EXISTENCE THEOREMS FOR 90° VORTEX-VORTEX SCATTERING

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of M.SC. is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

Signed: ____________ ID No. 91700604

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Dedicated

To my mother
Mabrouka

Abstract

The scattering of magnetic flux tubes in superconductors is studied. First, we introduce the Abelian-Higgs model, which describes vortices in a superconductor, and the Euler-Lagrange equations which minimize the energy density given by this model. Static vortex solutions satisfying these equations are reviewed. A technique proposed by on Manton [1] in which slowly changing solutions are approximated by a special family of time-independent solutions is described. Time-dependent solutions over small intervals are also studied. Then the existence and the symmetries of the time-dependent solutions are studied. This analysis rules out all cases other than 0°, 90° or 180° scattering of two vortices. The proof of the Cauchy-Kowalewskyi theorem for a system of first order quasi-linear partial differential equations of (n+1) independent variables and m unknown functions is given. The Taylor expansion of the initial data near the origin is studied. The Cauchy Kowalewskyi theorem is applied to find the solutions of the time-dependent Euler-Lagrange equations near the origin. This study proves that our solution describes 90° scattering. Mathematica programs to calculate the series solutions are also supplied.

Contents

Chapter 1

Introduction

Over the years, solitons and soliton-like solutions of non-linear partial differential equations have been studied in great detail One of the most important results of these studies was the discovery of the unusual behavior of solitons in a scattering process. In recent years, mainly based on an idea by Manton [1], results for the scattering of soliton-like objects, like magnetic monopoles [2], CP^1 skyrmions [3-6], and cosmic strings or vortices [7] have been obtained. Important numerical work has also been done for example on cosmic strings or vortices [8-13] and skyrmions in (2+1) dimensions [14-16]. We consider the work on the scattering of vortices to be of particular importance because, unlike the other soliton-like objects mentioned, vortices can be produced in the laboratory and with conventional techniques [17], it may be possible to study their collisions experimentally.

Among the theoretical predictions for the scattering of soliton-like objects scattering at 90° is one of the most exciting. For slowly moving vortices at the point between type I and type II superconductivity, there is analytic evidence, based on the slow-motion approximation, for scattering at right angle [7]. If the repulsion between the vortices increases and they cannot come very close anymore, we would expect to see a switch over to backscattering at a certain value of the repulsion. There is numerical evidence that for fixed repulsion an increase in the velocity can bring the vortices close enough together again to produce scattering at right angles. In ref

[18], an approximation method, which involves linearization of the equations, has been used to show 90° scattering. This work is continued and brought to a conclusion in this thesis, where 90° scattering for certain initial data is shown mathematically rigorously on the level of the Ginzburg-Landau equations.

In the second chapter, we introduce the Abelian Higgs model and discuss previous studies to find time-independent solutions which minimize the energy density. In the third chapter, we discuss two approximation techniques for time-dependent solutions One of the techniques is based on Manton's work [1] in which a slowly changing solution is approximated by a special family of time-independent solutions The second technique studies the time-dependent solution over a small time interval only, i.e, we study the scattering of slowly moving vortices from shortly before to shortly after their collision. In the forth chapter, we study the existence and the symmetries of solutions of the Cauchy problem with initial conditions constructed from static solutions and approximate time-dependent solutions. We find that, for our initial conditions, only 0°,90° or 180° scattering is possible. In the fifth chapter, we rewrite the time-dependent Euler-Lagrange equations as a system of first order quasi-linear partial differential equations and discuss the proof of the Cauchy-Kowalewskyi theorem for a system of first order quasi-linear partial differential equations of (n+1) independent variables and m unknown functions In the sixth chapter, we give the Taylor expansion of the initial data and apply the Cauchy-Kowalewskyi theorem to find a series solutions near the origin. This solution shows 90° scattering.

Chapter 2

The Abelian Higgs Model

In this chapter we discuss the Abelian Higgs model in general and in particular, the Euler-Lagrange equations which minimize the action of this theory. We will introduce the Lagrangian and the energy density, and study the static solutions which satisfy the equations of motion and give finite energy. The static solution which is of particular interest describes two vortices sitting on top of each other. We will also show that the Abelian Higgs model is invariant under a U(1) gauge transformation

2.1 Lagrangian and Euler-Lagrange Equations

The Abelian Higgs model describes a superconductor in a magnetic field in z- direction. The Lagrangian density is given by

$$\mathcal{L}=\frac{1}{2}\left(D_\mu\Phi\right)\left(D^\mu\Phi\right)^* -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} -\frac{\lambda}{8}\left(|\Phi|^2-1\right)^2, \tag{2.1}$$

where Φ is the complex Higgs field,

$$D_\mu\Phi = \partial_\mu\Phi - iA_\mu\Phi, \qquad \mu=0,1,2, \tag{2.2}$$

is the covariant derivative, and the gauge fields $F_{\mu\nu}$ are defined in terms of the real gauge potentials A_μ as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \mu,\nu=0,1,2. \tag{2 3}$$

The indices are lowered and raised with the metric tensor g=diag(+1,-1,-1) This model is related to the Ginzburg-Landau model. For the special class of configurations which are constant in one direction (say z) and under the assumption that the gauge potential A_3 is zero, the Ginzburg-Landau model reduces to the two dimension Abelian Higgs model which is given by the Lagrangian (2 1)

The equations of motion can be derived from the Lagrangian (2 1) by using the usual variational technique. In our case, we have the equations

$$\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial \mathcal{L}}{\partial A_{\nu,\mu}}\right)-\frac{\partial \mathcal{L}}{\partial A_{\nu}}=0,$$
$$\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial \mathcal{L}}{\partial \Phi_{,\mu}}\right)-\frac{\partial \mathcal{L}}{\partial \Phi}=0, \tag{2.4}$$

where

$$A_{\nu,\mu}=\frac{\partial A_{\nu}}{\partial x^{\mu}}, \qquad \Phi_{,\mu}=\frac{\partial \Phi}{\partial x^{\mu}}$$

These yield the equations of motion (Euler-Lagrange equations)

$$D^{\mu}D_{\mu}\Phi+\frac{\lambda}{2}\Phi\left(|\Phi|^{2}-1\right)=0, \tag{2.5}$$

$$\partial_{\mu}F^{\mu\nu}+\frac{i}{2}\left(\Phi^{*}(D^{\nu}\Phi)-\Phi(D^{\nu}\Phi)^{*}\right)=0. \tag{2.6}$$

The Abelian Higgs theory given by (2.1) represents a classical gauge field theory which is characterized by a group of symmetries not associated with any physical coordinate transformation in space-time. The property of a gauge theory is gauge invariance, i.e, the invariance of the Lagrangian under a group of transformations which can be different at different points in space-time This implies that if the original fields are a solution of the equations of motion, so are the gauge transformed fields. In our case the Lagrangian (2.1) is invariant under the gauge transformation

$$\Phi \to \Phi' = e^{-i\varphi(x)}\Phi, \qquad A_{\mu} \to A'_{\mu} = A_{\mu} - \partial_{\mu}\varphi(x), \tag{2.7}$$

where

$$\varphi(x) = \varphi(t, x_1, x_2), \qquad e^{-i\varphi(x)} \in U(1).$$

Since it is easy to show that

$$(D_\mu \Phi)' = e^{-i\varphi(x)}(D_\mu \Phi), \qquad |\Phi|' = |\Phi|,$$

$$F'_{\mu\nu} = F_{\mu\nu}, \tag{2.8}$$

we can establish the invariance of the Lagrangian given by (2.1) under the gauge transformation (2.7). We also see that if (Φ, A_μ) is a solution of the equation of motion (2.5), (2.6), so is the transformed solution (Φ', A'_μ).

2.2 Time-Independent Solutions

We will discuss in this section special static solutions of the equations of motion (2.5), (2.6) with $A_0 = 0$, which minimize the potential energy. The existence of these solutions has been proven by Plohr [19]. Plohr has proven that these equations have n-vortex solutions which minimize the potential energy given by

$$E = \int \left[\tfrac{1}{2}(D_i\Phi)(D_i\Phi)^* + \tfrac{1}{4}\left(F_{ij}\right)^2 + \tfrac{\lambda}{8}\left(|\Phi|^2 - 1\right)^2 \right] d^2x. \tag{2.9}$$

To find static solutions of the equations of motion, let us consider functions of the form

$$A_i(r,\theta) = -\varepsilon_{ij} x_j n a(r)/r^2,$$

$$\Phi(r,\theta) = e^{in\theta} f(r), \qquad i,j=1,2 \tag{2.10}$$

where

$$\varepsilon_{11} = \varepsilon_{22} = 0, \qquad \varepsilon_{12} = -\varepsilon_{21} = 1.$$

We substitute (2.10) into the time-independent Euler-Lagrange equations

$$D_i D^i \Phi + \frac{\lambda}{2}\Phi\left(|\Phi|^2 - 1\right) = 0, \tag{2.11}$$

$$\partial_i F^{ij} + \tfrac{1}{2}\left(\Phi^*(D^i\Phi) - \Phi(D^i\Phi)^*\right) = 0 \tag{2.12}$$

Using

$$\partial_i = \left(\frac{x_i}{r}\partial_r - \varepsilon_{ij}\frac{x_j}{r^2}\partial_\theta\right),$$

$$\varepsilon_{jk}x_i + \varepsilon_{ki}x_j + \varepsilon_{ij}x_k = 0, \qquad i,j,k=1,2$$

we can derive

$$\partial_i F^{ij} = \frac{n}{r^2}x_i\varepsilon^{ij}\left(\frac{a'(r)}{r}\right)', \tag{2.13}$$

$$D_i D^i \Phi = \frac{-1}{r}\left([rf'(r)]' - \frac{n^2 f(r)[a(r)-1]^2}{r}\right)e^{in\theta}, \tag{2.14}$$

$$\Phi^*\left(D^j\Phi\right) - \Phi\left(D^j\Phi\right)^* = 2ix_i n\varepsilon^{ij}\frac{f^2(r)[a(r)-1]}{r} \tag{2.15}$$

From (2.12), (2.13) and (2.15) we obtain

$$[a'(r)/r]' - \frac{f^2(r)[a(r)-1]}{r} = 0, \tag{2.16}$$

and from (2.11) and (2.14) we can derive

$$[rf'(r)]' - \frac{n^2 f(r)[a(r)-1]^2}{r} - \frac{\lambda r}{2}f(r)[f^2(r)-1] = 0. \tag{2.17}$$

According to Plohr [19], there exist functions $a(r)$ and $f(r)$ which satisfy the above equations and minimize the potential energy (2.9).

For $\lambda=1$, there actually exist first order equations whose solutions automatically solve the second order equations (2.11) and (2.12). To see this we set $\Phi = \Phi_1 + i\Phi_2$ and $\lambda=1$ in (2.9) and integrate by parts, which yields

$$\begin{aligned} \mathrm{E} = \int \varepsilon d^2x = \frac{1}{2}\int d^2x\Big[&[(\partial_1\Phi_1 + A_1\Phi_2) \mp (\partial_2\Phi_2 - A_2\Phi_1)]^2 + \\ &[(\partial_2\Phi_1 + A_2\Phi_2) \pm (\partial_1\Phi_2 - A_1\Phi_1)]^2 + \\ &[F_{12} \pm (\Phi_1^2 + \Phi_2^2 - 1)/2]^2\Big] \pm \frac{1}{2}\int d^2x F_{12}, \end{aligned} \tag{2.18}$$

where $\mathcal{E}$ is the energy density. The upper sign and lower sign is taken according to whether the winding number n, which is given by

$$n = \frac{1}{2\pi}\int d^2x\ F_{12}, \tag{2 19}$$

is positive or negative. Jaffe and Taubes [20] have shown, that n measures the number of times

$$\Phi_\infty(\theta) = \lim_{r\to\infty}\Phi(r,\theta), \tag{2.20}$$

which is a unimodular complex number for each θ, winds around the unit circle in the complex plane while θ goes from 0 to 2π n is therefore an integer that does not change when finite smooth energy configurations are changed continously, and this is why the number (2.19) occurs in the functions (2 10) The sets of finite-energy functions with different winding numbers n are called topological sectors

Now the integral (2.18) gives a potential energy greater than or equal to $2|n|\pi$ with equality if and only if

$$(D_1 \pm iD_2)\Phi = 0, \qquad F_{12} = \mp(|\Phi|^2 - 1). \tag{2.21}$$

These equations are known as the Bogomol'nyi equations. It is easy to see that solutions of these equations satisfy the Euler-Lagrange equations (2.11) and (2.12) for $\lambda=1$. It has also been shown [20] that the Plohr solutions [19] satisfy the Bogomol'nyi equations $\lambda=1$. To evaluate the functions $a(r)$ and $f(r)$, let us substitute the solution (2.10) into the Bogomol'nyi equations, which yields

$$\begin{aligned} f'(r) &= \pm\frac{nf(r)[1-a(r)]}{r}, \\ na'(r) &= \mp\frac{r[f^2(r)-1]}{2}, \end{aligned} \tag{2.22}$$

where the upper sign is taken if n is positive, and the lower sign is taken if n is negative We will come back to these equations when we use the functions (2 10) for n=2 as part of our initial data.

Chapter 3

Approximate Time-Dependent Solutions

In this chapter we discuss two approximation techniques for time-dependent solutions. One of the techniques is based on Manton's approach [1] in which slowly changing solutions are approximated by a special family of time-independent solutions. For simplicity, this technique is illustrated in the context of the CP^1 model The second technique studies the time-dependent solution over a small time interval only, so that in this interval the solution does not differ much from the solution at t=0.

3.1 The Slow-Motion Approximation

The slow-motion approximation for vortex scattering was discussd by Ruback [7]. Ruback applied the idea, originally proposed by Manton [1] in the context of SU(2) monopoles, that for $\lambda = 1$ at low energies the Bogomol'nyi solutions can be used to approximate time-dependent solutions. As we have seen the potential energy is bounded below by a positive topological charge, and for a given topological sector, this bound is saturated if and only if a certain system of first order non-linear equations (Bogomol'nyi equations) is satisfied. It can also be shown that the submanifold or moduli space of these minimal energy solutions has dimension 2n In the slow-motion approximation it is assumed that the approximate time-dependent solution is a family of time-independent solutions which minimize the potential energy in a given topological sector. The action is then minimized for this 2n parameter family of solutions to the Bogomol'nyi equations with time-dependent parameters

For the U(1) model this calculation is not explicit To illustrate the method we briefly digress from the U(1) model and discuss this approximation for the CP^1 model following Ward [3]. The CP^1 model in (2+1) dimensions is given by the Lagrangian

$$\mathcal{L} = (1+|u|^2)^{-2}(\partial_\mu u)^*(\partial^\mu u), \qquad \mu=0,1,2. \tag{3 1}$$

If we use the Euler-Lagrange equation

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial u_{,\mu}}\right)=\frac{\partial\mathcal{L}}{\partial u},$$

we derive the classical equation of motion arising from (3 1),

$$(1+|u|^2)\partial_\mu\partial^\mu u = 2u^*(\partial_\mu u)(\partial^\mu u). \tag{3.2}$$

This model again has different topological sectors.

