SINGULARLY PERTURBED PROBLEMS MODELLING REACTION-CONVECTION-DIFFUSION PROCESSES

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy in Applied Mathematical Sciences is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

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To Mum and Dad

Contents

Abstract

In this thesis, parameter-uniform numerical methods for certain classes of singularly perturbed differential equations with two small parameters are studied We initially consider a class of two-parameter ordinary differential equations Parameter explicit bounds on the solution and its derivatives are derived The solution is decomposed into a sum of regular and singular components and based on this decomposition we construct a numerical algorithm consisting of an upwind finite difference operator and an appropriately chosen piecewise-uniform mesh Parameter-uniform convergence of the numerical approximations is established Some numerical results are given to illustrate this convergence

Two-parameter parabolic and elliptic partial differential equations are considered We derive parameter explicit bounds on the solutions and their derivatives for both problems, these bounds are analogous to those obtained for the ordinary differential equation The solutions are decomposed into a sum of regular and singular components but for both problems this decomposition differs from that for the ordinary differential equation In both cases a numerical algorithm based on an upwind finite difference operator and an appropriate piecewise-uniform mesh is constructed In the case of the parabolic problem, parameter-uniform error bounds for the numerical approximations are established and numerical results illustrating this convergence are given With the elliptic problem, we show that, given certain assumptions and conjectures our numerical method is parameter-uniform

List of Tables

List of Figures

Chapter 1

Introduction

1 1 Introduction to numerical methods for singularly perturbed differential equations

Singularly perturbed differential equations arise in many areas of applied mathematics They commonly appear in fluid dynamics, modelling of semiconductor devices and financial modelling (see Morton [18]) Such differential equations typically involve a small positive parameter ε $(0 < \varepsilon \leq 1)$ multiplying the highest order derivative, and their solutions exhibit layers as ε tends to zero

We are concerned with parameter-uniform numerical methods for singulary perturbed differential equations By parameter-uniform, we mean that the numerical approximations converge to the solution of the problem independently of the small parameter More exactly (see for example [16]),

Definition 1 1 1 *Suppose u_ε is the solution to a problem that is parameterized by a singular perturbation parameter ε where $0 < \varepsilon \leq 1$ We approximate u_ε by a sequence of numerical solutions $\{(U_\varepsilon, \bar{\Omega}^N)\}_{N=1}^{\infty}$ where U_ε is defined on the mesh Ω^N and N is a discretization parameter This sequence of functions $\{(U_\varepsilon, \bar{\Omega}^N)\}_{N=1}^{\infty}$ is said to converge ε-uniformly (of order p) to the exact solution u_ε if there exists N_0, C, and p all independent of ε, such that for all $N \geq N_0$,*

$$\sup_{0<\varepsilon\leq 1} ||U_\varepsilon - u_\varepsilon||_{\bar{\Omega}^N} \leq CN^{-p},$$

where N_0, C and p are all positive numbers with N_0 an integer

This thesis is concerned with the method of finite differences Within the area of finite differences, there are two main approaches to generate parameter-uniform numerical methods for singularly perturbed problems Firstly there are fitted operator methods, where, as the name suggests, the operator is fitted to resolve the singularity and therefore capture the layer behaviour Such operators are usually combined with uniform meshes Secondly there are fitted mesh methods where standard finite difference operators are applied on a mesh that has been fitted to resolve the layer We are concerned with this latter class of numerical methods

We must fit our mesh to resolve the layers [3] After a uniform mesh, the next simplest mesh to consider is a piecewise-uniform mesh In [29], Shishkin showed that such a fitted mesh was sufficient to obtain a parameter-uniform numerical method for many linear partial differential equations One of the main advantages of using these *Shishkin meshes* is that results obtained in one-dimension can be extended to higher dimensions more easily then with other approaches When working with such methods, the location and width of the boundary layers must be known *a priori*

The choice of norm to use is especially important when analysing the error in the numerical approximations for problems that exhibit layers For a discussion and a comparison of the various norms that one might consider using when undertaking such analysis see [3, 16] The conclusion reached is that in order to capture correctly boundary layer functions, the appropriate norm to use is the L_∞-norm (maximum pointwise norm)

1 2 Types of singularly perturbed problems

We now examine some examples of singularly perturbed differential equations Consider the following two classes of singularly perturbed ordinary differential equations (ODEs),

- One-dimensional convection-diffusion problem

$$\begin{aligned} \varepsilon y'' + ay' - by = f, \qquad &\text{on} \qquad \Omega = (0,1), \\ y(0) = \gamma_0, \qquad & y(1) = \gamma_1, \\ a \geq \alpha > 0, \quad b \geq \beta > 0, \quad & 0 < \varepsilon \leq 1 \end{aligned} \tag{1 2 1}$$

- One-dimensional reaction-diffusion problem

$$\begin{aligned} \varepsilon y'' - by = f, \qquad &\text{on} \qquad \Omega = (0,1), \\ y(0) = \gamma_0, \qquad & y(1) = \gamma_1 \\ b \geq \beta > 0, \qquad & 0 < \varepsilon \leq 1 \end{aligned} \tag{1 2 2}$$

Solutions of (1 2 1) typically exhibit boundary layers with width of order ε in the neighbourhood of $x = 0$ Solutions of the reaction-diffusion problem (1 2 2) exhibit layers of width of order $\sqrt{\varepsilon}$ in the neighbourhood of both $x = 0$ and $x = 1$ There is much literature already available for various methods to find the numerical solution of both of these singularly perturbed ODEs [3, 25] Fitted operator methods based on exponentially fitted finite difference operators have been developed for both problems [16, 25] Parameter-uniform numerical methods composed of finite difference operators and Shishkin meshes have also been established (see [3, 8, 16, 25, 29] and the survey articles [11, 24]) Using standard finite difference operators, it has been shown [3, 17, 27] that the error in the numerical approximations to the solution of (1 2 1) is of order $CN^{-1}\ln N$ and the error in approximating (1 2 2) is of the form

$$\|u - U^N\|_{\Omega^N} \leq C(N^{-1}\ln N)^2$$

Higher order methods also exist for these problems, see for example [4, 27, 32, 33]

We now introduce a dependence on time Consider the parabolic versions of the above problems,

- Parabolic convection-diffusion

$$\begin{gathered} \varepsilon u_{xx} + au_x - bu - du_t = f, \qquad \text{on} \qquad G = (0,1) \times (0,T], \\ u(0,t) = \gamma_0(t), \qquad u(1,t) = \gamma_1(t), \\ u(x,0) = \phi(x), \\ a \geq \alpha > 0, \quad b \geq \beta > 0, \quad d \geq \delta > 0, \quad 0 < \varepsilon \leq 1 \end{gathered} \tag{1 2 3}$$

- Parabolic reaction-diffusion

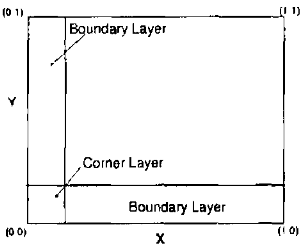
$$\begin{gathered} \varepsilon u_{xx} - bu - du_t = f, \qquad \text{on} \qquad G = (0,1) \times (0,T], \\ u(0,t) = \gamma_0(t), \qquad u(1,t) = \gamma_1(t), \\ u(x,0) = \phi(x), \\ b \geq \beta > 0, \quad d \geq \delta > 0, \quad 0 < \varepsilon \leq 1 \end{gathered} \tag{1 2 4}$$

Problem (1 2 3) typically exhibits layers in the neighbourhood of the edge $x = 0$ Solutions to (1 2 4) exhibit layers in the neighbourhood of both $x = 0$ and $x = 1$ Numerical methods for equation (1 2 3) have been considered in [8, 25, 29, 31] The reaction-diffusion problem (1 2 4) has been analysed in [17, 29]

For the convection-diffusion type problem (1 2 3), fitted operator methods were derived in [31] However, Shishkin [28] established that in order to obtain a parameter-uniform numerical method, it is necessary to fit the mesh when parabolic boundary layers are present This implies that we cannot use fitted operators on a uniform mesh to obtain parameter-uniform convergence in the case of (1 2 4) Parameter-uniform numerical methods consisting of standard finite difference operators and piecewise-uniform meshes [8, 25, 29] have been established for both (1 2 3) and (1 2 4)

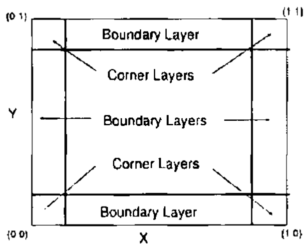
The final classes of singularly perturbed differential equations we will examine in this section are the two-dimensional versions of problems (1 2 1) and (1 2 2),

- Elliptic convection-diffusion



$$\varepsilon \triangle u + a \quad \nabla u - bu = f, \quad \text{on} \quad \Omega = (0,1)^2, \qquad (1\ 2\ 5)$$
$$u(x,0) = \gamma_0(x), \qquad u(x,1) = \gamma_1(x),$$
$$u(0,y) = \gamma_2(y), \qquad u(1,y) = \gamma_3(y),$$
$$a \geq \alpha > 0, \quad b \geq 2\beta > 0, \quad 0 < \varepsilon \leq 1$$

- Elliptic reaction-diffusion



$$\varepsilon \triangle u - bu = f, \quad \text{on} \quad \Omega = (0,1)^2, \qquad (1\ 2\ 6)$$
$$u(x,0) = \gamma_0(x), \qquad u(x,1) = \gamma_1(x),$$
$$u(0,y) = \gamma_2(y), \qquad u(1,y) = \gamma_3(y),$$
$$b \geq 2\beta > 0, \qquad 0 < \varepsilon \leq 1$$

Numerical methods for such problems have been considered in the books [3, 16, 25, 29] The analysis for such equations poses compatibility issues not encountered with the ODE or parabolic PDE

Linß and Stynes [14] analyse *Shishkin-type* decompositions for (1 2 5) Using such decompositions they obtain sharp bounds on the solution u of (1 2 5) and its derivatives The same authors consider a first-order convergent parameter-uniform numerical method for this problem in [13] The authors use a special difference scheme on a *Shishkin mesh*, the theoretical results in [14] are essential to showing convergence of this method The article [15] contains a comparison of the performance of several different numerical methods on Shishkin meshes for problem (1 2 5) In [10] numerical methods for (1 2 5) are considered on *modified Shishkin meshes* A parameter-uniform second-order finite difference scheme for the reaction-diffusion problem (1 2 6) is discussed in [2] The book [29], is concerned with parameter-uniform numerical methods on *Shishkin meshes* for linear differential equations The classes of problems considered in this book are vast and include both (1 2 5) and (1 2 6) The more complicated *N-dimension* versions of these problems are also examined

1 3 Two-parameter differential equations

The differential equations in the last section can be though of as one-parameter problems as they depend on the small positive parameter ε multiplying the highest order derivative We now introduce a second parameter μ multiplying the convective term Such equations are therefore known as two-parameter problems This thesis is concerned with numerical methods for a certain class of two-parameter differential equations This class of differential equations includes both the convection-diffusion and reaction-diffusion type problems described in the previous section and it also covers the transition from reaction-diffusion to convection-diffusion type

Consider the following classes of two-parameter singularly perturbed differential equations

- Two-parameter ODE

$$\begin{gathered} \varepsilon y'' + \mu a y' - b y = f, \quad \text{on} \quad \Omega, \\ y(0) = \gamma_0, \quad y(1) = \gamma_1, \\ a \geq \alpha > 0, \quad b \geq \beta > 0, \quad 0 < \varepsilon \leq 1, \quad 0 \leq \mu \leq 1 \end{gathered} \tag{1 3 1}$$

- Two-parameter parabolic PDE

$$\begin{gathered} \varepsilon u_{xx} + \mu a u_x - b u - d u_t = f, \qquad \text{on} \qquad G, \\ u(0,t) = \gamma_0(t), \qquad u(1,t) = \gamma_1(t), \\ u(x,0) = \phi(x), \\ a \geq \alpha > 0, \quad b \geq \beta > 0, \quad d \geq \delta > 0, \quad 0 < \varepsilon \leq 1, \quad 0 \leq \mu \leq 1 \end{gathered} \tag{1 3 2}$$

- Two-parameter elliptic PDE

$$\varepsilon \triangle u + \mu a \;\; \nabla u - bu = f, \quad \text{on} \quad \Omega = (0,1)^2, \tag{1 3 3}$$
$$u(x,0) = \gamma_0(x), \qquad u(x,1) = \gamma_1(x),$$
$$u(0,y) = \gamma_2(y), \qquad u(1,y) = \gamma_3(y),$$
$$a \geq \alpha > 0, \quad b \geq 2\beta > 0, \quad 0 < \varepsilon \leq 1, \quad 0 \leq \mu \leq 1$$

When $\mu = 1$ we have convection-diffusion problems, and when $\mu = 0$ the equations are of reaction-diffusion type In the past the special cases of $\mu = 0$ and $\mu = 1$ have been considered separately (see previous section) The aim of this thesis is to take this analysis and adapt it to deal with the two-parameter problem, thus obtaining one approach that deals with a wider class of problems including both special cases

There is comparatively little literature available on parameter-uniform numerical methods for problems with two small parameters Most of the articles published to date deal with the two-parameter ODE (1 3 1) The asymptotic structure of the solutions to (1 3 1) was examined by O'Malley [19, 20], where the ratio of μ to $\sqrt{\varepsilon}$ was identified as significant Vulanovic [34] considered finite difference methods in the case of $\mu = \varepsilon^{\frac{1}{2}+\lambda}$, $\lambda > 0$, however, as we will see later, with this restriction the problem behaves similarly to one-dimensional reaction-diffusion problems

Recently, parameter-uniform numerical methods for problem (1 3 1) were examined by Linß and Roos [12], Roos and Uzelac [26] and O'Riordan et al [21] The main results of Chapter 2 of this thesis have appeared in [21] Both [12] and [21] are concerned with finite difference methods and apply standard finite difference operators on special piecewise-uniform meshes The method of analysis and the choice of transition points used to generate the mesh differs in these two papers In [26] the ODE (1 3 1) is solved using the streamline-diffusion finite element method on a piecewise-uniform mesh and the operators are adapted in order to achieve a higher order scheme The analysis in this paper follows from the analysis in [12] Higher order schemes for problem (1 3 1) are also considered in [5], where the approach follows that taken in [21] and [22]

Significantly less literature is available on the two-parameter parabolic and elliptic PDEs Shishkin considered two-parameter elliptic problems in [30], however, these problems are different to those studied in this thesis Equation (1 3 2) is considered in [22] where a numerical method consisting of standard finite difference operators applied on a piecewise-uniform mesh is constructed A form of the material in Chapter 3 of this thesis has appeared in [22] Equation (1 3 3) has been considered in [23] and the main results from Chapter 4 have appeared in this article

1.4 Numerical methods for two-parameter differential equations

The analysis in this thesis is based on the principles laid down in [29] and in the books [3] and [16] for a single parameter singularly perturbed problem. The argument consists of firstly establishing a maximum principle, and then decomposing the solution into regular and layer components and deriving sharp parameter-explicit bounds on these components and their derivatives. The discrete solution is decomposed in an analogous fashion, and the numerical error between the discrete and continuous components are analysed separately using discrete maximum principle, truncation error analysis and appropriate barrier functions.

The analysis of equations (1.3.1), (1.3.2) and (1.3.3) naturally splits into the two cases of $\mu^2 \leq C\varepsilon$ and $\mu^2 \geq C\varepsilon$. In the first case the analysis follows closely that of reaction-diffusion when $\mu = 0$, however, in the second case the analysis is more intricate. Considering (1.3.1) and (1.3.2), when $\mu^2 \leq C\varepsilon$ an $O(\sqrt{\varepsilon})$ layer appears in the neighbourhood of $x = 0$ and $x = 1$. In the other case of $\mu^2 \geq C\varepsilon$, a layer of width $O(\frac{\varepsilon}{\mu})$ appears in the neighbourhood of $x = 0$ and a layer of width $O(\mu)$ appears near $x = 1$. With (1.3.3), when $\mu^2 \leq C\varepsilon$, an $O(\sqrt{\varepsilon})$ layer appears in the neighbourhood of all four edges. When $\mu^2 \geq C\varepsilon$, we get layers of width $O(\frac{\varepsilon}{\mu})$ in the neighbourhood of $x = 0$ and $y = 0$ and layers of width $O(\mu)$ in the neighbourhood of the other two edges.

In Chapter 2, the two-parameter ODE (1.3.1) is examined. We derive parameter explicit bounds on the solution of this problem and its derivatives. The solution is decomposed into regular and layer components and sharp bounds are obtained on these components and their derivatives. Using these bounds a numerical algorithm based on an upwind finite difference operator and an appropriately chosen piecewise uniform mesh is constructed. The method is then shown to converge independently of both perturbation parameters. Numerical results are given to illustrate this convergence.

Chapter 3 is concerned with the two-parameter parabolic problem. Difficulties arose when attempting to extend some of the techniques of analysis used in Chapter 2 in order to deal with the parabolic PDE. It became clear that some changes had to be made so that the parabolic problem, and the more difficult elliptic PDE, could be considered. The method of analysis in this chapter is similar to that in the previous chapter apart from a few notable exceptions.

- The analysis in Chapter 3 splits entirely into two cases depending on the ratio of μ to $\sqrt{\varepsilon}$

- The transition points used in defining the Shishkin mesh also depend on this ratio and are simpler then those used in Chapter 2

- When $\mu^2 \geq C\varepsilon$, we define the regular component v using a double expansion, first in ε and then a further expansion in μ

- In the case of $\mu^2 \geq C\varepsilon$, the definition of the right singular layer component w_R in Chapter 2 does not quite isolate the layer In Chapter 2 we manage to overcome this problem in the error analysis, but in order to analyse the two-parameter parabolic or elliptic differential equations, we need to define w_R so that its effect is felt only near $x = 1$ Hence we decompose w_R

A numerical method consisting of finite difference operators applied on a piecewise-uniform mesh obtained with these new simpler transition points is constructed, and the numerical approximations are shown to converge independently of the small parameters Numerical results are given to illustrate this convergence The main results in the final section of this chapter have appeared in [5] We apply the new approach detailed above to the regular component and right singular layer component of (1 3 1) The bounds obtained in this section are needed in [5] when analysing higher order methods for (1 3 1)

In Chapter 4, we extend the approach used in Chapter 3 to elliptic problems in the case of $\mu^2 \leq C\varepsilon$ Compatibility is now an issue and the extension idea of Shishkin's [29] is vital to ensure no overly artificial compatibility conditions are imposed A numerical method is constructed and parameter-uniform error bounds are established

Chapter 5 deals with elliptic two-parameter problems in the case of $\mu^2 \geq C\varepsilon$, the style of this chapter is different from that of the previous chapters The solution is decomposed into regular and layer components Parameter-explicit bounds are obtained on the regular and boundary layer components and their derivatives It is when we consider the corner layer functions that the style of the thesis changes Bounds on these components and their derivatives are required for the error analysis We state and motivate conjectures on the bounds of these functions, however, we leave rigorous proofs for future work A numerical method is constructed and, assuming the conjectures on the bounds on corner layer functions are true, parameter-uniform error bounds are established

The main findings of this thesis are as follows

- The original aim of this thesis was to take the literature for the convection-diffusion and reaction-diffusion problems and adapt it to create one approach that dealt with the two-parameter problem We now realise that the simplest and most extendable approach to the two-parameter problem is to consider separately the cases of $\mu^2 \leq C\varepsilon$ and $\mu^2 \geq C\varepsilon$

- The analysis in this thesis highlights the importance of using decompositions to define the regular and layer components of the solution The key advantage of such an approach is its extendability to problems of higher dimension

- Ensuring that the layer functions are defined so as to correctly isolate the singularities of our solution proved to be essential The order in which these components are defined is also shown to be important When the regular and layer functions are defined correctly, the choice of piecewise-uniform mesh for our numerical method is clear and the ensuing error analysis is relatively straight forward

1 5 Notation

- Throughout the thesis, $0 < \varepsilon \leq 1$ is a parameter multiplying all second order derivatives and $0 \leq \mu \leq 1$ is a parameter multiplying all first order space derivatives

- We adopt the following notation

$$\|f\|_D = \max_{\vec{x} \in D} |f(\vec{x})|,$$

  and when the norm is not subscripted, the maximum is over the entire domain

- In Chapter 2 and Chapter 3, we take

$$\alpha = \min_{\bar{D}} a, \qquad \beta = \min_{\bar{D}} b, \quad \text{and} \quad \gamma < \min_{\bar{D}} \left\{ \frac{b}{a} \right\},$$

  while in the elliptic problem it is taken (for notational simplicity) as

$$\alpha = \min_{\bar{D}} \{\alpha_1, \alpha_2\}, \qquad \beta = \frac{1}{2} \min_{\bar{D}} b, \quad \text{and} \qquad \gamma < \min_{\bar{D}} \left\{ \frac{b}{2a_1}, \frac{b}{2a_2} \right\}$$

- The superscript $*$ notation denotes an extended domain or an extended function (for example Ω^*, f^*) Superscripts such as $[*, TB]$ also tell us the direction in which the domain or the functions are extended ($[*, TB]$ implying that we extend to the top and bottom of the original domain)

- We use capital letters to denote discrete functions and small letters for continuous functions

- Throughout this thesis, C (sometimes subscripted) will denote a generic constant independent of the parameters ε and μ and the dimensions of the discrete problem (N,M)

Chapter 2

Ordinary differential equations

2 1 Introduction

Consider the following two-parameter singularly perturbed boundary value problem

$$L_{\varepsilon,\mu}u = \varepsilon u''(x) + \mu a(x)u'(x) - b(x)u(x) = f(x), \quad x \in \Omega = (0,1), \qquad (2\,1\,1)$$
$$u(0), u(1) \quad \text{given},$$

where $a, b, f \in C^4(\Omega)$, $0 < \varepsilon \leq 1$, $0 \leq \mu \leq 1$, $0 < \alpha \leq a(x)$ and $0 < \beta \leq b(x)$ When the parameter $\mu = 1$, the problem is the well-studied one-dimensional convection-diffusion problem ([16],[25]) In this case, a boundary layer of width $O(\varepsilon)$ appears in a neighbourhood of the point $x = 0$ When the parameter $\mu = 0$, the problem is called reaction-diffusion and boundary layers of width $O(\sqrt{\varepsilon})$ appear at both $x = 0$ and $x = 1$ A discussion of these special cases and the two-parameter problem (2 1 1) can be found in Chapter 1

In this chapter we construct and analyse a numerical method for this problem class We show that the convergence of the numerical approximations to the exact solution is independent of both small parameters The main results in this chapter have appeared in [21]

In Section 2 2 we obtain parameter-explicit *a priori* bounds on the solution u of (2 1 1) and its derivatives In Section 2 3 we decompose the solution of (2 1 1) into regular and layer components These components are then analysed separately and sharp parameter explicit bounds are obtained on the components themselves and their derivatives Our numerical method is defined in Section 2 4 We decompose our discrete solution U into components analogous to those in the continuous case and obtain bounds on these discrete functions Section 2 5 is concerned with analysing the error between the continuous solution u of (2 1 1) and the discrete solution U This is achieved by analysing the error in the regular and singular components separately We show that we have a parameter-uniform numerical method Finally, Section 2 6 contains numerical results to support the theoretical proofs given in the previous section

Notation particular to this chapter We define the zero order, first order and second order differential operators L_0, L_μ and $L_{\varepsilon,\mu}$ as follows

$$\begin{aligned} L_0 z &= -bz, \\ L_\mu z &= a\mu z_x + L_0 z, \\ L_{\varepsilon,\mu} z &= \varepsilon z_{xx} + L_\mu z \end{aligned}$$

We should also note the following notation

$$\partial\Omega = \{0,1\}, \qquad ||u||_{\bar{\Omega}} = \max_{\bar{\Omega}} |u(x)|,$$

and if the norm is not subscripted we can assume $||\ || = ||\ ||_{\bar{\Omega}}$

2 2 Bounds on the solution u and its derivatives

In this section we will establish *a priori* bounds on the solution of (2 1 1) and its derivatives These bounds will be used in the error analysis in later sections We start by stating a continuous minimum principle for the differential operator in (2 1 1), whose proof is standard

Minimum Principle 1 *If $w \in C^2[0,1]$ such that $L_{\varepsilon,\mu} w\ |_\Omega \leq 0$ and $w\ |_{\partial\Omega} \geq 0$ then $w\ |_{\bar{\Omega}} \geq 0$*

Lemma 2 2 1 *The solution u of the differential equation (2 1 1), satisfies the following bound*

$$||u||_{\bar{\Omega}} \leq \max\{|u(0)|, |u(1)|\} + \frac{1}{\beta}||f||$$

Proof Let us consider the following barrier functions

$$\psi^{\pm}(x) = \max\{|u(0)|, |u(1)|\} + \frac{1}{\beta}||f|| \pm u(x)$$

Clearly the functions $\psi^{\pm}(x)$ are nonnegative at $x = 0$ and $x = 1$ Also since

$$L_{\varepsilon,\mu}\psi^{\pm}(x) = -b(x)\max\{|u(0)|, |u(1)|\} - \frac{b(x)}{\beta}||f|| \pm f(x) \leq 0,$$

we can apply Minimum Principle 1 to show that $\psi^{\pm}(x) \geq 0$ for all $x \in \bar{\Omega}$ The required result follows □

Lemma 2 2 2 *The derivatives $\frac{d^k u}{dx^k}$ of the solution u of (2 1 1) satisfy the following bounds*

$$\left\|\frac{d^k u}{dx^k}\right\|_{\bar{\Omega}} \leq \frac{C}{(\sqrt{\varepsilon})^k}\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^k\right)\max\{||u||, ||f||\}, \qquad k = 1, 2,$$

$$\left\|\frac{d^3 u}{dx^3}\right\|_{\bar{\Omega}} \leq \frac{C}{(\sqrt{\varepsilon})^3}\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^3\right)\max\{||u||, ||f||, ||f'||\},$$

where C depends only on $||a||$, $||a'||$, $||b||$ and $||b'||$

Proof Given any $x \in (0,1)$ we can construct a neighbourhood $N_x = (p, p+r)$ (where r is some combination of ε and μ yet to be determined and $0 < p < x < 1$) such that $x \in N_x$ and $N_x \subset (0,1)$ The mean value theorem implies that there exists $y \in N_x$ such that

$$u'(y) = \frac{u(p+r) - u(p)}{r}$$

It follows that

$$|u'(y)| \leq 2\frac{||u||}{r} \tag{2 2 2}$$

We have

$$u'(x) = u'(y) + \int_y^x u''(\xi)\, d\xi,$$

and therefore from the original differential equation (2 1 1) and using integration by parts we obtain

$$\begin{aligned} u'(x) = u'(y) \quad &+ \quad \varepsilon^{-1}\int_y^x f(\xi)\, d\xi + \varepsilon^{-1}\int_y^x b(\xi)u(\xi)\, d\xi \\ &- \quad \varepsilon^{-1}\left(\mu a u\Big|_y^x - \mu\int_y^x a'(\xi)u(\xi)\, d\xi\right) \end{aligned}$$

Using (2 2 2) and the fact that $x - y \leq r$, we have

$$|u'(x)| \leq \frac{2}{r}||u|| + \frac{r}{\varepsilon}||f|| + \frac{C_1 r}{\varepsilon}||u|| + \frac{2C_2\mu}{\varepsilon}||u|| + \frac{C_3 r\mu}{\varepsilon}||u||$$

We obtain the following bound,

$$|u'(x)| \le C\left(\frac{1}{r}+\frac{r}{\varepsilon}+\frac{\mu}{\varepsilon}\right)||u|| + \frac{r}{\varepsilon}||f|| \le C\left(\frac{1}{r}+\frac{r}{\varepsilon}+\frac{\mu}{\varepsilon}\right)\max\{||u||,||f||\}$$

If we choose $r = \sqrt{\varepsilon}$, then the right hand side of the above expression is minimised with respect to r and we obtain the required result for $k = 1$ Using the differential equation (2 1 1) we can obtain the required bounds for $k = 2$ and by differentiating (2 1 1) the result for $k = 3$ follows □

2 3 Decomposition of the solution

In order to obtain parameter-uniform error estimates we decompose the solution of (2 1 1) into regular and singular components Firstly we want to show that there exists a function v (regular component) where the boundary conditions can be chosen such that

$$L_{\varepsilon,\mu}v = f \text{ on } (0,1) \quad \text{ and } \quad \left\|\frac{d^i v}{dx^i}\right\| \le C \text{ for } i = 0,1,2$$

The analysis splits into two cases depending on the ratio of μ to $\sqrt{\varepsilon}$

Starting with $\mu^2 \le C_1\varepsilon$ we consider the following differential equation

$$L_{\varepsilon,\mu}v = f \text{ on } (0,1) \tag{2 3 1}$$

We decompose v as follows

$$v = v_0(x) + \sqrt{\varepsilon}v_1(x\ \ \varepsilon,\mu) + \varepsilon v_2(x,\varepsilon,\mu), \tag{2 3 2a}$$

where

$$\begin{aligned} L_0 v_0 &= f, && \text{(2 3 2b)} \\ \sqrt{\varepsilon}L_0 v_1 &= (L_0 - L_{\varepsilon,\mu})v_0 && \text{(2 3 2c)} \\ \varepsilon L_{\varepsilon,\mu}v_2 &= \sqrt{\varepsilon}(L_0 - L_{\varepsilon,\mu})v_1 \text{ on } (0,1), \quad v_2(0) = v_2(1) = 0 && \text{(2 3 2d)} \end{aligned}$$

We know that

$$\left\|\frac{d^i v_0}{dx^i}\right\| \le C \qquad \text{if} \qquad \left\|\frac{d^i\ (f/b)}{dx^i}\right\| \le C,$$

and since $\mu^2 \le C_1\varepsilon$, we also have

$$\left\|\frac{d^i v_1}{dx^i}\right\| \le C \qquad \text{if} \qquad \left\|\frac{d^{i+2}(f/b)}{dx^{i+2}}\right\| \le C, \quad \left\|\frac{d^i a}{dx^i}\right\| \le C, \quad \text{and} \quad \left\|\frac{d^i b}{dx^i}\right\| \le C$$

Hence, if we have $f, b \in C^4$ and $a \in C^2$, we can use Lemma 2 2 2 in order to obtain

$$\left\|\frac{d^i v_2}{dx^i}\right\| \le C\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^i\right)\left(\frac{1}{\sqrt{\varepsilon}}\right)^i \le C\left(\frac{1}{\sqrt{\varepsilon}}\right)^i, \qquad i \le 2$$

Therefore using the decomposition (2 3 2), we conclude

$$\left\|\frac{d^i v}{dx^i}\right\| \le C \text{ for } i = 0, 1, 2$$

In the second case $\mu^2 \ge C_2\varepsilon$ where $C_2 < C_1$ and $C_2 \le \frac{\gamma}{\alpha}$, ($\gamma < \min_{\bar{\Omega}}\{\frac{b}{a}\}$), we consider the differential equation

$$\widehat{L}_{\varepsilon,\mu}\hat{v} = \hat{f} \text{ on } (0,d) \qquad d \ge 1 \qquad \hat{v}(d) = 1, \quad \hat{v}(0) \text{ chosen in } (2\ 3\ 4), \tag{2 3 3}$$

where the differential operators $\widehat{L}_{\varepsilon,\mu}$ and $\widehat{L}_\mu$ coincide with $L_{\varepsilon,\mu}$ and L_μ respectively on the interval $(0,1)$ and $\hat{a}$, $\hat{b}$ and $\hat{f}$ are extensions of the functions a, b and f to the interval $(0,d)$ (they have the same properties as a, b and f and also coincide with the functions on the interval $(0,1)$) We extend the functions in such a way that $||\hat{a}|| > ||a||$, $||\hat{b}|| > ||b||$ and $\gamma < \min_{\bar{\Omega}}\{\frac{b}{a}\}$ Let us now decompose $\hat{v}$ as follows

$$\hat{v} = \hat{v}_0 + \varepsilon\hat{v}_1 + \varepsilon^2\hat{v}_2 \tag{2 3 4a}$$

where

$$\widehat{L}_\mu\hat{v}_0 = \hat{f} \text{ on } [0,d), \qquad \hat{v}_0(d) = 1, \tag{2 3 4b}$$

$$\varepsilon\widehat{L}_\mu\hat{v}_1 = (\widehat{L}_\mu - \widehat{L}_{\varepsilon,\mu})\hat{v}_0 \text{ on } [0,d), \qquad \hat{v}_1(d) = 0, \tag{2 3 4c}$$

$$\varepsilon^2\widehat{L}_{\varepsilon,\mu}\hat{v}_2 = \varepsilon(\widehat{L}_\mu - \widehat{L}_{\varepsilon,\mu})\hat{v}_1 \text{ on } (0,d), \qquad \hat{v}_2(0) = \hat{v}_2(d) = 0 \tag{2 3 4d}$$

We note that $\hat{v}(0) = \hat{v}_0(0) + \varepsilon\hat{v}_1(0)$

In order to establish bounds on derivatives of the components $\hat{v}_0$ and $\hat{v}_1$, we first need the following lemma on the first order singularly perturbed operator $\widehat{L}_\mu$

Lemma 2 3 1 *Let y be the solution of the first order differential equation*

$$\widehat{L}_\mu^{[1]} y(x) = \mu y'(x) - k y(x) = g(x,\mu), \quad 0 \le x < d,$$
$$|y(d)| \le \frac{C}{\mu^p}, \quad p \ge 0,$$

where

$$\left|\frac{d^i g}{d x^i}\right| \le C\left(1 + \frac{1}{\mu^{p+i}} e^{-\frac{\gamma^*}{\mu}(d-x)}\right), \quad i = 0,1, \quad x \le d,$$

and for all $x \in [0,d]$

$$k(x) > \gamma^* > 0, \quad \left\|\frac{d^i k}{d x^i}\right\| \le C, \quad i = 0,1$$

then

$$\left|\frac{d^i y}{d x^i}\right| \le C\left(1 + \frac{1}{\mu^{p+i}} e^{-\frac{\gamma^*}{\mu}(d-x)}\right), \quad i = 0,1, \quad x \le d$$

Proof Suppose $z \in C^0([0,d])$, we first note the following property can be established using a simple proof by contradiction argument

$$\text{If} \quad \widehat{L}_\mu^{[1]} z\Big|_{[0,d)} \le 0 \quad \text{and} \quad z\Big|_{d} \ge 0 \quad \text{then} \quad z\Big|_{[0,d]} \ge 0 \tag{2 3 5}$$

Consider the following barrier functions

$$\psi^{\pm}(x) = C_1\left(1 + \frac{1}{\mu^p} e^{-\frac{\gamma^*}{\mu}(d-x)}\right) \pm y(x)$$

Clearly the functions $\psi^{\pm}(x)$ are nonnegative at $x = d$ for C_1 large enough We also have

$$\hat{L}_\mu^{[1]} \psi^{\pm} = \frac{C_1}{\mu^p}(\gamma^* - k) e^{-\frac{\gamma^*}{\mu}(d-x)} - k C_1 \pm g(x,\mu)$$

Since $k > \gamma^*$ we can choose C_1 such that $\widehat{L}_\mu^{[1]} \psi^{\pm} \le 0$ and therefore we can apply (2 3 5) in order to obtain

$$|y(x)| \le C\left(1 + \frac{1}{\mu^p} e^{-\frac{\gamma^*}{\mu}(d-x)}\right) \tag{2 3 6}$$

To derive the required bounds on the derivative of y, we decompose the solution as follows

$$y(x) = -\frac{g(x,\mu)}{k(x)} + \left(y(d) + \frac{g(d,\mu)}{k(d)}\right) s(x) + \mu z(x), \tag{2 3 7a}$$

where

$$\widehat{L}_{\mu}^{[1]} s = 0 \text{ on } [0,d), \qquad s(d) = 1, \tag{2 3 7b}$$

$$\widehat{L}_{\mu}^{[1]} z = \left(\frac{g}{k}\right)' \text{ on } [0,d), \quad z(d) = 0 \tag{2 3 7c}$$

Starting with (2 3 7b), we can use $\psi^{\pm}(x) = Ce^{-\frac{\gamma^*}{\mu}(d-x)} \pm s(x)$ as our barrier functions in order to obtain

$$\widehat{L}_{\mu}^{[1]} \psi^{\pm}(x) = C(\gamma^* - k)e^{-\frac{\gamma^*}{\mu}(d-x)} \pm 0$$

Again since $k(x) > \gamma^*$ we find that the above expression is always nonpositive and we therefore can apply (2 3 5) in order to obtain the following bound on the function s,

$$|s(x)| \leq Ce^{-\frac{\gamma^*}{\mu}(d-x)}$$

Using the above bound and (2 3 7b), we obtain

$$|s'(x)| \leq \frac{C}{\mu} e^{-\frac{\gamma^*}{\mu}(d-x)}$$

Next, since z satisfies a similar equation to y we have from (2 3 6) that

$$|z(x)| \leq C\left(1 + \frac{1}{\mu^{p+1}} e^{-\frac{\gamma^*}{\mu}(d-x)}\right)$$

The bounds on the derivative of z can be derived using (2 3 7c) and the above result We obtain

$$|\mu z'(x)| \leq C\left(1 + \frac{1}{\mu^{p+1}} e^{-\frac{\gamma^*}{\mu}(d-x)}\right)$$

Combining this with (2 3 7a) we now have

$$|y'(x)| \leq C\left(1 + \frac{1}{\mu^{p+1}} e^{-\frac{\gamma^*}{\mu}(d-x)}\right)$$

□

Lemma 2 3 2 *If $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$, $\gamma < \min\{\frac{b}{a}\}$ and $\hat{f}, \hat{a}, \hat{b} \in C^4$ then the solution $\hat{v}$ of (2 3 3) satisfies the following bounds*

$$\left|\frac{d^i \hat{v}}{d x^i}\right| \leq C\left(1 + \frac{1}{\mu^i} e^{-\frac{\gamma}{2\mu}(d-x)}\right), \qquad i = 0,1,2,$$

where C depends only on $||a||, ||a'||$, $||b||$ and $||b'||$

Proof Note that $\hat{v} = \hat{v}_0 + \varepsilon\hat{v}_1 + \varepsilon^2\hat{v}_2$ We first consider $\hat{v}_0$ which is the solution of (2 3 4b) Since $\hat{v}_0(d) = 1$ and $\left\|\frac{d^i (f/a)}{dx^i}\right\| \leq C$ for $i = 0, 1$ we apply Lemma 2 3 1 with $p = 0$ in order to obtain

$$\left|\frac{d^i \hat{v}_0}{dx^i}\right| \leq C\left(1 + \frac{1}{\mu^i} e^{-\frac{\gamma}{\mu}(d-x)}\right), \qquad i = 0, 1, \qquad \frac{\hat{b}}{\hat{a}} > \gamma$$

Differentiating (2 3 4b) we have

$$\mu(\hat{v}_0')' - \frac{\hat{b}}{\hat{a}}\hat{v}_0' = \left(\frac{\hat{f}}{\hat{a}}\right)' + \left(\frac{\hat{b}}{\hat{a}}\right)'\hat{v}_0 = g_1(x)$$

In this case $|\hat{v}_0'(d)| \leq \frac{c}{\mu}$ and we also know $\left|\frac{d^i g_1}{dx^i}\right| \leq C(1 + \frac{1}{\mu^i} e^{-\frac{\gamma}{\mu}(d-x)})$ for $i = 0, 1$ We therefore can apply Lemma 2 3 1 with $p = 1$ in order to obtain

$$|\hat{v}_0''(x)| \leq C\left(1 + \frac{1}{\mu^2} e^{-\frac{\gamma}{\mu}(d-x)}\right)$$

Continuing in this way (differentiating (2 3 4b) and applying Lemma 2 3 1 to differential equations involving derivatives of $\hat{v}_0$ for the appropriate value of p), we obtain

$$\left|\frac{d^i \hat{v}_0}{dx^i}\right| \leq C\left(1 + \frac{1}{\mu^i} e^{-\frac{\gamma}{\mu}(d-x)}\right) \qquad i = 0, 1, 2, 3, 4$$

Next we consider $\hat{v}_1$ which is the solution of (2 3 4c) Letting $g_2(x) = -\frac{\hat{v}_0''(x)}{a(x)}$, we find that $\hat{v}_1(d) = 0$ and $|g_2^{(i)}(x)| \leq C(1 + \frac{1}{\mu^{2+i}} e^{-\frac{\gamma}{\mu}(d-x)})$ We therefore start by applying Lemma 2 3 1 with $p = 2$ We now have the following

$$\left|\frac{d^i \hat{v}_1}{dx^i}\right| \leq C\left(1 + \frac{1}{\mu^{2+i}} e^{-\frac{\gamma}{\mu}(d-x)}\right) \qquad i = 0, 1$$

As with $\hat{v}_0$, we differentiate (2 3 4c) in order to obtain

$$\mu(\hat{v}_1')' - \frac{\hat{b}}{\hat{a}}\hat{v}_1' = -\left(\frac{\hat{v}_0''}{\hat{a}}\right)' + \left(\frac{\hat{b}}{\hat{a}}\right)'\hat{v}_1$$

Applying the lemma with $p = 3$, we now have

$$\left|\frac{d^i \hat{v}_1}{dx^i}\right| \leq C\left(1 + \frac{1}{\mu^{2+i}} e^{-\frac{\gamma}{\mu}(d-x)}\right), \qquad i = 0, 1, 2 \qquad (2\ 3\ 8)$$

Finally we consider $\hat{v}_2$. Choosing $\psi^{\pm}(x) = C_1\left(1 + \frac{1}{\mu^4}e^{-\frac{\gamma}{2\mu}(d-x)}\right) \pm \hat{v}_2$ as our barrier functions we see that both are nonnegative at $x = d$. We also have

$$\widehat{L}_{\varepsilon,\mu}\psi^{\pm}(x) = -C_1\hat{b} + \frac{C_1}{\mu^4}\left(\frac{\varepsilon\gamma^2}{4\mu^2} + \frac{\hat{a}\gamma}{2} - \hat{b}\right)e^{-\frac{\gamma}{2\mu}(d-x)} \pm \hat{v}_1''$$

If we take $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$ we can show that the above expression is negative if C_1 is large enough (since $\gamma < \min\{\frac{b}{a}\}$). We can therefore apply the minimum principle in order to obtain

$$|\hat{v}_2(x)| \leq C\left(1 + \frac{1}{\mu^4}e^{-\frac{\gamma}{2\mu}(d-x)}\right) \tag{2.3.9}$$

We now need to bound the derivatives of $\hat{v}_2$. Given any $x \in (0, d)$ we can construct a neighbourhood $N_x = (p, p+\sqrt{\varepsilon})$, where $x \in N_x$ and $N_x \subset \hat{\Omega}$. The mean value theorem implies there exists $y \in N_x$ such that

$$\hat{v}_2'(y) = \frac{\hat{v}_2(p+\sqrt{\varepsilon}) - \hat{v}_2(p)}{\sqrt{\varepsilon}}$$

Using (2.3.9) we now obtain

$$|\hat{v}_2'(y)| \leq \frac{C}{\sqrt{\varepsilon}}\left(1 + \frac{1}{\mu^4}e^{-\frac{\gamma}{2\mu}(d-(p+\sqrt{\varepsilon}))}\right) \leq \frac{C}{\sqrt{\varepsilon}}\left(1 + \frac{1}{\mu^4}e^{-\frac{\gamma}{2\mu}(d-(x+2\sqrt{\varepsilon}))}\right)$$

However this can be simplified to

$$|\hat{v}_2'(y)| \leq \frac{C}{\sqrt{\varepsilon}}\left(1 + \frac{1}{\mu^4}e^{-\frac{\gamma}{2\mu}(d-x)}e^{\frac{\gamma\sqrt{\varepsilon}}{\mu}}\right)$$

Since $\gamma < \min\{\frac{b}{a}\}$ and using $\mu^2 \geq C_2\varepsilon$, we know that $e^{\frac{\sqrt{\varepsilon}\gamma}{\mu}} \leq C$. We therefore obtain

$$|\hat{v}_2'(y)| \leq \frac{C}{\sqrt{\varepsilon}}\left(1 + \frac{1}{\mu^4}e^{-\frac{\gamma}{2\mu}(d-x)}\right)$$

From the original differential equation for $\hat{v}_2$, we have

$$\hat{v}_2'(x) = \hat{v}_2'(y) + \int_y^x \hat{v}_2''(\xi)\,d\xi,$$

and using the bounds on $\hat{v}_2$ above and (2 3 8) we find (as in the proof of Lemma 2 2)

$$|\hat{v}_2'(x)| \leq \quad \frac{C_1}{\sqrt{\varepsilon}}\left(1+\frac{1}{\mu^4}e^{-\frac{\gamma}{2\mu}(d-x)}\right)+\frac{C_2}{\varepsilon}\int_y^x\left(1+\frac{1}{\mu^4}e^{-\frac{\gamma}{2\mu}(d-\xi)}\right)d\xi$$
$$+\frac{C_3\mu}{\varepsilon}\left(1+\frac{1}{\mu^4}e^{-\frac{\gamma}{2\mu}(d-x)}\right)$$

Integrating, and remembering $x-y \leq \sqrt{\varepsilon}$, we see

$$|\hat{v}_2'(x)| \leq \quad \frac{C_1}{\sqrt{\varepsilon}}\left(1+\frac{\mu}{\sqrt{\varepsilon}}\right)\left(1+\frac{1}{\mu^4}e^{-\frac{\gamma}{2\mu}(d-x)}\right)+\frac{C_2}{\sqrt{\varepsilon}}$$
$$+\frac{C_3\mu\left(\frac{\gamma(x-y)}{2\mu}\right)}{\varepsilon\mu^4}e^{-\frac{\gamma}{2\mu}(d-x)}\left(\frac{1-e^{-\frac{\gamma}{2\mu}(x-y)}}{\left(\frac{\gamma(x-y)}{2\mu}\right)}\right)$$

Using the inequality $\frac{1-e^{-t}}{t} \leq C$ we see

$$|\hat{v}_2'(x)| \leq C\left(\frac{\mu+\sqrt{\varepsilon}}{\varepsilon}\right)\left(1+\frac{1}{\mu^4}e^{-\frac{\gamma}{2\mu}(d-x)}\right)$$

Also given that $\mu^2 \geq C_2\varepsilon$ this can be simplified in order to obtain

$$|\hat{v}_2'(x)| \leq C\left(\frac{\mu}{\varepsilon}\right)\left(1+\frac{1}{\mu^4}e^{-\frac{\gamma}{2\mu}(d-x)}\right) \tag{2 3 10}$$

Substituting (2 3 9) and (2 3 10) into (2 3 4d),we now have the following bounds for $\hat{v}_2''$,

$$|\hat{v}_2''(x)| \leq C\left(\frac{\mu^2}{\varepsilon^2}+\frac{1}{\varepsilon}\right)\left(1+\frac{1}{\mu^4}e^{-\frac{\gamma}{2\mu}(d-x)}\right)$$

Finally we use the bounds for $\hat{v}_0$, $\hat{v}_1$ and $\hat{v}_2$ and their derivatives to obtain the required result □

Using Lemma 2 3 2, we conclude that $\hat{v}$ is bounded above away from $x = d$, and imposing the condition that $d > 1$, we know $\exists\ \hat{v} \in C^3(0,1)$ such that $L_{\varepsilon,\mu}\hat{v} = f$ and $\left\|\frac{d^i v}{dx^i}\right\| \leq C$ on $(0,1)$ for $i = 0,1,2$ In this case we define the regular component v as the solution to the following problem

$$L_{\varepsilon,\mu}v = f \text{ on } (0,1), \qquad v(0) = \hat{v}(0), \qquad v(1) = \hat{v}(1)$$

Remark 2 3 1 *When analysing the two-parameter ode* (2 1 1), *attention was always* *given to constructing proofs and using analytical tools that are extendable to problems of higher dimensions However, one would encounter significant difficulties in an attempt to extend the approach taken in Lemma 2 3 2 A new and more extendable approach to define the regular component is needed when considering the two-parameter parabolic problem Such an approach is detailed in Chapter 3*

In both cases we now have the following decomposition of the solution u

$$u = v + w_L + w_R, \tag{2 3 11a}$$

where

$$\begin{aligned} L_{\varepsilon,\mu} v &= f \text{ on } (0,1), & v(0), v(1) \text{ chosen in } (2\ 3\ 2) \text{ or } (2\ 3\ 4), \quad & (2\ 3\ 11\text{b}) \\ L_{\varepsilon,\mu} w_L &= 0 \text{ on } (0,1), & w_L(0) = u(0) - v(0), \quad w_L(1) = 0, \quad & (2\ 3\ 11\text{c}) \\ L_{\varepsilon,\mu} w_R &= 0 \text{ on } (0,1), & w_R(0) = 0, \quad w_R(1) = u(1) - v(1) \quad & (2\ 3\ 11\text{d}) \end{aligned}$$

The boundary conditions of v are chosen (as above) so that it satisfies the bounds

$$\left\| \frac{d^i v}{d x^i} \right\| \leq C \quad i = 0, 1, 2 \quad \text{and} \quad \left\| \frac{d^3 v}{d x^3} \right\| \leq \frac{C}{\varepsilon}, \tag{2 3 12}$$

and therefore we call v the regular component of the solution The singular components w_L and w_R satisfy the bounds in Lemma 2 2 2 However, we can also obtain the following sharper bounds on the exponential character of the two components

Lemma 2 3 3 *When the solution of (2 1 1) is decomposed as in (2 3 11a), the singular components w_L and w_R satisfy the following bounds*

$$\begin{aligned} |w_L(x)| &\leq Ce^{-\theta_1 x}, \\ |w_R(x)| &\leq Ce^{-\theta_2(1-x)}, \end{aligned}$$

where

$$\theta_1 = \frac{\mu\alpha + \sqrt{\mu^2\alpha^2 + 4\varepsilon\beta}}{2\varepsilon},$$

and

$$\theta_2 = \frac{-\mu A + \sqrt{\mu^2 A^2 + 4\varepsilon\beta}}{2\varepsilon}$$

