Classification of the Asymptotic Behaviour of Solutions of Stochastic Differential Equations with State Independent Noise.

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Declaration

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Contents

Abstract

We investigate the asymptotic behaviour of solution of differential equation with state–independent perturbation. The differential equation studied is a perturbed version of a globally stable autonomous equation with unique equilibrium where the diffusion coefficient is independent of the state.

Perturbed differential equation is widely applied to model natural phenomena, in Finance, Engineering, Physics and other disciplines. Real–world processes are often subjected to interference in the form of random external perturbations. This could lead to a dramatic effect on the behaviour of these processes. Therefore it is important to analyse these equations.

We start by considering an additive deterministic perturbation in Chapter 1. It is assumed that the restoring force is asymptotically negligible as the solution becomes large, and that the perturbation tends to zero as time becomes indefinitely large. It is shown that solutions are always locally stable, and that solutions either tend to zero or to infinity as time tends to infinity. In Chapter 2 and 4, we each explore a linear and nonlinear equation with stochastic perturbation in finite dimensions. We find necessary and sufficient conditions on the rate of decay of the noise intensity for the solution of the equations to be globally asymptotically stable, bounded, or unstable. In Chapter 3 we concentrate on a scalar nonlinear stochastic differential equation. As well as the necessary and sufficient condition, we also explore the simple sufficient conditions and the connections between the conditions which characterise the various classes of long–run behaviour. To facilitate the analysis, we investigate using Split–Step method the difference equations both in the scalar case and the finite dimensional case in Chapter 5 and 6. We can mimic the exact asymptotic behaviour of the solution of the stochastic differential equation under the same conditions in discrete time.

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Introduction and Notations

0.1 Introduction

0.1.1 Overview and highlights of the work

It is a natural problem in deterministic dynamical systems to ask under what conditions is there a unique globally asymptotic stable solution of the equation

$$x'(t) = -f(x(t)), \quad t > 0, \quad x(0) = \xi,$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$, and this problem has been studied for general f since the 1960's. Without loss of generality, we always take this unique equilibrium to lie at $x = 0$. In the 1960's and 1970's especially, this question was of great interest in dynamic macroeconomics, as it corresponds to the notion of the "invisible hand" that prices and outputs of various commodities in an economy come to a unique set of equilibrium outputs and prices without external intervention. However, it is likely that such economic systems are subjected to persistent time–varying shocks, which fade over time. Such shocks may be deterministic or stochastic in nature.

Therefore, it is equally natural to suppose that the equation is (somehow) perturbed by adding a function g to the righthand side. Now the question is: what is the maximal size of the perturbation for which the stable solution preserves its stability (or does any perturbation cause the loss of stability)? What happens if the perturbation becomes bigger? The structure of the perturbation should also be important. For example, the perturbation may depend on the state (e.g., there are higher order nonlinear added to an already linear problem). We call such a perturbation *state dependent*. On the other hand, the perturbation may model a purely external force, in which case we may view g as simply a function of time. We would call such a perturbation *state independent*. Another possibility is that the perturbation is stochastic rather than deterministic, so the equation becomes

$$dX(t) = -f(X(t))\,dt + \sigma(t, X(t))\,dB(t),$$

where B is a standard Brownian motion, and we understand this *stochastic differential equation* (or SDE for short) as an integral equation of the form

$$X(t) = \xi - \int_0^t f(X(s))\,ds + \int_0^t \sigma(s, X(s))\,dB(s), \quad t \geq 0,$$

where the last term is an Itô integral. In this case, we say the perturbation is state–independent if $\sigma(x,t) = \sigma(t)$, and state dependent otherwise.

In this thesis we deal with deterministic and stochastic differential equations with state–independent perturbations. Our perspective is that we will assume that the perturbation is such that the equilibrium is not preserved. However, it can still be the case that the solutions of the perturbed equation are attracted to the equilibrium state of the the original unperturbed problem. For example, if y is given by $y'(t) = -ay(t)$, for $t \geq 0$, then $y(t) \to 0$ as $t \to \infty$. Suppose now that this equation is perturbed by a state–independent term so that now it reads $x'(t) = -ax(t) + p(t)$. If $p(t) \neq 0$ but $p(t) \to 0$ as $t \to \infty$, then $x(t) \to 0$ as $t \to \infty$, so the solution converges to the original equilibrium state of the unperturbed equation.

We want our results to hold for a very large class of f, and to investigate the relationship between the strength of mean reversion characterised by the nonlinearity of f, and the intensity of the perturbation (g or σ). We wish to ask: what is the difference between the perturbation being "stochastic" or "deterministic"? Given these criteria, we are led to study the equations

$$x'(t) = -f(x(t)) + g(t), \tag{0.1.1}$$

and

$$dX(t) = -f(X(t))dt + \sigma(t)dB(t).$$

For equation (0.1.1)(especially in the scalar case with $g(t) > 0$ for all $t \geq 0$), we can develop a condition on f and g which discriminates between cases where $x(t) \to 0$ as $t \to \infty$ and $x(t) \to \infty$ as $t \to \infty$. It is notable that we can have $x(t) \to \infty$ as $t \to \infty$ even if $g(t) \to 0$ as $t \to \infty$. This happens when $f(x) \to 0$ as $x \to \infty$, and g does not decay sufficiently rapidly, so that the strength of the mean reversion is weak. One reason to include such deterministic analysis is to enable us to see the very different impact of a state–independent stochastic term, in which σ tends to zero in some sense. For scalar

SDEs The situation under which $X(t,\xi) \to 0$ as $t \to \infty$ with probability one is equivalent to the convergence of any solution with non-zero probability, and this is characterised by a condition which involves the size of the perturbation σ only. Furthermore, if the perturbation exceeds this size, irrespective of the strength of the mean-reverting force, the solutions will be unbounded.

Furthermore, we can classify entirely the asymptotic behaviour if $|f(x)| \to \infty$ as $|x| \to \infty$, depending only on a function which depends *entirely* on the perturbation intensity σ. This function allows for only three types of behaviour in the solution X: it is either convergent with probability one; it is bounded but not convergent, with probability one; or it is unbounded, with probability one. Other than $|f(x)| \to \infty$ as $|x| \to \infty$, we make no further assumption about the nature of nonlinearity (e.g., there are no linear, polynomial, or exponential bounds on f).

It is interesting, however, to analyse the linear case in both one and arbitrarily many dimensions, so the equation is

$$dX(t) = AX(t)\,dt + \sigma(t)\,dB(t).$$

Here A can be a $d \times d$ matrix, B is an r–dimensional Brownian motion, and σ a matrix–valued function. The linear analysis turns out to be very helpful in understanding the asymptotic behaviour of the nonlinear equation. For all linear equations, irrespective of dimension, we can classify the asymptotic behaviour and characterise the conditions under which solutions are asymptotically convergent to the equilibrium. Stability ensues whenever the underlying deterministic equation is globally stable, and the noise fades sufficiently rapidly; once it exceeds a critical level, solutions do not converge to the underlying deterministic equilibrium. A classification of the asymptotic behaviour into stable, bounded and unbounded solutions can be performed, and the conditions under which each type of asymptotic behaviour is observed is independent of A, provided that all eigenvalues of A have negative real part. Moreover, we don't see any change in these conditions as the dimension d changes.

When we come to deal with the general nonlinear problem, however, in the case when the solution may only weakly revert to the mean (which means that, in some sense $\|f(x)\| \to 0$

as $\|x\| \to \infty$), there appears to be a distinction between scalar and finite dimensional nonlinear cases. Stronger sufficient conditions are needed on f in order to guarantee convergence, even though, granted these stronger conditions, the behaviour of $\|\sigma\|$ characterises the asymptotic behaviour. We speculate that stronger conditions on f are needed in finite dimensions to ensure that the stability of the equilibrium is preserved, or that solutions remain bounded, because of the transience of Brownian motion in higher dimensions.

We are unaware of a necessary and sufficient condition on f which would guarantee the global stability of the zero equilibrium in finite dimensions. However, we employ a simple and widely–used condition namely

$$\langle x, f(x) \rangle > 0, \quad x \neq 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product. This condition is often called the *dissipative condition* in the literature. This condition is always employed in our analysis of the stochastic equation. We show that the dissipative condition is slightly strengthened to

$$\liminf_{y \to \infty} \inf_{\|x\|=y} \langle x, f(x) \rangle > 0,$$

then we can characterise the convergence to the equilibrium in a manner otherwise independent of f. Therefore, if the strengthened dissipative condition holds, we have stability for all nonlinear problems under the same condition on σ, no matter how strong the mean reverting force is. If the dissipative condition is strengthened yet further to

$$\liminf_{y \to \infty} \inf_{\|x\|=y} \frac{\langle x, f(x) \rangle}{\|x\|} = +\infty > 0,$$

then we can classify the asymptotic behaviour as being convergent, bounded but not convergent, or unbounded, in a manner which depends solely on σ through its norm. Once again, therefore, once this second strengthening of the dissipative condition has been made, the stability or boundedness of the solution are independent of the strength of the mean reverting force.

We notice that the second strengthening of the dissipative condition implies that

$$\lim_{\|x\| \to \infty} \|f(x)\| = +\infty. \tag{0.1.2}$$

We conjecture, based on the linear case, and the scalar nonlinear case, that the global stability of the deterministic equation, together with a condition of the form (0.1.2), would enable the asymptotic behaviour to be classified. Similarly, we would expect that a weaker condition on the size of f for large $\|x\|$, together with the global stability of the unperturbed deterministic equation, would enable us to characterise the convergence of all solutions in a manner depending only on σ.

The necessary and sufficient conditions which yield stable, bounded or unbounded solutions are quite complex, involving hard–to compute integrals or summations. However, simple sufficient conditions on σ are available which describe very well the classification of the asymptotic behaviour. Roughly speaking, if there exists $L \in [0, \infty]$ such that

$$\lim_{t\to\infty} \|\sigma(t)\|_F^2 \log t = L,$$

we have convergent solutions if $L = 0$; bounded but not convergent solutions if $L \in (0, \infty)$; and unbounded solutions if $L = +\infty$.

Finally, because f and σ are general functions, it is usually impossible to write down a formula for the solution of the SDE. Therefore, if one want to have quantitative information about solutions it is necessary to simulate them on a computer. For this reason, we must design reliable numerical methods for their simulation, and demonstrate that the important properties of the solutions hold. In particular, any successful numerical method should preserve the stability, boundedness or unboundedness of solutions of the equation. Accordingly, we demonstrate that the asymptotic behaviour of the continuous problem can be recovered completely by applying an appropriate implicit discretisation scheme. Moreover, the same scheme works for all the problems that are considered in this thesis.

0.1.2 Relevant literature

The topic of this thesis is the asymptotic behaviour of stochastic differential equations. This constitutes a large field of research. A number of important textbooks and monographs have been written on the subject. Classical work on the asymptotic behaviour, especially asymptotic stability of stochastic differential equations, was undertaken in Gikhman and Skorohod [43] and in Khas'minski [45]. The work of Skorohod emphasised linear stochastic equations [73]. Mao has made a number of important contributions, particularly with regard to the exponential stability of solutions in [53], with further developments, including extensions to functional and neutral equations appearing in Mao [55]. A very comprehensive monograph on stochastic functional differential equations is Kolmanovskii and Myshkis [46], which devotes a lot of space to different modes of convergence, especially in p–th mean. Extensions of the results of these works, with particular emphasis on SDEs with Markovian switching, appear in Mao and Yuan [58]. Further results on the asymptotic behaviour and stability of stochastic partial differential equations and stochastic delay partial equations are in the book of Liu [52].

This thesis is especially interested in studying the asymptotic behaviour of stochastic differential equations with state–independent noise. Such equations have attracted a lot of attention. Liapunov function techniques have been applied to study their asymptotic stability in Khas'minski [45], with a lot of emphasis given to equations with perturbations σ being in $L^2(0,\infty)$. However, in a pair of papers in 1989, Chan and Williams [31] and Chan[30] demonstrated that the stability of global equilibria in these systems could be preserved with a much slower rate of decay in σ: in fact, they showed that provided the noise perturbation decayed monotonically in its intensity, then solutions converged to the equilibrium with probability one if and only if

$$\lim_{t\to\infty} \sigma^2(t)\log t = 0.$$

These results also required strong assumptions on the strength of the nonlinear feedback. Shortly thereafter, Rajeev [64] demonstrated that these results could be generalised to equations with some non–autonomous features, and some results on bounded solutions were obtained. In parallel, Mao demonstrated in [54] that a polynomial rate of decay of solutions was possible if the perturbation intensity decayed at a polynomial rate. These results were extended to neutral functional differential equations by Mao and Liao in [50], with exponential decaying upper bounds on the intensity giving rise to an exponential convergence rate in the solution.

After this, Appleby and his co–authors extended Chan and Williams' results to stochastic functional differential equations [15] and to Volterra equations especially (see Appleby and Appleby and Riedle [1, 14]), with extensions to discrete Volterra equations appearing in Appleby, Riedle and Rodkina [20]. Necessary and sufficient conditions for exponential stability in linear Volterra equation in the presence of fading noise was studied in [9].

One of the handful of papers which has most influence on this thesis is Appleby, Gleeson and Rodkina [10], which returns directly to the equation studied by Chan and Williams in [31]. In it, the monotonicity assumptions on f and σ were completely relaxed, and the mean reversion strength was also considerably weakened. Moreover, results on unbounded and unstable solutions also appeared for the first time. However, the finite dimensional case was not addressed, nor was a complete classification of the dynamics presented. Furthermore, it remained an open question as to whether the weaker mean–reverting assumption on f

$$\liminf_{|x|\to\infty} |f(x)| =: \phi > 0$$

was necessary to prove stability results. The goal of the thesis is to address each of these shortcomings, and some papers, whose results are recorded in this thesis, have already been published. See Appleby, Cheng and Rodkina [4, 5] and Appleby and Cheng [3]. The first of these papers covers work presented in Chapter 2; the second forms the basis of Chapter 5; and the third is, almost verbatim, Chapter 1. The other Chapters form the basis of three preprints: Appleby, Cheng and Rodkina [6] covers the scalar nonlinear SDE studied in Chapter 3; Appleby, Cheng and Rodkina [7] deals with its extension to finite dimensions in Chapter 4; and Appleby, Cheng and Rodkina [8] deals with the numerical methods for finite dimensional SDEs covered in Chapter 6.

Since most stochastic differential equations cannot be solved in closed form, there is an obvious importance in developing reliable methods for their numerical simulation. In particular, much interest has centered on the question of preserving the asymptotic behaviour of solutions when they are discretised. For deterministic equations, this approach is advocated in the book of Stuart and Humphries. [76], for instance. For stochastic equations, when this programme of research started, the major emphasis was on the mean square asymptotic behaviour of linear SDEs. Among the early and important contributions we highlight work of Saito and Mitsui [61], Schurz [71] and [72] and Higham [38]. The papers of Schurz and Higham also demonstrate the usefulness of implicit methods for dealing with problems in which the continuous solutions converge to the equilibrium state. These early works on SDEs were extended to study p–th mean exponential stability in stochastic delay differential equations in Baker and Buckwar [26]; necessary and sufficient conditions for exponential stability in the solution of SDEs and the corresponding discretisation were given in Higham, Mao, Stuart [40].

More recently, attention has focused on preserving the pathwise stability and asymptotic properties of solutions of stochastic differential and delay differential equations. Close to necessary and sufficient conditions for pathwise stability of discretisation of nonlinear scalar SDEs appears in Appleby, Mao and Rodkina [13]. Almost sure exponential stability has been studied extensively too. The literature is expanding rapidly, for SDDEs, exponential stability of numerical solutions has been established in Wu, Mao, and Szpruch [80]. For equations with Markov switching, the a.s. exponential stability of numerical solutions has been examined in Yin, Mao, Yuan, and Cao [81] and in S. Pang, F. Deng and X. Mao [63]. On the other hand, non–exponential rates of convergence to equilibria of discretisations of SDEs, which arise due to nonlinear drift and diffusion coefficients have also been investigated. Examples of papers in this direction include Appleby, Rodkina, and Berkolaiko [17] and Appleby, Rodkina and Mackey [19]. The latter paper is interesting in the context of this thesis, as it concerns equations with state–independent perturbations. However, what is notable in all these papers is that additional assumptions on the size of the coefficients are needed, whether these are (essentially) linear or nonlinear (particularly polynomial), in order to determine the rate of convergence. Instead, we wish to proceed for equations with state independent noise in a manner analogous to Szpruch and Mao [77, 78] in the state–dependent case by determining convergence to equilibria by making minimal assumptions on the size of the drift and diffusion coefficients. Of course, in relaxing these assumptions, we expect only to demonstrate convergence, but not to determine an upper bound on its rate.

Recently, the limitations of using explicit Euler methods for simulating stochastic differential equations have been explored. For the equation studied in this work, the paper of Appleby, Rodkina, Berkolaiko [16] demonstrates that if f does not obey a global linear bound (in the sense that $\lim_{|x|\to\infty} |f(x)|/|x| = +\infty$), then for sufficiently large initial conditions, the solution will oscillate unboundedly with probability arbitrarily close to unity, even though all solutions of the corresponding continuous equation tend to zero with probability one. However, local stability is preserved, in the sense that if the noise intensity remains arbitrarily small and the initial condition is sufficiently small, then solutions of the explicit scheme will converge with probability arbitrarily close to unity. These results were extended to equations with state–dependent noise in Appleby, Kelly, Mao, and Rodkina [11]. Examples which demonstrate that explicit Euler methods will suffer from these problems when it is desired to preserve stationarity in SDEs, are presented in Mattingly, Stuart, and Higham [60].

Given, therefore, that we desire to preserve the asymptotic behaviour for general nonlinear f (which need not obey global linear bounds), it becomes necessary to use a method other than explicit Euler–Maruyama. Implicit methods have been recognised as performing well in these circumstances, as evidence by work of Schurz [70] and Rodkina and Schurz [66]. Among possible implicit methods, a good candidate would appear to be the split step backward Euler method (SSBE) developed by Higham, Mao, and Stuart in [39] and in [60], as it has been shown to ensure convergence of numerical approximations of solutions of SDEs on compact intervals, and preserves a.s. exponential stability in SDEs (see e.g., Higham, Mao, and Yuan [41]) and in hybrid SDEs (see e.g., Mao, Shen, Gray [57]), and stationarity in SDEs with and without Markovian switching (see Yuan and Mao [82] and Mao, Yuan and Yin [59]). General stability under weaker assumptions on the drift and diffusion coefficient is in Szpruch and Mao [77] and [78] in which a dissipative condition on f is used. However, in contrast to the equations studied here, the diffusion term depends only on the state, and equilibria are preserved by the stochastic perturbation.

In each case, the algorithms perform well with weak or no restrictions on the uniform step size, in contrast to explicit Euler methods. In this thesis, we show with very weak assumptions on the nonlinear function f and the perturbation σ, that the SSBE method preserves all possible types of asymptotic behaviour, without restriction being made on the step size $h > 0$.

0.1.3 Technical synopsis of the thesis

The purpose of this research is to study the asymptotic behaviour of solutions of a class of differential equations with perturbations. With the results obtained from the continuous–time equations, we also investigate if similar behaviour of the solutions are preserved by discretisation. We also extend our results in the finite–dimensional case.

Before studying the differential equations with stochastic perturbation, we consider non-linear differential equation with deterministic perturbation independent of the state which do not involve any randomness. These results are presented in Chapter 1. The equation in question is a perturbed version of equation

$$y'(t) = -f(y(t)), \quad t \geq 0.$$

It is presumed that the unperturbed equation has a globally stable and unique equilibrium at zero. Therefore the question arises as whether stability is preserved when the perturbation g is asymptotically small. We already know that we have stability when the perturbation is integrable. Also, if f obeys $\liminf_{x\to\infty} f(x) > 0$ and $g(t) \to 0$ as $t \to \infty$ we have stability. Therefore we confine our attention in the case when g is not integrable and $f(x) \to 0$ as $x \to \infty$. It is shown that the solutions are locally stable, and that the solutions either tend to zero or to infinity as time tends to infinity. We also give the critical rate of decay of the perturbation g which depends on the strength of the restoring force f, for which the solution will go to zero or to infinity.

In Chapter 2, we then apply the knowledge obtained from analysing the stability and instability of deterministic differential equations to characterize the asymptotic behaviour of solutions of linear stochastic differential equation in finite–dimensions. The equation in question is a perturbed version of a linear deterministic differential equation with a globally stable equilibrium at zero,

$$y'(t) = Ay(t), \quad t \geq 0.$$

where A is a matrix whose eigenvalues all have negative real parts. We want to answer the question under what condition on the perturbation intensity σ would we preserve stability. We first completely characterize the asymptotic stability, boundedness and unboundedness of the solution of the linear stochastic differential equation whose diffusion coefficient is state–independent. In fact, everything can be inferred from the sum

$$S(\epsilon) = \sum_{n=0}^{\infty} \left\{ 1 - \Phi\left(\frac{\epsilon}{\sqrt{\int_n^{n+1} \|\sigma(s)\|_F^2 ds}} \right) \right\} < +\infty, \quad \text{for every} \quad \epsilon > 0, \tag{0.1.3}$$

where Φ is the distribution function of a standardised normal random variable, or a related integral. If the sum is finite for all ϵ, solutions tend to zero with probability one; if it is always infinite, then solutions are unbounded; the third possibility, that S is finite for some values of ϵ, but infinite for others, leads to the solution being bounded but not convergent. In each case, we see the specified behaviour with probability one.