In the charge-two sector, the family of static finite-energy solutions (static lumps) can be written in the form

$$u = \alpha + (\beta z + \gamma)(z^2 + \delta z + \varepsilon)^{-1}, \tag{3.3}$$

where $\alpha, \beta, \gamma, \delta$ and ε are complex parameters and $z = (x^1 + ix^2)/2$. The idea of the approximation for slowly moving lumps is as follows· We assume that the solutions of the equations of motion (3.2) are of the form (3 3) where the parameters depend on t We then substitute (3 3) into the action which is then minimized This leads to ordinary differential equations for the parameters as functions of t. Solving these equations, yields the approximate time-dependent solutions

Before we persue this idea, we use the requirement of finite energy and certain symmetries to set $\alpha = \beta = \delta = 0$. Then we change the form of the parameters $(\gamma, \gamma^*, \varepsilon, \varepsilon^*)$ to (R, ψ, θ, ϕ) according to the equations

$$\begin{aligned} \gamma(t) &= \mathrm{R}e^{i\phi}\sin\psi, \\ \varepsilon(t) &= \mathrm{R}e^{i\theta}\cos\psi. \end{aligned} \tag{3.4}$$

Next if we substitute a solution of the form (3.3) with parameters given by (3.4) into the kinetic energy functional given by the Lagrangian (3.1)

$$T = \int (1+|u|^2)^{-2}|\dot{u}|^2 d^2x. \tag{3.5}$$

For functions of the form (3.3) the rest of the action is just a multiple of the winding number and does not contribute to the Euler-Lagrange equations Thus we obtain

$$\begin{aligned} T = \int\Big[& \dot{R}^2 R^{-2}|z|^4 + \dot{\phi}^2\left|z^2+\varepsilon\right|^2 + \dot{\theta}^2|\varepsilon|^2 + \\ & \dot{\psi}^2\left(|z|^4\cot^2\psi + |\varepsilon|^2\sec^2\psi\csc^2\psi + 2\csc^2\psi\,\mathrm{Re}(z^2\varepsilon^*)\right) + \\ & 2\dot{R}\dot{\psi}R^{-1}\left(|z|^4\cot\psi + \sec\psi\csc\psi\,\mathrm{Re}(z^2\varepsilon^*)\right) - \\ & 2\dot{\phi}\dot{\theta}\left(|\varepsilon|^2 + \mathrm{Re}(z^2\varepsilon^*)\right) - 2\dot{R}\dot{\phi}R^{-1}\,\mathrm{Im}(z^2\varepsilon^*) - \\ & 2\dot{R}\dot{\theta}R^{-1}\,\mathrm{Im}(z^2\varepsilon^*) - 2\dot{\psi}\dot{\phi}\tan\psi\,\mathrm{Im}(z^2\varepsilon^*) - \\ & 2\dot{\psi}\dot{\theta}\cot\psi\,\mathrm{Im}(z^2\varepsilon^*)\Big] A\,d^2x, \end{aligned} \tag{3 6}$$

where we have used the abbreviation

$$A = |\gamma|^2 / \left(|\gamma|^2 + \left|z^2+\varepsilon\right|^2\right)^2. \tag{3 7}$$

The integration can now be performed and the kinetic energy can be used to define a metric on the four dimensional parameter space. First one obtains

$$T = \xi R^{-1}\dot{R}^2 + \mu R\dot{\psi} + \nu R\dot{\psi}^2 + R(\tau\dot{\phi}^2 + \sigma\dot{\theta}\dot{\phi} + \omega\dot{\theta}^2), \tag{3.8}$$

where $\xi, \mu, \nu, \tau, \sigma$ and ω are functions of ψ only and are given by

$$\begin{aligned} &\xi = E/2, && \mu = (K-E)s/c, \\ &\nu = K-(E/2), && \tau = s^2\nu, \\ &\sigma = -sc\mu, && \omega = c^2\xi, \end{aligned} \tag{3.9}$$

with $s = \sin\psi$, $c = \cos\psi$ and $K = K(\cos\psi)$, $E = E(\cos\psi)$ are complete elliptic integrals of the first and second kind, respectively. The metric G is defined by

$$T = G_{ij}\dot{S}^i\dot{S}^j,$$

where $S^1 = R$, $S^2 = \phi$, $S^3 = \theta$, $S^4 = \psi$. The geodesic equations which minimize the time-integral of the kinetic energy (3.8) can be written as

$$G_{kj}\ddot{S}^j + G_{kj,i}\dot{S}^i\dot{S}^j - (G_{ij,k}\dot{S}^i\dot{S}^j)/2 = 0, \tag{3 10}$$

where $G_{kj,i} = \partial G_{kj} / \partial S^i$ and $G_{ij} = G_{ji}$ for i,j,k=1,2,3,4. In our case we have to solve these equations for

$$G_{12} = G_{13} = G_{24} = G_{34} = 0$$

and

$$\begin{aligned} G_{11} &= E/2R, & G_{14} &= s(K-E)/2c, \\ G_{22} &= Rs^2(K - \tfrac{1}{2}E), & G_{23} &= -Rs^2(K-E)/2, \\ G_{33} &= Rc^2E/2, & G_{44} &= R(K - \tfrac{1}{2}E). \end{aligned} \tag{3 11}$$

Only some of the solutions to the geodesic equations have been found explicitly One family of solutions is

$$\psi = \psi_0, \qquad R = T(Q^2 + t^2)/\xi_0, \qquad \phi = \theta = \tan^{-1}\left(\frac{2Qt}{Q^2 - t^2}\right) \tag{3 12}$$

As is expected, the functions (3 12) do not belong to a solution which satisfy the Euler-Lagrange equations. Furthermore, although this might be plausible, it is by no means proven that (3 12) leads to an approximate solution for slowly moving lumps

For the Abelian Higgs model the situation is more complicated The 2n-parameter family of 2-vortex solutions is not known explicitly, i e, there is no analogue of (3.3) Ruback [7] has used symmetries of the Lagrangian to find constraints on the metric Furthermore, his examination of the metric indicates that a certain angle which parameterizes the parameter space has be identified modulo π. This implies 90° scattering for head-on-collisions.

3.2 Approximate Solutions for Small Time Intervals

In ref [18] the functions, which were used by Ruback to study the metric and by Weinberg [21] to find the zero modes of static solution, were used to show the existence of time-dependent solutions to the full Euler-Lagrange equations that describe 90° scattering. In this approach we consider an approximate solution of the Euler-Lagrange equations (2 5), (2.6) of the form

$$
\begin{aligned}
&\Phi(t,\bar{x}) = \hat{\Phi}(\bar{x}) + \tilde{\Phi}(t,\bar{x}), \\
&A_i(t,\bar{x}) = \hat{A}_i(\bar{x}) + \tilde{A}_i(t,\bar{x}), \qquad A_0(t,\bar{x}) = 0,
\end{aligned}
\tag{3.13}
$$

where $(\hat{A}_i,\hat{\Phi})$ is the static solution for two vortices sitting on top of each other. The perturbations $(\tilde{A}_i,\tilde{\Phi})$ on the static solution are represented by $(\tilde{\lambda}a_i(\bar{x})+tB_i,\tilde{\lambda}\varphi(\bar{x})+t\zeta)$ which is small because it is assumed that $\lambda = 1+\tilde{\lambda}$, $0<\tilde{\lambda}<<1$, $t\in(-\varepsilon,\varepsilon)$, $\varepsilon<<1$, where $(\hat{\phi}+\tilde{\lambda}\varphi,\hat{A}_i+\tilde{\lambda}a_i)$ satisfy the static equations of motion linearized in $\tilde{\lambda}$ Hence the equations for (B_i,ζ) can be linearized The idea is to study the scattering of slowly moving vortices from shortly before to shortly after their collision.

If we substitute (3.13) into the equations of motion (2.5) and (2 6), using the fact that $(\hat{A}_i,\hat{\Phi})$ are the static solutions of the time-independent Euler-Lagrange equations (2.11), (2.12), and keeping only the linear terms in $(\tilde{A}_i,\tilde{\Phi})$, we can derive

$$
\begin{aligned}
&\hat{D}^i\hat{D}_i\tilde{\Phi} - 2i\tilde{A}^i\hat{D}_i\hat{\Phi} - i\hat{\Phi}\partial^i\tilde{A}_i + \frac{1}{2}\tilde{\Phi}\left(|\hat{\Phi}|^2-1\right) \\
&\qquad + \frac{1}{2}\hat{\Phi}(\hat{\Phi}\tilde{\Phi}^* + \hat{\Phi}^*\tilde{\Phi}) + \frac{1}{2}\tilde{\lambda}\hat{\Phi}\left(|\hat{\Phi}|^2-1\right) = 0, \\
&\partial^i\tilde{F}_{ij} + \tilde{A}_j|\hat{\Phi}|^2 + \frac{i}{2}\left[\tilde{\Phi}^*(\hat{D}_j\hat{\Phi}) - \tilde{\Phi}(\hat{D}_j\hat{\Phi})^*\right] + \\
&\qquad + \frac{i}{2}\left[\hat{\Phi}^*(\hat{D}_j\tilde{\Phi}) - \hat{\Phi}(\hat{D}_j\tilde{\Phi})^*\right] = 0, \\
&\partial^i\partial_0\tilde{A}_i + \frac{i}{2}\left(\hat{\Phi}^*\partial_0\tilde{\Phi} - \hat{\Phi}\partial_0\tilde{\Phi}^*\right) = 0,
\end{aligned}
\tag{3.14}
$$

where $t\in(\varepsilon,\varepsilon)$, $\varepsilon<<1$ and

$$
\hat{D}_i = \partial_i - i\hat{A}_i, \qquad \tilde{F}_{ij} = \partial_i\tilde{A}_j - \partial_j\tilde{A}_i.
$$

Equations (3.14) are satisfied if

$$\xi = 2f(r)k(r),$$
$$(B_1, B_2) = \left(\frac{-2\sin\theta}{r}[rk'(r) + 2k(r)], \frac{-2\cos\theta}{r}[rk'(r) + 2k(r)] \right), \tag{3.15}$$

is chosen, where k satisfies the equation

$$r^2 k''(r) + rk'(r) - k(r)\left[4 + r^2 f^2(r)\right] = 0. \tag{3.16}$$

Studying the zeros of $|\Phi|^2$ reveals that this solution describes 90° scattering. The problem with this approach is that this linearization has not been justified in a mathematically rigorous fashion In this thesis we will bring this approach to a mathematically rigorous conclusion.

Chapter 4

Global Existence and Symmetry of Solutions

In this chapter we will study the solution of the equations (2.5), (2.6) for certain initial data, and show following ref. [22] that a unique global time-dependent solution exists. For the existence proof, the equations (2.5), (2.6) are rewritten as a system of first order partial differential equations and an iteration formula is applied We use the iteration formula to show that the solution of the Cauchy problem has a left-right symmetry and an up-down symmetry.

4.1 Global Existence

In this section we will show that a unique global time-dependent solution of the equations (2.5), (2.6) for certain initial data exists, by showing that the assumption of ref. [22] are satisfied. To do this let us first subtract a background field $(\hat{\Phi}, \hat{A}_\mu)$ and write

$$\begin{aligned} \Phi(t,\vec{x}) &= \hat{\Phi}(\vec{x}) + \varphi(t,\vec{x}), \\ A_\mu(t,\vec{x}) &= \hat{A}_\mu(\vec{x}) + a_\mu(t,\vec{x}). \end{aligned} \tag{4.1}$$

Substitution into the Euler-Lagrange equations (2 5), (2.6), yields

$$D^\mu(D_\mu\hat{\Phi}) + D^\mu(D_\mu\varphi) + \frac{\lambda}{2}(\hat{\Phi}+\varphi)\left(\left|\hat{\Phi}+\varphi\right|^2 - 1\right) = 0, \tag{4 2-a}$$

$$\partial_\mu F^{\mu\nu} + \frac{i}{2}(\Phi+\varphi)^*\left[D^\nu(\hat{\Phi}+\varphi)\right] - \frac{i}{2}(\Phi+\varphi)\left[D^\nu(\hat{\Phi}+\varphi)\right]^* = 0, \tag{4.2-b}$$

where

$$F_{\mu\nu} = \partial_\mu(\hat{A}_\nu + a_\nu) - \partial_\nu(\hat{A}_\mu + a_\mu). \tag{4.3}$$

For the background field we choose the static solution (2.10) with n=2,

$$\hat{\Phi}(\bar{x}) = e^{2i\theta} f(r), \qquad \hat{A}_i(\bar{x}) = \frac{-2\varepsilon_{ij} x_j a(r)}{r^2},$$

$$\hat{A}_0(\bar{x}) = 0, \qquad i,j=1,2.$$

As initial data we choose

$$\begin{aligned}
&\varphi(0,\bar{x}) = 0, \qquad a_0(0,\bar{x}) = 0,\\
&a_i(0,\bar{x}) = 0, \qquad i=1,2,\\
&\partial_t a_0(0,\bar{x}) = 0,\\
&\partial_t \varphi(0,\bar{x}) = 2f(r)k(r),\\
&\partial_t a_1(0,\bar{x}) = \frac{-2\sin\theta}{r}\left[rk'(r) + 2k(r)\right],\\
&\partial_t a_2(0,\bar{x}) = \frac{-2\cos\theta}{r}\left[rk'(r) + 2k(r)\right].
\end{aligned} \tag{4.4}$$

To show that a unique global time-dependent solution of the Cauchy problem (4.2a-b), (4.4) exists, we will show that the background field satisfies the following conditions:

$$\hat{A}_0 = \partial_t \hat{A}_i = \partial_t \hat{\Phi} = 0, \qquad \partial_i \hat{A}_i = 0 \qquad i=1,2 \tag{4.5}$$

$$\sup_{x\in R^2} \left| \underbrace{\partial_j \partial_k \cdots}_{m} \hat{A}_i \right| < \infty, \qquad m=0,1, \tag{4.6}$$

$$\sup_{x\in R^2}\Big|\underbrace{\partial_i\partial_j\cdots}_{m}\hat{\Phi}\Big|<\infty, \qquad m=0,1,2, \tag{4.7}$$

$$\begin{aligned} &\left(|\hat{\Phi}|^2-1\right)\in L^2, && \hat{\nabla}_i\hat{\Phi}=\left(\partial_i\hat{\Phi}-i\hat{A}_i\hat{\Phi}\right)\in\mathcal{H}_2 \\ &\hat{F}_{ij}\in\mathcal{H}_2, && \partial_i^2\hat{A}_j\in\mathcal{H}_1. \end{aligned} \tag{4.8}$$

For the subtracted field

$$\begin{aligned} &\Psi^t=(a_0,p_0,a_1,p_1,a_2,p_2,\varphi,\pi^*), \\ &p_\mu=\partial_t a_\mu, \qquad \pi^*=\partial_0\varphi-ia_0\varphi, \end{aligned} \tag{4.9}$$

our initial data satisfy

$$\Psi^t\in\mathcal{H}^{(2)}:=(\mathcal{H}_3\times\mathcal{H}_2)^4. \tag{4.10}$$

Moreover, the Lorentz condition

$$\partial_\mu a^\mu=0, \tag{4.11}$$

and the Gauss equation

$$\Delta a_0-\partial_i\partial_i a_i=\frac{i}{2}[(\hat{\Phi}+\varphi)(\pi+ia_0\hat{\Phi}^*)-(\hat{\Phi}^*+\varphi^*)(\pi^*-ia_0\hat{\Phi})], \tag{4.12}$$

hold at t=0 Here $\mathcal{H}_s$ is the Sobolev space of distributions f with finite norm

$$\|f\|^2_{\mathcal{H}_s}=\|f\|^2_{L^2}+\|\partial_i f\|^2_{L^2}+\cdots+\Big\|\underbrace{\partial_i\partial_j\cdots}_{s}f\Big\|^2_{L^2}, \tag{4.13}$$

and $\mathcal{H}_0$ denotes L^2, i.e, if $f\in\mathcal{H}_s$ then $f\in L^2$ and its derivatives are also in L^2.