($A = ||a||_{\bar\Omega}$ and θ_1 and θ_2 are respectively the positive roots of the equations $\varepsilon\theta_1^2 - \mu\alpha\theta_1 - \beta = 0$ and $\varepsilon\theta_2^2 + \mu A\theta_2 - \beta = 0$)

Proof Consider the following barrier functions

$$\psi^{\pm}(x) = Ce^{-\theta_1 x} \pm w_L(x),$$

where θ_1 is as stated We find that for C large enough, the functions are both nonnegative at $x = 0$ and $x = 1$, and after a simple calculation we also find that $L_{\varepsilon,\mu}\psi^{\pm}(x) \leq 0$ We therefore can apply the minimum principle in order to obtain

$$|w_L(x)| \leq Ce^{-\theta_1 x}$$

The proof in the case of w_R is similar □

Remark 2 3 2 *The following properties of θ_1 and θ_2 can easily be established They will be required in order to analyse the error in the numerical approximations to the solution*

$$\theta_1 \geq \max\left\{\frac{\sqrt{\beta}}{\sqrt{\varepsilon}}, \frac{\alpha\mu}{\varepsilon}\right\}, \tag{2 3 13a}$$

$$\textit{if } \mu^2 \leq C\varepsilon \textit{ then } \theta_2 \geq \frac{C}{\sqrt{\varepsilon}}, \qquad \textit{if } \mu^2 \geq C\varepsilon \textit{ then } \theta_2 \geq \frac{C}{\mu} \tag{2 3 13b}$$

2 4 Discrete problem

Consider the following upwind finite difference scheme

$$L^N U(x_i) = \varepsilon\delta^2 U(x_i) + \mu a(x_i) D^+ U(x_i) - b(x_i)U(x_i) = f(x_i), \qquad x_i \in \Omega^N, \tag{2 4 1a}$$

where

$$D^+U(x_i) = \frac{U(x_{i+1}) - U(x_i)}{x_{i+1} - x_i}, \qquad D^-U(x_i) = \frac{U(x_i) - U(x_{i-1})}{x_i - x_{i-1}},$$

and

$$\delta^2 U(x_i) = \frac{D^+U(x_i) - D^-U(x_i)}{(x_{i+1} - x_{i-1})/2}$$

The piecewise-uniform mesh, Ω^N, on which we apply the above finite difference operator consists of two transition points

$$\begin{aligned} \sigma_1 &= \min\left\{\frac{1}{4}, \frac{2}{\theta_1}\ln N\right\}, \\ \sigma_2 &= \min\left\{\frac{1}{4}, \frac{2}{\theta_2}\ln N\right\} \end{aligned} \tag{2 4 1b}$$

More specifically

$$\Omega^N = \left\{ x_i | x_i = \left\{ \begin{array}{lll} \frac{4\sigma_1 i}{N}, & \text{if} & i \le \frac{N}{4} \\ \sigma_1 + (i - \frac{N}{4})H, & \text{if} & \frac{N}{4} \le i \le \frac{3N}{4} \\ 1 - \sigma_2 + (i - \frac{3N}{4})\frac{4\sigma_2}{N}, & \text{if} & \frac{3N}{4} \le i \le N \end{array} \right. \right\}, \tag{2 4 1c}$$

where $NH = 2(1 - \sigma_1 - \sigma_2)$ We now state a discrete comparison principle for (2 4 1a), whose proof is standard

Discrete Minimum Principle *If W is any mesh function and $L^N W \mid_{\Omega^N} \le 0$ and $W \mid_{\partial\Omega^N} \ge 0$, then $W \mid_{\bar{\Omega}^N} \ge 0$*

We have the following discrete decomposition

$$U = V + W_L + W_R, \tag{2 4 2a}$$

where the components are the solutions of the following

$$L^N V = f(x_i), \qquad V(0) = v(0), \quad V(1) = v(1), \tag{2 4 2b}$$

$$L^N W_L = 0, \qquad W_L(0) = w_L(0), \quad W_L(1) = 0, \tag{2 4 2c}$$

$$L^N W_R = 0, \qquad W_R(0) = 0, \quad W_R(1) = w_R(1) \tag{2 4 2d}$$

We can prove the following bounds on the discrete counterparts of the singular components w_L and w_R

Theorem 2 4 1 *We have the following bounds on W_L and W_R*

$$|W_L(x_j)| \le C \prod_{i=1}^{j} (1 + \theta_L h_i)^{-1} = \Psi_{L,j}, \qquad \Psi_{L,0} = C, \tag{2 4 3a}$$

$$|W_R(x_j)| \le C \prod_{i=j+1}^{N} (1 + \theta_R h_i)^{-1} = \Psi_{R,j}, \qquad \Psi_{R,N} = C, \tag{2 4 3b}$$

where W_L and W_R are solutions of (2 4 2c) and (2 4 2d) respectively and $h_i = x_i - x_{i-1}$ The parameters θ_L and θ_R are defined to be the positive roots of the following equations

$$2\varepsilon\theta_L^2 - \mu\alpha\theta_L - \beta = 0 \quad \textit{and} \quad 2\varepsilon\theta_R^2 + \mu A\theta_R - \beta = 0, \qquad (A = ||a||)$$

Proof We start with W_L Consider $\Phi^{\pm}_{L,j} = \Psi_{L,j} \pm W_L(x_j)$ Now $L^N\Phi^{\pm}_{L,j} = \varepsilon\delta^2\Psi_{L,j} + \mu a D^+\Psi_{L,j} - b\Psi_{L,j} \pm 0$, and using

$$\Psi_{L,j} > 0, \quad D^+\Psi_{L,j} = -\theta_L \Psi_{L,j+1} < 0 \quad \text{and} \quad \delta^2\Psi_{L,j} = \theta_L{}^2 \Psi_{L,j+1}\tfrac{h_{j+1}}{\bar{h}_j} > 0,$$

we obtain

$$L^N \Phi^{\pm}_{L,j} \leq \varepsilon \theta_L{}^2 \Psi_{L,j+1} \frac{h_{j+1}}{\bar{h}_j} - \mu\alpha\theta_L \Psi_{L,j+1} - \beta\Psi_{L,j},$$

where $\bar{h}_j = \frac{h_{j+1}+h_j}{2}$ Rewriting the right hand side of this equation we have

$$L^N \Phi^{\pm}_{L,j} \leq \Psi_{L,j+1}\left(2\varepsilon\theta_L{}^2\left(\frac{h_{j+1}}{2\bar{h}_j} - 1\right) + (2\varepsilon\theta_L{}^2 - \mu\alpha\theta_L - \beta) - \beta\theta_L h_{j+1}\right) \leq 0$$

Using the discrete minimum principle we obtain the required result

The same idea is applied to W_R We consider $\Phi^{\pm}_{R,j} = \Psi_{R,j} \pm W_R(x_j)$ Now $L^N\Phi^{\pm}_{R,j} = \varepsilon\delta^2\Psi_{R,j} + \mu a D^+\Psi_{R,j} - b\Psi_{R,j} \pm 0$, and using

$$\Psi_{R,j} \leq \Psi_{R,j+1}, \; \Psi_{R,j} > 0, \; D^+\Psi_{R,j} = \theta_R\Psi_{R,j} \text{ and } \delta^2\Psi_{R,j} = \frac{\theta_R{}^2}{(1+\theta_R h_j)}\Psi_{R,j}\frac{h_j}{\bar{h}_j},$$

we obtain

$$L^N\Phi^{\pm}_{R,j} \leq \frac{\Psi_{R,j}}{(1+\theta_R h_j)}\left(\varepsilon\theta_R{}^2\left(\frac{h_j}{\bar{h}_j} - 2\right) + 2\varepsilon\theta_R^2 + \mu A\theta_R(1+\theta_R h_j) - \beta(1+\theta_R h_j)\right)$$

Rewriting the right hand side of this inequality we have

$$L^N\Phi^{\pm}_{R,j} \leq \frac{\Psi_{R,j}}{(1+\theta_R h_j)}\left(\varepsilon\theta_R{}^2\left(\frac{h_j}{\bar{h}_j} - 2\right) + (2\varepsilon\theta_R^2 + \mu A\theta_R - \beta)(1+\theta_R h_j) - 2\varepsilon\theta_R^3 h_j\right) \leq 0,$$

and again we use the discrete minimum principle to finish □

2.5 Error analysis

We now wish to analyse the bounds on the error between the discrete solution and the continuous solution

Lemma 2 5 1 *At each mesh point $x_i \in \bar{\Omega}^N$ the regular component of the error satisfies the following estimate*

$$|(V - v)(x_i)| \leq CN^{-1},$$

where v is the solution of (2 3 11b) and V is the solution of (2 4 2b)

Proof Using the usual truncation error argument and (2 3 12) we have

$$|L^N(V-v)(x_i)| \le CH\left(\varepsilon||v'''|| + \mu||v''||\right) \le CH \le CN^{-1},$$

where H is the maximum step size If we choose $\psi^{\pm}(x_i) = C_1 N^{-1} \pm (V-v)(x_i)$ as our barrier functions, we know that these functions are both nonnegative at $x=0$ and $x=1$ We also find that $L^N\psi^{\pm} \le 0$ for C_1 large enough and therefore we can apply the discrete minimum principle in order to obtain the required result □

Lemma 2 5 2 *At each mesh point $x_i \in \bar{\Omega}^N$ the left singular component of the error satisfies the following estimate*

$$|(W_L - w_L)(x_i)| \le CN^{-1}(\ln N)^2,$$

where w_L is the solution of (2 3 11c) and W_L is the solution of (2 4 2c)

Proof We can use a classical argument in order to obtain the following truncation error bounds

$$|L^N(W_L - w_L)(x_i)| \le C(h_{i+1}+h_i)\left(\varepsilon||w'''|| + \mu||w''||\right)$$

Since w_L satisfies a similar equation to u, we can use Lemma 2 2 2 to obtain

$$|L^N(W_L - w_L)(x_i)| \le C(h_{i+1}+h_i)\left(\tfrac{1}{\sqrt{\varepsilon}}\left(1+\left(\tfrac{\mu}{\sqrt{\varepsilon}}\right)^3\right) + \tfrac{\mu}{\varepsilon}\left(1+\left(\tfrac{\mu}{\sqrt{\varepsilon}}\right)^2\right)\right)$$

Simplifying the right hand side of this expression we have

$$|L^N(W_L - w_L)(x_i)| \le C(h_{i+1}+h_i)\left(\frac{1}{\sqrt{\varepsilon}}\left(1+\left(\frac{\mu}{\sqrt{\varepsilon}}\right)^3\right)\right) \qquad (2\ 5\ 1)$$

Starting with when $\sigma_1 = \frac{1}{4}$, we can show that in this case $\theta_1 \le 8\ln N$ and therefore using (2 3 13a) our bound for the truncation error now becomes

$$|L^N(W_L - w_L)(x_i)| \le CN^{-1}(\ln N)^2, \quad \text{if } \sigma_1 = \frac{1}{4}$$

If we choose $\psi^{\pm}(x_i) = CN^{-1}(\ln N)^2 \pm (W_L - w_L)(x_i)$ as our barrier functions we find that we can apply the discrete minimum principle in order to obtain

$$|(W_L - w_L)(x_i)| \le CN^{-1}(\ln N)^2, \quad \text{if } \sigma_1 = \frac{1}{4} \qquad (2\ 5\ 2)$$

The next case to consider is $\sigma_1 < \frac{1}{4}$ In this case the mesh is piecewise uniform We firstly analyse the error in the coarse mesh region $[\sigma_1, 1)$ and then we proceed to analyse the fine mesh on $(0, \sigma_1)$ With the coarse mesh region, instead of using the usual truncation error argument, we will use Lemma 2 3 3 and (2 4 3a) to obtain the required error bounds From (2 4 3a) we have

$$|W_L(x_{\frac{N}{4}})| \leq C(1 + \theta_L h_L)^{-\frac{N}{4}}$$

where $h_L = \frac{4\sigma_1}{N}$ When $\sigma_1 < \frac{1}{4}$, we can prove that $\theta_L h_L \geq 4N^{-1} \ln N$ We obtain the following

$$|W_L(x_{\frac{N}{4}})| \leq C(1 + 4N^{-1} \ln N)^{-\frac{N}{4}}$$

Using the standard inequality $\ln(1+t) > t(1 - \frac{t}{2})$ and letting $t = 4N^{-1} \ln N$, we can show that $(1 + 4N^{-1} \ln N)^{-\frac{N}{4}} \leq 4N^{-1}$ and therefore we conclude that on the interval $[\sigma_1, 1)$ we have

$$|W_L(x_i)| \leq CN^{-1}$$

Looking at the continuous solution in this region we have

$$|w_L(x)| \leq Ce^{-\theta_1 x} \leq Ce^{-\theta_1(\frac{2}{\theta_1} \ln N)} \leq CN^{-2}$$

Combining these two results we now obtain the following error bounds

$$|(W_L - w_L)(x_i)| \leq CN^{-1}, \qquad x_i \in [\sigma_1, 1) \quad \text{and} \quad \sigma_1 < \frac{1}{4} \tag{2 5 3}$$

We now consider the fine mesh region The bound (2 5 1) on the truncation error still holds and since we are in the fine mesh region with $\sigma_1 < \frac{1}{4}$ we know that $h_{i+1} = h_i = \frac{8}{\theta_1} N^{-1} \ln N$ We can therefore use (2 3 13a) in order to obtain

$$|L^N(W_L - w_L)(x_i)| \leq C_1 N^{-1} \ln N + C_2 N^{-1} \frac{\mu^2}{\varepsilon} \ln N$$

If we choose $\psi^{\pm}(x_i) = C_3 N^{-1} \ln N + C_4 N^{-1}(\sigma_1 - x_i)\left(\frac{\mu}{\varepsilon}\right) \ln N \pm (W_L - w_L)(x_i)$ as our barrier functions we find that both functions are nonnegative at x_0 and $x_{\frac{N}{4}}$ C_3 and C_4 can be chosen so that $L^N \psi^{\pm} \leq 0$ and therefore applying the discrete minimum principle we obtain

$$|(W_L - w_L)(x_i)| \leq C_3 N^{-1} \ln N + C_4(\sigma_1 - x_i)\left(\frac{\mu}{\varepsilon}\right) N^{-1} \ln N$$

Therefore using $\sigma_1 = \frac{2}{\theta_1} \ln N$ and $\frac{\mu}{\varepsilon}\frac{1}{\theta_1} \leq C$ (see (2 3 13a)), we obtain

$$|(W_L - w_L)(x_i)| \leq CN^{-1}(\ln N)^2, \qquad x_i \in (0, \sigma_1) \quad \text{and} \quad \sigma_1 < \frac{1}{4} \tag{2 5 4}$$

Combining the bounds (2 5 2), (2 5 3) and (2 5 4) gives us the required result □

Lemma 2 5 3 *At each mesh point $x_i \in \bar{\Omega}^N$ the right singular component of the error satisfies the following estimate*

$$|(W_R - w_R)(x_i)| \leq CN^{-1}(\ln N)^2, \tag{2 5 5}$$

where w_R is the solution of (2 3 11d) and W_R is the solution of (2 4 2d)

Proof We start with the case $\mu^2 \leq C\varepsilon$ We again use a classical argument and Lemma 2 2 2 in order to obtain the following

$$|L^N(W_R - w_R)(x_i)| \leq C(h_{i+1} + h_i)\left(\frac{1}{\sqrt{\varepsilon}}\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^3\right)\right) \tag{2 5 6}$$

However, in the case $\mu^2 \leq C\varepsilon$, this simplifies to

$$|L^N(W_R - w_R)(x_i)| \leq \frac{C}{\sqrt{\varepsilon}}(h_{i+1} + h_i) \tag{2 5 7}$$

If $\sigma_2 = \frac{1}{4}$ and $\mu^2 \leq C\varepsilon$, we can use (2 3 13b) to show $\frac{C}{\sqrt{\varepsilon}} \leq \theta_2 \leq 8 \ln N$ We now obtain the following bounds on the truncation error

$$|L^N(W_R - w_R)(x_i)| \leq CN^{-1} \ln N$$

If we choose $\psi^{\pm}(x_i) = CN^{-1} \ln N \pm (W_R - w_R)(x_i)$ as our barrier functions on the entire interval $[0, 1]$, we obtain

$$|(W_R - w_R)(x_i)| \leq CN^{-1} \ln N, \quad \mu^2 \leq C\varepsilon \text{ and } \sigma_2 = \frac{1}{4} \tag{2 5 8}$$

In the case where $\sigma_2 < \frac{1}{4}$, we have to analyse the error in the fine and coarse mesh regions separately As with w_L we will start by examining the coarse mesh region $(0, 1-\sigma_2]$ Using (2 4 3b) we have

$$|W_R(x_{\frac{3N}{4}})| \leq C(1 + \theta_R h_R)^{\frac{-N}{4}}$$

where $h_R = \frac{4\sigma_2}{N}$ In this case we can prove that $\theta_R h_R \geq 4N^{-1} \ln N$ so, as with W_L, we obtain (after some calculations) $|W_R(x_{\frac{3N}{4}})| \leq CN^{-1}$ Therefore on the interval $(0, 1-\sigma_2]$ we have

$$|W_R(x_i)| \leq CN^{-1}$$

Using the fact that on the interval $(0, 1-\sigma_2]$ with $\sigma_2 = \frac{2}{\theta_2}\ln N$ we have

$$|w_R(x)| \leq Ce^{-\theta_2(1-x)} \leq CN^{-2},$$

we now obtain the following bounds on the error

$$|(W_R - w_R)(x_i)| \leq CN^{-1}, \quad \mu^2 \leq C\varepsilon, \quad x_i \in (0, 1-\sigma_2] \text{ and } \sigma_2 < \frac{1}{4} \qquad (2\ 5\ 9)$$

We should note that this result in the coarse mesh region still holds when $\mu^2 \geq C\varepsilon$ and $\sigma_2 < \frac{1}{4}$ We now continue to the fine mesh region $(1-\sigma_2, 1)$ The bounds on the truncation error in (2 5 7) still hold and given that we are in the fine mesh region we have $h_{i+1} = h_i = \frac{8}{\theta_2}N^{-1}\ln N$ Using (2 3 13b) we now obtain $\frac{C}{\sqrt{\varepsilon}}(h_{i+1} + h_i) \leq CN^{-1}\ln N$ and hence

$$|L^N(W_R - w_R)(x_i)| \leq CN^{-1}\ln N$$

As before, choosing $\psi^{\pm}(x_i) = CN^{-1}\ln N \pm (W_R - w_R)(x_i)$ as our barrier functions we obtain the following error bounds

$$|(W_R - w_R)(x_i)| \leq CN^{-1}\ln N, \quad \mu^2 \leq C\varepsilon, \quad x_i \in (1-\sigma_2, 1) \text{ and } \sigma_2 < \frac{1}{4} \qquad (2\ 5\ 10)$$

In the case $\mu^2 \geq C\varepsilon$, we need to look at w_R differently We can decompose w_R as follows

$$w_R(x) = y(x) - \frac{y(0)}{w_L(0)}w_L(x) \qquad (2\ 5\ 11a)$$

where

$$L_{\varepsilon,\mu}y(x) = 0, \qquad y(1) = w_R(1), \qquad (2\ 5\ 11b)$$

and $w_L(x)$ is defined as in (2 3 11c) Using this decomposition we have

$$|(W_R - w_R)(x_i)| \leq |(Y-y)(x_i)| + C|(W_L - w_L)(x_i)|,$$

where W_R, Y, W_L are the discrete counterparts of w_R, y and w_L respectively We see that y satisfies a similar equation to $\hat{v}$ in (2 3 3), therefore appropriately choosing $y(0)$

and setting $d = 1$, we can use Lemma (2 3 2) to obtain the following bounds for y

$$\left|\frac{d^i y}{d x^i}\right| \leq C\left(1 + \frac{1}{\mu^i} e^{-\frac{\gamma}{2\mu}(1-x)}\right), \qquad i = 0, 1, 2$$

More simply

$$\left\|\frac{d^i y}{d x^i}\right\| \leq \frac{C}{\mu^i}, \qquad i = 0, 1, 2 \tag{2 5 12}$$

We know that $|(W_L - w_L)(x_i)| \leq CN^{-1}(\ln N)^2$ at each mesh point $x_i \in \Omega^N$, so we therefore only need to consider the error y generates In the case $\sigma_2 = \frac{1}{4}$ we know that $\theta_2 \leq 8 \ln N$ and using (2 3 13b) we can therefore show that $\frac{1}{\mu} \leq C \ln N$ Using the usual truncation error argument (noting that $\varepsilon y''' = (by)' - \mu(ay')')$ and a suitable barrier function, we find that

$$|(Y - y)(x_i)| \leq CN^{-1}(\ln N)^2$$

Combining this with the bound obtained on the left singular component of the error we have

$$|(W_R - w_R)(x_i)| \leq CN^{-1}(\ln N)^2, \quad \mu^2 \geq C\varepsilon \text{ and } \sigma_2 = \frac{1}{4}$$

In the case of $\sigma_2 < \frac{1}{4}$, the bound in the coarse mesh region $(0, 1 - \sigma_2]$, obtained in the case $\mu^2 \leq C\varepsilon$, still holds In the fine mesh region $(1 - \sigma_2, 1)$ we use (2 5 12) again in order to obtain

$$|L^N(Y - y)(x_i)| \leq C\frac{h_{i+1} + h_i}{\mu}$$

In this case we know that $h_{i+1} = h_i = \frac{8}{\theta_2} N^{-1} \ln N$ and using (2 3 13b) we can prove that

$$|L^N(Y - y)(x_i)| \leq CN^{-1} \ln N$$

Therefore using a suitable barrier function we obtain

$$|(Y - y)(x_i)| \leq CN^{-1} \ln N$$

Hence, we now have the following bound on the error

$$|(W_R - w_R)(x_i)| \leq CN^{-1}(\ln N)^2 \quad \mu^2 \geq C\varepsilon, \ x_i \in (1 - \sigma_2, 1), \text{ and } \sigma_2 < \frac{1}{4}$$

Combining all the error bounds for w_R in the different cases gives the required result □

Remark 2 5 1 *Such a decomposition of* w_R *in* (2 5 11) *suggests that in this case of* $\mu^2 \geq$

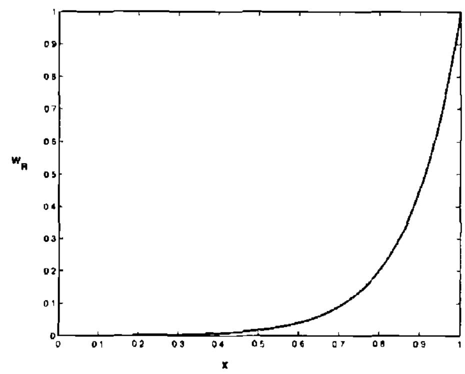


Figure 2 1 A plot of the solution of (2 5 13) *when* $\mu = 2^{-3}$ *and* $\varepsilon = 2^{-18}$

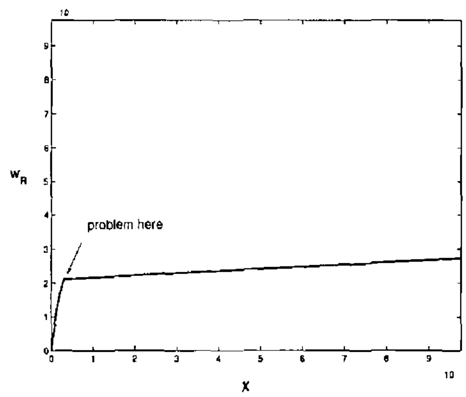


Figure 2 2 A zoom in to the bottom-left corner of Figure 2 1

$C\varepsilon$*, the definition of* w_R *in* (2 3 11d) *does not correctly isolate the right layer component See for example the following sample problem*

$$\begin{aligned} \varepsilon \tilde{w}_R'' + \mu \tilde{w}_R' - \tilde{w}_R &= 0, \\ \tilde{w}_R(0) = 0, \quad \tilde{w}_R(1) &= 1 \end{aligned} \tag{2 5 13}$$

Figure 2 1 is the solution of (2 5 13) *when* $\mu = 2^{-3}$ *and* $\varepsilon = 2^{-18}$ *Such a plot might lead us to naively believe there is just a layer on the right, however, in Figure 2 2 we zoom in to the bottom-left corner of Figure 2 1 and we see there is a problem We have not isolated our right layer component*

Since, when $\mu^2 \geq C\varepsilon$*, there is a layer of width* $O(\mu)$ *on the right, it seems more* *natural to assume that the function* y *defined in* (2 5 11b) *behaves like the right singular component However, as previously discussed, such an approach to bound* $y(x)$ *as detailed in Lemma 2 3 2 may pose difficult to extend to higher dimensions For these reasons a new approach to correctly define the right layer component* w_R *in the case of* $\mu^2 \geq C\varepsilon$ *is constructed in Chapter 3*

Theorem 2 5 1 *Let* u *be the solution of the differential equation (2 1 1) and* U *be the solution of (2 4 1) Then at each mesh point* $x_i \in \bar{\Omega}^N$ *we have*

$$|(U-u)(x_i)| \leq CN^{-1}(\ln N)^2 \qquad (2\ 5\ 14)$$

Proof This result immediately follows from Lemmas 2 5 1, 2 5 2 and 2 5 3 □

2 6 Numerical results

The scheme (the upwind finite difference operator (2 4 1a) applied on the mesh (2 4 1c)) has been tested with the following constant coefficient problem

$$\varepsilon u''_{\varepsilon,\mu}(x) + \mu u'_{\varepsilon,\mu}(x) - u_{\varepsilon,\mu}(x) = 1, \qquad u_{\varepsilon,\mu}(0) = u_{\varepsilon,\mu}(1) = 1 \qquad (2\ 6\ 1)$$

Figures (2 3) and (2 4) are graphs of the exact solution of the above problem The progressively lower graphs in these figures correspond to progressively smaller values of the parameter ε Note that in Figure (2 3b), the layer on the left is obvious while the layer

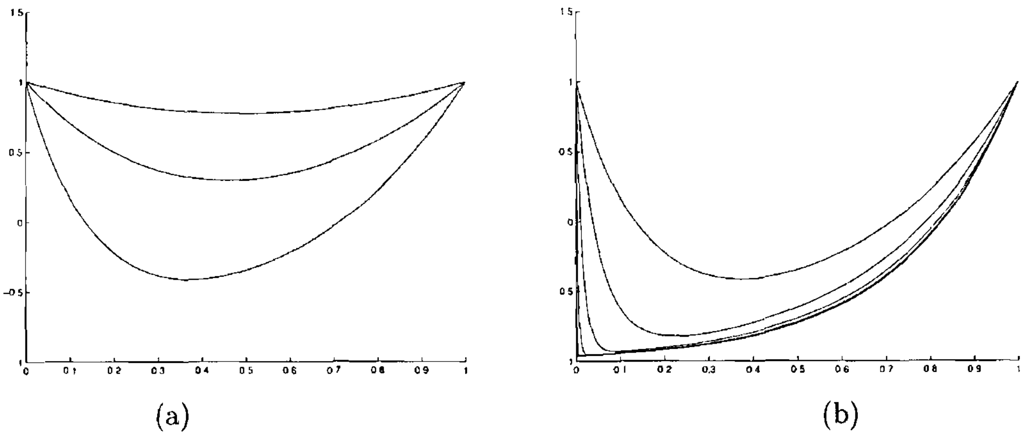
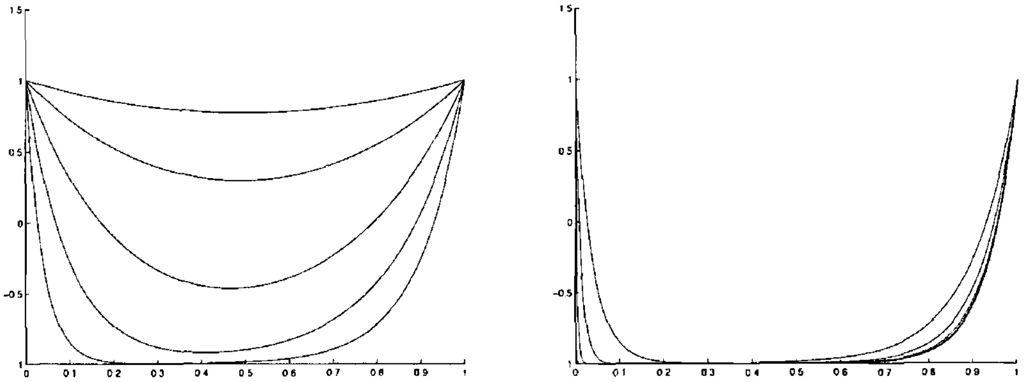


Figure 2 3 Exact solutions of 2 6 1 with $\mu = 2^{-2}$ for $2^{-32} \leq \varepsilon \leq 1$ when (a) $\mu^2 \leq \varepsilon$ and (b) $\mu^2 \geq 0\ 75\varepsilon$



(a) (b)

Figure 2 4 Exact solutions of 2 6 1 with $\mu = 2^{-4}$ for $2^{-32} \leq \varepsilon \leq 1$ when (a) $\mu^2 \leq \varepsilon$ and (b) $\mu^2 \geq 0\ 75\varepsilon$

on the right is notably weaker However, in Figure (2 4b) we see that as μ is reduced the layer on the right does in fact become more pronounced

We define the exact maximum pointwise error by

$$E^N_{\varepsilon,\mu,exact} = ||U^N_{\varepsilon,\mu} - u_{\varepsilon,\mu}||_{\Omega^N}$$

We also can find the maximum pointwise ε-uniform errors using

$$E^N_{\mu,exact} = \max_{2^{-\rho}\leq\varepsilon\leq 1} ||U^N_{\varepsilon,\mu} - u_{\varepsilon,\mu}||_{\Omega^N},$$

and finally we define the maximum pointwise (ε,μ)-uniform errors by

$$E^N_{exact} = \max_{2^{-32}\leq\mu\leq 1} \{ \max_{2^{-\rho}\leq\varepsilon\leq 1} ||U^N_{\varepsilon,\mu} - u_{\varepsilon,\mu}||_{\Omega^N} \},$$

where ρ is chosen in order to achieve stability of $E^N_{\mu,exact}$ with respect to ε As μ decreases, we must also consider progressively smaller values of ε (larger values of ρ) in order to reach this stability (e g when $\mu = 2^{-32}$ we must let ε decrease to 2^{-80}) Similarly we find the exact order of convergence using

$$p^N_{\varepsilon,\mu,exact} = \log_2 \frac{E^N_{\varepsilon,\mu,exact}}{E^{2N}_{\varepsilon,\mu,exact}}$$

We define the exact ε-uniform order of convergence by

$$p^N_{\mu,exact} = \log_2 \frac{E^N_{\mu,exact}}{E^{2N}_{\mu,exact}},$$

and finally we define the exact (ε,μ)-uniform order of convergence by

$$p^N_{exact} = \log_2 \frac{E^N_{exact}}{E^{2N}_{exact}}$$

Table 2 1 contains values of $E^N_{\varepsilon,\mu,exact}$ and $E^N_{\mu,exact}$ for $\mu = 2^{-16}$ and various values of ε The range in ε we present is from 1 to 2^{-60}, however, we can see that the errors have stabilised with respect to ε after $\varepsilon = 2^{-46}$ The vertical dots in the $N = 16, 32, \quad, 2048$ columns indicate that the values in these columns remain unchanged (the only exception to this being the $N = 8$ case) and similar notation is used in Tables 2 2, 2 3 and 2 4 Table 2 2 contains values of $p^N_{\varepsilon,\mu,exact}$ and $p^N_{\mu,exact}$ for $\mu = 2^{-16}$ and various values of ε and N Note that when $\mu > \sqrt{\varepsilon}$ the orders are approaching first order, however, in the region where $\mu < \sqrt{\varepsilon}$ we observe rates of second order appearing

Table 2 3 contains the values of $E^N_{\mu,exact}$ and E^N_{exact} for various values of μ and N An interesting effect to note is how quickly the error stablises with respect to μ Finally Table 2 4 contains the values of $p^N_{\mu,exact}$ and p^N_{exact} for various values of μ and N We can see that this table validates the theory given in Theorem 2 5 1 Note that in this theorem, theoretical error bounds of $N^{-1}(\ln N)^2$ were obtained, however, the numerical orders suggest a rate of $N^{-1} \ln N$ It is expected that more sophisticated barrier function techniques could be used to achieve this result

Table 2 1 The maximum pointwise errors $E^N_{\varepsilon,\mu,exact}$ and the ε-uniform maximum pointwise errors $E^N_{\mu,exact}$ generated by the upwind finite difference operator (2 4 1a) and the mesh (2 4 1c) applied to problem (2 6 1) for $\mu = 2^{-16}$ and for various values of ε and N

	Number of intervals N						
ε	32	64	128	256	512	1024	2048
2^{0}	1 67e-05	4 19e-06	1 05e-06	2 67e-07	6 82e-08	1 78e-08	4 84e-09
2^{-2}	1 61e-04	4 04e-05	1 02e-05	2 57e-06	6 57e-07	1 72e-07	4 66e-08
2^{-4}	6 68e-04	1 68e-04	4 22e-05	1 07e-05	2 73e-06	7 13e-07	1 93e-07
2^{-6}	1 94e-03	4 88e-04	1 23e-04	3 11e-05	7 95e-06	2 08e-06	5 64e-07
2^{-8}	7 52e-03	1 92e-03	4 84e-04	1 23e-04	3 13e-05	8 19e-06	2 22e-06
2^{-10}	2 14e-02	7 54e-03	1 93e-03	4 90e-04	1 25e-04	3 27e-05	8 89e-06
2^{-12}	2 15e-02	8 19e-03	2 84e-03	9 50e-04	3 09e-04	9 97e-05	3 26e-05
2^{-14}	2 17e-02	8 29e-03	2 90e-03	9 82e-04	3 27e-04	1 10e-04	3 80e-05
2^{-16}	2 20e-02	8 49e-03	3 01e-03	1 05e-03	3 63e-04	1 29e-04	4 87e-05
2^{-18}	2 27e-02	8 91e-03	3 25e-03	1 18e-03	4 34e-04	1 69e-04	7 03e-05
2^{-20}	2 41e-02	9 73e-03	3 71e-03	1 43e-03	5 77e-04	2 47e-04	1 13e-04
2^{-22}	2 69e-02	1 14e-02	4 63e-03	1 95e-03	8 64e-04	4 05e-04	2 00e-04
2^{-24}	3 23e-02	1 46e-02	6 46e-03	2 98e-03	1 44e-03	7 20e-04	3 72e-04
2^{-26}	4 28e-02	2 10e-02	1 01e-02	5 03e-03	2 57e-03	1 35e-03	7 15e-04
2^{-28}	6 25e-02	3 32e-02	1 71e-02	9 04e-03	4 81e-03	2 58e-03	1 39e-03
2^{-30}	9 64e-02	5 53e-02	3 02e-02	1 66e-02	9 07e-03	4 95e-03	2 69e-03
2^{-32}	1 45e-01	8 93e-02	5 12e-02	2 92e-02	1 63e-02	8 98e-03	4 92e-03
2^{-34}	1 91e-01	1 26e-01	7 50e-02	4 41e-02	2 50e-02	1 40e-02	7 72e-03
2^{-36}	2 18e-01	1 48e-01	9 05e-02	5 42e-02	3 11e-02	1 75e-02	9 71e-03
2^{-38}	2 27e-01	1 56e-01	9 64e-02	5 81e-02	3 35e-02	1 89e-02	1 05e-02
2^{-40}	2 30e-01	1 58e-01	9 81e-02	5 92e-02	3 42e-02	1 93e-02	1 07e-02
2^{-42}	2 30e-01	1 59e-01	9 85e-02	5 95e-02	3 44e-02	1 94e-02	1 08e-02
2^{-44}	2 31e-01	1 59e-01	9 87e-02	5 95e-02	3 44e-02	1 95e-02	1 08e-02
2^{-46}	2 31e-01	1 59e-01	9 87e-02	5 96e-02	3 44e-02	1 95e-02	1 08e-02
2^{-60}	2 31e-01	1 59e-01	9 87e-02	5 96e-02	3 44e-02	1 95e-02	1 08e-02
$E^N_{\mu,exact}$	2 31e-01	1 59e-01	9 87e-02	5 96e-02	3 44e-02	1 95e-02	1 08e-02

Table 2 2 Exact orders of convergence $p^N_{\varepsilon,\mu,exact}$ and ε-uniform exact orders of convergence $p^N_{\mu,exact}$ generated by the upwind finite difference operator (2 4 1a) and the mesh (2 4 1c) applied to problem (2 6 1) for $\mu = 2^{-16}$ and for various values of ε and N

ε	Number of intervals N							
	8	16	32	64	128	256	512	1024
2^0	2 00	2 00	2 00	1 99	1 98	1 97	1 94	1 88
2^{-2}	1 99	2 00	2 00	1 99	1 98	1 97	1 94	1 88
2^{-4}	1 98	1 99	1 99	1 99	1 98	1 97	1 94	1 88
2^{-6}	1 91	1 97	1 99	1 99	1 98	1 97	1 94	1 88
2^{-8}	1 36	1 91	1 97	1 99	1 98	1 97	1 94	1 88
2^{-10}	0 57	1 18	1 51	1 97	1 98	1 97	1 94	1 88
2^{-12}	0 52	1 17	1 39	1 53	1 58	1 62	1 63	1 61
2^{-14}	0 49	1 17	1 39	1 52	1 56	1 59	1 58	1 53
2^{-16}	0 45	1 16	1 37	1 49	1 53	1 53	1 49	1 41
2^{-18}	0 46	1 15	1 35	1 46	1 46	1 44	1 36	1 26
2^{-20}	0 46	1 12	1 31	1 39	1 37	1 31	1 22	1 13
2^{-22}	0 45	1 08	1 24	1 30	1 25	1 18	1 09	1 02
2^{-24}	0 43	1 02	1 14	1 18	1 12	1 06	1 00	0 95
2^{-26}	0 41	0 93	1 03	1 06	1 00	0 97	0 93	0 91
2^{-28}	0 38	0 84	0 91	0 95	0 92	0 91	0 90	0 89
2^{-30}	0 34	0 74	0 80	0 87	0 86	0 87	0 87	0 88
2^{-32}	-0 00	0 66	0 70	0 80	0 81	0 84	0 86	0 87
2^{-34}	0 27	0 59	0 61	0 74	0 77	0 82	0 84	0 86
2^{-36}	0 25	0 57	0 56	0 70	0 74	0 80	0 83	0 85
2^{-38}	0 24	0 56	0 55	0 69	0 73	0 79	0 82	0 85
2^{-40}	0 24	0 55	0 54	0 69	0 73	0 79	0 82	0 85
2^-60	0 08	0 55	0 54	0 69	0 73	0 79	0 82	0 85
$p^N_{\mu,exact}$	0 24	0 55	0 54	0 69	0 73	0 79	0 82	0 85

Table 2 3 The ε-uniform maximum pointwise errors $E^N_{\mu,exact}$ and the (ε, μ)-uniform maximum pointwise errors E^N_{exact} generated by the upwind finite difference operator (2 4 1a) and the mesh (2 4 1c) applied to problem (2 6 1) for various values of μ and N

μ	Number of intervals N						
	32	64	128	256	512	1024	2048
2^0	1 57e-01	1 06e-01	6 51e-02	3 89e-02	2 24e-02	1 26e-02	7 00e-03
2^{-2}	2 37e-01	1 61e-01	9 92e-02	5 95e-02	3 43e-02	1 94e-002	1 07e-02
2^{-4}	2 32e-01	1 59e-01	9 89e-02	5 96e-02	3 44e-02	1 95e-02	1 08e-02
2^{-6}	2 31e-01	1 59e-01	9 87e-02	5 96e-02	3 44e-02	1 95e-02	1 08e-02
2^{-32}	2 31e-01	1 59e-01	9 87e-02	5 96e-02	3 44e-02	1 95e-02	1 08e-02
E^N_{exact}	2 37e-01	1 61e-01	9 92e-02	5 96e-02	3 44e-02	1 95e-02	1 08e-02

Table 2 4 Exact ε-uniform orders of convergence $p^N_{\mu,exact}$ and the exact (ε,μ)-uniform orders of convergence p^N_{exact} generated by the upwind finite difference operator (2 4 1a) and the mesh (2 4 1c) applied to problem (2 6 1) for various values of μ and N

	Number of intervals N							
μ	8	16	32	64	128	256	512	1024
2^{0}	0 48	0 67	0 57	0 70	0 74	0 80	0 83	0 85
2^{-2}	0 31	0 62	0 56	0 69	0 74	0 79	0 82	0 85
2^{-4}	0 26	0 52	0 54	0 69	0 73	0 79	0 82	0 85
2^{-6}	0 24	0 55	0 54	0 69	0 73	0 79	0 82	0 85
2^{-32}	0 24	0 55	0 54	0 69	0 73	0 79	0 82	0 85
p^N_{exact}	0 31	0 62	0 56	0 69	0 74	0 79	0 82	0 85

Chapter 3

Parabolic problems

3 1 Introduction

Consider the following class of singularly perturbed parabolic problems posed on the domain $G = \Omega \times (0,T]$, $\quad \Omega = (0,1)$, $\quad \Gamma = \bar{G}\backslash G$

$$L_{\varepsilon,\mu}u \equiv \varepsilon u_{xx} + \mu a u_x - bu - du_t = f(x,t), \qquad \text{in} \quad G, \tag{3 1 1a}$$

$$u = s(x), \qquad \text{on } \Gamma_B, \tag{3 1 1b}$$

$$u = q_1(t), \quad \text{on } \Gamma_L, \qquad u = q_2(t), \quad \text{on } \Gamma_R, \tag{3 1 1c}$$

$$a(x,t) \geq \alpha > 0, \; b(x,t) \geq \beta > 0, \; d(x,t) \geq \delta > 0, \tag{3 1 1d}$$

where $\Gamma_B = \{(x,0) \mid 0 \leq x \leq 1\}$, $\Gamma_L = \{(0,t) \mid 0 \leq t \leq T\}$ and $\Gamma_R = \{(1,t) \mid 0 \leq t \leq T\}$ We assume sufficient regularity and compatibility at the corners so that the solution and its regular component are sufficiently smooth for our analysis In this chapter we construct a parameter-uniform numerical method [3] for this class of singularly perturbed problems

When the parameter $\mu = 1$, the problem is the well-studied parabolic convection-diffusion problem [8, 25, 31], when $\mu = 0$ we have a parabolic reaction-diffusion problem [17] Parameter-uniform numerical methods composed of standard finite difference operators and piecewise-uniform meshes have been established [8, 25] for both the steady-state and the time dependent versions of (3 1 1) in the two special cases of $\mu = 0$ and $\mu = 1$ These methods have been discussed in Chapter 1

When considering the two-parameter parabolic problem (3 1 1), the initial aim was to take the analysis in Chapter 2 and extend it to deal with the time-dependent problem Difficulties were encountered when attempting this extension, therefore some new ideas were needed

- The analysis in this chapter splits completely into the two cases of $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$ and $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$

- New analytical approaches have been developed in this chapter to define the regular component v and the right layer component w_R in the case of $\mu^2 \geq C\varepsilon$

In Section 3 2, we derive parameter-explicit theoretical bounds on the solution of (3 1 1) and its derivatives We decompose the solution into regular and singular components The definition of these components differ depending on the ratio of μ to $\sqrt{\varepsilon}$ Sharp parameter-explicit bounds on these components and their derivatives are obtained in Section 3 3 In Section 3 4, we apply an upwind finite difference operator on a piecewise uniform mesh in the construction of our numerical algorithm to solve (3 1 1) for all values of the parameters in the range $\mu \in [0,1]$ and $\varepsilon \in (0,1]$ In Chapter 2, the piecewise uniform mesh constructed consisted of the two transition points

$$\sigma_1 = \min\{\frac{1}{4}, \frac{2\ln N}{\eta_1}\} \quad \text{and} \quad \sigma_2 = \min\{\frac{1}{4}, \frac{2\ln N}{\eta_2}\}, \tag{3 1 2}$$

where η_1 is the positive root of the quadratic equation $\varepsilon\eta_1^2 - \mu\alpha\eta_1 - \beta = 0$ and similarly η_2 is the positive root of the quadratic equation $\varepsilon\eta_2^2 + \mu\|a\|\eta_2 - \beta = 0$ In this chapter the choice of transition points in (3 4 1b) is simpler then those given in (3 1 2) and depends on the ratio of μ to $\sqrt{\varepsilon}$ In [12], the similar problem of

$$-\varepsilon u'' + \mu b u' + cu = f \quad \text{in} \quad (0,1), \quad u(0) = v_0, \; u(1) = v_1,$$

is examined These new transition points in (3 4 1b) are also notably simpler then those given in [12] where the piecewise uniform mesh consists of two transition points,

$$\sigma_1 = \min\{\frac{1}{4}, \frac{2\ln N}{\varrho_0}\} \quad \text{and} \quad \sigma_2 = \min\{\frac{1}{4}, \frac{2\ln N}{\varrho_1}\},$$

where

$$\varrho_0 = \max_{x\in[0,1]} \lambda_0(x) < 0 \qquad \text{and} \qquad \varrho_1 = \min_{x\in[0,1]} \lambda_1(x) > 0,$$

with $\lambda_1(x)$ and $\lambda_2(x)$ defined to be the solutions of the charactersitic equation

$$-\varepsilon\lambda(x)^2 + \mu b(x)\lambda(x) + c(x) = 0$$

The error between the continuous and discrete solution is analysed in Section 3 5 and some numerical results are given to illustrate the parameter uniform convergence of the numerical approximations The main results of this chapter have appeared in [22]

These new analytical techniques designed for the two-parameter parabolic problem, can also be applied when considering the ODE in Chapter 2 The final section of this chapter is concerned with higher order methods for (2 1 1) We use the new approach developed for (3 1 1) to define and bound the regular component v, the right layer component w_R, and their derivatives The results of this section were used in [5] to prove parameter-uniform asymptotic error bounds which are essentially second order

Notation particular to this chapter We define the zero order, first order and second order differential operators L_0, L_μ and $L_{\varepsilon,\mu}$ as follows

$$\begin{aligned} L_0 z &= -bz - dz_t, \\ L_\mu z &= a\mu z_x + L_0 z, \\ L_{\varepsilon,\mu} z &= \varepsilon z_{xx} + L_\mu z \end{aligned}$$

We let $\gamma < \min_{\bar{G}}\{\frac{b}{a}\}$ and we also adopt the following notation

$$||u||_{\bar{G}} = \max_{\bar{G}} |u(x,t)|$$

and if the norm is not subscripted then $|| \ || = || \ ||_{\bar{G}}$

3 2 Bounds on the solution u and its derivatives

We will establish *a priori* bounds on the solution of (3 1 1) and its derivatives These bounds will be needed in the error analysis in later sections We start by stating a continuous minimum principle for the differential operator in (3 1 1), whose proof is standard

Minimum Principle 2 *If $w \in C^2(G) \cap C^0(\bar{G})$ such that $L_{\varepsilon,\mu} w \mid_G \leq 0$ and $w \mid_\Gamma \geq 0$ then $w \mid_{\bar{G}} \geq 0$*

The following lemma follows immediately from the above minimum principle and its proof again is standard

Lemma 3 2 1 *The solution u of problem (3 1 1), satisfies the following bound*

$$||u|| \leq ||s||_{\Gamma_B} + ||q_1||_{\Gamma_L} + ||q_2||_{\Gamma_R} + \frac{1}{\beta}||f||$$

Lemma 3 2 2 *Assuming sufficient compatibility, the derivatives of the solution u of (3 1 1) satisfy the following bounds for all nonnegative integers k,m, such that* $1 \le k + 2m \le 3$, *if* $\mu^2 \le C\varepsilon$ *then*

$$\left\|\frac{\partial^{k+m}u}{\partial x^k \partial t^m}\right\| \le \frac{C}{(\sqrt{\varepsilon})^k}\max\left\{||u||, \sum_{k+2m=0}^{2}(\sqrt{\varepsilon})^k\left\|\frac{\partial^{k+m}f}{\partial x^k\partial t^m}\right\|,\right.$$
$$\left.\sum_{i=0}^{4}\left\|\frac{d^{\,i}s}{dx^i}\right\|_{\Gamma_B}, \sum_{i=0}^{4}\left\|\frac{d^{\,i}q_1}{dt^i}\right\|_{\Gamma_L}, \sum_{i=0}^{4}\left\|\frac{d^{\,i}q_2}{dt^i}\right\|_{\Gamma_R}\right\},$$

and if $\mu^2 \ge C\varepsilon$ *then*

$$\left\|\frac{\partial^{k+m}u}{\partial x^k \partial t^m}\right\| \le C\left(\frac{\mu}{\varepsilon}\right)^k\left(\frac{\mu^2}{\varepsilon}\right)^m\max\left\{||u||, \sum_{k+2m=0}^{2}\left(\frac{\varepsilon}{\mu}\right)^k\left(\frac{\varepsilon}{\mu^2}\right)^{m+1}\left\|\frac{\partial^{k+m}f}{\partial x^k\partial t^m}\right\|,\right.$$
$$\left.\sum_{i=0}^{4}\left\|\frac{d^{\,i}s}{dx^i}\right\|_{\Gamma_B}, \sum_{i=0}^{4}\left\|\frac{d^{\,i}q_1}{dt^i}\right\|_{\Gamma_L}, \sum_{i=0}^{4}\left\|\frac{d^{\,i}q_2}{dt^i}\right\|_{\Gamma_R}\right\},$$

where C depends only on the coefficients a, b,and d and their derivatives

Proof The proof of such bounds follows a standard argument (see [17] for example) We start by making a stretching of variables to transform our problem Local estimates in [9] are then applied to this transformed problem and we obtain bounds on the solution and its derivatives We then transform back to our original variables in order to obtain bounds on the solution of the original differential equation and its derivatives

