^j g_j(t,s,u)\quad \text{as } \varepsilon\to 0$$

Here each $g_j(t,s,u)$ should satisfy ($\mathbf{H_3}$) and $g_0(t,s,u)$ satisfies both ($\mathbf{H_4}$) and ($\mathbf{H_5}$)

Proposition 5 7 *Suppose that* (H_3), (H_4) *and* (H_5) *hold Then* (5 3 2) *and* (5 2 14) *have a* C^∞ *solution* z_0 *satisfying*

$$|z_0(\tau)| \leq c_0 e^{-\alpha\tau}, \quad \tau \geq 0, \tag{5 3 7}$$

for some positive constant c_0

Proof The problem of solving (5 3 2) subject to (5 2 14) is equivalent to the initial-value problem

$$z_0'(\tau) = g(0,0,y_0(0) + z_0(\tau)) - g(0,0,y_0(0)), \quad z_0(0) = f_1(0) - y_0(0) \tag{5 3 8}$$

By standard theory of ordinary differential equations (see, for example, Hirsch and Smale [12], Ch 8), (5 3 8) has a unique continuous solution defined on a maximal interval $[0, S)$ such that $\lim_{\tau \uparrow S} |z_0(\tau)| = \infty$ if $S < \infty$ By the Mean Value Theorem there is a function $\omega(\tau)$ such that

$$z_0'(\tau) = \partial_3 g(0,0,(1-\omega(\tau))y_0(0) + \omega(\tau)z_0(\tau))z_0(\tau)$$

Assumption (H_5) implies that $z_0(\tau)$ decreases if $z_0(0) > 0$ and increases if $z_0(0) < 0$ and that $z_0(\tau) + y_0(0)$ lies between $y_0(0)$ and $y_0(0) + f_1(0)$ Therefore

$$z_0'(\tau)z_0(\tau) \leq -\alpha z_0(\tau)^2,$$

and hence $|z_0(\tau)| \leq |z_0(0)|e^{-\alpha\tau}$ for all $0 \leq \tau < S$ Hence $S = \infty$ and (5 3 7) holds □

Proposition 5 8 *Suppose that* (H_3), (H_4) *and* (H_5) *hold Then for every integer* $j \geq 1$, (5 3 3) *has a* C^∞ *solution* $y_j(t)$ *on* $[0,T]$, *and equations* (5 3 4) *and* (5 2 14) *have a* C^∞ *solution* z_j *on* $[0,\infty)$ *satisfying*

$$|z_j(\tau)| \leq c_j e^{-\beta\tau}, \quad \tau \geq 0, \tag{5 3 9}$$

for some positive constants c_j *and* $\beta < \alpha$

Proof Consider the hypothesis that there is an integer $N \geq 0$ such that there are C^∞ solutions $y_j(t)$ of (5 3 3) for $0 \leq j \leq N$ and C^∞ solutions $z_j(\tau)$ for $0 \leq j \leq N$ of (5 3 4) and (5 2 14) such that

$$|z_j(\tau)| \leq c_j e^{-\beta\tau}, \quad \tau \geq 0 \tag{5 3 10}$$

Due to Proposition 5 7 and $(\mathbf{H}_4)$, this hypothesis is true for $N = 0$

Suppose now this hypothesis is true for $M > 0$ Then $\Phi_{M+1}(t,s)$ and $k_M(t,\sigma)$ are determined and, by (5 3 5), $\phi_{M+1}(t)$ is a well-defined C^∞ function on $[0,T]$ Assumption $(\mathbf{H}_4)$ implies that $\partial_3 g(t,t,y_0(t)) \neq 0$ for all $0 \leq t \leq T$ Then it makes sense to consider the differentiated version of (5 3 3), namely

$$y_{M+1}(t) = -\frac{\phi'_{M+1}(t)}{\partial_3 g(t,t,y_0(t))} - \frac{1}{\partial_3 g(t,t,y_0(t))}\int_0^t \partial_3\partial_1 g(t,s,y_0(s))y_{M+1}(s)\,ds \tag{5 3 11}$$

This is a linear Volterra integral equation of the second kind in y_{M+1} and has a C^∞ solution on $[0,T]$, which can be written in terms of the resolvent kernel The theory can be found for example in Ch 2 of Gripenberg, Londen and Staffan [10] or Ch IV of Miller [19] It follows from (5 3 11) that

$$\text{constant} = \phi_{M+1}(t) + \int_0^t \partial_3 g(t,s,y_0(s))y_{M+1}(s)\,ds \tag{5 3 12}$$

But since $z_M(0) = f_{M+1}(0) - y_M(0)$ and $l_M(0,\sigma) = k_M(0,\sigma)$, (5 3 4) implies that

$$\begin{aligned}\phi_{M+1}(0) &= f_{M+1}(0) - y_M(0) + \int_0^\infty k_M(0,\sigma)\,d\sigma \\ &= z_M(0) + \int_0^\infty k_M(0,\sigma)\,d\sigma = 0\end{aligned}$$