Obviously $\hat{A}_0=\partial_t\hat{A}_i=\partial_t\hat{\Phi}=0$ and a short calculation shows that

$$\partial_i\hat{A}_i=-2\varepsilon_{ij}x_j\frac{x_i}{r}(a(r)/r)'=0,$$

since $\varepsilon_{ij}x_i x_j = 0$. From ref. [19] we also know that the functions f and a are C^∞ maps on $[0,\infty)$. Their asymptotic behavior at the origin is

$$f \sim \alpha r^2, \qquad\qquad a \sim \beta r^2 + \gamma r^4. \tag{4.14}$$

At infinity, a-1, f-1, and all their derivatives decay exponentially These properties guarantee that the conditions (4.6)-(4.8) hold.

The function k that satisfies equation (3.16) has the following asymptotic behavior at the origin:

$$k \sim c_1 r^{-2} + c_2 r^2. \tag{4.15}$$

It also decays exponentially at infinity. This implies that the condition (4.10) holds. From $a_0 = a_1 = a_2 = 0$ at t=0, it is clear that the Lorentz condition (4.11) holds at t=0 By substituting the initial conditions (4.4) into the Gauss equation (4.12) and using the equation (3.16), we can easily prove that these initial conditions will satisfy the Gauss equation (4.11) at t=0. Now we have proved that all the conditions are satisfied to guarantee the existence of a unique global solution of the Cauchy problem We can also easily show that the energy is initially, and is therefore always, finite.

An essential element of the proof in ref. [22], which is based on Segal's existence and uniqueness theorem [23], is an iteration method. The method starts with rewriting the Cauchy problem (4.2), (4.4) in the form

$$\partial_t \Psi = -i\tilde{A}\Psi + J, \tag{4 16}$$

where the operator $\tilde{A}$ is defined by

$$\tilde{A} = i\begin{bmatrix} \Gamma & 0 & 0 & 0 \\ 0 & \Gamma & 0 & 0 \\ 0 & 0 & \Gamma & 0 \\ 0 & 0 & 0 & \Gamma \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} 0 & 1 \\ \Delta - m^2 & 0 \end{bmatrix}.$$

The vector J is given by

$$J^t = (J_1, J_2, J_3, J_4, J_5, J_6, J_7, J_8), \tag{4.17}$$

with $J_1 = J_3 = J_5 = J_7 = 0$, and

$$J_2 = m^2 a_0 - \frac{\imath}{2}\left[(\hat{\Phi}^* + \varphi^*)\pi^* - (\hat{\Phi} + \varphi)\pi\right] -$$
$$a_0\left(\hat{\Phi}\varphi^* + 2|\hat{\Phi}|^2 + \dot{\Phi}^*\varphi\right)/2, \tag{4.18}$$

$$J_{2\imath+2} = -\frac{\imath}{2}[\hat{\Phi}^*(\hat{\nabla}_\imath\hat{\Phi}) - \hat{\Phi}(\hat{\nabla}_\imath\hat{\Phi})^*] + m^2 a_\imath + \Delta\hat{A}_\imath +$$
$$\frac{1}{2}[(\hat{\Phi}^* + \varphi^*)(\hat{A}_\imath\varphi + a_\imath\hat{\Phi}) + (\hat{\Phi} + \varphi)(\hat{A}_\imath\varphi^* + a_\imath\hat{\Phi}^*)] -$$
$$\frac{\imath}{2}\left[(\hat{\Phi}^* + \varphi^*)(\nabla_\imath\varphi) - (\hat{\Phi} + \varphi)(\nabla_\imath\varphi)^*\right] - \tag{4.19}$$
$$\frac{\imath}{2}\left[\varphi^*(\hat{\nabla}_\imath\hat{\Phi}) - \varphi(\hat{\nabla}_\imath\hat{\Phi})^*\right], \qquad \text{i=1,2,}$$

$$J_8 = m^2\varphi - i\partial_\imath\left(\hat{A}_\imath\hat{\Phi}\right) - i\hat{A}_\imath\left(\hat{\nabla}_\imath\hat{\Phi}\right) + \Delta\hat{\Phi} - \imath\partial_\imath\left(a_\imath\varphi\right) - ia_\imath\left(\nabla_\imath\varphi\right) +$$
$$\imath a_0\pi^* - 2\imath\hat{A}_\imath(\nabla_\imath\varphi) - \hat{A}_\imath^2\varphi - 2a_\imath\left(\hat{\nabla}_\imath\hat{\Phi}\right) - \imath\hat{\Phi}(\partial_\imath a_\imath) +$$
$$\left(a_\mu a^\mu\right)\hat{\Phi} + ip_0\hat{\Phi} - \frac{\lambda}{2}\hat{\Phi}\left(|\hat{\Phi}|^2 - 1\right) - \tag{4.20}$$
$$\frac{\lambda}{2}\left[\varphi\left(|\hat{\Phi} + \varphi|^2 - 1\right) + \hat{\Phi}\left(|\varphi|^2 + \hat{\Phi}^*\varphi + \hat{\Phi}\varphi^*\right)\right],$$

where $\nabla_\imath = \partial_\imath - \imath a_\imath$.

4.2 Reflection and Rotation Symmetries

In the this section we will use (4.16) to discuss the symmetry of the solution The solution of the Cauchy problem (4.16) can be obtained as the solution of the following integro-differential equation

$$\Psi(t,\vec{x}) = e^{-\imath\hat{A}t}\Psi(0,\vec{x}) + \int_0^t ds\{e^{-\imath\hat{A}(t-s)}J(\Psi(s))\}. \tag{4.21}$$

In turn we can solve this integro-differential equation by using the Picard Method [23]. The Picard procedure for solving (4.21) is to set up a sequence of successive approximations Ψ_n defined by the formula

$$\Psi_{n+1}(t,\bar{x}) = e^{-\iota\tilde{A}t}\Psi_n(0,\bar{x}) + \int_0^t ds\{e^{-\iota\tilde{A}(t-s)}J(\Psi_n(s))\}, \tag{4.22}$$

where

$$\Psi_0^t(0,\bar{x}) = (a_0, p_0, a_1, p_1, a_2, p_2, \varphi, \pi^*)(0,\bar{x}), \tag{4.23}$$

with the initial data (4.4). We now establish certain symmetries of the initial data Ψ_0 and use (4.22) to establish these symmetries for the successive approximations Ψ_n, and finally for the solution of (4.21).

The first transformation we study is $(x_1, x_2) \to (-x_1, -x_2)$. Under this transformation the initial data change as follows:

$$\Psi(0,-\bar{x}) = \hat{\Psi}(0,\bar{x}), \tag{4.24}$$

where

$$\hat{\Psi}^t = (a_0, p_0, -a_1, -p_1, -a_2, -p_2, \varphi, \pi^*),$$

which can be written as

$$\Psi(0,-\bar{x}) = M_1\Psi(0,\bar{x}), \tag{4 25}$$

where

$$M_1 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4 26}$$

We see that

$$J(\Psi(0,-\bar{x})) = M_1 J(\Psi(0,\bar{x})), \qquad \left[M_1, \tilde{A}\right] = 0, \tag{4.27}$$

and

$$\exp\{-\iota\tilde{A}t\}M_1\Psi_n(0,\bar{x}) = M_1\exp\{-\iota\tilde{A}t\}\Psi_n(0,\bar{x}). \tag{4.28}$$

Which implies that $\Psi_n(t,-\bar{x}) = M_1\Psi_n(t,\bar{x})$ for all $n \in N$. From this follows

$$\Psi(t,-\bar{x}) = M_1\Psi(t,\bar{x}), \tag{4 29}$$

for the solution Ψ.

Next we study the reflection $(x_1,x_2) \to (-x_1,x_2)$. Under this transformation the initial data change as follows

$$\Psi(t,-x_1,x_2) = M_2\Psi(t,x_1,x_2), \tag{4.30}$$

where

$$M_2 = \begin{bmatrix} -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & C \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.31}$$

and $CV = V^*$. Furthermore

$$\begin{aligned} J_i(0,-x_1,x_2) &= -J_i(0,x_1,x_2), \qquad i=2,6 \\ J_4(0,-x_1,x_2) &= J_4(0,x_1,x_2), \\ J_8(0,-x_1,x_2) &= J^*{}_8(0,x_1,x_2), \end{aligned} \tag{4.32}$$

which implies

$$J(\Psi(0,-x_1,x_2)) = M_2 J(\Psi(0,x_1,x_2)). \tag{4.33}$$

Again, we have $\left[M_2,\tilde{A}\right] = 0$ and $\Psi_n(t,-x_1,x_2) = M_2\Psi_n(t,x_1,x_2)$. From this follows

$$\Psi(t,-x_1,x_2) = M_2\Psi(t,x_1,x_2), \tag{4.34}$$

for the solution Ψ.

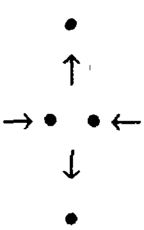
By combining the two transformations we can also study the reflection $(x_1,x_2) \to (x_1,-x_2)$. We find that $\Psi(t,x_1,-x_2) = M_3\Psi(t,x_1,x_2)$, where

$$M_3 = M_2 M_1 = \begin{bmatrix} -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & C \end{bmatrix}. \tag{4.35}$$

Under the transformation considered the energy density

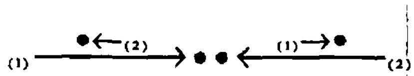
$$\varepsilon = \frac{1}{2}|D_0\Phi|^2 + \frac{1}{2}|D_i\Phi|^2 + \frac{1}{4}F_{ij}^2 + \frac{1}{2}F_{0i}^2 + \frac{\lambda}{2}\left(|\Phi|^2 - 1\right)^2,$$

is invariant. This means that the solution which satisfies our initial conditions has left-right symmetry and up-down symmetry for all time t. Hence there exist only three possibilities when two vortices collide in a head-on collision such that our initial data are realized at t=0. We describe these three cases in the following diagram:



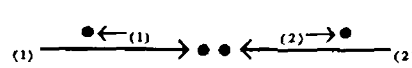
case(1)

90° *Scattering*



case(2)

180° *Scattering*



case(3)

0° *Scattering*.

diagram(I)

Chapter 5

The Cauchy-Problem

In this chapter we will rewrite the time-dependent Euler-Lagrange equations as a system of first-order quasi-linear partial differential equations with coefficients which depend only on the unknown functions. We will then discuss the proof of the Cauchy-Kowalewskyi theorem [24] for a system of first-order quasi-linear partial differential equations of (n+1) independent variables and m unknown functions.