The argument splits into two cases $\mu^2 \le C\varepsilon$ and $\mu^2 \ge C\varepsilon$ If $\mu^2 \le C\varepsilon$ consider the transformation $\xi = \frac{x}{\sqrt{\varepsilon}}$ Our transformed domain is given by $\tilde{G} = \left(0, \frac{1}{\sqrt{\varepsilon}}\right) \times (0,T]$ Also we have $\tilde{u}(\xi,t) = u(x,t)$ with $\tilde{a}$, $\tilde{b}$, $\tilde{d}$ and $\tilde{f}$ defined similarly Applying this transformation to (3 1 1) we obtain

$$\tilde{u}_{\xi\xi} + \frac{\mu}{\sqrt{\varepsilon}}\tilde{a}\tilde{u}_\xi - \tilde{b}\tilde{u} - \tilde{d}\tilde{u}_t = \tilde{f}, \text{ on } \tilde{G}$$

Then for every $\zeta \in \left(0, \frac{1}{\sqrt{\varepsilon}}\right)$ and $\delta > 0$, we denote the rectangle $((\zeta-\delta,\zeta+\delta)\times(0,T]) \cap \tilde{G}$ by $R_{\zeta,\delta}$ The closure of $R_{\zeta,\delta}$ is denoted $\bar{R}_{\zeta,\delta}$ For each $(\zeta,t) \in \tilde{G}$, we use [9] (Lemma 10 1 pg 352) to obtain the following bounds for $1 \le k+2m \le 3$

$$\left\|\frac{\partial^{k+m}\tilde{u}}{\partial\xi^k\partial t^m}\right\|_{\bar{R}_{\zeta\,\delta}} \le C\max\left\{||\tilde{u}||, \sum_{k+2m=0}^{2}\left\|\frac{\partial^{k+m}\tilde{f}}{\partial\xi^k\partial t^m}\right\|, \sum_{i=0}^{4}\left\|\frac{d^{\,i}\tilde{s}}{d\xi^i}\right\|_{\Gamma'_B},\right.$$
$$\left.\sum_{i=0}^{4}\left\|\frac{d^{\,i}\tilde{q_1}}{dt^i}\right\|_{\Gamma'_L}, \sum_{i=0}^{4}\left\|\frac{d^{\,i}\tilde{q_2}}{dt^i}\right\|_{\Gamma'_R}\right\},$$

where $\Gamma'_B = \bar{R}_{\zeta,2\delta} \cap \Gamma_B$, $\Gamma'_L = \bar{R}_{\zeta,2\delta} \cap \Gamma_L$, $\Gamma'_R = \bar{R}_{\zeta,2\delta} \cap \Gamma_R$ and C is independent of the rectangle $R_{\zeta,\delta}$ These bounds hold for any point $(\zeta, t) \in \tilde{G}$ Transforming back to the original (x, t) variables gives us the required result If $\mu^2 \geq C\varepsilon$, then we are required to stretch in time also Introduce the transformation $\varrho = \frac{\mu x}{\varepsilon}, \tau = \frac{\mu^2 t}{\varepsilon}$ Applying this transformation to (3 1 1) we obtain for $\hat{u}(\varrho, \tau) = u(x, t)$

$$\hat{u}_{\varrho\varrho} + \hat{a}\hat{u}_\varrho - \frac{\varepsilon}{\mu^2}\hat{b}\hat{u} - \hat{d}\hat{u}_\tau = \frac{\varepsilon}{\mu^2}\hat{f}, \text{ on } \hat{G}$$

Our transformed domain is given by $\hat{G} = \left(0, \frac{\mu}{\varepsilon}\right) \times \left(0, \frac{\mu^2 T}{\varepsilon}\right]$ Repeat the argument for the previous case to obtain the result □

Corollary 3 2 1 *Assuming sufficient smoothness of f, s, q_1 and q_2, the second order time derivative of the solution of (1 1) satisfies the following bound*

$$\|u_{tt}\| \leq \begin{cases} C, & \text{if} \quad \mu^2 \leq C\varepsilon \\ C\mu^4\varepsilon^{-2}, & \text{if} \quad \mu^2 \geq C\varepsilon \end{cases}$$

Proof Follows using the same argument as in Lemma 3 2 2 □

Note that similar parameter-dependent bounds on the time derivatives also appear in Hemker et al [7] for the case of $\mu = 1$

3 3 Decomposition of the solution

In order to obtain parameter-uniform error estimates, the solution of (3 1 1) is decomposed into a sum of regular and singular components The regular component will be constructed so that the first two space derivatives of this component will be bounded independently of both small parameters Consider the following differential equation

$$L_{\varepsilon,\mu} v = f \text{ on } G \tag{3 3 1}$$

In the case of $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, we decompose v as follows

$$v(x, t, \varepsilon, \mu) = v_0(x, t) + \sqrt{\varepsilon} v_1(x, t, \varepsilon, \mu) + \varepsilon v_2(x, t, \varepsilon, \mu) \tag{3 3 2a}$$

where

$$\begin{aligned}
L_0 v_0 &= f \quad \text{on} \quad \bar{G} \setminus \Gamma_B, & v_0(x,0) = u(x,0), \qquad & (3\ 3\ 2b)\\
\sqrt{\varepsilon} L_0 v_1 &= (L_0 - L_{\varepsilon,\mu}) v_0 \quad \text{on} \quad \bar{G} \setminus \Gamma_B, & v_1(x,0,\varepsilon,\mu) = 0, \qquad & (3\ 3\ 2c)\\
\varepsilon L_{\varepsilon,\mu} v_2 &= \sqrt{\varepsilon}(L_0 - L_{\varepsilon,\mu}) v_1 \quad \text{on } G & v_2|_\Gamma = 0 \qquad & (3\ 3\ 2d)
\end{aligned}$$

We see that $v(0,t,\varepsilon,\mu) = v_0(0,t) + \sqrt{\varepsilon} v_1(0,t,\varepsilon,\mu)$ and $v(1,t,\varepsilon,\mu) = v_0(1,t) + \sqrt{\varepsilon} v_1(1,t,\varepsilon,\mu)$ Assuming sufficient smoothness on the coefficients (a, b, d, $f \in C^6$) and the initial condition $v_0(x,0)$ and noting that $\alpha\mu^2 \leq \gamma\varepsilon$, we see that v_0 and its derivatives with respect to x and t up to sixth order and v_1 and its derivatives with respect to x and t up to fourth order are bounded independently of ε and μ

Since v_2 satisfies a similar equation to u we can apply Lemma 3 2 1 and Lemma 3 2 2 to problem (3 3 2d) We obtain for $0 \leq k + 2m \leq 3$,

$$\left\| \frac{\partial^{k+m} v_2}{\partial x^k \partial t^m} \right\| \leq C \left(\frac{1}{\sqrt{\varepsilon}} \right)^k$$

We conclude that when $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, there exists a function v satisfying (3 3 1) where the boundary conditions of v can be chosen so that it satisfies the following bounds for $0 \leq k + 2m \leq 3$,

$$\left\| \frac{\partial^{k+m} v}{\partial x^k \partial t^m} \right\| \leq C \left(1 + \varepsilon^{\frac{2-k}{2}}\right)$$

From Corollary 3 2 1 we deduce that

$$\|v_{tt}\| \leq C, \quad \text{if} \quad \mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$$

We consider the case of $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$ We again consider the differential equation (3 3 1), however, we decompose v as follows

$$v(x,t,\varepsilon,\mu) = v_0(x,t,\mu) + \varepsilon v_1(x,t,\varepsilon,\mu) + \varepsilon^2 v_2(x,t,\varepsilon,\mu) \qquad (3\ 3\ 3a)$$

where

$$\begin{aligned}
L_\mu v_0 &= f \quad \text{on } G_1, & v_0(x,0,\mu) = u(x,0), \quad v_0(1,t,\mu) \text{ chosen in } (3\ 3\ 6), \qquad & (3\ 3\ 3b)\\
\varepsilon L_\mu v_1 &= (L_\mu - L_{\varepsilon,\mu}) v_0 \quad \text{on } G_1, & v_1(x,0,\varepsilon,\mu) = v_1(1,t,\varepsilon,\mu) = 0, \qquad & (3\ 3\ 3c)\\
\varepsilon^2 L_{\varepsilon,\mu} v_2 &= \varepsilon(L_\mu - L_{\varepsilon,\mu}) v_1 \quad \text{on } G, & v_2(x,t,\varepsilon,\mu)|_\Gamma = 0 \qquad & (3\ 3\ 3d)
\end{aligned}$$

Note that $G_1 = [0,1) \times (0,T]$ We can establish the following for the differential operator L_μ by considering the transformation $w = e^{\beta_1 t} z$ $(\beta_1 < \frac{b}{d})$ and using a proof by contradiction argument Suppose $z \in C^1(G_1) \cap C^0(\bar{G}_1)$ then

$$\text{If} \quad L_\mu z\Big|_{G_1} \leq 0 \quad \text{and} \quad z\Big|_{\Gamma_1} \geq 0, \quad \text{then} \quad z\Big|_{\bar{G}_1} \geq 0, \tag{3 3 4}$$

where $L_\mu z = a\mu z_x - bz - dz_t$, $\Gamma_1 = \Gamma_B \cup \Gamma_R$ and $G_1 = [0,1) \times (0,T]$ We note that the proof only requires that a and d are strictly positive

We will now state and prove the following technical lemmas that are needed when examining the dependence of the components v_0 and v_1 on the parameter μ

Lemma 3 3 1 *Suppose $z(x,t) \in C^1(G_1) \cap C^0(\bar{G}_1)$ satisfies the first order initial-boundary value problem*

$$\begin{aligned} L_\mu z = a\mu z_x - bz - dz_t = f \quad (x,t) \in [0,1) \times [0,T], \\ z(x,0) = g_1(x), \qquad z(1,t) = g_2(t), \end{aligned} \tag{3 3 5}$$

where $a > 0$, $d > 0$ and $b > 0$, then

$$||z|| \leq \frac{1}{\beta}||f|| + ||g_1||_{\Gamma_B} + ||g_2||_{\Gamma_R}$$

Proof Consider $\psi^{\pm}(x,t) = \frac{1}{\beta}||f|| + ||g_1||_{\Gamma_B} + ||g_2||_{\Gamma_R} \pm z(x,t)$ We see that the functions $\psi^{\pm}(x,t)$ are nonnegative for $(x,t) \in \Gamma_1$ Also

$$L_\mu \psi^{\pm}(x,t) = -b(\frac{1}{\beta}||f|| + ||g_1||_{\Gamma_B} + ||g_2||_{\Gamma_R}) \pm f \leq 0,$$

and the required bound on $||z||$ follows by applying (3 3 4)

□

Lemma 3 3 2 *Suppose $z(x,t) \in C^{k+m}(\bar{G}_1)$ satisfies the differential equation (3 3 5), assuming sufficient regularity of the coefficients, its derivatives satisfy the following bounds* *for positive integers k and m,*

$$\left\|\frac{\partial^{k+m} z}{\partial x^k \partial t^m}\right\| \leq \frac{C}{\mu^k}\left(\left\|\frac{\partial^{k+m} f}{\partial t^{k+m}}\right\| + \sum_{r+s=0}^{k+m-1} \mu^r \left\|\frac{\partial^{r+s} f}{\partial x^r \partial t^s}\right\| + \sum_{j=0}^{k+m}\left\|\frac{d^j g_1}{d x^j}\right\| \right.$$
$$\left. + \sum_{j=0}^{k+m}\left\|\frac{d^j g_2}{d t^j}\right\| + ||z||\right) e^{-(k+m)AT},$$

where $A = \min\{0, \left(\frac{a}{d}\right)\left(\frac{d}{a}\right)_t\}$ and the constant C depends only on the coefficients a, b, d and their derivatives

Proof Differentiating (3 3 5) with respect to t, we obtain

$$L_\mu^{[1]} z_t = \mu z_{tx} - \left(\tfrac{b}{a} + \left(\tfrac{d}{a}\right)_t\right) z_t - \tfrac{d}{a} z_{tt} = \left(\tfrac{f}{a}\right)_t + \left(\tfrac{b}{a}\right)_t z,$$
$$z_t(1,t) = g_2'(t), \quad z_t(x,0) = \phi_1(x),$$

where $\phi_1(x)$ can be expressed in terms of g_1, g_1', f and the coefficients of (3 3 5) Consider the barrier functions $\psi_1^{\pm}(x,t) = C(||f|| + ||f_t|| + ||g_1|| + ||g_1'|| + ||g_2'|| + ||z||)e^{-At} \pm z_t$ with A as above For C large enough the functions $\psi_1^{\pm}$ are nonnegative for $(x,t) \in \Gamma_1$ Also

$$\begin{aligned} L_\mu^{[1]}\psi_1^{\pm}(x,t) &= -C\left(\tfrac{b}{a} + \left(\tfrac{d}{a}\right)_t - \tfrac{d}{a}A\right)(||f|| + ||f_t|| + ||g_1|| + ||g_1'|| + ||g_2'|| + ||z||)e^{-At} \\ &\quad \pm \left(\left(\tfrac{f}{a}\right)_t + \left(\tfrac{b}{a}\right)_t z\right), \end{aligned}$$

and, using the definition of A, we see that for C chosen correctly we have $L_\mu^{[1]}\psi_1^{\pm}(x,t) \leq 0$ Therefore using (3 3 4) we obtain

$$||z_t|| \leq C(||f|| + ||f_t|| + ||g_1|| + ||g_1'|| + ||g_2'|| + ||z||)e^{-AT},$$

and using (3 3 5) we have that

$$||z_x|| \leq \frac{C}{\mu}(||f|| + ||f_t|| + ||g_1|| + ||g_1'|| + ||g_2'|| + ||z||)e^{-AT}$$

Proceed by induction Assume the statement true for $0 \leq k + m \leq l$ Differentiate

(3 3 5) $l+1$ times with respect to t to obtain

$$L_\mu^{[l+1]}\frac{\partial^{l+1}z}{\partial t^{l+1}} = \mu\left(\frac{\partial^{l+1}z}{\partial t^{l+1}}\right)_x - \left(\frac{b}{a}+(l+1)\left(\frac{d}{a}\right)_t\right)\left(\frac{\partial^{l+1}z}{\partial t^{l+1}}\right) - \frac{d}{a}\left(\frac{\partial^{l+1}z}{\partial t^{l+1}}\right)_t = \rho(x,t),$$
$$\frac{\partial^{l+1}z}{\partial t^{l+1}}(1,t) = \frac{d^{l+1}g_2}{dt^{l+1}}, \qquad \frac{\partial^{l+1}z}{\partial t^{l+1}}(x,0) = \phi_{l+1}(x)$$

The expression $\rho(x,t)$ involves z and its t derivatives up to order l, f and its t derivatives up to order $l+1$ and the coefficients and their derivatives The function $\phi_{l+1}(x)$ involves g_1 and all its derivatives up to order $l+1$, the derivatives of f of the form $\mu^r\frac{\partial^{r+s}f}{\partial x^r\partial t^s}$ up to order l and the coefficients and their derivatives

Consider the barrier functions

$$\psi_{l+1}^{\pm}(x,t) = C\left(\left|\left|\frac{\partial^{l+1}f}{\partial t^{l+1}}\right|\right| + \sum_{r+s=0}^{l}\mu^r\left|\left|\frac{\partial^{r+s}f}{\partial x^r\partial t^s}\right|\right| + \sum_{j=0}^{l+1}\left|\left|\frac{d^jg_1}{d\,x^j}\right|\right| + \sum_{j=0}^{l+1}\left|\left|\frac{d^jg_2}{d\,t^j}\right|\right| + ||z||\right)e^{-(l+1)At} \pm \frac{\partial^{l+1}z}{\partial t^{l+1}}$$

We see that for C large enough $\psi_{l+1}^{\pm}(x,t)$ are nonnegative for $(x,t)\in\Gamma_1$ Also for C chosen correctly we see that $L_\mu^{[l+1]}\psi_{l+1}^{\pm}(x,t)\le 0$, therefore using (3 3 4) we obtain

$$\left|\left|\frac{\partial^{l+1}z}{\partial t^{l+1}}\right|\right| \le C\left(\left|\left|\frac{\partial^{l+1}f}{\partial t^{l+1}}\right|\right| + \sum_{r+s=0}^{l}\mu^r\left|\left|\frac{\partial^{r+s}f}{\partial x^r\partial t^s}\right|\right| + \sum_{j=0}^{l+1}\left|\left|\frac{d^jg_1}{d\,x^j}\right|\right| + \sum_{j=0}^{l+1}\left|\left|\frac{d^jg_2}{d\,t^j}\right|\right| + ||z||\right)e^{-(l+1)AT}$$

Differentiate (3 3 5) appropriately to obtain the required result for $k+m=l+1$ □

We now continue with our analysis of v_0 and v_1 The following two Lemmas establish that when the boundary condition $v_0(1,t,\mu)$ is chosen correctly, the first two space derivatives of $v_0(x,t,\mu)$ are bounded independent of μ and the space derivatives of $v_1(x,t,\mu)$ are bounded by inverse powers of μ

Lemma 3 3 3 *If v_0 satisfies the first order differential equation (3 3 3b) then there exists a value for $v_0(1,t,\mu)$ such that the following bounds hold for $0\le k+m\le 6$*

$$\left|\left|\frac{\partial^{k+m}v_0}{\partial x^k\partial t^m}\right|\right| \le C(1+\mu^{2-k})$$

Proof We further decompose $v_0(x,t,\mu)$ as follows

$$v_0(x,t,\mu) = s_0(x,t) + \mu s_1(x,t) + \mu^2 s_2(x,t,\mu) \tag{3 3 6a}$$

where

$$\begin{aligned} L_0 s_0 &= f \quad \text{on} \quad \bar{G}\setminus\Gamma_B, \qquad s_0(x,0) = u(x,0), && (3\ 3\ 6b)\\ \mu L_0 s_1 &= (L_0 - L_\mu)s_0 \quad \text{on} \quad \bar{G}\setminus\Gamma_B, \qquad s_1(x,0)=0, && (3\ 3\ 6c)\\ \mu^2 L_\mu s_2 &= \mu(L_0 - L_\mu)s_1 \quad \text{on} \quad G_1 = [0,1)\times(0,T], \qquad s_2|_{\Gamma_1} = 0 && (3\ 3\ 6d)\end{aligned}$$

We see that $v_0(1,t,\mu) = s_0(1,t) + \mu s_1(1,t)$ and if a, b, d, $f \in C^7(G)$ and $u(x,0) \in C^7(\Gamma_B)$, we have

$$\begin{aligned} \left\|\frac{\partial^{k+m} s_0}{\partial x^k \partial t^m}\right\| &\le C \qquad \text{for} \quad 0 \le k+m \le 7, && (3\ 3\ 7)\\ \left\|\frac{\partial^{k+m} s_1}{\partial x^k \partial t^m}\right\| &\le C \qquad \text{for} \quad 0 \le k+m \le 6 \quad \text{and} \quad \left\|\frac{\partial^7 s_1}{\partial x \partial t^6}\right\| \le C && (3\ 3\ 8)\end{aligned}$$

Next we apply Lemma 3 3 1 and Lemma 3 3 2 to obtain for $0 \le k+m \le 6$

$$\left\|\frac{\partial^{k+m} s_2}{\partial x^k \partial t^m}\right\| \le \frac{C}{\mu^k} e^{-(k+m)AT}, \tag{3 3 9}$$

where $A = \min\left\{0, \frac{a}{d}\left(\frac{d}{a}\right)_t\right\}$ Using the decomposition (3 3 6) and the bounds on the components of this decomposition given in (3 3 7), (3 3 8) and (3 3 9), we obtain the required result □

Lemma 3 3 4 *If v_1 satisfies the first order differential equation (3 3 3c) then the following bounds hold for $0 \le k+m \le 4$*

$$\left\|\frac{\partial^{k+m} v_1}{\partial x^k \partial t^m}\right\| \le \frac{C}{\mu^k}$$

Proof We simply apply Lemma 3 3 1 and Lemma 3 3 2 to (3 3 3c) □

Lemma 3 3 5 *If $v_2(x,t,\varepsilon,\mu)$ satisfies the differential equation (3 3 3d) then the following bounds hold for $0 \le k+m \le 3$*

$$\left\|\frac{\partial^{k+m} v_2}{\partial x^k \partial t^m}\right\| \le \frac{C}{\mu^2}\left(\frac{\mu}{\varepsilon}\right)^k \left(\frac{\mu^2}{\varepsilon}\right)^m, \quad \text{if} \quad \mu^2 \ge C\varepsilon$$

Proof Since v_2 satisfies a similar equation to u, we use Lemma 3 2 1 to obtain

$$\|v_2(x,t,\varepsilon,\mu)\| \le \|v_2\|_\Gamma + \frac{1}{\beta}\|v_{1xx}\|$$

Applying the bounds in Lemma 3 3 4 we therefore have

$$\|v_2\| \le \frac{C}{\mu^2}$$

Finally noting that the equation for v_2 has zero boundary conditions, we use Lemma 3 2 2, the bounds for v_1 and the fact that

$$\left(\frac{\varepsilon}{\mu}\right)^k \left\| \frac{\partial^{k+m} v_{1xx}}{\partial x^k \partial t^m} \right\| \le C\mu^{-2}\left(\frac{\varepsilon}{\mu^2}\right)^k \le C\mu^{-2}$$

to obtain the required result □

Substituting these bounds for $v_0(x,t,\mu)$, $v_1(x,t,\mu)$ and $v_2(x,t,\varepsilon,\mu)$ into (3 3 3) and noting that $\mu^2 \ge C\varepsilon$, we conclude that, in this case, there exists a function v satisfying (3 3 1) where the boundary conditions of v can be chosen so that the following bounds holds for $0 \le k + 2m \le 3$,

$$\left\| \frac{\partial^{k+m} v}{\partial x^k \partial t^m} \right\| \le C\left(1 + \left(\frac{\mu}{\varepsilon}\right)^{k-2}\right)$$

Assuming sufficient smoothness of the data, from Corollary 3 2 1 and extending the argument in the previous lemma to the case of $k + 2m = 4$ we deduce that

$$\|v_{tt}\| \le C(1 + \varepsilon^2\mu^{-2}\mu^4\varepsilon^{-2}) \le C, \quad \text{if} \quad \mu^2 \ge \frac{\gamma\varepsilon}{\alpha}$$

In both cases we now have the following decomposition of the solution u into regular and singular components,

$$u(x,t) = v(x,t) + w_L(x,t) + w_R(x,t) \tag{3 3 10a}$$

where w_L and w_R satisfy homogeneous differential equations and

$$
\begin{aligned}
L_{\varepsilon,\mu} v &= f \text{ on } G, \quad v(x,0) = u(x,0), && (3\ 3\ 10\text{b}) \\
& \qquad\qquad v(0,t) \text{ and } v(1,t) \text{ chosen in } (3\ 3\ 2) \text{ or } (3\ 3\ 3), \\
L_{\varepsilon,\mu} w_L &= 0 \text{ on } G, \quad w_L(x,0) = w_L(1,t) = 0, && (3\ 3\ 10\text{c}) \\
& \qquad\qquad w_L(0,t) = u(0,t) - v(0,t) - w_R(0,t), \\
L_{\varepsilon,\mu} w_R &= 0 \text{ on } G, \quad w_R(x,0) = 0, \ w_R(1,t) = u(1,t) - v(1,t), && (3\ 3\ 10\text{d}) \\
& \qquad\qquad \text{if } \mu^2 \le \frac{\gamma\varepsilon}{\alpha}, \text{ then } w_R(0,t) = 0, \\
& \qquad\qquad \text{else } w_R(0,t) \text{ is chosen in } (3\ 3\ 12)
\end{aligned}
$$

The boundary conditions of v are chosen in (3 3 2) or (3 3 3) so that the regular component satisfies the bounds

$$
\left\| \frac{\partial^{k+m} v}{\partial x^k \partial t^m} \right\| \le C\left(1 + \varepsilon^{2-k}\right), \qquad \text{for} \quad 0 \le k + 2m \le 3, \qquad \|v_{tt}\| \le C \qquad (3\ 3\ 11)
$$

When $\mu^2 \le \frac{\gamma\varepsilon}{\alpha}$, the singular components w_L and w_R satisfy the bounds in Lemma 3 2 2 and Corollary 3 2 1 When $\mu^2 \ge \frac{\gamma\varepsilon}{\alpha}$, the value for $w_R(0,t)$ is taken from the following decomposition

$$
w_R(x,t,\varepsilon,\mu) = w_0(x,t,\mu) + \varepsilon w_1(x,t,\mu) + \varepsilon^2 w_2(x,t,\varepsilon,\mu) \qquad (3\ 3\ 12\text{a})
$$

where $v(1,t) = v_0(1,t)$ is given in (3 3 6) and

$$
\begin{aligned}
L_\mu w_0 &= 0 \quad \text{on} \quad G_1, \qquad w_0(x,0,\mu) = 0, \ \ w_0(1,t,\mu) = u(1,t) - v_0(1,t), && (3\ 3\ 12\text{b}) \\
\varepsilon L_\mu w_1 &= (L_\mu - L_{\varepsilon,\mu}) w_0 \quad \text{on} \quad G_1, \qquad w_1(x,0,\mu) = w_1(1,t,\mu) = 0, && (3\ 3\ 12\text{c}) \\
\varepsilon^2 L_{\varepsilon,\mu} w_2 &= \varepsilon (L_\mu - L_{\varepsilon,\mu}) w_1 \quad \text{on} \quad G, \qquad w_2(x,t,\varepsilon,\mu)|_\Gamma = 0 && (3\ 3\ 12\text{d})
\end{aligned}
$$

Lemma 3 3 6 *When $w_R(x,t)$ is defined as in* (3 3 10d), *the following bound holds*

$$
|w_R(0,t)|_{\Gamma_L} \le e^{-2Bt} e^{\frac{-\gamma}{\mu}},
$$

where $B < A = \min\left\{0, \frac{a}{d}\left(\frac{d}{a}\right)_t\right\}$

Proof When $\mu^2 \le \frac{\gamma\varepsilon}{\alpha}$, the result is trivial Consider the case of $\mu^2 \ge \frac{\gamma\varepsilon}{\alpha}$ Using the decomposition (3 3 12), we see that $w_R(0,t) = w_0(0,t) + \varepsilon w_1(0,t)$ We start by analysing $w_0(x,t)$ Consider the barrier functions $\psi^{\pm}(x,t) = Ce^{-\frac{\gamma}{\mu}(1-x)} \pm w_0(x,t)$ We can show that for C large enough $\psi^{\pm}|_{\Gamma_B \cup \Gamma_R} \geq 0$ and we have

$$L_\mu \psi^{\pm}(x,t) = C(a\gamma - b)e^{-\frac{\gamma}{\mu}(1-x)} \leq 0$$

We can therefore apply (3 3 4) in order to obtain

$$|w_0(x,t)| \leq Ce^{\frac{-\gamma}{\mu}(1-x)} \tag{3 3 13}$$

In order to analyse $w_1(x,t)$, we first obtain sharp bounds on $w_{0xx}(x,t)$ Differentiate (3 3 12b) with respect to t to obtain

$$L_\mu^{[1]}(w_{0t}) = \mu(w_{0t})_x - \left(\frac{b}{a} + \left(\frac{d}{a}\right)_t\right)w_{0t} - \frac{d}{a}(w_{0t})_t = \left(\frac{b}{a}\right)_t w_0, \quad \begin{aligned} w_{0t}(x,0) &= 0, \\ w_{0t}(1,t) &= (w_R(1,t))_t \end{aligned}$$

Consider the barrier functions $\psi_1^{\pm}(x,t) = Ce^{-Bt}e^{-\frac{\gamma}{\mu}(1-x)} \pm w_{0t}(x,t)$, where B is as defined We can show that for C large enough $\psi_1^{\pm}|_{\Gamma_B \cup \Gamma_R} \geq 0$ and $L_\mu^{[1]}\psi_1^{\pm}(x,t) \leq 0$ Apply (3 3 4) in order to obtain

$$|w_{0t}(x,t)| \leq Ce^{-Bt}e^{-\frac{\gamma}{\mu}(1-x)}$$

Using the equation for w_0, (3 3 12b), this implies that

$$|w_{0x}(x,t)| \leq \frac{C}{\mu}e^{-Bt}e^{-\frac{\gamma}{\mu}(1-x)}$$

If we differentiate (3 3 12b) twice with respect to t and apply the same argument we obtain

$$|w_{0tt}(x,t)| \leq Ce^{-2Bt}e^{-\frac{\gamma}{\mu}(1-x)},$$

Using the equation for w_0, (3 3 12b), this implies that

$$|w_{0xt}(x,t)| \leq \frac{C}{\mu}e^{-2Bt}e^{-\frac{\gamma}{\mu}(1-x)} \qquad \text{and} \qquad |w_{0xx}(x,t)| \leq \frac{C}{\mu^2}e^{-2Bt}e^{-\frac{\gamma}{\mu}(1-x)}$$

Since we have exponential bounds on w_0 and its derivatives, we can now examine how $w_1(x,t)$ depends on μ Consider the barrier functions $\psi_2^{\pm}(x,t) = \frac{C}{\mu^2}e^{-2Bt}e^{-\frac{\gamma}{\mu}(1-x)} \pm w_1(x,t)$ Note that $\psi_2^{\pm}(x,t)|_{\Gamma_B \cup \Gamma_R} \geq 0$, also for C large enough

$$L_\mu \psi_2^{\pm}(x,t) = C\big[\gamma a - b + Bd\big]\frac{1}{\mu^2}e^{-2Bt}e^{-\frac{\gamma}{\mu}(1-x)} \pm w_{0xx}$$

Therefore using the definitions of γ and B we find $L_\mu \psi_2^{\pm}(x,t) \le 0$, and using (3 3 4), we have

$$|w_1(x,t)| \le \frac{C}{\mu^2} e^{-2Bt} e^{-\frac{\gamma}{\mu}(1-x)} \tag{3 3 14}$$

Since $\mu^2 \ge \frac{\gamma\varepsilon}{\alpha}$ we can use (3 3 12d), (3 3 13) and (3 3 14) to obtain

$$|w_R(0,t)| \le C e^{-2Bt} e^{-\frac{\gamma}{\mu}}$$

$\square$

Lemma 3 3 7 *When the solution of (3 1 1) is decomposed as in (3 3 10a), the singular components w_L and w_R satisfy the following bounds*

$$|w_L(x,t)| \le C e^{-\theta_1 x},$$

$$|w_R(x,t)| \le C e^{-\theta_2 (1-x)},$$

where

$$\theta_1 = \begin{cases} \frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}, & \text{if} \quad \mu^2 \le \frac{\gamma\varepsilon}{\alpha} \\ \frac{\alpha\mu}{\varepsilon}, & \text{if} \quad \mu^2 \ge \frac{\gamma\varepsilon}{\alpha} \end{cases}, \qquad \theta_2 = \begin{cases} \frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}, & \text{if} \quad \mu^2 \le \frac{\gamma\varepsilon}{\alpha} \\ \frac{\gamma}{2\mu}, & \text{if} \quad \mu^2 \ge \frac{\gamma\varepsilon}{\alpha} \end{cases}$$

Proof Consider the following barrier functions

$$\psi^{\pm}(x,t) = C e^{-\theta_1 x} \pm w_L(x,t),$$

In both cases, we find that for C large enough $\psi^{\pm}(x,t)\big|_\Gamma \ge 0$ and $L_{\varepsilon,\mu}\psi^{\pm}(x,t) \le 0$ We apply the Minimum Principle in order to obtain the required bound on $|w_L(x,t)|$

When $\mu^2 \le \frac{\gamma\varepsilon}{\alpha}$, the proof in the case of w_R is similar We consider the barrier functions $\psi^{\pm}(x,t) = C e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)} \pm w_R(x,t)$ Again we find that for C large enough $\psi^{\pm}(x,t)\big|_\Gamma \ge 0$ and, using the definition of γ,

$$\begin{aligned} L_{\varepsilon,\mu}\psi^{\pm}(x,t) &= C\left(\frac{\gamma\alpha}{4} + \frac{\mu a\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}} - b\right) e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)} \\ &\le C\left(\frac{\gamma a}{4} + \frac{\gamma a}{2} - b\right) e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)} \le 0 \end{aligned}$$

Since $w_R(0,t) \ne 0$ in the case of $\mu^2 \ge \frac{\gamma\varepsilon}{\alpha}$, we have to be more careful Consider the barrier functions

$$\psi_1^{\pm}(x,t) = C e^{-2At} e^{\frac{-\gamma}{2\mu}(1-x)} \pm w_R(x,t),$$

where $A = \min\left\{0, \frac{a}{d}\left(\frac{d}{a}\right)_t\right\}$ Using the previous lemma we have that $\psi_1^{\pm}(x,t)\big|_\Gamma \ge 0$ for C

large enough, we also find $L_{\varepsilon,\mu}\psi_1^{\pm}(x,t) \leq 0$ Use the Minimum Principle and the fact that $t \in (0,T]$ to obtain the required bound □

Lemma 3 3 8 *When $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$, w_R the solution of (3 3 10d), satisfies the following bounds*

$$\left\|\frac{\partial^k w_R}{\partial x^k}\right\| \leq C\left(\mu^{-k} + \mu^{-1}\varepsilon^{2-k}\right),\ 1 \leq k \leq 3 \quad and \quad \left\|\frac{\partial^m w_R}{\partial t^m}\right\| \leq C,\ m = 1, 2$$

Proof Consider the decomposition (3 3 12), we start by analysing $w_0(x,t)$ Using the same method as used for v_1 in Lemma 3 3 4 we obtain for $0 \leq k+m \leq 6$

$$\left|\frac{\partial^{k+m} w_0}{\partial x^k \partial t^m}\right| \leq \frac{C}{\mu^k},$$

Using this method again for $w_1(x,t)$ we obtain for $0 \leq k+m \leq 4$

$$\left|\frac{\partial^{k+m} w_1}{\partial x^k \partial t^m}\right| \leq \frac{C}{\mu^{k+2}}$$

We can apply Lemma 3 2 1 to obtain

$$||w_2||_{\bar{G}} \leq ||w_2||_\Gamma + \frac{1}{\beta}||w_{1xx}||_{\bar{G}} \leq \frac{C}{\mu^4}$$

Finally from Lemma 3 2 2 we obtain for $1 \leq k + 2m \leq 3$

$$\left\|\frac{\partial^{k+m} w_2}{\partial x^k \partial t^m}\right\|_{\bar{G}} \leq C\mu^{-4}\left(\frac{\mu}{\varepsilon}\right)^k \left(\frac{\mu^2}{\varepsilon}\right)^m$$

and by Corollary 3 2 1

$$\left\|\frac{\partial^2 w_2}{\partial t^2}\right\|_{\bar{G}} \leq C\mu^{-4}\mu^4\varepsilon^{-2}$$

Using (3 3 12) and $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$ gives us the required result □

Lemma 3 3 9 *When $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$, w_L the solution of (3 3 10c), satisfies the following bounds*

$$\left\|\frac{\partial^k w_L}{\partial x^k}\right\| \leq C\left(\frac{\mu}{\varepsilon}\right)^k,\ 1 \leq k \leq 3 \quad and \quad \left\|\frac{\partial^2 w_L}{\partial t^2}\right\| \leq C(1 + \mu^2\varepsilon^{-1})$$

Proof The bounds on the derivatives of the space derivatives follow from Lemma 3 2 2 and the fact that

$$w_L(0,t) = (u - v_0 - w_0)(0,t) - \varepsilon(v_1 + w_1)(0,t)$$

To obtain the bound on the time derivative we introduce the decomposition

$$w_L(x,t) = w_L(0,t)\phi(x,t) + \varepsilon\mu^{-2}R(x,t)$$

where the function ϕ is the solution of the boundary value problem

$$\varepsilon\phi_{xx} + \mu a(0,t)\phi_x = 0,\ x \in (0,1), \quad \phi(0,t) = 1,\ \phi(1,t) = 0$$

Note that, by using $z^n e^{-z} \le Ce^{-z/2}$, $n \ge 1$, $z \ge 0$, we have

$$\left|\frac{\partial^{k+m}\phi}{\partial x^k \partial t^m}\right| \le C\left(\frac{\mu}{\varepsilon}\right)^k e^{-\frac{\mu\alpha x}{(m+1)\varepsilon}}$$

Note that $R = 0$ on Γ and

$$\mu^{-2}\varepsilon L_{\varepsilon,\mu}R = w_L(0,t)(\mu(a(0,t) - a(x,t))\phi_x + b\phi) + d(w_L(0,t)\phi)_t$$

Thus using

$$|L_{\varepsilon,\mu}R(x,t)| \le \frac{C\mu^2}{\varepsilon}\left(1 + \frac{\mu^2 x}{\varepsilon}\right)e^{-\frac{\mu\alpha x}{\varepsilon}} + C\frac{\mu^2}{\varepsilon}e^{-\frac{\mu\alpha x}{2\varepsilon}} \le \frac{C\mu^2}{\varepsilon}e^{-\frac{\mu\alpha x}{2\varepsilon}}$$

one can deduce that

$$|R(x,t)| \le Ce^{-\frac{\mu\alpha x}{2\varepsilon}}$$

Finally note that for $1 \le k + 2m \le 3$

$$\left\|\frac{\partial^{k+m}(L_{\varepsilon,\mu}R)}{\partial x^k \partial t^m}\right\|_{\bar{G}} \le C\frac{\mu^2}{\varepsilon}\left(\frac{\mu}{\varepsilon}\right)^k$$

Using Lemma 3 2 2 (extended to the case of $k + 2m = 4$) and noting the exponent of $(m+1)$ this implies that

$$\left\|\frac{\partial^2 R}{\partial t^2}\right\| \le C\varepsilon^{-2}\mu^4$$

□

Remark 3 3 1 *When considering the parabolic problem* (3 1 1), *compatibility is an issue Let us consider the following problem with zero boundary conditions*

$$L_{\varepsilon,\mu}u = f \quad on\ G, \qquad u|_\Gamma = 0 \tag{3 3 16}$$

We note that any parabolic problem of the form (3 1 1) *can be transformed into a problem of the form* (3 3 16) *with zero boundary and initial conditions (see [31] for example) Using [9, 17] it can be shown that if*

$$\frac{\partial^{i+j} f}{\partial x^i \partial t^j}(1,0) = \frac{\partial^{i+j} f}{\partial x^i \partial t^j}(0,0) = 0, \quad 0 \le i + 2j \le 2$$

then $u \in C^4(\bar{G})$

Since our method of analysis involves decomposing the solution of (3 1 1) *into a sum of various components, we also need to ensure that each of the components considered satisfy sufficient compatibility conditions However, in the case of zero boundary conditions, all of these components can be traced back to depend on* f *Sufficient compatibility conditions for these components therefore involve ensuring that* f *and a sufficient number of its derivatives are zero at the corners* $(0,0)$ *and* $(1,0)$ *We should note that additional compatibility is required at the corner* $(1,0)$*, since for example in the case of* $\mu^2 \ge \frac{\gamma\varepsilon}{\alpha}$, s_2 *is defined in* (3 3 6d) *to be the solution of a first order problem We need* $s_2 \in C^6(\bar{G}_1)$ *therefore we must impose the condition that* f *and a sufficient number of its derivatives are zero at that corner (see for example [1, 14]) To be specific in the case of* (3 3 6d)*, by assuming that*

$$\frac{\partial^{i+j} f}{\partial x^i \partial t^j}(1,0) = 0, \quad 0 \le i + j \le 7$$

then

$$\frac{\partial^{i+j} s_0}{\partial x^i \partial t^j}(1,0) = 0, \quad 0 \le i + j \le 7 \quad \textit{and} \quad \frac{\partial^{i+j} s_1}{\partial x^i \partial t^j}(1,0) = 0, \quad 0 \le i + j \le 6$$

which (given sufficient regularity of the data) suffices for $s_2 \in C^6(\bar{G})$

It should be noted that this issue of compatibility, while obviously important, is not the main thrust of this thesis Zero order compatibility conditions have been checked in the case of all the components in this chapter

3 4 Discrete problem

We discretize (3 1 1) using a numerical method that is composed of a fully implicit in time and upwinded in space finite difference operator $L^{N,M}$ on a tensor product mesh $\bar{G}^{N,M} = \{(x_i, t_j)\}_{i=0,j=0}^{N,M}$, which is piecewise-uniform in space and uniform in time We have the following discrete problem,

$$L^{N,M}U(x_i,t_j) = \varepsilon\delta_x^2 U + \mu a D_x^+ U - bU - dD_t^- U = f, \qquad (x_i,t_j) \in G^{N,M}$$
$$U = u, \qquad (x_i,t_j) \in \Gamma^{N,M} = \bar{G}^{N,M} \cap \Gamma \tag{3 4 1a}$$

where the finite difference operators D_x^+, D_t^- and δ_x^2 are

$$D_x^+ U(x_i,t_j) = \frac{U(x_{i+1},t_j) - U(x_i,t_j)}{x_{i+1} - x_i}, \qquad D_x^- U(x_i,t_j) = \frac{U(x_i,t_j) - U(x_{i-1},t_j)}{x_i - x_{i-1}},$$

$$D_t^- U(x_i,t_j) = \frac{U(x_i,t_j) - U(x_i,t_{j-1})}{t_j - t_{j-1}} \quad \text{and} \quad \delta_x^2 U(x_i,t_j) = \frac{D_x^+ U(x_i,t_j) - D_x^- U(x_i,t_j)}{(x_{i+1} - x_{i-1})/2}$$

The piecewise-uniform mesh in space Ω^N consists of two transition points

$$\sigma_1 = \begin{cases} \min\left\{\frac{1}{4}, \frac{2\sqrt{\varepsilon}}{\sqrt{\gamma\alpha}} \ln N\right\}, & \text{if} \quad \mu^2 \le \frac{\gamma\varepsilon}{\alpha} \\ \min\left\{\frac{1}{4}, \frac{2\varepsilon}{\mu\alpha} \ln N\right\}, & \text{if} \quad \mu^2 \ge \frac{\gamma\varepsilon}{\alpha} \end{cases},$$

$$\sigma_2 = \begin{cases} \min\left\{\frac{1}{4}, \frac{2\sqrt{\varepsilon}}{\sqrt{\gamma\alpha}} \ln N\right\}, & \text{if} \quad \mu^2 \le \frac{\gamma\varepsilon}{\alpha} \\ \min\left\{\frac{1}{4}, \frac{2\mu}{\gamma} \ln N\right\}, & \text{if} \quad \mu^2 \ge \frac{\gamma\varepsilon}{\alpha} \end{cases} \tag{3 4 1b}$$

More specifically

$$\Omega^N = \left\{ x_i | x_i = \begin{cases} \frac{4\sigma_1 i}{N}, & \text{if} \quad i \le \frac{N}{4} \\ \sigma_1 + (i - \frac{N}{4})H, & \text{if} \quad \frac{N}{4} \le i \le \frac{3N}{4} \\ 1 - \sigma_2 + (i - \frac{3N}{4})\frac{4\sigma_2}{N}, & \text{if} \quad \frac{3N}{4} \le i \le N \end{cases} \right\}, \tag{3 4 1c}$$

where $NH = 2(1 - \sigma_1 - \sigma_2)$ and the mesh in time is taken to be uniform with $t_j = \frac{j}{M}$, $j = 0, \quad M$ We now state a discrete comparison principle for the finite difference operator in (3 4 1a), whose proof is standard

Discrete Minimum Principle *If W is any mesh function and $L^{N,M}W\,|_{G^{N\,M}} \le 0$ and $W\,|_{\Gamma^{N\,M}} \ge 0$, then $W\,|_{\bar{G}^{N\,M}} \ge 0$*

A standard corollary to this is that For any mesh function Z

$$||Z|| \le C||L^{N,M}Z|| + ||Z||_{\Gamma^{N\,M}} \tag{3 4 2}$$

The discrete solution can be decomposed in an analogous fashion to the continuous solution We have the sum

$$U = V + W_L + W_R \tag{3 4 3a}$$

where the components V, W_L and W_R are the solutions of the following

$$L^{N,M}V = f, \qquad V|_{\Gamma^{N M}} = v|_{\Gamma^{N M}}, \tag{3 4 3b}$$

$$L^{N,M}W_L = 0, \qquad W_L|_{\Gamma^{N M}} = w_L|_{\Gamma^{N M}}, \tag{3 4 3c}$$

$$L^{N,M}W_R = 0, \qquad W_R|_{\Gamma^{N M}} = w_R|_{\Gamma^{N M}} \tag{3 4 3d}$$

Theorem 3 4 1 *We have the following bounds on W_L and W_R*

$$|W_L(x_j, t_k)| \leq C \prod_{i=1}^{j} (1 + \theta_L h_i)^{-1} = \Psi_{L,j}, \qquad \Psi_{L,0} = C \tag{3 4 4a}$$

$$|W_R(x_j, t_k)| \leq C \prod_{i=j+1}^{N} (1 + \theta_R h_i)^{-1} = \Psi_{R,j}, \qquad \Psi_{R,N} = C \tag{3 4 4b}$$

where W_L and W_R are solutions of (3 4 3c) and (3 4 3d) respectively, $0 \leq j \leq N$, $0 \leq k \leq M$, $h_i = x_i - x_{i-1}$ *and the parameters θ_L and θ_R are defined as follows*

$$\theta_L = \begin{cases} \frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}, & if \quad \mu^2 \leq \frac{\gamma\varepsilon}{\alpha} \\ \frac{\mu\alpha}{2\varepsilon}, & if \quad \mu^2 \geq \frac{\gamma\varepsilon}{\alpha} \end{cases}, \qquad \theta_R = \begin{cases} \frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}, & if \quad \mu^2 \leq \frac{\gamma\varepsilon}{\alpha} \\ \frac{\gamma}{2\mu}, & if \quad \mu^2 \geq \frac{\gamma\varepsilon}{\alpha} \end{cases} \tag{3 4 4c}$$

We note that $\theta_L = \frac{\theta_1}{2}$ and $\theta_R = \theta_2$, where θ_1 and θ_2 are defined in Lemma 3 3 7

Proof We start with W_L Consider $\Phi_L^{\pm}(x_j, t_k) = \Psi_{L,j} \pm W_L(x_j, t_k)$ We have $L^{N,M}\Phi_L^{\pm}(x_j, t_k) = \varepsilon\delta_x^2\Psi_{L,j} + \mu a D_x^+\Psi_{L,j} - b\Psi_{L,j}$ Using the properties

$$\Psi_{L,j} > 0, \quad D_x^+\Psi_{L,j} = -\theta_L\Psi_{L,j+1} < 0, \quad \text{and} \quad \delta_x^2\Psi_{L,j} = {\theta_L}^2\Psi_{L,j+1}\frac{h_{j+1}}{\bar{h}_j} > 0,$$

we obtain

$$L^{N,M}\Phi_L^{\pm}(x_j, t_k) = \varepsilon{\theta_L}^2\Psi_{L,j+1}\frac{h_{j+1}}{\bar{h}_j} - \mu a\theta_L\Psi_{L,j+1} - b\Psi_{L,j},$$

where $\bar{h}_j = \frac{h_{j+1}+h_j}{2}$ Rewriting the right hand side of this equation we have

$$L^{N,M}\Phi_{L,j}^{\pm} \leq \Psi_{L,j+1}\left(2\varepsilon{\theta_L}^2\left(\frac{h_{j+1}}{2\bar{h}_j} - 1\right) + (2\varepsilon{\theta_L}^2 - \mu a\theta_L - b) - \beta\theta_L h_{j+1}\right)$$

Using this expression we can show that for both values of θ_L, $L^{N,M}\Phi^{\pm}_{L,j} \leq 0$ Now using the discrete minimum principle we obtain the required bound (3 4 4a)

The same idea is applied to W_R Consider $\Phi^{\pm}_R(x_j, t_k) = \Psi_{R,j} \pm W_R(x_j, t_k)$ If $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, it is easy to see that $\Phi^{\pm}_R(0, t_k) \geq 0$, $\Phi^{\pm}_R(1, t_k) \geq 0$ and $\Phi^{\pm}_R(x_j, 0) \geq 0$ However in the other case we need to look at $\Phi^{\pm}_R(0, t_k)$ in more detail We know that

$$\Phi^{\pm}_R(0, t_k) = C\prod_{i=1}^{N}(1 + \frac{\gamma}{2\mu}h_i)^{-1} \pm W_R(0, t_k)$$

However, given that $e^{-\frac{\gamma}{\mu}h_i} \leq (1 + \frac{\gamma}{2\mu}h_i)^{-1}$ and $e^{-\frac{\gamma}{\mu}} = e^{-\frac{\gamma}{\mu}\sum_{i=1}^{N} h_i} = \prod_{i=1}^{N} e^{-\frac{\gamma}{\mu}h_i}$, we see using Lemma 3 3 6 that $\Phi^{\pm}_R(0, t_k) \geq 0$

Considering both cases together again, $L^{N,M}\Phi^{\pm}_R(x_j, t_k) = \varepsilon\delta^2_x\Psi_{R,j} + \mu a D^+_x\Psi_{R,j} - b\Psi_{R,j}$, and using

$$\Psi_{R,j} \leq \Psi_{R,j+1},\ \Psi_{R,j} > 0,\ D^+_x\Psi_{R,j} = \theta_R\Psi_{R,j},\ \text{and } \delta^2_x\Psi_{R,j} = \frac{\theta_R{}^2}{(1+\theta_R h_j)}\Psi_{R,j}\frac{h_j}{\bar{h}_j},$$

we obtain

$$L^{N,M}\Phi^{\pm}_R(x_j, t_k) \leq \frac{\Psi_{R,j}}{(1+\theta_R h_j)}\left(2\varepsilon\theta_R{}^2\left(\frac{h_j}{2\bar{h}_j} - 1\right) + (2\varepsilon\theta_R^2 + \mu a\theta_R - b)(1+\theta_R h_j) - 2\varepsilon\theta_R^3 h_j\right)$$

Again, we can see that for both values of θ_R, that $L^{N,M}\Phi^{\pm}_R(x_j, t_k) \leq 0$ Therefore we apply the discrete minimum principle to obtain the required bound (3 4 4b) □

3 5 Error analysis

In this section, we analyse the error between the continuous solution of (3 1 1) and the discrete solution of (3 4 1) This is done by analysing the error in approximating each of the components in the decomposition (3 3 10a) separately

Lemma 3 5 1 *At each mesh point $(x_i, t_j) \in \bar{G}^{N,M}$ the regular component of the error satisfies the following estimate*

$$|(V - v)(x_i, t_j)| \leq C(N^{-1} + M^{-1}),$$

where v is the solution of (3 3 10b) and V is the solution of (3 4 3b)