Thus the constant in (5 3 12) vanishes and (5 3 3) holds in the case $j = M+1$

Now that $y_{M+1}(t)$ has been found, it follows from (5 3 6) that $\psi_{M+1}(\tau)$ is a well-defined C^∞ function An argument like that of O'Malley [20] pp 84–85 shows that

$$|\psi_j(\tau)| \leq \gamma_j e^{-\beta\tau}, \quad \tau \geq 0, \tag{5 3 13}$$

can be deduced from (5 3 10) for $0 \leq j \leq M$ The details are omitted Equation (5 3 4) is equivalent to the linear scalar equation

$$z'_{M+1}(\tau) = \partial_3 g(0,0,y_0(0) + z_0(\tau))z_{M+1}(\tau) + \psi'_{M+1}(\tau),$$

$$z_{M+1}(0) = f_{M+1}(0) - y_M(0)$$

It easily follows from the exact solution, $(\mathbf{H}_5)$ and (5 3 13) that (5 3 10) holds for $j = M+1$ This completes our proof that the induction hypothesis holds for $M+1$ The proposition then follows □

Lemma 5 9 *Suppose that* $(\mathbf{H_3})$, $(\mathbf{H_4})$ *and* $(\mathbf{H_5})$ *hold Then the residual* ρ_N *given by (5 2 10) satisfies*

$$|\rho_N(t,\varepsilon)| = O(\varepsilon^{N+1}) \quad \text{as } \varepsilon \to 0, \tag{5 3 14}$$

uniformly for all $0 \le t \le T$ *Moreover*

$$|\rho_N'(t,\varepsilon)| = O(\varepsilon^{N+1}) \quad \text{as } \varepsilon \to 0, \tag{5 3 15}$$

uniformly for all $0 \le t \le T$, *and*

$$|\rho_N(0,\varepsilon)| = O(\varepsilon^{N+2}) \tag{5 3 16}$$

Proof Since Propositions 5 7 and 5 8 have established (5 2 3), the proof of (5 3 14) follows from Remark 5 3 To prove (5 3 16)

$$\rho_N(0,\varepsilon) = f(0,\varepsilon) - \varepsilon U_N(0,\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j f_j(0) - \sum_{j=0}^{N} \varepsilon^{j+1}(y_j(0) + z_j(0))$$

Using the initial conditions in (5 2 14) and the fact that $f_0(0) = 0$, we have

$$\rho_N(0,\varepsilon) = \sum_{j=N+1}^{\infty} f_{j+1}(0)\varepsilon^{j+1} = O(\varepsilon^{N+2})$$

Differentiation of (5 2 10) gives

$$\begin{aligned}\rho_N'(t,\varepsilon) =& f'(t,\varepsilon) - \sum_{j=0}^{N} \varepsilon^{j+1} y_j'(t) - \sum_{j=0}^{N} \varepsilon^j z_j'(t/\varepsilon) \\ &+ g(t,t,\sum_{j=0}^{N} \varepsilon^j (y_j(t) + z_j(t/\varepsilon))) + \int_0^t \partial_1 g(t,s,\sum_{j=0}^{N} \varepsilon^j (y_j(s) + z_j(s/\varepsilon)))\, ds\end{aligned}$$

Introducing the new notations

$$H^\star(t,s,\varepsilon) = \partial_1 g(t,s,\sum_{j=0}^{N} \varepsilon^j y_j(s)),$$

$$K^\star(t,s,\sigma,\varepsilon) = \partial_1 g(t,s,\sum_{j=0}^{N} \varepsilon^j (y_j(s) + z_j(\sigma))) - \partial_1 g(t,s,\sum_{j=0}^{N} \varepsilon^j y_j(s)),$$

we have

$$\begin{aligned}\rho_N'(t,\varepsilon) =& \sum_{j=0}^{N} \varepsilon^j f_j'(t) - \sum_{j=0}^{N} \varepsilon^{j+1} y_j'(t) - \sum_{j=0}^{N} \varepsilon^j z_j'(t/\varepsilon) + H(t,t,\varepsilon) + K(t,t,t/\varepsilon,\varepsilon) \\ &+ \int_0^t H^\star(t,s,\varepsilon)\, ds + \varepsilon \int_0^{t/\varepsilon} K^\star(t,\varepsilon\sigma,\sigma,\varepsilon)\, d\sigma + O(\varepsilon^{N+1})\end{aligned} \tag{5 3 17}$$

Two useful Taylor expansions are

$$\begin{aligned}
H^\star(t,s,\varepsilon) &= \sum_{j=0}^{N} \varepsilon^j h_j^\star(t,s) + O(\varepsilon^{N+1}) \\
K^\star(t,\varepsilon\sigma,\sigma,\varepsilon) &= \sum_{j=0}^{N} \varepsilon^j k_j^\star(t,\sigma) + O(\varepsilon^{N+1}),
\end{aligned}$$

where the coefficients satisfy

$$k_j^\star(t,\sigma) = \partial_1 k_j(t,\sigma), \quad h_j^\star(t,s) = \partial_1 h_j(t,s)$$