5.1 Associated First-Order Quasi-Linear System

In this section we will show that the time-dependent Euler-Lagrange equations can be rewritten as a system of first-order quasi-linear partial differential equations To do this, let us substitute

$$\Phi = u_1 + iu_2, \qquad A_1 = u_3, \qquad A_2 = u_4, \qquad A_0 = u_5, \tag{5.1}$$

into the equations (2.5), (2.6), which yields

$$\begin{aligned} \partial_0^2 u_1 = {} & \partial_1^2 u_1 + \partial_2^2 u_1 - u_1 u_3^2 - u_1 u_4^2 + u_2 \partial_1 u_3 + u_2 \partial_2 u_4 + \\ & 2u_3 \partial_1 u_2 + 2u_4 \partial_2 u_2 - \frac{\lambda}{2} u_1^3 - \frac{\lambda}{2} u_1 u_2^2 + \frac{\lambda}{2} u_1 + \\ & u_1 u_5^2 - 2u_5 \partial_0 u_2 - u_2 \partial_0 u_5, \end{aligned} \tag{5.2}$$

$$\partial_0^2 u_2 = \partial_1^2 u_2 + \partial_2^2 u_2 - u_2 u_3^2 - u_2 u_4^2 - u_1 \partial_1 u_3 - u_1 \partial_2 u_4 -$$

$$2u_3 \partial_1 u_1 - 2u_4 \partial_2 u_1 - \frac{\lambda}{2} u_2^3 - \frac{\lambda}{2} u_2 u_1^2 + \frac{\lambda}{2} u_2 +$$

$$u_2 u_5^2 + 2u_5 \partial_0 u_1 + u_1 \partial_0 u_5,$$

$$\partial_0^2 u_3 = \partial_2^2 u_3 - \partial_2 \partial_1 u_4 + u_1 \partial_1 u_2 - u_2 \partial_1 u_1 - u_3 u_1^2 - u_3 u_2^2 + \partial_1 \partial_0 u_5,$$

$$\partial_0^2 u_4 = \partial_1^2 u_4 - \partial_1 \partial_2 u_3 + u_1 \partial_2 u_2 - u_2 \partial_2 u_1 - u_4 u_1^2 - u_4 u_2^2 + \partial_2 \partial_0 u_5,$$

$$\partial_0^2 u_5 = \partial_1 \partial_0 u_3 + \partial_2 \partial_0 u_4,$$

where

$$\partial_\mu = \partial / \partial x_\mu, \qquad \partial_\mu^2 = \partial^2 / \partial x_\mu^2, \qquad \mu = 0, 1, 2.$$

We can write our initial data in the form

$$u_i(0, \bar{x}) = \alpha_i(\bar{x}) \qquad \partial_0 u_i(0, \bar{x}) = \beta_i(\bar{x}), \tag{5.3}$$

where

$$\begin{aligned}
&\alpha_1(\bar{x}) = f(r)\cos 2\theta, && \alpha_2(\bar{x}) = f(r)\sin 2\theta, \\
&\alpha_3(\bar{x}) = \frac{-2}{r} a(r)\sin\theta, && \alpha_4(\bar{x}) = \frac{2}{r} a(r)\cos\theta, \\
&\alpha_5(\bar{x}) = 0, && \\
&\beta_1(\bar{x}) = 2f(r)k(r), && \beta_2(\bar{x}) = 0, \\
&\beta_3(\bar{x}) = \frac{-2}{r}\sin\theta(rk' + 2k), && \beta_4(\bar{x}) = \frac{-2}{r}\cos\theta(rk' + 2k). \\
&\beta_5(\bar{x}) = 0 &&
\end{aligned} \tag{5.4}$$

To reduce the order of the Cauchy problem (5.2),(5 3), let us assume

$$R^{(00i)} = \partial_0^2 u_i, \qquad P^{(0i)} = \partial_0 u_i, \qquad q^{(1i)} = \partial_1 u_i,$$

$$q^{(2i)} = \partial_2 u_i, \qquad S^{(01i)} = \partial_0\partial_1 u_i, \qquad S^{(02i)} = \partial_0\partial_2 u_i \tag{5.5}$$

$$T^{(11i)} = \partial_1^2 u_i, \qquad T^{(22i)} = \partial_2^2 u_i, \qquad \text{i,j=1,2,3,4,5.}$$

Then the equations (5.2) can be rewritten as

$$R^{(00i)} = G_i(u_j, P^{(0j)}, q^{(1j)}, q^{(2j)}, T^{(11j)}, T^{(22j)}, S^{(01j)}, S^{(12j)}, q_{x_2}^{(1j)}, P_{x_1}^{(0j)}, P_{x_2}^{(0j)}), \tag{5 6}$$

where

$$\begin{aligned}
G_1 &= T^{(111)} + T^{(221)} - u_1 u_3^2 - u_1 u_4^2 + u_2 q^{(13)} + u_2 q^{(24)} + \\
&\quad 2u_3 q^{(12)} + 2u_4 q^{(22)} - \frac{\lambda}{2} u_1^3 - \frac{\lambda}{2} u_1 u_2^2 + \frac{\lambda}{2} u_1 + \\
&\quad u_1 u_5^2 - 2u_5 P^{(02)} - u_2 P^{(05)}, \\
G_2 &= T^{(112)} + T^{(222)} - u_2 u_3^2 - u_2 u_4^2 + u_1 q^{(13)} - u_1 q^{(24)} - \\
&\quad 2u_3 q^{(11)} - 2u_4 q^{(21)} - \frac{\lambda}{2} u_2^3 - \frac{\lambda}{2} u_2 u_1^2 + \frac{\lambda}{2} u_2 + \\
&\quad u_2 u_5^2 + 2u_5 P^{(01)} - u_1 P^{(05)}, \\
G_3 &= T^{(223)} - q_{x_2}^{(14)} + u_1 q^{(12)} - u_2 q^{(11)} - u_3 u_1^2 - u_3 u_2^2 + P_{x_1}^{(05)}, \\
G_4 &= T^{(114)} - q_{x_2}^{(13)} + u_1 q^{(22)} - u_2 q^{(21)} - u_4 u_1^2 - u_4 u_2^2 + P_{x_2}^{(05)}, \\
G_5 &= P_{x_1}^{(03)} + P_{x_2}^{(04)}
\end{aligned} \tag{5.7}$$

And the initial conditions (5.3) will take the form

$$u_i(0, \bar{x}) = \alpha_i(\bar{x}), \qquad P^{(0i)}(0, \bar{x}) = \beta_i(\bar{x}) \tag{5.8}$$

If we differentiate the equations (5 5) with respect to t, the Cauchy problem (5.2), (5.3) can be rewritten in the form

$$\frac{\partial u_i}{\partial t} = P^{(0i)}, \qquad q_t^{(1i)} = P_{x_1}^{(0i)}, \qquad q_t^{(2i)} = P_{x_2}^{(0i)},$$

$$S_t^{(01i)} = R_{x_1}^{(00i)}, \qquad S_t^{(02i)} = R_{x_2}^{(00i)}, \qquad T_t^{(11i)} = S_{x_1}^{(01i)}$$

$$T_t^{(22i)} = S_{x_2}^{(02i)}, \qquad P_t^{(0i)} = R^{(00i)}, \tag{5 9}$$

$$R_t^{(00i)} = F_i(u_j, P^{(0j)}, q^{(1j)}, q^{(2j)}, T^{(11j)}, T^{(22j)},$$

$$S^{(01j)}, S^{(02j)}, R^{(00j)}, P_{x_k}^{(0j)}, S_{x_k}^{(01j)}, S_{x_k}^{(02j)}, R_{x_k}^{(00j)}), \qquad k=1,2$$

with the initial conditions

$$u_i(0,\vec{x}) = \alpha_i(\vec{x}), \qquad P^{(0i)}(0,\vec{x}) = \beta_i(\vec{x}),$$

$$q^{(1i)}(0,\vec{x}) = \alpha_{i,x_1}(\vec{x}), \qquad q^{(2i)}(0,\vec{x}) = \alpha_{i,x_2}(\vec{x}), \tag{5 10}$$

$$T^{(11i)}(0,\vec{x}) = \alpha_{i,x_1,x_1}(\vec{x}), \qquad T^{(22i)}(0,\vec{x}) = \alpha_{i,x_2,x_2}(\vec{x}),$$

$$S^{(01i)}(0,\vec{x}) = \beta_{i,x_1}(\vec{x}), \qquad S^{(02i)}(0,\vec{x}) = \beta_{i,x_2}(\vec{x}),$$

$$R^{(00i)}(0,\vec{X}) = F_i(\alpha_j, \beta_j, \alpha_{j,x_1}, \alpha_{j,x_2}, \alpha_{j,x_1,x_1}, \alpha_{j,x_2,x_2}, \beta_{j,x_1}, \beta_{j,x_2}, \alpha_{j,x_1,x_2}, \beta_{j,x_1}, \beta_{j,x_2}),$$

$$i,j=1,2\ldots,5$$

where

$$F_1 = S_{x_1}^{(011)} + S_{x_2}^{(021)} - 2u_1u_3P^{(03)} - u_3^2P^{(01)} - 2u_1u_4P^{(04)} - u_4^2P^{(01)} +$$

$$u_2P_{x_1}^{(03)} + P^{(02)}q^{(13)} + u_2P_{x_2}^{(04)} + P^{(02)}q^{(24)} + 2u_3P_{x_1}^{(02)} + 2P^{(03)}q^{(12)} +$$

$$2u_4P_{x_2}^{(02)} + 2P^{(04)}q^{(22)} - \frac{3\lambda}{2}u_1^2P^{(01)} - \frac{\lambda}{2}u_2^2P^{(01)} - \lambda u_1u_2P^{(02)} + \frac{\lambda}{2}P^{(01)} +$$

$$u_5^2P^{(01)} + 2u_1u_5P^{(05)} - 3P^{(05)}P^{(02)} - 2u_5R^{(002)} - u_2R^{(005)}, \tag{5.11-a}$$

$$F_2 = S_{x_1}^{(012)} + S_{x_2}^{(022)} - 2u_2u_3P^{(03)} - u_3^2P^{(02)} - 2u_2u_4P^{(04)} - u_4^2P^{(02)} -$$

$$u_1P_{x1}^{(03)} - P^{(01)}q^{(13)} - u_1P_{x_2}^{(04)} - P^{(01)}q^{(24)} - 2u_3P_{x_1}^{(01)} - 2P^{(03)}q^{(11)} -$$

$$2u_4P_{x_2}^{(01)} - 2P^{(04)}q^{(21)} - \frac{3\lambda}{2}u_2^2P^{(02)} - \frac{\lambda}{2}u_1^2P^{(02)} - \lambda u_2u_1P^{(01)} + \frac{\lambda}{2}P^{(02)} +$$

$$u_5^2P^{(02)} + 2u_2u_5P^{(05)} + 3P^{(05)}P^{(02)} + 2u_5R^{(001)} + u_1R^{(005)}, \tag{5.11-b}$$

$$F_3 = S_{x_1}^{(023)} - S_{x_2}^{(014)} + u_1P_{x_1}^{(02)} + P^{(01)}q^{(12)} - u_2P_{x_1}^{(01)} - P^{(02)}q^{(11)} -$$

$$u_1^2P^{(03)} - 2u_1u_3P^{(01)} - u_2^2P^{(03)} - 2u_3u_2P^{(02)} + R_{x_1}^{(005)}, \tag{5.11-c}$$

$$F_4 = S_{x_1}^{(014)} - S_{x_2}^{(013)} + u_1P_{x_2}^{(02)} + P^{(01)}q^{(22)} - u_2P_{x_2}^{(01)} - P^{(02)}q^{(21)} -$$

$$u_1^2P^{(04)} - 2u_1u_4P^{(01)} - u_2^2P^{(04)} - 2u_4u_2P^{(02)} + R_{x_2}^{(023)}, \tag{5.11-d}$$

$$F_5 = R_{x_1}^{(003)} + R_{x_2}^{(004)}. \tag{5 11-e}$$

To show that the Cauchy problem (5.2), (5.3) is equivalent to the new Cauchy problem (5.9), (5.10), we will prove that a solution $(u_i, P^{(0i)}, q^{(1i)}, q^{(2i)}, T^{(11i)}, T^{(22i)}, S^{(01i)}, S^{(02i)}, R^{(00i)})$ satisfies the Cauchy problem (5.2), (5.3). It is clear from (5.9) that $\partial_0 u_i = P^{(0i)}$ and $\partial_0\partial_2 u_i = P_{x_2}^{(0i)} = q_t^{(2i)}$, which implies $\partial_2 u_i - q^{(2i)} = \Omega(\bar{x})$. But at t=0, $\Omega(\bar{x}) = \alpha_{i,x_2}(\bar{x}) - \alpha_{i,x_2}(\bar{x}) = 0$, and this implies $\partial_2 u_i = q^{(2i)}$ Analogously, we can prove that $\partial_1 u_i = q^{(1i)}$. Also $S_t^{(01i)} - R_{x_1}^{(00i)} = 0$ implies $\partial_0\partial_1 u_i - S^{(01i)} = \Pi(\bar{x})$. But the initial condition at t=0 implies that $\Pi(\bar{x})=0$, and this proves that $\partial_0\partial_1 u_i = S^{(01i)}$. Analogously, we can also prove that $\partial_0\partial_2 u_i = S^{(02i)}$ Similarly by using the same technique as above we can prove $\partial_1^2 u_i = T^{(11i)}$, $\partial_2^2 u_i = T^{(22i)}$, $\partial_0^2 u_i = R^{(00i)}$. We have rewritten our problem as a first order system of quasi-linear partial differential equations (5.9) with initial conditions given by (5.10). Note that each term on the right-hand side of the equations (5.9) contains either one first-order derivative of an unknown function or no derivative at all.

To rewrite the terms which do not contain a derivative we introduce the function V which satisfies

$$V_t = 0, \qquad V(0,\bar{x}) = x_1 \tag{5 12}$$

Clearly $V = x_1$, and we can multiply each term, which before did not contain a derivative, by V_{x_1}. Now the problem (5.9), (5.10), (5.12) is of the form

$$\frac{\partial u_i}{\partial t} = \sum_{p=1}^{2} \sum_{j=1}^{46} H_{i,j,p}(u_1, \ldots, u_{46}) \frac{\partial u_j}{\partial x_p}, \tag{5 13}$$

with the intial conditions given by

$$u_i(0, \bar{x}) = \Phi_i(\bar{x}), \qquad i=1,2,\ldots,46. \tag{5,14}$$

By using the substitution $u_i - \Phi_i(0,..,0)$ for u_i, we can always arrange that the initial conditions give zero at the origin

5.2 The Cauchy-Kowalewskyi Theorem

We include in this section the discussion of a fundamental theorem due to Cauchy and Kowalewskyi assuring that there exists an unique analytic solution of a certain class of Cauchy problems which contains our Cauchy problem. The Cauchy problem which we consider is a system of first order partial differential equations of the form

$$\frac{\partial}{\partial \xi} u_i = \sum_{\nu=1}^{n} \sum_{j=1}^{m} G_{ij\nu}(u_1, \ldots, u_m) \frac{\partial}{\partial \eta_\nu} u_j \tag{5 15}$$

with the initial conditions

$$u_i(0, \eta_1, \ldots, \eta_n) = \Phi_i(\eta_1, \ldots, \eta_n), \tag{5 16}$$

with

$i=1,2,\ldots,m$

where $G_{ij\nu}(u_1, \ldots, u_m)$ and $\Phi_i(\eta_1, \ldots, \eta_n)$ are analytic functions with respect to all their arguments in some neighbourhood of the origin. Furthermore, $\Phi_i(0,\ldots,0) = 0$ for $i=1,\ldots,m$ In section 5.1 it was shown that our vortex-vortex scattering problem is of the form (5.15), (5.16) with analytic functions $G_{ij\nu}$, and in section 6 1 it will be shown that the initial data have the required analyticity.

The idea behind the Cauchy-Kowalewskyi theorem is to consider a related Cauchy problem which has a unique formal power series solution as well as the Cauchy problem (5.15), (5.16), and to prove the following two important facts:

(i) The formal power series solution of the related problem is a majorant of the formal power series solution of the original problem (5.15), (5 16); (ii) The formal power series solution of the related problem is in fact a solution (in the rigorous sense) in some neighbourhood of the origin. This shows that the formal power series solution of the problem (5 15), (5.16) is in fact a solution.