Proof Using the usual truncation error argument and (3 3 11) we have

$$|L^{N,M}(V-v)(x_i,t_j)| \leq C_1 N^{-1}\left(\varepsilon||v_{xxx}|| + \mu||v_{xx}||\right) + C_2 M^{-1}||v_{tt}|| \leq C(N^{-1}+M^{-1}),$$

and we apply (3 4 2) to obtain the required result □

Lemma 3 5 2 *At each mesh point $(x_i,t_j) \in \bar{G}^{N,M}$ the left singular component of the error satisfies the following estimate*

$$|(W_L - w_L)(x_i,t_j)| \leq \begin{cases} C(N^{-1}(\ln N) + M^{-1}), & \text{if} \quad \mu^2 \leq C\varepsilon \\ C(N^{-1}(\ln N)^2 + M^{-1}\ln N), & \text{if} \quad \mu^2 \geq C\varepsilon \end{cases}$$

where w_L is the solution of (3 3 10c) and W_L is the solution of (3 4 3c)

Proof We use a classical argument in order to obtain the following truncation error bounds

$$|L^{N,M}(W_L - w_L)(x_i,t_j)| \leq C_1(h_{i+1}+h_i)\left(\varepsilon||w_{Lxxx}|| + \mu||w_{Lxx}||\right) + C_2 M^{-1}||w_{Ltt}|| \quad (3\ 5\ 1)$$

The proof splits into the two cases of (a) $\sigma_1 < \frac{1}{4}$ and (b) $\sigma_1 = \frac{1}{4}$

(a) We consider the case of $\sigma_1 < \frac{1}{4}$ In this case the mesh Ω^N, is piecewise uniform We firstly analyse the error in the region $[\sigma_1, 1) \times (0,T]$ and then we proceed to analyse the fine mesh on $(0,\sigma_1) \times (0,T]$ To obtain the required error bounds in $[\sigma_1,1) \times (0,T]$, we will use Lemma 3 3 7 and (3 4 4a) instead of the usual truncation error argument From (3 4 4a) we have

$$|W_L(x_{\frac{N}{4}},t_j)| \leq C\Big(1+\theta_L \frac{4\sigma_1}{N}\Big)^{-\frac{N}{4}},$$

where θ_L and σ_1 depend on the ratio of μ^2 to ε and are given in (3 4 4c) and (3 4 1b) respectively For both these choices of θ_L and σ_1 we can show that

$$|W_L(x_{\frac{N}{4}},t_j)| \leq C(1+4N^{-1}\ln N)^{-\frac{N}{4}}$$

Letting $t = 4N^{-1}\ln N$ in the inequality $\ln(1+t) > t(1-\frac{t}{2})$, it follows that $(1+4N^{-1}\ln N)^{-\frac{N}{4}} \leq 4N^{-1}$ Therefore

$$|W_L(x_i,t_j)| \leq CN^{-1}, \qquad (x_i,t_j) \in [\sigma_1,1) \times (0,T]$$

Looking at the continuous solution in this region we have from Lemma 3 3 7

$$|w_L(x_i,t_j)| \leq Ce^{-\theta_1 x_i} \leq Ce^{\theta_1\sigma_1} \leq CN^{-2},$$

for both choices of σ_1 and θ_1 Combining these two results we obtain the following error bounds in the region $[\sigma_1, 1) \times (0, T]$ when $\sigma_1 < \frac{1}{4}$

$$|(W_L - w_L)(x_i, t_j)| \leq CN^{-1}$$

We now consider the fine mesh region $(0, \sigma_1) \times (0, T]$ We start with the case $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$ In this case the truncation error bound (3 5 1) simplifies to

$$|L^{N,M}(W_L - w_L)(x_i, t_j)| \leq \frac{C_1}{\sqrt{\varepsilon}}(h_{i+1} + h_i) + C_2 M^{-1} \tag{3 5 2}$$

Since $\sigma_1 < \frac{1}{4}$, using (3 4 1b) and (3 4 1c), we know that $h_{i+1} = h_i = \frac{8\sqrt{\varepsilon}}{\sqrt{\gamma\alpha}} N^{-1} \ln N$ and therefore we obtain

$$|L^{N,M}(W_L - w_L)(x_i, t_j)| \leq C_1(N^{-1} \ln N + M^{-1})$$

Finish using (3 4 2) to obtain the required error bound Next we consider the case of $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$ Here we know that $h_{i+1} = h_i = \frac{8\varepsilon}{\mu\alpha} N^{-1} \ln N$ The bound on the truncation error given in (3 5 1) still holds and therefore using Lemma 3 3 9 we obtain

$$|L^{N,M}(W_L - w_L)(x_i, t_j)| \leq C_1 N^{-1} \ln N + C_2 N^{-1} \frac{\mu^2}{\varepsilon} \ln N + C_3 M^{-1}(1 + \mu^2 \varepsilon^{-1})$$

Choosing

$$\Psi^{\pm}(x_i, t_j) = C\Big(N^{-1} \ln N + M^{-1} + \Big((\sigma_1 - x_i)\frac{\mu}{\varepsilon}\Big)(N^{-1} \ln N + M^{-1})\Big) \pm (W_L - w_L)(x_i, t_j)$$

as our barrier functions, we find that we can choose C large enough so that both functions are nonnegative at all points in $G^{N,M}$ of the form $(0, t_j)$, $(x_{\frac{N}{4}}, t_j)$ and $(x_i, 0)$ and $L^{N,M}\Psi^{\pm}(x_i, t_j) \leq 0$ Therefore applying the discrete minimum principle we obtain

$$|(W_L - w_L)(x_i, t_j)| \leq C\Big(N^{-1} \ln N + M^{-1} + \Big((\sigma_1 - x_i)\frac{\mu}{\varepsilon}\Big)(N^{-1} \ln N + M^{-1})\Big)$$

Finally using $\sigma_1 = \frac{2\varepsilon}{\mu\alpha} \ln N$ we have

$$|(W_L - w_L)(x_i, t_j)| \leq C(N^{-1}(\ln N)^2 + M^{-1} \ln N), \tag{3 5 3}$$

(b) If $\sigma_1 = \frac{1}{4}$ and $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$ then $\sqrt{\frac{\gamma\alpha}{\varepsilon}} \leq 8 \ln N$ The truncation error bound (3 5 2) still holds, and we obtain

$$|L^{N,M}(W_L - w_L)(x_i, t_j)| \leq C_1(N^{-1}\ln N + M^{-1})$$

When $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$ and $\sigma_1 = \frac{1}{4}$ we have $\frac{\mu\alpha}{\varepsilon} \leq 8\ln N$ Our bound (3 5 1) for the truncation error becomes

$$|L^{N,M}(W_L - w_L)(x_i, t_j)| \leq C(N^{-1}(\ln N)^2 + M^{-1}\ln N)$$

In both cases above, we use (3 4 2) to finish

$\square$

Lemma 3 5 3 *At each mesh point $(x_i, t_j) \in \bar{G}^{N,M}$ the right singular component of the error satisfies the following estimate*

$$|(W_R - w_R)(x_i, t_j)| \leq C(N^{-1}\ln N + M^{-1}),$$

where w_R is the solution of (3 3 10d) and W_R is the solution of (3 4 3d)

Proof (**a**) The analysis of this component splits depending on the value of σ_2 We consider the case of $\sigma_2 < \frac{1}{4}$ We will start by examining the region $(0, 1-\sigma_2] \times (0, T]$ Using the discrete bounds (3 4 4b) we obtain

$$|W_R(x_{\frac{3N}{4}}, t_j)| \leq C\Big(1 + \theta_R \frac{4\sigma_2}{N}\Big)^{-\frac{N}{4}},$$

where θ_R and σ_2 depend on the ratio of μ^2 to ε and are given in (3 4 4c) and (3 4 1b) respectively We can show that for both choices of θ_R and σ_2 we have

$$|W_R(x_{\frac{3N}{4}}, t_j)| \leq C\Big(1 + 4N^{-1}\ln N\Big)^{-\frac{N}{4}},$$

and using the same argument as with W_L, we conclude that if $(x_i, t_j) \in (0, 1-\sigma_2] \times (0, T]$, then

$$|W_R(x_i, t_j)| \leq CN^{-1}$$

Next, looking at the continuous solution in this region, we use Lemma 3 3 7, to obtain

$$|w_R(x_i, t_j)| \leq Ce^{-\theta_2(1-x_i)} \leq Ce^{-\theta_2\sigma_2} \leq CN^{-1},$$

for both choices of σ_2 and θ_2 We therefore have the following bounds on the error in the region $(0, 1-\sigma_2] \times (0, T]$ when $\sigma_2 < \frac{1}{4}$

$$|(W_R - w_R)(x_i, t_j)| \leq CN^{-1} \tag{3 5 4}$$

We consider the mesh region $(1-\sigma_2, 1) \times (0, T]$, we have a similar truncation error bound to that in (3 5 1) We start with the case of $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, we can show (3 5 1) simplifies to

$$|L^{N,M}(W_R - w_R)(x_i, t_j)| \leq \frac{C_1}{\sqrt{\varepsilon}}(h_{i+1} + h_i) + C_2 M^{-1} \tag{3 5 5}$$

Since we are in the fine mesh region we have $h_{i+1} = h_i = \frac{8\sqrt{\varepsilon}}{\sqrt{\alpha\gamma}} N^{-1} \ln N$ and using (3 5 5) we now obtain

$$|L^{N,M}(W_R - w_R)(x_i, t_j)| \leq C_1 N^{-1} \ln N + C_2 M^{-1}$$

If $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$, using classical analysis we can obtain the following truncation error bounds

$$|L^{N,M}(W_R - w_R)(x_i, t_j)| \leq C_1(h_{i+1} + h_i)\left(\varepsilon||w_{Rxxx}|| + \mu||w_{Rxx}||\right) + C_2 M^{-1}||w_{Rtt}||$$

Using the bounds on w_R in Lemma 3 3 8, we find that this simplifies to

$$|L^{N,M}(W_R - w_R)(x_i, t_j)| \leq \frac{C_1}{\mu}(h_{i+1} + h_i) + C_2 M^{-1} \tag{3 5 6}$$

Since we are in the fine mesh region we have $h_{i+1} = h_i = \frac{8\mu}{\gamma} N^{-1} \ln N$, and therefore we obtain

$$|L^{N,M}(W_R - w_R)(x_i, t_j)| \leq C_1 N^{-1} \ln N + C_2 M^{-1}$$

Use (3 4 2) to finish in both cases

(b) If $\sigma_2 = \frac{1}{4}$ and $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, then $\sqrt{\frac{\gamma\alpha}{\varepsilon}} \leq 8 \ln N$ and since (3 5 5) holds we have

$$|L^{N,M}(W_R - w_R)(x_i, t_j)| \leq C_1 N^{-1} \ln N + C_2 M^{-1}$$

If $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$ and $\sigma_2 = \frac{1}{4}$, then $\frac{\gamma}{\mu} \leq 8 \ln N$ and using (3 5 6) we obtain

$$|L^{N,M}(W_R - w_R)(x_i, t_j)| \leq C_1 N^{-1} \ln N + C_2 M^{-1}$$

In both cases, we use (3 4 2) to complete the proof

□

Theorem 3 5 1 *At each mesh point $(x_i, t_j) \in \bar{G}^{N,M}$ the maximum pointwise error satisfies the following parameter-uniform error bound*

$$\|U - u\|_{G^{N,M}} \leq \begin{cases} C(N^{-1}(\ln N) + M^{-1}), & \text{if} \quad \mu^2 \leq C\varepsilon \\ C(N^{-1}(\ln N)^2 + M^{-1}\ln N), & \text{if} \quad \mu^2 \geq C\varepsilon \end{cases} \qquad (3\ 5\ 7)$$

where u is the solution of (3 1 1) and U is the solution of (3 4 1)

Proof The proof follows from Lemma 3 5 1, Lemma 3 5 2 and Lemma 3 5 3 □

Remark 3 5 1 *It is worth noting that the error bound (3 5 7) extends to the case of $-1 \leq \mu \leq 1$, where the discrete problem is defined to be*

$$L^{N,M}U(x_i, t_j) = \varepsilon\delta^2 U + \mu a D_x U - bU - dD_t^- U = f, \qquad (x_i, t_j) \in G^{N,M}, \qquad (3\ 5\ 8a)$$

$$D_x = \begin{cases} D_x^- & \text{if} \quad \mu < 0 \\ D_x^+ & \text{if} \quad \mu \geq 0 \end{cases},$$

and the transition points in the piecewise-uniform mesh in space are taken to be

$$\sigma_1 = \begin{cases} \min\left\{\frac{1}{4}, \frac{2|\mu|}{\gamma}\ln N\right\}, & \text{if} \quad \mu \leq -\sqrt{\frac{\gamma\varepsilon}{\mu}} \\ \min\left\{\frac{1}{4}, \frac{2\sqrt{\varepsilon}}{\sqrt{\gamma\alpha}}\ln N\right\}, & \text{if} \quad |\mu| \leq \sqrt{\frac{\gamma\varepsilon}{\alpha}} \\ \min\left\{\frac{1}{4}, \frac{2\varepsilon}{\mu\alpha}\ln N\right\}, & \text{if} \quad \mu \geq \sqrt{\frac{\gamma\varepsilon}{\alpha}} \end{cases}, \qquad (3\ 5\ 8b)$$

$$\sigma_2 = \begin{cases} \min\left\{\frac{1}{4}, \frac{2\varepsilon}{|\mu|\alpha}\ln N\right\}, & \text{if} \quad \mu \leq -\sqrt{\frac{\gamma\varepsilon}{\alpha}} \\ \min\left\{\frac{1}{4}, \frac{2\sqrt{\varepsilon}}{\sqrt{\gamma\alpha}}\ln N\right\}, & \text{if} \quad |\mu| \leq \sqrt{\frac{\gamma\varepsilon}{\alpha}} \\ \min\left\{\frac{1}{4}, \frac{2\mu}{\gamma}\ln N\right\}, & \text{if} \quad \mu \geq \sqrt{\frac{\gamma\varepsilon}{\alpha}} \end{cases} \qquad (3\ 5\ 8c)$$

3 6 Numerical results

The numerical method (3 4 1), has been applied to the following particular problem

$$(\varepsilon u_{xx} + \mu(1+x)u_x - u - u_t)(x,t) = 16x^2(1-x)^2, (x,t) \in (0,1) \times (0,1], \qquad (3\ 6\ 1)$$
$$u|_\Gamma = 0$$

In the numerical experiments, we have taken $N = M$ We define the maximum pointwise two-mesh differences to be

$$D^N_{\varepsilon,\mu} = \|U^N_{\varepsilon,\mu} - \overline{U}^{2N}_{\varepsilon,\mu}\|_{G^{N,M}},$$

where $\overline{U}_{\varepsilon,\mu}^{N}$ is the piecewise linear interpolants of the numerical solutions $U_{\varepsilon,\mu}^{N}$ From these values one can compute the ε-uniform maximum pointwise two-mesh differences D_{μ}^{N} and the (ε, μ)-uniform maximum pointwise two-mesh differences D^N, which are defined by

$$D_{\mu}^{N} = \max_{\varepsilon \in R_\varepsilon} D_{\varepsilon,\mu}^{N}, \qquad D^N = \max_{\mu \in R_\mu} \max_{\varepsilon \in R_\varepsilon} D_{\varepsilon,\mu}^{N},$$

where $R_\varepsilon = [2^{-26}, 1]$ and $R_\mu = [2^{-22}, 1]$ Approximations for the order of local convergence $p_{\varepsilon,\mu}^{N}$, the ε-uniform order of local convergence p_{μ}^{N} and the (ε, μ)-uniform order of convergence p^N are computed from

$$p_{\varepsilon,\mu}^{N} = \log_2 \frac{D_{\varepsilon,\mu}^{N}}{D_{\varepsilon,\mu}^{2N}}, \quad p_{\mu}^{N} = \log_2 \frac{D_{\mu}^{N}}{D_{\mu}^{2N}}, \qquad \text{and} \quad p^N = \log_2 \frac{D^N}{D^{2N}}$$

The numerical results presented in Table 3 1, Table 3 2 and Table 3 3 are in agreement with the theoretical asymptotic error bound (3 5 7)

	Number of intervals $N(=M)$					
ε	8	16	32	64	128	256
2^{0}	0 62	0 76	0 87	0 93	0 96	0 98
2^{-2}	0 76	0 89	0 95	0 97	0 99	0 99
2^{-4}	0 80	0 90	0 95	0 97	0 99	0 99
2^{-6}	0 78	0 85	0 92	0 95	0 98	0 99
2^{-8}	0 68	0 76	0 90	0 97	1 00	1 02
2^{-10}	0 65	0 76	0 86	0 93	0 97	0 99
2^{-12}	0 61	0 75	0 86	0 93	0 97	0 98
2^{-14}	0 60	0 75	0 86	0 93	0 96	0 98
2^{-16}	0 59	0 75	0 86	0 93	0 96	0 98
2^{-18}	0 59	0 75	0 86	0 93	0 96	0 98
2^{-20}	0 59	0 75	0 86	0 93	0 96	0 98
2^{-22}	0 59	0 75	0 86	0 93	0 96	0 98
2^{-24}	0 59	0 75	0 86	0 93	0 96	0 98
2^{-26}	0 59	0 75	0 86	0 93	0 96	0 98
$p_{\mu=2^{-2}}^{N}$	0 59	0 75	0 86	0 93	0 96	0 98

Table 3 1 The orders of local convergence $p_{\varepsilon,\mu}^{N}$ and the ε-uniform orders of local convergence p_{μ}^{N} generated by the upwind finite difference operator (3 4 1a) and the mesh (3 4 1c) applied to problem (3 6 1) for $\mu = 2^{-2}$ and for various values of ε and $N(= M)$

ε	Number of intervals $N(= M)$					
	8	16	32	64	128	256
2^{0}	0 61	0 75	0 87	0 93	0 96	0 98
2^{-2}	0 75	0 88	0 94	0 97	0 98	0 99
2^{-4}	0 80	0 90	0 95	0 98	0 99	0 99
2^{-6}	0 86	0 93	0 97	0 98	0 99	1 00
2^{-8}	0 92	0 96	0 98	0 99	0 99	1 00
2^{-10}	0 93	0 97	0 99	0 99	1 00	1 00
2^{-12}	0 94	0 97	0 99	0 99	1 00	1 00
2^{-14}	0 94	0 97	0 99	0 99	1 00	1 00
2^{-16}	0 94	0 97	0 99	0 99	1 00	1 00
2^{-18}	0 94	0 97	0 99	0 99	1 00	1 00
2^{-20}	0 94	0 97	0 99	0 99	1 00	1 00
2^{-22}	0 94	0 97	0 99	0 99	0 99	0 99
2^{-24}	0 94	0 97	0 98	0 99	0 99	0 99
2^{-26}	0 94	0 97	0 98	0 99	0 99	0 99
$p^N_{\mu=2^{-10}}$	0 94	0 97	0 99	0 99	1 00	1 00

Table 3 2 The orders of local convergence $p^N_{\varepsilon,\mu}$ and the ε-uniform orders of local convergence p^N_μ generated by the upwind finite difference operator (3 4 1a) and the mesh (3 4 1c) applied to problem (3 6 1) for $\mu = 2^{-10}$ and for various values of ε and $N(= M)$

3 7 Higher order methods

This method for the parabolic differential equation can also be applied to the ODE (2 1 1) Moreover, the analysis can be extended in order to allow us obtain a higher order numerical method for (2 1 1) We decompose the solution u of (2 1 1) into regular and singular components This section is concerned with obtaining bounds on these components and their derivatives, these bounds are then used in [5] to prove that the numerical method proposed in this article is of almost second order

The following notation is particular to this section We define the zero order, first order and second order differential operators L_0, L_μ and $L_{\varepsilon,\mu}$ as follows

$$\begin{aligned} L_0 z &= -bz, \\ L_\mu z &= a\mu z_x + L_0 z, \\ L_{\varepsilon,\mu} z &= \varepsilon z_{xx} + L_\mu z \end{aligned}$$

μ	Number of intervals $N(=M)$ 8	16	32	64	128	256
2^0	0 41	0 46	0 58	0 66	0 71	0 80
2^{-2}	0 59	0 75	0 86	0 93	0 96	0 98
2^{-4}	0 85	0 91	0 97	0 98	0 99	1 00
2^{-6}	0 89	0 98	0 97	0 98	1 01	1 00
2^{-10}	0 94	0 97	0 99	0 99	1 00	1 00
2^{-14}	0 95	0 97	0 99	0 99	1 00	1 00
2^{-18}	0 95	0 97	0 99	0 99	1 00	1 00
2^{-22}	0 95	0 97	0 99	0 99	1 00	1 00
p^N	0 95	0 97	0 99	0 99	1 00	1 00

Table 3 3 The orders of ε-uniform local convergence p_μ^N and the (ε, μ)-uniform orders of local convergence p^N generated by the upwind finite difference operator (3 4 1a) and the mesh (3 4 1c) applied to problem (3 6 1) for various values of ε, μ and $N(=M)$

Analogous to (3 3 10a), we have the following decomposition of u

$$u(x) = v(x) + w_L(x) + w_R(x), \tag{3 7 2a}$$

where w_L and w_R satisfy homogeneous differential equations and

$$\begin{aligned}
L_{\varepsilon,\mu} v &= f \text{ on } (0,1), \quad v(0) \text{ and } v(1) \text{ chosen in (3 7 3) or (3 7 4)}, && \text{(3 7 2b)}\\
L_{\varepsilon,\mu} w_L &= 0 \text{ on } (0,1), \quad w_L(0) = u(0) - v(0) - w_R(0), \quad w_L(1) = 0, && \text{(3 7 2c)}\\
L_{\varepsilon,\mu} w_R &= 0 \text{ on } (0,1), \quad w_R(1) = u(1) - v(1), \quad \text{if } \mu^2 \le \frac{\gamma\varepsilon}{\alpha}, \text{ then } w_R(0) = 0, \\
& \qquad \text{else } w_R(0) \text{ is chosen in (3 7 8)} && \text{(3 7 2d)}
\end{aligned}$$

Let us first consider the regular component v in the case of $\mu^2 \le \frac{\gamma\varepsilon}{\alpha}$ We have the following decomposition

$$v(x, \varepsilon, \mu) = v_0(x) + \sqrt{\varepsilon} v_1(x, \varepsilon, \mu) + (\sqrt{\varepsilon})^2 v_2(x, \varepsilon, \mu) + (\sqrt{\varepsilon})^3 v_3(x, \varepsilon, \mu) \tag{3 7 3a}$$

where

$$
\begin{aligned}
-bv_0 &= f, && (3\ 7\ 3b)\\
bv_1 &= \sqrt{\varepsilon}v_0'' + \frac{\mu}{\sqrt{\varepsilon}}av_0', && (3\ 7\ 3c)\\
bv_2 &= \sqrt{\varepsilon}v_1'' + \frac{\mu}{\sqrt{\varepsilon}}av_1', && (3\ 7\ 3d)\\
L_{\varepsilon,\mu}v_3 &= -\sqrt{\varepsilon}v_2'' - \frac{\mu}{\sqrt{\varepsilon}}av_2' \text{ on } (0,1) \qquad v_3(0,\varepsilon,\mu) = v_3(1,\varepsilon,\mu) = 0 && (3\ 7\ 3e)
\end{aligned}
$$

We see that $v(0,\varepsilon,\mu) = v_0(0) + \sqrt{\varepsilon}v_1(0,\varepsilon,\mu) + \varepsilon v_2(0,\varepsilon,\mu)$ and $v(1,\varepsilon,\mu) = v_0(1) + \sqrt{\varepsilon}v_1(1,\varepsilon,\mu) + \varepsilon v_2(1,\varepsilon,\mu)$ Assuming sufficient smoothness on the coefficients (a, b, d, $f \in C^8$) and noting that $\alpha\mu^2 \leq \gamma\varepsilon$, we see that v_0 and its derivatives up to order eight, v_1 and its derivatives up to sixth order and v_2 and its derivatives up to order four are bounded independently of ε and μ

Next we proceed to analyse $v_3(x,\varepsilon,\mu)$ Using the minimum principle for $L_{\varepsilon,\mu}$ and a suitable barrier function we obtain (see Chapter 2, Lemma 2 2 1)

$$\|v_3\| \leq \max\left\{|v_3(0)|, |v_3(1)|\right\} + \frac{1}{\beta}(\|v_2''\| + \|v_2'\|)$$

Applying the bounds on v_2 we therefore have

$$\|v_3\| \leq C$$

Using the differential equation (3 7 3e) and the mean value theorem on an interval of width $\sqrt{\varepsilon}$ and noting that $\mu^2 \leq C\varepsilon$, we obtain (see Chapter 2, Lemma 2 2 2),

$$\left\|\frac{d^k v_3}{d\,x^k}\right\| \leq \frac{C}{(\sqrt{\varepsilon})^k}\max\left\{\|v_3\|, \|v_2''\|, \|v_2'\|\right\} \leq \frac{C}{(\sqrt{\varepsilon})^k} \qquad k = 1,2$$

Differentiating (3 7 3e) and using the above bounds we also obtain

$$\left\|\frac{d^k v_3}{d\,x^k}\right\| \leq \frac{C}{(\sqrt{\varepsilon})^k} \qquad k = 3,4$$

Substituting all of these bounds for $v_0(x,\mu)$, $v_1(x,\mu)$, $v_2(x,\mu)$ and $v_3(x,\varepsilon,\mu)$ into the equation for $v(x,\varepsilon,\mu)$ gives us

$$\left\|\frac{d^i v}{d\,x^i}\right\| \leq C(1 + \sqrt{\varepsilon}^{(3-k)}), \qquad i = 0,1,2,3,4$$

When $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$, we consider the following decomposition

$$v(x,\varepsilon,\mu) = v_0(x,\mu) + \varepsilon v_1(x,\mu) + \varepsilon^2 v_2(x,\mu) + \varepsilon^3 v_3(x,\varepsilon,\mu) \qquad (3\ 7\ 4a)$$

where

$$\begin{aligned}
L_\mu v_0 &= f(x) \ \text{ on } [0,1), & v_0(1,\mu) \text{ chosen in } (3\ 7\ 6), && (3\ 7\ 4b)\\
L_\mu v_1 &= -v_0''(x,\mu) \ \text{ on } [0,1), & v_1(1,\mu) \text{ chosen in } (3\ 7\ 7), && (3\ 7\ 4c)\\
L_\mu v_2 &= -v_1''(x,\mu) \ \text{ on } [0,1), & v_2(1,\mu) = 0, && (3\ 7\ 4d)\\
L_{\varepsilon,\mu} v_3(x,\varepsilon,\mu) &= -v_2''(x,\mu) \ \text{ on } (0,1), & v_3(0,\varepsilon,\mu) = v_3(1,\varepsilon,\mu) = 0 && (3\ 7\ 4e)
\end{aligned}$$

We see that $v(0,\varepsilon,\mu) = v_0(0,\mu) + \varepsilon v_1(0,\mu) + \varepsilon^2 v_2(0,\mu)$ The following lemmas establish that when $v_0(1,\mu)$ and $v_1(1,\mu)$ are chosen correctly, the first three derivatives of $v_0(x,\mu)$ and the first derivative of $v_1(x,\mu)$ are bounded independent of μ

Lemma 3 7 1 *If v_0 satisfies the first order differential equation (3 7 4b) then there exists a value for $v_0(1,\mu)$ such that the following bounds hold for* $0 \leq i \leq 7$

$$\left\| \frac{d^i v_0}{dx^i} \right\| \leq C\left(1 + \frac{1}{\mu^{i-3}}\right)$$

Proof Suppose $z \in C^0([0,1])$, we start by noting that since $a > 0$ and $b > 0$ we can establish the following

$$\text{If} \quad L_\mu z\Big|_{[0,1)} \leq 0 \quad \text{and} \quad z(1) \geq 0, \quad \text{then} \quad z\Big|_{[0,1]} \geq 0, \qquad (3\ 7\ 5)$$

using a simple proof by contradiction argument We decompose $v_0(x,\mu)$ as follows

$$v_0(x,\mu) = s_0(x) + \mu s_1(x) + \mu^2 s_2(x) + \mu^3 s_3(x) \qquad (3\ 7\ 6a)$$

where

$$\begin{aligned}
s_0(x) &= -\frac{f}{b}, && (3\ 7\ 6b)\\
s_1(x) &= \frac{a s_0'(x)}{b}, && (3\ 7\ 6c)\\
s_2(x) &= \frac{a s_1'(x)}{b}, && (3\ 7\ 6d)\\
L_\mu s_3(x,\mu) &= -a s_2'(x) \ \text{ on } [0,1), \qquad s_3(1,\mu) = 0 && (3\ 7\ 6e)
\end{aligned}$$

We see that $v_0(1,\mu) = s_0(1) + \mu s_1(1) + \mu^2 s_2(1)$ and assuming sufficient smoothness of the coefficients, we have

$$\left\|\frac{d^i s_0}{dx^i}\right\| \le C, \quad \left\|\frac{d^i s_1}{dx^i}\right\| \le C \quad \text{and} \quad \left\|\frac{d^i s_2}{dx^i}\right\| \le C \quad \text{for} \quad 0 \le i \le 3$$

Using (3 7 5) and (3 7 6e) we can also obtain

$$\left\|\frac{d^i s_3}{dx^i}\right\| \le \frac{C}{\mu^i} \qquad \text{for} \qquad 0 \le i \le 3$$

We use these bounds for $s_0(x)$, $s_1(x)$, $s_2(x)$ and $s_3(x)$ to obtain $\left\|\frac{d^i v_0}{dx^i}\right\| \le C$ for $0 \le i \le 3$ Differentiate (3 7 4b) to obtain the required result □

Lemma 3 7 2 *If* v_1 *satisfies the first order differential equation (3 7 4c) then there exists a value for* $v_1(1,\mu)$ *such that the following bounds hold for* $0 \le i \le 5$

$$\left\|\frac{d^i v_1}{dx^i}\right\| \le C\left(1 + \frac{1}{\mu^{i-1}}\right)$$

Proof We decompose $v_1(x,\mu)$ as follows

$$v_1(x,\mu) = \rho_0(x) + \mu\rho_1(x) + \mu^2\rho_2(x,\mu) \tag{3 7 7a}$$

where

$$\rho_0(x) = -\frac{v_{0xx}}{b}, \tag{3 7 7b}$$

$$\rho_1(x) = \frac{a\rho_0'(x)}{b}, \tag{3 7 7c}$$

$$L_\mu \rho_2(x,\mu) = -a\rho_1'(x) \text{ on } [0,1), \qquad \rho_2(1,\mu) = 0 \tag{3 7 7d}$$

We see that $v_1(1,\mu) = \rho_0(1) + \mu\rho_1(1)$ and assuming sufficient smoothness of the coefficients, we have

$$\left\|\frac{d^i \rho_0}{dx^i}\right\| \le C\left(1 + \frac{1}{\mu^{i-1}}\right) \quad \text{and} \quad \left\|\frac{d^i \rho_1}{dx^i}\right\| \le C\left(\frac{1}{\mu^i}\right) \quad \text{for} \quad 0 \le i \le 2$$

Using (3 7 5) and (3 7 7d) we can also obtain

$$\left\|\frac{d^i \rho_2}{dx^i}\right\| \le \frac{C}{\mu^{i+1}} \qquad \text{for} \qquad 0 \le i \le 2$$

We use these bounds for $\rho_0(x)$, $\rho_1(x)$, and $\rho_2(x,\mu)$ and their derivatives to obtain $\left\|\frac{d^i v_1}{dx^i}\right\| \le C(1+\mu^{i-1})$ for $i = 0,1,2$ The required result for $0 \le i \le 5$ follows by differentiating the differential equation for v_1 □

Lemma 3 7 3 *If v_2 satisfies the first order differential equation (3 7 4d) then the following bounds hold for $0 \le i \le 4$*

$$\left\|\frac{d^i v_2}{dx^i}\right\| \le C\left(\frac{1}{\mu^{i+1}}\right)$$

Proof The proof follows using (3 7 5), the differential equation (3 7 4d) and the bounds in Lemma 3 7 2 □

Lemma 3 7 4 *If v_3 satisfies the differential equation (3 7 4e) then the following bounds hold for $0 \le i \le 4$,*

$$\left\|\frac{d^i v_3}{dx^i}\right\| \le C\left(\frac{\mu}{\varepsilon}\right)^i\left(\frac{1}{\mu^3}\right)$$

Proof Using the minimum principle for $L_{\varepsilon,\mu}$ (Minimum Principle 1) and a suitable barrier function we obtain (see Chapter 2, Lemma 2 2 1),

$$\|v_3\| \le \max\{|v_3(0)|, |v_3(1)|\} + \frac{1}{\beta}\|v_2''\|$$

Applying the bounds in Lemma 3 7 3 we therefore have

$$\|v_3\| \le \frac{C}{\mu^3}$$

Using the differential equation (3 7 4e) and the mean value theorem on an interval of width $\sqrt{\varepsilon}$ we obtain (see Chapter 2, Lemma 2 2 2),

$$\left\|\frac{d^k v_3}{dx^k}\right\| \le \frac{C}{(\sqrt{\varepsilon})^k}\left(1+\left(\frac{\mu}{\sqrt{\varepsilon}}\right)^k\right)\max\{\|v_3\|, \|v_2''\|\} \qquad k = 1,2$$

Simplifying this expression using Lemma 3 7 3

$$\left\|\frac{d^k v_3}{dx^k}\right\| \le \frac{C}{(\sqrt{\varepsilon})^k}\left(1+\left(\frac{\mu}{\sqrt{\varepsilon}}\right)^k\right)\frac{1}{\mu^3} \qquad k = 1,2$$

Differentiating the equation for v_3 and applying these bounds gives

$$\|v_3'''\| \le \frac{C}{\varepsilon^3} \qquad \text{and} \qquad \|v_3''''\| \le \frac{C\mu}{\varepsilon^4}$$

$\square$

Substituting all of these bounds for $v_0(x,\mu)$, $v_1(x,\mu)$, $v_2(x,\mu)$ and $v_3(x,\varepsilon,\mu)$ into the equation for $v(x,\varepsilon,\mu)$ and noting that $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$ gives

$$\left\|\frac{d^i v}{dx^i}\right\| \leq C\left(1+\left(\frac{\varepsilon}{\mu}\right)^{(3-k)}\right), \qquad i=0,1,2,3,4$$

We next consider the layer components defined in (3 7 2c) and (3 7 2d) The definition of the left-layer component w_L is similar to that in Chapter 2 (see 2 3 11c) In the case of $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, we define w_R as in (2 3 11d) Hence, we need only consider the right layer component w_R in the case of $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$ We have the following lemma

Lemma 3 7 5 *When $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$, w_R, the solution of (3 7 2d), satisfies the following bounds for $0 \leq i \leq 3$,*

$$\left\|\frac{d^i w_R}{dx^i}\right\| \leq \frac{C}{\mu^i}$$

Proof Consider the following decomposition

$$w_R(x,\varepsilon,\mu) = w_0(x,\mu) + \varepsilon w_1(x,\mu) + \varepsilon^2 w_2(x,\mu) + \varepsilon^3 w_3(x,\varepsilon,\mu) \qquad (3\ 7\ 8a)$$

where $v(1) = v_0(1,\mu) + \varepsilon v_1(1,\mu)$ given in (3 7 6) and (3 7 7), and

$$\begin{aligned}
L_\mu w_0 &= 0 \text{ on } [0,1), & w_0(1,\mu) &= u(1) - v(1), & (3\ 7\ 8b)\\
\varepsilon L_\mu w_1 &= (L_\mu - L_{\varepsilon,\mu}) w_0 \text{ on } [0,1), & w_1(1,\mu) &= 0, & (3\ 7\ 8c)\\
\varepsilon^2 L_\mu w_2 &= \varepsilon(L_\mu - L_{\varepsilon,\mu}) w_1 \text{ on } [0,1), & w_2(1,\mu) &= 0, & (3\ 7\ 8d)\\
\varepsilon^3 L_{\varepsilon,\mu} w_3 &= \varepsilon^2(L_\mu - L_{\varepsilon,\mu}) w_2 \text{ on } (0,1), & w_3(x,\varepsilon,\mu)|_\Gamma &= 0 & (3\ 7\ 8e)
\end{aligned}$$

We start by analysing $w_0(x)$ Using (3 7 5) and (3 7 8b) we obtain the following bounds for $0 \leq i \leq 5$

$$\left\|\frac{d^i w_0}{d\,x^i}\right\| \leq \frac{C}{\mu^i} \qquad (3\ 7\ 9)$$

Using this method again for $w_1(x)$ and $w_2(x)$ we obtain

$$\left\|\frac{d^i w_1}{d\,x^i}\right\| \leq \frac{C}{\mu^{i+2}}, \quad 0 \leq i \leq 4, \qquad \text{and} \qquad \left\|\frac{d^i w_2}{d\,x^i}\right\| \leq \frac{C}{\mu^{i+4}} \quad 0 \leq i \leq 3 \qquad (3\ 7\ 10)$$

Finally we consider w_3, we can apply Lemma 2 2 1 to obtain

$$||w_3|| \leq \frac{C}{\mu^6}$$

From Lemma 2 2 2 we have the following bounds for $1 \leq i \leq 2$

$$\left\| \frac{d^i w_3}{dx^i} \right\| \leq \frac{C}{(\sqrt{\varepsilon})^i} \left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^i\right) \frac{C}{\mu^6}$$

Finally differentiating (3 7 8e) we obtain

$$\left\| \frac{d^3 w_3}{dx^3} \right\| \leq \frac{C}{(\sqrt{\varepsilon})^3} \left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^3\right) \frac{C}{\mu^6} + \frac{1}{\varepsilon} \frac{1}{\mu^7}$$

The required bounds follow using (3 7 8) and the inequality $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$ $\square$

Chapter 4

Elliptic PDE's - reaction dominated case

4 1 Introduction

Consider the following class of singularly perturbed elliptic problems posed on the unit square $\Omega = (0,1)^2$,

$$L_{\varepsilon,\mu} u = \varepsilon(u_{xx} + u_{yy}) + \mu(a_1 u_x + a_2 u_y) - bu = f \qquad \text{in } \Omega, \tag{4 1 1a}$$

$$u = s_1(x) \text{ on } \Gamma_B, \qquad u = s_2(x) \text{ on } \Gamma_T, \tag{4 1 1b}$$

$$u = q_1(y) \text{ on } \Gamma_L, \qquad u = q_2(y) \text{ on } \Gamma_R, \tag{4 1 1c}$$

$$a_1(x,y) \geq \alpha_1 > 0, \quad a_2(x,y) \geq \alpha_2 > 0, \quad b(x,y) \geq 2\beta > 0, \tag{4 1 1d}$$

where Γ_B, Γ_T, Γ_L and Γ_R are all subsets of the boundary $\partial\Omega$ and are defined as follows

$$\Gamma_B = \{(x,0) \mid 0 \leq x \leq 1\}, \qquad \Gamma_T = \{(x,1) \mid 0 \leq x \leq 1\},$$
$$\Gamma_L = \{(0,y) \mid 0 \leq y \leq 1\}, \qquad \Gamma_R = \{(1,y) \mid 0 \leq y \leq 1\}$$

We note that $0 < \varepsilon \leq 1$ and $0 \leq \mu \leq 1$ are perturbation parameters Throughout this chapter we consider the case of $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$ ($\gamma < \min_{\bar{D}} \left\{\frac{b}{2a_1}, \frac{b}{2a_2}\right\}$) and we assume sufficient regularity and compatibility so that the solution is sufficiently regular for the following analysis to be valid

There is very little literature available dealing with problems of this type When $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, an $O(\sqrt{\varepsilon})$ layer appears in the neighbourhood of all four edges When $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$

we get layers of width $O(\frac{\varepsilon}{\mu})$ in the neighbourhood of $x=0$ and $y=0$ and layers of width $O(\mu)$ in the neighbourhood of the other two edges The aim of this chapter is to extend the analytical techniques used in Chapter 3, so as to deal with the two-parameter elliptic problem (4 1 1) in the case of $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$ A form of the material in this chapter has appeared in [23]

In Section 4 2, we use a classical argument to obtain parameter-explicit bounds on the solution of (4 1 1) and its derivatives when $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$ We then decompose the solution into regular and singular components Section 4 3 is concerned with the definition of the smooth or regular component v of the solution The layer components are defined in Sections 4 4 and 4 5 Sharp parameter-explicit bounds are obtained on these components and their derivatives In Section 4 6, we propose a numerical method We decompose the discrete solution U in an analogous fashion to the continuous solution u The final section of this chapter is concerned with error analysis We prove that, when $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, we have a parameter uniform numerical method for (4 1 1)

Notation particular to this chapter We define the zero order, first order and second order differential operators L_0, L_μ and $L_{\varepsilon,\mu}$ as follows

$$\begin{aligned} L_0 z &= -bz, \\ L_\mu z &= \mu a_1 z_x + \mu a_2 z_y + L_0 z, \\ L_{\varepsilon,\mu} z &= \varepsilon(z_{xx} + z_{yy}) + L_\mu z \end{aligned}$$

We let

$$\gamma < \min_{\bar{\Omega}}\{\frac{b}{2a_1}, \frac{b}{2a_2}\}$$

and we also adopt the following notation

$$||u||_{\bar{\Omega}} = \max_{\bar{\Omega}} |u(x)| \tag{4 1 3}$$

If the norm is not subscripted then $|| \ \ || = || \ \ ||_{\bar{\Omega}}$

For nonnegative integers k, we define the semi-norms on $C^k(D)$ by

$$|u|_{k,D} = \sum_{i+j=k} \sup_{(x,y)\in D} \left| \frac{\partial^{i+j} u}{\partial x^i \partial y^j} \right|,$$

and the related norms using

$$||u||_{k,D} = \sum_{0 \le j \le k} |u|_{j,D}$$

When $D = \Omega$ we omit the D, and when the norm is not subscripted, we presume that it is the norm with $k = 0$ as defined in (4 1 3) We next consider $C^{k,\lambda}(D)$, the space of functions in $C^k(D)$ whose derivatives of order k are Holder continuous of degree λ We define the associated Holder norms and Holder semi-norms by

$$|u|_{k,\lambda,D} = \sum_{i+j=k} \left| \frac{\partial^{i+j} u}{\partial x^i \partial y^j} \right|_{0,\lambda,D} \qquad \text{and} \qquad ||u||_{k,\lambda,D} = \sum_{0 \le j \le k} |u|_{j,D} + |u|_{k,\lambda,D}$$

4 2 Bounds on the solution u and its derivatives

In this section we will establish *a priori* bounds on the solution of (4 1 1) and its derivatives These bounds are essential for the error analysis in subsequent sections We begin by stating a continuous minimum principle for the differential operator in (4 1 1) The proof of this comparison principle is standard

Minimum Principle 3 *If $w \in C^2(\Omega) \cap C^0(\bar{\Omega})$ such that $L_{\epsilon,\mu} w|_\Omega \le 0$ and $w|_{\partial\Omega} \ge 0$, then $w|_{\bar{\Omega}} \ge 0$*

The following lemma follows directly from the above comparison principle The proof of this lemma is again standard

Lemma 4 2 1 *The solution u of (4 1 1) satisfies the following bound*

$$||u|| \le ||s_1||_{\Gamma_B} + ||s_2||_{\Gamma_T} + ||q_1||_{\Gamma_L} + ||q_2||_{\Gamma_R} + \frac{1}{2\beta}||f||$$

Lemma 4 2 2 *If $f \in C^{1,\lambda}(\bar{\Omega})$, $s, q \in C^{3,\lambda}(0,1)$ are independent of ε and μ, and assuming sufficient compatibility of the boundary data at the corners, the derivatives of the solution of (4 1 1) satisfy the following bounds for all nonnegative integers k and m, where $1 \le k + m \le 3$*

$$\left\| \frac{\partial^{k+m} u}{\partial x^k \partial y^m} \right\| \le C \left(\frac{1}{\sqrt{\varepsilon}} \right)^{k+m} (1 + ||u||), \qquad (4\ 2\ 1)$$

where C depends on the coefficients a_1, a_2 and b and their derivatives

Proof Firstly we consider the following function

$$h(x,y) = \big(s_1(x) - s_1(0)(1-x)\big)(1-y) + \big(s_2(x) - s_2(1)x\big)y$$
$$+\big(q_1(y) - q_1(1)y\big)(1-x) + \big(q_2(y) - q_2(0)(1-y)\big)x$$

Assuming the boundary data of (4 1 1) is continuous at the four corners, we see that h interpolates to the boundary conditions Consider $\omega = u - h$ It is clear that ω satisfies an equation similar to (4 1 1) with zero boundary conditions We have

$$L_{\varepsilon,\mu}\omega = f - L_{\varepsilon,\mu}h = \hat{f} \qquad \text{on} \quad \Omega, \tag{4 2 2a}$$
$$\omega \equiv 0 \qquad \text{on} \quad \partial\Omega \tag{4 2 2b}$$

Consider the transformation $\xi = \frac{(\mu+\sqrt{\varepsilon})x}{\varepsilon}$ and $\eta = \frac{(\mu+\sqrt{\varepsilon})y}{\varepsilon}$ The transformed domain $\tilde{\Omega}$ is given by $\tilde{\Omega} = (0, \frac{\mu+\sqrt{\varepsilon}}{\varepsilon})^2$ Applying this transformation, (4 2 2) now becomes

$$\tilde{\omega}_{\xi\xi} + \tilde{\omega}_{\eta\eta} + \frac{\mu}{\sqrt{\varepsilon}+\mu}\tilde{a_1}\tilde{\omega}_\xi + \frac{\mu}{\sqrt{\varepsilon}+\mu}\tilde{a_2}\tilde{\omega}_\eta - \frac{\varepsilon}{(\sqrt{\varepsilon}+\mu)^2}\tilde{b}\tilde{\omega} = \tilde{f}, \qquad \text{on } \tilde{\Omega},$$

where $\tilde{\omega}(\xi,\eta) = \omega(x,y)$, $\tilde{a_1}$, $\tilde{a_2}$, $\tilde{b}$ are defined similarly and $\tilde{f}(\xi,\eta) = \frac{\varepsilon}{(\sqrt{\varepsilon}+\mu)^2}\hat{f}(x,y)$ For each $(\zeta_1,\zeta_2) \in \tilde{\Omega}$, we denote the rectangle $((\zeta_1-\delta,\zeta_1+\delta)\times(\zeta_2-\delta,\zeta_2+\delta)) \cap \tilde{\Omega}$ by $\tilde{R}_\delta(\zeta_1,\zeta_2)$ Using [14] we see that for all $(\xi,\eta) \in \tilde{\Omega}$ and $\tilde{R}_\delta$ we have

$$|\tilde{\omega}|_{1,\lambda,\tilde{R}_\delta} \le C(||\tilde{f}||_{0,\lambda\tilde{R}_{2\delta}} + ||\tilde{\omega}||_{\tilde{R}_{2\delta}}),$$

and for $l = 0,1$

$$|\tilde{\omega}|_{l+2,\lambda,\tilde{R}_{2\delta}} \le C(||\tilde{f}||_{l,\lambda,\tilde{R}_{2\delta}} + ||\tilde{\omega}||_{\tilde{R}_{2\delta}})$$

Since we know that $|\omega|_{k,\Omega} \le |\omega|_{k,\lambda,\Omega}$, we obtain

$$|\tilde{\omega}|_{1,\tilde{R}_\delta} \le |\omega|_{1,\lambda,\tilde{R}_\delta} \le C(||\tilde{f}||_{0,\lambda,\tilde{R}_{2\delta}} + ||\tilde{\omega}||_{\tilde{R}_{2\delta}}), \tag{4 2 3a}$$

and for $l = 0,1$

$$|\tilde{\omega}|_{l+2,\tilde{R}_\delta} \le |\omega|_{l+2,\lambda,\tilde{R}_\delta} \le C(||\tilde{f}||_{l,\lambda,\tilde{R}_{2\delta}} + ||\tilde{\omega}||_{\tilde{R}_{2\delta}}) \tag{4 2 3b}$$

Transforming back to the original variables this implies for all $(x,y) \in \Omega$ and $R_\delta = R_\delta(x,y) = ((x-\delta,x+\delta)\times(y-\delta,y+\delta)) \cap \Omega$

$$\left(\frac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right)|\omega|_{1,R_{2\delta}} \le C\left(\frac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2}\left(\frac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right)^\lambda ||\hat{f}||_{0,\lambda,R_{2\delta}} + ||\omega||_{R_{2\delta}}\right),$$

and for $l = 0, 1$

$$\begin{aligned}
\left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right)^{l+2} |\omega|_{l+2,R_\delta} \leq\ & C\Bigg(\sum_{v=0}^{l} \left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right)^{v} \left(\tfrac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2}\right) |\hat{f}|_{v,R_{2\delta}} \\
& + \left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right)^{l+\lambda} \left(\tfrac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2}\right) |\hat{f}|_{l,\lambda,R_{2\delta}} + ||\omega||_{R_{2\delta}} \Bigg)
\end{aligned}$$