Although this is a necessary and sufficient condition for stability, boundedness or unboundedness, it can be hard to apply in practice. Therefore we also deduce a sharp sufficient condition on σ to obtain the appropriate asymptotic behaviour. Perhaps the most simple but comprehensive sufficient condition, involving the existence and size of the limit $\lim_{t\to\infty} \|\sigma(t)\|_F^2 \log t$ has already been stated earlier.

We next move to characterize the global stability, global boundedness and recurrence of solutions of a scalar nonlinear stochastic differential equation in Chapter 3. It is also a perturbed version of a globally stable autonomous equation with unique equilibrium where the diffusion coefficient is independent of the state. To be precise, we look at the equation

$$dX(t) = -f(X(t))dt + \sigma(t)dB(t), \quad t \geq 0; \quad X(0) = \xi \in \mathbb{R}.$$

We give conditions which depend on the rate of decay of the noise intensity under which solutions either (a) tend to the equilibrium almost surely, (b) are bounded almost surely but tend to zero with probability zero, (c) or are recurrent on the real line almost surely. We also show that no other types of asymptotic behaviour are possible. The condition which characterises these types of behaviour uses (0.1.3) in the scalar case, once $|f(x)| \to \infty$ as $|x| \to \infty$. For stability, the necessary and sufficient condition is simply the finiteness of $S(\epsilon)$ in (0.1.3) for all ϵ. Therefore, all conditions regarding monotonicity of f and σ (as required in [31]), or requiring f to satisfy extra conditions at infinity (as in [10]) have been removed. Moreover, our results apply with only continuity of f, rather than stronger conditions such as locally Lipschitz continuity. Under this relaxation, it may be that solutions are no longer unique, but nonetheless all solutions have the same asymptotic behaviour.

These results are then extended to the finite–dimensional case in Chapter 4, where stability and instability results are obtained under the equivalent conditions. We study the same equation as in Chapter 3, with $X(0) = \xi \in \mathbb{R}^d$, B being an r–dimensional standard Brownian motion, $f : \mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz continuous function and $\sigma \in C([0,\infty);\mathbb{R}^{d\times r})$. We obtain stability under the finiteness of $S(\epsilon)$ in (0.1.3), provided that f obeys $\liminf_{x\to\infty}\inf_{|y|=x}\langle y, f(y)\rangle > 0$. If we strengthen the dissipative condition yet further, as indicated above, then we can classify the long run behaviour using $S(\epsilon)$ as we did for the linear equation in Chapter 2.

The last two chapters are devoted to the numerical analysis of solutions of the SDE. We consider the scalar equation, under monotonicity conditions, in Chapter 5. We investigate there the possibility of the preservation of the behaviour of the solutions of the scalar stochastic differential equation under discretization. We consider a special Euler–type discretisation called the Split-step backward Euler method. It takes the form the Split-Step backward Euler method:

$$\begin{aligned} X_h^*(n) &= X_h(n) - hf(X_h^*(n)), \quad n \geq 0, \\ X_h(n+1) &= X_h^*(n) + \sqrt{h}\sigma(nh)\xi(n+1), \quad n \geq 0. \end{aligned}$$

Our main result is that the SSBE method preserves the asymptotic behaviour of the solution of the SDE under the monotonicity assumptions imposed by Chan and Williams in [31].

We use such an implicit scheme rather than an explicit scheme because we are interested in the long–run behaviour of the solution. We can use explicit scheme if we are working on finite time intervals, provided the size of the step is sufficiently small. If we want to study the long run behaviour of the solution, an explicit scheme will still work reliably if f has a global linear bound and is Lipschitz continuous. But there are drawbacks: if we are interested in controlling the error of the solution, the explicit scheme requires a smaller and smaller step size to maintain a particular error. Therefore, the cost of continuously reducing the step size is very large. Moreover, it is known when f does not obey a global linear bound, that the explicit scheme predicts unboundedly oscillating solutions even when the true solution is known to be asymptotically stable. However, using the implicit scheme, we do not need to worry about the step size, and moreover, are ensured that the solutions inherit the appropriate asymptotic behaviour.

We extend the numerical results in finite–dimensions in Chapter 6. The same Split-Step backward Euler method is used with $\xi = \xi(n) : n \geq 1$ being a sequence of r–dimensional independent and identically distributed Gaussian vectors. We are able to classify the pathwise stability, and more generally, the pathwise asymptotic behaviour of the discretisation.

We can impose that f be locally Lipschitz continuous on $\mathbb{R}^d$, or that f satisfies a global one–sided Lipschitz condition, to guarantee the existence of a unique strong solution of the SDE. However, if we do not impose these conditions on f, the continuity of f and σ would guarantee the existence of a local solution of the SDE, and together with the condition (4.1.9), we have global existence of a unique solution. In the main result, it is shown that when the split–step–method is applied to the resulting stochastic differential equation, and the stochastic intensity is decreasing, the solutions of the discretised equation inherit the asymptotic behaviour of the continuous equation, regardless of whether the continuous equation has stable, bounded but unstable, or unbounded solutions. Classification of the long run behaviour of the numerical solutions is also possible when $\|\sigma\|$ is not monotone. If σ obeys $\lim_{t\to\infty} \|\sigma(t)\|_F^2 \log t = L$ for some $L \in [0, \infty]$, then the discretisation has stable, bounded but unstable, or unbounded solutions, if and only if the continuous equations has solutions with the corresponding asymptotic behaviour.

0.2 Notations

We introduce some standard notation used in this thesis:

$\mathbb{N}$ denotes the natural numbers.

$\mathbb{Q}^+$ denotes the positive rational numbers.

$\mathbb{R}$: set of real numbers.

$\mathbb{R}^+$: set of non-negative real numbers.

$\mathbb{R}^d$: d-dimensional Euclidean space.

$\mathbb{C}$: set of complex numbers.

$C(I;J)$: space of continuous functions $f : I \to J$ where I and J are intervals contained in $\mathbb{R}$.

$C^1(I;J)$: space of differentiable functions $f : I \to J$ where $f' \in C(I;J)$.

$L^1(0,\infty)$: the space of Lebesgue integrable functions $f : [0,\infty) \to \mathbb{R}$ such that

$$\int_0^\infty |f(s)|ds < +\infty.$$

$L^2(0,\infty)$: the space of Lebesgue square integrable functions $f : [0,\infty) \to \mathbb{R}$ such that

$$\int_0^\infty |f(s)|^2 ds < +\infty.$$

$x \vee y$: the maximum value between x and y.

$x \wedge y$: the minimum value between x and y.

$g \circ f : I \to K : x \to (g \circ f)(x) := g(f(x))$: composition of two functions g and f.

$h \in RV_\infty(\alpha)$: we say that a function $h : [0,\infty) \to (0,\infty)$ is regularly varying at infinity with index $\alpha \in \mathbb{R}$ if

$$\lim_{x\to\infty} \frac{h(\lambda x)}{h(x)} = \lambda^\alpha.$$

$\mathbb{R}^{d\times r}$: set of d by r matrices.

A^T : the transpose of $A \in \mathbb{R}^{d\times r}$.

$\det A$: the determinate of a square matrix A.

$\langle \cdot, \cdot \rangle$: the standard inner product on $\mathbb{R}^d$.

$\| \cdot \|$: the Euclidean norm on a row or column vector.

For $A \in \mathbb{R}^{d\times r}$, we denote the entry in the i–th row and j–th column by A_{ij}, we denote the Frobenius norm of A by

$$\|A\|_F = \left(\sum_{j=1}^{r} \sum_{i=1}^{d} \|A_{ij}\|^2 \right)^{1/2}.$$

$\mathbf{e}_i$: the i-th standard basis vector in $\mathbb{R}^d$.

$\mathcal{N}(a,b)$: normal distribution with mean a and standard deviation b.

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$: a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.

0.3 Important results from stochastic Analysis

In this thesis, the stochastic differential equations studied are driven by Brownian motions. They are often expressed in their integral form, where the stochastic integrals are continuous martingales. And it is often convenient to understand the behaviour of continuous martingales in terms of standard Brownian motions, particularly when dealing with asymptotic results. Therefore we first establish a few definitions:

A stochastic process, $\{B(t) : 0 \le t \le \infty\}$, is a *standard Brownian motion* if

- $B(0) = 0$,
- It has continuous sample paths,
- It has independent, stationary and normally-distributed increments.

Often we write $\mathcal{F}^B(t) = \sigma(\{B(s) : 0 \le s \le t\})$, which is the so–called natural filtration of Brownian motion.

If $(\mathcal{F}(t))_{t\ge 0}$ is a filtration, an $\mathbb{R}^d$–valued $\mathcal{F}(t)$–adapted integrable process $\{M(t)\}_{t\ge 0}$ is called a *martingale with respect to* $\{\mathcal{F}(t)\}$(or simply, *martingale*) if

$$\mathbb{E}[M(t)|\mathcal{F}(s)] = M(s) \quad a.s. \quad \text{for all } 0 \le s < t < \infty. \tag{0.3.1}$$

A right–continuous adapted process $M = \{M(t)\}_{t\ge 0}$ is called a *local martingale* if there exists a nondecreasing sequence $\{\tau_k\}_{k\ge 1}$ of stopping times with $\tau_k \uparrow \infty$ a.s. such that every $\{M(\tau_k \wedge t) - M(0)\}_{t\ge 0}$ is a martingale.

A stochastic process $X = \{(X(t), \mathcal{F}(t))_{t\ge 0}\}$ is called a *semi-martingale* if its trajectories are cadlag(right-continuous and have left limits), and if it can be represented as the sum of a local martingale and a process of locally bounded variation, i.e. in the form $X(t) = M(t) + V(t)$, where $M(t)$ is a local martingale and $V(t)$ is a process of locally bounded variation, that is, $\int_0^t |dV(s,\omega)| < +\infty, \quad t > 0, \quad \omega \in \Omega$.

In this thesis, the martingales that we encounter are almost always *Itô integral.* We do not give a precise definition(see [44]) but note that these are processes of the form

$$I(t) = \int_0^t H(s)dB(s), t \ge 0 \tag{0.3.2}$$

where H is a $\mathcal{F}^B$–adapted process. It transpires that if H is, for example, continuous or bounded, that I is a local martingale. An important measure of the variability of the path of a martingale is given by its quadratic variation. If M is a scalar martingale, its quadratic variation $\langle M \rangle$ is the unique continuous adapted process vanishing at 0, for which $M^2 - \langle M \rangle$ is a martingale. We notice that if I in (0.3.2) is a martingale, then this implies that

$$\langle I \rangle(t) = \int_0^t H^2(s)ds, \quad t \geq 0.$$

This fact is used repeatedly throughout our work. Indeed, the quadratic variation assists us in expressing Brownian motions in terms of continuous martingales and vice versa. This is particularly useful when dealing with asymptotic results. The *martingale time change theorem*, stated below, helps greatly in this direction. On the other hand, it can sometimes happen that a martingale is standard Brownian motion, and Lévy's martingale characterisation gives us easily checked conditions under which this can arise.

Accordingly, we state without proof these important results below. First we state martingale time change theorem [65, Theorem V.1.6]:

Theorem 0.3.1. *If M is a $(\mathcal{F}(t), \mathbb{P})$–continuous local martingale vanishing at* 0 *and such that $\langle M \rangle(\infty) = \infty$ and if we set*

$$T(t) = \inf \{s : \langle M \rangle(s) = t\},$$

then, $B(t) = M(T(t))$ is a $(\mathcal{F}(T(t)))$–Brownian motion and $M(t) = B(\langle M \rangle(t))$ for all $t \geq 0$.

We then have the martingale convergence theorem [65, Proposition IV.1.2.6]:

Theorem 0.3.2. *A continuous local martingale M converges a.s. as $t \to \infty$, on the set $\{\langle M \rangle(\infty) < \infty\}$.*

Another useful result in proving a continuous, adapted stochastic process is a Brownian motion is Lévy's characterisation of Brownian motion given in [44, Theorem 3.3.16]. It is

Theorem 0.3.3. *Let* $\{M(t), \mathcal{F}(t), 0 \le t < +\infty\}$ *be a continuous, adapted process in* $\mathbb{R}$ *such that,* $M(t)$ *is a continuous local martingale relative to* $\{\mathcal{F}(t)\}$*. Then* $\{M(t), \mathcal{F}(t), 0 \le t < +\infty\}$ *is a one–dimensional Brownian motion.*

In view of Theorem 0.3.1, we see that precise asymptotic information about standard Brownian motion would lead to precise asymptotic information about continuous martingales. The following result, which is called the *law of iterated logarithm for standard Brownian motions*, characterise the fluctuations of standard Brownian motion.

Theorem 0.3.4. *For the standard Brownian motion* B,

$$\limsup_{t\to\infty} \frac{B(t)}{\sqrt{2t\log\log t}} = 1, \quad \liminf_{t\to\infty} \frac{B(t)}{\sqrt{2t\log\log t}} = -1.$$

One consequence of the law of the iterated logarithm for standard Brownian motions and the martingale time change theorem is that an law of iterated logarithm result holds for continuous time martingales. It is

Theorem 0.3.5. *Let* M *be a continuous local martingale such that* $\lim_{t\to\infty}\langle M\rangle(t) = +\infty$ *a.s. on an event* A*. Then*

$$\limsup_{t\to\infty} \frac{M(t)}{\sqrt{2\langle M\rangle(t)\log\log\langle M\rangle(t)}} = 1, \quad \liminf_{t\to\infty} \frac{M(t)}{\sqrt{2\langle M\rangle(t)\log\log\langle M\rangle(t)}} = -1 \quad a.s. \quad on \quad A.$$

To conclude this section, we mention an important analogue of Theorem 0.3.2 for non–negative semi–martingale. It is given in [55, Theorem 3.9]:

Theorem 0.3.6. *Let* $\{A(t)\}_{t\ge 0}$ *and* $\{U(t)\}_{t\ge 0}$ *be two continuous adapted increasing process with* $A(0) = U(0) = 0$ *a.s. Let* $\{M(t)\}_{t\ge 0}$ *be a real–valued continuous local martingale with* $M(0) = 0$ *a.s. Let* ξ *be a nonnegative* $\mathcal{F}_0$*–measurable random variable. Define*

$$X(t) = \xi + A(t) - U(t) + M(t) \quad for \quad t \ge 0.$$

If $X(t)$ *is nonnegative, then*

$$\left\{\lim_{t\to\infty} A(t) < \infty\right\} \subset \left\{\lim_{t\to\infty} X(t) \text{ exists and is finite}\right\} \cap \left\{\lim_{t\to\infty} U(t) < \infty\right\} a.s.$$

where $B \subset D$ *a.s. means* $\mathbb{P}(B \cap D^c) = 0$. *In particular, if* $\lim_{t\to\infty} A(t) < \infty$ *a.s., then for almost all* $\omega \in \Omega$

$$\lim_{t\to\infty} X(t)(\omega) \textit{exists and is finite, and} \lim_{t\to\infty} U(t)(\omega) < \infty.$$

Chapter 1

Asymptotic Stability of Perturbed ODEs with Weak Asymptotic Mean Reversion

1.1 Introduction and Connection with the Literature

Mainly in this thesis, we investigate the asymptotic behaviour of solutions of differential equations with stochastic perturbations. However, in order to see the effect of random perturbations, we first ask what can happen if there are fading deterministic perturbations, and in particular to study the relationship between the nonlinear restoring force and the rate of decay of the deterministic perturbation.

In this chapter we consider the global and local stability and instability of solutions of the perturbed scalar differential equation

$$x'(t) = -f(x(t)) + g(t), \quad t \geq 0; \quad x(0) = \xi. \tag{1.1.1}$$

It is presumed that the underlying unperturbed equation $y'(t) = -f(y(t))$ for $t \geq 0$ has a globally stable and unique equilibrium at zero. It is a natural question to ask whether stability is preserved in the case when g is asymptotically small. In the case when g is integrable, it is known that

$$\lim_{t\to\infty} x(t,\xi) = 0, \quad \text{for all } \xi \neq 0. \tag{1.1.2}$$

However, when g is not integrable, and $f(x) \to 0$ as $x \to \infty$ examples of equations are known for $x(t,\xi) \to \infty$ as $t \to \infty$. However, if we know only that $g(t) \to 0$ as $t \to \infty$, but that $\liminf_{|x|\to\infty} |f(x)| > 0$, then all solutions obey (1.1.2).

In this chapter, we investigate the asymptotic behaviour of solutions of (1.1.1) under the assumption that $f(x) \to 0$ as $x \to \infty$ and $g \notin L^1(0,\infty)$, but that $g(t) \to 0$ as $t \to \infty$. In order to characterise critical rates of decay of g for which stability still pertains we stipulate that $\xi > 0$ and $g(t) > 0$ for all $t \geq 0$, so that solutions always lie above the zero equilibrium.

As might be expected, such a critical rate depends on the rate at which $f(x)$ tends to zero as $x \to \infty$, and the more rapidly that f decays, the more rapidly that g needs to decay in order to guarantee that x obeys (1.1.2). Furthermore, regardless of how rapidly f decays to zero, there are still a class of non–integrable g for which solutions obey (1.1.2), and regardless of how slowly g tends to zero, there are a class of f for which $f(x) \to 0$ as $x \to \infty$ for which (1.1.2) still pertains.

More precisely, if we define by F the invertible function

$$F(x) = \int_1^x \frac{1}{f(u)}\, du, \quad x > 0,$$

it is shown that provided f is ultimately decreasing on $[0, \infty)$, and g decays to zero more rapidly than the non–integrable function $f \circ F^{-1}$, then solutions are globally stable (i.e., they obey (1.1.2)). This rate of decay of g is essentially the slowest possible, for it can be shown in the case when f decays either very slowly or very rapidly, that for every initial condition there exists a perturbation g which tends to zero more slowly than $f \circ F^{-1}$, for which solutions of (1.1.1) actually obey $x(t) \to \infty$ as $t \to \infty$. Moreover it can be shown under a slight strengthening of the decay hypothesis on g that for every g decaying more slowly than $f \circ F^{-1}$ that all solutions of (1.1.1) obey $x(t) \to \infty$ as $t \to \infty$, provided the initial condition is large enough. In the intermediate case when f tends to zero like $x^{-\beta}$ for $\beta > 0$ as $x \to \infty$ (modulo a slowly varying factor) a similar situation pertains, except that the critical rate of decay to zero of g is $\lambda f \circ F^{-1}$, where $\lambda > 1$ is a constant which depends purely on β.

The question addressed in this chapter is classical; under the assumptions here, we note that the autonomous differential equation

$$x'(t) = -f(x(t)) \tag{1.1.3}$$

is the unique positive limiting equation of the differential equation (1.1.1) if either $g(t) \to 0$ as $t \to \infty$ or if $g \in L^1(0, \infty)$. Therefore the problem studied here is connected strongly with work which relates the asymptotic behaviour of original non–autonomous equations to their limiting equations. Especially interesting work in this direction is due to Artstein in a series of papers [23, 24, 25]. Among the major conclusions of his work show that in some sense asymptotic stability and attracting regions of the limiting equation are synonymous with the asymptotic stability and attracting regions of the original non–autonomous equation. However, these results do not apply directly to the problems considered here, because the non–autonomous differential equation (1.1.1) does not have zero as a solution. Moreover, equation (1.1.1) does not exhibit the property that its limiting equation is not an ordinary differential equation, so the extension of the limiting equation theory expounded in e.g., [23] is not needed to explain the difference in the asymptotic behaviour between the original equation and its limiting equation. Other interesting works on asymptotically autonomous equations in this direction include Strauss and Yorke [74, 75] and D'Anna, Maio and Moauro [32].

Another approach which seems to generate good results involves Liapunov functions. Since the equation (1.1.1) is non–autonomous, we are inspired by the works of LaSalle (especially [49] and [48]), in which ideas from Liapunov's direct method, as well inspiration from the limiting equation approach are combined. In our case, however, it seems that the only possible ω–limit set is zero, the equilibrium point of the limiting equation, and once more the fact that zero is not an equilibrium of (1.1.1) makes it difficult to determine a t–independent lower bound on the derivative of the Liapunov function. Some Liapunov–like results are presented here in order to compare the results with those achieved using comparison approaches. However, the methods using comparison arguments to which the bulk of this paper is devoted, seem at this point to generate a more precise characterisation of the asymptotic behaviour of (1.1.1).

The motivation for this chapter originates from work on the asymptotic behaviour of stochastic differential equations with state independent perturbations, for which the underlying deterministic equation is globally asymptotically stable. In the case when f has relatively strong mean reversion, it is shown in [10], for a sufficiently rapidly decaying noise intensity, that solutions are still asymptotically stable, but that slower convergence leads to unbounded solutions. A complete categorisation of the asymptotic behaviour in the linear case is given in Chapter 2. It appears that the situation in the scalar case for

Itô stochastic equations differs from the ordinary case (see Chapter 3), even in the case when there is weak mean–reversion, but the situation in finite dimensions may differ. The Liapunov–like approach we have applied here is also partly inspired by work of Mao, who presented work on a version of LaSalle's invariance principle for Itô stochastic equations in [56], partly because the intrinsically non–autonomous character of the stochastic equation leads the author to allow for the presence of an integrable t–dependent function on the righthand side of the inequality for the "derivative" of the Liapunov function. A similar relaxation of the conditions on the "derivative" of the Liapunov function for Itô equations can be seen in [45, Chapter 7.4] of Hasminskii when considering the asymptotic behaviour of so–called damped stochastic differential equations, which also form the subject of [10] and Chapters 2, 3 and 4 of this thesis.