Therefore (5 3 17) is equivalent to

$$\begin{aligned}
\rho_N'(t,\varepsilon) = &\sum_{j=0}^{N} \varepsilon^j f_j'(t) - \sum_{j=0}^{N} \varepsilon^{j+1} y_j'(t) - \sum_{j=0}^{N} \varepsilon^j z_j'(t/\varepsilon) \\
&+ \sum_{j=0}^{N} \varepsilon^j h_j(t,t) + \sum_{j=0}^{N} \varepsilon^j k_j(t,t/\varepsilon) + \sum_{j=0}^{N} \varepsilon^j \int_0^t h_j^\star(t,s)\,ds \\
&+ \sum_{j=0}^{N} \varepsilon^{j+1} \int_0^\infty k_j^\star(t,\sigma)\,d\sigma - \sum_{j=0}^{N} \varepsilon^{j+1} \int_{t/\varepsilon}^\infty k_j^\star(t,\sigma)\,d\sigma + O(\varepsilon^{N+1})
\end{aligned}$$

Then substituting the differentiated version of (5 2 12) we get

$$\begin{aligned}
\rho_N'(t,\varepsilon) = &\varepsilon^{N+1} \left(\int_0^\infty k_N^\star(t,\sigma)\,d\sigma - y_N'(t) \right) - \sum_{j=0}^{N} \varepsilon^j z_j'(t/\varepsilon) \\
&+ \sum_{j=0}^{N} \varepsilon^j k_j(t,t/\varepsilon) - \sum_{j=0}^{N} \varepsilon^{j+1} \int_{t/\varepsilon}^\infty k_j^\star(t,\sigma)\,d\sigma + O(\varepsilon^{N+1})
\end{aligned} \tag{5 3 18}$$

By substituting the differentiated version of (5 2 13) one gets

$$\begin{aligned}
\rho_N'(t,\varepsilon) = &\varepsilon^{N+1} \left(\int_0^\infty k_N^\star(t,\sigma)\,d\sigma - y_N'(t) \right) + \sum_{j=0}^{N} \varepsilon^j k_j(t,t/\varepsilon) - \sum_{j=0}^{N} \varepsilon^j l_j(t/\varepsilon,t/\varepsilon) \\
&+ \sum_{j=0}^{N} \varepsilon^j \int_{t/\varepsilon}^\infty \partial_1 l_j(t/\varepsilon,\sigma)\,d\sigma - \sum_{j=0}^{N} \varepsilon^{j+1} \int_{t/\varepsilon}^\infty k_j^\star(t,\sigma)\,d\sigma + O(\varepsilon^{N+1})
\end{aligned}$$

Using Lemma 5 1,

$$\begin{aligned}
\rho_N'(t,\varepsilon) = &\varepsilon^{N+1} \left(\int_0^\infty k_N^\star(t,\sigma)\,d\sigma - y_N'(t) \right) + \sum_{j=0}^{N} \varepsilon^j \sum_{i=0}^{j} \varepsilon^i \tilde{k}_{j,i}(t/\varepsilon,t/\varepsilon) - \sum_{j=0}^{N} \varepsilon^j l_j(t/\varepsilon,t/\varepsilon) \\
&+ \sum_{j=0}^{N} \varepsilon^j \int_{t/\varepsilon}^\infty \partial_1 l_j(t/\varepsilon,\sigma)\,d\sigma - \sum_{j=0}^{N} \varepsilon^{j+1} \int_{t/\varepsilon}^\infty \sum_{i=0}^{j} \varepsilon^i \tilde{k}_{j,i}^\star(t/\varepsilon,\sigma)\,d\sigma + O(\varepsilon^{N+1})
\end{aligned}$$

Collecting terms together using (5 2 9) gives

$$\begin{aligned}
\rho_N'(t,\varepsilon) = &\varepsilon^{N+1} \left(\int_0^\infty k_N^\star(t,\sigma)\,d\sigma - y_N'(t) \right) + \sum_{j=1}^{N} \varepsilon^j \int_{t/\varepsilon}^\infty \partial_1 l_j(t/\varepsilon,\sigma)\,d\sigma \\
&- \sum_{j=1}^{N+1} \varepsilon^j \int_{t/\varepsilon}^\infty \sum_{i=0}^{j-1} \tilde{k}_{j-i-1\,i}^\star(t/\varepsilon,\sigma)\,d\sigma + O(\varepsilon^{N+1})
\end{aligned}$$