Replacing $\hat{f}$ by $f - L_{\varepsilon,\mu}h$ and using the definition of h gives us

$$\begin{aligned}
\left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right) |\omega|_{1,R_\delta} \leq\ & C\left(\tfrac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2} \left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right)^{\lambda} ||f - L_{\varepsilon,\mu}h||_{0,\lambda,R_{2\delta}} + ||\omega||_{R_{2\delta}} \right), \\
\leq\ & C\Bigg(\tfrac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2} \left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right)^{\lambda} \big(||f||_{0,\lambda,R_{2\delta}} + ||s_1||_{2,\lambda,R_{2\delta}} + ||s_2||_{2,\lambda,R_{2\delta}} \\
& + ||q_1||_{2,\lambda,R_{2\delta}} + ||q_2||_{2,\lambda,R_{2\delta}} \big) + ||\omega||_{R_{2\delta}} \Bigg),
\end{aligned}$$

and for $l = 0, 1$

$$\begin{aligned}
\left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right)^{l+2} |\omega|_{l+2,R_\delta} \leq\ & C\Bigg(\sum_{v=0}^{l} \left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right)^{v} \left(\tfrac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2}\right) |f - L_{\varepsilon,\mu}h|_{v,R_{2\delta}} \\
& + \left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right)^{l+\lambda} \left(\tfrac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2}\right) |f - L_{\varepsilon,\mu}h|_{l,\lambda,R_{2\delta}} + ||\omega||_{R_{2\delta}} \Bigg) \\
\leq\ & C\Bigg(\sum_{v=0}^{l} \left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right)^{v} \left(\tfrac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2}\right) \big(|f|_{v,R_{2\delta}} + |s_1|_{v+2,R_{2\delta}} + |s_2|_{v+2,R_{2\delta}} \\
& + |q_1|_{v+2,R_{2\delta}} + |q_2|_{v+2,R_{2\delta}} \big) + \left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right)^{l+\lambda} \left(\tfrac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2}\right) \big(|f|_{l,\lambda,R_{2\delta}} \\
& + ||s_1||_{l+2,\lambda,R_{2\delta}} + ||s_2||_{l+2,\lambda,R_{2\delta}} + ||q_1||_{l+2,\lambda,R_{2\delta}} + ||q_2||_{l+2,\lambda,R_{2\delta}} \big) \\
& + ||\omega||_{R_{2\delta}} \Bigg)
\end{aligned}$$

Rearranging these equations, we obtain

$$\begin{aligned}
|\omega|_{1,R_\delta} \leq\ & C\Bigg(\tfrac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2} \left(\tfrac{\mu+\sqrt{\varepsilon}}{\varepsilon}\right)^{1-\lambda} \big(||f||_{0,\lambda,R_{2\delta}} + ||s_1||_{2,\lambda,R_{2\delta}} + ||s_2||_{2,\lambda,R_{2\delta}} + ||q_1||_{2,\lambda,R_{2\delta}} \\
& + ||q_2||_{2,\lambda,R_{2\delta}} \big) + \left(\tfrac{\mu+\sqrt{\varepsilon}}{\varepsilon}\right) ||\omega||_{R_{2\delta}} \Bigg),
\end{aligned}$$

and for $l = 0, 1$

$$\begin{aligned}|\omega|_{l+2,R_\delta} \leq\ & C\Big(\sum_{v=0}^{l}\left(\tfrac{\mu+\sqrt{\varepsilon}}{\varepsilon}\right)^{l+2-v}\left(\tfrac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2}\right)(|f|_{v,R_{2\delta}} + |s_1|_{v+2,R_{2\delta}} + |s_2|_{v+2,R_{2\delta}} \\ & + |q_1|_{v+2,R_{2\delta}} + |q_2|_{v+2,R_{2\delta}}) + \left(\tfrac{\mu+\sqrt{\varepsilon}}{\varepsilon}\right)^{2-\lambda}\left(\tfrac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2}\right)(|f|_{l,\lambda,R_{2\delta}} \\ & + \|s_1\|_{l+2,\lambda,R_{2\delta}} + \|s_2\|_{l+2,\lambda,R_{2\delta}} + \|q_1\|_{l+2,\lambda,R_{2\delta}} + \|q_2\|_{l+2,\lambda,R_{2\delta}}) \\ & + \left(\tfrac{\mu+\sqrt{\varepsilon}}{\varepsilon}\right)^{l+2}\|\omega\|_{R_{2\delta}}\Big)\end{aligned}$$

When $f \in C^{1,\lambda}(\bar{\Omega})$ and $s_1, s_2, q_1, q_2 \in C^{3,\lambda}(0,1)$ are independent of both small parameters, we use the above to obtain

$$\left\|\frac{\partial^{k+m} w}{\partial x^k \partial y^m}\right\| \leq C\left(\frac{\mu+\sqrt{\varepsilon}}{\varepsilon}\right)^{k+m}(1 + \|w\|)$$

Finally, noting that $u = w + h$ and using $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$ we obtain the result □

Remark 4 2 1 *Compatibility conditions to ensure $u \in C^{3,\lambda}(\Omega)$ are given in [6] Han and Kellogg [6] also indicate that for variable coefficient convection-diffusion problems, compatibility conditions to ensure that $u \in C^{k,\lambda}(\Omega)$ for $k > 3$ are in general not available The layer components and the boundary layer components are defined on extended domains such that there are no compatibility issues when a_1, a_2, b and f are extended to be constant in neighbourhoods of the extended-domain corners It can also be shown that the corner layer functions, which are defined on the original domain, inherit their compatibility from u*

4 3 Definition of regular component

In order to obtain parameter-uniform error bounds, the solution of (4 1 1) is decomposed into the sum of regular and layer components The extension idea of Shishkin [29] is essential to ensure no overly artificial compatibility conditions are imposed

The regular component will now be constructed so that its derivatives up to second order are bounded independently of both small parameters Consider the extended domain $\Omega^* = (-d, 1+d) \times (-d, 1+d) \supset \Omega, d > 0$ The differential operators $L^*_{\varepsilon,\mu}$ and L^*_0 coincide with the operators $L_{\varepsilon,\mu}$ and L_0 respectively in Ω We also define smooth extensions a_1^*, a_2^*, b^* and f^* of the functions a_1, a_2, b and f to Ω^*

We consider the differential equation

$$L^*_{\varepsilon,\mu}v^* = f^* \text{ on } \Omega^* \tag{4 3 1}$$

We decompose v^* as follows

$$v^*(x,y,\varepsilon,\mu) = v_0^*(x,y) + \sqrt{\varepsilon}v_1^*(x,y,\varepsilon,\mu) + \varepsilon v_2^*(x,y,\varepsilon,\mu) \tag{4 3 2a}$$

where

$$\begin{aligned} L_0^* v_0^* &= f^*, & (4\ 3\ 2\text{b})\\ \sqrt{\varepsilon}L_0^* v_1^* &= (L_0^* - L^*_{\varepsilon,\mu})v_0^*, & (4\ 3\ 2\text{c})\\ \varepsilon L^*_{\varepsilon,\mu}v_2^* &= \sqrt{\varepsilon}(L_0^* - L^*_{\varepsilon,\mu})v_1^*, \quad \text{on } \Omega^* \quad v_2^*|_{\partial\Omega^*} = 0 & (4\ 3\ 2\text{d}) \end{aligned}$$

Note that v_0^* and v_1^* satisfy zero order differential equations so they pose no compatibility issues Given $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, we see the functions v_0^*, v_1^* and their derivatives are bounded independently of both small parameters We need to be more careful with compatibility when looking at v_2^* We construct our extensions of the functions a_1, a_2, f and b so that $a_1^* \geq 0$, $a_2^* \geq 0$, and $b^* \geq \beta > 0$ at all points in the extended domain Ω^*, and

$$f^* = a_1^* = a_2^* = 0 \qquad b^* = 2\beta, \quad (x,y) \in \Omega^* \backslash D,$$

where D is an open set such that $\bar{\Omega} \subset D \subset \Omega^*$ This ensures the function $g^* = \sqrt{\varepsilon}(L_0^* - L^*_{\varepsilon,\mu})v_1^*$ is zero at the corners of the extended domain We also assume the functions a_1^*, a_2^*, f^* and b^* are sufficiently regular so that we have $g^* \in C^{1,\lambda}(\Omega^*)$ We conclude that $v_2^* \in C^{3,\lambda}(\Omega^*)$, and is therefore sufficiently regular for our analysis

Since v_2^* satisfies a similar equation to (4 1 1), we can apply Lemma 4 2 1 and Lemma 4 2 2 to obtain for $0 \leq k+m \leq 3$

$$\left\| \frac{\partial^{k+m} v_2^*}{\partial x^k \partial y^m} \right\| \leq C \left(\frac{1}{\sqrt{\varepsilon}} \right)^{k+m}$$

We conclude that if we take the regular component v to be the solution of

$$L_{\varepsilon,\mu}v = f, \ (x,y) \in \Omega, \quad v = v^*, \ (x,y) \in \partial\Omega, \tag{4 3 3}$$

assuming the coefficients are sufficiently smooth, we have the following bounds for $0 \leq$

$k+m \leq 3,$

$$\left\|\frac{\partial^{k+m} v}{\partial x^k \partial y^m}\right\| \leq C(1+\varepsilon^{\frac{2-k-m}{2}}), \qquad \mu^2 \leq \frac{\gamma\varepsilon}{\alpha} \tag{4 3 4}$$

4 4 Definition of boundary layer functions

We consider the boundary layer function w_L, associated with the left edge Γ_L In order to obtain bounds on w_L we consider the extended domain $\Omega^{[*,\mathrm{TB}]} = (0,1) \times (-d, 1+d)$ with $0\ 5 > d > 0$ We define w_L^* to be the solution of

$$L_{\varepsilon,\mu}^{[*,\mathrm{TB}]} w_L^* = 0, \qquad (x,y) \in \Omega^{[*,\mathrm{TB}]}, \tag{4 4 1a}$$

$$w_L^*(0,y) = (u-v)^*(0,y), \qquad y \in [-d, 1+d], \tag{4 4 1b}$$

$$w_L^*(1,y) = 0, \qquad y \in [-d, 1+d], \tag{4 4 1c}$$

$$w_L^*(x,-d) = w_L^*(x, 1+d) = 0, \qquad x \in [0,1] \tag{4 4 1d}$$

We define smooth extensions of the coefficients a_1, a_2 and b to the domain $\Omega^{[*,\mathrm{TB}]}$ so that we have

$$\left|\frac{\partial^k a_i^*}{\partial y^k}\right| \leq C(d+y)(1+d-y), \qquad \text{for} \quad i = 1,2 \quad \text{and} \quad k = 0,1,2, \tag{4 4 2a}$$

and

$$\left|\frac{\partial b^*}{\partial y}\right| \leq C(d+y)(1+d-y) \tag{4 4 2b}$$

We also extend the boundary function $(u-v)(0,y)$ so that $(u-v)^*(0,y) = 0$ for $y < -\frac{d}{2}$ and $y > 1 + \frac{d}{2}$, we therefore can show that $|w_L^*(0,y)| \leq C(d+y)(1+d-y)$

Lemma 4 4 1 *Given $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, the left layer function w_L^*, satisfies the following bounds*

$$|w_L^*(x,y)| \leq Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}x} \quad \textit{and} \quad \left\|\frac{\partial^i w_L^*}{\partial y^i}\right\| \leq C(1+\sqrt{\varepsilon}^{1-i}), \quad 0 \leq i \leq 3$$

Proof Consider the barrier functions

$$\psi^{\pm}(x,y) = Ce^{-\sqrt{\frac{\gamma\alpha}{\varepsilon}}x} \pm w_L^*$$

We can see that these functions are nonnegative on the boundary $\partial\Omega^{[*,\mathrm{TB}]}$ Also

$$L_{\varepsilon,\mu}^{[*,\mathrm{TB}]} \psi^{\pm}(x,y) = C\left(\gamma\alpha - \frac{\mu}{\sqrt{\varepsilon}} a_1^* \sqrt{\alpha\gamma} - b^*\right) e^{-\sqrt{\frac{\gamma\alpha}{\varepsilon}}x} \leq 0$$

The exponential bound on w_L^* follows using the comparison principle

It can be shown that the crude bounds in Lemma 4 2 2 hold for w_L^*, for $0 \leq k+m \leq 3$,

$$\left\| \frac{\partial^{k+m} w_L^*}{\partial x^k \partial y^m} \right\| \leq \frac{C}{(\sqrt{\varepsilon})^{k+m}} \tag{4 4 3}$$

Remark 4 4 1 *Note that the boundary data for u are independent of the singular perturbation parameters However, this is not the case for w_L Nevertheless, even though the third derivatives of v may depend adversely on the parameters, this does not change the validity of the above bounds on the derivatives of w_L*

In the direction orthogonal to the layer we need to sharpen these bounds We refer to derivatives in this direction as orthogonal derivatives Consider the barrier functions $\psi^{\pm}(x,y) = C(d+y)(1+d-y) \pm w_L^*$ We see that these functions are nonnegative on the boundary $\partial\Omega^{[*,\mathrm{TB}]}$ for C correctly chosen Also

$$L_{\varepsilon,\mu}^{[*,\mathrm{TB}]}\psi^{\pm}(x,y) = C(-2\varepsilon + \mu(1-2y)a_2^* - b^*(d+y)(1+d-y))$$

Using (4 4 2) and assuming μ is sufficiently small ($C\mu(1+2d) - b < 0$), we obtain $L_{\varepsilon,\mu}^{[*,\mathrm{TB}]}\psi^{\pm}(x,y) \leq 0$ The comparison principle gives us

$$|w_L^*(x,y)| \leq C(d+y)(1+d-y), \quad (x,y) \in \bar{\Omega}^{[*,\mathrm{TB}]} \tag{4 4 4}$$

We can show that $\left|\frac{\partial w_L^*}{\partial y}(0,y)\right| \leq C$ and $\frac{\partial w_L^*}{\partial y}(1,y) = 0$ Using (4 4 4) and the fact that $w_L^*(x,-d) = 0$ and $w_L^*(x,1+d) = 0$, we also obtain

$$\left|\frac{\partial w_L^*}{\partial y}(x,-d)\right| \leq C, \quad \text{and} \quad \left|\frac{\partial w_L^*}{\partial y}(x,d+1)\right| \leq C$$

Differentiate the equation (4 4 1a) with respect to y, we obtain

$$L_{\varepsilon,\mu}^{[*,\mathrm{TB}]}\frac{\partial w_L^*}{\partial y} = -\mu\frac{\partial a_1^*}{\partial y}\frac{\partial w_L^*}{\partial x} - \mu\frac{\partial a_2^*}{\partial y}\frac{\partial w_L^*}{\partial y} + \frac{\partial b^*}{\partial y}w_L^* = \tilde{f} \qquad (x,y) \in \Omega^{[*,\mathrm{TB}]}$$

Using the bounds (4 4 3) and $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, we see that $||\tilde{f}|| \leq C$ The comparison principle along with suitably chosen barrier functions yields the bound

$$\left\|\frac{\partial w_L^*}{\partial y}\right\| \leq C$$

We continue this approach so as to obtain sharper bounds on the higher orthogonal derivatives of w_L^*. Using (4.4.1a), (4.4.1d) and $a_2^*(x,1+d) = a_2^*(x,d) = 0$, we see that $\frac{\partial^2 w_L^*}{\partial y^2}(x,1+d) = \frac{\partial^2 w_L^*}{\partial y^2}(x,-d) = 0$. Also we note that $\left\|\frac{\partial^2 w_L^*}{\partial y^2}(0,y)\right\| \leq C$ and $\frac{\partial^2 w_L^*}{\partial y^2}(1,y) = 0$.

Using Taylor expansions and the bounds (4.3.4), we obtain

$$\left|\frac{\partial^2 w_L^*}{\partial y^2}(0,y)\right| \leq \frac{C}{\sqrt{\varepsilon}}(d+y)(1+d-y)$$

Differentiate (4.4.1a) twice with respect to y, we have

$$L_{\varepsilon,\mu}^{[*,\mathrm{TB}]}\frac{\partial^2 w_L^*}{\partial y^2} = -2\mu\frac{\partial a_1^*}{\partial y}\frac{\partial^2 w_L^*}{\partial x\partial y} - 2\mu\frac{\partial a_2^*}{\partial y}\frac{\partial^2 w_L^*}{\partial y^2}+(2\frac{\partial b^*}{\partial y} - \mu\frac{\partial^2 a_2^*}{\partial y^2})\frac{\partial w_L^*}{\partial y}$$
$$- \mu\frac{\partial^2 a_1^*}{\partial y^2}\frac{\partial w_L^*}{\partial x} + \frac{\partial^2 b^*}{\partial y^2}w_L^* = \tilde{f}_1 \ (x,y) \in \Omega^{[*,\mathrm{TB}]}$$

Again using (4.4.3) and the properties of a_1^*, a_2^* and b^* in (4.4.2), we can show that $|\tilde{f}_1| \leq \frac{C}{\sqrt{\varepsilon}}(d+y)(1+d-y)$. Consider the barrier functions $\psi^{\pm}(x,y) = \frac{C}{\sqrt{\varepsilon}}(d+y)(1+d-y) \pm \frac{\partial^2 w_L^*}{\partial y^2}$. We can see that both these functions are nonnegative on $\partial\Omega^{[*,\mathrm{TB}]}$, and using the conditions $|a_2^*| \leq C_1(d+y)(1+d-y)$ and $C_1\mu(1+2d) - b^* < 0$, we obtain $L_{\varepsilon,\mu}^{[*,\mathrm{TB}]}\psi^{\pm}(x,y) \leq 0$. We conclude

$$\left|\frac{\partial^2 w_L^*}{\partial y^2}(x,y)\right| \leq \frac{C}{\sqrt{\varepsilon}}(d+y)(1+d-y), \qquad \text{on} \quad \Omega^{[*,TB]}$$

Using this bound we obtain

$$\left|\frac{\partial^3 w_L^*}{\partial y^3}(x,-d)\right| \leq \frac{C}{\sqrt{\varepsilon}} \quad \text{and} \quad \left|\frac{\partial^3 w_L^*}{\partial y^3}(x,1+d)\right| \leq \frac{C}{\sqrt{\varepsilon}}$$

We also have $\left|\frac{\partial^3 w_L^*}{\partial y^3}(0,y)\right| \leq \frac{C}{\sqrt{\varepsilon}}$ and $\frac{\partial^3 w_L^*}{\partial y^3}(1,y) = 0$.

Differentiate (4.4.1a) three times with respect to y to obtain $L_{\varepsilon,\mu}^{[*,\mathrm{TB}]}\frac{\partial^3 w_L^*}{\partial y^3} = \tilde{f}_2$. We can show that $||\tilde{f}_2|| \leq \frac{C}{\varepsilon}$ and using suitable barrier functions and the minimum principle for $L_{\varepsilon,\mu}^{[*,\mathrm{TB}]}$, we obtain

$$\left\|\frac{\partial^3 w_L^*}{\partial y^3}\right\| \leq \frac{C}{\varepsilon}$$

□

Define the boundary layer function w_L associated with the left edge Γ_L by

$$L_{\varepsilon,\mu}w_L = 0, \ (x,y) \in \Omega, \tag{4.4.5a}$$

$$w_L = u - v, \ (x,y) \in \Gamma_L, \quad w_L = 0, \ (x,y) \in \Gamma_R, \tag{4.4.5b}$$

$$w_L(x,0) = w_L^*(x,0), \quad w_L(x,1) = w_L^*(x,1) \tag{4.4.5c}$$

Remark 4 4 2 *The condition* $C_1\mu(1+2d) - b^* < 0$ *is a reasonable assumption to make in the case of* $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$ *This is because if* $\mu > C$, *we also have* $\varepsilon > \frac{C^2\alpha}{\gamma}$ *and we are in the non-singularly perturbed case where all the derivatives of the solution are bounded independently of both* ε *and* μ

We now consider w_T, the boundary layer function associated with the top edge Γ_T Our extended domain is given by $\Omega^{[*,\mathrm{LR}]} = (-d, 1+d) \times (0,1)$ and we define w_T using

$$L_{\varepsilon,\mu}^{[*,\mathrm{LR}]} w_T^* = 0, \qquad (x,y) \in \Omega^{[*,\mathrm{LR}]}, \tag{4 4 6a}$$

$$w_T^*(x,1) = (u-v) * (x,1), \qquad x \in [-d, 1+d], \tag{4 4 6b}$$

$$w_T^*(x,0) = 0, \qquad x \in [-d, 1+d], \tag{4 4 6c}$$

$$w_T^*(-d,y) = w_T^*(1+d,y) = 0, \qquad y \in [0,1] \tag{4 4 6d}$$

We have the following lemma analogous to that for w_L^*

Lemma 4 4 2 *Given* $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, *the top layer function* w_T^* *satisfies the following bounds*

$$|w_T^*(x,y)| \leq Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}(1-y)} \quad \text{and} \quad \left\| \frac{\partial^i w_T^*}{\partial x^i} \right\| \leq C(1 + \sqrt{\varepsilon}^{1-i}), \quad 0 \leq i \leq 3$$

Proof The proof is similar to that in Lemma 4 4 1 We consider the barrier functions $\psi^{\pm}(x,y) = Ce^{-\sqrt{\frac{\gamma\alpha}{\varepsilon}}(1-y)} \pm w_T^*$ These functions are nonnegative on the boundary $\partial\Omega^{[*,\mathrm{LR}]}$ Also

$$L_{\varepsilon,\mu}^{[*,\mathrm{LR}]}\psi^{\pm}(x,y) = C\left(\gamma\alpha + \frac{\mu}{\sqrt{\varepsilon}} a_1^* \sqrt{\alpha\gamma} - b^*\right) e^{-\sqrt{\frac{\gamma\alpha}{\varepsilon}}(1-y)} \leq 0,$$

and we obtain the required result

Extensions of a_1, a_2 and b to $\Omega^{[*,\mathrm{LR}]}$ are constructed so that

$$\left| \frac{\partial^k a_i^*}{\partial x^k} \right| \leq C(d+x)(1+d-x), \qquad \text{for} \quad i = 1,2 \quad \text{and} \quad k = 0,1,2,$$

and

$$\left| \frac{\partial b^*}{\partial x} \right| \leq C(d+x)(1+d-x)$$

We can then use the same approach as for w_L^* in Lemma 4 4 1 to obtain the required orthogonal derivative bounds □

Define the boundary layer function w_T associated with the top edge Γ_T by

$$L_{\varepsilon,\mu} w_T = 0, \ (x,y) \in \Omega, \tag{4 4 7a}$$

$$w_T = u - v, \ (x,y) \in \Gamma_T, \quad w_L = 0, \ (x,y) \in \Gamma_B, \tag{4 4 7b}$$

$$w_T(0,y) = w_T^*(0,y), \quad w_T(1,y) = w_T^*(1,y) \tag{4 4 7c}$$

We define the other two layer functions w_R and w_B analogously and obtain corresponding bounds on the functions and their derivatives

4 5 Definition of corner layer functions

We now define our corner layer functions Note that compatibility is now more of an issue as the equations defining these functions are all posed on the non-extended original domain Ω

Consider the corner layer function w_{LB} associated with the corner $\Gamma_{LB} = \Gamma_L \cap \Gamma_B$ We define w_{LB} to be the solution of

$$L_{\varepsilon,\mu} w_{LB} = 0 \ (x,y) \in \Omega, \tag{4 5 1a}$$

$$w_{LB} = -w_B, \ (x,y) \in \Gamma_L, \quad w_{LB} = -w_L, \ (x,y) \in \Gamma_B, \tag{4 5 1b}$$

$$w_{LB} = 0, \ (x,y) \in \Gamma_R, \quad w_{LB} = 0, \ (x,y) \in \Gamma_T \tag{4 5 1c}$$

Note at the corner $(0,0)$, $w_L(x,0)$ is equal to $w_L(0,y) = (u-v)(0,y)$, which is equal to $(u-v)(x,0) = w_B(x,0)$ which in turn is equal to $w_B(0,y)$ Hence $w_L(x,0)$ matches with $w_B(0,y)$ at $(0,0)$

Consider the barrier functions $\psi^{\pm}(x,y) = Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}x} e^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}y} \pm w_{LB}$ Using the exponential bounds on w_L and w_B we see that both functions are nonnegative on Γ Also

$$L_{\varepsilon,\mu}\psi^{\pm}(x,y) = C\left(\left(\gamma\alpha - \mu\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}a_1 - \frac{b}{2}\right) + \left(\gamma\alpha - \mu\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}a_2 - \frac{b}{2}\right)\right) e^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}x} e^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}y},$$

and using the definitions of γ and α we see that $L_{\varepsilon,\mu}\psi^{\pm}(x,y) \leq 0$ Using the minimum principle we therefore obtain

$$|w_{LB}(x,y)| \leq Ce^{-\sqrt{\frac{\gamma\alpha}{\varepsilon}}x} e^{-\sqrt{\frac{\gamma\alpha}{\varepsilon}}y} \tag{4 5 2}$$

Associated with the corner $\Gamma_{RT} = \Gamma_R \cap \Gamma_T$ we define a corner layer function w_{RT}

$$L_{\varepsilon,\mu} w_{RT} = 0 \; (x,y) \in \Omega, \tag{4 5 3a}$$

$$w_{RT} = 0, \; (x,y) \in \Gamma_L, \quad w_{RT} = 0, \; (x,y) \in \Gamma_B, \tag{4 5 3b}$$

$$w_{RT} = -w_T, \; (x,y) \in \Gamma_R, \quad w_{RT} = -w_R, \; (x,y) \in \Gamma_T \tag{4 5 3c}$$

Considering the barrier functions $\psi^{\pm}(x,y) = Ce^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)} e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-y)} \pm w_{RT}$, and noting that

$$L_{\varepsilon,\mu}\psi^{\pm}(x,y) = C\left(\left(\tfrac{\gamma\alpha}{4} + \mu\tfrac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}a_1 - \tfrac{b}{2}\right) + \left(\tfrac{\gamma\alpha}{4} + \mu\tfrac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}a_2 - \tfrac{b}{2}\right)\right) e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)} e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-y)} \leq 0,$$

we establish the bound

$$|w_{RT}| \leq Ce^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)} e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-y)} \tag{4 5 4}$$

Analogous bounds hold for the other corner layer functions w_{LT} and w_{RB}

Remark 4 5 1 *Since the corner layer functions satisfy similar equations to u in* (4 1 1)*, an analogous argument to that in Lemma 4 2 2 holds to obtain bounds on their derivatives We continue from* (4 2 3) *and note that when considering the corner layer functions the $\tilde{f}$ in this equation depends on the transformed boundary data of the corner layer functions and their derivatives For all four corner layers, we can show that $||\tilde{f}||_{0,\lambda,\tilde{R}_{2\delta}} \leq C$ and $||\tilde{f}||_{1,\lambda,\tilde{R}_{2\delta}} \leq C$ Transforming back to the original variables and using the crude bounds on the layer functions in* (4 4 3)*, we obtain the following bounds for all the corner layer components*

$$\left\|\frac{\partial^{k+m} w}{\partial x^k \partial y^m}\right\| \leq C\left(\frac{1}{\sqrt{\varepsilon}}\right)^{k+m}, \quad 0 \leq k+m \leq 3$$

Theorem 4 5 1 *When $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$ the solution u of* (4 1 1) *can be decomposed as*

$$u = v + w_L + w_R + w_T + w_B + w_{LB} + w_{LT} + w_{RB} + w_{RT}$$

where $L_{\varepsilon,\mu} v = f$, and the layer and corner layer functions are each solutions of the homogenous equation $L_{\varepsilon,\mu} w = 0$ Boundary conditions for these functions can be specified so that the bounds on the components and their derivatives given below hold

$$\left\|\frac{\partial^{k+m} v}{\partial x^k \partial y^m}\right\| \le C\left(1+\varepsilon^{\frac{2-k-m}{2}}\right), \quad 0 \le k+m \le 3, \qquad (4\ 5\ 5a)$$

$$|w_L(x,y)| \le Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}x}, \ |w_B(x,y)| \le Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}y}, \qquad (4\ 5\ 5b)$$

$$|w_R(x,y)| \le Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}(1-x)}, \quad |w_T(x,y)| \le Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}(1-y)}, \qquad (4\ 5\ 5c)$$

$$\left\|\frac{\partial^k w_L}{\partial y^k}\right\| \le C\left(1+\sqrt{\varepsilon}^{1-k}\right), \qquad \left\|\frac{\partial^k w_R}{\partial y^k}\right\| \le C\left(1+\sqrt{\varepsilon}^{1-k}\right), \qquad (4\ 5\ 5d)$$

$$\left\|\frac{\partial^k w_B}{\partial x^k}\right\| \le C\left(1+\sqrt{\varepsilon}^{1-k}\right), \qquad \left\|\frac{\partial^k w_T}{\partial x^k}\right\| \le C\left(1+\sqrt{\varepsilon}^{1-k}\right), \qquad (4\ 5\ 5e)$$

$$|w_{LB}(x,y)| \le Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}x}e^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}y}, \qquad (4\ 5\ 5f)$$

$$|w_{LT}(x,y)| \le Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}x}e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-y)}, \qquad (4\ 5\ 5g)$$

$$|w_{RB}(x,y)| \le Ce^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)}e^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}y}, \qquad (4\ 5\ 5h)$$

$$|w_{RT}(x,y)| \le Ce^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)}e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-y)}, \qquad (4\ 5\ 5i)$$

and for all the layer components we have

$$\left\|\frac{\partial^{k+m} w}{\partial x^k \partial y^m}\right\| \le C\left(\frac{1}{\sqrt{\varepsilon}}\right)^{k+m} \quad 0 \le k+m \le 3 \qquad (4\ 5\ 5j)$$

Proof The result follows Lemma 4 2 2, Lemma 4 4 1, Lemma 4 4 2 and equations (4 3 4), (4 5 2) and (4 5 4) □

Remark 4 5 2 *We should note that even though the case of $\mu^2 \le \frac{\gamma\varepsilon}{\alpha}$ behaves similarly to that of reaction diffusion ($\mu = 0$), the analysis and the resulting bounds on the components and their derivatives are not exactly alike One difference we should note is the 2's appearing in the exponential bounds of the corner layer functions associated with the right and top edges These extra 2's are a result of the fact there is a convective term present in* (4 1 1) *These bounds therefore differ slightly from those obtained for the reaction-diffusion problem*

4 6 Discrete problem

In order to discretise (4 1 1), we use a numerical method that is composed of an upwind finite difference scheme applied on a mesh $\Omega^{N,M}$ Consider the following discrete problem

$$
\begin{aligned}
L^{N,M}U(x_i, y_j) &= \varepsilon\delta_x^2 U + \varepsilon\delta_y^2 U + \mu a_1 D_x^+ U + \mu a_2 D_y^+ U - bU \\
&= f, \qquad (x_i, y_j) \in \Omega^{N,M}
\end{aligned}
\tag{4 6 1a}
$$

where D_x^+ and δ_x^2 are the standard forward difference operator and second order centered difference operator respectively (D_y^+ and δ_y^2 defined analogously) The mesh $\Omega^{N,M}$ is defined to be the tensor product of two piecewise-uniform meshes Ω^N and Ω^M Ω^N is divided into three subregions $[0, \sigma^N]$, $[\sigma^N, 1-\sigma^N]$ and $[1-\sigma^N, 1]$ In each of these regions a uniform mesh is placed The transition point σ^N is defined by

$$
\sigma^N = \min\left\{\frac{1}{4}, \frac{2\sqrt{\varepsilon}}{\sqrt{\gamma\alpha}} \ln N\right\}
\tag{4 6 1b}
$$

More specifically

$$
\Omega^N = \left\{ x_i | x_i = \begin{cases} \frac{4\sigma^N i}{N}, & \text{if} \quad i \le \frac{N}{4} \\ \sigma^N + (i - \frac{N}{4})H, & \text{if} \quad \frac{N}{4} \le i \le \frac{3N}{4} \\ 1 - \sigma^N + (i - \frac{3N}{4})\frac{4\sigma^N}{N}, & \text{if} \quad \frac{3N}{4} \le i \le N \end{cases} \right\},
\tag{4 6 1c}
$$

where $NH = 2(1 - 2\sigma^N)$ and Ω^M is defined analogously with transition point σ^M

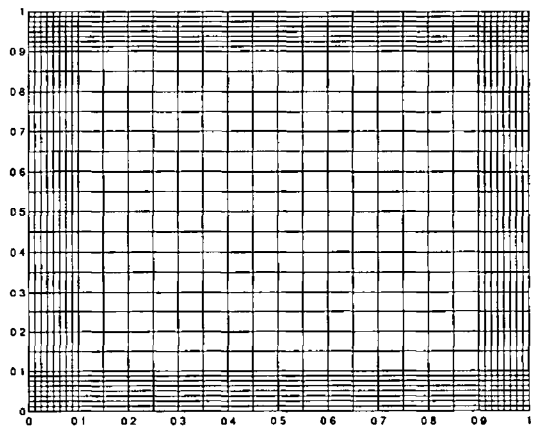


Figure 4 1 A sample piecewise-uniform mesh $\Omega^{N,M}$

Discrete Minimum Principle *If W is any mesh function and $L^{N,M}W|_{\Omega^{N,M}} \le 0$ and $W|_{\Gamma^{N,M}} \ge 0$ then $W|_{\bar{\Omega}^{N,M}} \ge 0$*

We decompose the discrete solution U into the following sum

$$U = V + W_L + W_R + W_B + W_T + W_{LB} + W_{LT} + W_{RB} + W_{RT} \tag{4 6 2a}$$

where

$$L^{N,M}V = f, \qquad V|_{\Gamma^{N M}} = v|_{\Gamma^{N M}}, \tag{4 6 2b}$$

$$L^{N,M}W_L = 0, \qquad W_L|_{\Gamma^{N M}} = w_L|_{\Gamma^{N M}}, \tag{4 6 2c}$$

$$L^{N,M}W_{LB} = 0, \qquad W_{LB}|_{\Gamma^{N M}} = w_{LB}|_{\Gamma^{N M}}, \tag{4 6 2d}$$

with the other layer functions defined similarly

Theorem 4 6 1 *We have the following bounds on the discrete boundary layer function W_L and discrete corner layer function W_{LB},*

$$|W_L(x_i, y_j)| \le C \prod_{s=1}^{i} \left(1 + \frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}} h_s\right)^{-1} = \Psi_{L,i}, \qquad \Psi_{L,0} = C, \tag{4 6 3a}$$

$$|W_{LB}(x_i, y_j)| \le C \prod_{s=1}^{i} \left(1 + h_s \frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\right)^{-1} \prod_{r=1}^{j} \left(1 + k_r \frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\right)^{-1} = \Psi_{L,i}\Psi_{B,j}, \tag{4 6 3b}$$

where W_L and W_{LB} are solutions of (4 6 2c) and (4 6 2d) respectively, $h_s = x_s - x_{s-1}$ and $k_r = y_r - y_{r-1}$

Proof We start with W_L Consider the discrete barrier functions

$$\Phi_L^{\pm}(x_i, y_j) = \Psi_{L,i} \pm W_L(x_i, y_j)$$

We see that $\Phi_L^{\pm}(x_N, y_j) \ge 0$ and $\Phi_L^{\pm}(0, y_j) \ge 0$ for C large enough Looking at $\Phi_L^{\pm}(x_i, 0) = C\prod_{s=1}^{i}\left(1 + \frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}} h_s\right)^{-1} \pm w_L(x_i, 0)$, using Theorem 4 5 1, we see that $|W_L(x_i, 0)| = |w_L(x_i, 0)| \le Ce^{-\sqrt{\frac{\gamma\alpha}{\varepsilon}}x_i}$ However

$$e^{-\sqrt{\frac{\gamma\alpha}{\varepsilon}}x_i} \le e^{-\sqrt{\frac{\gamma\alpha}{4\varepsilon}}x_i} = \prod_{s=1}^{i} e^{-\sqrt{\frac{\gamma\alpha}{4\varepsilon}}h_s} \le \prod_{s=1}^{i}\left(1 + h_s\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\right)^{-1}$$

A similar argument holds for $\Phi_L^{\pm}(x_i, y_N)$ and we conclude that for C large enough $\Phi_L^{\pm}|_{\Gamma^{N M}} \ge 0$

Note that $L^{N,M}\Phi_L^{\pm}(x_i, y_j) = \varepsilon\delta_x^2\Psi_{L,i} + \varepsilon\delta_y^2\Psi_{L,i} + \mu a_1 D_x^+\Psi_{L,i} + \mu a_2 D_y^+\Psi_{L,i} - b\Psi_{L,i}$

Since $\Psi_{L,i} > 0$ we can show that

$$D_x^+\Psi_{L,i} = -\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\Psi_{L,i+1} < 0, \qquad D_x^-\Psi_{L,i} = -\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\Psi_{L,i+1}\left(1+\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}h_{i+1}\right),$$
$$\delta_x^2\Psi_{L,i} = \frac{\gamma\alpha}{4\varepsilon}\Psi_{L,i+1}\frac{h_{i+1}}{\bar{h}_i} > 0,$$

where $\bar{h}_i = \frac{h_{i+1}+h_i}{2}$ Also we have that $D_y^+\Psi_{L,i} = \delta_y^2\Psi_{L,i} = 0$ We see that

$$L^{N,M}\Phi_L^{\pm}(x_i,y_j) = \varepsilon\frac{\gamma\alpha}{4\varepsilon}\Psi_{L,i+1}\frac{h_{i+1}}{\bar{h}_i} - \mu a_1\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\Psi_{L,i+1} - b\Psi_{L,i} \pm 0$$

Rearranging this equation we have

$$L^{N,M}\Phi_L^{\pm}(x_i,y_j) = \left(2\varepsilon\frac{\gamma\alpha}{4\varepsilon}\left(\frac{h_{i+1}}{2\bar{h}_i}-1\right)+\left(2\varepsilon\frac{\gamma\alpha}{4\varepsilon}-b\right)-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}h_{i+1}b-\mu a_1\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\right)\Psi_{L,i+1}$$

Because of the definition of γ, we see that $L^{N,M}\Phi_L^{\pm}(x_i,y_j)|_{\Omega^{N,M}} \leq 0$ and using the discrete minimum principle we obtain the required result

We next consider W_{LB} We use the barrier functions $\Phi_{LB}^{\pm}(x_i,y_j) = \Psi_{L,i}\Psi_{B,j} \pm W_{LB}(x_i,y_j)$ It is clear that $\Phi_{LB}^{\pm}(x_i,y_N) > 0$ and $\Phi_{LB}^{\pm}(x_N,y_j) > 0$, and using (4 6 2d) and Theorem 4 5 1, we also know that $|W_{LB}(x_i,0)| \leq Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}x_i} \leq C\prod_{s=1}^{i}\left(1+\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}h_s\right)^{-1}$ This implies that $\Phi_{LB}^{\pm}(x_i,0) \geq 0$ for C large enough and a similar argument holds for $\Phi_{LB}^{\pm}(0,y_j)$ We can show that

$$D_x^+\Psi_{L,i}\Psi_{B,j} = -\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\Psi_{L,i+1}\Psi_{B,j} < 0, \qquad \delta_x^2\Psi_{L,i}\Psi_{B,j} = \frac{\gamma\alpha}{4\varepsilon}\Psi_{L,i+1}\Psi_{B,j}\frac{h_{i+1}}{\bar{h}_i} > 0,$$
$$D_y^+\Psi_{L,i}\Psi_{B,j} = -\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\Psi_{L,i}\Psi_{B,j+1} < 0, \qquad \delta_y^2\Psi_{L,i}\Psi_{B,j} = \frac{\gamma\alpha}{4\varepsilon}\Psi_{L,i}\Psi_{B,j+1}\frac{k_{j+1}}{\bar{k}_j} > 0$$

We therefore obtain

$$L^{N,M}\Phi_{LB}^{\pm}(x_i,y_j) = \varepsilon\frac{\gamma\alpha}{4\varepsilon}\Psi_{L,i+1}\Psi_{B,j}\frac{h_{i+1}}{\bar{h}_i} + \varepsilon\frac{\gamma\alpha}{4\varepsilon}\Psi_{L,i}\Psi_{B,j+1}\frac{k_{j+1}}{\bar{k}_i} - \mu a_1\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\Psi_{L,i+1}\Psi_{B,j}$$
$$-\mu a_2\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\Psi_{L,i}\Psi_{B,j+1} - b\Psi_{L,i}\Psi_{B,j} \pm 0$$

Rearranging this equation we have

$$\begin{aligned}L^{N,M}\Phi^{\pm}_{LB}(x_i,y_j) &= \Bigg(\Psi_{L,i+1}\Psi_{B,j}\Bigg(2\varepsilon\frac{\gamma\alpha}{4\varepsilon}\left(\frac{h_{i+1}}{2\bar{h}_i}-1\right)+\left(2\varepsilon\frac{\gamma\alpha}{4\varepsilon}-\frac{b}{2}\right)-\mu a_1\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\\ &\quad -\frac{b}{4}\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}h_{i+1}\Bigg)+\Psi_{L,i}\Psi_{B,j+1}\Bigg(2\varepsilon\frac{\gamma\alpha}{4\varepsilon}\left(\frac{k_{j+1}}{2\bar{k}_j}-1\right)+\left(2\varepsilon\frac{\gamma\alpha}{4\varepsilon}-\frac{b}{2}\right)\\ &\quad -\mu a_2\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}-\frac{b}{4}\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}k_{j+1}\Bigg)\Bigg)\end{aligned}$$

We see from the above expression that $L^{N,M}\Phi^{\pm}_{LB}\leq 0$ for C large enough and we use the discrete minimum principle to finish □

The other discrete layer functions satisfy analogous bounds to those in Theorem 4 6 1 We note that, using the definition of γ, the expressions

$$\left(2\varepsilon\frac{\gamma\alpha}{4\varepsilon}\left(\frac{h_{i+1}}{2\bar{h}_i}-1\right)+\left(2\varepsilon\frac{\gamma\alpha}{4\varepsilon}+\mu a_1\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}-\frac{b}{2}\right)\left(1+\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}h_i\right)-2\varepsilon\left(\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\right)^3 h_i\right),$$

and

$$\left(2\varepsilon\frac{\gamma\alpha}{4\varepsilon}\left(\frac{k_{j+1}}{2\bar{k}_j}-1\right)+\left(2\varepsilon\frac{\gamma\alpha}{4\varepsilon}+\mu a_2\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}-\frac{b}{2}\right)\left(1+\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}k_j\right)-2\varepsilon\left(\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\right)^3 k_j\right),$$

can be shown to be non-positive in the case of $\mu^2\leq\frac{\gamma\varepsilon}{\alpha}$

4 7 Error analysis

We now analyse the error between the continuous solution of (4 1 1) and the discrete solution of (4 6 1) in the case $\mu^2\leq\frac{\gamma\varepsilon}{\alpha}$

Lemma 4 7 1 *At each mesh point $(x_i,y_j)\in\bar{\Omega}^{N,M}$ the regular component of the error satisfies the following estimate*

$$|(V-v)(x_i,y_j)|\leq C(N^{-1}+M^{-1})\sqrt{\varepsilon},$$

where v is the solution of (4 3 3) *and V is the solution of* (4 6 2b)

Proof Using the usual truncation error argument and (4 3 4) we have

$$\begin{aligned}
|L^{N,M}(V-v)(x_i,y_j)| &\le C_1N^{-1}\left(\varepsilon||v_{xxx}||+\mu||v_{xx}||\right)+C_2M^{-1}\left(\varepsilon||v_{yyy}||+\mu||v_{yy}||\right)\\
&\le C(N^{-1}+M^{-1})\sqrt{\varepsilon}
\end{aligned}$$

We consider the barrier functions $\Phi^{\pm}(x_i,y_j)=C_1(N^{-1}+M^{-1})\sqrt{\varepsilon}\pm(V-v)$ We see that these functions are nonnegative on the boundary $\Gamma^{N,M}$, also we find $L^{N,M}\Phi^{\pm}(x_i,y_j)\le 0$ for C_1 large enough We apply the discrete minimum principle to obtain the required result □

Lemma 4 7 2 *At each mesh point $(x_i,y_j)\in\bar{\Omega}^{N,M}$, the left singular component of the error satisfies the following estimate*

$$|(W_L-w_L)(x_i,y_j)|\le C(N^{-1}\ln N+M^{-1}),$$

where w_L is the solution of (4 4 5) *and W_L is the solution of* (4 6 2c)

Proof We can use a classical argument to obtain the following truncation error bounds

$$\begin{aligned}
|L^{N,M}(W_L-w_L)(x_i,y_j)|\le{}& C_1(h_{i+1}+h_i)\left(\varepsilon||w_{Lxxx}||+\mu||w_{Lxx}||\right)\\
&+C_2(k_{j+1}+k_j)\left(\varepsilon||w_{Lyyy}||+\mu||w_{Lyy}||\right)
\end{aligned}$$

We use the bounds in Theorem 4 5 1 to obtain

$$|L^{N,M}(W_L-w_L)(x_i,y_j)|\le\frac{C_1}{\sqrt{\varepsilon}}(h_{i+1}+h_i)+C_2M^{-1} \tag{4 7 1}$$

The proof splits into the two cases of $\sigma^N<\frac{1}{4}$ and $\sigma^N=\frac{1}{4}$ Starting with the former, we consider the region $[\sigma^N,1)\times(0,1)$ Using Theorem 4 6 1 we have

$$|W_L(x_{\frac{N}{4}},y_j)|\le C\left(1+\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\frac{4\sigma^N}{N}\right)^{-\frac{N}{4}}$$

Using (4 6 1b) we see that $\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\sigma^N=\ln N$, and therefore

$$|W_L(x_{\frac{N}{4}},y_j)|\le C(1+4N^{-1}\ln N)^{-\frac{N}{4}}$$

Letting $t=4N^{-1}\ln N$ in the inequality $\ln(1+t)>t\left(1-\frac{t}{2}\right)$, we see that $|W_L(x_{\frac{N}{4}},y_j)|\le$

CN^{-1} Therefore in the region $[\sigma^N, 1) \times (0,1)$ we have

$$|W_L(x_i, y_j)| \leq CN^{-1}$$

Considering the continuous solution in this region, from Theorem 4 5 1 we have

$$|w_L(x_i, y_j)| \leq e^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}\sigma^N} \leq CN^{-2}, \qquad x_i \geq \sigma^N$$

Combining these results we have the following bound in the region $[\sigma^N, 1) \times (0,1)$ when $\sigma^N < \frac{1}{4}$

$$|(W_L - w_L)(x_i, y_j)| \leq CN^{-1}$$

We next consider the region $(0, \sigma^N) \times (0,1)$ Since $\sigma^N < \frac{1}{4}$, we have $h_i = h_{i+1} = \frac{8\sqrt{\varepsilon}}{\sqrt{\gamma\alpha}} N^{-1} \ln N$ We then use (4 7 1) and obtain

$$|L^{N,M}(W_L - w_L)| \leq C(N^{-1} \ln N + M^{-1})$$

Using an appropriately chosen barrier function and the discrete minimum principle we obtain the required result in this region

We finally consider the case of $\sigma^N = \frac{1}{4}$ We find $\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}} \leq 8 \ln N$ and using the truncation error bound (4 7 1) we obtain

$$|L^{N,M}(W_L - w_L)| \leq C(N^{-1} \ln N + M^{-1})$$

Using a suitable barrier function we achieve the required result $\square$

We note that similar proofs hold for the error components $|(W_B - w_B)|$, $|(W_R - w_R)|$ and $|(W_T - w_T)|$ We therefore have the following lemma

Lemma 4 7 3 *At each mesh point* $(x_i, y_j) \in \bar{\Omega}^{N,M}$, *the bottom, right and top singular components of the error satisfies the following estimates*

$$\begin{aligned}
|(W_B - w_B)(x_i, y_j)| &\leq C(N^{-1} + M^{-1} \ln M), \\
|(W_R - w_R)(x_i, y_j)| &\leq C(N^{-1} \ln N + M^{-1}), \\
|(W_T - w_T)(x_i, y_j)| &\leq C(N^{-1} + M^{-1} \ln M),
\end{aligned}$$

where w_B, w_R *and* w_T *are defined analogously to* (4 4 5) *and* W_B, W_R *and* W_T *are defined analogously to* (4 6 2c)

Proof See Lemma 4 7 2 □

Lemma 4 7 4 *At each mesh point $(x_i, y_j) \in \bar{\Omega}^{N,M}$, the bottom-left corner singular component of the error satisfies the following estimate*

$$|(W_{LB} - w_{LB})(x_i, y_j)| \leq C(N^{-1} \ln N + M^{-1} \ln M),$$

where w_{LB} is the solution of (4 5 1) *and W_{LB} is the solution of* (4 6 2d)

Proof We can obtain the following truncation error bounds

$$\begin{aligned} |L^{N,M}(W_{LB} - w_{LB})(x_i, y_j)| \leq\ & C_1(h_{i+1} + h_i)\left(\varepsilon||w_{LBxxx}|| + \mu||w_{LBxx}||\right) \\ & + C_2(k_{j+1} + k_j)\left(\varepsilon||w_{LByyy}|| + \mu||w_{LByy}||\right) \end{aligned}$$

Since w_{LB} satisfies a similar equation to u, we apply Lemma 4 2 2 to obtain (see Remark 4 5 1)

$$|L^{N,M}(W_{LB} - w_{LB})(x_i, y_j)| \leq \frac{C_1}{\sqrt{\varepsilon}}(h_{i+1} + h_i) + \frac{C_2}{\sqrt{\varepsilon}}(k_{j+1} + k_j) \tag{4 7 2}$$

We start by considering the case $\sigma^N < \frac{1}{4}$ and $\sigma^M < \frac{1}{4}$ We consider the region $\Omega^{N,M} \backslash (0, \sigma^N) \times (0, \sigma^M)$ Using Theorem 4 6 1 we have

$$|W_{LB}(x_{\frac{N}{4}}, y_j)| \leq C \prod_{s=1}^{\frac{N}{4}} \left(1 + \frac{4\sigma^N}{N} \frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\right)^{-1},$$

and

$$|W_{LB}(x_i, y_{\frac{M}{4}})| \leq C \prod_{r=1}^{\frac{M}{4}} \left(1 + \frac{4\sigma^M}{M} \frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\right)^{-1}$$

Using (4 6 1b) we see that $\sigma^N \frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}} = \ln N$ and similarly $\sigma^M \frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}} = \ln M$, we therefore obtain

$$|W_{LB}(x_{\frac{N}{4}}, y_j)| \leq C(1 + 4N^{-1} \ln N)^{-\frac{N}{4}},$$

and

$$|W_{LB}(x_i, y_{\frac{M}{4}})| \leq C(1 + 4M^{-1} \ln M)^{-\frac{M}{4}}$$

In an analogous fashion to w_L, we can therefore prove that in this region we have

$$|W_{LB}(x_i, y_j)| \leq C(N^{-1} + M^{-1}), \qquad x_i \geq \sigma^N \text{ and/or } y_j \geq \sigma^M$$

Consider the continuous solution in this region Using Theorem 4 5 1 we obtain

$$|w_{LB}(x_i,y_j)| \le Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}x_i}e^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}y_j} \le e^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}\sigma^N} \le CN^{-2}, \qquad x_i > \sigma^N,$$

and

$$|w_{LB}(x_i,y)| \le Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}x_i}e^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}y_j} \le e^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}\sigma^M} \le CM^{-2}, \qquad y_j > \sigma^M$$