The Chapter is organised as follows. Section 2 contains preliminaries, introduces the equation to be studied, and states explicitly the hypotheses to be studied. Section 3 lists the main results of the paper. In Section 4 a number of examples are given which illustrate the main results. Section 5 considers extensions to the results indicated above to include finite–dimensional equations or equations in which the perturbation changes sign. A Liapunov–style stability theorem is given in Section 6, along with some examples. The proofs of the results are given in the remaining Sections 7–13.

1.2 Mathematical Preliminaries

1.2.1 Notation

In advance of stating and discussing our main results, we introduce some standard notation. We denote the maximum of the real numbers x and y by $x \vee y$. Let $C(I;J)$ denote the space of continuous functions $f : I \to J$ where I and J are intervals contained in $\mathbb{R}$. Similarly, we let $C^1(I;J)$ denote the space of differentiable functions $f : I \to J$ where $f' \in C(I;J)$. We denote by $L^1(0,\infty)$ the space of Lebesgue integrable functions $f : [0,\infty) \to \mathbb{R}$ such that

$$\int_0^\infty |f(s)|\, ds < +\infty.$$

If I, J and K are intervals in $\mathbb{R}$ and $f : I \to J$ and $g : J \to K$, we define the composition $g \circ f : I \to K : x \mapsto (g \circ f)(x) := g(f(x))$. If $g : [0, \infty) \to \mathbb{R}$ and $h : [0, \infty) \to (0, \infty)$ are such that

$$\lim_{x\to\infty} \frac{g(x)}{h(x)} = 1,$$

we sometimes write $g(x) \sim h(x)$ as $x \to \infty$.

1.2.2 Regularly varying functions

In this short section we introduce the class of slowly growing and decaying functions called regularly varying functions. The results and definition given here may all be found in e.g., Bingham, Goldie and Teugels [27].

We say that a function $h : [0, \infty) \to (0, \infty)$ is regularly varying at infinity with index $\alpha \in \mathbb{R}$ if

$$\lim_{x\to\infty} \frac{h(\lambda x)}{h(x)} = \lambda^\alpha.$$

We write $h \in \mathrm{RV}_\infty(\alpha)$.

We record some useful and well–known facts about regularly varying functions that will be used throughout the chapter. If h is invertible, and $\alpha \neq 0$ we have that $h^{-1} \in \mathrm{RV}_\infty(1/\alpha)$. If h is continuous, and $\alpha > -1$ it follows that the function $H : [0, \infty) \to \mathbb{R}$ defined by

$$H(x) = \int_1^x h(u)\, du, \quad x \geq 0$$

obeys $H \in \mathrm{RV}_\infty(\alpha + 1)$ and in fact we have that

$$\lim_{x\to\infty} \frac{H(x)}{xh(x)} = \frac{1}{\alpha + 1}.$$

If $h_1 \in \mathrm{RV}_\infty(\alpha_1)$ and $h_2 \in \mathrm{RV}_\infty(\alpha_2)$, then the composition $h_1 \circ h_2$ is in $\mathrm{RV}_\infty(\alpha_1\alpha_2)$.

1.2.3 Set-up of problem and statement and discussion of hypotheses

We consider the perturbed ordinary differential equation

$$x'(t) = -f(x(t)) + g(t), \quad t > 0; \quad x(0) = \xi. \tag{1.2.1}$$

We suppose that

$$f \in C(\mathbb{R};\mathbb{R}); \quad xf(x) > 0, \quad x \neq 0; \quad f(0) = 0. \tag{1.2.2}$$

and that g obeys

$$g \in C([0,\infty);\mathbb{R}). \tag{1.2.3}$$

To simplify the existence and uniqueness of a continuous solutions on $[0,\infty)$, we assume that

$$f \text{ is locally Lipschitz continuous.} \tag{1.2.4}$$

In the case when g is identically zero, it follows under the hypothesis (1.2.2) that the solution x of (1.2.1) i.e.,

$$x'(t) = -f(x(t)), \quad t > 0; \quad x(0) = \xi, \tag{1.2.5}$$

obeys

$$\lim_{t\to\infty} x(t;\xi) = 0 \text{ for all } \xi \neq 0. \tag{1.2.6}$$

Clearly $x(t) = 0$ for all $t \geq 0$ if $\xi = 0$. The convergence phenomenon captured in (1.2.6) for the solution of (1.2.1) is often called *global convergence* (or *global stability* for the solution of (1.2.5)), because the solution of the perturbed equation (1.2.1) converges to the zero equilibrium solution of the underlying unperturbed equation (1.2.5). We see that if g obeys

$$g \in L^1(0,\infty), \tag{1.2.7}$$

then (1.2.2) still suffices to ensure that the solution x of (1.2.1) obeys (1.2.6). On the other hand if we assume only that

$$\lim_{t\to\infty} g(t) = 0, \tag{1.2.8}$$

but that $g \notin L^1(0,\infty)$, (1.2.2) is not sufficient to ensure that x obeys (1.2.6). Under (1.2.8), it is only true in general that

$$\lim_{t\to\infty} x(t,\xi) = 0, \quad \text{for all } |\xi|, \sup_{t\geq 0}|g(t)| \text{ sufficiently small.} \tag{1.2.9}$$

This convergence phenomenon is referred to as *local stability with respect to perturbations*, and is established in this chapter.

An example which show that some solutions of (1.2.1) even obey

$$\lim_{t\to\infty} x(t) = \infty \tag{1.2.10}$$

in the case when g obeys (1.2.8) but $g \notin L^1(0,\infty)$ and when f obeys (1.2.2) but the restoring force $f(x)$ as $x \to \infty$ is so weak that

$$\lim_{x\to\infty} f(x) = 0 \tag{1.2.11}$$

are presented in Appleby, Gleeson and Rodkina [10].

However, when (1.2.11) is strengthened so that in addition to (1.2.2), f also obeys

$$\text{There exists } \phi > 0 \text{ such that } \phi := \liminf_{|x|\to\infty} |f(x)|, \tag{1.2.12}$$

then the condition (1.2.8) on g suffices to ensure that the solution x of (1.2.1) obeys (1.2.6). See also [10]. For this reason, we restrict our focus in this paper to the case when f obeys (1.2.11).

The question therefore arises: if f obeys (1.2.11), is the condition (1.2.7) *necessary* in order for solutions of (1.2.1) to obey (1.2.6), or does a weaker condition suffice. In this paper we give a relatively sharp characterisation of conditions on g under which solutions of (1.2.1) obey (1.2.6) or (1.2.10). In general, we focus on the case where $g \notin L^1(0,\infty)$, once we have shown that x obeys (1.2.6) when $g \in L^1(0,\infty)$.

To capture these critical rates of decay of the perturbation g, we constrain it obey

$$g(t) > 0, \quad t \geq 0, \tag{1.2.13}$$

Our purpose here is not to simplify the analysis, but rather to try to obtain a good lower bound on a critical rate of decay of the perturbation. To see why choosing g to be positive may help in this direction, suppose momentarily that $g(t)$ tends to zero in such a way that it experiences relatively large but rapid fluctuations around zero. In this case, it is possible that the "positive" and "negative" fluctuations cancel. Therefore an upper bound on the rate of decay of the perturbation to zero, which must majorise the amplitude of the fluctuations of g, is likely to give a conservative estimate on the rate of decay. Hence it may be difficult to ascertain whether a given upper bound on the rate of decay of g is sharp in this case. Similarly, we constrain the initial condition ξ to obey

$$\xi > 0, \tag{1.2.14}$$

as this in conjunction with the positivity of g and the condition (1.2.2) on f will prevent the solution of (1.2.1) from oscillating around the zero equilibrium of (1.2.5): indeed these conditions force $x(t) > 0$ for all $t \geq 0$. This positivity enables us to get lower as well as upper bounds on the solution.

Many stability results in the case when ξ and g do not satisfy these sign constraints can be inferred by applying a comparison argument to a related equation which does possess a positive initial condition and g. Details of some representative results, and extensions of our analysis to systems of equations is given in Section 1.5.

To determine the critical rate of decay to zero of g, we introduce the invertible function F, given by

$$F(x) = \int_1^x \frac{1}{f(u)}\,du, \quad x > 0. \tag{1.2.15}$$

Roughly speaking, we show here that provided $g(t)$ decays to zero according to

$$\limsup_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} < 1, \tag{1.2.16}$$

and

$$\text{There exists } x^* \geq 0 \text{ such that } f \text{ is non–increasing on } (x^*, \infty) \tag{1.2.17}$$

then the solution x of (1.2.1) obeys (1.2.6). The condition (1.2.16) forces $g(t) \to 0$ as $t \to \infty$. To see this note that the fact that f obeys (1.2.17), and (1.2.2) implies that $F(t) \to \infty$ as $t \to \infty$ and therefore $F^{-1}(t) \to \infty$ as $t \to \infty$. Since f obeys (1.2.11), we have $f(F^{-1}(t)) \to 0$ as $t \to \infty$. This implies that $g(t) \to 0$ as $t \to \infty$. We note also that (1.2.16) allows for g to be non–integrable, because $t \mapsto f(F^{-1}(t))$ is non–integrable, owing to the identity

$$\int_0^t f(F^{-1}(s))\,ds = \int_{F^{-1}(0)}^{F^{-1}(t)} f(u) \cdot F'(u)\,du = F^{-1}(t) - 1,$$

which tends to $+\infty$ as $t \to \infty$. Careful scrutiny of the proofs reveals that the condition (1.2.17) can be relaxed to the hypothesis that f is asymptotic to a function which obeys

(1.2.17). However, for simplicity of exposition, we prefer the stronger (1.2.17) when it is required.

On the other hand, the condition (1.2.16) is sharp when f decays either very rapidly or very slowly to zero. We make this claim precise. When f decays so rapidly that

$$f \circ F^{-1} \in \mathrm{RV}_\infty(-1) \tag{1.2.18}$$

or f decays to zero so slowly that

$$f \in \mathrm{RV}_\infty(0) \tag{1.2.19}$$

then *for every* $\xi > 0$ *there exists* a g which obeys

$$\limsup_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} > 1, \tag{1.2.20}$$

for which the solution x of (1.2.1) obeys (1.2.10). In fact we can construct explicitly such a g. Moreover, under either (1.2.18) or (1.2.19), it follows that *for every* g for which

$$\liminf_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} > 1, \tag{1.2.21}$$

there exists $\bar{x} > 0$ such that the solution x of (1.2.1) obeys (1.2.10) for all $\xi > \bar{x}$. We observe that (1.2.21) implies that $g \notin L^1(0,\infty)$. We note that the condition (1.2.18) automatically implies that f obeys (1.2.11) and also that f is asymptotic to a function which obeys (1.2.17).

In the case when f decays to zero "polynomially" we can still characterise quite precisely the critical rate of decay. Once again, what matters is the relative rate of convergence of $g(t)$ and of $f(F^{-1}(t))$ to 0 as $t \to \infty$. Suppose that f obeys

$$\text{There exists } \beta > 0 \text{ such that } f \in \mathrm{RV}_\infty(-\beta). \tag{1.2.22}$$

This condition automatically implies that f obeys (1.2.11) and moreover that it is asymptotic to a function which obeys (1.2.17). In the case that

$$\limsup_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} < \lambda(\beta) := \beta^{\frac{1}{\beta+1}} \left(1 + \beta^{-1}\right), \tag{1.2.23}$$

and f obeys (1.2.17), we have that the solution x of (1.2.1) obeys (1.2.6). On the other hand if f obeys (1.2.22), then *for every* $\xi > 0$ *there exists* a g which obeys

$$\limsup_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} \geq \lambda(\beta), \tag{1.2.24}$$

where $\lambda(\beta)$ is defined in (1.2.23) for which the solution x of (1.2.1) obeys (1.2.10). Moreover, when f obeys (1.2.22), it follows that *for every* g for which

$$\liminf_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} > \lambda(\beta), \tag{1.2.25}$$

that *there exists* $\bar{x} > 0$ such that the solution x of (1.2.1) obeys (1.2.10) for all $\xi > \bar{x}$. We note that (1.2.25) implies that $g \notin L^1(0,\infty)$.

In the next section, we state precisely the results proven in the paper, referring to the above hypotheses. Although the hypotheses (1.2.19), (1.2.18) and (1.2.22) do not cover all possible modes of convergence of $f(x) \to 0$ as $x \to \infty$, we find in practice that collectively they cover many functions f which decay monotonically to zero.

1.3 Precise Statement of Main Results

In this section we list our main results, and demonstrate that for any non–integrable g that it is possible to find an f for which solutions of (1.2.1) are globally stable. We also find the maximal size of perturbation g which is permissible for a given f so that solutions of (1.2.1) are globally stable.

1.3.1 List of main results

In our first result, we show that when $g \in L^1(0,\infty)$, then x obeys (1.2.6) even when f obeys (1.2.11).

Theorem 1.3.1. *Suppose that f obeys* (1.2.2) *and that g obeys* (1.2.3) *and* (1.2.7)*. Let x be the unique continuous solution of* (1.2.1)*. Then x obeys* (1.2.6)*.*

As a result of Theorem 1.3.1 we confine attention when f obeys (1.2.11) to the case in which g is not integrable. We assume instead that $g(t) \to 0$ as $t \to \infty$ and try to identify the appropriate non–integrable and f–dependent pointwise rate of decay which ensures that x obeys (1.2.6). Our first result shows that the non–negativity of g and global stability of the zero solution of the underlying equation (1.2.5) ensure that solutions x of the perturbed equation (1.2.1) obey either $\lim_{t\to\infty} x(t) = 0$ or $\lim_{t\to\infty} x(t) = \infty$.

Theorem 1.3.2. *Suppose that* g *obeys* (1.2.3), (1.2.8), *and* g *is non–negative. Suppose that* f *obeys* (1.2.2) *and that* x *is the unique continuous solution* x *of* (1.2.1). *Then either* $\lim_{t\to\infty} x(t) = 0$ *or* $\lim_{t\to\infty} x(t) = +\infty$.

Of course, Theorem 1.3.2 does not tell us into which category of asymptotic behaviour a particular initial value problem will fall, or whether either asymptotic behaviour is possible under certain asymptotic assumptions on f and g.

We first show that when the initial condition ξ is sufficiently small and $\sup_{t\geq 0} g(t)$ is sufficiently small (in addition to g obeying (1.2.8)), then the zero solution of the underlying unperturbed equation is *locally* stable and we have that the solution x of (1.2.1) obeys $x(t) \to 0$ as $t \to \infty$.

Theorem 1.3.3. *Suppose that* f *obeys* (1.2.2) *and that* g *obeys* (1.2.8). *Then for every* $\epsilon > 0$ *sufficiently small there exists a number* $x_1(\epsilon) > 0$ *such that* $g(t) \leq \epsilon$ *for all* $t \geq 0$ *and* $\xi \in (0, x_1(\epsilon))$ *implies* $x(t, \xi) \to 0$ *as* $t \to \infty$.

In the case when $f(x) \to 0$ as $x \to \infty$ and f is ultimately monotone, our most general *global* stability result states that if g decays to zero so rapidly that (1.2.16) is true, then we have that the solution x of (1.2.1) obeys $x(t) \to 0$ as $t \to \infty$.

Moreover, instead of the pointwise rate of decay (1.2.16), we can provide a slightly sharper condition, that is if g decays to zero so rapidly that

$$\limsup_{t\to\infty} \frac{\int_0^t g(s)\, ds}{F^{-1}(t)} < 1, \tag{1.3.1}$$

then we have that the solution x of (1.2.1) obeys $x(t) \to 0$ as $t \to \infty$.

Theorem 1.3.4. *Suppose that* f *obeys* (1.2.2) *and* g *obeys* (1.2.3). *Suppose that* x *is the unique continuous solution of* (1.2.1). *Suppose that* f *obeys* (1.2.11) *and* (1.2.17) *and let* F *be defined by* (1.2.15). *If* g *and* f *are such that* (1.3.1) *holds, then the solution* x *of* (1.2.1) *obeys* (1.2.6).

Therefore we can think of the following Theorem as a Corollary of Theorem 1.3.4.

Theorem 1.3.5. *Suppose that f obeys* (1.2.2) *and g obeys* (1.2.3)*. Suppose that x is the unique continuous solution of* (1.2.1)*. Suppose that f obeys* (1.2.11) *and* (1.2.17) *and let F be defined by* (1.2.15)*. If g and f are such that* (1.2.16) *holds, then the solution x of* (1.2.1) *obeys* (1.2.6)*.*

We have some partial converses to this result. If it is supposed that for every f which decays to zero so slowly that $f \in \text{RV}_\infty(0)$, and for every initial condition $\xi > 0$ there exists g which violates (1.2.16) (and *a fortiori* obeys (1.2.20)) for which the solution of (1.2.1) obeys $x(t) \to \infty$ as $t \to \infty$.

Theorem 1.3.6. *Suppose that f obeys* (1.2.2) *and g obeys* (1.2.3)*. Suppose that x is the unique continuous solution of* (1.2.1)*. Suppose that f obeys* (1.2.11) *and* (1.2.19) *and let F be defined by* (1.2.15)*. For every $\xi > 0$ there is a g which obeys* (1.2.20) *such that the solution x of* (1.2.1) *obeys* (1.2.10)*.*

Moreover, we have that the solution $x(\cdot, \xi)$ of (1.2.1) obeys $x(t, \xi) \to \infty$ as $t \to \infty$ for any g obeying an asymptotic condition slightly stronger than the negation of (1.2.20), provided the initial condition ξ is sufficiently large. More precisely the asymptotic condition on g is (1.2.21).

Theorem 1.3.7. *Suppose that f obeys* (1.2.2)*, g obeys* (1.2.3)*, and that f obeys* (1.2.19) *and g and f obey* (1.2.21)*. Suppose that x is the unique continuous solution of* (1.2.1)*. Then there exists $\bar{x} > 0$ such that for all $\xi > \bar{x}$ we have $\lim_{t\to\infty} x(t, \xi) = \infty$.*

Similar converses to Theorem 1.3.4 exist in the case that $f(x)$ decays so rapidly to zero as $x \to \infty$ that $f \circ F^{-1}$ is in $\text{RV}_\infty(-1)$. We first note that for every initial condition, a destabilising perturbation can be found.

Theorem 1.3.8. *Suppose that f obeys* (1.2.2) *and g obeys* (1.2.3)*. Suppose that x is the unique continuous solution of* (1.2.1)*. Suppose that f obeys* (1.2.11) *and* (1.2.18) *where F is defined by* (1.2.15)*. For every $\xi > 0$ there is a g which obeys* (1.2.20) *such that the solution x of* (1.2.1) *obeys* (1.2.10)*.*

Once again, if the initial condition is sufficiently large, and g obeys an asymptotic condition slightly stronger than the negation of (1.2.20) (viz., (1.2.21)), then once again solutions tend to infinity.

Theorem 1.3.9. *Suppose that f obeys* (1.2.2), *g obeys* (1.2.3), *and that f obeys* (1.2.18) *and g and f obey* (1.2.21). *Suppose that x is the unique continuous solution of* (1.2.1). *Then there exists $\bar{x} > 0$ such that for all $\xi > \bar{x}$ we have $\lim_{t\to\infty} x(t,\xi) = \infty$.*

In the case where f is in $\text{RV}_\infty(-\beta)$ for some $\beta > 0$ we have the following case distinction. If g decays to zero so slowly that (1.2.23) holds, then $x(t) \to 0$ as $t \to \infty$. Moreover, analogously to Theorem 1.3.4, instead of the pointwise rate of decay (1.2.23), if we impose the weaker condition

$$\limsup_{t\to\infty} \frac{\int_0^t g(s)\,ds}{F^{-1}(t)} \le \lambda < \lambda(\beta), \tag{1.3.2}$$

then we have that the solution x of (1.2.1) obeys $x(t) \to 0$ as $t \to \infty$.

Theorem 1.3.10. *Suppose that f obeys* (1.2.2) *and g obeys* (1.2.3). *Suppose that x is the unique continuous solution of* (1.2.1). *Suppose that there is $\beta > 0$ such that f obeys* (1.2.17) *and* (1.2.22) *and let F be defined by* (1.2.15). *If g and f are such that* (1.3.2) *holds, then the solution x of* (1.2.1) *obeys* (1.2.6).

Therefore the following Theorem is a direct corollary of Theorem 1.3.10.

Theorem 1.3.11. *Suppose that f obeys* (1.2.2) *and g obeys* (1.2.3). *Suppose that x is the unique continuous solution of* (1.2.1). *Suppose that there is $\beta > 0$ such that f obeys* (1.2.17) *and* (1.2.22) *and let F be defined by* (1.2.15). *If g and f are such that* (1.2.23) *holds, then the solution x of* (1.2.1) *obeys* (1.2.6).