We also see that if

$$K^\star(\varepsilon\tau,\varepsilon\sigma,\sigma,\varepsilon)=\sum_{j=0}^{\infty}\varepsilon^j l_j^\star(\tau,\sigma),$$

then

$$\sum_{i=0}^{j}\tilde{k}^\star_{j-i,i}(\tau,\sigma)=l_j^\star(\tau,\sigma),$$

where the coefficients obey

$$l^\star_{j-1}(\tau,\sigma)=\partial_1 l_j(\tau,\sigma),\quad j\geq 1$$

Therefore

$$|\rho'_N(t,\varepsilon)|=O(\varepsilon^{N+1}),\tag{5 3 19}$$

uniformly for all $0\leq t\leq T$ □

5.4 Existence of Asymptotic Solution

In this section we establish that $U_N(t,\varepsilon)$ defined in Section 5 1 is an asymptotic solution Our method is to adapt the theory in §6 3 of Smith [25] for systems of singularly perturbed ordinary differential equations Skinner [24] employed a similar method The analysis here has also benefited from the general discussion in §6 1 of Eckhaus [5] on developing a rigorous theory of singular perturbation The main result in this chapter is the following

Theorem 5 10 *Suppose that* $(\mathbf{H}_3)$, $(\mathbf{H}_4)$ *and* $(\mathbf{H}_5)$ *hold Then (5 1 1) has a continuous solution* $u(t,\varepsilon)$ *with the property that there are constants* C_N *and* ε_N^* *such that*

$$|u(t,\varepsilon)-U_N(t,\varepsilon)|\leq C_N\varepsilon^{N+1}$$

for all $0\leq t\leq T$ *and* $0<\varepsilon\leq\varepsilon_N^*$

It is natural to introduce $r_N(t,\varepsilon)=u(t,\varepsilon)-U_N(t,\varepsilon)$ which satisfies the equation

$$\varepsilon r_N(t,\varepsilon)=\rho_N(t,\varepsilon)+\int_0^t[g(t,s,U_N(s,\varepsilon)+r_N(s,\varepsilon))-g(t,s,U_N(s,\varepsilon))]\,ds\tag{5 4 1}$$

However, if the functions r_N and ρ_N are scaled, a mapping considered later becomes a *uniform* contraction rather than just a contraction For this reason let

$$\theta(t,\varepsilon)=\varepsilon^{-(N+1)}\rho_N(t,\varepsilon),\quad x(t,\varepsilon)=\varepsilon^{-(N+1)}r_N(t,\varepsilon),$$

where, for simplicity, the dependence on the fixed integer N is omitted from the notation Then for $\varepsilon > 0$, (5 4 1) is equivalent to

$$\varepsilon x(t,\varepsilon) = \theta(t,\varepsilon) + \int_0^t \partial_3 g(t,s,U_N(s,\varepsilon))x(s,\varepsilon)\,ds + \int_0^t h(t,s,x(s,\varepsilon),\varepsilon)\,ds, \qquad (5\ 4\ 2)$$

where

$$h(t,s,x,\varepsilon) = \varepsilon^{-(N+1)}g(t,s,U_N(s,\varepsilon)+x) - \varepsilon^{-(N+1)}g(t,s,U_N(s,\varepsilon)) - \partial_3 g(t,s,U_N(s,\varepsilon))x$$

By Taylor's theorem $h(t,s,x,\varepsilon) = \varepsilon^{(N+1)}h_1(t,s,x,\varepsilon)$, where

$$h_1(t,s,x,\varepsilon) = x^2 \int_0^1 (1-v)\partial_3^2 g(t,s,U_N(s,\varepsilon) + v\varepsilon^{(N+1)}x)\,dv$$

Hence, because $|\theta(t,\varepsilon)| = O(1)$ as $\varepsilon \to 0$ uniformly by Lemma 5 9, we expect the nonlinear term

$$\int_0^t h(t,s,x(s,\varepsilon),\varepsilon)\,ds$$

to be of higher order than other terms in (5 4 2) Therefore we first consider the approximate equation

$$\varepsilon w(t,\varepsilon) = \xi(t,\varepsilon) + \int_0^t \partial_3 g(t,s,U_N(s,\varepsilon))w(s,\varepsilon)\,ds, \qquad (5\ 4\ 3)$$

where $\xi(t,\varepsilon) = O(1)$ uniformly as $\varepsilon \to 0$ and $\xi(0,\varepsilon) = O(\varepsilon)$

Lemma 5 11 *Suppose that* $(\mathbf{H_3})$, $(\mathbf{H_4})$ *and* $(\mathbf{H_5})$ *hold for each* $0 < \varepsilon \le \varepsilon_0$ *Also suppose that* $\xi(\,,\varepsilon)$ $[0,T] \to \mathbb{R}$ *is a continuously differentiable function with* $\|\xi'(\,,\varepsilon)\| = O(1)$ *and* $|\xi(0,\varepsilon)| = O(\varepsilon)$ *Then (5 4 3) has a unique continuous solution* $w(\,,\varepsilon)$ *satisfying* $\|w(\,,\varepsilon)\| = O(1)$ *for all* ε *in some interval* $(0,\varepsilon_1] \subset (0,\varepsilon_0]$