Now if we assume that the problem (5.15), (5.16) has a formal power series solution of the form

$$u_i(\xi,\eta_1,\dots,\eta_n)=\sum_{k_0,k_1,\dots,k_n=0}^{\infty} c^i_{k_0k_1\dots k_n}\xi^{k_0}\eta_1^{k_1}\dots\eta_n^{k_n}, \tag{5.17}$$

then it is easily to prove the uniquence of this solution. The initial conditions (5.16) are a condition on $c^i_{0k_1\dots k_n}$. Then the equation (5.15) at $\xi=0$ is equivalent to conditions on $c^i_{1k_1\dots k_n}$. If we differentiating (5.15) with respect to ξ we can find recursively all the coefficients $c^i_{k_0\dots k_n}$.

Next we will introduce the Cauchy problem related to the Cauchy problem (5.15), (5.16) While we do this we will also show the first fact, namely, that the formal power series solution of the related problem is a majorant of the formal power series solution of the original problem. To do this let us consider a function $f(x_1,x_2,\dots\dots,x_n)$ at the point $(0,\dots\dots,0)$. If we assume that the function $f(x_1,x_2,\dots\dots,x_n)$ is analytic at this point then there exists a neighbourhood N(0) wherein f can be represented by a convergent power series of the form

$$f(x_1,x_2,\dots\dots,x_n)=\sum_{k_1,\dots k_n=0}^{\infty} a_{k_1\dots k_n}x_1^{k_1}\dots x_n^{k_n}, \tag{5.18}$$

where

$$a_{k_1\dots k_n}=\frac{1}{k_1!\dots k_n!}\frac{\partial^{k_1+\dots+k_n}f(0)}{\partial x_1^{k_1}\dots\partial x_n^{k_n}}. \tag{5.19}$$

If we assume also that the power series given by (5.18) is convergent at the point $x_1=\dots=x_n=\rho$, where $\rho>0$, then for any set of non-negative integer $k_1,\dots,k_n$ there exists a number M such that

$$\left|a_{k_1\dots k_n}\right|\le\frac{M}{\rho^{k_1+\dots+k_n}} \tag{5.20}$$

This inequality implies that the series

$$S = M\sum_{n=0}^{\infty}\left(\frac{x_1+\dots+x_n}{\rho^n}\right)^n, \tag{5.21}$$

for

$$\left|(x_1+\cdots+x_n)/\rho^n\right| < 1$$

is a majorant of the series (5.18). This can be easily seen as follows: S can be written in the form

$$S = M\sum_{k_1,\dots,k_n=0}^{\infty}\alpha_{k_1\dots k_n}\frac{x_1^{k_1}\cdots x_n^{k_n}}{\rho^{k_1+\dots+k_n}}, \tag{5.22}$$

where the coefficients $\alpha_{k_1\dots k_n}$ are positive integers. This fact together with the inequality (5.20) yields that

$$\left|a_{k_1\dots k_n}\right| \le \frac{M\alpha_{k_1\dots k_n}}{\rho^{k_1+\dots+k_n}}, \tag{5.23}$$

and this proves that the series (5.22) is also a majorant of (5.18).

To proceed with our discussion of the Cauchy-Kowalewskyi theorem, we will use the analyticity of our data to define the functions $G_{ij\nu}$ and Φ_i in terms of power series as

$$\Phi_i(\eta_1,\dots,\eta_n) = \sum_{\nu_1,\dots,\nu_n=1}^{\infty}a^i_{\nu_1\dots\nu_n}\eta_1^{\nu_1}\dots\dots\eta_n^{\nu_n}, \tag{5.24}$$

and

$$G_{ij\nu}(u_1,\dots,u_m) = \sum_{\nu_1,\dots,\nu_m=0}^{\infty}b^{ij\nu}_{\nu_1\dots\nu_m}u_1^{\nu_1}\dots u_m^{\nu_m}. \tag{5.25}$$

These power series are convergent in the region

$$\left|\eta_1+\cdots+\eta_n\right| \le \rho, \qquad \left|u_1+\cdots+u_m\right| \le \rho,$$

for small ρ.

Using the result which has been derived above, we can easily show that the power series (5.24), (5.25) are majorized by the power series

$$\Psi_i(\eta_1,\ldots,\eta_n) = M\sum_{K=1}^{\infty}\left(\frac{\eta_1+\ldots+\eta_n}{\rho}\right)^K, \tag{5.26}$$

and

$$\overline{G_{ij\nu}}(u_1,\ldots,u_m) = M\sum_{K=1}^{\infty}\left(\frac{u_1+\ldots+u_m}{\rho}\right)^K, \tag{5.27}$$

respectively, which yields

$$\Psi_i(\eta_1,\cdots,\eta_n) = \frac{M(\eta_1+\cdots+\eta_n)}{\rho-\eta\cdots-\eta_n}, \tag{5.28}$$

and

$$\overline{G}_{ij\nu}(u_1,\ldots,u_m) = \frac{M}{1-\dfrac{u_1+\ldots+u_m}{\rho}}. \tag{5.29}$$

Now if we assume that $A^i_{\nu_1\ \nu_n}$ and $B^{ij\nu}_{\nu_1\ \nu_m}$ are the coefficients of the above power series then

$$\left|a^i_{\nu_1\ \nu_n}\right| \le A^i_{\nu_1\ \nu_n}, \qquad \left|b^{ij\nu}_{\nu_1\ \nu_m}\right| \le B^{ij\nu}_{\nu_1\ \nu_m}. \tag{5.30}$$

In other words, the coefficients of the power series (5.28), (5.29) are non-negative and not smaller than the absolute values of the corresponding coefficients of the power series (5.24), (5.25).

We now consider the related Cauchy problem

$$\frac{\partial}{\partial x}V_i = \sum_{\nu=1}^{n}\sum_{j=1}^{m}\overline{G}_{ij\nu}\frac{\partial}{\partial y_\nu}V_j, \tag{5.31}$$

with the initial condition

$$V_i(0,\eta_1,\ldots,\eta_n) = \Psi_i(\eta_1,\ldots,\eta_n), \tag{5.32}$$

as a majorant Cauchy problem of the original Cauchy problem (5.15), (5.16). Again let us assume that the above problem has a series solution of the form

$$V_i(\xi,\eta_1,\ldots,\eta_n) = \sum_{k_0,k_1,\ldots,k_n=0}^{\infty} C^i_{k_0k_1\ k_n}\xi^{k_0}\eta_1^{k_1}\ldots\eta_n^{k_n}, \tag{5.33}$$

where the new quantities $C^{\iota}_{k_0 \ldots k_n}$ can be evaluated from $A^{\iota}_{\nu_1 \ldots \nu_n}$ and $B^{ij\nu}_{\nu_1 \ldots \nu_m}$ in the way the original coefficients $c^{\iota}_{k_0 k_1 \ldots k_n}$ in (5.17) were obtained from the $a^{\iota}_{\nu_1 \ldots \nu_n}$ and $b^{ij\nu}_{\nu_1 \ldots \nu_m}$. In other words,

$$C^{\iota}_{k_0 k_1 \ldots k_n} = P^{\iota}_{k_0 k_1 \ldots k_n}\left(A^{\iota}_{\nu_1 \ldots \nu_n}, B^{ij\nu}_{\nu_1 \ldots \nu_m}\right), \tag{5.34}$$

where the polynomials $P^{\iota}_{k_0 k_1 \ldots k_n}$ have non-negative coefficients. This yields

$$\left|c^{\iota}_{k_0 k_1 \ldots k_n}\right| \le C^{\iota}_{k_0 k_1 \ldots k_n}, \tag{5.35}$$

i.e., the power series (5.33) is a majorant of (5.17).

Next we will show that the initial value problem (5.31), (5.32) has a solution which can be expanded into the power series (5.33) which will prove the convergence of the power series. To do this, let us assume that

$$\begin{aligned} V_1(\xi,\eta_1,\ldots,\eta_n) \equiv V_2(\xi,\eta_1,\ldots,\eta_n) \equiv \cdots \equiv V_m(\xi,\eta_1,\ldots,\eta_n) \\ \equiv V(\xi,\eta_1,\ldots,\eta_n) \\ = V(\xi,z), \end{aligned} \tag{5.36}$$

where $z = \eta_1 + \ldots + \eta_n$. Substituting this solution into (5.31), (5.32), we will get the first order partial differential equation

$$[(\rho - mV)/\rho]V_\xi(\xi,z) - (nmM)V_z(\xi,z) = 0, \tag{5.37}$$

with the initial condition

$$V(0,z) = \Psi(0,z), \tag{5.38}$$

where

$$\Psi(0,z) = \frac{Mz}{\rho - z}.$$

The first order partial differential equation (5.37) has the form

$$A(V)V_\xi + B(V)V_z = 0, \tag{5.39}$$

with the initial condition

$$V(0,z) = \Psi(0,z). \tag{5.40}$$

This first order partial differential equation has a solutions which satisfies

$$A(V)z - B(V)\xi = C(V), \tag{5 41}$$

where C(V) is an arbitrary function that can be evaluated by using the initial condition (5.40). If we substite $\xi = 0$ and $V = \Psi(z)$ into the equation (5.41) we will get $C(\Psi) = A(\Psi)z$, and if we invert the function $V = \Psi(z)$ to obtain $z = \Omega(V)$, we see that the function $C(V)$ is determined by the relation $C(\Psi) = A(\Psi)\Omega(\Psi)$. This shown that the solution satisfies

$$V(\xi,z) = \Psi\left(z - \frac{B(V)}{A(V)}\xi\right). \tag{5.42}$$

Applying this method, the solution of our first order partial differential equation (5.37) can be written in the form

$$V(\xi,z) = \Psi\left(z + \frac{(nmM)}{(1-(m/\rho)V)}\right). \tag{5.43}$$

From the initial condition (5.38) we can find the solution in the form

$$V(\xi,z) = M\left[\left[z + (nmM/(1-\frac{V}{\rho}))\xi\right] \middle/ \left[\rho - [z + (nmM/(1-\frac{V}{\rho}))\xi\right]\right], \tag{5.44}$$

which can be written as the quadratic equation

$$\left(1-\frac{z}{\rho}\right)V^2 - \left[\frac{M}{\rho}(z - n\rho\xi) + \frac{r}{m}\left(1-\frac{z}{\rho}\right)\right]V + \frac{\rho M}{\rho m}(z + nmM\xi) = 0. \tag{5.45}$$

This quadratic equation has the root

•EMBED Equation $$V(\xi,z)=\frac{\rho}{2(\rho-z)}\left[\frac{M}{\rho}(z-n\rho\xi)+\frac{\rho}{m}((\rho-z)/\rho)\right.$$

$$\left.-\left[\left(\frac{M}{\rho}(z+n\rho\xi)-\frac{\rho}{m}((\rho-z)/\rho)\right)^2-4M^2\frac{n\rho\xi}{\rho}\right]^{1/2}\right] \tag{5.46}$$

which gives V(ξ,z)=0 at ξ=0, z=0. Finally, because the quadratic equation (5.45) has a discriminant different from zero at the origin and in a neighbourhood of the origin where the root can be expanded into a convergent power series in ξ and z. Thus the convergence of the majorant series (5.33) and hence the convergence of the original series (5.17) is proved in a certain neighbourhood of the origin and the existence of an analytic solution of our Cauchy problem (5.15), (5.16) is completely established.

Chapter 6

Time-Dependent Series Solutions

In this chapter we will study the Taylor series expansion of our initial data which was used in the previous chapter. We will use these series as initial conditions to the equations (5.2), which are the Euler-Lagrange equations (2.5), (2.6). Next we will find the series solutions of this Cauchy problem near the origin which exist due to the Cauchy Kowalewskyi theorem.

6.1 Taylor Expansion For The Initial Data

In this section we will find series solutions of the functions $a(r)$, $f(r)$ and k(r) near the origin by using Taylor series expansion. To do this we investigate the series solution of the second order couple partial differential equations

$$r^2 f'' + rf' - 4f - \frac{\lambda}{2} r^2 f(f^2 - 1) - 4fa(a-2) = 0, \tag{6.1}$$

$$r^2 a'' - ra' - r^2 f^2 (a-1) = 0, \tag{6.2}$$

and the second order differential equations (3.16). To find the solution of the equations (6.1), (6.2) and (3.16), we will first investigate the series solution at the origin by using Taylor series of the form

$$f(r) = \sum_{n=2}^{\infty} f_n r^n, \qquad a(r) = \sum_{n=2}^{\infty} a_n r^n, \qquad k(r) = \sum_{n=-2}^{\infty} k_n r^n. \tag{6.3}$$

From ref. 19, we know that f and a start with r^2-terms and equation (3.16) shows that $k_{-1} = k_0 = k_1 = 0$.

If we substitute the Taylor series (6.3) into the equations (6.1), (6.2), (3.16), and solve it for the respective coefficients, we will find that the coefficients of the odd powers of r are equal to zero. Hence the the series solutions (6.3) can be written as

$$f(r) = \sum_{n=1}^{\infty} f_n r^{2n}, \qquad a(r) = \sum_{n=1}^{\infty} a_n r^{2n},$$
$$k(r) = \sum_{n=-1}^{\infty} k_n r^{2n}. \tag{6.4}$$

To prove that only even powers of r appear, let us substitute

$$f(r) = \sum_{n=1}^{N} f_n r^{2n} + \tilde{f}_n r^{2N+1} + \ldots, \tag{6.5}$$

$$a(r) = \sum_{n=1}^{N} a_n r^{2n} + \tilde{a}_n r^{2N+1} + \ldots, \tag{6.6}$$

and

$$k(r) = \sum_{n=-1}^{N} k_n r^{2n} + \tilde{k}_n r^{2N+1} + \ldots, \tag{6.7}$$

into the equations (6.1), (6.2) and (3 16) respectively. Comparing the coefficients of the r^{2N+1}-terms in (6.1) yields $(4N^2 + 4N - 3)\tilde{f} = 0$ which implies $\tilde{f} = 0$. The same arguments for equations (6.2) and (3.16) yield $\tilde{a} = \tilde{k} = 0$. In this way, we proved (6 4) by induction.