We conclude that when $\sigma^N < \frac{1}{4}$ and $\sigma^M < \frac{1}{4}$, we have the following error bound in the region $\Omega^{N,M}\backslash(0,\sigma^N)\times(0,\sigma^M)$

$$|(W_{LB} - w_{LB})(x_i,y_j)| \le C(N^{-1} + M^{-1})$$

Next, we consider the region $(0,\sigma^N)\times(0,\sigma^M)$ We know that $h_i = h_{i+1} = \frac{8\sqrt{\varepsilon}}{\sqrt{\gamma\alpha}}N^{-1}\ln N$ and $k_j = k_{j+1} = \frac{8\sqrt{\varepsilon}}{\sqrt{\gamma\alpha}}M^{-1}\ln M$ Using the truncation error bound (4 7 2) we obtain

$$|L^{N,M}(W_{LB} - w_{LB})(x_i,y_j)| \le C(N^{-1}\ln N + M^{-1}\ln M)$$

Therefore using a suitably chosen barrier function and the discrete minimum principle we obtain

$$|(W_{LB} - w_{LB})(x_i,y_j)| \le C(N^{-1}\ln N + M^{-1}\ln M)$$

Finally we consider the case of $\sigma^N = \frac{1}{4}$ and $\sigma^M = \frac{1}{4}$ In this case, we know that $\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}} \le 8\ln N$ and $\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}} \le 8\ln M$ and using (4 7 2) and a suitable barrier function we obtain

$$|(W_{LB} - w_{LB})(x_i,y_j)| \le C(N^{-1}\ln N + M^{-1}\ln M)$$

Combining these results for the different cases in the different regions gives us the required result □

We note that similar proofs hold for the error components $|(W_{RB} - w_{RB})|$, $|(W_{RT} - w_{RT})|$ and $|(W_{LT} - w_{LT})|$ We therefore have the following lemma

Lemma 4 7 5 *At each mesh point $(x_i,y_j) \in \bar{\Omega}^{N,M}$, the right-bottom, right-top and left-top singular components of the error satisfies the following estimates*

$$|(W_{RB} - w_{RB})(x_i,y_j)| \le C(N^{-1}\ln N + M^{-1}\ln M),$$
$$|(W_{RT} - w_{RT})(x_i,y_j)| \le C(N^{-1}\ln N + M^{-1}\ln M),$$
$$|(W_{LT} - w_{LT})(x_i,y_j)| \le C(N^{-1}\ln N + M^{-1}\ln M),$$

where w_{RB}, w_{RT} and w_{LT} are defined analogously to w_{LB} in (4 5 1) *and W_{RB}, W_{RT} and W_{LT} are defined analogously to W_{LB} in* (4 6 2d)

Proof See Lemma 4 7 4 □

Theorem 4 7 1 *At each mesh point $(x_i, y_j) \in \bar{\Omega}^{N,M}$ the maximum pointwise error satisfies the following parameter–uniform error bound when $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$,*

$$\|U - u\|_{\bar{\Omega}^{N,M}} \leq C(N^{-1}\ln N + M^{-1}\ln M),$$

where u is the solution of (4 1 1) and U is the solution of (4 6 1)

Proof The proof follows from Lemma 4 7 2, Lemma 4 7 3, Lemma 4 7 4 and Lemma 4 7 5 □

Chapter 5

Elliptic PDE's - the case of $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$

5 1 Introduction

This final chapter is different in style to the previous chapters The analysis relies on various assumptions and conjectures and is more exploratory in spirit We consider the same class of problems as (4 1 1), however this time we examine the more complex case of $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$ The minimum principle and the bounds given in Lemma 4 2 1 and Lemma 4 2 2 still hold This case is significantly more complicated than that of $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, and this analysis is seen merely as a starting point for those wishing to study this problem There are possibly significant compatibility issues with our approach, although the extension idea of Shishkin [29] plays an essential part in minimising these difficulties We ignore these issues of compatibility and assume sufficient regularity for the analysis to be valid The notation in this chapter is as defined in Chapter 4

The assumptions given below restrict the class of problems that we are considering and are sufficient to define and bound the regular component v and all four boundary layer components

Assumption 1 *Arbitrary regularity and compatibility assumed throughout*

We note that the assumption of constant coefficients would reduce complications with compatibility The following assumption is also used when necessary (We will state explicitly in the text when this assumption is used)

Assumption 2 $a_1(x, y) = a_1(x)$ *and* $a_2(x, y) = a_2(y)$

The case of $\mu \geq \gamma_1$, where γ_1 is some constant (convection diffusion) is a subset of this present case and will be dealt with in the final section of this chapter Parameter-explicit bounds on the derivatives of (4 1 1) are derived in Section 5 2 when $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$ The solution is decomposed into a sum of regular and singular components In Section 5 3, we define a regular component v The boundary layer components are discussed in Sections 5 4 and 5 5 It is when considering the corner layer functions in Section 5 6 that the style of the thesis really changes We state and motivate a series of conjectures on the corner layer functions The validity of these conjectures remain open questions The numerical method is then proposed and the discrete solution is decomposed in an analogous fashion to the continuous solution The error between the solutions of the discrete and continuous problems is then analysed We show that given the various assumptions and conjectures made in this chapter, we have a parameter-uniform numerical method

5 2 Parameter-explicit bounds on the derivatives

We need to first obtain crude bounds on the continuous solution u of (4 1 1) and its derivatives Such bounds were discussed in Lemma 4 2 2 in Chapter 4 However, in that proof we concentrated on obtaining bounds with the minimal amount of regularity assumptions on f and the boundary data In this chapter, we focus more on identifying the dependence on the parameters ε and μ, and less on minimising the regularity requirements

Lemma 5 2 1 *The derivatives of the solution u of* (4 1 1) *satisfy the following bounds*

$$\begin{aligned}|u|_1 \;\leq\; & C\Bigg(\frac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2}\sum_{v=0}^{1}\left(\frac{\mu+\sqrt{\varepsilon}}{\varepsilon}\right)^{1-v}\Big(|f|_v + \varepsilon|s_1|_{v+2}, + \mu|s_1|_{v+1} + |s_1|_v, + \varepsilon|s_2|_{v+2},\\ & +\mu|s_2|_{v+1} + |s_2|_v, + \varepsilon|q_1|_{v+2} + \mu|q_1|_{v+1} + |q_1|_v + \varepsilon|q_2|_{v+2} + \mu|q_2|_{v+1} + |q_2|_v\Big)\\ & +|s_1|_1 + |s_2|_1 + |q_1|_1 + |q_2|_1 + \left(\frac{\mu+\sqrt{\varepsilon}}{\varepsilon}\right)||u||\Bigg)\end{aligned}$$

and for $l = 0, 1$

$$\begin{aligned}|u|_{l+2} \;\leq\; & C\Bigg(\frac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2}\sum_{v=0}^{l+1}\left(\frac{\mu+\sqrt{\varepsilon}}{\varepsilon}\right)^{l+2-v}\Big(|f|_v + \varepsilon|s_1|_{v+2} + \mu|s_1|_{v+1} + |s_1|_v + \varepsilon|s_2|_{v+2}\\ & +\mu|s_2|_{v+1} + |s_2|_v + \varepsilon|q_1|_{v+2} + \mu|q_1|_{v+1} + |q_1|_v + \varepsilon|q_2|_{v+2} + \mu|q_2|_{v+1} + |q_2|_v\Big)\\ & +|s_1|_{l+2} + |s_2|_{l+2} + |q_1|_{1+2} + |q_2|_{1+2} + \left(\frac{\mu+\sqrt{\varepsilon}}{\varepsilon}\right)^{l+2}||u||\Bigg),\end{aligned}$$

where C depends on the coefficients a_1, a_2 and b and their derivatives

Proof We continue from equation (4 2 3) in Chapter 4, simplifying the RHS of these equations, we obtain

$$|\tilde{\omega}|_{1,\tilde{R}_\delta} \leq C(||\tilde{f}||_{1,\tilde{R}_{2\delta}} + ||\omega||_{\tilde{R}_{2\delta}}),$$

and for $l = 0, 1$

$$|\tilde{\omega}|_{l+2,\tilde{R}_\delta} \leq C(||\tilde{f}||_{l+1,\tilde{R}_{2\delta}} + ||\omega||_{\tilde{R}_{2\delta}})$$

Transforming back to the original variables this implies for all $(x, y) \in \Omega$ and $R_\delta = R_\delta(x, y)$

$$\left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right) |\omega|_{1,R_\delta} \leq C\left(\frac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2}\sum_{v=0}^{1}\left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right)^v |\hat{f}|_{v,R_{2\delta}} + ||\omega||_{R_{2\delta}}\right),$$

and for $l = 0, 1$

$$\left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right)^{l+2} |\omega|_{l+2,R_\delta} \leq C\left(\frac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2}\sum_{v=0}^{l+1}\left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right)^v |\hat{f}|_{v,R_{2\delta}} + ||\omega||_{R_{2\delta}}\right)$$

Replacing $\hat{f}$ by $f - L_{\varepsilon,\mu}h$ and using the definition of h gives us

$$\begin{aligned}
\left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right) |\omega|_{1,R_\delta} \leq\ & C\left(\tfrac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2}\sum_{v=0}^{1}\left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right)^v |f - L_{\varepsilon,\mu}h|_{v,R_{2\delta}} + ||\omega||_{R_{2\delta}}\right)\\
\leq\ & C\Bigg(\tfrac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2}\sum_{v=0}^{1}\left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right)^v \big(|f|_{v,R_{2\delta}} + \varepsilon|s_1|_{v+2,R_{2\delta}} + \mu|s_1|_{v+1,R_{2\delta}}\\
&+|s_1|_{v,R_{2\delta}} + \varepsilon|s_2|_{v+2,R_{2\delta}} + \mu|s_2|_{v+1,R_{2\delta}} + |s_2|_{v,R_{2\delta}} + \varepsilon|q_1|_{v+2,R_{2\delta}}\\
&+\mu|q_1|_{v+1,R_{2\delta}} + |q_1|_{v,R_{2\delta}} + \varepsilon|q_2|_{v+2,R_{2\delta}} + \mu|q_2|_{v+1,R_{2\delta}} + |q_2|_{v,R_{2\delta}}\big)\\
&+||\omega||_{R_{2\delta}}\Bigg)
\end{aligned}$$

and for $l = 0, 1$

$$\begin{aligned}
\left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right)^{l+2} |\omega|_{l+2,R_\delta} \leq\ & C\Bigg(\tfrac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2}\sum_{v=0}^{l+1}\left(\tfrac{\varepsilon}{\mu+\sqrt{\varepsilon}}\right)^v \big(|f|_{v,R_{2\delta}} + \varepsilon|s_1|_{v+2,R_{2\delta}} + \mu|s_1|_{v+1,R_{2\delta}}\\
&+|s_1|_{v,R_{2\delta}} + \varepsilon|s_2|_{v+2,R_{2\delta}} + \mu|s_2|_{v+1,R_{2\delta}} + |s_2|_{v,R_{2\delta}}\\
&+\varepsilon|q_1|_{v+2,R_{2\delta}} + \mu|q_1|_{v+1,R_{2\delta}} + |q_1|_{v,R_{2\delta}} + \varepsilon|q_2|_{v+2,R_{2\delta}}\\
&+\mu|q_2|_{v+1,R_{2\delta}} + |q_2|_{v,R_{2\delta}}\big) + ||\omega||_{R_{2\delta}}\Bigg)
\end{aligned}$$

Rearranging these equations, we obtain

$$\begin{aligned}|\omega|_{1,R_\delta} \;\leq\; & C\Bigg(\frac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2}\sum_{v=0}^{1}\left(\frac{\mu+\sqrt{\varepsilon}}{\varepsilon}\right)^{1-v}\Big(|f|_{v,R_{2\delta}}+\varepsilon|s_1|_{v+2,R_{2\delta}}+\mu|s_1|_{v+1,R_{2\delta}}+|s_1|_{v,R_{2\delta}}\\ &+\varepsilon|s_2|_{v+2,R_{2\delta}}+\mu|s_2|_{v+1,R_{2\delta}}+|s_2|_{v,R_{2\delta}}+\varepsilon|q_1|_{v+2,R_{2\delta}}+\mu|q_1|_{v+1,R_{2\delta}}\\ &+|q_1|_{v,R_{2\delta}}+\varepsilon|q_2|_{v+2,R_{2\delta}}+\mu|q_2|_{v+1,R_{2\delta}}+|q_2|_{v,R_{2\delta}}\Big)+\left(\frac{\mu+\sqrt{\varepsilon}}{\varepsilon}\right)||\omega||_{R_{2\delta}}\Bigg)\end{aligned}$$

and for $l = 0, 1$

$$\begin{aligned}|\omega|_{l+2,R_\delta} \;\leq\; & C\Bigg(\frac{\varepsilon}{(\mu+\sqrt{\varepsilon})^2}\sum_{v=0}^{l+1}\left(\frac{\mu+\sqrt{\varepsilon}}{\varepsilon}\right)^{l+2-v}\Big(|f|_{v,R_{2\delta}}+\varepsilon|s_1|_{v+2,R_{2\delta}}+\mu|s_1|_{v+1,R_{2\delta}}+|s_1|_{v,R_{2\delta}}\\ &+\varepsilon|s_2|_{v+2,R_{2\delta}}+\mu|s_2|_{v+1,R_{2\delta}}+|s_2|_{v,R_{2\delta}}+\varepsilon|q_1|_{v+2,R_{2\delta}}+\mu|q_1|_{v+1,R_{2\delta}}\\ &+|q_1|_{v,R_{2\delta}}+\varepsilon|q_2|_{v+2,R_{2\delta}}+\mu|q_2|_{v+1,R_{2\delta}}+|q_2|_{v,R_{2\delta}}\Big)\\ &+\left(\frac{\mu+\sqrt{\varepsilon}}{\varepsilon}\right)^{l+2}||\omega||_{R_{2\delta}}\Bigg)\end{aligned}$$

Since $\bar{\Omega}$ can be covered by the neighbourhoods N_δ of a finite number of points and noting that $u = w + h$, the result follows □

Remark 5 2 1 *In the case where $f \in C^2(\bar{\Omega})$, $s, q \in C^4([0,1])$ are independent of ε and μ, we obtain for $1 \leq k+m \leq 3$*

$$\left\|\frac{\partial^{k+m}u}{\partial x^k \partial y^m}\right\| \leq C\left(\frac{\mu}{\varepsilon}\right)^{k+m}(1+||u||),$$

where C depends on f, s, and q and the coefficients a_1, a_2 and b and their derivatives

5.3 Regular component in case of $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$

In order to obtain parameter-uniform error bounds for the numerical approximations generated in the final sections of this chapter, we decompose the solution u of (4 1 1) into a sum of regular and singular components Consider the differential equation (4 3 1) in the extended domain $\Omega^{[*,LB]} = (-d,1)^2$ Decompose v^* as follows,

$$v^*(x,y,\varepsilon,\mu) = v_0^*(x,y,\mu) + \varepsilon v_1^*(x,y,\mu) + \varepsilon^2 v_2^*(x,y,\varepsilon,\mu) \tag{5 3 1a}$$

where

$$\begin{aligned}
L_\mu^* v_0^* &= f^* \quad \text{on} \quad \Omega_1^{[*,LB]}, \qquad v_0^*|_{\partial\Omega_1^{[*,LB]}} \text{ chosen in (5 3 4)}, && (5\ 3\ 1\text{b}) \\
\varepsilon L_\mu^* v_1^* &= (L_\mu^* - L_{\varepsilon,\mu}^*) v_0^*, \quad \text{on} \quad \Omega_1^{[*,LB]}, \quad v_1^*|_{\partial\Omega_1^{[*,LB]}} = 0, && (5\ 3\ 1\text{c}) \\
\varepsilon^2 L_{\varepsilon,\mu}^* v_2^* &= \varepsilon(L_\mu^* - L_{\varepsilon,\mu}^*) v_1^*, \quad \text{on} \quad \Omega^{[*,LB]}, \quad v_2^*(x,y,\varepsilon,\mu)|_{\partial\Omega^{[*,LB]}} = 0 && (5\ 3\ 1\text{d})
\end{aligned}$$

Note that $\Omega_1^{[*,LB]} = [-d,1)^2$ and $\partial\Omega_1^{[*,LB]} = \Gamma_T^{[*,L]} \cup \Gamma_R^{[*,B]}$ When $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, v_0^* and v_1^* were defined as solutions of reduced problems obtained by setting both ε and μ to zero in the elliptic differential equation In this case, we see that v_0^* and v_1^* are solutions of singularly perturbed first order differential equations obtained by letting just ε be zero in the elliptic problem Since v_2^* satisfies an elliptic problem, there are potential issues in relation to compatibility at the inflow corner $(1,1)$ We do not address this concern

We can establish the following for the first order differential operator L_μ using a proof by contradiction argument Note $b \geq 2\beta > 0$ is not used in the proof

Lemma 5 3 1 *Let $\Omega_1 = [0,1)^2$ and $\partial\Omega_1 = \Gamma_T \cup \Gamma_R$ Suppose $z \in C^1(\Omega_1) \cap C^0(\bar{\Omega}_1)$,*

$$\text{If} \qquad L_\mu z\Big|_{\Omega_1} \leq 0 \qquad \text{and} \qquad z\Big|_{\partial\Omega_1} \geq 0, \qquad \text{then} \qquad z\Big|_{\bar{\Omega}_1} \geq 0$$

Proof Let $z = e^{\frac{\beta_1}{\mu}x} w$, where $\beta_1 < \min_{\Omega_1} \frac{b}{a_1}$ Assume that $\min_{\Omega_1} z < 0$, this implies that $\min_{\Omega_1} w < 0$ Consider a point $p = (x_0, y_0)$ such that $w(p) = \min_{\Omega_1} w < 0$ At this point p we know that $w_x(p) \geq 0$ and $w_y(p) \geq 0$ We see that

$$L_\mu z(p) = e^{\frac{\beta_1}{\mu}x_0} \left(\mu a_1 w_x(p) + \mu a_2 w_y(p) - (b - \beta_1 a_1) w(p)\right) > 0,$$

which is a contradiction $\square$

Lemma 5 3 2 *If $z(x,y)$ satisfies the first order problem*

$$L_\mu z = a_1 \mu z_x + a_2 \mu z_y - bz = f \quad (x,y) \in \Omega_1 = [0,1)^2, \qquad z|_{\partial\Omega_1} = 0, \qquad (5\ 3\ 2)$$

where $a_1 > 0$, $a_2 > 0$ and $b \geq 2\beta > 0$, then we have the following bounds on z and its derivatives

$$\|z\| \leq \frac{1}{2\beta} \|f\|,$$

and

$$\left\|\frac{\partial^{k+m}z}{\partial x^k \partial y^m}\right\| \le C\left(\left(\frac{1}{\mu^{\min\{k,m\}}}\right)||z|| + \left\|\frac{\partial^{k+m}f}{\partial x^{k+m}}\right\| + \left\|\frac{\partial^{k+m}f}{\partial y^{k+m}}\right\| + \frac{1}{\mu^{k+m}}\sum_{r+s=0}^{k+m-1}\mu^{r+s}\left\|\frac{\partial^{r+s}f}{\partial x^r \partial y^s}\right\|\right)e^{(k+m)A},$$

where $A = \max\left\{0, \left(\frac{a_2}{a_1}\right)\left(\frac{a_1}{a_2}\right)_x, \left(\frac{a_1}{a_2}\right)\left(\frac{a_2}{a_1}\right)_y\right\}$ *and the constant* C *depends only on the coefficients* a_1, a_2, b *and their derivatives*

Proof Consider the barrier functions $\psi^{\pm}(x,y) = \frac{1}{2\beta}||f|| \pm z$ We see that these functions are nonnegative for $(x,y) \in \partial\Omega_1$ We also have

$$L_\mu\psi^{\pm}(x,y) = -\frac{b}{2\beta}||f|| \pm f \le 0$$

Apply Lemma 5 3 1 to obtain the required bound on z We will establish by induction that

$$\left\|\frac{\partial^{k+m}z}{\partial x^k \partial y^m}\right\| \le C\left(\left(\frac{1}{\mu^m}\right)||z|| + \left\|\frac{\partial^{k+m}f}{\partial x^{k+m}}\right\| + \frac{1}{\mu^{k+m}}\sum_{r+s=0}^{k+m-1}\mu^{r+s}\left\|\frac{\partial^{r+s}f}{\partial x^r \partial y^s}\right\|\right)e^{(k+m)A} \quad (5\ 3\ 3)$$

Differentiating equation (5 3 2) with respect to x we obtain

$$L_\mu^{[1]}z_x = \mu\frac{a_1}{a_2}(z_x)_x + \mu(z_x)_y - \left(\frac{b}{a_2} - \mu\left(\frac{a_1}{a_2}\right)_x\right)(z_x) = \left(\frac{f}{a_2}\right)_x + \left(\frac{b}{a_2}\right)_x z,$$

where $z_x(x,1) = 0$ and using the differential equation (5 3 2) we have $||z_x(1,y)|| \le \frac{C}{\mu}||f||$ Consider the barrier functions

$$\psi^{\pm}(x,y) = C\left(\frac{1}{\mu}||f|| + \left\|\frac{\partial f}{\partial x}\right\| + ||z||\right)e^{A(1-x)} \pm z_x,$$

where A is defined as above We see that for C large enough the functions $\psi^{\pm}(x,y)$ are nonnegative on the boundary $\partial\Omega_1$ Also

$$\begin{aligned} L_\mu^{[1]}\psi^{\pm}(x,y) &= C\left(-\mu\left(\frac{a_1}{a_2}\right)A - \frac{b}{a_2} + \mu\left(\frac{a_1}{a_2}\right)_x\right)\left(\frac{1}{\mu}||f|| + \left\|\frac{\partial f}{\partial x}\right\| + ||z||\right)e^{A(1-x)} \\ &\quad \pm\left(\left(\frac{f}{a_2}\right)_x + \left(\frac{b}{a_2}\right)_x z\right) \end{aligned}$$

We see that for C chosen correctly $L_\mu^{[1]}\psi^\pm(x,y) \le 0$, therefore applying Lemma 5 3 1 and using (5 3 2), we obtain (5 3 3) for $k+m=1$

We now prove the more general result (5 3 3) by induction We assume that the lemma is true for $0 \le k+m \le l$ Differentiate (5 3 2) $l+1$ times with respect to x to obtain

$$\begin{aligned} L_\mu^{[l+1]}\left(\frac{\partial^{l+1}z}{\partial x^{l+1}}\right) &= \mu\frac{a_1}{a_2}\left(\frac{\partial^{l+1}z}{\partial x^{l+1}}\right)_x + \mu\left(\frac{\partial^{l+1}z}{\partial x^{l+1}}\right)_y - \left(\frac{b}{a_2} - (l+1)\mu\left(\frac{a_1}{a_2}\right)_x\right)\left(\frac{\partial^{l+1}z}{\partial x^{l+1}}\right) \\ &= \rho(x,y), \end{aligned}$$

where $\rho(x,y)$ involves f and its derivatives with respect to x up to order $l+1$, z and its derivatives with respect to x up to order l and the coefficients and their derivatives We see that $\frac{\partial^{l+1}z}{\partial x^{l+1}}(x,1)=0$ and $\frac{\partial^{l+1}z}{\partial x^{l+1}}(1,y)=\phi(x,y)$ Using the differential equation (5 3 2), we can show that

$$|\phi(x,y)| \le C\sum_{r+s=0}^{l}\frac{1}{\mu^{l+1-r-s}}\left\|\frac{\partial^{r+s}f}{\partial x^r\partial y^s}\right\|$$

Consider the barrier functions

$$\psi^\pm(x,y) = C\left(||z|| + \left\|\frac{\partial^{l+1}f}{\partial x^{l+1}}\right\| + \frac{1}{\mu^{l+1}}\sum_{r+s=0}^{l}\mu^{r+s}\left\|\frac{\partial^{r+s}f}{\partial x^r\partial y^s}\right\|\right)e^{(l+1)A(1-x)} \pm \left(\frac{\partial^{l+1}z}{\partial x^{l+1}}\right)$$

We see that for C large enough the functions are both nonnegative on $\partial\Omega_1$ Also we have

$$\begin{aligned} L_\mu^{[l+1]}\psi^\pm(x,y) = C\left(-(l+1)\mu\left(\frac{a_1}{a_2}\right)A - \frac{b}{a_2} + (l+1)\mu\left(\frac{a_1}{a_2}\right)_x\right)\left(||z|| + \left\|\frac{\partial^{l+1}f}{\partial x^{l+1}}\right\| +\right. \\ \left. +\frac{1}{\mu^{l+1}}\sum_{r+s=0}^{l}\mu^{r+s}\left\|\frac{\partial^{r+s}f}{\partial x^r\partial y^s}\right\|\right)e^{(l+1)A(1-x)} \pm \rho(x,y) \end{aligned}$$

Using our induction assumption and the definition of A, we see that $L_\mu^{[l+1]}\psi^\pm(x,y) \le 0$ for C chosen correctly We therefore obtain

$$\left\|\frac{\partial^{l+1}z}{\partial x^{l+1}}\right\| \le C\left(||z|| + \left\|\frac{\partial^{l+1}f}{\partial x^{l+1}}\right\| + \frac{1}{\mu^{l+1}}\sum_{r+s=0}^{l}\mu^{r+s}\left\|\frac{\partial^{r+s}f}{\partial x^r\partial y^s}\right\|\right)e^{(l+1)A}$$

Differentiating (5 3 2) k times with respect to x and m times with respect to y, we obtain for $k+m=l+1$

$$\left\|\frac{\partial^{k+m}z}{\partial x^k\partial y^m}\right\| \le C\left(\left(\frac{1}{\mu^m}\right)\|z\| + \left\|\frac{\partial^{k+m}f}{\partial x^{k+m}}\right\| + \frac{1}{\mu^{k+m}}\sum_{r+s=0}^{k+m-1}\mu^{r+s}\left\|\frac{\partial^{r+s}f}{\partial x^r\partial y^s}\right\|\right)e^{(k+m)A}$$

Similarly if we started the proof by differentiating (5 3 2) with respect to y, we would obtain

$$\left\|\frac{\partial^{k+m}z}{\partial x^k\partial y^m}\right\| \le C\left(\left(\frac{1}{\mu^k}\right)\|z\| + \left\|\frac{\partial^{k+m}f}{\partial y^{k+m}}\right\| + \frac{1}{\mu^{k+m}}\sum_{r+s=0}^{k+m-1}\mu^{r+s}\left\|\frac{\partial^{r+s}f}{\partial x^r\partial y^s}\right\|\right)e^{(k+m)A},$$

and combining these two bounds gives the required result □

With this lemma, we can analyse the reduced solution v_0^*, the solution of (5 3 1b) We show that if the inflow boundary conditions $v_0^*(x,1)$ and $v_0^*(1,y)$ are chosen correctly, then all the derivatives up to second order of v_0^* are bounded independently of μ (and obviously ε) We note that Lemma 5 3 1 and Lemma 5 3 2 also hold for the differential operator $L_\mu^{[*,LB]}$ and the domain $\Omega_1^{[*,LB]}$ defined as before

Lemma 5 3 3 *When the boundary conditions $v_0^*|_{\partial\Omega_1^{[*,LB]}}$ are chosen correctly, the solution v_0^* of the differential equation (5 3 1b) satisfies the following bounds for $0 \le k+m \le 6$,*

$$\left\|\frac{\partial^{k+m}v_0^*}{\partial x^k\partial y^m}\right\| \le C(1+\mu^{2-k-m})$$

Proof Consider the following secondary decomposition of $v_0^*(x,y,\mu)$

$$v_0^*(x,y,\mu) = s_0^*(x,y) + \mu s_1^*(x,y) + \mu^2 s_2^*(x,y,\mu) \tag{5 3 4a}$$

where

$$\begin{aligned}
L_0^* s_0^* &= f^*, && (5\ 3\ 4\text{b})\\
\mu L_0^* s_1^* &= (L_0^* - L_\mu^*)s_0^*, && (5\ 3\ 4\text{c})\\
\mu^2 L_\mu^* s_2^* &= \mu(L_0^* - L_\mu^*)s_1^* \quad \text{on} \quad \Omega_1^{[*,LB]}, \quad s_2^*|_{\partial\Omega_1^{[*,LB]}} = 0 && (5\ 3\ 4\text{d})
\end{aligned}$$

Since s_0^* and s_1^* do not depend on μ, we have

$$\left\|\frac{\partial^{k+m} s_0^*}{\partial x^k \partial y^m}\right\| \leq C \quad \text{for} \quad 0 \leq k+m \leq 8, \tag{5 3 5}$$

$$\left\|\frac{\partial^{k+m} s_1^*}{\partial x^k \partial y^m}\right\| \leq C \quad \text{for} \quad 0 \leq k+m \leq 7 \tag{5 3 6}$$

The function s_2^* satisfies a similar equation to z in (5 3 2) We can apply Lemma 5 3 2 and the bounds above to obtain

$$||s_2^*|| \leq \frac{1}{2\beta}\left(\left\|\frac{\partial s_1^*}{\partial x}\right\| + \left\|\frac{\partial s_1^*}{\partial y}\right\|\right) \leq C,$$

and for $1 \leq k+m \leq 6$

$$\begin{aligned} \left\|\frac{\partial^{k+m} s_2^*}{\partial x^k \partial y^m}\right\| &\leq C\left(\left(\frac{1}{\mu^{\min\{k,m\}}}\right)||s_2^*|| + \left\|\frac{\partial^{k+m} s_{1x}^*}{\partial x^{k+m}}\right\| + \left\|\frac{\partial^{k+m} s_{1y}^*}{\partial x^{k+m}}\right\| + \left\|\frac{\partial^{k+m} s_{1x}^*}{\partial y^{k+m}}\right\|\right. \\ &+ \left.\left\|\frac{\partial^{k+m} s_{1y}^*}{\partial y^{k+m}}\right\| + \frac{1}{\mu^{k+m}}\sum_{r+s=0}^{k+m-1}\mu^{r+s}\left(\left\|\frac{\partial^{r+s} s_{1x}^*}{\partial x^r \partial y^s}\right\| + \left\|\frac{\partial^{r+s} s_{1y}^*}{\partial x^r \partial y^s}\right\|\right)\right) e^{(k+m)A} \end{aligned}$$

Therefore, using the fact that s_1^* and its derivatives are bounded independent of μ we obtain for $0 \leq k+m \leq 6$,

$$\left\|\frac{\partial^{k+m} s_2^*}{\partial x^k \partial y^m}\right\| \leq \frac{C}{\mu^{k+m}} \tag{5 3 7}$$

Using the decomposition (5 3 4) and the bounds (5 3 5), (5 3 6) and (5 3 7) gives us the required result □

Lemma 5 3 4 *If v_1^* satisfies the first order differential equation (5 3 1c) then the following bounds hold for* $0 \leq k+m \leq 4$,

$$\left\|\frac{\partial^{k+m} v_1^*}{\partial x^k \partial y^m}\right\| \leq \frac{C}{\mu^{k+m}}$$

Proof Since v_1^* satisfies a similar equation to z in (5 3 2), we can apply Lemma 5 3 2 and the bounds above to obtain

$$||v_1^*|| \leq C\left(\left\|\frac{\partial^2 v_0^*}{\partial x^2}\right\| + \left\|\frac{\partial^2 v_0^*}{\partial y^2}\right\|\right) \leq C,$$

and for $1 \leq k+m \leq 4$

$$\left\|\frac{\partial^{k+m} v_1^*}{\partial x^k \partial y^m}\right\| \leq C\left(\left(\frac{1}{\mu^{\min\{k,m\}}}\right)\|v_1^*\| + \left\|\frac{\partial^{k+m} v_{0xx}^*}{\partial x^{k+m}}\right\| + \left\|\frac{\partial^{k+m} v_{0yy}^*}{\partial x^{k+m}}\right\| + \left\|\frac{\partial^{k+m} v_{0xx}^*}{\partial y^{k+m}}\right\| + \right.$$
$$\left.\left\|\frac{\partial^{k+m} v_{0yy}^*}{\partial y^{k+m}}\right\| + \frac{1}{\mu^{k+m}} \sum_{r+s=0}^{k+m-1} \mu^{r+s}\left(\left\|\frac{\partial^{r+s} v_{0xx}^*}{\partial x^r \partial y^s}\right\| + \left\|\frac{\partial^{r+s} v_{0yy}^*}{\partial x^r \partial y^s}\right\|\right)\right) e^{(k+m)A}$$

Using the bounds on v_0^* in Lemma 5 3 3 we obtain the required result □

Lemma 5 3 5 *If $v_2^*(x, y, \varepsilon, \mu)$ satisfies the differential equation (5 3 1d) then we have the following bounds for* $0 \leq k+m \leq 3$

$$\left\|\frac{\partial^{k+m} v_2^*}{\partial x^k \partial y^m}\right\| \leq \frac{C}{\mu^2}\left(\frac{1}{(\sqrt{\varepsilon})^{k+m}}\right)\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^{k+m}\right)$$

Proof Since v_2^* satisfies a similar equation to u, applying Lemma 4 2 1 we obtain

$$\|v_2^*(x, t, \varepsilon, \mu)\| \leq \|v_2\|_{\partial\Omega[\cdot \text{ LB}]} + \frac{1}{2\beta}\left(\left\|\frac{\partial^2 v_1}{\partial x^2}\right\| + \left\|\frac{\partial^2 v_1^*}{\partial y^2}\right\|\right)$$

Using Lemma 5 3 4 we have

$$\|v_2^*\| \leq \frac{C}{\mu^2}$$

Finally we use Lemma 5 2 1 to obtain for $1 \leq k+m \leq 3$,

$$\left\|\frac{\partial^{k+m} v_2^*}{\partial x^k \partial y^m}\right\|_{\bar{\Omega}[\cdot \text{ LB}]} \leq \frac{C}{(\sqrt{\varepsilon})^{k+m}}\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^{k+m}\right) \max\left\{\|v_2^*\|_{\bar{\Omega}[\cdot \text{ LB}]}, \sum_{r+s=0}^{2} (\sqrt{\varepsilon})^{r+s}\left\|\frac{\partial^{r+s} v_{1xx}^*}{\partial x^r \partial y^s}\right\|, \right.$$
$$\left.\sum_{r+s=0}^{2} (\sqrt{\varepsilon})^{r+s}\left\|\frac{\partial^{r+s} v_{1yy}^*}{\partial x^r \partial y^s}\right\|\right\}$$

and applying the bounds for v_1^* in Lemma 5 3 4 we obtain the required result □

Combining the results of Lemma 5 3 3, Lemma 5 3 4 and Lemma 5 3 5, we see that if we take the regular solution v to be the solution of

$$L_{\varepsilon,\mu} v = f \ (x, y) \in \Omega, \quad v = v^* \ (x, y) \in \partial\Omega, \tag{5 3 8}$$

then when $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$, v satisfies the following bounds for $0 \leq k+m \leq 3$,

$$\left\|\frac{\partial^{k+m}v}{\partial x^k \partial y^m}\right\| \leq C\left(1+\left(\frac{\mu}{\varepsilon}\right)^{k+m-2}\right), \qquad (5\ 3\ 9)$$

where v^* is defined in the decomposition (5 3 1)

5 4 Boundary layer components at the inflow

In this section, we define the boundary layer functions w_R and w_T associated with the right and top edges respectively In the case of $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$, the order in which we define the layer functions is crucial to correctly isolating the singularities of the solution u

We start by analysing w_R, the layer function associated with the right edge Γ_R Consider the extended domain $\Omega^{[*,\mathrm{T}]} = (0,1) \times (0, 1+d)$, $d > 0$ We define w_R^* to be the solution of $L_{\varepsilon,\mu}^{[*,\mathrm{T}]} w_R^* = 0$ We need to chose the boundary conditions for w_R^* so as to isolate the layer on the right Consider the following decomposition of w_R^*

$$w_R^*(x,y,\varepsilon,\mu) = w_0^*(x,y,\mu) + \varepsilon w_1^*(x,y,\mu) + \varepsilon^2 w_2^*(x,y,\varepsilon,\mu), \qquad (5\ 4\ 1\mathrm{a})$$

where $v(1,y) = v_0(1,y) = \left(\frac{f}{b} - \left(\frac{\mu}{b}\right) a \quad \nabla\left(\frac{f}{b}\right)\right)(1,y)$ is given in (5 3 4) and

$$\begin{aligned}
L_{\mu}^{[*,\mathrm{T}]} w_0^* &= 0 \text{ on } \Omega_1^{[*,T]}, \ \ w_0^*(x,1+d,\mu) = 0, \ \ w_0^*(1,y,\mu) = (u(1,y) - v(1,y))^*, && (5\ 4\ 1\mathrm{b})\\
\varepsilon L_{\mu}^{[*,\mathrm{T}]} w_1^* &= (L_{\mu}^{[*,\mathrm{T}]} - L_{\varepsilon,\mu}^{[*,\mathrm{T}]}) w_0^* \text{ on } \Omega_1^{[*,T]}, \ \ w_1^*(x,1+d,\mu) = w_1^*(1,y,\mu) = 0, && (5\ 4\ 1\mathrm{c})\\
\varepsilon^2 L_{\varepsilon,\mu}^{[*,\mathrm{TB}]} w_2^* &= (\varepsilon(L_{\mu}^{[*,\mathrm{T}]} - L_{\varepsilon,\mu}^{[*,\mathrm{T}]}) w_1^*)^*, \quad \text{on } \Omega^{[*,TB]} \ \ w_2^*(x,y,\varepsilon,\mu)|_{\partial\Omega[\ \mathrm{TB}]} = 0 && (5\ 4\ 1\mathrm{d})
\end{aligned}$$

Remark 5 4 1 *We should note that the last function w_2^* is defined on the extended domain $\Omega^{[*,TB]} = (0,1) \times (-d, 1+d)$ This domain is obtained by extending to the top and bottom of the original domain, while w_0^* and w_1^* are defined on the smaller extended domain $\Omega_1^{[*,T]} = [0,1) \times [0,1+d)$*

The following lemmas prove parameter-explicit bounds on the components w_0^*, w_1^*, w_2^* and their derivatives These results are then used to bound the layer function w_R^* and its derivatives

Lemma 5 4 1 *When w_0^* is defined as in (5 4 1b), given $\mu < \gamma_1$, the function and its derivatives satisfy the following bounds for any positive integer k (assuming sufficient* *regularity and compatibility)*

$$\left|\frac{\partial^k w_0^*}{\partial x^i \partial y^j}(x,y)\right| \le \frac{C_1}{\mu^i}(d+1-y)e^{-\frac{\gamma}{\mu}(1-x)}, \quad (x,y)\in\Omega_1^{[*,T]}, \qquad \left(C_1 \le Ce^{j\left\|\left(\frac{a_2}{a_1}\right)_y\right\|}\right)$$

Proof Since $w_0^*(x,1+d) = 0$, we can show that $|w_0^*(1,y)| \le C(1+d-y)$ Consider the barrier function $\psi^{\pm}(x,y) = C(1+d-y)e^{-\frac{\gamma}{\mu}(1-x)} \pm w_0^*$, we see that the functions $\psi^{\pm}(x,y)|_{\Gamma^{[*,T]}}$ are nonnegative for C large enough Also

$$L_\mu^{[*,T]}\psi^{\pm}(x,y) = C\left(\gamma a_1^*(d+1-y) - \mu a_2^* - b^*(d+1-y)\right)e^{-\frac{\gamma}{\mu}(1-x)} \pm 0,$$

and using our definition of γ, we see that $L_\mu^{[*,T]}\psi^{\pm}(x,y) \le 0$ for C chosen correctly Apply Lemma 5 3 1 to obtain

$$|w_0^*| \le C(d+1-y)e^{-\frac{\gamma}{\mu}(1-x)}$$

Differentiate equation (5 4 1b) with respect to y , we have

$$L_\mu^{[*,T,1]}\frac{\partial w_0^*}{\partial y} = \mu\left(\frac{\partial w_0^*}{\partial y}\right)_x + \mu\frac{a_2^*}{a_1^*}\left(\frac{\partial w_0^*}{\partial y}\right)_y - \left(\frac{b^*}{a_1^*} - \mu\left(\frac{a_2^*}{a_1^*}\right)_y\right)\frac{\partial w_0^*}{\partial y} = \left(\frac{b^*}{a_1^*}\right)_y w_0^*$$

Clearly $\frac{\partial w_0^*}{\partial y}(1,y) = ((u-v)(1,y))_y$ and since w_0^* satisfies a homogenous first order problem, using $\frac{\partial w_0^*}{\partial x}(x,1+d) = 0$, we see that $\frac{\partial w_0^*}{\partial y}(x,1+d) = 0$ Taylor expansions give $\left|\frac{\partial w_0^*}{\partial y}(1,y)\right| \le C(1+d-y)$ Consider the barrier functions

$$\psi^{\pm}(x,y) = C(1+d-y)e^{\left(\left\|\left(\frac{a_2^*}{a_1^*}\right)_y\right\| - \frac{\gamma}{\mu}\right)(1-x)} \pm \frac{\partial w_0^*}{\partial y}$$

We see that the functions $\psi^{\pm}(x,y)$ are nonnegative on the boundary for C large enough Also

$$\begin{aligned} L_\mu^{[*,T,1]}\psi^{\pm}(x,y) = C\Bigg(\Bigg(\left(\gamma - \frac{b^*}{a_1^*}\right) + \mu\left(\left(\frac{a_2^*}{a_1^*}\right)_y - \left\|\left(\frac{a_2^*}{a_1^*}\right)\right\|\right)\Bigg)(1+d-y) \\ - \mu\left(\frac{a_2^*}{a_1^*}\right)\Bigg)e^{\left(\left\|\left(\frac{a_2^*}{a_1^*}\right)_y\right\| - \frac{\gamma}{\mu}\right)(1-x)} \pm \left(\frac{b^*}{a_1^*}\right)_y w_0^*, \end{aligned}$$

and we can see that $L_\mu^{[*,T,1]}\psi^{\pm}(x,y) \le 0$ for C chosen correctly Applying Lemma 5 3 1 we have

$$\left|\frac{\partial w_0^*}{\partial y}(x,y)\right| \leq C(1+d-y)e^{\left(\left\|\left(\frac{a_2^*}{a_1^*}\right)_y\right\|-\frac{\gamma}{\mu}\right)(1-x)} \leq C_1(1+d-y)e^{-\frac{\gamma}{\mu}(1-x)}$$

Using the differential equation (5 4 1b) we therefore obtain

$$\left|\frac{\partial w_0^*}{\partial x}\right| \leq \frac{C}{\mu}(1+d-y)e^{-\frac{\gamma}{\mu}(1-x)}$$

We now continue by induction Assume for $k \leq l$

$$\left\|\frac{\partial^k w_0^*}{\partial x^i \partial y^j}\right\| \leq \frac{C_1}{\mu^i}(d+1-y)e^{-\frac{\gamma}{\mu}(1-x)}, \qquad \left(C_1 \leq Ce^{k\left\|\left(\frac{a_2^*}{a_1^*}\right)_y\right\|}\right)$$

We wish to prove true for $k = l+1$ Differentiating (5 4 1b) $l+1$ times with respect to y, we obtain

$$\begin{aligned} L_\mu^{[*,\mathrm{T},l+1]}\left(\frac{\partial^{l+1} w_0^*}{\partial y^{l+1}}\right) &= \mu\left(\frac{\partial^{l+1} w_0^*}{\partial y^{l+1}}\right)_x + \mu\frac{a_2^*}{a_1^*}\left(\frac{\partial^{l+1} w_0^*}{\partial y^{l+1}}\right)_y \\ &\quad -\left(\frac{b^*}{a_1^*} - (l+1)\mu\left(\frac{a_2^*}{a_1^*}\right)_y\right)\left(\frac{\partial^{l+1} w_0^*}{\partial y^{l+1}}\right) = \rho(x,y), \end{aligned}$$

where $\rho(x,y)$ contains w_0^* and its derivatives with respect to y up to order l and the coefficients and their derivatives Using the differential equation and its derivatives with respect to x and y we can express $\frac{\partial^{l+1} w_0^*}{\partial y^{l+1}}(x, 1+d)$ in terms the functions $\frac{\partial^i w_0}{\partial x^i}(x, 1+d)$ where $i \leq l+1$ Since $\frac{\partial^k w_0^*}{\partial x^k}(x, 1+d) = 0$ for all k, we obtain $\frac{\partial^{l+1} w_0^*}{\partial y^{l+1}}(x, 1+d) = 0$ Using this result and the fact that assuming sufficient regularity we have $\left|\frac{\partial^{l+2} w_0^*}{\partial y^{l+2}}(1,y)\right| \leq C$, we obtain $\left|\frac{\partial^{l+1} w_0^*}{\partial y^{l+1}}(1,y)\right| \leq C(1+d-y)$

Consider the barrier functions

$$\psi^{\pm}(x,y) = C(1+d-y)e^{\left((l+1)\left\|\left(\frac{a_2^*}{a_1^*}\right)_y\right\|-\frac{\gamma}{\mu}\right)(1-x)} \pm \frac{\partial^{l+1} w_0^*}{\partial y^{l+1}}$$

We see that the functions $\psi^{\pm}(x,y)$ are nonnegative on the boundaries for C large enough

Also

$$L_\mu^{[*,T,l+1]}\psi^\pm(x,y) = C\left(\left(\left(\gamma - \tfrac{b^*}{a_1^*}\right) + \mu\left((l+1)\left(\tfrac{a_2^*}{a_1^*}\right)_y - (l+1)\left\|\left(\tfrac{a_2^*}{a_1^*}\right)_y\right\|\right)\right)(1+d-y)\right.$$
$$\left. - \mu\left(\tfrac{a_2^*}{a_1^*}\right)\right)e^{\left((l+1)\left\|\left(\frac{a_2^*}{a_1^*}\right)_y\right\| - \frac{\gamma_1}{\mu}\right)(1-x)} \pm \rho(x,y)$$

Choosing C correctly, we find $L_\mu^{[*,T,l+1]}\psi^\pm(x,y) \le 0$ and therefore applying Lemma 5 3 1 we have

$$\left|\frac{\partial^{l+1} w_0^*}{\partial y^{l+1}}(x,y)\right| \le C(1+d-y)e^{\left((l+1)\left\|\left(\frac{a_2^*}{a_1^*}\right)_y\right\| - \frac{\gamma_1}{\mu}\right)(1-x)} \le C_1(1+d-y)e^{-\frac{\gamma_1}{\mu}(1-x)}$$

Using the differential equation (5 4 1b) and its derivatives with respect to x and y we can obtain the required result for $k = l+1$ □

Lemma 5 4 2 *When w_1^* is defined as in (5 4 1c), then given $\mu < \gamma_1$ and assuming sufficient regularity of the coefficients and the boundary data, the solution and its derivatives satisfy the following bounds for any positive integer k,*

$$\left|\frac{\partial^k w_1^*}{\partial x^i \partial y^j}(x,y)\right| \le \frac{C_1}{\mu^{i+2}}(d+1-y)e^{-\frac{\gamma_1}{\mu}(1-x)}, \quad (x,y) \in \Omega^{[*,TB]}, \qquad \left(C_1 \le Ce^{(k+2)\left\|\left(\frac{a_2^*}{a_1^*}\right)_y\right\|}\right)$$

Proof Let $f_1^* = w_{0xx}^* + w_{0yy}^*$ Using Lemma (5 4 1) we see that

$$\left|\frac{\partial^k f_1^*(x,y)}{\partial x^i \partial y^j}\right| \le \frac{C_1}{\mu^{i+2}}(d+1-y)e^{-\frac{\gamma_1}{\mu}(1-x)}, \qquad \left(C_1 \le Ce^{(k+2)\left\|\left(\frac{a_2^*}{a_1^*}\right)_y\right\|}\right)$$

Consider the barrier functions $\psi^\pm(x,y) = \frac{C}{\mu^2}(1+d-y)e^{-\frac{\gamma_1}{\mu}(1-x)} \pm w_1^*$ Since $w_1^*|_{\partial\Omega_1^{[*,T]}} = 0$, we see that the functions are nonnegative on the boundary Also

$$L_\mu^{[*,T]}\psi^\pm(x,y) = \frac{C}{\mu^2}\left(\gamma a_1^*(d+1-y) - \mu a_2^* - b^*(d+1-y)\right)e^{-\frac{\gamma_1}{\mu}(1-x)} \pm f_1^*$$

and for C large enough $L_\mu^{[*,T]}\psi^\pm(x,y) \le 0$ Using Lemma 5 3 1 we can therefore conclude that

$$|w_1^*| \le \frac{C}{\mu^2}(d+1-y)e^{-\frac{\gamma}{\mu}(1-x)}$$

As with w_0^*, we proceed by induction Assume the lemma is true for $0 \le k \le l$,

$$\left\| \frac{\partial^k w_1^*}{\partial x^i \partial y^j} \right\| \le \frac{C_1}{\mu^{i+2}}(d+1-y)e^{-\frac{\gamma}{\mu}(1-x)}, \qquad \left(C_1 \le Ce^{k+2\left\|\left(\frac{a_2^*}{a_1^*}\right)_y\right\|} \right)$$