Proof The standard theory of linear Volterra equations of the second kind ensures that for each $0 < \varepsilon < \varepsilon_0$ (5 4 3) has a continuous solution $t \mapsto w(t,\varepsilon)$ on $[0,T]$ and that $w(\,,\varepsilon)$ is continuously differentiable because $\xi(\,,\varepsilon)$ is Let $0 < \beta < \alpha$ If follows from $(\mathbf{H_5})$ that there is a number $0 < \varepsilon_1 \le \varepsilon_0$ such that

$$p(t,\varepsilon) = \partial_3 g(t,t,U_N(t,\varepsilon)) \le -\beta$$

for all $0 \le t \le T$ and $0 \le \varepsilon \le \varepsilon_1$ Equation (5 4 3) can be differentiated to get an equation of the form

$$\varepsilon w'(t,\varepsilon) - p(t,\varepsilon)w(t,\varepsilon) = \xi_1(t,\varepsilon), \qquad (5\ 4\ 4)$$

where $w(0,\varepsilon) = \xi(0,\varepsilon)/\varepsilon$ and

$$\xi_1(t,\varepsilon) = \xi'(t,\varepsilon) + \int_0^t \partial_1\partial_3 g(t,s,U_N(s,\varepsilon))w(s,\varepsilon)\,ds$$

Since the solution of (5 4 4) satisfies

$$w(t,\varepsilon) = w(0,\varepsilon)e^{\frac{1}{\varepsilon}\int_0^t p(v,\varepsilon)\,dv} + \frac{1}{\varepsilon}\int_0^t e^{\frac{1}{\varepsilon}\int_s^t p(v,\varepsilon)\,dv}\xi_1(s,\varepsilon)\,ds$$

and

$$e^{\frac{1}{\varepsilon}\int_0^t p(v,\varepsilon)\,dv} \le e^{-\beta t/\varepsilon}, \quad e^{\frac{1}{\varepsilon}\int_s^t p(v,\varepsilon)\,dv} \le e^{-\beta(t-s)/\varepsilon},$$

we see that

$$|w(t,\varepsilon)| \le C_1 + \frac{C_2}{\beta} + \frac{M}{\beta}\int_0^t |w(s,\varepsilon)|\,ds,$$

where

$$C_1 = \sup_{0<\varepsilon\le\varepsilon_0} |\xi(0,\varepsilon)|/\varepsilon, \qquad C_2 = \sup_{0<\varepsilon\le\varepsilon_0} \|\xi'(\,,\varepsilon)\|, \qquad M = \sup_{\substack{(t,s)\in\Delta_T \\ 0\le\varepsilon\le\varepsilon_0}} |\partial_1\partial_3 g(t,s,U_N(s,\varepsilon))|$$

By Gronwall's inequality

$$|w(t,\varepsilon)| \le \left(C_1 + \frac{C_2}{\beta}e^{\frac{Mt}{\beta}}\right),$$

and the lemma is proved □

Equation (5 4 2) can be written as

$$\mathcal{L}(x,\varepsilon) = \theta(\,,\varepsilon) + \mathcal{N}(x,\varepsilon), \tag{5 4 5}$$

where $\mathcal{L},\mathcal{N}$ $C[0,T]\times[0,\varepsilon_1] \to C[0,T]$ are defined by

$$\mathcal{L}(x,\varepsilon)(t) = \varepsilon x(t) - \int_0^t \partial_3 g(t,s,U_N(s,\varepsilon))x(s)\,ds,$$
$$\mathcal{N}(x,\varepsilon)(t) = \varepsilon^{(N+1)}\int_0^t h_1(t,s,x(s),\varepsilon)\,ds$$

It is convenient to introduce the space $\mathcal{X}$ of functions $(t,\varepsilon)\mapsto\xi(t,\varepsilon)$ on $[0,T]\times[0,\varepsilon_1]$ with $t\mapsto\xi(t,\varepsilon)$ continuously differentiable and $\|\xi'(\,,\varepsilon)\|$ and $\xi(0,\varepsilon)/\varepsilon$ are uniformly bounded on $[0,\varepsilon_0]$ and $(0,\varepsilon_0]$ respectively $\mathcal{X}$ is given the norm

$$\|\xi\|_{\mathcal{X}} = \sup_{0<\varepsilon\le\varepsilon_1} |\xi(0,\varepsilon)/\varepsilon| + \sup_{0<\varepsilon\le\varepsilon_1} \|\xi'(\,,\varepsilon)\|$$

Then $(t,\varepsilon) \mapsto \mathcal{L}(x,\varepsilon)(t)$, $(t,\varepsilon) \mapsto \mathcal{N}(x,\varepsilon)(t)$ and $(t,\varepsilon) \mapsto \theta(t,\varepsilon)$ are in $\mathcal{X}$