The condition (1.2.23), which is sufficient for stability in the case when $f \in \text{RV}_\infty(-\beta)$ is weaker than (1.2.16). However, it is difficult to relax it further. For every f in $\text{RV}_\infty(-\beta)$ and every initial condition ξ it is possible to find a g which violates (1.2.23) (and therefore obeys (1.2.24)) for which the solution obeys $x(t) \to \infty$ as $t \to \infty$.

Theorem 1.3.12. *Suppose that f obeys* (1.2.2) *and g obeys* (1.2.3)*. Suppose that x is the unique continuous solution of* (1.2.1)*. Suppose that there is $\beta > 0$ such that f obeys* (1.2.22) *and let F be defined by* (1.2.15)*. Then for every $\xi > 0$ there is a g which obeys* (1.2.24) *such that the solution x of* (1.2.1) *obeys* (1.2.10).

On the other hand, we have that the solution $x(\cdot, \xi)$ of (1.2.1) obeys $x(t, \xi) \to \infty$ as $t \to \infty$ for any g obeying an asymptotic condition slightly stronger than the negation of (1.2.24), provided the initial condition ξ is sufficiently large. More precisely the asymptotic condition on g is (1.2.25), where $\lambda(\beta)$ is as defined by (1.2.23).

Theorem 1.3.13. *Suppose that f obeys* (1.2.2)*, g obeys* (1.2.3)*, and that f obeys* (1.2.22) *and g and f obey* (1.2.25)*. Suppose that x is the unique continuous solution of* (1.2.1)*. Then there exists $\bar{x} > 0$ such that for all $\xi > \bar{x}$ we have $\lim_{t\to\infty} x(t, \xi) = \infty$.*

1.3.2 Minimal conditions for global stability

In this short subsection we address two questions: given any non–integrable g, we show that it is possible to find an f for which the solution of (1.2.1) is globally stable. And given an f, we determine how large is the largest possible perturbation g that is permissible so that the solution is globally stable.

We also consider two extreme cases: when g just fails to be integrable $g \in \text{RV}_\infty(-1)$, and when g tends to zero so slowly that $g \in \text{RV}_\infty(0)$. In the case when g just fails to be integrable (so that $g \in \text{RV}_\infty(-1)$), we can choose an f which decays to zero so rapidly that $f \circ F^{-1} \in \text{RV}_\infty(-1)$ while at the same time ensuring that solutions of (1.2.1) are globally asymptotically stable. On the other hand, if g decays to zero so slowly that $g \in \text{RV}_\infty(0)$, we choose f to decay to zero slowly also while preserving global stability. In particular, it transpires that f is in $\text{RV}_\infty(0)$.

Consider first the general question. Suppose that $g(t) \to 0$ as $t \to \infty$ in such a way that $g \notin L^1(0, \infty)$. If moreover g is ultimately decreasing, the next Proposition show that it is possible to find an f,which satisfies all the conditions of Theorem 1.3.4, so that the solution f of (1.2.1) x obeys (1.2.6). Therefore, there is no rate of decay of g to zero, however slow, that cannot be stabilised by an f for which $f(x) \to 0$ as $x \to \infty$. Therefore, it is possible for g to be very far from being integrable, and $f(x) \to 0$ as $x \to \infty$, but provided that this rate of decay of f is not too fast, then solutions of (1.2.1) can still be globally stable.

Proposition 1.3.1. *Suppose that g is positive, continuous and obeys* (1.2.8) *and $g \notin L^1(0,\infty)$. Let $\lambda > 0$. Then there exists a continuous f which obeys* (1.2.2), (1.2.11) *and also obeys*

$$\lim_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} = \lambda. \tag{1.3.3}$$

Moreover, if g is decreasing on $[\tau,\infty)$ for some $\tau \geq 0$, then f obeys (1.2.17).

Proof. Suppose that f is such that $f(0) = 0$, $f(x) > 0$ for all $x \in (0,1]$ and that

$$\lim_{x\to 1^-} f(x) = \frac{1}{\lambda} g(0) > 0.$$

Define

$$G_\lambda(x) = \frac{1}{\lambda}\int_1^x g(s)\,ds, \quad x \geq 0. \tag{1.3.4}$$

Then G_λ is increasing and therefore G_λ^{-1} exists. Moreover since $g \notin L^1(0,\infty)$, we have that $G_\lambda(x) \to \infty$ as $x \to \infty$, so $G_\lambda^{-1}(x) \to \infty$ as $x \to \infty$. Define also

$$f(x) = \frac{1}{\lambda} g(G_\lambda^{-1}(x - 1 + G_\lambda(0))), \quad x \geq 1.$$

For $x \geq 1$ we have that $x - 1 + G_\lambda(0) \geq G_\lambda(0)$, so $G_\lambda^{-1}(x - 1 + G_\lambda(0)) \geq 0$. Therefore f is well–defined. Moreover, since g is positive, we have that $f(x) > 0$ for all $x > 0$. Note that $f(1) = g(0)/\lambda$, and g and G_λ are continuous, we have that $f : [0,\infty) \to [0,\infty)$ is continuous. Since $g(t) \to 0$ as $t \to \infty$ and $G_\lambda^{-1}(t) \to \infty$ as $t \to \infty$, it follows that $f(x) \to 0$ as $x \to \infty$. We see also that if g is ultimately decreasing, that f must obey (1.2.17), because G_λ^{-1} is increasing.

Finally, notice that

$$F(x) = \int_1^x \frac{1}{f(u)}\,du = \int_0^{G_\lambda^{-1}(x-1+G_\lambda(0))} \frac{1}{1/\lambda \cdot g(s)} \frac{1}{\lambda} g(s)\,ds = G_\lambda^{-1}(x-1+G_\lambda(0)).$$

Therefore for $x \geq 1$ we have $g(F(x)) = \lambda f(x)$. Now $F(x) \geq 0$ for $x \geq 1$, so we have $g(y) = \lambda f(F^{-1}(y))$ for $y \geq 0$, so clearly we have that (1.3.3) holds. □

Suppose next that g tends to zero arbitrarily slowly (restricted to the class of $\text{RV}_\infty(0)$). Then it is possible to find an f (also in $\text{RV}_\infty(0)$) which satisfies all the conditions of Theorem 1.3.4, so that x obeys (1.2.6).

Proposition 1.3.2. *Suppose that $g \in RV_\infty(0)$ is continuous, positive and decreasing and obeys* (1.2.8)*. Define*

$$G(t) = \int_1^t g(s)\,ds, \quad t \geq 0. \tag{1.3.5}$$

Let $\lambda > 0$. Suppose that f is continuous and obeys (1.2.2)*, as well as*

$$f(x) \sim \frac{1}{\lambda} g(G^{-1}(x)), \quad x \to \infty. \tag{1.3.6}$$

Then f obeys (1.2.11)*, f is asymptotic to a decreasing function, $f \in RV_\infty(0)$ and* (1.3.3)*.*

As an example, suppose that $n \in \mathbb{N}$ and that $g(x) \sim 1/(\log_n x)$ as $x \to \infty$. It can then be shown that $G^{-1}(x) \sim x \log_n x$ as $x \to \infty$. Therefore we have

$$g(G^{-1}(x)) \sim \frac{1}{\log_n x}$$

Hence if $f(x) \sim \lambda^{-1}/\log_n x$ as $x \to \infty$, we have that g and f obey (1.3.3).

Remark 1.3.1. If f tends to zero very slowly, we can still have g tending to zero very slowly, and yet have solutions of (1.2.1) obeying (1.2.6). Indeed, suppose that $f \in \text{RV}_\infty(0)$. Then $F \in \text{RV}_\infty(1)$ so $F^{-1} \in \text{RV}_\infty(1)$. Therefore $f \circ F^{-1} \in \text{RV}_\infty(0)$. Hence if g obeys (1.3.3) with $\lambda < 1$, we have that $g \in \text{RV}_\infty(0)$.

Remark 1.3.2. We note that if f tends to zero very rapidly, so that $f \circ F^{-1}$ is in $\text{RV}_\infty(-1)$, then g must be dominated by a function in $\text{RV}_\infty(-1)$. Therefore, if f tends to zero very rapidly, it can be seen that g must be close to being integrable. This is related to the fact that however rapidly f tends to zero (in the sense that $f \circ F^{-1}$ is in $\text{RV}_\infty(-1)$), it is always possible to find non–integrable g for which solutions of (1.2.1) are globally asymptotically stable and obey (1.2.6).

Remark 1.3.3. Suppose conversely that $g \in \text{RV}_\infty(-1)$ in such a way that $g \notin L^1(0,\infty)$. Then we can find an f which decays so quickly to zero as $x \to \infty$ that $f \circ F^{-1} \in \text{RV}_\infty(-1)$ while f and g obey (1.3.3). Therefore, if g tends to zero in such a way that it is close to being integrable (but is non–integrable), then solutions of (1.2.1) are globally asymptotically stable provided f exhibits very weak mean reversion.

To see this let $\lambda > 0$. Then it can be shown in a manner similar to Proposition 1.3.1 that if f is defined by

$$f(x) = \frac{1}{\lambda} g(G_\lambda^{-1}(x)), \quad x \geq 1$$

where G_λ is defined by (1.3.4), then f and g obey (1.3.3). Moreover, if F is defined by (1.2.15), for this choice of f we have $F(x) = G_\lambda^{-1}(x) - G_\lambda^{-1}(1)$ for $x \geq 1$. Rearranging yields $F^{-1}(x) = G_\lambda(x + G')$ for $x \geq 0$, where we define $G' := G_\lambda^{-1}(1)$. Hence

$$f(F^{-1}(x)) = \frac{1}{\lambda} g(G_\lambda^{-1}(F^{-1}(x))) = \frac{1}{\lambda} g(x + G').$$

Since $g \in \text{RV}_\infty(-1)$ it follows that $f \circ F^{-1} \in \text{RV}_\infty(-1)$.

Example 1.3.1. In the case when $g(t) \sim 1/(t \log t)$ as $t \to \infty$, we have

$$G_\lambda(t) \sim \frac{1}{\lambda} \log_2 t, \quad \text{as } t \to \infty.$$

Therefore can see (formally) that $\log G_\lambda^{-1}$ behaves asymptotically like $e^{\lambda t}$ and that $G_\lambda^{-1}(t)$ behaves like $\exp(e^{\lambda t})$ as $t \to \infty$. Hence a good candidate for f is

$$f(x) = \frac{1}{\lambda} e^{-\lambda x} \exp(-e^{\lambda x}), \quad x \geq 1.$$

Then, with $x' = \exp(e^{\lambda})$, we have $F(x) = \exp(e^{\lambda x}) - x'$. Therefore we have $F^{-1}(x) = \log_2(x + x')/\lambda$. Hence

$$f(F^{-1}(x)) = \frac{1}{\lambda}\frac{1}{x+x'}\frac{1}{\log(x+x')}.$$

Therefore we have that g and f obey (1.3.3). Note moreover that $f \circ F^{-1}$ is in $\mathrm{RV}_\infty(-1)$.

1.4 Examples

In this section we give examples of equations covered by Theorems 1.3.2—1.3.13 above.

Example 1.4.1. Let $a > 0$ and $\beta > 0$. Suppose that $f(x) = ax(1+x)^{-(\beta+1)}$ for $x \geq 0$. Then f obeys (1.2.2) and (1.2.17). We have that $f \in \mathrm{RV}_\infty(-\beta)$. Now as $x \to \infty$ we have

$$F(x) = \int_1^x \frac{1}{f(u)}\,du \sim \int_1^x 1/au^\beta\,du = \frac{1/a}{\beta+1}x^{\beta+1}.$$

Then $F^{-1}(x) \sim [a(1+\beta)x]^{1/(\beta+1)}$ as $x \to \infty$. Therefore as $x \to \infty$ we have

$$f(F^{-1}(x)) \sim a[a(1+\beta)x]^{-\beta/(\beta+1)} = a^{1/(\beta+1)}(1+\beta)^{-\beta/(\beta+1)}x^{-\beta/(\beta+1)}.$$

Suppose that

$$\limsup_{t\to\infty} \frac{g(t)}{a^{1/(\beta+1)}(1+\beta)^{-\beta/(\beta+1)}t^{-\beta/(\beta+1)}} < \beta^{1/(\beta+1)}(1+\beta^{-1})$$

Then for every $\xi > 0$ we have $x(t,\xi) \to 0$ as $t \to \infty$. On the other hand, for every $\xi > 0$, there is a g which obeys

$$\limsup_{t\to\infty} \frac{g(t)}{a^{1/(\beta+1)}(1+\beta)^{-\beta/(\beta+1)}t^{-\beta/(\beta+1)}} \geq \beta^{1/(\beta+1)}(1+\beta^{-1})$$

such that $x(t,\xi) \to \infty$ as $t \to \infty$. Finally, for every g which obeys

$$\liminf_{t\to\infty} \frac{g(t)}{a^{1/(\beta+1)}(1+\beta)^{-\beta/(\beta+1)}t^{-\beta/(\beta+1)}} > \beta^{1/(\beta+1)}(1+\beta^{-1})$$

there is an $\bar{x} > 0$ such that for all $\xi > \bar{x}$ we have $x(t,\xi) \to \infty$ as $t \to \infty$.

Example 1.4.2. Let $a>0$ and suppose that

$$f(x)=\frac{ax}{(1+x)\log(e+x)}, \quad x\geq 0.$$

Then f obeys (1.2.2) and (1.2.17). Moreover, we have that $f\in \text{RV}_\infty(0)$. Hence as $x\to\infty$ we have

$$F(x)\sim\int_1^x \frac{1}{a}\log(e+u)\,du\sim\frac{1}{a}x\log x.$$

Therefore we have $F^{-1}(x)\sim ax/\log x$ as $x\to\infty$. Thus as $x\to\infty$ we have

$$f(F^{-1}(x))\sim a/\log F^{-1}(x)\sim a/\log x.$$

Therefore if

$$\limsup_{t\to\infty} g(t)\log t<a,$$

we have $x(t,\xi)\to 0$ for all $\xi>0$. On the other hand for every $\xi>0$ there is a g which obeys

$$\limsup_{t\to\infty} g(t)\log t>a,$$

for which $x(t,\xi)\to\infty$. Finally, for every g which obeys

$$\liminf_{t\to\infty} g(t)\log t>a,$$

there is a $\bar{x}>0$ such that for all $\xi>\bar{x}$ we have $x(t,\xi)\to\infty$.

Example 1.4.3. Let $a>0$, $\beta>0$ and $\delta>0$ and suppose that

$$f(x)=axe^{-\delta x^\beta}, \quad x\geq 1,$$

where $f(0)=0$, $f(x)>0$ for $x\in(0,1)$ and f is continuous on $[0,1)$ with $\lim_{x\to 1^-}f(x)=ae^{-\delta}$. Then f obeys (1.2.2) and (1.2.17). By L'Hôpital's rule we have

$$\lim_{x\to\infty}\frac{F(x)}{e^{\delta x^\beta}/x^\beta}=\frac{1}{a}\lim_{x\to\infty}\frac{x^{-1}}{-\beta x^{-\beta-1}+\delta\beta x^{-1}}=\frac{1}{a\delta\beta}.$$

Therefore we have

$$\lim_{x\to\infty} \frac{x}{e^{\delta F^{-1}(x)^\beta}/F^{-1}(x)^\beta} = \frac{1}{a\delta\beta}.$$

From this it can be inferred that

$$\lim_{x\to\infty} \frac{e^{\delta F^{-1}(x)^\beta}/F^{-1}(x)^\beta}{x} = a\delta\beta.$$

Now we have $e^{\delta F^{-1}(x)^\beta} \sim a\delta\beta x F^{-1}(x)^\beta$ as $x \to \infty$. Therefore as $x \to \infty$ we get

$$xf(F^{-1}(x)) = xaF^{-1}(x)/e^{\delta F^{-1}(x)^\beta} \sim \frac{xaF^{-1}(x)}{a\delta\beta x F^{-1}(x)^\beta} = \frac{1}{\delta\beta} \cdot F^{-1}(x)^{1-\beta}.$$

It remains to estimate the asymptotic behaviour of $F^{-1}(x)$ as $x \to \infty$. Since $\delta F^{-1}(x)^\beta - \beta \log F^{-1}(x) - \log x \to \log(a\delta\beta)$ as $x \to \infty$, we therefore obtain

$$\lim_{x\to\infty} \frac{\delta F^{-1}(x)^\beta}{\log x} = 1.$$

Hence

$$\lim_{x\to\infty} \frac{F^{-1}(x)^{1-\beta}}{(\log x)^{(1-\beta)/\beta}} = \left(\frac{1}{\delta}\right)^{(1-\beta)/\beta}.$$

Thus $(F^{-1})^{1-\beta}$ is in $\mathrm{RV}_\infty(0)$ and thus $f \circ F^{-1} \in \mathrm{RV}_\infty(-1)$. Moreover as $x \to \infty$ we have

$$f(F^{-1}(x)) \sim \frac{1}{\delta\beta} \cdot \frac{1}{x} \cdot F^{-1}(x)^{1-\beta} \sim \frac{1}{\beta\delta^{1/\beta}} \cdot \frac{1}{x} \cdot \frac{1}{(\log x)^{-1/\beta+1}}.$$

Therefore if

$$\limsup_{t\to\infty} \frac{g(t)}{\frac{1}{\beta\delta^{1/\beta}} \cdot \frac{1}{t} \cdot \frac{1}{(\log t)^{-1/\beta+1}}} < 1,$$

we have $x(t,\xi) \to 0$ for all $\xi > 0$. On the other hand for every $\xi > 0$ there is a g which obeys

$$\limsup_{t\to\infty} \frac{g(t)}{\frac{1}{\beta\delta^{1/\beta}} \cdot \frac{1}{t} \cdot \frac{1}{(\log t)^{-1/\beta+1}}} > 1,$$

for which $x(t,\xi) \to \infty$. Finally, for every g which obeys

$$\liminf_{t\to\infty} \frac{g(t)}{\frac{1}{\beta\delta^{1/\beta}} \cdot \frac{1}{t} \cdot \frac{1}{(\log t)^{-1/\beta+1}}} > 1,$$

there is a $\bar{x} > 0$ such that for all $\xi > \bar{x}$ we have $x(t,\xi) \to \infty$.

1.5 Extensions to General Scalar Equations and Finite–Dimensional Equations

We have formulated and discussed our main results for scalar equations where the solutions remain of a single sign. This restriction has enabled us to achieve sharp results on the asymptotic stability and instability. However, it is also of interest to investigate asymptotic behaviour of equations of a similar form in which changes in the sign of g lead to changes in the sign of the solution, or to equations in finite dimensions. In this section, we demonstrate that results giving sufficient conditions for global stability can be obtained for these wider classes of equation, by means of appropriate comparison arguments. In this section, we denote by $\langle x, y\rangle$ the standard innerproduct of the vectors $x, y \in \mathbb{R}^d$, and let $\|x\|$ denote the standard Euclidean norm of $x \in \mathbb{R}^d$ induced from this innerproduct.

1.5.1 Finite–dimensional equations

In this section, we first discuss appropriate hypotheses under which the d–dimensional ordinary differential equation

$$x'(t) = -\phi(x(t)) + \gamma(t), \quad t > 0; \quad x(0) = \xi \in \mathbb{R}^d \tag{1.5.1}$$

will exhibit asymptotically convergent solutions under conditions of weak asymptotic mean reversion. Here, we assume that $\phi : \mathbb{R}^d \to \mathbb{R}^d$ and that $\gamma : [0, \infty) \to \mathbb{R}^d$. Therefore, if there is a solution x, $x(t) \in \mathbb{R}^d$ for any $t \geq 0$ for which x exists. In order to simplify matters, we assume once again that ϕ is locally Lipschitz on $\mathbb{R}^d$ and that γ is continuous, as these assumptions guarantee the existence of a unique continuous solution, defined on $[0, T)$ for some $T > 0$. In order that solutions be global (i.e., that $T = +\infty$), we need to show that there does not exist $T < +\infty$ such that

$$\lim_{t \uparrow T} \|x(t)\| = +\infty.$$

In the scalar setting, this is ensured by the global stability condition (1.2.2). We need a natural analogue of this condition, as well as the condition that 0 is the unique solution of the underlying unperturbed equation

$$z'(t) = -\phi(z(t)), \quad t > 0; \quad z(0) = \xi. \tag{1.5.2}$$

A suitable and simple condition which achieves all these ends is

$$\phi \text{ is locally Lipschitz continuous, } \phi(0) = 0, \quad \langle \phi(x), x \rangle > 0 \text{ for all } x \neq 0. \tag{1.5.3}$$

We also find it convenient to introduce a function φ_0 given by

$$\varphi_0(x) = \begin{cases} \inf_{\|u\|=x} \frac{\langle u, \phi(u) \rangle}{\|u\|}, & x > 0, \\ 0, & x = 0. \end{cases} \tag{1.5.4}$$

It turns out that the function φ_0 is important in several of our proofs. For this reason, we list here its relevant properties.

Lemma 1.5.1. *Let $\varphi_0 : [0, \infty) \to \mathbb{R}$ be the function defined in* (1.5.4)*. Then*

$$\varphi_0(x) = \inf_{\|u\|=1} \langle u, \phi(xu) \rangle, \quad x \geq 0. \tag{1.5.5}$$

If ϕ obeys (1.5.3)*, then $\varphi_0(0) = 0$, $\varphi_0(x) > 0$ for $x > 0$ and φ_0 is locally Lipschitz continuous. Moreover, if $\phi(x) \to 0$ as $\|x\| \to \infty$, then $\varphi_0(x) \to 0$ as $x \to \infty$.*

In the scalar case when ϕ is an odd function, we note that φ_0 collapses to ϕ itself. The proof of Lemma 1.5.1 is presented in the final section.