We wish to prove the result true for $k = l+1$ Differentiating (5 4 1c) $l+1$ times with respect to y, we have

$$\begin{aligned} L_\mu^{[*,\mathrm{T},l+1]}\left(\frac{\partial^{l+1} w_1^*}{\partial y^{l+1}}\right) &= \mu\left(\frac{\partial^{l+1} w_1^*}{\partial y^{l+1}}\right)_x + \mu\frac{a_2^*}{a_1^*}\left(\frac{\partial^{l+1} w_1^*}{\partial y^{l+1}}\right)_y \\ &\quad -\left(\frac{b^*}{a_1^*} - (l+1)\mu\left(\frac{a_2^*}{a_1^*}\right)_y\right)\left(\frac{\partial^{l+1} w_1^*}{\partial y^{l+1}}\right) = \rho(x,y), \end{aligned}$$

where $\rho(x,y)$ contains w_1^* and its derivatives with respect to y up to order l, f_1^* and its derivatives with respect to y up to order $l+1$ and the coefficients and their derivatives Since for all k we have $\frac{\partial^k w_0^*}{\partial x^i \partial y^j}(x, 1+d) = 0$ we can use equation (5 4 1c) and its derivatives to obtain $\frac{\partial^{l+1} w_1^*}{\partial y^{l+1}}(x, 1+d) = 0$ Clearly we also have $\frac{\partial^{l+1} w_1^*}{\partial y^{l+1}}(1,y) = 0$ Consider the following barrier functions

$$\psi^\pm(x,y) = \frac{C}{\mu^2}(1+d-y)e^{\left((l+1)\left\|\left(\frac{a_2^*}{a_1^*}\right)_y\right\| - \frac{\gamma}{\mu}\right)(1-x)} \pm \frac{\partial^{l+1} w_1^*}{\partial y^{l+1}}$$

We see that the functions $\psi^\pm(x,y)$ are nonnegative on the boundaries, also

$$\begin{aligned} L_\mu^{[*,\mathrm{T},l+1]}\psi^\pm(x,y) = \frac{C}{\mu^2}\Bigg(\left(\left(\gamma - \frac{b^*}{a_1^*}\right) + \mu\left((l+1)\left(\frac{a_2^*}{a_1}\right)_y - (l+1)\left\|\left(\frac{a_2}{a_1^*}\right)_y\right\|\right)\right)(1+d-y) \\ -\mu\left(\frac{a_2^*}{a_1^*}\right)\Bigg) e^{\left((l+1)\left\|\left(\frac{a_2^*}{a_1^*}\right)_y\right\| - \frac{\gamma}{\mu}\right)(1-x)} \pm \rho(x,y) \end{aligned}$$

For C large enough $L_\mu^{[*,\mathrm{T},l+1]}\psi^\pm(x,y) \le 0$ and therefore using Lemma 5 3 1 we see that

$$\left|\frac{\partial^{l+1} w_1^*}{\partial y^{l+1}}\right| \le \frac{C}{\mu^2}(1+d-y)e^{\left((l+1)\left\|\left(\frac{a_2^*}{a_1^*}\right)_y\right\| - \frac{\gamma}{\mu}\right)(1-x)} \le \frac{C_1}{\mu^2}(1+d-y)e^{-\frac{\gamma}{\mu}(1-x)}$$

The differential equation (5 4 1c) and its derivatives with respect to x and y give the required result for $k = l + 1$ $\square$

Lemma 5 4 3 *Given $\mu < \gamma_1$, when w_2^* is defined as in* (5 4 1d), *then the solution and its derivatives satisfy the following bounds for* $0 \leq k \leq 3$,

$$|w_2^*| \leq \frac{C_1}{\mu^4} e^{-\frac{\gamma}{\mu}(1-x)},$$

$$\left\| \frac{\partial^k w_2^*}{\partial x^i \partial y^j} \right\| \leq \frac{C}{\mu^4} \left(\frac{\mu}{\varepsilon}\right)^k \quad \textit{and} \quad \left\| \frac{\partial^k w_2^*}{\partial y^k} \right\| \leq \frac{C}{\mu^4} \left(1 + \mu \left(\frac{\mu}{\varepsilon}\right)^k\right)$$

Proof On $\Omega^{[*,\mathrm{T}]}$, using Lemma 5 4 2, we know $\left\| \frac{\partial^2 w_1^*}{\partial x^2} \right\| + \left\| \frac{\partial^2 w_1^*}{\partial y^2} \right\| \leq \frac{C}{\mu^4}(1 + d - y) e^{-\frac{\gamma}{\mu}(1-x)}$ We extend $f^* = \frac{\partial^2 w_1^*}{\partial x^2} + \frac{\partial^2 w_1^*}{\partial y^2}$ to $\Omega^{[*,\mathrm{TB}]}$ so that $f^*(x, -d) = 0$ We therefore obtain

$$|f^*| \leq \frac{C}{\mu^4}(1 + d - y)(y + d) e^{-\frac{\gamma}{\mu}(1-x)}$$

We define smooth extensions of the coefficients a_1, a_2 and b to the domain $\Omega^{[*,\mathrm{TB}]}$ so that we have

$$\left| \frac{\partial^k a_i^*}{\partial y^k} \right| \leq C(d + y)(1 + d - y), \qquad \text{for} \quad i = 1, 2 \quad \text{and} \quad k = 0, 1, 2, \tag{5 4 2a}$$

and

$$\left| \frac{\partial b^*}{\partial y} \right| \leq C(d + y)(1 + d - y) \tag{5 4 2b}$$

Consider the barrier functions

$$\psi^\pm(x, y) = \frac{C}{\mu^4} e^{-\frac{\gamma}{\mu}(1-x)} \pm w_2^*$$

We see these functions are nonnegative on the boundary and using Lemma 5 4 2, we see that $L_{\varepsilon,\mu}^{[*,\mathrm{TB}]} \psi^\pm(x, y) \leq 0$ Applying the elliptic comparison principle gives the required exponential bound Since w_2^* satisfies a similar equation to u, we use Lemma 5 2 1 to obtain for $1 \leq i + j \leq 3$,

$$\left\| \frac{\partial^k w_2^*}{\partial x^i \partial y^j} \right\| \leq \frac{C}{\mu^4} \left(\frac{\mu}{\varepsilon}\right)^k \tag{5 4 3}$$

We need to sharpen these bounds in the direction orthogonal to the layer Consider the barrier functions

$$\psi^\pm(x, y) = \frac{C}{\mu^4}(d + y)(1 + d - y) \pm w_2$$

Clearly $\psi^{\pm}(x,y) \geq 0$ on the boundaries Also

$$L^{[*,\mathrm{TB}]}_{\varepsilon,\mu}\psi^{\pm} = \frac{C}{\mu^4}\left(-2\varepsilon - b^*(d+y)(1+d-y) + (1-2y)\mu a_2^*\right) \pm f^*$$

Given (5 4 2), and the fact that $\mu < \gamma_1$, we see that $L^{[*,\mathrm{TB}]}\psi^{\pm}(x,y) \leq 0$ for C large enough We can therefore apply the minimum principle to obtain

$$|w_2^*(x,y)| \leq \frac{C}{\mu^4}(d+y)(1+d-y)$$

Using the above bound we have

$$\left|\frac{\partial w_2^*}{\partial y}(x,1+d)\right| \leq \frac{C}{\mu^4} \quad \text{and} \quad \left|\frac{\partial w_2^*}{\partial y}(x,-d)\right| \leq \frac{C}{\mu^4}$$

We also note that $\frac{\partial w_2^*}{\partial y}(1,y) = \frac{\partial w_2^*}{\partial y}(0,y) = 0$

Differentiate (5 4 1d) with respect to y to obtain

$$L^{[*,\mathrm{TB}]}_{\varepsilon,\mu}\frac{\partial w_2^*}{\partial y} = -\mu\left(\frac{\partial a_1^*}{\partial y}\right)\frac{\partial w_2^*}{\partial x} - \mu\left(\frac{\partial a_2^*}{\partial y}\right)\frac{\partial w_2^*}{\partial y} + \left(\frac{\partial b^*}{\partial y}\right)w_2^* + \frac{\partial^3 w_1^*}{\partial x^2 \partial y} + \frac{\partial^3 w_1^*}{\partial y^3} = f^{**}$$

Using (5 4 3) and Lemma 5 4 2 we see that $|f^{**}| \leq C\left(\frac{1}{\mu^4} + \frac{1}{\mu^2\varepsilon}\right)$ Consider the barrier functions $\psi^{\pm}(x,y) = C_1\left(\frac{1}{\mu^4} + \frac{1}{\mu^2\varepsilon}\right) \pm \frac{\partial w_2^*}{\partial y}$ We see that the functions $\psi^{\pm}(x,y)$ are nonnegative on $\partial\Omega^{[*,\mathrm{TB}]}$ for C_1 large enough Also

$$L^{[*,\mathrm{TB}]}_{\varepsilon,\mu}\psi^{\pm}(x,y) = -bC_1\left(\frac{1}{\mu^4} + \frac{1}{\varepsilon\mu^2}\right) \pm f^{**} \leq 0,$$

for C_1 chosen correctly Therefore using the minimum principle we obtain

$$\left\|\frac{\partial w_2^*}{\partial y}\right\| \leq C\left(\frac{1}{\mu^4} + \frac{1}{\mu^2\varepsilon}\right)$$

Now we need to find $\frac{\partial^2 w_2^*}{\partial y^2}|_{\partial\Omega^{[*,\mathrm{TB}]}}$ Clearly $\frac{\partial^2 w_2^*}{\partial y^2}(1,y) = \frac{\partial^2 w_2}{\partial y^2}(0,y) = 0$ Using (5 4 1d) and our extension of a_2 and f^* we also find $\frac{\partial^2 w_2^*}{\partial y^2}(x,1+d) = \frac{\partial^2 w_2^*}{\partial y^2}(x,-d) = 0$ We differentiate (5 4 1d) twice with respect to y,

$$L_{\varepsilon,\mu}^{[*,\mathrm{TB}]}\frac{\partial^2 w_2^*}{\partial y^2} = -2\mu\left(\frac{\partial a_1^*}{\partial y}\right)\frac{\partial^2 w_2^*}{\partial x\partial y} - \mu\left(\frac{\partial^2 a_1^*}{\partial y^2}\right)\frac{\partial w_2^*}{\partial x} - 2\left(\frac{\partial a_2^*}{\partial y}\right)\frac{\partial^2 w_2^*}{\partial y^2}$$
$$+\left(2\left(\frac{\partial b^*}{\partial y}\right) - \mu\left(\frac{\partial^2 a_2^*}{\partial y^2}\right)\right)\frac{\partial w_2^*}{\partial y} - \left(\frac{\partial^2 b^*}{\partial y^2}\right)w_2^* + \frac{\partial^4 w_1^*}{\partial x^2\partial y^2} + \frac{\partial^4 w_1^*}{\partial y^4} = f^{***}$$

Using the crude bounds (5 4 3) and the bounds on $\frac{\partial w_2^*}{\partial y}$ above, we see that $|f^{***}| \le \frac{C}{\mu^4}\left(1+\frac{\mu^3}{\varepsilon^2}\right)$ Also, using the extension of the coefficients in (5 4 2) and the extension of the function w_1^*, we find

$$|f^{***}| \le \frac{C}{\mu^4}\left(1+\frac{\mu^3}{\varepsilon^2}\right)(y+d)(1+d-y)$$

Consider the barrier functions $\psi^\pm(x,y) = \frac{C_1}{\mu^4}\left(1+\frac{\mu^3}{\varepsilon^2}\right)(y+d)(1+d-y) \pm \frac{\partial^2 w_2^*}{\partial y^2}$ Both these functions are nonnegative on $\Omega^{[*,\mathrm{TB}]}$ Given $\mu < \gamma_1$, we have

$$L_{\varepsilon,\mu}^{[*,\mathrm{TB}]}\psi^\pm = \frac{C_1}{\mu^4}\left(1+\frac{\mu^3}{\varepsilon^2}\right)\Bigg(-2\varepsilon + \mu(1-2y)a_2^* - b^*(y+d)(1+d-y)\Bigg) \pm f^{***} \le 0$$

We apply the minimum principle to obtain

$$\left|\frac{\partial^2 w_2^*}{\partial y^2}\right| \le \frac{C_1}{\mu^4}\left(1+\frac{\mu^3}{\varepsilon^2}\right)(d+y)(1+d-y)$$

Therefore, we have

$$\left|\frac{\partial^3 w_2^*}{\partial y^3}(x,-d)\right| = \left|\frac{\frac{\partial^2 w_2^*}{\partial y^2}(x,y) - \frac{\partial^2 w_2^*}{\partial y^2}(x,-d)}{d+y}\right| \le \frac{C}{\mu^4}\left(1+\frac{\mu^3}{\varepsilon^2}\right)$$

Similarly we obtain $\left|\frac{\partial^3 w_2}{\partial y^2}(x,1+d)\right| \le \frac{C}{\mu^4}\left(1+\frac{\mu^3}{\varepsilon^2}\right)$ and we also have $\frac{\partial^3 w_2}{\partial y^3}(0,y) = 0$ and $\frac{\partial^3 w_2^*}{\partial y^3}(1,y) = 0$

Differentiate (5 4 1d) three times with respect to y to obtain

$$L^{[*,\mathrm{TB}]}\frac{\partial^3 w_2^*}{\partial y^3} = \frac{\partial^3 b^*}{\partial y^3}w_2^* + \left(3\frac{\partial^2 b^*}{\partial y^2} - \mu\left(\frac{\partial^3 a_2^*}{\partial y^3}\right)\right)\frac{\partial w_2^*}{\partial y} + \left(3\left(\frac{\partial b^*}{\partial y}\right) - 3\mu\left(\frac{\partial^2 a_2^*}{\partial y^2}\right)\right)\frac{\partial^2 w_2^*}{\partial y^2}$$
$$-3\mu\left(\frac{\partial a_2^*}{\partial y}\right)\frac{\partial^3 w_2^*}{\partial y^3} - 3\mu\left(\frac{\partial a_1^*}{\partial y^2}\right)\frac{\partial^3 w_2^*}{\partial x\partial y^2} - 3\mu\left(\frac{\partial^2 a_1^*}{\partial y^2}\right)\frac{\partial^2 w_2^*}{\partial x\partial y} - \mu\left(\frac{\partial^3 a_1^*}{\partial y^3}\right)\frac{\partial w_2^*}{\partial x}$$
$$+\frac{\partial^5 w_1^*}{\partial x^2\partial y^3} + \frac{\partial^5 w_1^*}{\partial y^5} = f^{****}\ (x,y) \in \Omega^{[*,\mathrm{TB}]}$$

We see that $\|f^{****}\| \le \frac{C_1}{\mu^4}\left(\frac{\mu^4}{\varepsilon^3}+1\right)$ and we can use barrier functions and the minimum principle to obtain

$$\left\|\frac{\partial^3 w_2^*}{\partial y^3}\right\| \le \frac{C}{\mu^4}\left(1+\frac{\mu^4}{\varepsilon^3}\right)$$

Combining all the above bounds, we obtain the required result $\square$

Lemma 5 4 4 *When w_R^* is defined as in (5 4 1), given $\mu < \gamma_1$, we see that*

$$|w_R^*(x,y)| \le Ce^{-\frac{\gamma}{\mu}(1-x)}$$

and its derivatives satisfy

$$\left\|\frac{\partial^k w_R^*}{\partial x^i \partial y^j}\right\| \le \frac{C}{\mu^k} \quad \textit{for} \quad 0 \le k \le 2, \qquad \left\|\frac{\partial^k w_R^*}{\partial x^i \partial y^j}\right\| \le \frac{C}{\varepsilon\mu}, \quad k=3$$

Moreover, in the direction orthogonal to the layer

$$\left\|\frac{\partial w_R^*}{\partial y}\right\| \le C, \qquad \left\|\frac{\partial^2 w_R^*}{\partial y^2}\right\| \le \frac{C}{\mu} \quad \textit{and} \quad \left\|\frac{\partial^3 w_R^*}{\partial y^3}\right\| \le \frac{C}{\varepsilon}$$

Proof This result follows from the decomposition (5 4 1) using $\mu^2 \ge \frac{\gamma\varepsilon}{\alpha}$ and the bounds on w_0^*, w_1^* and w_2^* their derivatives given respectively in Lemma 5 4 1, Lemma 5 4 2, and Lemma 5 4 3 $\square$

Define the boundary layer function w_R associated with the right edge Γ_R by

$$L_{\varepsilon,\mu} w_R = 0, \ (x,y) \in \Omega, \tag{5 4 4a}$$

$$w_R = u - v, \ (x,y) \in \Gamma_R, \quad w_R(0,y) = w_R^*(0,y), \tag{5 4 4b}$$

$$w_R(x,0) = w_R^*(x,0), \quad w_R(x,1) = w_R^*(x,1) \tag{5 4 4c}$$

Since $w_R = w_R^*$ on Ω the bounds in Lemma 5 4 4 transfer across

We now consider w_T the boundary layer function associated with the top edge Γ_T Our extended domain is given by $\Omega^{[*,R]} = (0,1+d) \times (0,1)$ (with $\Omega_1^{[*,R]} = [0,1+d) \times [0,1))$ and we define w_T^* to be the solution of $L_{\varepsilon,\mu}^{[*,R]} w_T^* = 0$, where the boundary data is chosen in the following decomposition

$$w_T^*(x,y,\varepsilon,\mu) = \tilde{w}_0^*(x,y,\mu) + \varepsilon\tilde{w}_1^*(x,y,\mu) + \varepsilon^2\tilde{w}_2^*(x,y,\varepsilon,\mu), \tag{5 4 5a}$$

where $v(x,1) = v_0(x,1) = \left(\frac{f}{b} - \left(\frac{\mu}{b}\right) a \cdot \nabla\left(\frac{f}{b}\right)\right)(x,1)$ is given in (5 3 4) and

$$\begin{aligned}
L_{\mu}^{[*,R]}\tilde{w}_0^* &= 0 \text{ on } \Omega_1^{[*,R]}, \quad \tilde{w}_0^*(1+d,y) = 0, \quad w_0^*(x,1) = (u(x,1) - v(x,1))^*, &(5\ 4\ 5\text{b}) \\
\varepsilon L_{\mu}^{[*,R]}\tilde{w}_1^* &= (L_{\mu}^{[*,R]} - L_{\varepsilon,\mu}^{[*,R]})\tilde{w}_0^* \text{ on } \Omega_1^{[*,R]}, \quad \tilde{w}_1^*(1+d,y,\mu) = \tilde{w}_1^*(x,1,\mu) = 0, &(5\ 4\ 5\text{c}) \\
\varepsilon^2 L_{\varepsilon,\mu}^{[*,LR]}\tilde{w}_2^* &= (\varepsilon(L_{\mu}^{[*,R]} - L_{\varepsilon,\mu}^{[*,R]})\tilde{w}_1^*)^* \text{ on } \Omega^{[*,LR]}, \quad \tilde{w}_2^*(x,y,\varepsilon,\mu)|_{\partial\Omega^{[*,LR]}} = 0 &(5\ 4\ 5\text{d})
\end{aligned}$$

We have the following lemma analogous to that for w_R

Lemma 5 4 5 *Given $\mu < \gamma_1$, the top layer function w_T^* defined in* (5 4 5), *satisfies the following bounds*

$$|w_T^*(x,y)| \le Ce^{-\frac{\gamma}{\mu}(1-y)}$$

and its derivatives satisfy

$$\left\|\frac{\partial^k w_T^*}{\partial x^i \partial y^j}\right\| \le \frac{C}{\mu^k} \quad \textit{for} \quad 0 \le k \le 2, \qquad \left\|\frac{\partial^k w_T^*}{\partial x^i \partial y^j}\right\| \le \frac{C}{\varepsilon\mu}, \quad k = 3$$

Moreover, in the direction orthogonal to the layer

$$\left\|\frac{\partial w_T^*}{\partial x}\right\| \le C, \qquad \left\|\frac{\partial^2 w_T^*}{\partial x^2}\right\| \le \frac{C}{\mu} \quad \textit{and} \quad \left\|\frac{\partial^3 w_T^*}{\partial x^3}\right\| \le \frac{C}{\varepsilon}$$

Proof The proof is similar to that in Lemma 5 4 4 Bounding each of the components $\tilde{w}_0^*$, $\tilde{w}_1^*$ and $\tilde{w}_2^*$ and their derivatives separately, we obtain the required exponential bounds and bounds on the derivatives of w_T^* These derivative bounds need to be sharpened in the direction orthogonal to the layer Extensions of a_1, a_2 and b to $\Omega^{[*,LR]}$ are constructed so that

$$\left|\frac{\partial^k a_i^*}{\partial x^k}\right| \le C(d+x)(1+d-x), \qquad \text{for} \quad i = 1,2 \quad \text{and} \quad k = 0,1,2,$$

and

$$\left|\frac{\partial b^*}{\partial x}\right| \le C(d+x)(1+d-x)$$

We can then use the same approach as for w_R^* in Lemma 5 4 4 to obtain the required orthogonal derivative bounds □

Define the boundary layer function w_T associated with the top edge Γ_T by

$$L_{\varepsilon,\mu} w_T = 0, \ (x,y) \in \Omega, \tag{5 4 6a}$$

$$w_T = u - v, \ (x,y) \in \Gamma_T, \quad w_L(x,0) = w_T^*(x,0), \tag{5 4 6b}$$

$$w_T(0,y) = w_T^*(0,y), \quad w_T(1,y) = w_T^*(1,y) \tag{5 4 6c}$$

5 5 Boundary layer components at the outflow

Consider w_L, the boundary layer function associated with the left edge Γ_L In order to obtain bounds on w_L we consider the extended domain $\Omega^{[*,\mathrm{TB}]} = (0,1) \times (-d, 1+d)$, $d > 0$ We define w_L^* to be the solution of

$$L_{\varepsilon,\mu}^{[*,\mathrm{TB}]} w_L^* = 0, \qquad (x,y) \in \Omega^{[*,\mathrm{TB}]}, \tag{5 5 1a}$$

$$w_L^*(0,y) = (u - v - w_R)^*(0,y), \qquad y \in [-d, 1+d], \tag{5 5 1b}$$

$$w_L^*(1,y) = 0, \qquad y \in [-d, 1+d], \tag{5 5 1c}$$

$$w_L^*(x,-d) = w_L^*(x, 1+d) = 0, \qquad x \in [0,1], \tag{5 5 1d}$$

and we extend $(u - y - w_R)(0,y)$ to $\Omega^{[*,\mathrm{TB}]}$ so that sufficient compatibility conditions are satisfied

Lemma 5 5 1 *Assuming $a_1(x,y) = a_1(x)$ and $\mu < \gamma_1$, when w_L^* is defined as in (5 5 1) we see that*

$$|w_L^*(x,y)| \leq C e^{-\frac{\mu\alpha}{\varepsilon} x}$$

Its derivatives satisfy

$$\left\| \frac{\partial^k w_L^*}{\partial x^i \partial y^j} \right\| \leq \left(\frac{\mu}{\varepsilon}\right)^k \quad \textit{for} \ \ 0 \leq k \leq 3,$$

and in the direction orthogonal to the layer

$$\left\| \frac{\partial w_L^*}{\partial y} \right\| \leq C, \qquad \left\| \frac{\partial^2 w_L^*}{\partial y^2} \right\| \leq \frac{C}{\mu} \quad \textit{and} \quad \left\| \frac{\partial^3 w_L^*}{\partial y^3} \right\| \leq \frac{C}{\varepsilon}$$

Proof We proceed as in the case of $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$ Using a suitably chosen barrier function, the exponential bounds can be shown Using Lemma 4 2 2 and Remark 4 5 1, we can show that when $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$ we have for $0 \leq k+m \leq 3$

$$\left|\left|\frac{\partial^{k+m} w_L^*}{\partial x^k \partial y^m}\right|\right| \leq C\left(\frac{\mu}{\varepsilon}\right)^{k+m} \tag{5 5 2}$$

In the direction orthogonal to the layer we must sharpen these bounds We only consider functions a_1 where $a_1(x,y) = a_1(x)$, and we smoothly extend a_1 to $\Omega^{[*,\mathrm{TB}]}$ so it is identically zero on Γ_T^* and Γ_B^* We extend the coefficients so that

$$\left|\frac{\partial^k a_2^*}{\partial y^k}\right| \leq C(d+y)(1+d-y), \qquad \text{for} \qquad k = 0,1,2, \tag{5 5 3a}$$

and

$$\left|\frac{\partial b^*}{\partial y}\right| \leq C(d+y)(1+d-y) \tag{5 5 3b}$$

Using the definition of $u(0,y)$, the bounds on v in (5 3 9) and the bounds on w_R in Lemma 5 4 4, we can show using a Taylor series expansion that $|w_L^*(0,y)| \leq C(d+y)(1+d-y)$ Consider the barrier functions

$$\psi^{\pm}(x,y) = C(d+y)(1+d-y) \pm w_L^*$$

The functions $\psi^{\pm}(x,y)$ are nonnegative on $\partial\Omega^{[*,\mathrm{TB}]}$ Since $\mu < \gamma_1$ and $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$, we see that $L_{\varepsilon,\mu}^{[*,\mathrm{TB}]}\psi^{\pm}(x,y) \leq 0$ and therefore using the minimum principle we obtain

$$|w_L^*(x,y)| \leq C(d+y)(1+d-y), \quad (x,y) \in \bar{\Omega}^{[*,\mathrm{TB}]} \tag{5 5 4}$$

Equation (5 3 9) and Lemma 5 4 4 gives $\left|\frac{\partial w_L^*}{\partial y}(0,y)\right| \leq C$ and $\frac{\partial w_L^*}{\partial y}(1,y) = 0$ Using (5 5 4) and the fact that $w_L^*(x,-d) = 0$ and $w_L^*(x,1+d) = 0$ we also obtain

$$\left|\frac{\partial w_L^*}{\partial y}(x,-d)\right| \leq C \quad \text{and} \quad \left|\frac{\partial w_L^*}{\partial y}(x,1+d)\right| \leq C$$

Differentiate (5 5 1) with respect to y, remembering that $a_1^*(x,y) = a_1^*(x)$, we obtain

$$\begin{aligned}\varepsilon(w_{Ly}^*)_{xx} + \varepsilon(w_{Ly}^*)_{yy} + \mu a_1^*(w_{Ly}^*)_x + \mu a_2^*(w_{Ly}^*)_y - (b^* - \mu a_{2y}^*)w_{Ly}^* &= b_y^* w_L^* \\ &= f^*, \qquad (x,y) \in \Omega^{[*,\mathrm{TB}]}\end{aligned}$$

Using (5 5 4) we see $||f^*|| \leq C$ Since $\mu < \gamma_1$, using barrier functions and the minimum principle we can show that

$$\left\| \frac{\partial w_L^*}{\partial y} \right\| \leq C$$

Equation (5 5 1) and the properties of a_2^* give us $\frac{\partial^2 w_L^*}{\partial y^2}(x, 1+d) = \frac{\partial^2 w_L^*}{\partial y^2}(x, -d) = 0$ Also using (5 3 9) and Lemma 5 4 4 we obtain $\left\| \frac{\partial^2 w_L^*}{\partial y^2}(0, y) \right\| \leq \frac{C}{\mu}$ and $\frac{\partial^2 w_L^*}{\partial y^2}(1, y) = 0$ Differentiating (5 5 1) twice with respect to y, remembering that a_1^* is a function of x alone, we obtain

$$\begin{aligned} \varepsilon(w_{Lyy}^*)_{xx} + \varepsilon(w_{Lyy}^*)_{yy} + \mu a_1^*(w_{Lyy}^*)_x + \mu a_2^*(w_{Lyy})_y - (b^* - 2\mu a_{2y}^*)w_{Lyy}^* \\ = (2b_y^* - \mu a_{2yy}^*)w_{Ly}^* - b_{yy}^* w_L^* = f^{**} \; (x, y) \in \Omega^{[*,\mathrm{TB}]} \end{aligned}$$

We see that $||f^{**}|| \leq C$ Using a suitable barrier function we can show that

$$\left\| \frac{\partial^2 w_L^*}{\partial y^2} \right\| \leq \frac{C}{\mu}$$

In order to obtain bounds on the third derivative of w_L^* in the direction orthogonal to the layer, we need sharper bounds on the second derivative above Using Taylor expansions, equation (5 3 9) and Lemma 5 4 4 we can show that $\left| \frac{\partial^2 w_L^*}{\partial y^2}(0, y) \right| \leq \frac{C}{\varepsilon}(d+y)(1+d-y)$ Also we can show that $|f^{**}| \leq C(d+y)(1+d-y)$ Consider the barrier functions $\psi^{\pm}(x, y) = \frac{C}{\varepsilon}(d+y)(1+d-y) \pm \frac{\partial^2 w_L^*}{\partial y^2}$ We can see that, choosing C large enough, both these functions are nonnegative on $\Omega^{[*,\mathrm{TB}]}$ Using the condition that $\mu < \gamma_1$, we obtain $L_{\varepsilon,\mu}^{[*,\mathrm{TB}]}\psi^{\pm}(x, y) \leq 0$ and applying the minimum principle, we therefore conclude

$$\left| \frac{\partial^2 w_L^*}{\partial y^2} \right| \leq \frac{C}{\varepsilon}(d+y)(1+d-y)$$

Since $\frac{\partial^2 w_L^*}{\partial y^2}(x, -d) = 0$, we have

$$\left\| \frac{\partial^3 w_L^*}{\partial y^3}(x, -d) \right\| = \left\| \frac{\frac{\partial^2 w_L^*}{\partial y^2}(x, y) - \frac{\partial^2 w_L^*}{\partial y^2}(x, -d)}{d+y} \right\| \leq \frac{C}{\varepsilon}$$

Similarly we obtain $\left\| \frac{\partial^3 w_L^*}{\partial y^3}(x, 1+d) \right\| \leq \frac{C}{\varepsilon}$ and we also have $\left\| \frac{\partial^3 w_L^*}{\partial y^3}(0, y) \right\| \leq \frac{C}{\varepsilon}$ and

$\frac{\partial^3 w_L^*}{\partial y^3}(1,y) = 0$ We differentiate (5 5 1) three times with respect to y to obtain

$$\begin{aligned}&\varepsilon(w^*_{Lyyy})_{xx} + \varepsilon(w^*_{Lyyy})_{yy} + \mu a_1^*(w^*_{Lyyy})_x + \mu a_2^*(w^*_{Lyyy})_y - (b^* - 3\mu a^*_{2y})w^*_{Lyyy} \\ &= b^*_{yyy}w_L^* + (3b^*_{yy} - \mu a^*_{2yyy})w^*_{Ly} + (3b^*_y - 3\mu a^*_{2yy})w^*_{Lyy} = f^{***} \ (x,y) \in \Omega^{[*,\mathrm{TB}]}\end{aligned}$$

We see that $\|f^{***}\| \leq \frac{C}{\mu}$, and noting $\mu < \gamma_1$ we can use barrier functions and the minimum principle to obtain

$$\left\|\frac{\partial^3 w_L^*}{\partial y^3}\right\| \leq \frac{C}{\varepsilon}$$

This concludes our proof □

We therefore define the boundary layer function w_L associated with the left edge Γ_L by

$$L_{\varepsilon,\mu} w_L = 0, \ (x,y) \in \Omega, \tag{5 5 5a}$$

$$w_L = u - v - w_R, \ (x,y) \in \Gamma_L, \quad w_L = 0, \ (x,y) \in \Gamma_R, \tag{5 5 5b}$$

$$w_L(x,0) = w_L^*(x,0), \quad w_L(x,1) = w_L^*(x,1) \tag{5 5 5c}$$

The layer component w_B is defined similarly We consider the extended domain $\Omega^{[*,\mathrm{LR}]}$ and we define w_B^* to be the solution of

$$L^{[*,\mathrm{LR}]}_{\varepsilon,\mu} w_B^* = 0, \qquad (x,y) \in \Omega^{[*,\mathrm{LR}]}, \tag{5 5 6a}$$

$$w_B^*(x,0) = (u - v - w_T)^*(x,0), \qquad x \in [-d, 1+d], \tag{5 5 6b}$$

$$w_B^*(x,1) = 0, \qquad x \in [-d, 1+d], \tag{5 5 6c}$$

$$w_B^*(-d,y) = w_B^*(1+d,y) = 0, \qquad y \in [0,1], \tag{5 5 6d}$$

and we extend $(u - y - w_T)(x,0)$ to $\Omega^{[*,\mathrm{TB}]}$ so that sufficient compatibility conditions are satisfied

Lemma 5 5 2 *Assuming $a_2(x,y) = a_2(y)$ and $\mu < \gamma_1$, when w_B^* is defined as in* (5 5 6) *we see that*

$$|w_B^*(x,y)| \leq Ce^{-\frac{\mu\alpha}{\varepsilon}y}$$

Its derivatives satisfy

$$\left\|\frac{\partial^k w_B^*}{\partial x^i \partial y^j}\right\| \leq \left(\frac{\mu}{\varepsilon}\right)^k \quad \textit{for} \ \ 0 \leq k \leq 3$$

Moreover, in the direction orthogonal to the layer

$$\left\|\frac{\partial w_B^*}{\partial x}\right\| \leq C, \qquad \left\|\frac{\partial^2 w_B^*}{\partial x^2}\right\| \leq \frac{C}{\mu} \quad \textit{and} \quad \left\|\frac{\partial^3 w_B^*}{\partial x^3}\right\| \leq \frac{C}{\varepsilon}$$

Proof The proof is similar to that in Lemma 5 5 1 We consider the barrier functions $\psi^{\pm}(x,y) = Ce^{-\frac{\mu\alpha}{\varepsilon}y} \pm w_B^*$ These functions are nonnegative on the boundary $\partial\Omega^{[*,\mathrm{LR}]}$ Also for C chosen correctly, $L_{\varepsilon,\mu}^{[*,\mathrm{LR}]}\psi^{\pm}(x,y) \leq 0$, and we obtain the required exponential bound Using Lemma 4 2 2 and Remark 4 5 1, we can show that when $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$ we have for $0 \leq k+m \leq 3$

$$\left\|\frac{\partial^{k+m} w_B^*}{\partial x^k \partial y^m}\right\| \leq C\left(\frac{\mu}{\varepsilon}\right)^{k+m} \tag{5 5 7}$$

In order to obtain the sharp orthogonal derivative bounds, extensions of a_1 and b to $\Omega^{[*,\mathrm{LR}]}$ are constructed so that

$$\left|\frac{\partial^k a_1^*}{\partial x^k}\right| \leq C(d+x)(1+d-x), \qquad \text{for} \qquad k = 0,1,2,$$

and

$$\left|\frac{\partial b^*}{\partial x}\right| \leq C(d+x)(1+d-x)$$

Assuming that $a_2(x,y) = a_2(y)$, we extend a_2 so that a_2^* is identically zero on Γ_L and Γ_R We then use the same approach as for w_L^* in Lemma 5 5 1 to obtain the required orthogonal derivative bounds □

We therefore describe the boundary layer function associated with the bottom edge Γ_B by

$$L_{\varepsilon,\mu} w_B = 0, \ (x,y) \in \Omega, \tag{5 5 8a}$$

$$w_B = (u-v) - w_T, \ (x,y) \in \Gamma_B, \quad w_B = 0, \ (x,y) \in \Gamma_T, \tag{5 5 8b}$$

$$w_B(0,y) = w_B^*(0,y), \quad w_B(1,y) = w_B^*(1,y) \tag{5 5 8c}$$

Remark 5 5 1 *Since we have defined all of the above boundary layer functions on extended domains, we are not imposing overly artificial compatibility conditions at the corners When we move to the analysis of the corner layer functions we sometimes will be considering elliptic problems on the non-extended original domain where compatibility may be an issue*

5 6 Corner layer components

The order in which we define the corner layer functions is vital to obtaining the correct bounds on the components and their derivatives required for the error analysis The four corners are treated differently in our analysis In order to correctly isolate the corner layer components, we have to be careful about the boundary data chosen for each of the functions As with w_R and w_T, we use decompositions to chose these boundary conditions so as to correctly isolate the corner singularities In order to isolate the top-right corner layer function w_{RT}, we use a decomposition of w_{RT} into a sum of solutions to first order problems and the solution of an elliptic problem The top-left and bottom-right layer components are both decomposed into a sum of a solution to a parabolic problem and the solution of an elliptic problem It is not necessary to decompose w_{LB}

In this section, we show how we believe the corner layer functions should be defined In order to prove parameter-uniform convergence of our numerical method, we need to obtain bounds on these components and their derivatives However, at present we do not have a rigorous proof of these bounds Instead, we state a series of conjectures, the validity of which remain an open question These conjectures are motivated using arguments similar to those in the previous sections but the proofs of such bounds are left for future work

Starting with the corner layer function associated with the top right corner, we define w_{RT} by

$$L_{\varepsilon,\mu} w_{RT} = 0, \ (x,y) \in \Omega,$$
$$w_{RT} = -w_T, \ (x,y) \in \Gamma_R, \quad w_{RT} = -w_R, \ (x,y) \in \Gamma_T,$$
$$w_R(x,0), \ w_R(0,y) \qquad \text{defined in } (5\ 6\ 1)$$

In order to determine appropriate values for $w_{RT}(0,y)$ and $w_{RT}(1,y)$, we decompose w_{RT} as follows,

$$w_{RT}(x,y,\varepsilon,\mu) = \bar{w}_0(x,y,\mu) + \varepsilon \bar{w}_1(x,y,\mu) + \varepsilon^2 \bar{w}_2(x,y,\varepsilon,\mu) \qquad (5\ 6\ 1a)$$

where

$$\begin{aligned}
L_\mu \bar{w}_0 &= 0 \text{ on } \Omega_1 = [0,1)^2, \ \ \bar{w}_0(x,1) = -w_R(x,1), \ \ \bar{w}_0(1,y) = -w_T(1,y), && (5\ 6\ 1b)\\
\varepsilon L_\mu \bar{w}_1 &= (L_\mu - L_{\varepsilon,\mu})\bar{w}_0 \ \text{ on } \ \Omega_1, \ \ \bar{w}_1(x,1,\mu) = \bar{w}_1(1,y,\mu) = 0, && (5\ 6\ 1c)\\
\varepsilon^2 L_{\varepsilon,\mu} \bar{w}_2 &= \varepsilon(L_\mu - L_{\varepsilon,\mu})\bar{w}_1 \quad \text{on } \ \Omega, \ \bar{w}_2(x,y,\varepsilon,\mu)|_{\partial\Omega} = 0 && (5\ 6\ 1d)
\end{aligned}$$

Conjecture 5 6 1 *When w_{RT} is defined as in the decomposition* (5 6 1), *we have the following bounds on the corner layer function associated with the top-right corner*

$$|w_{RT}(x,y)| \leq Ce^{-\frac{\gamma}{2\mu}(1-x)}e^{-\frac{\gamma}{2\mu}(1-y)},$$

and its derivatives satisfy

$$\left\|\frac{\partial^k w_{RT}}{\partial x^i \partial y^j}\right\| \leq \frac{C}{\mu^k}, \quad \textit{for} \quad 0 \leq k \leq 2,$$

and

$$\left\|\frac{\partial^k w_{RT}}{\partial x^i \partial y^j}\right\| \leq \frac{C}{\varepsilon\mu}, \qquad k = 3$$

Motivation In order to obtain the exponential character of the layer function w_{RT}, we must assume the boundary conditions $w_{RT}(0,y) = \bar{w}_0(0,y) + \varepsilon\bar{w}_1(0,y)$ and $w_{RT}(x,0) = \bar{w}_0(x,0) + \varepsilon\bar{w}_1(x,0)$, obtained using the decomposition (5 6 1), satisfy the bounds

$$|w_{RT}(0,y)| \leq Ce^{-\frac{\gamma}{2\mu}}e^{-\frac{\gamma}{2\mu}(1-y)} \quad \text{and} \quad |w_{RT}(x,0)| \leq Ce^{-\frac{\gamma}{2\mu}}e^{-\frac{\gamma}{2\mu}(1-x)}$$

Consider the barrier functions $\psi^{\pm}(x,y) = Ce^{-\frac{\gamma}{2\mu}(1-x)}e^{-\frac{\gamma}{2\mu}(1-y)} \pm w_{RT}$ Using Lemma 5 4 4, Lemma 5 4 5 and this assumption, we see $\psi^{\pm}(x,y)|_{\partial\Omega} \geq 0$ We also see that when $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$, for C large enough

$$L_{\varepsilon,\mu}\psi^{\pm}(x,y) = C\left(\frac{\varepsilon}{4\mu^2}\gamma^2 + \frac{\varepsilon}{4\mu^2}\gamma^2 + \frac{\gamma}{2}a_1 + \frac{\gamma}{2}a_2 - b\right)e^{-\frac{\gamma}{2\mu}(1-x)}e^{-\frac{\gamma}{2\mu}(1-y)} \leq 0$$

We therefore apply the minimum principle to obtain the result

To obtain the derivative bounds on $\bar{w}_0$, we could applying a similar argument to that in Lemma 5 4 1 to get for $0 \leq k \leq 6$

$$\left\|\frac{\partial^k \bar{w}_0}{\partial x^i \partial y^j}\right\| \leq \frac{C}{\mu^k}$$

We should note that the proof of such bounds would require us to extend the derivative bounds in Lemma 5 4 4 to give

$$\left\|\frac{\partial^k w_R}{\partial x^i \partial y^j}\right\| \leq \frac{C}{\mu^k} \text{ for } 0 \leq k \leq p \quad \text{and} \quad \left\|\frac{\partial^k w_T}{\partial x^i \partial y^j}\right\| \leq \frac{C}{\mu^k} \text{ for } 0 \leq k \leq p$$

These bounds can possibly be achieved by the more complex decomposition of the components into p terms, $p-1$ of which are solutions of first order differential equations and the final term a solution of an elliptic differential equation We do not discuss the resulting compatibility or regularity issues that arise from decomposing w_R^* and w_T^* into such sums of p terms

If the above bounds hold, we can show using Lemma 5 3 2 that for $0 \le k \le 4$ we have

$$\left\| \frac{\partial^k \bar{w}_1}{\partial x^i \partial y^j} \right\| \le \frac{C}{\mu^{k+2}}$$

Finally since $\bar{w}_2$ satisfies a similar equation to u, we can use Lemma 5 2 1 along with the above bounds to obtain the required derivative bounds on w_{RT}

The next component to consider is w_{LT}, the corner layer function associated with the top left corner Γ_{LT}

$$\begin{gathered} L_{\varepsilon,\mu} w_{LT} = 0, \ (x,y) \in \Omega, \\ w_{LT} = -w_T - w_{RT}, \ (x,y) \in \Gamma_L, \quad w_{LT} = -w_L, \ (x,y) \in \Gamma_T, \\ w_{LT}(1,y) = 0, \qquad w_{LT}(x,0) \quad \text{defined in (5 6 2)} \end{gathered}$$

In order to determine the appropriate value for $w_{LT}(x,0)$ so as to isolate the top-left singularity, we consider the extended domain $\Omega^{[*,\mathrm{B}]}$ and decompose w_{LT}^* into a sum of a solution to a parabolic problem and a solution of an elliptic problem as follows,

$$w_{LT}^*(x,y,\varepsilon,\mu) = \hat{w}_0^*(x,y,\varepsilon,\mu) + \varepsilon \hat{w}_1^*(x,y,\varepsilon,\mu), \tag{5 6 2a}$$

where

$$\begin{aligned} \hat{L}_{p,\varepsilon,\mu}^{[*,\mathrm{B}]} \hat{w}_0^* &= \varepsilon \hat{w}_{0xx}^* + \mu a_1^* \hat{w}_{0x}^* - b^* \hat{w}_0^* + \mu a_2^* \hat{w}_{0y}^* = 0, \quad \hat{w}_0^*(x,1) = -w_L(x,1), && \text{(5 6 2b)} \\ & \qquad \hat{w}_0^*(0,y) = (-w_T(0,y) - w_{RT}(0,y))^*, \qquad \hat{w}_0^*(1,y) = 0, && \\ \varepsilon L_{\varepsilon,\mu}^{[*,\mathrm{B}]} \hat{w}_1^* &= (L_{p,\varepsilon,\mu}^{[*,\mathrm{B}]} - L_{\varepsilon,\mu}^{[*,\mathrm{B}]}) \hat{w}_0^* \ \text{ on } \ \Omega^{[*,\mathrm{B}]}, \qquad \hat{w}_1^*(x,y,\varepsilon,\mu)|_{\partial\Omega^{[*,\mathrm{B}]}} = 0 && \text{(5 6 2c)} \end{aligned}$$

Remark 5 6 1 *We should note that, keeping with the style of the thesis, it would seem more natural for the above decomposition to have three terms in the expansion However, in this case such an expansion is not necessary for the discrete error analysis Having three terms in the expansion would also make the establishment of the bounds on the derivatives significantly more difficult We should also note that we are required to know $w_{RT}(0,y)$ before we define w_{LT}^* and for this reason it is essential to be extremely careful about the* *order in which these layer functions are defined*

Conjecture 5 6 2 *When w^*_{LT} is defined as in the decomposition* (5 6 2) *we have the following bounds on the corner layer function associated with the top-left corner*

$$|w^*_{LT}(x,y)| \le Ce^{-\frac{\gamma}{2\mu}(1-y)}e^{-\frac{\mu}{\varepsilon}\alpha x},$$

and its derivatives satisfy

$$\left\|\frac{\partial^k w^*_{LT}}{\partial y^k}\right\| \le C\left(\left(\frac{1}{\mu}\right)^k + \frac{\varepsilon}{\mu^2}\left(\frac{\mu}{\varepsilon}\right)^k\right), \qquad 0 \le k \le 3,$$

and

$$\left\|\frac{\partial^k w^*_{LT}}{\partial x^i \partial y^j}\right\| \le C\left(\frac{\mu}{\varepsilon}\right)^k, \qquad 0 \le k \le 3$$

Motivation In order to obtain the required exponential bounds on w^*_{LT}, we begin by analysing the component $\hat{w}^*_0$ We make a change of variables $t = \frac{1-y}{\mu}$ Letting $\hat{w}^*_0(x,y) = \eta_0(x,t)$, and $a^*_1(x,y) = \tilde{a}_1(x,t)$ with the other functions defined analogously, we obtain

$$L^{[*,B]}_{p,\varepsilon,\mu}\eta_0 = \varepsilon\eta_{0xx} + \mu\tilde{a}_1\eta_{0x} - \tilde{b}\eta_0 - \tilde{a}_2\eta_{0t} = 0, \quad \begin{aligned} \eta_0(x,0) &= -\tilde{w}_L(x,0), \quad \eta_0(1,t) = 0, \\ \eta_0(0,t) &= -\tilde{w}_T(0,t) - \tilde{w}_{RT}(0,t) \end{aligned}$$

Consider the barrier functions

$$\psi^{\pm}(x,y) = Ce^{-\frac{t\gamma}{2}}e^{\frac{-\mu\alpha}{\varepsilon}x} \pm \eta_0$$

Using the exponential bounds on w_L and w_T given in Lemma 5 5 1 and Lemma 5 4 5 and assuming Conjecture 5 6 1 holds, we see that $\psi(x,t)|_{\Gamma_p} \ge 0$ for C large enough ($\Gamma_p = \tilde{\Gamma}^{[*,B]}_L \cup \tilde{\Gamma}^{[*,B]}_R \cup \tilde{\Gamma}_T$) We also obtain

$$L^{[*,B]}_{p,\varepsilon,\mu}\psi^{\pm}(x,t) = \left(\varepsilon\left(\frac{\mu}{\varepsilon}\right)^2\alpha^2 - \frac{\mu^2}{\varepsilon}\alpha a_1 - b + a_2\frac{\gamma}{2}\right)e^{-\frac{t\gamma}{2}}e^{\frac{-\mu\alpha}{\varepsilon}x} \pm 0,$$

and we can show that $L^{[*,B]}_{p,\varepsilon,\mu}\psi^{\pm}(x,t) \le 0$ for C large enough Using the minimum principle for the parabolic problem, we obtain

$$|\eta_0(x,t)| \le Ce^{-\frac{t\gamma}{2}}e^{\frac{-\mu\alpha}{\varepsilon}x}$$

Therefore, transforming back, we have

$$|\hat{w}_0^*(x,y)| \leq Ce^{-\frac{\gamma(1-y)}{2\mu}} e^{\frac{-\mu\alpha}{\varepsilon}x} \tag{5 6 3}$$

To find the exponential character of the corner layer function w_{LT}^*, consider the following barrier functions on the extended domain $\Omega^{[*,B]}$,

$$\psi^{\pm}(x,y) = Ce^{-\frac{\gamma}{2\mu}(1-y)} e^{-\frac{\mu}{\varepsilon}\alpha x} \pm w_{LT}^*(x,y)$$

If the exponential bounds on $\hat{w}_0^*$ in (5 6 3) hold then we have $|\hat{w}_0^*(x,-d)| \leq Ce^{-\frac{\gamma}{2\mu}(1+d)} e^{-\frac{\mu}{\varepsilon}\alpha x}$ Using the exponential bounds in Lemma 5 5 1 and Lemma 5 4 5 and assuming Conjecture 5 6 1, we obtain $\psi^{\pm}(x,y)|_{\partial\Omega^{[*,B]}} \geq 0$ for C large enough We can also show for C chosen correctly we have $L_{\varepsilon,\mu}^{[*,B]}\psi^{\pm} \leq 0$ and therefore we obtain the required exponential bound

The required bounds on the derivatives of w_{LT}^* can possibly be obtained by analysing the each of its components separately Such a proof would however require that the bounds in Lemma 5 5 1 and Lemma 5 4 5 can be extended as follows

$$\left\|\frac{\partial^k w_T}{\partial x^i \partial y^j}\right\| \leq \frac{C}{\mu^k} \text{ for } 0 \leq k \leq p \quad \text{and} \quad \left\|\frac{\partial^k w_L}{\partial x^i \partial y^j}\right\| \leq C\left(\frac{\mu}{\varepsilon}\right)^k \text{ for } 0 \leq k \leq p \tag{5 6 4}$$

We define the boundary layer function w_{LT} associated with the top left corner Γ_{LT} by

$$L_{\varepsilon,\mu} w_{LT} = 0, \ (x,y) \in \Omega, \tag{5 6 5a}$$

$$w_{LT} = -w_T - w_{RT}, \ (x,y) \in \Gamma_L, \quad w_L = 0, \ (x,y) \in \Gamma_R, \tag{5 6 5b}$$

$$w_{LT}(x,0) = w_{LT}^*(x,0), \quad w_{LT}(x,1) = -w_L(x,1) \tag{5 6 5c}$$

We now consider w_{RB}, the corner layer function associated with the bottom-right corner Γ_{RB}

$$L_{\varepsilon,\mu} w_{RB} = 0, \ (x,y) \in \Omega,$$

$$w_{RB} = -w_B, \ (x,y) \in \Gamma_R, \quad w_{RB} = -w_R - w_{RT}, \ (x,y) \in \Gamma_B,$$

$$w_{RB}(x,1) = 0, \quad w_{RB}(0,y) \quad \text{defined in (5 6 6)}$$

We consider the extended domain $\Omega^{[*,L]}$ and decompose w_{RB}^* into a sum of a solution to a parabolic and a solution of an elliptic problem as follows ,

$$w_{RB}^*(x,y) = \hat{w}_0^*(x,y) + \varepsilon\hat{w}_1^*(x,y) \tag{5 6 6a}$$

where

$$\begin{aligned} \hat{L}_{p,\varepsilon,\mu}^{[*,L]}\hat{w}_0^* &= \varepsilon\hat{w}_{0yy}^* + \mu a_2^*\hat{w}_{0y}^* - b^*\hat{w}_0^* + \mu a_1^*\hat{w}_{0x}^* = 0, \quad \hat{w}_0^*(1,y) = -w_B(1,y), && (5\ 6\ 6b)\\ &\qquad \hat{w}_0^*(x,0) = -w_R(x,0) - w_{RT}(x,0), \qquad \hat{w}_0^*(x,1) = 0, \\ \varepsilon L_{\varepsilon,\mu}^{[*,L]}\hat{w}_1^* &= (L_{p,\varepsilon,\mu}^{[*,L]} - L_{\varepsilon,\mu}^{[*,L]})\hat{w}_0^*, \qquad \hat{w}_1^*(x,y,\varepsilon,\mu)|_{\partial\Omega^{[*,L]}} = 0 && (5\ 6\ 6c) \end{aligned}$$

Conjecture 5 6 3 *When w_{RB}^* is defined as in the decomposition* (5 6 6), *we have the following bounds on the corner layer function associated with the bottom-right corner*

$$|w_{RB}^*(x,y)| \le Ce^{-\frac{\gamma}{2\mu}(1-x)}e^{-\frac{\mu}{\varepsilon}\alpha y},$$

and its derivatives satisfy

$$\left\|\frac{\partial^k w_{RB}^*}{\partial x^k}\right\| \le C\left(\left(\frac{1}{\mu}\right)^k + \frac{\varepsilon}{\mu^2}\left(\frac{\mu}{\varepsilon}\right)^k\right), \qquad 0 \le k \le 3,$$

and

$$\left\|\frac{\partial^k w_{RB}^*}{\partial x^i \partial y^j}\right\| \le C\left(\frac{\mu}{\varepsilon}\right)^k, \qquad 0 \le k \le 3$$

Motivation The motivation for this result is analogous to that of Conjecture 5 6 2 We therefore define the boundary layer function w_{RB} associated with the bottom-right corner Γ_{RB} by

$$L_{\varepsilon,\mu}w_{RB} = 0, \ (x,y) \in \Omega, \tag{5 6 7a}$$

$$w_{LT} = -w_R - w_{RT}, \ (x,y) \in \Gamma_B, \quad w_L = 0, \ (x,y) \in \Gamma_T, \tag{5 6 7b}$$

$$w_{LT}(0,y) = w_{LT}^*(0,y), \quad w_{LT}(1,y) = -w_B(1,y) \tag{5 6 7c}$$

Finally we consider the corner layer function w_{LB} associated with the corner Γ_{LB} We define w_{LB} to be the solution of

$$L_{\varepsilon,\mu}w_{LB} = 0 \ (x,y) \in \Omega, \tag{5 6 8a}$$

$$w_{LB} = -w_B - w_{RB}, \ (x,y) \in \Gamma_L, \quad w_{LB} = -w_L - w_{LT}, \ (x,y) \in \Gamma_B, \tag{5 6 8b}$$

$$w_{LB} = 0, \ (x,y) \in \Gamma_R, \quad w_{LB} = 0, \ (x,y) \in \Gamma_T \tag{5 6 8c}$$

Conjecture 5.6.4 *When w_{LB} is defined as in (5.6.8), we have the following bounds on the corner layer function associated with the bottom-left corner*

$$|w_{LB}(x,y)| \leq Ce^{-\frac{\mu\alpha}{\varepsilon}x}e^{-\frac{\mu\alpha}{\varepsilon}y},$$

and its derivatives satisfy

$$\left\|\frac{\partial^k w_{LB}}{\partial x^i \partial y^j}\right\| \leq C\left(\frac{\mu}{\varepsilon}\right)^k$$

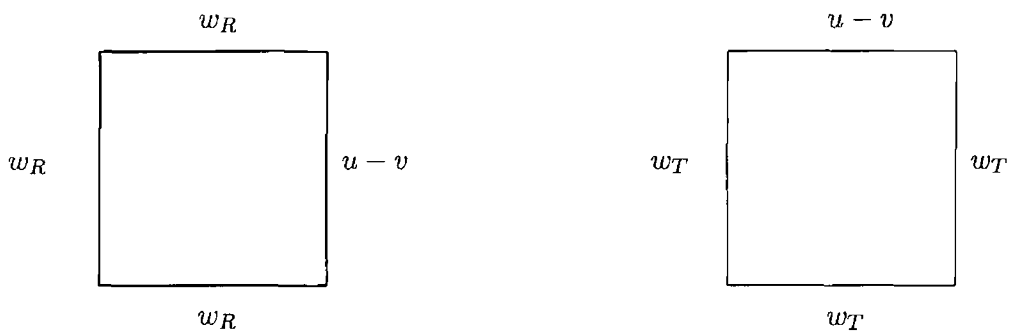
Motivation Consider the barrier functions $\psi^{\pm}(x,y) = Ce^{-\frac{\mu\alpha}{\varepsilon}x}e^{-\frac{\mu\alpha}{\varepsilon}y}$. Using the exponential bounds on w_L and w_B in Lemma 5.5.1 and Lemma 5.5.2 and assuming the exponential bounds on w_{LT} and w_{RB} in Conjecture 5.6.2 and Conjecture 5.6.3 hold, we see that both these functions are nonnegative on $\partial\Omega$. Also

$$L_{\varepsilon,\mu}\psi^{\pm}(x,y) = C\left(\frac{2\mu^2\alpha^2}{\varepsilon} - \frac{\alpha\mu^2 a_1}{\varepsilon} - \frac{\alpha\mu^2 a_2}{\varepsilon} - b\right)e^{-\frac{\mu\alpha}{\varepsilon}x}e^{-\frac{\mu\alpha}{\varepsilon}y} \pm 0,$$

and using the definition of α we see that $L_{\varepsilon,\mu}\psi^{\pm}(x,y) \leq 0$. Using the minimum principle we obtain the required exponential bounds.