Lemma 5 11 can be reinterpreted as asserting that for $\xi \in \mathcal{X}$ the equation $\mathcal{L}(w,\varepsilon) = \xi(\ ,\varepsilon)$ is equivalent to $w(\ ,\varepsilon) = \mathcal{M}(\ ,\varepsilon)\xi(\ ,\varepsilon)$ for some linear operator $\mathcal{M}(\ ,\varepsilon)\ \ \mathcal{X} \to C[0,T]$ and there is a constant μ such that $\|\mathcal{M}(\ ,\varepsilon)\xi(\ ,\varepsilon)\| \le \mu\|\xi\|_{\mathcal{X}}$ uniformly for $0 < \varepsilon \le \varepsilon_1$ Hence there is a number $\delta > 0$ such that

$$\|\mathcal{M}(\ ,\varepsilon)\theta(\ ,\varepsilon)\| \le \delta$$

Also (5 4 5) is equivalent to

$$x = \mathcal{M}(\ ,\varepsilon)\left[\theta(\ ,\varepsilon) + \mathcal{N}(x,\varepsilon)\right]$$

Thus the problem of finding solutions of (5 4 5) is equivalent to finding fixed points of a mapping

Let

$$\mathcal{B} = \{x \in C[0,T]\ \ \|x\| \le 2\delta\}$$

A simple calculation shows that if x is in $\mathcal{B}$ then

$$\|\mathcal{N}(x,\ )\|_{\mathcal{X}} \le \varepsilon^{N+1}TM_1,$$

where

$$M_1 = \max_{\substack{(t,s)\in\Delta_T \\ |x|\le 2\delta,\ 0\le\varepsilon\le\varepsilon_1}} |h_1(t,s,x,\varepsilon)|$$

Therefore for each x in $\mathcal{B}$

$$\|\mathcal{M}(\ ,\varepsilon)\left[\theta(\ ,\varepsilon) + \mathcal{N}(x,\varepsilon)\right]\| \le \delta + \mu TM_1\varepsilon^{N+1} \le 2\delta,$$

if ε is in some interval $(0,\varepsilon_2]$ It has been shown that the mapping $\mathcal{T}_\varepsilon\ \ \mathcal{B} \to \mathcal{B}$ given by

$$\mathcal{T}_\varepsilon(x) = \mathcal{M}(\ ,\varepsilon)\left[\theta(\ ,\varepsilon) + \mathcal{N}(x,\varepsilon)\right]$$

is well-defined

Next it is shown that $\mathcal{T}_\varepsilon$ is a contraction on $\mathcal{B}$ Note that $\mathcal{N}(x,\varepsilon)(0) = 0$ Let x_1, x_2 be in $\mathcal{B}$ Then

$$(\mathcal{N}(x_1,\varepsilon)'(t) - \mathcal{N}(x_2,\varepsilon)'(t)) = \varepsilon^{N+1}\big[h_1(t,t,x_1(t),\varepsilon) - h_1(t,t,x_2(t),\varepsilon)$$
$$+ \int_0^t \{\partial_1 h_1(t,s,x_1(s),\varepsilon) - \partial_1 h_1(t,s,x_2(s),\varepsilon)\}\, ds\big],$$

and, using the Mean Value Theorem,

$$|\mathcal{N}(x_1,\varepsilon)'(t) - \mathcal{N}(x_2,\varepsilon)'(t)| \le \varepsilon^{N+1}\left\{M_2|x_1(t)-x_2(t)| + M_3\int_0^t |x_1(s)-x_2(s)|\,ds\right\}$$

where

$$M_2 = \max_{\substack{0\le t\le T\\ |x|\le 2\delta,\ 0\le\varepsilon\le\varepsilon_0}} |\partial_3 h_1(t,t,x,\varepsilon)|, \quad M_3 = \max_{\substack{(t,s)\in\Delta_T\\ |x|\le 2\delta,\ 0\le\varepsilon\le\varepsilon_0}} |\partial_3\partial_1 h_1(t,s,x,\varepsilon)|$$

It follows that

$$\|\mathcal{N}(x_1,\varepsilon) - \mathcal{N}(x_2,\varepsilon)\|_{\mathcal{X}} \le \varepsilon^{N+1}(M_2 + M_3 T)\|x_1 - x_2\|$$

and hence that $\mathcal{T}_\varepsilon$ $\mathcal{B}\to\mathcal{B}$ is a uniform contraction for ε in some interval $(0,\varepsilon_3]$ with $0\le\varepsilon_3\le\varepsilon_2$ The Banach fixed point theorem implies the following result

Lemma 5 12 *Suppose that* $(\mathbf{H}_3)$, $(\mathbf{H}_4)$ *and* $(\mathbf{H}_5)$ *hold Then there is a number* $\varepsilon_3 > 0$ *such that (5 4 2) has a unique solution* $x(\varepsilon)$ *in* $\mathcal{B}$ *for all* $0<\varepsilon\le\varepsilon_3$