We consolidate the facts collected above regarding solutions of (1.5.2) and (1.5.1) into two propositions. Their proofs are standard, and are also relegated to the end.

Proposition 1.5.1. *Suppose that ϕ obeys* (1.5.3)*. Then $x = 0$ is the unique equilibrium solution of* (1.5.2)*. Moreover, the initial value problem* (1.5.2) *has a unique continuous solution defined on $[0, \infty)$ and for all initial conditions $z(t) \to 0$ as $t \to \infty$.*

Proposition 1.5.2. *Suppose that ϕ obeys* (1.5.3)*. Then, the initial value problem* (1.5.2) *has a unique continuous solution defined on $[0, \infty)$.*

1.5.2 Extension of Results

In order to compare solutions of finite–dimensional equations with scalar equations to which results in Section 1.3 can be applied, we make an additional hypotheses on ϕ.

$\varphi : [0,\infty) \to [0,\infty)$ is locally Lipschitz continuous where

$$\langle x, \phi(x)\rangle \geq \varphi(||x||) \text{ for all } x \in \mathbb{R}^d \setminus \{0\}, \quad \varphi(0) = 0, \quad \varphi(x) > 0 \text{ for all } x > 0. \tag{1.5.6}$$

Under (1.5.3), we observe by Lemma 1.5.1 that the function φ_0 introduced in (1.5.4) can play the role of φ in (1.5.6). Our comparison theorem is now stated.

Theorem 1.5.1. *Suppose that ϕ obeys* (1.5.3) *and* (1.5.6)*, and that γ is a continuous function. Let x be the unique continuous solution of* (1.5.1)*. Let $\epsilon > 0, \eta > 0$ and suppose that $x_{\epsilon,\eta}$ is the unique continuous solution of*

$$x'_{\eta,\epsilon}(t) = -\varphi(x_{\eta,\epsilon}(t)) + \|\gamma(t)\| + \frac{\epsilon}{2}e^{-t}, \quad t > 0; \quad x_{\eta,\epsilon}(0) = \|x(0)\| + \frac{\eta}{2}. \tag{1.5.7}$$

Then for every $\epsilon > 0, \eta > 0$, $\|x(t)\| \leq x_{\eta,\epsilon}(t)$ for all $t \geq 0$.

The proof is deferred to the end.

Scalar equations

We now consider the ramifications of Theorem 1.5.1 for scalar differential equations. Notice first that the function φ_0 introduced in (1.5.4) is very easily computed. Due to (1.5.5), we have that

$$\varphi_0(x) = \inf_{\|u\|=1} u\phi(xu) = \min_{u=\pm 1} u\phi(xu) = \min(\phi(x), -\phi(-x)). \tag{1.5.8}$$

We restate the hypothesis (1.5.3) for ϕ in scalar form:

$$\phi : \mathbb{R} \to \mathbb{R} \text{ is locally Lipschitz continuous, } x\phi(x) > 0 \text{ for } x \neq 0,\ \phi(0) = 0. \tag{1.5.9}$$

The following results are then direct corollaries of results in Section 1.3 and Theorem 1.5.1.

Theorem 1.5.2. *Suppose that ϕ obeys* (1.5.9) *and γ is continuous and in $L^1(0,\infty)$. Then the unique continuous solution x of* (1.5.1) *obeys $x(t) \to 0$ as $t \to \infty$.*

Proof. Let $\epsilon > 0$. Define $g(t) = |\gamma(t)| + \epsilon e^{-t}/2$ for $t \geq 0$. Then by hypothesis, g is continuous and positive on $[0, \infty)$, and $g \in L^1(0, \infty)$. By (1.5.9) and Lemma 1.5.1, the function φ_0 defined in (1.5.8) is locally Lipschitz continuous and obeys $\varphi_0(0) = 0$ and $\varphi_0(x) > 0$ for $x > 0$. Therefore for any $\epsilon > 0$ and $\eta > 0$, we may apply Theorem 1.3.1 to the solution $x_{\eta,\epsilon}$ of (1.5.7) and conclude that $x_{\eta,\epsilon}(t) \to 0$ as $t \to \infty$. By Theorem 1.5.1 we have that $x(t) \to 0$ as $t \to \infty$. □

Theorem 1.5.3. *Suppose that* ϕ *obeys* (1.5.9) *and* γ *is continuous and* $\gamma(t) \to 0$ *as* $t \to \infty$. *Then for every* $\epsilon > 0$ *sufficiently small there exists a number* $x_1(\epsilon) > 0$ *such that* $|\gamma(t)| \leq \epsilon/2$ *for all* $t \geq 0$ *and* $|\xi| < x_1(\epsilon)/2$ *implies that the unique continuous solution* x *of* (1.5.1) *obeys* $x(t, \xi) \to 0$ *as* $t \to \infty$.

Proof. Let $\epsilon > 0$. Define $g(t) = |\gamma(t)| + \epsilon e^{-t}/2$ for $t \geq 0$. Then by hypothesis, g is continuous and positive on $[0, \infty)$, obeys $g(t) \to 0$ as $t \to \infty$, and also $g(t) < \epsilon$ for all $t \geq 0$. By (1.5.9) and Lemma 1.5.1, the function φ_0 defined in (1.5.8) is locally Lipschitz continuous and obeys $\varphi_0(0) = 0$ and $\varphi_0(x) > 0$ for $x > 0$. There exists $\epsilon_0 > 0$ sufficiently small so that the set $\inf\{x > 0 : \varphi_0(x) = 2\epsilon_0\}$ is non–empty. For $\epsilon \in (0, \epsilon_0)$ define $x_1(\epsilon) = \inf\{x > 0 : \varphi_0(x) = 2\epsilon\}$. Then $\varphi_0(x) < 2\epsilon$ for all $x \in [0, x_1(\epsilon))$. Fix $\eta(\epsilon) = x_1(\epsilon) > 0$. Since $|\xi| < x_1(\epsilon)/2$, we have that $|x_{\eta(\epsilon),\epsilon}(0)| = |x(0)| + \eta(\epsilon)/2 < x_1(\epsilon)$. Suppose there is a finite $T_1(\epsilon) = \inf\{t > 0 : x_{\eta(\epsilon),\epsilon}(t) = x_1(\epsilon)\}$. Then $x'_{\eta(\epsilon),\epsilon}(T_1(\epsilon)) \geq 0$. Also

$$0 \leq x'_{\eta(\epsilon),\epsilon}(T_1(\epsilon)) = -\varphi_0(x_{\eta(\epsilon),\epsilon}(T_1(\epsilon))) + g(T_1(\epsilon)) \leq -\varphi_0(x_1(\epsilon)) + \epsilon = -\epsilon < 0,$$

a contradiction. Hence we have that $x_{\eta(\epsilon),\epsilon}(t) < x_1(\epsilon)$ for all $t \geq 0$. Now by Lemma 1.8.1 it follows that $x_{\eta(\epsilon),\epsilon}(t) \to 0$ as $t \to \infty$. Therefore, by Theorem 1.5.1, we have that $|x(t)| < x_1(\epsilon)$ for all $t \geq 0$ and that $x(t) \to 0$ as $t \to \infty$. □

Theorem 1.5.4. *Suppose that* ϕ *obeys* (1.5.9) *and* γ *is continuous and obeys* $\gamma(t) \to 0$ *as* $t \to \infty$. *Suppose also that* φ_0 *given by* (1.5.8) *is decreasing on* (x^*, ∞) *for some* $x^* > 0$. *If* Φ_0 *is defined by*

$$\Phi_0(x) = \int_1^x \frac{1}{\varphi_0(u)}\, du,$$

and

$$\limsup_{t\to\infty} \frac{\int_0^t |\gamma(s)|\, ds}{\Phi_0^{-1}(t)} < 1$$

then the unique continuous solution x *of* (1.5.1) *obeys* $x(t) \to 0$ *as* $t \to \infty$.

Proof. Let $\epsilon > 0$. Define $g(t) = |\gamma(t)| + \epsilon e^{-t}/2$ for $t \geq 0$. Then by hypothesis, g is continuous and positive on $[0, \infty)$, obeys $g(t) \to 0$ as $t \to \infty$, and also $g(t) < \epsilon$ for all $t \geq 0$. By (1.5.9) and Lemma 1.5.1, the function φ_0 defined in (1.5.8) is locally Lipschitz continuous and obeys $\varphi_0(0) = 0$ and $\varphi_0(x) > 0$ for $x > 0$. Therefore for every $\epsilon > 0$ and $\eta > 0$ the equation (1.5.7) is of the form of (1.2.1) with φ_0 in the role of f and Φ_0 in the role of F. Notice that the monotonicity of φ_0 implies that $\Phi_0(x) \to \infty$ as $x \to \infty$, and therefore that $\Phi_0^{-1}(x) \to \infty$ as $x \to \infty$. Therefore by hypothesis, we have

$$\limsup_{t\to\infty} \frac{\int_0^t g(s)\, ds}{\Phi_0^{-1}(t)} = \limsup_{t\to\infty} \frac{\int_0^t |\gamma(s)|\, ds}{\Phi_0^{-1}(t)} + \frac{\int_0^t \epsilon e^{-s}\, ds}{\Phi_0^{-1}(t)} = \limsup_{t\to\infty} \frac{\int_0^t |\gamma(s)|\, ds}{\Phi_0^{-1}(t)} < 1.$$

Therefore, by Theorem 1.3.4 we have that $x_{\eta,\epsilon}(t) \to 0$ as $t \to \infty$, and hence by Theorem 1.5.1, it follows that $x(t) \to 0$ as $t \to \infty$. □

A result analogous to Theorem 1.3.10 can be formulated even when γ changes sign. We state the result but do not provide a proof.

Theorem 1.5.5. *Suppose that* ϕ *obeys* (1.5.9) *and* γ *is continuous and obeys* $\gamma(t) \to 0$ *as* $t \to \infty$. *Suppose also that* φ_0 *given by* (1.5.8) *is in* $RV_\infty(-\beta)$ *for* $\beta > 0$. *If* Φ_0 *is defined by*

$$\Phi_0(x) = \int_1^x \frac{1}{\varphi_0(u)}\, du,$$

and

$$\limsup_{t\to\infty} \frac{\int_0^t |\gamma(s)|\,ds}{\Phi_0^{-1}(t)} < \lambda(\beta) = \beta^{1/(\beta+1)}(1+\beta^{-1}),$$

then the unique continuous solution x of (1.5.1) *obeys $x(t) \to 0$ as $t \to \infty$.*

Finite–dimensional results

In this section, we often request that the function φ introduced in (1.5.6) obeys a monotonicity restriction.

$$x \mapsto \varphi(x) \text{ is decreasing on } (x^*, \infty) \text{ for some } x^* > 0. \tag{1.5.10}$$

Results analogous to Theorems 1.5.2, 1.5.3, 1.5.4 and 1.5.5 can be stated for finite–dimensional systems. The proofs are very similar to those of the corresponding scalar results, and are therefore omitted.

Theorem 1.5.6. *Suppose that ϕ obeys* (1.5.3) *and γ is continuous and in $L^1(0,\infty)$. Then the unique continuous solution x of* (1.5.1) *obeys $x(t) \to 0$ as $t \to \infty$.*

Theorem 1.5.7. *Suppose that ϕ obeys* (1.5.3) *and that γ is continuous and $\gamma(t) \to 0$ as $t \to \infty$. Then for every $\epsilon > 0$ sufficiently small there exists a number $x_1(\epsilon) > 0$ such that $\|\gamma(t)\| \le \epsilon/2$ for all $t \ge 0$ and $\|\xi\| < x_1(\epsilon)/2$ implies that the unique continuous solution x of* (1.5.1) *obeys $x(t,\xi) \to 0$ as $t \to \infty$.*

Theorem 1.5.8. *Suppose that ϕ obeys* (1.5.3) *and that ϕ and φ obey* (1.5.6) *and* (1.5.10)*. Suppose that γ is continuous and that $\gamma(t) \to 0$ as $t \to \infty$. If Φ is defined by*

$$\Phi(x) = \int_1^x \frac{1}{\varphi(u)}\,du,$$

and

$$\limsup_{t\to\infty} \frac{\int_0^t \|\gamma(s)\|\,ds}{\Phi^{-1}(t)} < 1$$

then the unique continuous solution x of (1.5.1) *obeys $x(t) \to 0$ as $t \to \infty$.*

Theorem 1.5.9. *Suppose that ϕ obeys* (1.5.3) *and that ϕ and φ obey* (1.5.6)*. Suppose also that φ is in $RV_\infty(-\beta)$ for $\beta > 0$. Suppose that γ is continuous and that $\gamma(t) \to 0$ as $t \to \infty$. If Φ is defined by*

$$\Phi(x) = \int_1^x \frac{1}{\varphi(u)}\,du,$$

and

$$\limsup_{t\to\infty} \frac{\int_0^t \|\gamma(s)\|\,ds}{\Phi^{-1}(t)} < \beta^{1/(\beta+1)}(1+\beta^{-1}),$$

then the unique continuous solution x of (1.5.1) *obeys $x(t) \to 0$ as $t \to \infty$.*

1.6 A Liapunov Result

The main result of this section shows that if f has a certain rate of decay to zero, and g decays more rapidly than a certain rate which depends on f, then solutions of (1.2.1) can be shown to tend to 0 as $t \to \infty$ by means of a Liapunov–like technique. The results are not as sharp as those obtained in Section 3, and do not have anything to say about instability, but nonetheless the conditions do seem to identify, albeit crudely, the critical rate for g at which global stability is lost.

The conditions of the theorem appear forbidding in general, and the reader may doubt it is possible to construct auxiliary functions with the desired properties. However, by considering examples in which f decays either polynomially or exponentially, we demonstrate that the result can be applied in practice, and that the claims made above regarding the sharpness of the result are not unjustified.

Theorem 1.6.1. *Suppose that f obeys* (1.2.2) *and* (1.2.4) *and that $g \in C([0,\infty);(0,\infty))$ and $g(t) \to 0$ as $t \to \infty$. Let $\Theta \in C([0,\infty);[0,\infty))$ be a twice differentiable and increasing function such that $\Theta(0) = 0$. Define $\psi(x) = x\Theta^{-1}(x)$ for $x > 0$ and $\psi(0) = 0$, and suppose that ψ is an increasing and convex function on $(0,\infty)$ with $\lim_{x\to 0^+} x\psi'(x) = 0$. Define also $\theta : [0,\infty) \to [0,\infty)$ by*

$$\theta(x) = x(\psi')^{-1}(x) - (\psi \circ (\psi')^{-1})(x), \quad x > 0; \quad \theta(0) = 0.$$

Suppose that $\Theta \circ f \notin L^1(0,\infty)$ *and that* $\theta \circ g \in L^1(0,\infty)$*. Then the unique continuous solution* x *of* (1.2.1) *obeys* $x(t) \to 0$ *as* $t \to \infty$*.*

Proof. Since Θ is increasing, ψ is a well–defined function. Moreover, as Θ is twice differentiable, it follows that Θ^{-1} is twice differentiable, and therefore we have that $x \mapsto \psi'(x)$ is a continuous function and that $\psi''(x)$ is well–defined for all $x > 0$. In fact, by the assumption that ψ is increasing and convex, we have that $\psi'(x) > 0$ and that $\psi''(x) > 0$ for all $x > 0$. Let $\Psi : [0,\infty) \to \mathbb{R}$ be defined by $\Psi(x) = \psi'(x)$ for $x > 0$ and $\Psi(0) = 0$. Then Ψ is an increasing and continuous function on $[0,\infty)$ with $\Psi(0) = 0$. Therefore, by Young's inequality, for every $a, b > 0$ we have

$$ab \le \int_0^a \Psi(s)\,ds + \int_0^b \Psi^{-1}(s)\,ds = \psi(a) + H(b), \tag{1.6.1}$$

using the fact that ψ is continuous from the left at zero with $\psi(0) = 0$, and the definition

$$H(x) = \int_0^x \Psi^{-1}(s)\,ds, \quad x \ge 0. \tag{1.6.2}$$

Now for $x > 0$, using the fact that ψ is twice differentiable, and that $\psi'(0+) = 0$, we have

$$H(x) = \int_0^x \Psi^{-1}(s)\,ds = \int_0^x (\psi')^{-1}(s)\,ds = \int_{0+}^{(\psi')^{-1}(x)} w\psi''(u)\,dw.$$

Now, by integration by parts, and the definition of θ, we have

$$\begin{aligned} H(x) &= \int_{0+}^{(\psi')^{-1}(x)} w\psi''(w)\,dw \\ &= (\psi')^{-1}(x)\psi'((\psi')^{-1}(x)) - \lim_{w\to 0^+} w\psi'(w) - \int_{0+}^{(\psi')^{-1}(x)} \psi'(w)\,dw \\ &= (\psi')^{-1}(x)\psi'((\psi')^{-1}(x)) - \lim_{w\to 0^+} w\psi'(w) - \psi((\psi')^{-1}(x)) - \lim_{w\to 0^+} \psi(w) \\ &= \theta(x), \end{aligned}$$

since $\psi(w) \to 0$ as $w \to 0^+$ and $w\psi'(w) \to 0$ as $w \to 0^+$ by hypothesis. Therefore by (1.6.1) and the fact that $\psi(a) = a\Theta^{-1}(a)$ for $a > 0$, we have

$$ab \le a\Theta^{-1}(a) + \theta(b), \qquad \text{for all } a, b > 0. \tag{1.6.3}$$

We notice also that the definition of H forces $\theta(x) = H(x) > 0$ for all $x > 0$, and since Ψ^{-1} is a positive and increasing function, it follows that θ will be increasing and convex on $(0, \infty)$.

Now, define

$$I(x) = \int_0^x \Theta(f(s))\, ds, \quad x \geq 0 \tag{1.6.4}$$

Notice that $I(x) > 0$ for $x > 0$ because $\Theta(x) > 0$ and $f(x) > 0$ for $x > 0$. Also, $\Theta \circ f \notin L^1(0, \infty)$ is equivalent to $I(x) \to \infty$ as $x \to \infty$. Define also

$$V(t) = I(x(t)), \quad t \geq 0. \tag{1.6.5}$$

Since $\Theta \circ f$ is continuous on $[0, \infty)$ and the solution x of (1.2.1) is in $C^1(0, \infty)$, it follows that $V \in C^1(0, \infty)$ and moreover

$$V'(t) = \Theta(f(x(t)))x'(t) = -f(x(t))\Theta(f(x(t))) + g(t)\Theta(f(x(t))), \quad t > 0. \tag{1.6.6}$$

By hypothesis, $g(t) > 0$ for all $t \geq 0$. Also, it is a consequence of our hypotheses that $x(t) > 0$ for all $t > 0$, and so by (1.2.2) that $f(x(t)) > 0$ for all $t \geq 0$. Since $\Theta(0) = 0$ and Θ is increasing on $(0, \infty)$ by hypothesis, it follows that $\Theta(f(x(t))) > 0$ for all $t \geq 0$. Therefore we can apply (1.6.3) with $b := g(t) > 0$ and $a = \Theta(f(x(t))) > 0$ to get

$$\begin{aligned}\Theta(f(x(t)))g(t) &\leq \Theta(f(x(t)))\Theta^{-1}(\Theta(f(x(t)))) + \theta(g(t)) \\ &= f(x(t))\Theta(f(x(t))) + (\theta \circ g)(t), \quad t \geq 0.\end{aligned}$$

Inserting this estimate into (1.6.6) we get

$$V'(t) = -f(x(t))\Theta(f(x(t))) + g(t)\Theta(f(x(t))) \leq (\theta \circ g)(t), \quad t > 0.$$

Therefore by (1.6.4) and (1.6.5) we get

$$I(x(t)) = V(t) = V(0) + \int_0^t V'(s)\, ds \leq V(0) + \int_0^t (\theta \circ g)(s)\, ds = I(\xi) + \int_0^t (\theta \circ g)(s)\, ds$$

for all $t \geq 0$. Since $\theta \circ g \in L^1(0,\infty)$ by hypothesis, we have that there is a finite $K > 0$ such that

$$I(x(t)) \leq I(\xi) + \int_0^\infty (\theta \circ g)(s)\, ds =: K, \quad t \geq 0.$$

The positivity of K is guaranteed by the fact that $I(x) > 0$ for $x > 0$, and the fact that $\theta(x) > 0$ for $x > 0$ and $g(t) > 0$ for $t > 0$. Suppose now that $\limsup_{t\to\infty} x(t) = +\infty$, so by the continuity of $t \mapsto x(t)$, there is an increasing sequence of times $t_n \to \infty$ such that $x(t_n) = n$. Then $I(n) \leq K$ for all $n \in \mathbb{N}$ sufficiently large. Since $I(n) \to +\infty$ as $n \to \infty$, we have $\infty = \lim_{n\to\infty} I(n) \leq K < +\infty$, a contradiction. Therefore, it follows that $\limsup_{t\to\infty} x(t)$ is finite and non–negative. Therefore by (1.8.3), we have that $x(t) \to 0$ as $t \to \infty$, as required. □

The next result is a corollary of Theorem 1.6.1 which is of utility when $f(x)$ decays like a power of x for large x. In this case, we know from our earlier analysis that g must also exhibit a power law decay. Our Liapunov–like result also reflects this fact.