Next by substituting (6 4) into (6.1) and comparing the coefficients of r on both sides, we can evaluate the coefficients of the Taylor series for $f(r)$ in the form

$$f_n = \frac{-1}{4(n^2-1)} \left\{ \frac{\lambda}{2} \left[f_{n-1} - \sum_{\iota=2,m=1}^{n-2,\iota-1} f_m f_{\iota-m} f_{n-\iota-1} \right] - \right.$$
$$\left. 4 \sum_{\iota=2,m=1}^{n-1,\iota-1} f_m a_{\iota-m} a_{n-\iota} + 8 \sum_{m=1}^{n-1} f_m a_{n-m} \right\}, \qquad (n>3), \tag{6.8}$$

with, e.g., $f_2 = \frac{-\lambda}{24} f_1 - \frac{2}{3} f_1 a_1$, and $f_3 = \frac{1}{8} f_1 a_1^2 - \frac{1}{4} f_1 a_2 - \frac{1}{4} f_2 a_1 - \frac{\lambda}{64} f_2$.

Similarly by substituting (6.4) into (6.2) and again comparing the coefficients of r on both sides, we can evaluate the coefficients of the Taylor series for $a(r)$ in the form

$$a_n = \frac{1}{4n(n-1)}\left[\sum_{i=2,m=1}^{n-2,i-1} f_m f_{i-m} a_{n-i-1} - \sum_{m=1}^{n-2} f_m f_{n-m-1}\right], \qquad (n>3), \qquad (6.9)$$

with, e.g., $a_2 = 0$ and $a_3 = \frac{-\lambda f_1^2}{24}$. Finally we substitute (6.4) into (3.16) and compare the coefficients of r on both sides Thus we can evaluate the coefficients of the Taylor series for $k(r)$ in the form

$$k_n = \frac{-1}{4(n^2-1)}\left[\sum_{i=2,m=1}^{n,i-1} f_m f_{i-m} k_{n-i-1}\right] \qquad (n>1), \qquad (6.10)$$

with $k_0 = 0$.

We will prove by inductions that the inequalities

$$|f_k| \le \frac{M^k}{(k+1)^2}, \qquad (6.11)$$

$$|a_k| \le \frac{M^k}{(k+1)^2}, \qquad (6.12)$$

$$|k_k| \le \frac{M^k}{(k+1)^2}, \qquad (6.13)$$

hold for sufficiently large k and $M \ge 1$, which will establish the convergence of the Taylor series solution (6.4) near the origin. Using the inequality

$$\sum_{n_1=1}^{n-1} \frac{1}{(n_1+1)^2}\frac{1}{(n-n_1+1)^2} \le \int_{\frac{1}{2}}^{n-\frac{1}{2}} \frac{dx}{(x+1)^2(n-x+1)^2},$$

$$\le \frac{4}{(n+2)^2}\left[\frac{1}{3} - \frac{1}{2n+1} + \frac{1}{n+2}\ln\left(\frac{2n+1}{3}\right)\right]$$

$$\le \frac{1}{(n+2)^2}o(1), \qquad (6.14)$$

and taking the absolute value of (6.8), we can prove the inequalty (6 11) as follows,

$$|f_n| = \left|\frac{1}{4(n^2-1)}\right| \left|\left\{\frac{\lambda}{2}\left[\frac{M^{n-1}}{n^2} - \sum_{\iota=2,m=1}^{n-2,\iota-1} \frac{M^m}{(m+1)^2}\frac{M^{\iota-m}}{(\iota-m+1)^2}\frac{M^{n-\iota-1}}{(n-\iota)^2}\right] - \right.\right.$$

$$\left.\left. 4\sum_{\iota=2,m=1}^{n-1,\iota-1} \frac{M^m}{(m+1)^2}\frac{M^{\iota-m}}{(\iota-m+1)^2}\frac{M^{n-\iota}}{(n-\iota+1)^2} + 8\sum_{m=1}^{n-1}\frac{M^m}{(m+1)^2}\frac{M^{n-m}}{(n-m+1)^2}\right\}\right|$$

$$\leq \left|\frac{1}{4(n^2-1)}\right| \left|\left\{\frac{\lambda}{2}\left[\frac{M^{n-1}}{n^2} - \sum_{m=1}^{\iota-1}\frac{M^m}{(m+1)^2}\frac{M^{n-m}}{(n-m+1)^2}\right] + \right.\right.$$

$$\left.\left. \frac{8M^n}{(n+1)^2} - 4\sum_{m=1}^{\iota-1}\frac{M^m}{(m+1)^2}\frac{M^{n-m}}{(n-m+1)^2}\right\}\right|$$

$$\leq \left|\frac{1}{4(n^2-1)}\right| \left|\frac{\lambda}{2}\left[\frac{M^{n-1}}{n^2} - \frac{M^{n-1}}{(n+2)^2}\right] + \left[\frac{8M^n}{(n+1)^2} - \frac{4M^n}{(n+3)^2}\right]\right|$$

$$\leq \left|\frac{1}{4(n^2-1)}\right| \left|\frac{\lambda}{2}\left(\frac{M^{n-1}}{n^2}\right) + \frac{8M^n}{(n+1)^2}\right| \leq \frac{M^n}{(n+1)^2}. \tag{6.15}$$

Similarly by taking the absolute value of (6.9), we can prove the inequalty (6 12) as follows,

$$|a_n| = \left|\frac{1}{4n(n-1)}\right| \left|\sum_{\iota=2,m=1}^{n-2,\iota-1}\frac{M^m}{(m+1)^2}\frac{M^{\iota-m}}{(i-m+1)^2}\frac{M^{n-\iota-1}}{(n-i)^2} - \sum_{m=1}^{n-2}\frac{M^m}{(m+1)^2}\frac{M^{n-m-1}}{(n-m)^2}\right|$$

$$\leq \left|\frac{1}{4n(n-1)}\right| \left|-\frac{M^{n-1}}{(n+1)^2} + \sum_{m=1}^{\iota-1}\frac{M^m}{(m+1)^2}\frac{M^{n-m-1}}{(n-m+1)^2}\right|$$

$$\leq \left|\frac{1}{4n(n-1)}\right| \left|\frac{M^{n-1}}{(m+2)^2} - \frac{M^{n-1}}{(n+1)^2}\right|$$

$$\leq \left|\frac{1}{4n(n-1)}\right| \frac{M^n}{(m+2)^2}$$

$$\leq \frac{M^n}{(n+1)^2}. \tag{6 16}$$

Finally, by taking the absolute value of (6.10), we can prove the inequalty (6.13) as follows,

$$|k_n| = \left|\frac{1}{4(n^2-1)}\right| \left|\sum_{i=2,m=1}^{n,i-1} \frac{M^m}{(m+1)^2}\frac{M^{i-m}}{(i-m+1)^2}\frac{M^{n-i-1}}{(n-i)^2}\right|$$

$$\leq \left|\frac{1}{4(n^2-1)}\right| \left|\sum_{m=1}^{i-1}\frac{M^m}{(m+1)^2}\frac{M^{n-m-1}}{(n-m+1)^2}\right|$$

$$\leq \left|\frac{1}{4(n^2-1)}\right| \left|\frac{M^{n-1}}{(n+2)^2}\right|$$

$$\leq \frac{M^n}{(n+1)^2}. \tag{6.17}$$

From the series representation (6.4) of the functions f, a and k follows the analyticity of the initial data (5.3).

6.2 Local Series Solutions

In this section we use Mathematica to find the series solutions of the the time-dependent Euler-Lagrange equations (5.2) near the origin up to any order, and show that the solutions describe 90° scattering. Let us assume that the series are of the form

$$\begin{aligned}
u_1(t,\vec{x}) &= \sum_{i,j,p=0}^{\infty} u_1[i,j,p]x_1^i x_2^j t^p, \\
u_2(t,\vec{x}) &= \sum_{i,j,p=0}^{\infty} u_2[i,j,p]x_1^i x_2^j t^p, \\
u_3(t,\vec{x}) &= \sum_{i,j,p=0}^{\infty} u_3[i,j,p]x_1^i x_2^j t^p, \\
u_4(t,\vec{x}) &= \sum_{i,j,p=0}^{\infty} u_4[i,j,p]x_1^i x_2^j t^p, \\
u_5(t,\vec{x}) &= \sum_{i,j,p=0}^{\infty} u_5[i,j,p]x_1^i x_2^j t^p,
\end{aligned} \tag{6.18}$$

To evaluate the coefficients of these series, we substitute (6.18) into (5.2). Solving this equations for its coefficients yields

$$u_1[m,n,k+2]=\{(m+1)(m+2)u_1[m+2,n,k]+(n+1)(n+2)u_1[m,n+2,k]+$$

$$\sum_{i_1,j_1,p_1=0}^{m,n,k}\ \sum_{m_1,n_1,k_1=0}^{m-i_1,n-j_1,k-p_1}\big[u_1[i_1,j_1,p_1]*$$

$$u_5[m_1,n_1,k_1]u_5[m-i_1-m_1,n-j_1-n_1,k-p_1-k_1]\big]-$$

$$\sum_{i_1,j_1,p_1=0}^{m,n,k}\ \sum_{m_1,n_1,k_1=0}^{m-i_1,n-j_1,k-p_1}\big[u_1[i_1,j_1,p_1]*$$

$$u_3[m_1,n_1,k_1]u_3[m-i_1-m_1,n-j_1-n_1,k-p_1-k_1]\big]-$$

$$\sum_{i_1,j_1,p_1=0}^{m,n,k}\ \sum_{m_1,n_1,k_1=0}^{m-i_1,n-j_1,k-p_1}\big[u_1[i_1,j_1,p_1]*$$

$$u_4[m_1,n_1,k_1]u_4[m-i_1-m_1,n-j_1-n_1,k-p_1-k_1]\big]-$$

$$\sum_{m_1,n_1,k_1=0}^{m,n,k}\big[(k_1+1)u_5[m_1,n_1,k_1+1]u_1[m-m_1,n-n_1,k-k_1]\big]+$$

$$\sum_{m_1,n_1,k_1=0}^{m,n,k}\big[(m_1+1)u_3[m_1+1,n_1,k_1]u_2[m-m_1,n-n_1,k-k_1]\big]+$$

$$\sum_{m_1,n_1,k_1=0}^{m,n,k}\big[(n_1+1)u_4[m_1,n_1+1,k_1]u_2[m-m_1,n-n_1,k-k_1]\big]-$$

$$2\sum_{m_1,n_1,k_1=0}^{m,n,k}\big[(k_1+1)u_2[m_1,n_1,k_1+1]u_5[m-m_1,n-n_1,k-k_1]\big]+$$

$$2\sum_{m_1,n_1,k_1=0}^{m,n,k}\big[(m_1+1)u_2[m_1+1,n_1,k_1]u_3[m-m_1,n-n_1,k-k_1]\big]+$$

$$2\sum_{m_1,n_1,k_1=0}^{m,n,k}\big[(n_1+1)u_2[m_1,n_1+1,k_1]u_4[m-m_1,n-n_1,k-k_1]\big]+$$

$$\tfrac{\lambda}{2}u_1[m,n,k]-\tfrac{\lambda}{2}\sum_{i_1,j_1,p_1=0}^{m,n,k}\ \sum_{m_1,n_1,k_1=0}^{m-i_1,n-j_1,k-p_1}\big[u_1[i_1,j_1,p_1]*$$

$$u_1[m_1,n_1,k_1]u_1[m-i_1-m_1,n-j_1-n_1,k-p_1-k_1]\big]-$$

$$\frac{\lambda}{2}\sum_{i_1,j_1,p_1=0}^{m,n,k}\sum_{m_1,n_1,k_1=0}^{m-i_1,n-j_1,k-p_1}\big[u_1[i_1,j_1,p_1]u_2[m_1,n_1,k_1]*$$

$$u_2[m-i_1-m_1,n-j_1-n_1,k-p_1-k_1]\big]\big\}/(k+1)(k+2), \qquad (6.19)$$

$$u_2[m,n,k+2]=\big\{(m+1)(m+2)u_2[m+2,n,k]+(n+1)(n+2)u_2[m,n+2,k]+$$

$$\sum_{i_1,j_1,p_1=0}^{m,n,k}\sum_{m_1,n_1,k_1=0}^{m-i_1,n-j_1,k-p_1}\big[u_2[i_1,j_1,p_1]*$$

$$u_5[m_1,n_1,k_1]u_5[m-i_1-m_1,n-j_1-n_1,k-p_1-k_1]\big]-$$

$$\sum_{i_1,j_1,p_1=0}^{m,n,k}\sum_{m_1,n_1,k_1=0}^{m-i_1,n-j_1,k-p_1}\big[u_2[i_1,j_1,p_1]*$$

$$u_3[m_1,n_1,k_1]u_3[m-i_1-m_1,n-j_1-n_1,k-p_1-k_1]\big]-$$

$$\sum_{i_1,j_1,p_1=0}^{m,n,k}\sum_{m_1,n_1,k_1=0}^{m-i_1,n-j_1,k-p_1}\big[u_2[i_1,j_1,p_1]$$

$$u_4[m_1,n_1,k_1]u_4[m-i_1-m_1,n-j_1-n_1,k-p_1-k_1]\big]+$$

$$\sum_{m_1,n_1,k_1=0}^{m,n,k}\big[(m_1+1)u_5[m_1+1,n_1,k_1]u_1[m-m_1,n-n_1,k-k_1]\big]-$$

$$\sum_{m_1,n_1,k_1=0}^{m,n,k}\big[(m_1+1)u_3[m_1+1,n_1,k_1]u_1[m-m_1,n-n_1,k-k_1]\big]-$$

$$\sum_{m_1,n_1,k_1=0}^{m,n,k}\big[(n_1+1)u_4[m_1,n_1+1,k_1]u_1[m-m_1,n-n_1,k-k_1]\big]+$$

$$2\sum_{m_1,n_1,k_1=0}^{m,n,k}\big[(k_1+1)u_1[m_1,n_1,k_1+1]u_5[m-m_1,n-n_1,k-k_1]\big]-$$

$$2\sum_{m_1,n_1,k_1=0}^{m,n,k}\big[(m_1+1)u_1[m_1+1,n_1,k_1]u_3[m-m_1,n-n_1,k-k_1]\big]-$$

$$2\sum_{m_1,n_1,k_1=0}^{m,n,k}\left[(n_1+1)u_1[m_1,n_1+1,k_1]u_4[m-m_1,n-n_1,k-k_1]\right]+$$

$$\tfrac{\lambda}{2}u_2[m,n,k]-\tfrac{\lambda}{2}\sum_{i_1,j_1,p_1=0}^{m,n,k}\ \sum_{m_1,n_1,k_1=0}^{m-i_1,n-j_1,k-p_1}\left[u_2[i_1,j_1,p_1]*\right.$$

$$u_1[m_1,n_1,k_1]u_1[m-i_1-m_1,n-j_1-n_1,k-p_1-k_1]]-$$

$$\tfrac{\lambda}{2}\sum_{i_1,j_1,p_1=0}^{m,n,k}\ \sum_{m_1,n_1,k_1=0}^{m-i_1,n-j_1,k-p_1}\left[u_2[i_1,j_1,p_1]u_2[m_1,n_1,k_1]*\right.$$