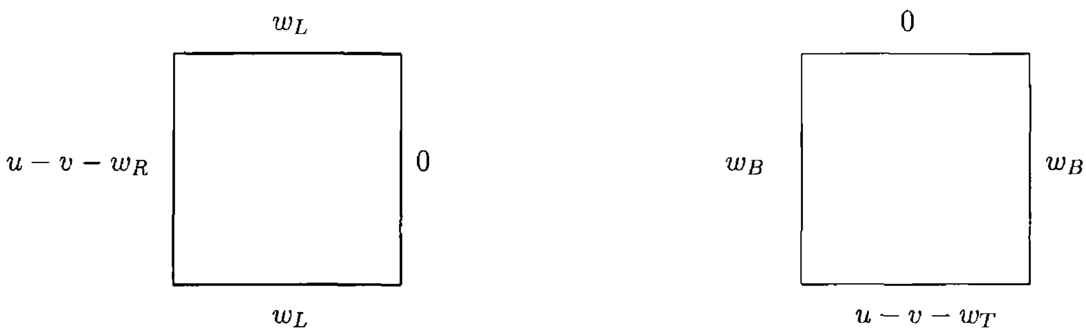
The bounds on derivatives of w_{LB} should follow using Lemma 5.2.1. However, such a proof would require extensions of the derivative bounds in Lemma 5.5.1, Lemma 5.5.2, Conjecture 5.6.2 and Conjecture 5.6.3.

Remark 5.6.2 *Figures 5.1-5.4 show the boundary data picked up by the layer functions defined in the previous sections. Since we see these functions are interdependent, the order in which they were defined was crucial in isolating the layers and obtaining the correct decomposition of u. The choice of boundary data for each function is also crucial to obtaining bounds on these components and their derivatives. With regards to compatibility, looking at Figure 5.4 (h) for example, we see that at the corner $(0,0)$, $(-w_B - w_{RB})(0,0)$ is equal to $(-u+v+w_T+w_R+w_{RT})(0,0)$ which is in turn equal to $(-w_L - w_{LT})(0,0)$. Similar arguments hold in the other three corners and for the other layer functions. However, we realise there are many compatibility issues that have not been addressed. We accept these issues are significant and we hope to examine them in some future publication.*



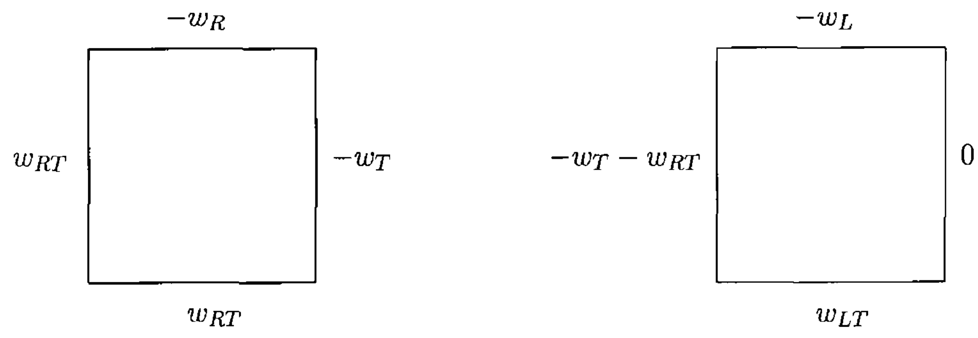
(a) (b)

Figure 5 1 Figures illustrating the boundary data of the functions (a) w_R and (b) w_T



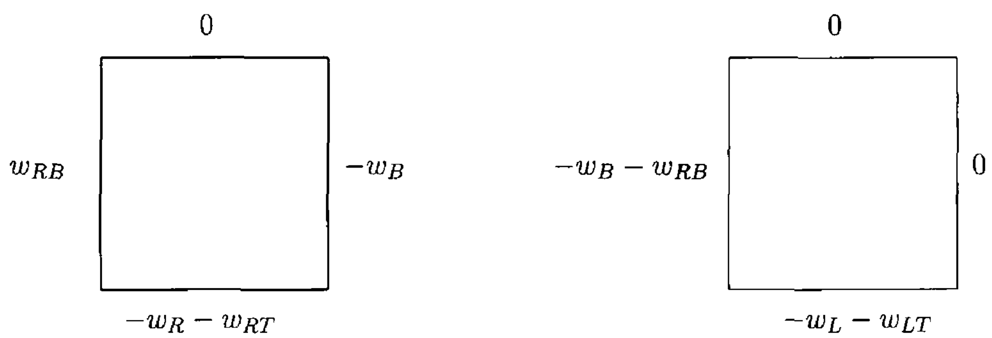
(c) (d)

Figure 5 2 Figures illustrating the boundary data of the functions (c) w_L and (d) w_B



(e) (f)

Figure 5 3 Figures illustrating the boundary data of the functions (e) w_{RT} and (f) w_{LT}



(g) (h)

Figure 5 4 Figures illustrating the boundary data of the functions (g) w_{RB} and (h) w_{LB}

Theorem 5 6 1 *When $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$ the solution u of* (4 1 1) *can be decomposed as*

$$u = v + w_L + w_R + w_T + w_B + w_{LB} + w_{LT} + w_{RB} + w_{RT} \tag{5 6 9}$$

where $L_{\varepsilon,\mu}v = f$, and the layer and corner layer functions are each solutions of the homogenous equation $L_{\varepsilon,\mu}w = 0$ Boundary conditions for these functions can be specified so that given Assumptions 1 *and* 2*, the bounds on the regular and boundary layer components and their derivatives given below hold*

$$\left\|\frac{\partial^{k+m}v}{\partial x^k \partial y^m}\right\| \leq C\left(1 + \left(\frac{\mu}{\varepsilon}\right)^{k+m-2}\right), \quad 0 \leq k+m \leq 3, \tag{5 6 10a}$$

$$|w_L(x,y)| \leq Ce^{-\frac{\mu\alpha}{\varepsilon}x}, \ |w_B(x,y)| \leq Ce^{-\frac{\mu\alpha}{\varepsilon}y}, \tag{5 6 10b}$$

$$|w_R(x,y)| \leq Ce^{-\frac{\gamma}{\mu}(1-x)}, \quad |w_T(x,y)| \leq Ce^{-\frac{\gamma}{\mu}(1-y)}, \tag{5 6 10c}$$

$$\left\|\frac{\partial^k w_L}{\partial x^i \partial y^j}\right\| \leq \left(\frac{\mu}{\varepsilon}\right)^k, \quad \left\|\frac{\partial^k w_B}{\partial x^i \partial y^j}\right\| \leq \left(\frac{\mu}{\varepsilon}\right)^k \ \textit{for} \ 0 \leq k \leq 3, \tag{5 6 10d}$$

$$\left\|\frac{\partial w_L}{\partial y}\right\| \leq C, \quad \left\|\frac{\partial^2 w_L}{\partial y^2}\right\| \leq \frac{C}{\mu}, \quad \left\|\frac{\partial^3 w_L}{\partial y^3}\right\| \leq \frac{C}{\varepsilon}, \tag{5 6 10e}$$

$$\left\|\frac{\partial w_B}{\partial x}\right\| \leq C, \quad \left\|\frac{\partial^2 w_B}{\partial x^2}\right\| \leq \frac{C}{\mu}, \quad \left\|\frac{\partial^3 w_B}{\partial x^3}\right\| \leq \frac{C}{\varepsilon}, \tag{5 6 10f}$$

$$\left\|\frac{\partial^k w_R}{\partial x^i \partial y^j}\right\| \leq \frac{C}{\mu^k} \ \textit{for} \ 0 \leq k \leq 2, \quad \left\|\frac{\partial^k w_R}{\partial x^i \partial y^j}\right\| \leq \frac{C}{\varepsilon\mu}, \ k = 3, \tag{5 6 10g}$$

$$\left\|\frac{\partial w_R}{\partial y}\right\| \leq C, \quad \left\|\frac{\partial^2 w_R}{\partial y^2}\right\| \leq \frac{C}{\mu}, \quad \left\|\frac{\partial^3 w_R}{\partial y^3}\right\| \leq \frac{C}{\varepsilon}, \tag{5 6 10h}$$

$$\left\|\frac{\partial^k w_T}{\partial x^i \partial y^j}\right\| \leq \frac{C}{\mu^k} \ \textit{for} \ 0 \leq k \leq 2, \quad \left\|\frac{\partial^k w_T}{\partial x^i \partial y^j}\right\| \leq \frac{C}{\varepsilon\mu}, \ k = 3, \tag{5 6 10i}$$

$$\left\|\frac{\partial w_T}{\partial x}\right\| \leq C, \quad \left\|\frac{\partial^2 w_T}{\partial x^2}\right\| \leq \frac{C}{\mu}, \quad \left\|\frac{\partial^3 w_T}{\partial x^3}\right\| \leq \frac{C}{\varepsilon} \tag{5 6 10j}$$

Conjecture 5 6 5 *When $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$, the solution u of (4 1 1) can be decomposed as in (5 6 9), we conjecture that the following bounds on the corner layer components and their derivatives hold*

$$|w_{LB}| \leq Ce^{-\frac{\mu\alpha}{\varepsilon}x}e^{-\frac{\mu\alpha}{\varepsilon}y}, \quad |w_{LT}| \leq Ce^{-\frac{\mu\alpha}{\varepsilon}x}e^{-\frac{\gamma}{2\mu}(1-y)}, \tag{5 6 11a}$$

$$|w_{RB}| \leq Ce^{-\frac{\gamma}{2\mu}(1-x)}e^{-\frac{\mu\alpha}{\varepsilon}y}, \quad |w_{RT}| \leq Ce^{-\frac{\gamma}{2\mu}(1-x)}e^{-\frac{\gamma}{2\mu}(1-y)}, \tag{5 6 11b}$$

$$\left\|\frac{\partial^k w_{LB}}{\partial x^i \partial y^j}\right\| \leq \left(\frac{\mu}{\varepsilon}\right)^k \quad for \quad 0 \leq k \leq 3, \tag{5 6 11c}$$

$$\left\|\frac{\partial^k w_{RT}}{\partial x^i \partial y^j}\right\| \leq \frac{C}{\mu^k}, \quad 0 \leq k \leq 2, \qquad \left\|\frac{\partial^k w_{RT}}{\partial x^i \partial y^j}\right\| \leq \frac{C}{\varepsilon\mu}, \quad k = 3, \tag{5 6 11d}$$

$$\left\|\frac{\partial^k w_{LT}}{\partial x^i \partial y^j}\right\| \leq C\left(\frac{\mu}{\varepsilon}\right)^k, \quad for \quad 0 \leq k \leq 3, \tag{5 6 11e}$$

$$\left\|\frac{\partial^k w_{LT}}{\partial y^k}\right\| \leq C\left(\left(\frac{1}{\mu}\right)^k + \frac{\varepsilon}{\mu^2}\left(\frac{\mu}{\varepsilon}\right)^k\right), \quad 0 \leq k \leq 3, \tag{5 6 11f}$$

$$\left\|\frac{\partial^k w_{RB}}{\partial x^i \partial y^j}\right\| \leq C\left(\frac{\mu}{\varepsilon}\right)^k, \quad for \quad 0 \leq k \leq 3, \tag{5 6 11g}$$

$$\left\|\frac{\partial^k w_{RB}}{\partial x^k}\right\| \leq C\left(\left(\frac{1}{\mu}\right)^k + \frac{\varepsilon}{\mu^2}\left(\frac{\mu}{\varepsilon}\right)^k\right), \quad 0 \leq k \leq 3 \tag{5 6 11h}$$

5.7 Discrete problem

As with the case of $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, we consider the following discrete problem

$$\begin{aligned} L^{N,M}U(x_i, y_j) &= \varepsilon\delta_x^2 U + \varepsilon\delta_y^2 U + \mu a_1 D_x^+ U + \mu a_2 D_y^+ U - bU \\ &= f, \qquad (x_i, y_j) \in \Omega^{N,M}, \end{aligned} \tag{5 7 1a}$$

where $\Omega^{N,M}$ is defined to be the tensor product of two piecewise uniform meshes Ω^N and Ω^M In this case, the mesh Ω^N consists of two transition points, σ_1^N and σ_2^N, where

$$\sigma_1^N = \min\left\{\frac{1}{4}, \frac{2\varepsilon}{\mu\alpha}\ln N\right\} \quad \text{and} \quad \sigma_2^N = \min\left\{\frac{1}{4}, \frac{2\mu}{\gamma}\ln N\right\} \tag{5 7 1b}$$

More specifically

$$\Omega^N = \left\{ x_i | x_i = \begin{cases} \frac{4\sigma_1^N i}{N}, & i \leq \frac{N}{4} \\ \sigma_1^N + (i - \frac{N}{4})H, & \frac{N}{4} \leq i \leq \frac{3N}{4} \\ 1 - \sigma_2^N + (i - \frac{3N}{4})\frac{4\sigma_2^N}{N}, & \frac{3N}{4} \leq i \leq N \end{cases} \right\}, \tag{5 7 1c}$$

where $NH = 2(1 - \sigma_1^N - \sigma_2^N)$ and Ω^M is defined analogously with transition points σ_1^M and σ_2^M

The discrete minimum principle in the previous chapter still holds and we have the following analogous decomposition

$$U = V + W_L + W_R + W_B + W_T + W_{LB} + W_{LT} + W_{RB} + W_{RT} \tag{5 7 2a}$$

where

$$\begin{aligned} L^{N,M}V &= f, \qquad V|_{\Gamma^{N\,M}} = v|_{\Gamma^{N\,M}}, && (5\ 7\ 2b) \\ L^{N,M}W_L &= 0, \qquad W_L|_{\Gamma^{N\,M}} = w_L|_{\Gamma^{N\,M}}, && (5\ 7\ 2c) \\ L^{N,M}W_{LB} &= 0, \qquad W_{LB}|_{\Gamma^{N\,M}} = w_{LB}|_{\Gamma^{N\,M}}, && (5\ 7\ 2d) \end{aligned}$$

with the other layer functions defined similarly

Theorem 5 7 1 *We have the following bounds on discrete boundary layer functions,*

$$|W_L(x_i, y_j)| \leq C \prod_{s=1}^{i} \left(1 + \frac{\mu\alpha}{2\varepsilon} h_s\right)^{-1} = \Psi_{L,i}, \qquad \Psi_{L,0} = C,$$

$$|W_R(x_i, y_j)| \leq C \prod_{s=i+1}^{N} \left(1 + \frac{\gamma}{2\mu} h_s\right)^{-1} = \Psi_{R,i}, \qquad \Psi_{R,N} = C,$$

$$|W_B(x_i, y_j)| \leq C \prod_{r=1}^{j} \left(1 + \frac{\mu\alpha}{2\varepsilon} k_r\right)^{-1} = \Psi_{B,j}, \qquad \Psi_{B,0} = C,$$

$$|W_T(x_i, y_j)| \leq C \prod_{r=j+1}^{M} \left(1 + \frac{\gamma}{2\mu} k_r\right)^{-1} = \Psi_{T,j}, \qquad \Psi_{T,M} = C,$$

where $h_s = x_s - x_{s-1}$ *and* $k_r = y_r - y_{r-1}$

Proof We start by considering W_L The proof follows a similar argument to that in

Theorem 4 6 1 in the case of $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$ We consider the barrier functions

$$\Phi_L^{\pm}(x_i, y_j) = \Psi_{L,i} \pm W_L(x_i, y_j)$$

We can show that for C large enough $\Phi_L^{\pm}(x_i, y_j)|_{\Gamma^{N\,M}} \geq 0$ Also we obtain

$$L^{N,M}\Phi_L^{\pm}(x_i, y_j) = \left(2\varepsilon\frac{\mu^2\alpha^2}{4\varepsilon^2}\left(\frac{h_{i+1}}{2\bar{h}_i} - 1\right) + \left(\frac{\mu^2\alpha}{\varepsilon} - \frac{\mu^2 a_1}{\varepsilon} - b\right) - \frac{\mu\alpha}{2\varepsilon}h_{i+1}b\right)\Psi_{L,i+1} \leq 0,$$

and we use the discrete minimum principle to obtain the required result The proof in the case of W_B is analogous

Let us now look at W_R We consider similar barrier functions

$$\Phi_R^{\pm}(x_i, y_j) = \Psi_{R,i} \pm W_R(x_i, y_j)$$

We need to check how the functions $\Phi_R^{\pm}(x_i, y_j)$ behave on the boundary Using a similar argument to that for W_L in Theorem 4 6 1 we can show that for C large enough $\Phi_R^{\pm}(1, y_j) \geq 0$, $\Phi_R^{\pm}(x_i, 0) \geq 0$ and $\Phi_R^{\pm}(x_i, 1) \geq 0$ It remains to consider $\Phi_R^{\pm}(0, y_j)$ Using the exponential bounds in Lemma 5 4 4 we see that $|W_R(0, y)| = |w_R(0, y)| \leq Ce^{-\frac{\gamma}{2\mu}}$ We have

$$\Phi_R^{\pm}(0, y_j) = C\prod_{s=1}^{N}\left(1 + \frac{\gamma}{2\mu}h_s\right)^{-1} \pm W_R(0, y_j)$$

however,

$$e^{-\frac{\gamma}{2\mu}} = e^{-\frac{\gamma}{2\mu}\sum_{s=1}^{N}h_s} = \prod_{s=1}^{N}e^{-\frac{\gamma}{2\mu}h_s} \leq \prod_{s=1}^{N}\left(1 + \frac{\gamma}{2\mu}h_s\right)^{-1}$$

We conclude that $\Phi_R^{\pm}(0, y_j) \geq 0$ for C large enough We also obtain

$$L^{N,M}\Phi_R^{\pm}(x_i, y_j) = \frac{\Psi_{R,i}}{\left(1 + \frac{\gamma}{2\mu}h_i\right)}\left(2\varepsilon\left(\frac{\gamma}{2\mu}\right)^2\left(\frac{h_i}{2\bar{h}_i} - 1\right) + \left(2\varepsilon\left(\frac{\gamma}{2\mu}\right)^2 + \mu a_1\frac{\gamma}{2\mu}\right.\right.$$
$$\left.\left. - b\right)\left(1 + \frac{\gamma}{2\mu}h_i\right) - 2\varepsilon\left(\frac{\gamma}{2\mu}\right)^3 h_i\right)$$

Using $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$ and the definitions of α and γ we see that the above quantity is non-positive and therefore we use the discrete minimum principle to obtain the required result The proof for W_T is similar to the above and analogous bounds hold ☐

Theorem 5 7 2 *Assuming Conjecture 5 6 5 is true, we have the following bounds on* *discrete corner layer functions*

$$|W_{LB}(x_i,y_j)| \le C\prod_{s=1}^{i}\left(1+\frac{\mu\alpha}{2\varepsilon}h_s\right)^{-1}\prod_{r=1}^{j}\left(1+\frac{\mu\alpha}{2\varepsilon}k_r\right)^{-1} = \Psi_{L,i}\Psi_{B,j},$$
$$|W_{LT}(x_i,y_j)| \le C\prod_{s=1}^{i}\left(1+\frac{\mu\alpha}{2\varepsilon}h_s\right)^{-1}\prod_{r=j+1}^{M}\left(1+\frac{\gamma}{2\mu}k_r\right)^{-1} = \Psi_{L,i}\Psi_{T,j},$$
$$|W_{RB}(x_i,y_j)| \le C\prod_{s=i+1}^{N}\left(1+\frac{\gamma}{2\mu}h_s\right)^{-1}\prod_{r=1}^{j}\left(1+\frac{\mu\alpha}{2\varepsilon}k_r\right)^{-1} = \Psi_{R,i}\Psi_{B,j},$$
$$|W_{RT}(x_i,y_j)| \le C\prod_{s=i+1}^{N}\left(1+\frac{\gamma}{2\mu}h_s\right)^{-1}\prod_{r=j+1}^{M}\left(1+\frac{\gamma}{2\mu}k_r\right)^{-1} = \Psi_{R,i}\Psi_{T,j},$$

where h_s and k_r are as previously defined

Proof The proof of the bounds for the corner functions follow the same method for W_{LB} in Theorem 4 6 1 A little more work is needed in some functions to show that the barrier functions are nonnegative on the boundary and to show that after we apply the discrete operator to the barrier function the resulting expression is non-positive □

5 8 Error analysis

We now analyse the error between the continuous solution of (4 1 1) and the discrete solution of (5 7 1) in the case $\mu^2 \ge \frac{\gamma\varepsilon}{\alpha}$

Lemma 5 8 1 *At each mesh point $(x_i, y_j) \in \bar{\Omega}^{N,M}$, the regular component of the error satisfies the following estimate*

$$|(V-v)(x_i,y_j)| \le C(N^{-1}+M^{-1}),$$

where v is the solution of (5 3 8) *and V is the solution of* (5 7 2b)

Proof Using the usual truncation error argument and (5 3 9) we have

$$\begin{aligned}|L^{N,M}(V-v)(x_i,y_j)| &\le C_1N^{-1}\left(\varepsilon||v_{xxx}||+\mu||v_{xx}||\right)+C_2M^{-1}\left(\varepsilon||v_{yyy}||+\mu||v_{yy}||\right)\\ &\le C(N^{-1}+M^{-1})\mu\end{aligned}$$

We consider the barrier functions $\Psi^{\pm}(x_i,y_j) = C_1(N^{-1}+M^{-1}) \pm (V-v)$ We see that these functions are nonnegative on the boundary $\Gamma_{N,M}$, also we find $L^{N,M}\Psi^{\pm}(x_i,y_j) \le 0$

for C_1 large enough We apply the discrete minimum principle to obtain the required result $\square$

Lemma 5 8 2 *Given Assumption 2, at each mesh point $(x_i, y_j) \in \bar{\Omega}^{N,M}$, the left singular component of the error satisfies the following estimate*

$$|(W_L - w_L)(x_i, y_j)| \leq C(N^{-1}(\ln N)^2 + M^{-1}),$$

where w_L is the solution of (5 5 5) *and W_L is the solution of* (5 7 2c)

Proof We can use a classical argument to obtain the following truncation error bounds

$$\begin{aligned} |L^{N,M}(W_L - w_L)(x_i, y_j)| \leq\ & C_1(h_{i+1} + h_i)\,(\varepsilon||w_{Lxxx}|| + \mu||w_{Lxx}||) \\ & + C_2(k_{j+1} + k_j)\,(\varepsilon||w_{Lyyy}|| + \mu||w_{Lyy}||) \end{aligned} \qquad (5\ 8\ 1)$$

We use Theorem 5 6 1 and obtain

$$|L^{N,M}(W_L - w_L)(x_i, y_j)| \leq \frac{C_1}{\sqrt{\varepsilon}}(h_{i+1} + h_i)\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^3\right) + C_2 M^{-1} \qquad (5\ 8\ 2)$$

The proof splits into the two cases of $\sigma_1^N < \frac{1}{4}$ and $\sigma_1^N = \frac{1}{4}$ Starting with the former, we consider the region $[\sigma_1^N, 1) \times (0,1)$ Using Theorem 5 7 1, equation (5 7 1b) and a similar argument to that for W_L when $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, we see that in this region we have

$$|W_L(x_i, y_j)| \leq CN^{-1}$$

Considering the continuous solution in this region, from Theorem 5 6 1 we have

$$|w_L(x_i, y_j)| \leq e^{-\frac{\mu\alpha}{\varepsilon}\sigma_1^N} \leq CN^{-2}, \qquad x_i \geq \sigma_1^N$$

Combining these results we have the following in the region $[\sigma_1^N, 1) \times (0,1)$ when $\sigma_1^N < \frac{1}{4}$,

$$|(W_L - w_L)(x_i, y_j)| \leq CN^{-1}$$

We next consider the region $(0, \sigma_1^N) \times (0,1)$ We have $h_i = h_{i+1} = \frac{8\varepsilon}{\mu\alpha}N^{-1}\ln N$ We then use (5 8 2) and obtain

$$|L^{N,M}(W_L - w_L)| \leq C_1 N^{-1}\ln N + C_2 N^{-1}\ln N\frac{\mu^2}{\varepsilon} + C_3 M^{-1}$$

We consider the barrier functions

$$\Psi^{\pm}(x_i, y_j) = C(N^{-1}\ln N + N^{-1}(\sigma_1^N - x_i)\ln N\frac{\mu}{\varepsilon} + M^{-1}) \pm (W_L - w_L)$$

We can show that for C sufficiently large $\Psi^{\pm}(x_i, y_j) \geq 0$ on the boundary Also

$$L^{N,M}\Psi^{\pm}(x_i, y_j) = -bC(N^{-1}\ln N + N^{-1}(\sigma_1^N - x_i)\ln N\frac{\mu}{\varepsilon} + M^{-1}) - \frac{\mu^2}{\varepsilon}a_1(N^{-1}\ln N)$$
$$\pm(L^{N,M}(W_L - w_L)) \leq 0,$$

for C chosen correctly Using the discrete minimum principle we obtain

$$|(W_L - w_L)| \leq C(N^{-1}\ln N + N^{-1}(\sigma_1^N - x_i)\ln N\frac{\mu}{\varepsilon} + M^{-1}),$$

and simplifying even further using the definition of σ_1^N in (5 7 1b),

$$|(W_L - w_L)| \leq C(N^{-1}\ln N + N^{-1}(\ln N)^2 + M^{-1})$$

The last case to consider is that of $\sigma_1^N = \frac{1}{4}$ Here we find $\frac{\mu\alpha}{\varepsilon} \leq 8\ln N$ and using the truncation error bound (5 8 2) we obtain

$$|L^{N,M}(W_L - w_L)| \leq C(N^{-1}\ln N + \mu N^{-1}(\ln N)^2 + M^{-1})$$

Using a suitable barrier function we achieve the required result □

A proof analogous to the above holds for the error bound $|(W_B - w_B)|$

Lemma 5 8 3 *At each mesh point $(x_i, y_j) \in \bar{\Omega}^{N,M}$, the right singular component of the error satisfies the following estimate*

$$|(W_R - w_R)(x_i, y_j)| \leq C(N^{-1}\ln N + M^{-1}),$$

where w_R is the solution of (5 4 4) *and W_R satisfies an analogous equation to W_L in* (5 7 2c)

Proof We can use a classical argument to obtain analogous truncation error bounds to those in (5 8 1) We use Lemma 5 4 4 to obtain,

$$|L^{N,M}(W_R - w_R)(x_i, y_j)| \leq \frac{C_1}{\mu}(h_{i+1} + h_i) + CM^{-1} \tag{5 8 3}$$

We first consider the case of $\sigma_2^N < \frac{1}{4}$ We consider the region $(0, 1-\sigma_2^N] \times (0,1)$ Using Theorem 5 7 1 and (5 7 1b) we have,

$$|W_R(x_i, y_j)| \leq CN^{-1}$$

Considering the continuous solution in this region, from Theorem 5 6 1 we obtain

$$|w_R(x_i, y_j)| \leq e^{-\frac{\gamma}{\mu}\sigma_2^N} \leq CN^{-2}, \qquad x_i \leq 1 - \sigma_2^N$$

Combining these results we have the following bound in the region $(0, 1-\sigma_2^N] \times (0,1)$ when $\sigma_2^N < \frac{1}{4}$

$$|(W_R - w_R)(x_i, y_j)| \leq CN^{-1}$$

We next consider the region $(1-\sigma_2^N, 1) \times (0,1)$ We have $h_i = h_{i+1} = \frac{8\mu}{\gamma} N^{-1} \ln N$ We can use (5 8 3) to obtain

$$|L^{N,M}(W_R - w_R)| \leq C_1 N^{-1} \ln N + C_2 M^{-1}$$

Using the discrete minimum principle and suitable barrier functions, we obtain the required result

We finally consider the case of $\sigma_2^N = \frac{1}{4}$ We see $\frac{\gamma}{\mu} \leq 8 \ln N$ and using the truncation error bound (5 8 3) we obtain,

$$|L^{N,M}(W_R - w_R)| \leq C(N^{-1} \ln N + M^{-1})$$

Again, using a suitable barrier function we achieve the required result □

A similar proof holds for the error bound $|(W_T - w_T)|$ We therefore have the following lemma

Lemma 5 8 4 *At each mesh point $(x_i, y_j) \in \bar{\Omega}^{N,M}$, the bottom and top singular components of the error satisfies the following estimates*

$$\begin{aligned} |(W_B - w_B)(x_i, y_j)| &\leq C(N^{-1} + M^{-1}(\ln M)^2), \\ |(W_T - w_T)(x_i, y_j)| &\leq C(N^{-1} + M^{-1} \ln M), \end{aligned}$$

where w_B and w_T are defined in (5 5 8a) *and* (5 4 6) *respectively and W_B and W_T are defined analogously to* (5 7 2c)

Proof See Lemma 5 8 2 and Lemma 5 8 3 □

Lemma 5 8 5 *At each mesh point $(x_i, y_j) \in \bar{\Omega}^{N,M}$, assuming Conjecture 5 6 5 and Assumption 2 are true, the bottom-left corner singular component of the error satisfies the following estimate*

$$|(W_{LB} - w_{LB})(x_i, y_j)| \leq C(N^{-1}(\ln N)^2 + M^{-1}(\ln M)^2),$$

where w_{LB} is the solution of (5 6 8) *and W_{LB} is the solution of* (5 7 2d)

Proof We can obtain the same truncation error bounds to those given for the left singular component in (5 8 2) We use the bounds on w_{LB} in Conjecture 5 6 5 to obtain,

$$|L^{N,M}W_{LB} - w_{LB})| \leq \frac{C_1}{\sqrt{\varepsilon}}(h_{i+1} + h_i)\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^3\right) + \frac{C_2}{\sqrt{\varepsilon}}(k_{j+1} + k_j)\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^3\right) \quad (5\ 8\ 4)$$

Consider the case $\sigma_1^N < \frac{1}{4}$ and $\sigma_1^M < \frac{1}{4}$ In the region $\Omega^{N,M} \backslash (0, \sigma_1^N) \times (0, \sigma_1^M)$, the proof follows the same method as when $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$ Therefore in this region we have

$$|W_{LB}(x_i, y_j)| \leq C(N^{-1} + M^{-1}), \qquad x_i \geq \sigma_1^N \text{ and/or } y_j \geq \sigma_1^M$$

Considering the continuous solution in this region, using Conjecture 5 6 5 and (5 7 1b) we obtain

$$|w_{LB}(x_i, y_j)| \leq Ce^{-\frac{\mu\alpha}{\varepsilon}x_i}e^{-\frac{\mu\alpha}{\varepsilon}y_j} \leq e^{-\frac{\mu\alpha}{\varepsilon}\sigma_1^N} \leq CN^{-2}, \qquad x_i > \sigma_1^N,$$

and

$$|w_{LB}(x_i, y_j)| \leq Ce^{-\frac{\mu\alpha}{\varepsilon}x_i}e^{-\frac{\mu\alpha}{\varepsilon}y_j} \leq e^{-\frac{\mu\alpha}{\varepsilon}\sigma_1^M} \leq CM^{-2}, \qquad y_j > \sigma_1^M,$$

We conclude that when $\sigma_1^N < \frac{1}{4}$ and $\sigma_1^M < \frac{1}{4}$, we have the following error bound in the region $\Omega^{N,M} \backslash (0, \sigma_1^N) \times (0, \sigma_1^M)$

$$|(W_{LB} - w_{LB})(x_i, y_j)| \leq C(N^{-1} + M^{-1})$$

We next consider the region $(0, \sigma_1^N) \times (0, \sigma_1^M)$ In this region we know that $h_i = h_{i+1} = \frac{8\varepsilon}{\mu\alpha}N^{-1}\ln N$ and $k_j = k_{j+1} = \frac{8\varepsilon}{\mu\alpha}M^{-1}\ln M$ Using the truncation error bound (5 8 4) we obtain

$$|L^{N,M}(W_{LB} - w_{LB})(x_i, y_j)| \leq C(N^{-1}\ln N + N^{-1}\ln N\frac{\mu^2}{\varepsilon} + M^{-1}\ln M + M^{-1}\ln M\frac{\mu^2}{\varepsilon})$$

Choosing similar barrier functions to those in Lemma 5 8 2 we obtain

$$|(W_{LB} - w_{LB})(x_i, y_j)| \leq C(N^{-1}(\ln N) + N^{-1}(\ln N)^2 + M^{-1}(\ln M) + M^{-1}(\ln M)^2)$$

We consider the case of $\sigma_1^N = \frac{1}{4}$ and $\sigma_1^M = \frac{1}{4}$ We know that $\frac{\mu\alpha}{\varepsilon} \leq 8\ln N$ and $\frac{\mu\alpha}{\varepsilon} \leq 8\ln M$ and using (5 8 4) and suitable barrier functions we obtain,

$$|(W_{LB} - w_{LB})(x_i, y_j)| \leq C(N^{-1}(\ln N)^2 + M^{-1}(\ln M)^2)$$

The other two possible combinations of σ_1^N and σ_1^M are trivial and give the same result when $N = M$ □

Remark 5 8 1 *When $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$ it is not sufficient to cover the error analysis for one corner layer function alone as it is not reflective of the error analysis of the other three corners*

Lemma 5 8 6 *At each mesh point $(x_i, y_j) \in \bar{\Omega}^{N,M}$, assuming Conjecture 5 6 5 and Assumption 2 are true, the top-left corner singular component of the error satisfies the following estimate*

$$|(W_{LT} - w_{LT})(x_i, y_j)| \leq C(N^{-1}(\ln N)^2 + M^{-1}(\ln M)(\ln N)),$$

where w_{LT} is the solution of (5 6 5) *and W_{LT} satisfies a similar equation to W_{LB} in* (5 7 2d)

Proof Using (5 8 1) and Conjecture 5 6 5, we have the following truncation error bounds

$$|L^{N,M}(W_{LT} - w_{LT})(x_i, y_j)| \leq \frac{C_1}{\sqrt{\varepsilon}}(h_{i+1} + h_i)\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^3\right)$$
$$+C_2(k_{j+1} + k_j)\left(\frac{1}{\mu} + \frac{\mu}{\varepsilon}\right) \qquad (5\ 8\ 6)$$

We consider the case of $\sigma_1^N < \frac{1}{4}$ and $\sigma_2^M < \frac{1}{4}$ and start with the region $\Omega^{N,M} \backslash (0, \sigma_1^N) \times (1 - \sigma_2^M, 1)$ Using Conjecture 5 6 5 and Theorem 5 7 2 we obtain as with W_{LB}

$$|(W_{LT} - w_{LT})(x_i, y_j)| \leq C(N^{-1} + M^{-1})$$

In the region $(0, \sigma_1^N) \times (1 - \sigma_2^M, 1)$ we have $h_i = h_{i+1} = \frac{8\varepsilon}{\mu\alpha}N^{-1}\ln N$ and $k_j = k_{j+1} = \frac{8\mu}{\gamma}M^{-1}\ln M$, therefore we obtain

$$|L^{N,M}(W_{LT}-w_{LT})(x_i, y_j)| \leq C_1\left(N^{-1}\ln N + \frac{\mu^2}{\varepsilon}N^{-1}\ln N\right) + C_2\left(M^{-1}\ln M + M^{-1}\frac{\mu^2}{\varepsilon}\ln M\right)$$

We consider the barrier functions

$$\Phi^{\pm}(x_i, y_j) = C_3\left(N^{-1}\ln N + M^{-1}\ln M\right) + C_4\frac{\mu}{\varepsilon}(\sigma_1^N - x_i)\left(N^{-1}\ln N + M^{-1}\ln M\right) \pm (W_{LT} - w_{LT})$$

We can show that these functions are nonnegative on the boundary and for C large enough we obtain $L^{N,M}\Phi^{\pm}(x_i, y_j) \leq 0$ Using the discrete minimum principle and the definition of σ_1^N in (5 7 1b) we obtain

$$|(W_{LT} - w_{LT})(x_i, y_j)| \leq C\left(N^{-1}\ln N + M^{-1}\ln M\right) + C_2\left(N^{-1}(\ln N)^2 + M^{-1}\ln N \ln M\right)$$

The case of $\sigma_1^N = \sigma_2^M = \frac{1}{4}$ follows closely that for layer function associated with the bottom-left corner We continue to the case of $\sigma_1^N < \frac{1}{4}$ and $\sigma_2^M = \frac{1}{4}$ We start with the region $[\sigma_1^N, 1) \times (0, 1)$ Using Theorem 5 7 2 we see that $|W_{LT}(x_i, y_j)| \leq CN^{-1}$ in this region Looking at Conjecture 5 6 5 we also obtain $|w_{LT}(x_i, y_j)| \leq CN^{-2}$ and combining these results we see that in $[\sigma_1^N, 1) \times (0, 1)$ we have

$$|W_{LT} - w_{LT}| \leq CN^{-1} \qquad (5\ 8\ 7)$$

Consider the region $(0, \sigma_1^N) \times (0, 1)$ Using (5 8 6) along with $\frac{1}{\mu} \leq C \ln M$ and $h_{i+1} = h_i = 2\frac{\varepsilon}{\mu\alpha}N^{-1}\ln N$ we obtain,

$$|L^{N,M}(W_{LT} - w_{LT})(x_i, y_j)| \leq C_1 N^{-1}\ln N\left(1 + \frac{\mu^2}{\varepsilon}\right) + C_2\left(M^{-1}\ln M + \frac{\mu^2}{\varepsilon}M^{-1}\ln M\right)$$

Using the barrier functions,

$$\Phi^{\pm}(x_i, y_j) = C_1(N^{-1}\ln N + M^{-1}\ln M) + C_2\frac{\mu}{\varepsilon}(\sigma_1^N - x_i)(N^{-1}\ln N + M^{-1}\ln M),$$

we see,

$$|W_{LT} - w_{LT}| \leq C_1(N^{-1}\ln N + M^{-1}\ln M) + C_2(N^{-1}(\ln N)^2 + M^{-1}\ln M \ln N)$$

Finally we consider the case of $\sigma_1^N = \frac{1}{4}$ and $\sigma_2^M < \frac{1}{4}$, using Conjecture 5 6 5 and Theorem 5 7 2, we see that in the region $(0, 1) \times (0, 1 - \sigma_2^M)$ we have

$$|W_{LT} - w_{LT}| \leq CM^{-1} \qquad (5\ 8\ 8)$$

Using (5 8 6), $\frac{\mu}{\epsilon} \leq C \ln N$ and $k_{j+1} = k_j = \frac{8\mu}{\gamma} M^{-1} \ln M$ we have

$$|L^{N,M}(W_{LT} - w_{LT})(x_i, y_j)| \leq C(N^{-1}(\ln N)^2 + M^{-1} \ln M + M^{-1} \ln N),$$

and using suitable barrier functions we obtain the required bounds We should note that when $N = M$ these bounds simplify to

$$|(W_{LT} - w_{LT})(x_i, y_j)| \leq CN^{-1}(\ln N)^2$$

This completes the error analysis for W_{LT} □

The analysis for $|W_{RB} - w_{RB}|$ follows a similar argument to the above We obtain the following lemma

Lemma 5 8 7 *At each mesh point $(x_i, y_j) \in \bar{\Omega}^{N,M}$, assuming Conjecture 5 6 5 and Assumption 2 are true, the bottom-right corner singular component of the error satisfies the following estimate*

$$|(W_{RB} - w_{RB})(x_i, y_j)| \leq C(N^{-1}(\ln N)(\ln M) + M^{-1}(\ln M)^2),$$

where w_{RB} is defined in (5 6 7) *and W_{RB} satisfies a similar equation to W_{LB} in* (5 7 2d)

The final error component to consider is the top-right corner layer

Lemma 5 8 8 *At each mesh point $(x_i, y_j) \in \bar{\Omega}^{N,M}$, assuming Conjecture 5 6 5 is true, the top-right corner singular component of the error satisfies the following estimate*

$$|(W_{RT} - w_{RT})(x_i, y_j)| \leq C(N^{-1}(\ln N) + M^{-1}(\ln M)),$$

where w_{RT} is defined in (5 6 1) *and W_{RT} satisfies a similar equation to W_{LB} in* (5 7 2d)

Proof Using (5 8 1), we obtain

$$|L^{N,M}(W_{RT} - w_{RT})(x_i, y_j)| \leq \frac{C_1}{\mu}(h_{i+1} + h_i) + \frac{C_2}{\mu}(k_{j+1} + k_j) \qquad (5\ 8\ 9)$$

By considering separately the cases of $\sigma_2^N < \frac{1}{4}$, $\sigma_2^M < \frac{1}{4}$ and $\sigma_2^N = \sigma_2^M = \frac{1}{4}$, we achieve the required result □

Theorem 5 8 1 *At each mesh point $(x_i, y_j) \in \bar{\Omega}^{N,M}$, assuming Conjecture 5 6 5 and*

Assumption 2 are true, the maximum pointwise error satisfies the following parameter-uniform error bound when $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$,

$$\|U - u\|_{\Omega^{N,M}} \leq C(N^{-1}(\ln N)^2 + M^{-1}(\ln M)^2 + N^{-1}\ln N \ln M + M^{-1}\ln N \ln M),$$

where u *is the solution of (4 1 1) and* U *is the solution of (5 7 1)*

Proof The proof follows from Lemma 5 8 2, Lemma 5 8 3, Lemma 5 8 4, Lemma 5 8 5, Lemma 5 8 6, Lemma 5 8 7 and Lemma 5 8 8 □

5.9 The case of $\mu \geq \gamma_1$

In the case of $\mu \geq \gamma_1$, the elliptic problem (4 1 1) is equivalent to a one-parameter convection-diffusion problem Such problems are not the main interest of this thesis Numerical methods for these differential equations have been considered in the books [3, 16, 25, 29] For a discussion of the literature see Chapter 1 Solutions to such problems exhibit boundary layers m the neighbourhood of the edges $x = 0$ and $y = 0$

We decompose u into a sum of regular and layer components as follows

$$u = v + w_L + w_B + w_{LB}$$

We define v^* on the extended domain $\Omega^{[*,LB]}$ as in (5 3 1), however, it is not necessary to further decompose the components in this decomposition as in (5 3 4) We let

$$v_0^* = u^* \quad \text{on} \quad \partial\Omega^{[*,LB]},$$

and it can be shown

$$\left\| \frac{\partial^{i+j} v}{\partial x^i \partial y^j} \right\| \leq C(1 + \varepsilon^{2-(i+j)})$$

We define the layer function w_L^* on the domain $(0,1) \times (-d,1)$ and the function w_B^* on the domain $(-d,1) \times (0,1)$ The corner layer function w_{LB} is defined on the original domain Ω We have

$$L_{\varepsilon,\mu} w_{LB} = 0 \ (x,y) \in \Omega, \tag{5 9 1}$$

$$w_{LB} = -w_B, \ (x,y) \in \Gamma_L, \quad w_{LB} = -w_L, \ (x,y) \in \Gamma_B, \tag{5 9 2}$$

$$w_{LB} = 0, \ (x,y) \in \Gamma_R, \quad w_{LB} = 0, \ (x,y) \in \Gamma_T \tag{5 9 3}$$

For all the layer components, we obtain the following bounds on the functions themselves

and their derivatives
$$\left\| \frac{\partial^{i+j} w}{\partial x^i \partial y^j} \right\| \leq C\varepsilon^{-(i+j)}, \quad i + j \leq 3$$

When considering the boundary layer functions w_L, and w_B, these bounds can be sharpened in the direction orthogonal to the layer

The numerical method used to solve such a problem consists of an upwind finite difference operator applied on a mesh $\Omega^{N,M}$ This mesh is the tensor product of two piecewise uniform meshes Ω^N and Ω^M In this case, Ω^N consists of one transition point, σ_1^N where

$$\sigma_1^N = \min \left\{ \frac{1}{2}, \frac{2\varepsilon}{\alpha} \ln N \right\}$$

We should note that when $\mu \geq \gamma_1$, the numerical method defined in (5 7 1) is equivalent to the above and therefore even though the analysis differs, the same numerical method as defined for the two-parameter problem works in this case

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