Corollary 1.6.1. *Suppose that f obeys* (1.2.2) *and* (1.2.4), *and $g \in C([0,\infty);(0,\infty))$ satisfies $g(t) \to 0$ as $t \to \infty$. Suppose that there is $\alpha > 0$ such that $f^\alpha \notin L^1(0,\infty)$ and $g^{1+\alpha} \in L^1(0,\infty)$. Then x, the unique continuous solution of* (1.2.1), *obeys $x(t) \to 0$ as $t \to \infty$.*

Proof. Suppose for all $x \geq 0$ that $\Theta(x) = x^\alpha$, where $\alpha > 0$. Then Θ is increasing on $(0,\infty)$ with $\Theta^{-1}(x) = x^{1/\alpha}$ for $x \geq 0$. Moreover, we have that Θ is in $C^2(0,\infty)$. Now, define $\psi(x) = x^{1+1/\alpha}$ for $x \geq 0$. Then $\psi(0) = 0$, $\psi'(x) = (1+1/\alpha)x^{1/\alpha} > 0$ for $x > 0$ and $\psi''(x) = \alpha^{-1}(1+\alpha^{-1})x^{1/\alpha-1} > 0$ for $x > 0$. Thus ψ is increasing and convex with $\lim_{x\to 0^+} x\psi'(x) = 0$. With $\psi'(x) = \Psi(x) = (1+1/\alpha)x^{1/\alpha}$ for $x > 0$, and $\Psi(0) = 0$, we have $\Psi^{-1}(x) = K_\alpha x^\alpha$ for $x \geq 0$, where $K_\alpha = 1/(1+\alpha^{-1})^\alpha > 0$. Therefore for $x \geq 0$, we have that $\theta(x) = \int_0^x \Psi^{-1}(s)\, ds = K_\alpha(1+\alpha)^{-1}x^{1+\alpha}$. Thus $g^{1+\alpha} \in L^1(0,\infty)$ implies that $\theta \circ g \in L^1(0,\infty)$. Moreover $\Theta \circ f = f^\alpha \notin L^1(0,\infty)$. Therefore, all the hypotheses of Theorem 1.6.1 are satisfied, and so $x(t) \to 0$ as $t \to \infty$, as claimed. □

An example illustrates the close connection between Corollary 1.6.1 and Theorem 1.3.10. In fact we see that the results are consistent in many cases.

Example 1.6.1. Suppose that there is $\beta > 0$ such that $f(x) \sim x^{-\beta}$ as $x \to \infty$ and that $g^{1+1/\beta} \in L^1(0,\infty)$. Let $\alpha = 1/\beta > 0$. Then $f^\alpha(x) \sim x^{-1}$ as $x \to \infty$, and thus $f^\alpha \notin L^1(0,\infty)$ and $g^{1+\alpha} \in L^1(0,\infty)$. Thus, by Corollary 1.6.1, we have that $x(t) \to 0$ as $t \to \infty$.

A condition that implies $g^{1+1/\beta} \in L^1(0,\infty)$ but $g \notin L^1(0,\infty)$ is $g(t) \sim t^{-\eta}$ as $t \to \infty$ for $\eta \in (\beta/(\beta+1), 1)$. Then

$$\int_0^t g(s)\,ds \sim \frac{1}{\eta} t^{1-\eta}, \text{ as } t \to \infty$$

while

$$F(x) = \int_1^x 1/f(u)\,du \sim \int_1^x u^\beta\,du = \frac{1}{1+\beta} x^{1+\beta}, \text{ as } x \to \infty.$$

Therefore $F^{-1}(x) = C_\beta x^{1/(\beta+1)}$ as $x \to \infty$. Hence

$$\lim_{t\to\infty} \frac{\int_0^t g(s)\,ds}{F^{-1}(t)} = \frac{1}{C_\beta \eta} \lim_{t\to\infty} \frac{t^{1-\eta}}{t^{1/(\beta+1)}} = 0.$$

By Theorem 1.3.10, we have that $x(t) \to 0$ as $t \to \infty$.

Therefore if $f(x) \sim x^{-\beta}$ for some $\beta > 0$ and $g(t) \sim t^{-\eta}$ as $t \to \infty$ for $\eta > \beta/(\beta+1)$, both Theorem 1.3.10 and Corollary 1.6.1 imply that $x(t) \to 0$ as $t \to \infty$. If $\eta > \beta/(\beta+1)$, we have that

$$\lim_{t\to\infty} \frac{\int_0^t g(s)\,ds}{F^{-1}(t)} = \frac{1}{C_\beta \eta} \lim_{t\to\infty} \frac{t^{1-\eta}}{t^{1/(\beta+1)}} = +\infty,$$

and so we know from Theorem 1.3.13 that $x(t,\xi) \to \infty$ as $t \to \infty$ for all initial conditions $\xi > 0$ that are sufficiently large. On the other hand, we see that the conditions of Corollary 1.6.1 do not hold if $\eta > \beta/(\beta+1)$, because $g^{1+1/\beta}(t) \sim t^{-\eta(\beta+1)/\beta}$ as $t \to \infty$, and so $g^{1+1/\beta} \notin L^1(0,\infty)$. Therefore, the conditions of Corollary 1.6.1 are quite sharp.

One reason to use the general form of Young's inequality in the proof of Theorem 1.6.1 is to enable us to prove stability results for differential equations in which g and f do not have power law asymptotic behaviour. The following example shows how Theorem 1.6.1 can be used in this situation.

Example 1.6.2. Suppose that $f(x) = e^{-x}$ for $x \geq 1$ and that $f(x) = xe^{-1}$ for $x \in [0, 1]$. Suppose that $g/\log(1/g) \in L^1(0, \infty)$. Let Θ be such that $\Theta(0) = 0$, $\Theta(y) = 1/\log(1/y)$ for $0 < y \leq 1/e$.

If we now suppose that we can extend Θ on $[1/e, \infty)$ so that Θ is twice differentiable and increasing on $[1/e, \infty)$ and $y \mapsto y\Theta^{-1}(y)$ is convex on $(1, \infty)$, Theorem 1.6.1 allows us to conclude that $x(t) \to 0$ as $t \to \infty$.

Notice that $\Theta^{-1}(y) = e^{-1/y}$ for $0 < y \leq 1$. Therefore for $y > 0$, we may define $\psi(y) = y\Theta^{-1}(y)$ with $\psi(0) = 0$. Since Θ is increasing, Θ^{-1} is increasing, and so ψ is increasing, and by hypothesis, ψ is convex on $[1, \infty)$.

In particular, for $y \in (0, 1]$ we have $\psi(y) = ye^{-1/y}$. Then $\psi'(y) = (1 + y^{-1})e^{-1/y} > 0$ for $y \in (0, 1)$ and

$$\psi''(y) = (1 + y^{-1})e^{-1/y} \cdot \frac{1}{y^2} - \frac{1}{y^2}e^{-1/y} = \frac{1}{y^3}e^{-1/y} > 0$$

for $y \in (0, 1)$. Therefore ψ is increasing and convex on $(0, \infty)$. Also, we have the limit $\lim_{y \to 0^+} y\psi'(y) = 0$. Now for x sufficiently small

$$\theta(x) = \int_0^{(\psi')^{-1}(x)} y\psi''(y)\, dy = \int_0^{(\psi')^{-1}(x)} \frac{1}{y^2}e^{-1/y}\, dy = \int_{1/(\psi')^{-1}(x)}^{\infty} e^{-u}\, du,$$

so $\theta(x) = e^{-1/(\psi')^{-1}(x)}$ for $x > 0$ sufficiently small. Now, using the formula for ψ', we have for $x > 0$ sufficiently small

$$x = \left(1 + \frac{1}{(\psi')^{-1}(x)}\right) e^{-1/(\psi')^{-1}(x)}.$$

Therefore we have $\log 1/x \sim 1/(\psi')^{-1}(x)$ as $x \to 0^+$, from which the limit

$$\lim_{x \to 0^+} \frac{\theta(x)}{x/\log(1/x)} = \lim_{x \to 0^+} \frac{e^{-1/(\psi')^{-1}(x)}}{x/\log(1/x)} = \lim_{x \to 0^+} \frac{x/\left(1 + \frac{1}{(\psi')^{-1}(x)}\right)}{x/\log(1/x)} = 1$$

can be inferred. Since $g(t) \to 0$ as $t \to \infty$ and $g/\log(1/g) \in L^1(0,\infty)$, we have that $\theta \circ g \in L^1(0,\infty)$. Also, because $\Theta(f(x)) = 1/x$ for $x \geq 1$ we have that $\Theta \circ f \notin L^1(0,\infty)$. Therefore all the hypotheses of Theorem 1.6.1 hold, and we conclude that $x(t) \to 0$ as $t \to \infty$.

In the case when $f(x) = e^{-x}$ for $x \geq 1$ and $g(t) \sim Ct^{-\eta}$ as $t \to \infty$ for any $\eta > 1$ and $C > 0$ we have that $g(t)/\log(1/g(t)) \sim t^{-\eta}/\log t$ as $t \to \infty$, and so $g/\log(1/g) \in L^1(0,\infty)$ and $\Theta \circ f \notin L^1(0,\infty)$. Therefore, by Theorem 1.6.1 we have that $x(t) \to 0$ as $t \to \infty$. If $\eta \leq 1$, then $g/\log(1/g) \notin L^1(0,\infty)$, and so the argument above does not apply.

On the other hand, we have for $x \geq 1$ that $F(x) = \int_1^x e^u\, du = e^x - e$, and so $F^{-1}(x) \sim \log(x)$ as $x \to \infty$. Then for $\eta > 1$, $g \in L^1(0,\infty)$, and so

$$\lim_{t\to\infty} \frac{\int_0^t g(s)\, ds}{F^{-1}(t)} = 0.$$

Therefore, by Theorem 1.3.4, we have that $x(t) \to 0$ as $t \to \infty$. If $\eta = 1$, we have that $\int_0^t g(s)\, ds \to C \log t$ as $t \to \infty$ and so

$$\lim_{t\to\infty} \frac{\int_0^t g(s)\, ds}{F^{-1}(t)} = C.$$

If $C < 1$, then $x(t) \to 0$ as $t \to \infty$; if $C > 1$ we have that $x(t) \to \infty$ for all initial conditions sufficiently large. If $\eta < 1$, we have that $\int_0^t g(s)\, ds$ grows polynomially fast as $t \to \infty$, and therefore

$$\lim_{t\to\infty} \frac{\int_0^t g(s)\, ds}{F^{-1}(t)} = +\infty.$$

Therefore, for all initial conditions sufficiently large, we have $x(t) \to \infty$ as $t \to \infty$.

This discussion once again shows how the results from Section 3 are consistent with the Liapunov stability result Theorem 1.6.1, and that moreover, Theorem 1.6.1 is quite sharp. The sharp results from Section 3 show that global asymptotic convergence holds for all $\eta > 1$, but that for $\eta \leq 1$, we can have $x(t) \to \infty$ for some initial conditions. On the other hand, Theorem 1.6.1 guarantees the global convergence of solutions for $\eta > 1$, but does not apply if $\eta \leq 1$.

1.7 Proof of Theorem 1.3.1

For all $t \geq 0$, $x(t) = \xi - \int_0^t f(x(s))ds + \int_0^t g(s)ds \leq \xi + \int_0^\infty g(s)ds := K$. Suppose $\liminf_{t\to\infty} x(t) = x^* > 0$. Clearly $x^* \leq K$. Now, as $f(x) > 0$ for $x > 0$

$$\inf_{x\in[\frac{x^*}{2},K]} f(x) := \phi > 0.$$

Therefore there exists $T > 0$ such that $x(t) \geq x^*/2$ for all $t \geq T$. Thus $x^*/2 \leq x(t) \leq K$ for all $t \geq T$ and so $f(x(t)) \geq \phi$ for all $t \geq T$. Therefore as $g \in L^1(0,\infty)$, for $t \geq T$ we have

$$\begin{aligned} x(t) &= x(T) - \int_T^t f(x(s))ds + \int_T^t g(s)ds \\ &\leq x(T) - \phi(t-T) + \int_T^\infty g(s)ds. \end{aligned}$$

Thus, as $\phi > 0$, we have $\liminf_{t\to\infty} x(t) = -\infty$, a contradiction. Therefore

$$\liminf_{t\to\infty} x(t) = 0 \tag{1.7.1}$$

Since $g \in L^1(0,\infty)$, for every $\epsilon > 0$, there is $T_1(\epsilon) > 0$ such that

$$\int_t^\infty g(s)ds < \epsilon \quad \text{for all } t > T_1(\epsilon).$$

(1.7.1) implies that there exists $t_n \to \infty$ such that $\lim_{n\to\infty} x(t_n) = 0$. Thus for every $\epsilon > 0$ there exists an $N_1(\epsilon) \in \mathbb{N}$ such that $x(t_n) \leq \epsilon$ for all $n \geq N_1(\epsilon)$. Clearly there exists $N_2(\epsilon)$ such that $t_{N_2(\epsilon)} \geq T_1(\epsilon) + 1$. Let $N_3(\epsilon) = \max[N_1(\epsilon), N_2(\epsilon)]$. Then $t_{N_3(\epsilon)} > T_1(\epsilon)$ and as $N_3(\epsilon) \geq N_1(\epsilon)$, $x(t_{N_3(\epsilon)}) \leq \epsilon$. Let $T_2(\epsilon) = t_{N_3(\epsilon)}$. Then for $t \geq T_2(\epsilon)$, we have

$$\begin{aligned} x(t) &= x(t_{N_3(\epsilon)}) - \int_{t_{N_3(\epsilon)}}^t f(x(s))ds + \int_{t_{N_3(\epsilon)}}^t g(s)\,ds \\ &\leq \epsilon + \int_{t_{N_3(\epsilon)}}^t g(s)\,ds \leq \epsilon + \int_{t_{N_3(\epsilon)}}^\infty g(s)\,ds < 2\epsilon. \end{aligned}$$

Thus for every $\epsilon > 0$, there is a $T_2(\epsilon) > 0$ such that $x(t) < 2\epsilon$ for all $t \geq T_2(\epsilon)$. Hence $x(t) \to 0$ as $t \to \infty$.

1.8 Finite liminf implies zero limit and Proof of Theorem 1.3.2

In this section, we show that whenever x had a finite liminf, it must have a zero limit.

Lemma 1.8.1. *Suppose that g obeys* (1.2.3), (1.2.8)*, and g is non–negative. Suppose that f obeys* (1.2.2) *and that the solution x of* (1.2.1) *obeys*

$$\liminf_{t\to\infty} x(t) \le x^* \tag{1.8.1}$$

for some $x^* > 0$. Then x obeys (1.2.6).

A consequence of Lemma 1.8.1 is that only two types of behaviour are possible for solutions of (1.2.1). Either solutions tend to zero, or they tend to infinity. This is nothing other than Theorem 1.3.2.

Proof of Theorem 1.3.2. Suppose that there exists $x^* > 0$ such that x obeys (1.8.1). Then by Lemma 1.8.1 it follows that $x(t) \to 0$ as $t \to \infty$. On the other hand, if there does not exist $x^* > 0$ such that $\liminf_{t\to\infty} x(t) \le x^*$, it follows that $\liminf_{t\to\infty} x(t) = +\infty$, which implies $x(t) \to \infty$ as $t \to \infty$. □

It remains to establish Lemma 1.8.1. In order to do so, we start by proving that (1.8.1) implies that x is bounded above.

Lemma 1.8.2. *Suppose that g obeys* (1.2.8)*, f obeys* (1.2.2) *and that the solution x of* (1.2.1) *obeys* (1.8.1)*. Then*

$$\limsup_{t\to\infty} x(t) \le 2x^*.$$

Proof. Suppose that $\limsup_{t\to\infty} x(t) > 2x^*$. Since f obeys (1.2.2), we may define $f^* = \min_{x\in[5x^*/4,3x^*/2]} f(x) > 0$. Let $\epsilon < f^*/2$. Since $g(t) \to 0$ as $t \to \infty$, there is $T_1(\epsilon) > 0$ such that $g(t) \le \epsilon$ for all $t \ge T_1(\epsilon)$. Let $T_2(\epsilon) = \inf\{t > T_1(\epsilon) : x(t) = 5x^*/4\}$ and

$T_3(\epsilon) = \inf\{t > T_2(\epsilon) : x(t) = 3x^*/2\}$. Then $x'(T_3) \geq 0$. Since $T_3 > T_2 > T_1$ we have

$$0 \leq x'(T_3) = -f(x(T_3)) + g(T_3) = -f(3x^*/2) + g(T_3) \leq -f^* + \epsilon < -f^* + f^*/2 < 0,$$

a contradiction. □

We next show that x has a zero liminf.

Lemma 1.8.3. *Suppose that g obeys* (1.2.8)*, f obeys* (1.2.2) *and that the solution x of* (1.2.1) *obeys* (1.8.1)*. Then*

$$\liminf_{t\to\infty} x(t) = 0.$$

Proof. Suppose that $\liminf_{t\to\infty} x(t) = c > 0$. By Lemma 1.8.2 it follows also that $c \leq \limsup_{t\to\infty} x(t) \leq 2x^*$. Therefore there exists $T_1 > 0$ such that $0 < c/2 \leq x(t) \leq 4x^*$ for all $t \geq T_1$. Define $c_1 = \min_{x\in[c/2,4x^*]} f(x) > 0$. Then $f(x(t)) \geq c_1$ for all $t \geq T_1$. Since $g(t) \to 0$ as $t \to \infty$, it follows that there exists $T_2 > 0$ such that that $g(t) \leq c_1/2$ for all $t \geq T_2$. Let $T_3 = \max(T_1, T_2)$. Then for all $t \geq T_3$ we have

$$x'(t) = -f(x(t)) + g(t) \leq -c_1 + \frac{c_1}{2} = -\frac{c_1}{2}.$$

Therefore we have that $x(t) \to -\infty$ as $t \to \infty$, which contradicts the fact that $x(t) \geq 0$ for all $t \geq 0$. □

We are now in a position to prove Lemma 1.8.1.

Proof of Lemma 1.8.1. By Lemma 1.8.2 we have that $\limsup_{t\to\infty} x(t) \leq 2x^*$ and by Lemma 1.8.3 we have that $\liminf_{t\to\infty} x(t) = 0$. Therefore we have $x(t) < x^{**}$ for all $t \geq 0$. Suppose that there is $c \in (0, x^{**})$ such that $\limsup_{t\to\infty} x(t) > c$. Fix $\eta \in (0, c)$. Since f obeys (1.2.2) and g obeys (1.2.8) we may define

$$0 < \epsilon_1(\eta) = \min_{x\in[\eta,x^{**}]} f(x),$$

$$T(\eta) = \sup\{t > 0 \,:\, g(t) > \epsilon_1(\eta)/2\}.$$

Define $T_1(\eta) = \inf\{t > T(\eta) : x(t) = \eta\}$. There exists $T^* > T_1(\eta)$ such that $x(t) > c > \eta$. Let $T_2 = \sup\{t < T^* : x(t) = \eta\}$. Then $T_2 \geq T_1$ and there is a $\delta > 0$ such that $x(t) > \eta$ for all $t \in (T_2, T_2 + \delta)$. However, for $t \in (T_2, T_2 + \delta)$ we have

$$\begin{aligned} x(t) &= x(T_2) - \int_{T_2}^{t} f(x(s))\,ds + \int_{T_2}^{t} g(s)\,ds \\ &\leq x(T_2) - \int_{T_2}^{t} \epsilon_1(\eta)\,ds + \int_{T_2}^{t} \frac{\epsilon_1(\eta)}{2}\,ds \\ &= x(T_2) - (t - T_2)\frac{\epsilon_1(\eta)}{2} < x(T_2) = \eta, \end{aligned}$$

which contradicts the definition of T_2. Therefore we have that $\lim_{t\to\infty} x(t) = 0$, as required. □

1.9 Proof of Theorem 1.3.4, 1.3.5, 1.3.10 and 1.3.11

1.9.1 Proof of Theorem 1.3.4

It is seen from Lemma 1.8.1 above that if we can show that there is an $x^* > 0$ such that the solution x of (1.2.1) obeys (1.8.1), then x obeys (1.2.6).