$$u_2[m-i_1-m_1,n-j_1-n_1,k-p_1-k_1]]\}/(k+1)(k+2), \qquad (6.20)$$

$$u_3[m,n,k+2]=\{(n+1)(n+2)u_3[m,n+2,k]+(m+1)(k+1)u_5[m,n+1,k+1]-$$

$$(n+1)(m+1)u_4[m+1,n+1,k]+$$

$$\sum_{m_1,n_1,k_1=0}^{m,n,k}(m_1+1)u_2[m_1+1,n_1,k_1]u_1[m-m_1,n-n_1,k-k_1]-$$

$$\sum_{m_1,n_1,k_1=0}^{m,n,k}(m_1+1)u_1[m_1+1,n_1,k_1]u_2[m-m_1,n-n_1,k-k_1]-$$

$$\sum_{i_1,j_1,p_1=0}^{m,n,k}\ \sum_{m_1,n_1,k_1=0}^{m-i_1,n-j_1,k-p_1}u_3[i_1,j_1,p_1]u_1[m_1,n_1,k_1]*$$

$$u_1[m-i_1-m_1,n-j_1-n_1,k-p_1-k_1]-$$

$$\sum_{i_1,j_1,p_1=0}^{m,n,k}\ \sum_{m_1,n_1,k_1=0}^{m-i_1,n-j_1,k-p_1}u_3[i_1,j_1,p_1]u_2[m_1,n_1,k_1]*$$

$$u_2[m-i_1-m_1,n-j_1-n_1,k-p_1-k_1]\}/(k+1)(k+2), \qquad (6.21)$$

$$u_4[m,n,k+2]=\{(m+1)(m+2)u_4[m+2,n,k]+(n+1)(k+1)u_5[m,n+1,k+1]-$$

$$(n+1)(m+1)u_3[m+1,n+1,k]+$$

$$\sum_{m_1,n_1,k_1=0}^{m,n,k}(n_1+1)u_2[m_1,n_1+1,k_1]u_1[m-m_1,n-n_1,k-k_1]-$$

$$\sum_{m_1,n_1,k_1=0}^{m,n,k}(n_1+1)u_1[m_1,n_1+1,k_1]u_2[m-m_1,n-n_1,k-k_1]-$$

$$\sum_{i_1,j_1,p_1=0}^{m,n,k}\ \sum_{m_1,n_1,k_1=0}^{m-i_1,n-j_1,k-p_1}u_4[i_1,j_1,p_1]u_1[m_1,n_1,k_1]*$$

$$u_1[m-i_1-m_1,n-j_1-n_1,k-p_1-k_1]-$$

$$\sum_{i_1,j_1,p_1=0}^{m,n,k}\ \sum_{m_1,n_1,k_1=0}^{m-i_1,n-j_1,k-p_1}u_4[i_1,j_1,p_1]u_2[m_1,n_1,k_1]*$$

$$u_2[m-i_1-m_1,n-j_1-n_1,k-p_1-k_1]\}/(k+1)(k+2), \tag{6.22}$$

$$u_5[m,n,k+2]=((m+1)(k+1)u_3[m+1,n,k+1]+$$

$$(n+1)(k+1)u_4[m,n+1,k+1])/(k+1)(k+2) \tag{6.23}$$

To solve these equations recursively we can use the initial data to find $u_n[i,j,p]$ for $i,j=0,1,2,..;\ p=0,1$. By backsubstitution we can find the other unknown coefficients of the series (6.18) up to any order

We first evaluate the unknown coefficients, with i,j,p=0,1,2,3,4, of the series (6.18). With the help of Mathematica, the functions u_1,u_2,u_3,u_4 and u_5 can be easily evaluated. We find, up to order 4,

$$u_1=x_1^2f_1-\frac{\lambda x_1^4f_1}{24}-x_2^2f_1+\frac{\lambda x_2^4f_1}{24}-\frac{2t^4a_1f_1}{3}-4t^2x_1^2a_1f_1-$$

$$\frac{2x_1^4a_1f_1}{3}+\frac{2x_2^4a_1f_1}{3}+2tf_1k_{-1}+\frac{\lambda t^3f_1k_{-1}}{9}-\frac{8t^3a_1f_1k_{-1}}{9}$$

$$-\frac{4tx_1^2a_1f_1k_{-1}}{3}-\frac{4tx_2^2a_1f_1k_{-1}}{3}-\frac{\lambda tx_1^2f_1k_{-1}}{12}-\frac{\lambda tx_2^2f_1k_{-1}}{12}, \tag{6.24}$$

$$u_2=2x_1x_2f_1-\frac{\lambda x_1^3x_2f_1}{12}-4t^2x_1x_2a_1f_1-\frac{4x_1^3x_2a_1f_1}{3}-\frac{4x_1x_2^3a_1f_1}{3}-\frac{\lambda x_1x_2^3f_1}{12}, \tag{6.25}$$

$$u_3 = -2x_2 a_1 - \frac{8t^3 x_2 f_1^2 k_{-1}}{3} - tx_1^2 x_2 f_1^2 k_{-1} - tx_2^3 x_2 f_1^2 k_{-1} - 8tx_2 k_1, \tag{6 26}$$

$$u_4 = \frac{-8t^2 x_1 f_1^2 k_{-1}}{3} - tx_1^3 f_1^2 k_{-1} - tx_1 x_2^2 f_1^2 k_{-1} - 8tx_1 k_1, \tag{6 27}$$

$$u_5 = -8t^2 x_1 x_2 f_1^2 k_{-1}. \tag{6 28}$$

These solutions have, of course, the symmetries discussed in section 4.2

In the same way $|\Phi|^2$ can be calculated. Up to order 4, we find

$$|\Phi|^2 = x_1^4 f_1^2 + 2x_1^2 x_2^2 f_1^2 + x_2^4 f_1^2 + 4tx_1^2 f_1^2 k_{-1} - 4tx_2^2 f_1^2 k_{-1} + 4t^2 f_1^2 k_{-1}^2 +$$
$$\frac{4\lambda t^4 f_1^2 k_{-1}^2}{9} - \frac{\lambda t^2 x_1^2 f_1^2 k_{-1}^2}{3} - \frac{\lambda t^2 x_2^2 f_1^2 k_{-1}^2}{3} - \frac{32t^4 a_1 f_1^2 k_{-1}^2}{9} -$$
$$\frac{16t^2 x_1^2 a_1 f_1^2 k_{-1}^2}{3} - \frac{16t^2 x_2^2 a_1 f_1^2 k_{-1}^2}{3}. \tag{6.29}$$

In addition to the symmetries of section 4.2, $|\Phi|^2$ is also invariant under the transformation $(t, x_1, x_2) \to (-t, x_2, -x_1)$.

Let us summarize the results we can obtain by considering the symmetries of the solution whose global existence was proven in Section 4.1: First, if by using functions like $|\Phi|^2$, F_{ij}^2, or ε, there is a way of defining the positions $(x_1^a(t), x_2^a(t))$, a=1,2, of exactly two separate vortices, these two position must lie either on the x_1-axis or the x_2-axis with equal distance from the origin (We will use the zeros of $|\Phi|^2$ to define these positions) Any vortex which does not lie on either axis immediately leads to three other vortices because of the left-right and up-down symmetry of our solution. Since our solution is continuous, these positions will change continuously such that at t=0 the two positions coincide, and after the collision the vortices move again on either the x_1-axis or x_2-axis. This only allows for $0°, 90°$, or $180°$ scattering. Any approximate solution can clearly distinguish between these three cases. We have calculated the analytic solution near the origin, which exists according to the Cauchy-Kowalewskyi theorem, and found a further symmetry. This symmetry tells us that $|\Phi|^2$ looks the same at times $\pm t$ before and after the collision if together with the transformation $\pm t \to \mp t$ we exchange the x_1 and the x_2-axis. This means we have $90°$ scattering.

All that is left to show is that there is in fact a way of defining the positions of vortices and that there are actually exactly two vortices before and after the collision Of course, we can use the zeros of $|\Phi|^2$ for our definition since we know that superconductivity is destroyed at the center of a vortex For approximate solutions we would look for the minima of $|\Phi|^2$. In our case the easiest way of finding these is to sum the time-independent terms and the linear terms in Φ, which are given by the initial data alone. This leads to the expression

$$|\Phi|^2 = f^2(1+4t\cos 2\theta + 4t^2k^2), \tag{6.30}$$

from ref. 18 For $t \neq 0$, expression (6.30) has exactly two zeros, namely at $r=\rho$, $\theta=\pi/2$ and $\theta=3\pi/2$ for $t>0$, and at $r=\rho$, $\theta=0$ and $\theta=\pi$ for $t<0$ Here ρ is the point where $k(\rho)=1/(2|t|)$. This complete our analysis.

Chapter 7

Conclusions

Our aim was to discuss vortex-vortex scattering in a mathematically rigorous way on the level of the Ginzburg-Landau equations. Guided by various results obtained previously, we have formulated a Cauchy problem with Cauchy data which describe two flux quanta both sitting at the origin. For this problem we have proven the following: First, a unique global finite-energy solution exists. This is the minimal requirement that for our data there is a solution for $-\infty < t < \infty$. The existence proof also shows how to construct the solution as a limit of a sequence. Using the symmetry of the initial data we can obtain our second main result which is a left-right and up-down symmetry, in particular, of the energy density and of $|\Phi|^2$. This rules out all cases other than 0°, 90° or 180° scattering of two vortices. Third, we have shown that a local solution exists near the origin, and have used this solution to establish 90° scattering. Our arguments depend on the fact that, using $|\Phi|^2$, we can define the location of the vortices and find that there are exactly two for $t \neq 0$.

Appendix A

Program Listings

Step[0]

```
Clear[a,f,k,g,z,d,x1,x2,t]
Clear[s1,s2,m,n,q,i1,j1,p1,m1,n1,q1,i,j,p,u1,u2,c0,c1,c2]
Clear[A1,A2,U,U1,U2,U3,U4,U5,Ut1,Ut2,At1,At2]
Clear[O1,O2,O3,O4,O5,O6,O7,O8,O9,O10,O11,O12,O13,O14,
                  O15,O16,O17,O18,O19,O20,O21,O22,O23]
```

Step[1]

```
f[1]:=f[1]
f[2]:=Expand[-(g/24)*f[1]-(2/3)f[1]*a[1]]
f[3]:=Expand[(1/8)*f[1]*(a[1])^2-(1/4)*f[1]*a[2]-(1/4)*f[2]*a[1]-
            (g/64)*f[2]]
f[n_Integer]:=Expand[(-1/(4(n^2-1)))*(g/2)*(f[n-1]-
            Sum[f[m]*f[i-m]*f[n-i-1],{i,2,n-2},{m,1,i-1}])+
            (1/(4(n^2-1)))*(4*Sum[f[m]*a[i-m]*a[n-i],{i,2,n-1},{m,1,i-1}]-
            8*Sum[f[m]*a[n-m],{m,1,n-1}])]             /,(n>3)

a[1]:=a[1]
a[2]:=0
a[3]:=Expand[(-1/24)*((f[1])^2)]
```

```
a[n_Integer]:=Expand[(1/(4n(n-1)))*(Sum[f[m]*f[i-m]*a[n-i-1],{i,2,n-2},
            {m,1,i-1}]-Sum[f[m]*f[n-m-1],{m,1,n-2}])]     /;(n>3)

k[-1]:=k[-1]
k[0]:=0
k[1]:=k[1]
k[n_Integer]:=Expand[(1/(4(n^2-1)))*Sum[Sum[f[i]*f[n-i-j-1]*k[j],
            {i,1,n-j-2}],{j,-1,n-3}]]                      /;(n>1)
```

Step[2]

[A]

```
O1[x1_,x2_,0_]=(((x1)^2-(x2)^2)/((x1)^2+(x2)^2))*
               Sum[f[i]*((x1)^2+(x2)^2)^i,{i,1,8}];
O2[x1_,x2_,0_]=Together[%],
U1[x1_,x2_,0_]=Expand[%],
O3[x1_,x2_,0_]=((2 x1 x2)/((x1)^2+(x2)^2))*
               Sum[f[i]*((x1)^2+(x2)^2)^i,{i,1,8}];
O4[x1_,x2_,0_]=Together[%],
U2[x1_,x2_,0_]=ExpandAll[%];
O5[x1_,x2_,0_]=2*(Sum[f[i]*((x1)^2+(x2)^2)^i,{i,1,8}]*
               (Sum[k[i]*((x1)^2+(x2)^2)^i,{i,1,8}]+
               k[-1]*((x1)^2+(x2)^2)^(-1))),
O6[x1_,x2_,0_]=Together[%],
Ut1[x1_,x2_,0_]=ExpandAll[%];
Ut2[x1_,x2_,0_]=0;
O7[x1_,x2_,0_]=-2*x2*Sum[a[i]*((x1)^2+(x2)^2)^(i-1),{i,1,8}];
O8[x1_,x2_,0_]=Together[%];
A1[x1_,x2_,0_]=Expand[%];
O9[x1_,x2_,0_]=2*x1*Sum[a[i]((x1)^2+(x2)^2)^(i-1),{i,1,8}],
O10[x1_,x2_,0_]=Together[%];
A2[x1_,x2_,0_]=Expand[%];
O11[x1_,x2_,0_]=-4*x2*Sum[(i+1)*k[i]*((x1)^2+(x2)^2)^(i-1),{i,1,8}],
O12[x1_,x2_,0_]=Together[%];
At1[x1_,x2_,0_]=Expand[%],
O13[x1_,x2_,0_]=-4*x1*Sum[(i+1)*k[i]*((x1)^2+(x2)^2)^(i-1),{i,1,8}],
```

```
O14[x1_,x2_,0_]=Together[%];
At2[x1_,x2_,0_]=Expand[%];
```