It is easy to show that since $x(\varepsilon)(t) = x(t,\varepsilon)$ satisfies (5 4 2)

$$u(t,\varepsilon) = U_N(t,\varepsilon) + \varepsilon^{N+1}x(t,\varepsilon)$$

is a solution of (5 1 1) Moreover

$$|u(t,\varepsilon) - U_N(t,\varepsilon)| = \varepsilon^{N+1}|x(t,\varepsilon)| \le 2\delta\varepsilon^{N+1}$$

for all $0\le t\le T$ This complete the proof of Theorem 5 10

5.5 Example

Let us consider the following example from Angell and Olmstead [2],

$$\varepsilon u(t) = \int_0^t e^{(t-s)}(u^2(s)-1)\,ds \qquad (5\ 5\ 1)$$

The exact solution of this determined by converting the integral equation to a nonlinear first order differential equation subject to the initial condition $u(0)=0$ is

$$u(t,\varepsilon) = \frac{2}{\varepsilon}\frac{1-e^{\gamma t}}{(\gamma-1)e^{\gamma t}+\gamma+1} \qquad (5\ 5\ 2)$$

where

$$\gamma = \frac{1}{\varepsilon}\sqrt{4+\varepsilon^2} \tag{5 5 3}$$

Example (5 5 1) corresponds to

$$f(t,\varepsilon) = 1 - \mathrm{e}^t, \quad g(t,s,u) = \mathrm{e}^{(t-s)}u^2,$$

which implies $\partial_3 g(t,t,u) = 2u$ It follows from (5 3 1) that the leading order outer solution satisfies

$$0 = \int_0^t \mathrm{e}^{(t-s)}(y_0^2(s) - 1)\,ds$$

which has solutions $y_0(t) = \pm 1$ But only one of these can be appropriate since (5 5 1) has a unique solution (H$_5$) cannot be satisfied with $y_0(t) = 1$, but with $y_0(t) = -1$ it holds with $\alpha = 2$, since $\partial_3 g(t,t,y_0) = -2$ Therefore

$$y_0(t) = -1, \quad t \geq 0$$

The leading order inner correction solution is given by the nonlinear ordinary differential equation

$$z_0'(\tau) = z_0^2(\tau) - 2z_0(\tau), \quad z_0(0) = 1,$$

which has a solution

$$z_0(\tau) = 1 - \tanh\tau, \quad \tau \geq 0$$

We see from this solution that $z_0(\tau)$ satisfies the requirement that

$$\lim_{\tau\to\infty} z_0(\tau) = 0$$

To the leading order, the asymptotic solution $U_0(t,\varepsilon)$ of (5 5 1) is given by

$$U_0(t,\varepsilon) = -\tanh\frac{t}{\varepsilon}$$

In general, for $j \geq 1$, the outer solution satisfies

$$y_{j-1}(t) = -2\int_0^t \mathrm{e}^{t-s}y_j(s)\,ds + \int_0^t \Phi_j(t,s)\,ds + \int_0^\infty k_{j-1}(t,\sigma)\,d\sigma,$$

where $k_{j-1}(t,\sigma)$ and $\Phi_j(t,s)$ are determined by $y_i(t)$ and $z_i(\tau)$ for $i \leq j-1$ Since

$$\Phi_1(t,s) = 0, \quad k_0(t,\sigma) = -\mathrm{e}^t\operatorname{sech}^2\sigma,$$

it follows that the first order outer solution satisfies the equation

$$2\int_0^t e^{t-s} y_1(s)\,ds = 1 - e^t$$

Solving this by differentiating once gives

$$y_1(t) = -\frac{1}{2}, \quad t \geq 0$$

From (5 3 4), the inner correction solution, in general satisfies

$$z_j'(\tau) = -2\tanh\tau z_j(\tau) + \psi_j'(\tau),$$

where

$$\psi_j(\tau) = -\int_\tau^\infty \Xi_j(\tau,\sigma)$$

is determined by $y_i(t)$ and $z_i(\tau)$ for $i \leq j$ respectively $i \leq j-1$ Then, since

$$\Xi_1(\tau,\sigma) = (\sigma - \tau)\operatorname{sech}^2\sigma + \tanh\sigma - 1,$$

the first order inner correction solution $z_1(\tau)$ satisfies

$$z_1'(\tau) = -2\tanh\tau z_1(\tau), \quad z_1(0) = \frac{1}{2}$$

Solving this gives

$$z_1(\tau) = \frac{1}{2}\operatorname{sech}^2\tau, \quad \tau \geq 0$$

Then to the first order, the asymptotic solution $U_1(t,\varepsilon)$ is given by

$$U_1(t,\varepsilon) = -\tanh\frac{t}{\varepsilon} - \frac{\varepsilon}{2}\tanh^2\frac{t}{\varepsilon}$$