Lemma 1.9.1. *Suppose that f obeys* (1.2.17) *and* (1.2.2) *and that g is continuous. Suppose that F is given by* (1.2.15) *and that f and g obey* (1.3.1)*. Let x be the unique continuous solution of* (1.2.1)*. Then it obeys* (1.8.1)*.*

Proof. Since g and f obey (1.3.1), there exists $\lambda < 1$ such that

$$\limsup_{t\to\infty} \frac{\int_0^t g(s)\,ds}{F^{-1}(t)} = \lambda < 1.$$

Choose $\epsilon \in (0, 2/3)$ so small that $\lambda(1+\epsilon) < 1 - \epsilon/2$. Therefore for every $\epsilon \in (0, 2/3)$ there exists $T(\epsilon) > 0$ such that

$$\frac{\int_0^t g(s)\,ds}{F^{-1}(t)} \leq \lambda(1+\epsilon) < 1 - \epsilon/2, \quad t \geq T(\epsilon).$$

Therefore $\int_0^t g(s)\,ds \le (1-\epsilon/2)F^{-1}(t)$ for all $t \ge T(\epsilon)$. Since f obeys (1.2.2), by defining $x_\epsilon = x(T(\epsilon))$, for all $t \ge T(\epsilon)$ we have

$$\begin{aligned} x(t) &= x_\epsilon - \int_{T(\epsilon)}^t f(x(s))\,ds + \int_{T(\epsilon)}^t g(s)\,ds \\ &\le x_\epsilon + \int_{T(\epsilon)}^t g(s)\,ds \\ &< x_\epsilon + (1-\epsilon/2)F^{-1}(t) := G(t). \end{aligned}$$

Suppose, in contradiction to the desired conclusion, that $\liminf_{t\to\infty} x(t) = x_1 > x^*$. Then there exists $T_2 > 0$ such that for all $t \ge T_2$ we have $x(t) > x^*$. Let $T_3(\epsilon) = \max(T(\epsilon), T_2)$. Then for $x^* < x(t) < G(t)$, so by (1.2.17) we have $f(x(t)) \ge f(G(t))$. Hence for $t \ge T_3(\epsilon)$ we have

$$\begin{aligned} x(t) &= x(T_3) - \int_{T_3}^t f(x(s))\,ds + \int_{T_3}^t g(s)\,ds \\ &< x(T_3) - \int_{T_3}^t f(G(s))\,ds + (1-\epsilon/2)F^{-1}(t) \\ &= x(T_3) + (1-\epsilon/2)F^{-1}(T_3) + \int_{T_3}^t [-f(G(s)) + (1-\epsilon/2)f(F^{-1}(s))]\,ds. \end{aligned}$$

Hence

$$x(t) < x(T_3) + (1-\epsilon/2)F^{-1}(T_3) + \int_{T_3}^t [-f(G(s)) + (1-\epsilon/2)f(F^{-1}(s))]\,ds, \quad t \ge T_3(\epsilon). \tag{1.9.1}$$

We next show that

For every $\epsilon \in (0, 2/3)$ there exists $\theta_3(\epsilon) > 0$ such that

$$(1-\epsilon/2)f(\theta) - f(x_\epsilon + (1-\epsilon/2)\theta) < -\epsilon/4 f(\theta), \quad \text{for all } \theta > \theta_3(\epsilon). \tag{1.9.2}$$

Now define $T_4(\epsilon) = F(\theta_3(\epsilon))$ and let $T_5(\epsilon) = \max(T_3(\epsilon), T_4(\epsilon)) + 1$. Therefore for $t \ge T_5(\epsilon) > T_4(\epsilon) = F(\theta_3(\epsilon))$ we have $F^{-1}(t) > \theta_3(\epsilon)$. Thus by (1.9.2) we have

$$(1-\epsilon/2)f(F^{-1}(t)) - f(G(t)) < -\epsilon/4 f(F^{-1}(t)), \quad \text{for all } t \ge T_5(\epsilon).$$

Since $T_5(\epsilon) > T_3(\epsilon)$, by (1.9.1) we have

$$\begin{aligned} x(t) &< x(T_3) + (1-\epsilon/2)F^{-1}(T_3) + \int_{T_5}^{t} [-f(G(s)) + (1-\epsilon/2)f(F^{-1}(s))]\,ds, \\ &< x(T_3) + (1-\epsilon/2)F^{-1}(T_3) - \epsilon/4\int_{T_5}^{t} f(F^{-1}(s))\,ds, \\ &= x(T_3) + (1-\epsilon/2)F^{-1}(T_3) - \epsilon/4[F^{-1}(t) - F^{-1}(T_5)], \end{aligned}$$

for all $t \geq T_5(\epsilon)$, therefore we have $\lim_{t\to\infty} x(t) = -\infty$. Since $\liminf_{t\to\infty} x(t) = x_1 > x^* > 0$ and $x'(t) < 0$ for all $t \geq T_5(\epsilon)$ it follows that $\lim_{t\to\infty} x(t) = x_1 > x^*$, a contradiction. Hence it follows that $\liminf_{t\to\infty} x(t) \leq x^*$.

It remains to prove (1.9.2). Since x_ϵ is fixed, for every $\epsilon \in (0, 4/3)$ there exists $\theta_1(\epsilon) > 0$ such that $-\epsilon\theta/4 < x_\epsilon < \epsilon\theta/4$ for all $\theta > \theta_1(\epsilon)$. Thus for $\theta > \theta_1(\epsilon)$ we have

$$0 < (1 - \frac{3\epsilon}{4})\theta < x_\epsilon + (1-\epsilon/2)\theta < (1-\epsilon/4)\theta.$$

Also, there exists $\theta_2(\epsilon) > 0$ such that $(1 - 3\epsilon/4)\theta_2(\epsilon) > x^*$. Define $\theta_3(\epsilon)$ by $\theta_3(\epsilon) = \max(\theta_1(\epsilon), \theta_2(\epsilon))$. Then for $\theta > \theta_3(\epsilon)$ we have

$$x^* < (1 - \frac{3\epsilon}{4})\theta < x_\epsilon + (1 - \frac{\epsilon}{2})\theta < (1-\epsilon/4)\theta < \theta.$$

Thus for $\theta > \theta_3(\epsilon)$, by (1.2.17) we have

$$f(x_\epsilon + (1-\epsilon/2)\theta) > f(\theta(1-\epsilon/4)) > f(\theta) > (1-\epsilon/4)f(\theta),$$

which proves (1.9.2). □

1.9.2 Proof of Theorem 1.3.10

It is seen from Lemma 1.8.1 above that if we can show that there is an $x^* > 0$ such that the solution x of (1.2.1) obeys (1.8.1), then x obeys (1.2.6). We next show that if g and f obey (1.3.2), then x does indeed obey (1.8.1).

Lemma 1.9.2. *Suppose that f obeys* (1.2.2), (1.2.17) *and* (1.2.22)*, and that g is continuous. Suppose that F is given by* (1.2.15) *and that f and g obey* (1.3.2)*. Let x be the unique continuous solution of* (1.2.1)*. Then x obeys* (1.8.1)*.*

In order to prove this result we require and auxiliary lemma.

Lemma 1.9.3. *Let $\beta > 0$. Let $\lambda \in (1, \lambda(\beta))$, where $\lambda(\beta)$ is given by* (1.2.23)*. Define $\Lambda(0) = \lambda$ and*

$$\Lambda(n+1) = \lambda - \Lambda(n)^{-\beta}, \quad 0 \leq n \leq n', \quad n' := \inf\{n \geq 1 \,:\, \Lambda(n+1) \leq 0\}. \tag{1.9.3}$$

Then n' is finite and $0 < \Lambda(n+1) < \Lambda(n)$ for $n = 0, \dots, n' - 1$.

Proof. We first note that because $\Lambda(0) = \lambda > 1$, we have $\Lambda(1) > 0$, so we can only have $\Lambda(n+1) \leq 0$ for $n \geq 1$. Hence n' is appropriately defined. Suppose that n' is infinite. Then we have that $\Lambda(n) > 0$ for all $n \geq 0$.

Define $k_\lambda(x) = x - \lambda + x^{-\beta}$ for $x > 0$ and $h_\lambda(x) = x^{\beta+1} - \lambda x^\beta + 1$ for $x \geq 0$. Then for $x > 0$ we have $k_\lambda(x) = x^{-\beta} h_\lambda(x)$. Clearly we have $h'_\lambda(x) = x^{\beta-1}\left((\beta+1)x - \lambda\beta\right)$ for $x > 0$. Define $x_* = \beta\lambda/(\beta+1)$. Then $x_* \in (0, \lambda)$ and we have that h_λ is increasing on $(0, x_*)$ and decreasing on (x_*, ∞). Therefore for all $x > 0$ we have

$$h_\lambda(x) \geq h_\lambda(x_*) = x_*^\beta (x_* - \lambda) + 1 = \frac{\beta^\beta \lambda^\beta}{(\beta+1)^\beta} \left(\frac{\beta\lambda}{\beta+1} - \lambda \right) + 1 = 1 - \frac{\beta^\beta \lambda^{\beta+1}}{(\beta+1)^{1+\beta}}.$$

Since $\lambda < \lambda(\beta)$, it follows that the righthand side is positive, and so we have $h_\lambda(x) > 0$ for all $x > 0$. Hence $k_\lambda(x) > 0$ for all $x > 0$.

Since $\Lambda(n) > 0$ for all $n \geq 0$, we have $k_\lambda(\Lambda(n)) > 0$ for all $n \geq 0$. Therefore $\Lambda(n) > \lambda - \Lambda(n)^{-\beta}$ for all $n \geq 0$. But $\Lambda(n+1) = \lambda - \Lambda(n)^{-\beta}$ for all $n \geq 0$, so we have $\Lambda(n+1) < \Lambda(n)$ for all $n \geq 0$. Therefore we have that $\Lambda(n) \to L \geq 0$ as $n \to \infty$. Suppose that $L > 0$. Then we have $L = \lambda - L^{-\beta}$, or $L^{\beta+1} - \lambda L^\beta + 1 = 0$. But this implies that

$h_\lambda(L) = 0$, a contradiction. Suppose that $L = 0$. Then we have

$$0 = \lim_{n\to\infty} \Lambda(n+1) = \lim_{n\to\infty} \lambda - \frac{1}{\Lambda(n)^\beta} = -\infty,$$

a contradiction. Therefore we must have that there is a finite $n' \geq 1$ such that $\Lambda(n) > 0$ for $n \leq n'$ and $\Lambda(n'+1) \leq 0$. Moreover, we note that $0 < \Lambda(n+1) < \Lambda(n)$ for $n = 0, \dots, n'-1$. □

Proof of Lemma 1.9.2. Without loss of generality, we may take λ in (1.2.23) to obey $\lambda > 1$, i.e.,

$$\limsup_{t\to\infty} \frac{\int_0^t g(s)\,ds}{F^{-1}(t)} \leq \lambda < \lambda(\beta) \tag{1.9.4}$$

From (1.2.1) and (1.9.4), we have for all $\epsilon > 0$, there exists $T(\epsilon)$ such that for all $t > T(\epsilon)$:

$$\int_0^t g(s)\,ds \leq \lambda(1+\epsilon)F^{-1}(t), \quad t \geq T(\epsilon),$$

and so

$$x(t) \leq x(T(\epsilon)) + \int_{T(\epsilon)}^t g(s)\,ds \leq x(T(\epsilon)) + \lambda(1+\epsilon)F^{-1}(t).$$

Therefore we have

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \leq \lambda =: \Lambda(0) > 1,$$

where Λ is the sequence defined in Lemma 1.9.3, so there is a $T_0(\epsilon) > 0$ such that $x(t) \leq \lambda(1+\epsilon)F^{-1}(t)$ for $t \geq T_0(\epsilon)$. Suppose, in contradiction to the desired conclusion, that $\liminf_{t\to\infty} x(t) = x_1 > x^*$. Then there exists $T_1 > 0$ such that $x(t) > x^*$ for all $t \geq T_1$. We have $x^* < x(t) \leq \lambda(1+\epsilon)F^{-1}(t)$ for $t \geq \max(T_0(\epsilon), T_1)$, which implies that

$$-f(x(t)) \leq -f(\lambda(1+\epsilon)F^{-1}(t)), \quad t \geq \max(T_0(\epsilon), T_1).$$

Therefore for $T_2(\epsilon) = \max(T(\epsilon), T_0(\epsilon), T_1)$, we have

$$\begin{aligned}
x(t) &\le x(T_2) - \int_{T_2}^{t} f(\lambda(1+\epsilon)F^{-1}(s))\,ds + \int_{T_2}^{t} g(s)\,ds \\
&\le x(T_2) - \int_{T_2}^{t} f(\lambda(1+\epsilon)F^{-1}(s)ds + \lambda(1+\epsilon)F^{-1}(t) \\
&= x(T_2) - \int_{F^{-1}(T_2)}^{F^{-1}(t)} \frac{f(\lambda(1+\epsilon)u)}{f(u)} du + \lambda(1+\epsilon)F^{-1}(t).
\end{aligned}$$

Therefore

$$\frac{x(t)}{F^{-1}(t)} \le \frac{x(T_2)}{F^{-1}(t)} - \frac{1}{F^{-1}(t)} \int_{F^{-1}(T_2)}^{F^{-1}(t)} \frac{f(\lambda(1+\epsilon)u)}{f(u)} du + \lambda(1+\epsilon).$$

Thus, as $f \in RV_\infty(-\beta)$ we have

$$\begin{aligned}
\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} &\le -\lim_{x\to\infty} \frac{1}{x} \int_{F^{-1}(T_3)}^{x} \frac{f(\lambda(1+\epsilon)s)}{f(s)} ds + \lambda(1+\epsilon), \\
&= -\lim_{x\to\infty} \frac{f(\lambda(1+\epsilon)x)}{f(x)} + \lambda(1+\epsilon), \\
&= -[\lambda(1+\epsilon)]^{-\beta} + \lambda(1+\epsilon).
\end{aligned}$$

Therefore by (1.9.3) we have

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \le \lambda - \Lambda(0)^{-\beta} = \Lambda(1).$$

Introduce the n-th level hypothesis for $n = 0, \dots, n'$:

$$\Lambda(n) > 0, \quad \limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \le \Lambda(n). \tag{1.9.5}$$

We have already that (1.9.5) is true for $n = 0$ and $n = 1$.

By Lemma 1.9.3, one of the following holds:

(a) There exists $n' \ge 1$ such that $\Lambda(n) > 0$ for $n \le n'$ and $\Lambda(n'+1) < 0$;

(b) There exists $n' \ge 1$ such that $\Lambda(n) > 0$ for $n \le n'$ and $\Lambda(n'+1) = 0$;

We show that (1.9.5) at level n implies (1.9.5) at level $n+1$ provided that $n = 0, \ldots, n'-1$. Therefore as (1.9.5) is true at level 0, we have that (1.9.5) is true at level n'. Hence

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \le \Lambda(n').$$

Since it is assumed that $x(t) > x^*$ for all $t \ge T_1$, for every $\epsilon > 0$ there exists a $T_3(\epsilon) = \max(T_1, T_2)$ such that $x^* < x(t) < \Lambda(n)(1+\epsilon)F^{-1}(t)$ for $t \ge T_3(\epsilon)$. We have

$$\begin{aligned} x(t) &= x(T_3) - \int_{T_3}^{t} f(x(s))ds + \int_{T_3}^{t} g(s)ds, \\ &< x(T_3) - \int_{T_3}^{t} f(\Lambda(n)(1+\epsilon)F^{-1}(s))ds + \lambda(1+\epsilon)F^{-1}(t), \\ &= x(T_3) - \int_{F^{-1}(T_3)}^{F^{-1}(t)} \frac{f(\Lambda(n)(1+\epsilon)u)}{f(u)}du + \lambda(1+\epsilon)F^{-1}(t). \end{aligned}$$

Therefore, we have

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \le \lambda(1+\epsilon) - (\Lambda(n)(1+\epsilon))^{-\beta}$$

Letting $\epsilon \to 0$ and using (1.9.3) yields

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \le \lambda - \Lambda(n)^{-\beta} = \Lambda(n+1),$$

which is simply (1.9.5) at level $n+1$.

We now consider the case distinctions $\Lambda(n'+1) < 0$ and $\Lambda(n'+1) = 0$. In the former case we have already shown that

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \le \Lambda(n'),$$

and this implies that

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \le \lambda - \Lambda(n')^{-\beta} = \Lambda(n'+1) < 0.$$

Since $F^{-1}(t) \to \infty$ as $t \to \infty$, therefore we have $\lim_{t\to\infty} x(t) = -\infty$, it follows that since for all $t > T_3$, $\liminf_{t\to\infty} x(t) = x_1 > x^* \ge 0$ and $x'(t) < 0$, it follows that $\lim_{t\to\infty} x(t) > x^*$, a contradiction. Hence we must have $\liminf_{t\to\infty} x(t) \le x^*$ and the proof is complete.

On the other hand, suppose that $\Lambda(n'+1) = 0$. Therefore we have $\Lambda(n') = \lambda^{-1/\beta} \in (0,1)$. Let $\epsilon' > 0$ be so small that $\epsilon' \in (0, \lambda^{1/\beta} - 1)$ and

$$(1+\epsilon')^\beta < \frac{1}{1-\lambda^{-(\beta+1)/\beta}}.$$

Now we have that

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \le \Lambda(n') = \lambda^{-1/\beta} < \lambda^{-1/\beta}(1+\epsilon') =: \lambda' < 1. \tag{1.9.6}$$

Now define

$$\lambda'' := \lambda - (\lambda')^{-\beta} = \lambda\left(1 - \frac{1}{(1+\epsilon')^\beta}\right) > 0, \tag{1.9.7}$$

Moreover as $(1+\epsilon')^{-\beta} > 1 - \lambda^{-(\beta+1)/\beta}$, we have $1 - (1+\epsilon')^{-\beta} < \lambda^{-(\beta+1)/\beta}$, so

$$\lambda'' = \lambda\left(1 - \frac{1}{(1+\epsilon')^\beta}\right) < \lambda^{-1/\beta}.$$

Thus $\lambda'' \in (0, \lambda^{-1/\beta})$, and we can prove that (1.9.6) and (1.9.7) together imply

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \le \lambda''. \tag{1.9.8}$$

Proceeding as in the case when $\Lambda(n'+1) < 0$ we arrive once more at the conclusion that $\liminf_{t\to\infty} x(t) \le x^*$. □

1.10 Proof of Theorem 1.3.6, 1.3.8, 1.3.12

Lemma 1.10.1. *Let $\alpha > 0$. Define $g \in C([0,\infty);(0,\infty))$ by*

$$g(t) = (1+\alpha) f(F^{-1}(\alpha t + F(\xi/2))) + e^{-t}, \quad t \ge 0. \tag{1.10.1}$$

Then the solution of (1.2.1) obeys $x(t) \to \infty$ as $t \to \infty$.

Proof. Define $x_L(t) = F^{-1}(\alpha t + F(\xi/2))$ for $t \ge 0$. Then $x_L(0) = \xi/2 < x(0)$. Clearly for $t \ge 0$ we have

$$x_L'(t) + f(x_L(t)) - g(t) = (1+\alpha) f(F^{-1}(\alpha t + F(\xi/2))) - g(t) < 0.$$

Then $x_L(t) < x(t)$ for all $t \geq 0$. Since $x_L(t) \to \infty$ as $t \to \infty$, we have that $x(t) \to \infty$ as $t \to \infty$. □

Lemma 1.10.2. *Let g be defined by* (1.10.1).

(i) If $f \circ F^{-1} \in RV_\infty(-1)$, then

$$\lim_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} = 1 + \alpha^{-1}.$$

(ii) If $\beta \geq 0$ and $f \in RV_\infty(-\beta)$, then

$$\lim_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} = (1+\alpha)\alpha^{-\beta/(\beta+1)}. \tag{1.10.2}$$

Proof. Since $f \circ F^{-1} \in \text{RV}_\infty(-1)$ we have that

$$\lim_{t\to\infty} \frac{(f \circ F^{-1})(\alpha t + F(\xi/2))}{(f \circ F^{-1})(\alpha t)} = 1.$$

Also as $f \circ F^{-1} \in \text{RV}_\infty(-1)$ we

$$\lim_{t\to\infty} \frac{(f \circ F^{-1})(\alpha t)}{(f \circ F^{-1})(t)} = \alpha^{-1}.$$

Since $f \circ F^{-1} \in \text{RV}_\infty(-1)$, we have $e^{-t}/(f \circ F^{-1})(t) \to 0$ as $t \to \infty$ and so g obeys

$$\lim_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} = (1+\alpha) \lim_{t\to\infty} \frac{(f \circ F^{-1})(\alpha t + F(\xi/2))}{(f \circ F^{-1})(\alpha t)} \frac{(f \circ F^{-1})(\alpha t)}{(f \circ F^{-1})(t)} = 1 + \alpha^{-1}.$$

Note that when $f \in \text{RV}_\infty(-\beta)$, we have $f \circ F^{-1} \in \text{RV}_\infty(-\beta/(\beta+1))$, so

$$\lim_{t\to\infty} \frac{f(F^{-1}(\alpha t))}{f(F^{-1}(t))} = \alpha^{-\beta/(\beta+1)}.$$

Since $f \circ F^{-1}$ is in $\text{RV}_\infty(-\beta/(\beta+1))$ we have that

$$\lim_{t\to\infty} \frac{(f \circ F^{-1})(\alpha t + F(\xi/2))}{(f \circ F^{-1})(\alpha t)} = 1.$$

Since $f \circ F^{-1} \in \text{RV}_\infty(-\beta/(\beta+1))$, we have $e^{-t}/(f \circ F^{-1})(t) \to 0$ as $t \to \infty$ and so g obeys

$$\lim_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} = (1+\alpha) \lim_{t\to\infty} \frac{f(F^{-1}(\alpha t + F(\xi/2))}{f(F^{-1}(\alpha t))} \frac{f(F^{-1}(\alpha t))}{f(F^{-1}(t))} = (1+\alpha)\alpha^{-\beta/(\beta+1)},$$

as required. □

Proof of Theorem 1.3.8. Let $\kappa > 1$. By hypothesis $f \circ F^{-1} \in \text{RV}_\infty(-1)$, and let $\alpha = 1/(\kappa - 1) > 0$. If g is defined by (1.10.1), then by part (i) of Lemma 1.10.2 we have

$$\lim_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} = \kappa > 1.$$

Moreover, by Lemma 1.10.1 we have that $x(t) \to \infty$. □

Proof of Theorem 1.3.6. Let $\kappa > 1$. By hypothesis $f \circ F^{-1} \in \text{RV}_\infty(0)$. Let $\alpha = \kappa - 1 > 0$. If g is defined by (1.10.1), then by part (ii) of Lemma 1.10.2 we have

$$\lim_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} = \kappa > 1.$$

Moreover, by Lemma 1.10.1 we have that $x(t) \to \infty$. □

Proof of Theorem 1.3.12. Let $\kappa \geq (1+\beta)\beta^{-\beta/(\beta+1)}$. Since $f \in \text{RV}_\infty(-\beta)$ we have $f \circ F^{-1} \in \text{RV}_\infty(-\beta/(\beta+1))$. Since $\kappa \geq (1+\beta)\beta^{-\beta/(\beta+1)}$ there exists a unique $\alpha \in (0, \beta]$ such that.

$$(1+\alpha)\alpha^{-\beta/(\beta+1)} = \kappa.$$

Since g is defined by (1.10.1), then by part (ii) of Lemma 1.10.2 we have

$$\lim_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} = (1+\alpha)\alpha^{-\beta/(\beta+1)} = \kappa.$$

Moreover, by Lemma 1.10.1 we have that $x(t) \to \infty$. □

1.11 Proof of Theorems 1.3.3, 1.3.7, 1.3.9 and 1.3.13

The proof of Theorem 1.3.3 is an easy consequence of Lemma 1.8.1, and is given next. We consider the proof of the other theorems in the second subsection.