[B]

```
u1[0,0,0]=Coefficient[z*U1[x1,x2,0],z]/. x1->0 / x2->0,
Do[If[i+j>0,u1[i,j,0]=Coefficient[U1[x1,x2,0],(x1)^i(x2)^j]
                    /.x1->0 /.x2->0,] ,{i,0,16},{j,0,16}],
u2[0,0,0]=Coefficient[z*U2[x1,x2,0],z]/. x1->0 /. x2->0;
Do[If[i+j>0,u2[i,j,0]=Coefficient[U2[x1,x2,0],(x1)^i(x2)^j]
                    /.x1->0 /.x2->0,]  ,{i,0,16},{j,0,16}],
u1[0,0,1]=Coefficient[z*Ut1[x1,x2,0],z]/. x1->0 / x2->0;
Do[If[i+j>0,u1[i,j,1]=Coefficient[Ut1[x1,x2,0],(x1)^i(x2)^j]
                    /. x1->0 /.x2->0 ,] ,{i,0,16},{j,0,16}];
u2[0,0,1]=Coefficient[z*Ut2[x1,x2,0],z]/. x1->0 /. x2->0;
Do[If[i+j>0,u2[i,j,1]=Coefficient[Ut2[x1,x2,0],(x1)^i(x2)^j]
                    /.x1->0 /.x2->0 ,] ,{i,0,16},{j,0,16}],
c1[0,0,0]=Coefficient[z*A1[x1,x2,0],z]/. x1->0 /. x2->0;
Do[If[i+j>0,c1[i,j,0]=Coefficient[A1[x1,x2,0],(x1)^i(x2)^j]
                    /.x1->0 /.x2->0  ,],{i,0,16},{j,0,16}];
c2[0,0,0]=Coefficient[z*A2[x1,x2,0],z]/. x1->0 /. x2->0;
Do[If[i+j>0,c2[i,j,0]=Coefficient[A2[x1,x2,0],(x1)^i(x2)^j]
                    /.x1->0 /.x2->0  ,],{i,0,16},{j,0,16}];
c1[0,0,1]=Coefficient[z*At1[x1,x2,0],z]/. x1->0 /. x2->0,
Do[If[i+j>0,c1[i,j,1]=Coefficient[At1[x1,x2,0],(x1)^i(x2)^j]
                    /. x1->0 /.x2->0 ,] ,{i,0,16},{j,0,16}],
c2[0,0,1]=Coefficient[z*At1[x1,x2,0],z]/. x1->0 /. x2->0,
Do[If[i+j>0,c2[i,j,1]=Coefficient[At2[x1,x2,0],(x1)^i(x2)^j]
                    /.x1->0 /.x2->0 ,] ,{i,0,16},{j,0,16}];
Do[c0[i,j,p]=0 ,{i,0,16},{j,0,16},{p,0,1}];
```

Step[3]

[A]

```
Do[Do[
u1[m,n,q+2]=(1/((q+1)(q+2)))*((m+1)*(m+2)*u1[m+2,n,q]+
        (n+1)*(n+2)*u1[m,n+2,q]+
```

```
        Expand[Sum[u1[i1,j1,p1]*c0[m1,n1,q1]*c0[m-i1-m1,n-j1-n1,q-p1-q1],
        {i1,0,m},{j1,0,n},{p1,0,q},{m1,0,m-i1},{n1,0,n-j1},{q1,0,q-p1}]]-
        Expand[Sum[u1[i1,j1,p1]*c1[m1,n1,q1]*c1[m-i1-m1,n-j1-n1,q-p1-q1],
        {i1,0,m},{j1,0,n},{p1,0,q},{m1,0,m-i1},{n1,0,n-j1},{q1,0,q-p1}]]-
        Expand[Sum[u1[i1,j1,p1]*c2[m1,n1,q1]*
        c2[m-i1-m1,n-j1-n1,q-p1-q1],{i1,0,m},{j1,0,n},{p1,0,q},
        {m1,0,m-i1},{n1,0,n-j1},{q1,0,q-p1}]]-
        Expand[Sum[(q1+1)*c0[m1,n1,q1+1]*u2[m-m1,n-n1,q-q1],
        {m1,0,m},{n1,0,n},{q1,0,q}]]+
        Expand[Sum[(m1+1)*c1[m1+1,n1,q1]*u2[m-m1,n-n1,q-q1],
        {m1,0,m},{n1,0,n},{q1,0,q}]]+
        Expand[Sum[(n1+1)*c2[m1,n1+1,q1]*u2[m-m1,n-n1,q-q1],
        {m1,0,m},{n1,0,n},{q1,0,q}]]-
        Expand[2*Sum[(q1+1)*u2[m1,n1,q1+1]*c0[m-m1,n-n1,q-q1],
        {m1,0,m},{n1,0,n},{q1,0,q}]]+
        Expand[2*Sum[(m1+1)*u2[m1+1,n1,q1]*c1[m-m1,n-n1,q-q1],
        {m1,0,m},{n1,0,n},{q1,0,q}]]+
        Expand[2*Sum[(n1+1)*u2[m1,n1+1,q1]*c2[m-m1,n-n1,q-q1],
        {m1,0,m},{n1,0,n},{q1,0,q}]]+Expand[(g/2)*u1[m,n,q]]-
        Expand[(g/2)*Sum[u1[i1,j1,p1]*u1[m1,n1,q1]*
        u1[m-i1-m1,n-j1-n1,q-p1-q1],{i1,0,m},{j1,0,n},{p1,0,q},
        {m1,0,m-i1},{n1,0,n-j1},{q1,0,q-p1}]]-
        Expand[(g/2)*Sum[u1[i1,j1,p1]*u2[m1,n1,q1]*
        u2[m-i1-m1,n-j1-n1,q-p1-q1],{i1,0,m},{j1,0,n},{p1,0,q},
        {m1,0,m-i1},{n1,0,n-j1},{q1,0,q-p1}]]);

u2[m,n,q+2]=(1/((q+1)(q+2)))*((m+1)*(m+2)*u2[m+2,n,q]+
        (n+1)*(n+2)*u2[m,n+2,q]+
        Expand[ Sum[u2[i1,j1,p1]*c0[m1,n1,q1]*
        c0[m-i1-m1,n-j1-n1,q-p1-q1],{i1,0,m},{j1,0,n},{p1,0,q},
        {m1,0,m-i1},{n1,0,n-j1},{q1,0,q-p1}]]-
        Expand[ Sum[u2[i1,j1,p1]*c1[m1,n1,q1]*
        c1[m-i1-m1,n-j1-n1,q-p1-q1],{i1,0,m},{j1,0,n},{p1,0,q},
        {m1,0,m-i1},{n1,0,n-j1},{q1,0,q-p1}]]-
        Expand[Sum[u2[i1,j1,p1]*c2[m1,n1,q1]*
        c2[m-i1-m1,n-j1-n1,q-p1-q1],{i1,0,m},{j1,0,n},{p1,0,q},
        {m1,0,m-i1},{n1,0,n-j1},{q1,0,q-p1}]]+
        Expand[Sum[(q1+1)*c0[m1,n1,q1+1]*u1[m-m1,n-n1,q-q1],
```

```
{m1,0,m},{n1,0,n},{q1,0,q}]]-
Expand[Sum[(m1+1)*c1[m1+1,n1,q1]*u1[m-m1,n-n1,q-q1],
{m1,0,m},{n1,0,n},{q1,0,q}]]-
Expand[Sum[(n1+1)*c2[m1,n1+1,q1]*u1[m-m1,n-n1,q-q1],
{m1,0,m},{n1,0,n},{q1,0,q}]]+
Expand[2*Sum[(q1+1)*u1[m1,n1,q1+1]*c0[m-m1,n-n1,q-q1],
{m1,0,m},{n1,0,n},{q1,0,q}]]-
Expand[2*Sum[(m1+1)*u1[m1+1,n1,q1]*c1[m-m1,n-n1,q-q1],
{m1,0,m},{n1,0,n},{q1,0,q}]]-
Expand[2*Sum[(n1+1)*u1[m1,n1+1,q1]*c2[m-m1,n-n1,q-q1],
{m1,0,m},{n1,0,n},{q1,0,q}]]+
Expand[(g/2)*u2[m,n,q]]-
Expand[(g/2)*Sum[u2[i1,j1,p1]*u1[m1,n1,q1]*
u1[m-i1-m1,n-j1-n1,q-p1-q1],{i1,0,m},{j1,0,n},{p1,0,q},
{m1,0,m-i1},{n1,0,n-j1},{q1,0,q-p1}]]-
Expand[(g/2)*Sum[u2[i1,j1,p1]*u2[m1,n1,q1]*
u2[m-i1-m1,n-j1-n1,q-p1-q1],{i1,0,m},{j1,0,n},{p1,0,q},
{m1,0,m-i1},{n1,0,n-j1},{q1,0,q-p1}]]);
```

```
c1[m,n,q+2]=(1/((q+1)(q+2)))*((n+1)*(n+2)*c1[m,n+2,q]-
    (n+1)*(m+1)*c2[m+1,n+1,q]+(m+1)*(q+1)*c0[m+1,n,q+1]+
    Expand[Sum[(m1+1)*u2[m1+1,n1,q1]*u1[m-m1,n-n1,q-q1],
    {m1,0,m},{n1,0,n},{q1,0,q}]]-
    Expand[Sum[(m1+1)*u1[m1+1,n1,q1]*u2[m-m1,n-n1,q-q1],
    {m1,0,m},{n1,0,n},{q1,0,q}]]-
    Expand[Sum[c1[i1,j1,p1]*u1[m1,n1,q1]*
    u1[m-i1-m1,n-j1-n1,q-p1-q1],{i1,0,m},{j1,0,n},{p1,0,q},
    {m1,0,m-i1},{n1,0,n-j1},{q1,0,q-p1}]]-
    Expand[Sum[c1[i1,j1,p1]*u2[m1,n1,q1]*
    u2[m-i1-m1,n-j1-n1,q-p1-q1],{i1,0,m},{j1,0,n},{p1,0,q},
    {m1,0,m-i1},{n1,0,n-j1},{q1,0,q-p1}]]),

c2[m,n,q+2]=(1/((q+1)(q+2)))*((m+1)*(m+2)*c2[m+2,n,q]-
    (n+1)*(m+1)*c1[m+1,n+1,q]+(q+1)*(n+1)*c0[m,n+1,q+1]+
    Expand[Sum[(n1+1)*u2[m1,n1+1,q1]*u1[m-m1,n-n1,q-q1],
    {m1,0,m},{n1,0,n},{q1,0,q}]]-
    Expand[Sum[(n1+1)*u1[m1,n1+1,q1]*u2[m-m1,n-n1,q-q1],
    {m1,0,m},{n1,0,n},{q1,0,q}]]-
```

```
Expand[Sum[c2[i1,j1,p1]*u1[m1,n1,q1]*
u1[m-i1-m1,n-j1-n1,q-p1-q1],{i1,0,m},{j1,0,n},{p1,0,q},
{m1,0,m-i1},{n1,0,n-j1},{q1,0,q-p1}]]-
Expand[Sum[c2[i1,j1,p1]*u2[m1,n1,q1]*
u2[m-i1-m1,n-j1-n1,q-p1-q1],{i1,0,m},{j1,0,n},{p1,0,q},
{m1,0,m-i1},{n1,0,n-j1},{q1,0,q-p1}]]);
```

```
c0[m,n,q+2]=Expand[(1/(q+1)*(q+2))*((m+1)*(q+1)*c1[m+1,n,q+1]+
    (n+1)*(q+1)*c2[m,n+1,q+1])],
    {m,0,6-q},{n,0,6-q}],{q,0,2}]
```

[B]

```
Do[If[(i+j+p)>4,u1[i,j,p]=u2[i,j,p]=0],{i,0,4},{j,0,4},{p,0,4}];
Do[If[(i+j+p)>4,c0[i,j,p]=c1[i,j,p]=c2[i,j,p]=0],{i,0,4},{j,0,4},{p,0,4}],
O15=Sum[u1[i,j,p]*(x1)^i(x2)^j(t)^p,{i,0,4},{j,0,4},{p,0,4}],
U1[x1_,x2_,t_]=Expand[%];
O16=Sum[u2[i,j,p]*(x1)^i(x2)^j(t)^p,{i,0,4},{j,0,4},{p,0,4}];
U2[x1_,x2_,t_]=Expand[%],
O17=Sum[c1[i,j,p]*(x1)^i(x2)^j(t)^p,{i,0,4},{j,0,4},{p,0,4}];
U3[x1_,x2_,t_]=Expand[%];
O18=Sum[c2[i,j,p]*(x1)^i(x2)^j(t)^p,{i,0,4},{j,0,4},{p,0,4}],
U4[x1_,x2_,t_]=Expand[%];
O19=Sum[c0[i,j,p]*(x1)^i(x2)^j(t)^p,{i,0,4},{j,0,4},{p,0,4}];
U5[x1_,x2_,t_]=Expand[%];
```

Step[4]

```
Length[U1[x1,x2,t]-U1[-x1,x2,t]];
Length[U1[x1,x2,t]-U1[x1,-x2,t]];
Length[U2[x1,x2,t]+U2[-x1,x2,t]];
Length[U2[x1,x2,t]+U2[x1,-x2,t]];
Length[U3[x1,x2,t]-U3[-x1,x2,t]],
Length[U3[x1,x2,t]+U3[x1,-x2,t]];
Length[U4[x1,x2,t]+U4[-x1,x2,t]];
Length[U4[x1,x2,t]-U4[x1,-x2,t]];
Length[U5[x1,x2,t]+U5[-x1,x2,t]];
Length[U5[x1,x2,t]+U5[x1,-x2,t]],
```

Step[5]

```
O20=Expand[(U1[x1,x2,t])^2],
O21=Expand[(U2[x1,x2,t])^2];
O22=Expand[O20+O21];
O23=Expand[%];
d[0,0,0]=Coefficient[z*O23,z]/. x1->0 /. x2->0  /.t->0,
Do[If[(i+j+p)>0,d[i,j,p]=Coefficient[O23,
(x1)^i(x2)^j(t)^p]/.x1->0 /.x2->0 /.t->0,] ,{i,0,4},{j,0,4},{p,0,4}];
Do[If[(i+j+p)>4,d[i,j,p]=0],{i,0,4},{j,0,4},{p,0,4}],
U[x1_,x2_,t_]=Sum[d[i,j,p]*(x1)^i(x2)^j(t)^p,{i,0,4},{j,0,4},{p,0,4}],
Length[U[x1,x2,t]-U[x2,-x1,-t]],
Length[U[x1,x2,t]-U[-x1,x2,t]],
Length[U[x1,x2,t]-U[x1,-x2,t]];
```

Step[6]

```
Print["U1[x1,x2,t]=",U1[x1,x2,t]];
Print["U2[x1,x2,t]=",U2[x1,x2,t]],
Print["U3[x1,x2,t]=",U3[x1,x2,t]];
Print["U4[x1,x2,t]=",U4[x1,x2,t]];
Print["U5[x1,x2,t]=",U5[x1,x2,t]];
Print["U[x1,x2,t]=",U[x1,x2,t]];
```

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