To verify that $U_0(t,\varepsilon)$ is a uniformly valid asymptotic solution, we consider the difference

$$\begin{aligned} u(t,\varepsilon) - U_0(t,\varepsilon) &= \frac{2/\varepsilon(1-e^{\gamma t})}{(\gamma-1)e^{\gamma t}+\gamma+1} + \frac{e^{2t/\varepsilon}-1}{e^{2t/\varepsilon}+1} \\ &= \frac{2/\varepsilon(1-e^{\gamma t})(e^{2t/\varepsilon}+1) + (e^{2t/\varepsilon}-1)\{(\gamma-1)e^{\gamma t}+\gamma+1\}}{(\gamma-1)e^{\gamma t}+\gamma+1(e^{2t/\varepsilon}+1)} \end{aligned} \quad (5\ 5\ 4)$$

Simplifying (5 5 4) gives

$$u(t,\varepsilon) - U_0(t,\varepsilon) = \frac{e^{\gamma t} + e^{2t/\varepsilon} - e^{\gamma t}e^{2t/\varepsilon} - 1}{\gamma e^{\gamma t} + \gamma e^{2t/\varepsilon} + (\gamma-1)e^{\gamma t}e^{2t/\varepsilon} + \gamma + 1}$$

We have from (5 5 3) that

$$\gamma \sim -\frac{2}{\varepsilon} + O(\varepsilon), \ \varepsilon \to 0$$

Therefore

$$u(t,\varepsilon) - U_0(t,\varepsilon) = \frac{2\varepsilon e^{2t/\varepsilon} - \varepsilon e^{4t/\varepsilon} - \varepsilon}{2\varepsilon e^{2t/\varepsilon} + (2-\varepsilon)e^{4t/\varepsilon} + 2 + \varepsilon}$$

and

$$|u(t,\varepsilon) - U_0(t,\varepsilon)| \le \left| \frac{2\varepsilon e^{-2t/\varepsilon} - \varepsilon e^{-4t/\varepsilon} - \varepsilon}{2\varepsilon e^{-2t/\varepsilon} + 2 - \varepsilon + (2+\varepsilon)e^{-4t/\varepsilon}} \right|$$

It therefore follows that for $0 < \varepsilon \le \varepsilon_0$, we have

$$|u(t,\varepsilon) - U_0(t,\varepsilon)| \le \frac{\varepsilon}{2},$$

for all $0 \le t \le T$ Similar calculations show that there exists a positive constant $c_1 > 0$ such that

$$|u(t,\varepsilon) - U_1(t,\varepsilon)| \le c_1 \varepsilon^2,$$

uniformly for all $0 \le t \le T$

5.6 Example from Population Growth

Consider the following example

$$\varepsilon u(t) = \varepsilon S(t) + \int_0^t S(t-s)u(s)(1-u(s)/c)\, ds, \tag{5 6 1}$$

where $c > 0$ is a constant Problem (5 6 1) is a model for the population growth The function $u(t)$ is the population size at time t The survival function $S(t)$ is the fraction of the initial population which is still alive at time t, so $S(0) = 1$ $u(1-u/c)$ is the rate of reproduction Since ε is small, (5 6 1) describes a rapid growing population

Problem (5 6 1) corresponds to

$$f(t,\varepsilon) = \varepsilon S(t), \quad g(t,s,u) = S(t-s)u(1-u/c)$$

The leading order outer solution, $y_0(t)$ is given by

$$0 = \int_0^t S(t-s)u(s)(1-u(s)/c)\, ds \tag{5 6 2}$$

which implies

$$y_0(t) = 0 \quad \text{or} \quad y_0(t) = c \tag{5 6 3}$$

To satisfy $(\mathbf{H}_5)$, the correct leading order outer solution is

$$y_0(t) = c,$$

since then $\partial_3 g(t, t, y_0(t)) = -1$ By (5 3 1) the leading order inner correction solution $z_0(\tau)$ is given by

$$z_0'(\tau) = -z_0(\tau)(1 - \frac{1}{c} z_0(\tau)), \quad z_0(0) = 1 - c, \tag{5 6 4}$$

which has solution

$$z_0(\tau) = \frac{c(1-c)\mathrm{e}^{-\tau}}{1 + (c-1)\mathrm{e}^{-\tau}} \tag{5 6 5}$$

This implies that $\lim_{\tau \to \infty} z_0(\tau) = 0$ and thus to the leading order, the asymptotic solution, $U_0(t, \varepsilon)$ of (5 6 1) is given by

$$U_0(t, \varepsilon) = \frac{c}{1 + (c-1)\mathrm{e}^{-t/\varepsilon}} \tag{5 6 6}$$

Thus on a time scale of order ε, the population increases rapidly Since (5 6 1) and $y_0(t)$ satisfy the hypotheses of this chapter, the unknown exact solution satisfies

$$|u(t, \varepsilon) - \frac{c}{1 + (c-1)\mathrm{e}^{-t/\varepsilon}}| = O(\varepsilon) \tag{5 6 7}$$

uniformly for $0 \leq t \leq T$

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