1.11.1 Proof of Theorem 1.3.3

There exists $\epsilon_0 > 0$ sufficiently small so that the set $\inf\{x > 0 : f(x) = 2\epsilon_0\}$ is non–empty. For $\epsilon \in (0, \epsilon_0)$ define $x_1(\epsilon) = \inf\{x > 0 : f(x) = 2\epsilon\}$. Then $f(x) < 2\epsilon$ for all $x \in [0, x_1(\epsilon))$. Suppose also that $g(t) \leq \epsilon$ for all $t \geq 0$.

Let $x(0) < x_1(\epsilon)$. Suppose there is a finite $T_1(\epsilon) = \inf\{t > 0 : x(t) = x_1(\epsilon)\}$. Then $x'(T_1(\epsilon)) \geq 0$. Also

$$0 \leq x'(T_1(\epsilon)) = -f(x(T_1(\epsilon))) + g(T_1(\epsilon)) \leq -f(x_1(\epsilon)) + \epsilon = -\epsilon < 0,$$

a contradiction. Hence we have that $x(t) < x_1(\epsilon)$ for all $t \geq 0$. Now by Lemma 1.8.1 it follows that $x(t) \to 0$ as $t \to \infty$.

1.11.2 Proof of Theorems 1.3.7, 1.3.9 and 1.3.13

In order to prove Theorems 1.3.7, 1.3.9 and 1.3.13, it proves convenient to establish the following condition on g and f:

$$\text{There exists } \alpha > 0 \text{ such that } \liminf_{t\to\infty} \frac{g(t)}{f(F^{-1}(\alpha t))} > 1 + \alpha. \tag{1.11.1}$$

We now show that (1.11.1) is satisfied under the conditions on g and f given in Theorems 1.3.7, 1.3.9 and 1.3.13.

Lemma 1.11.1. *Suppose that f obeys* (1.2.2) *and that g obeys* (1.2.3).

(i) Suppose that f obeys (1.2.18) *and that g and f obey* (1.2.21)*. Then g and f obeys* (1.11.1).

(ii) Suppose that f obeys (1.2.19) *and that g and f obey* (1.2.21)*. Then g and f obeys* (1.11.1).

(iii) Suppose that f obeys (1.2.22) *and that g and f obey* (1.2.25)*. Then g and f obeys* (1.11.1).

Proof. For part (i), by (1.2.21), there is $\kappa > 1$ be given by

$$\kappa = \liminf_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))}. \tag{1.11.2}$$

Then we may choose $\alpha > 1/(\kappa - 1) > 0$. Hence by (1.2.21) and (1.2.18) we have

$$\liminf_{t\to\infty} \frac{g(t)}{f(F^{-1}(\alpha t))} = \kappa / \lim_{t\to\infty} \frac{f(F^{-1}(\alpha t))}{f(F^{-1}(t))} = \frac{\kappa}{\alpha}.$$

Since $\alpha > 1/(\kappa-1)$, we have $\kappa\alpha > \alpha + 1$, so (1.11.1) holds.

For part (ii), once again there is $\kappa > 1$ which obeys (1.11.2). Then we may choose $\alpha \in (0, \kappa - 1) > 0$. Then $\alpha < \kappa - 1$. Hence by (1.2.21) and (1.2.19) we have

$$\liminf_{t\to\infty} \frac{g(t)}{f(F^{-1}(\alpha t))} = \kappa / \lim_{t\to\infty} \frac{f(F^{-1}(\alpha t))}{f(F^{-1}(t))} = \kappa.$$

Since $\kappa > \alpha + 1$, (1.11.1) holds.

For part (iii), there is $\lambda > \lambda(\beta) = (1+\beta)\beta^{-\beta/(1+\beta)}$ such that

$$\lambda = \liminf_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))}.$$

Let $\alpha = \beta > 0$. Since f is in $\text{RV}_\infty(-\beta)$, it follows that $f \circ F^{-1}$ is in $\text{RV}_\infty(-\beta/(\beta+1))$. Using this and the fact that f and g obey (1.2.25), we have

$$\begin{aligned}\liminf_{t\to\infty} \frac{g(t)}{f(F^{-1}(\alpha t))} &= \lambda / \lim_{t\to\infty} \frac{f(F^{-1}(\alpha t))}{f(F^{-1}(t))} = \lambda/\alpha^{-\beta/(\beta+1)} \\ &= \lambda\beta^{\beta/(\beta+1)} > \lambda(\beta)\beta^{\beta/(\beta+1)} = 1 + \beta = 1 + \alpha,\end{aligned}$$

which proves (1.11.1). □

Lemma 1.11.2. *Suppose that f obeys* (1.2.2) *and that g obeys* (1.2.3)*. Let x be the unique continuous solution of* (1.2.1)*. Suppose that g and f obey* (1.11.1)*. Then there exists $\bar{x} > 0$ such that for all $\xi > \bar{x}$, we have that* $\lim_{t\to\infty} x(t,\xi) = \infty$.

Proof. Define F by (1.2.15). By (1.11.1) there exists $\eta > 1 + \alpha$ such that

$$\liminf_{t\to\infty} \frac{g(t)}{f(F^{-1}(\alpha t))} = \eta.$$

Now for $\epsilon > 0$ sufficiently small we have $\eta(1-\epsilon) > (1+\alpha)(1+\epsilon)$. For such an $\epsilon > 0$ sufficiently small, there is $T(\epsilon) > 0$ such that

$$\frac{g(t)}{f(F^{-1}(\alpha t))} \geq \eta(1-\epsilon), \quad t \geq T(\epsilon),$$

and so

$$g(t) \geq (1+\epsilon)(1+\alpha)f(F^{-1}(\alpha t)), \quad t \geq T(\epsilon). \tag{1.11.3}$$

Next suppose that

$$\xi > F^{-1}((1+\alpha)T(\epsilon)), \quad \xi > F^{-1}(F(1)+T(\epsilon)). \tag{1.11.4}$$

Define

$$x_L(t) = F^{-1}(\alpha t), \quad t \geq T(\epsilon). \tag{1.11.5}$$

Define by y the solution of

$$y'(t) = -f(y(t)), \quad t \geq 0; \quad y(0) = \xi. \tag{1.11.6}$$

Since $g(t) \geq 0$ for all $t \geq 0$, we have $x'(t) \geq -f(x(t))$ for all $t \geq 0$. Then $x(t) \geq y(t)$ for all $t \geq 0$. Now, by (1.11.6) and (1.2.15) we have $y(t) = F^{-1}(F(\xi) - t)$ for all $t \in [0, T(\epsilon)]$, because the second part of (1.11.4) guarantees that $y(t) > 1$ for all $t \in [0, T(\epsilon)]$. Therefore by the first part of (1.11.4) and (1.11.5) we have

$$x_L(T(\epsilon)) = F^{-1}(\alpha T(\epsilon)) < F^{-1}(F(\xi) - T(\epsilon)) = y(T(\epsilon)) \leq x(T(\epsilon)). \tag{1.11.7}$$

Next note for $t \geq T(\epsilon)$ and by using (1.11.5) and (1.11.3) we have

$$\begin{aligned} x_L'(t) + f(x_L(t)) - g(t) &= (1+\alpha)f(F^{-1}(\alpha t)) - g(t) \\ &\leq (1+\alpha)f(F^{-1}(\alpha t)) - (1+\epsilon)(1+\alpha)f(F^{-1}(\alpha t)) \\ &= -\epsilon(1+\alpha)f(F^{-1}(\alpha t)) < 0. \end{aligned}$$

Therefore by this and (1.11.7) we have

$$x_L'(t) < -f(x_L(t)) + g(t), \quad t \geq T(\epsilon); \quad x_L(T(\epsilon)) < x(T(\epsilon)). \tag{1.11.8}$$

Hence $x_L(t) < x(t)$ for all $t \geq T(\epsilon)$. Therefore as $\alpha > 0$ we have $x_L(t) \to \infty$ as $t \to \infty$, and therefore it follows that $x(t) \to \infty$ as $t \to \infty$, as required. □

The proof of Theorem 1.3.7 is now a consequence of part (i) of Lemma 1.11.1 and Lemma 1.11.2. The proof of Theorem 1.3.9 is a consequence of part (ii) of Lemma 1.11.1 and Lemma 1.11.2. Finally, the proof of Theorem 1.3.13 is a consequence of part (iii) of Lemma 1.11.1 and Lemma 1.11.2.

1.12 Proof of Proposition 1.3.2

Note G is increasing. Moreover as $g \in \text{RV}_\infty(0)$, we have $G \in \text{RV}_\infty(1)$. Therefore $G^{-1} \in \text{RV}_\infty(1)$, and so $g \circ G^{-1} \in \text{RV}_\infty(0)$. By (1.3.6), we have that $f \in \text{RV}_\infty(0)$. Since $G^{-1}(x) \to \infty$ as $x \to \infty$ and g obeys (1.2.8), we have from (1.3.6) that f obeys (1.2.11). Since g is decreasing and G^{-1} is increasing, $x \mapsto g(G^{-1}(x))$ is decreasing, and so by (1.3.6), f is asymptotic to a decreasing function.

Define $G_\lambda(t) = G(t)/\lambda$ for $t \geq 0$. Then G_λ^{-1} exists and we have $G_\lambda^{-1}(t) = G^{-1}(t/\lambda)$. Since $g \circ G^{-1} \in \text{RV}_\infty(0)$, we have as $x \to \infty$ that

$$f(x) \sim \frac{1}{\lambda} g(G^{-1}(x)) \sim \frac{1}{\lambda} g(G^{-1}(x/\lambda)) = \frac{1}{\lambda} g(G_\lambda^{-1}(x)). \tag{1.12.1}$$

Now $g \in \text{RV}_\infty(0)$ implies that $G_\lambda(x) \sim x g(x)/\lambda$ as $x \to \infty$. Since $G_\lambda^{-1}(x) \to \infty$ as $x \to \infty$, we have that $\lambda x \sim G_\lambda^{-1}(x) g(G_\lambda^{-1}(x))$ as $x \to \infty$. Therefore we have that as $x \to \infty$

$$f(x) \sim \frac{1}{\lambda} g(G_\lambda^{-1}(x)) \sim \frac{1}{\lambda} \cdot \frac{\lambda x}{G_\lambda^{-1}(x)} = \frac{x}{G_\lambda^{-1}(x)}.$$

Since f is in $\text{RV}_\infty(0)$ we have as $x \to \infty$ that

$$F(x) = \int_1^x \frac{1}{f(u)}\, du \sim \frac{x}{f(x)} \sim G_\lambda^{-1}(x)$$

Since g is decreasing and g is in $\text{RV}_\infty(0)$ we have that $g(F(x)) \sim g(G_\lambda^{-1}(x))$ as $x \to \infty$. Therefore by (1.12.1) we have that as $x \to \infty$

$$g(F(x)) \sim g(G_\lambda^{-1}(x)) \sim \lambda f(x).$$

Since $F \in \text{RV}_\infty(1)$ we have that $F^{-1}(x) \to \infty$ as $x \to \infty$, and therefore it follows that (1.3.3) holds.

1.13 Proofs from Section 1.5

1.13.1 Proof of Lemma 1.5.1

For $x > 0$ we have that

$$\varphi_0(x) = \inf_{\|y\|=x} \langle \frac{y}{\|y\|}, \phi(y) \rangle = \inf_{\|u\|=1} \langle u, \phi(xu) \rangle.$$

Since $\phi(0) = 0$, we have that (1.5.5) holds. Moreover, (1.5.5) is equivalent to

$$-\varphi_0(x) = \sup_{\|u\|=1} -\langle u, \phi(xu) \rangle, \quad x \geq 0.$$

It is true for any $A, B : \mathbb{R}^d \to \mathbb{R}$ that

$$\left| \sup_{\|u\|=1} A(u) - \sup_{\|u\|=1} B(u) \right| \leq \sup_{\|u\|=1} |A(u) - B(u)|. \tag{1.13.1}$$

Let $x, y \in \mathbb{R}$ such that $|x| \vee |y| \leq n \in \mathbb{N}$. Since ϕ is locally Lipschitz continuous, for every $u \in \mathbb{R}^d$ with $\|u\| = 1$, we have

$$\|\phi(xu) - \phi(yu)\| \leq K_n |x - y| \tag{1.13.2}$$

for some $K_n > 0$. Therefore, for $|x| \vee |y| \leq n$, by using (1.13.1), the Cauchy–Schwarz inequality and (1.13.2) in turn, we get

$$\begin{aligned}
|\varphi_0(x) - \varphi_0(y)| &= \left| \sup_{\|u\|=1} -\langle u, \phi(xu) \rangle - \sup_{\|u\|=1} -\langle u, \phi(yu) \rangle \right| \\
&\leq \sup_{\|u\|=1} |\langle u, \phi(yu) \rangle - \langle u, \phi(xu) \rangle| \\
&= \sup_{\|u\|=1} |\langle u, \phi(yu) - \phi(xu) \rangle| \\
&\leq \sup_{\|u\|=1} \|u\| \|\phi(yu) - \phi(xu)\| \\
&\leq K_n |x - y|,
\end{aligned}$$

which establishes the local Lipschitz continuity of φ_0.

To show that $\varphi_0(x) > 0$ for $x > 0$, notice first by (1.5.3) that $\varphi_0(x) \geq 0$ for all $x > 0$. Suppose now that there is an $x_0 > 0$ such that $\varphi_0(x_0) = 0$. Then, by the continuity of φ_0, we have

$$0 = \varphi_0(x_0) = \inf_{\|u\|=x_0} \langle u, \phi(x_0 u) \rangle = \min_{\|u\|=1} \langle u, \phi(x_0 u) \rangle = \langle u^*, \phi(x_0 u^*) \rangle$$

for some $u^* \in \mathbb{R}^d$ such that $\|u^*\| = 1$. But then, with $x^* = x_0 u^* \neq 0$, we have $\langle x^*, \phi(x^*) \rangle = 0$, contradicting (1.5.3).

1.13.2 Proof of Proposition 1.5.1

It is easy to see that $\langle x, \phi(x)\rangle > 0$ for $x \neq 0$ ensures that $x = 0$ is the only equilibrium of the unperturbed equation (1.5.2). Suppose that $x_0 \neq 0$ is another equilibrium. Then $\phi(x_0) = 0$. But $0 = \langle x_0, \phi(x_0)\rangle > 0$ by (1.5.3), a contradiction. The global asymptotic stability of solutions is achieved by taking $u(t) = \|z(t)\|^2$ for $t \geq 0$.

If $z(0) = 0$, then $z(t) = 0$, for all $t \geq 0$. Otherwise, suppose $\dot{z}(0) \neq 0$, then $||z(0)|| > 0$ and $u'(t) = -2\langle z(t), \phi(z(t))\rangle \leq 0$, so $t \to u(t)$ is non-increasing. Either $u(t) \to 0$, as $t \to \infty$ or $u(t) \to L^2 > 0$ as $t \to \infty$. Suppose that the latter holds. We establish that

$$\liminf_{t\to\infty} \langle z(t), \phi(z(t))\rangle =: \lambda > 0$$

from which a contradiction will result.

Since $\|z(t)\| \to L > 0$ as $t \to \infty$, $\|z(t)\| > 0$ for all $t \geq 0$ and

$$\frac{\langle z(t), \phi(z(t))\rangle}{\|z(t)\|} \geq \inf_{\|u\|=\|z(t)\|} \frac{\langle u, \phi(u)\rangle}{\|u\|} = \varphi_0(\|z(t)\|).$$

By Lemma 1.5.1, φ_0 is locally Lipschitz continuous, so since $\|z(t)\| \to L$ as $t \to \infty$,

$$\liminf_{t\to\infty} \frac{\langle z(t), \phi(z(t))\rangle}{\|z(t)\|} \geq \liminf_{t\to\infty} \varphi_0(\|z(t)\|) = \varphi_0(L).$$

Also, as $L > 0$, Lemma 1.5.1 ensures that $\varphi_0(L) > 0$. Thus $\liminf_{t\to\infty}\langle z(t), \phi z(t))\rangle \geq L\varphi_0(L) > 0$. Recalling that $u'(t) \leq -2\langle z(t), \phi(z(t))\rangle$, we get

$$\limsup_{t\to\infty} u'(t) \leq \limsup_{t\to\infty} -2\langle z(t), \phi(z(t))\rangle \leq -2L\varphi_0(L) < 0.$$

Therefore $u(t) \to -\infty$ as $t \to \infty$, which contradicts the fact that $u(t) \geq 0$ for all $t \geq 0$.

1.13.3 Proof of Proposition 1.5.2

The local Lipschitz continuity of ϕ ensures that there is a unique continuous solution of (1.5.1), defined up to a maximal time $T > 0$ for which $\lim_{t\uparrow T} \|x(t)\| = +\infty$, or $x(t)$ is uniquely defined for all $t \geq 0$. Suppose the former and let $y(t) = \|x(t)\|^2$ for $t \in [0, T)$. Then for $t \in (0, T)$ we get

$$y'(t) = 2\langle -\phi(x(t)) + \gamma(t), x(t))\rangle \leq 2\langle \gamma(t), x(t)\rangle \leq 2\|\gamma(t)\|\|x(t)\|,$$

using (1.5.3) and the Cauchy Schwarz inequality. Since γ is continuous on $[0,T]$, we have that $\|\gamma(t)\| \le \Gamma$ for all $t \in [0,T]$ and some $\Gamma > 0$. Hence

$$y'(t) \le 2\Gamma\sqrt{y(t)}, \quad t \in (0,T), \quad y(0) = \|\xi\|^2 \ge 0.$$

Since $y(t) \to \infty$ as $t \uparrow T$ and y is continuous, there exists $T_1 \in (0,T)$ such that $y(t) \ge 1$ for all $t \in [T_1,T)$. Dividing by $\sqrt{y(t)}$ on both sides of this differential inequality for $t \in [T_1,T)$ and then integrating yields

$$y(t)^{1/2} - y(T_1)^{1/2} \le 2\Gamma(t - T_1), \quad t \in [T_1,T).$$

Letting $t \uparrow T$ on both sides of the inequality now leads to the desired contradiction.

1.13.4 Proof of Theorem 1.5.1

By hypothesis (1.5.6), φ is locally Lipschitz continuous and obeys $\varphi(0) = 0$. Therefore (1.5.7) has a unique continuous solution. Moreover, we see that $x_\epsilon(t) > 0$ for all $t \ge 0$, by considering $t_0 = \inf\{t > 0 : x_{\eta,\epsilon}(t) = 0\}$ and showing that such a t_0 cannot be finite. Clearly, we must have $x'_{\eta,\epsilon}(t_0) \le 0$, so that

$$0 \ge x'_{\eta,\epsilon}(t_0) = -\varphi(x_{\eta,\epsilon}(t_0)) + \|\gamma(t_0)\| + \frac{\epsilon}{2}e^{-t_0} = \|\gamma(t_0)\| + \frac{\epsilon}{2}e^{-t_0} \ge \epsilon e^{-t_0} > 0,$$

a contradiction. Thus $x_\epsilon(t) > 0$ for all $t \ge 0$.

Let $y(t) = \|x(t)\|^2$ and $y_\epsilon(t) = x_{\eta,\epsilon}(t)^2$ for $t \ge 0$. We show that $y(t) \le y_\epsilon(t)$ and this proves the result. Now as $y(t) = \langle x(t), x(t)\rangle$ and $x \in C^1([0,\infty);\mathbb{R}^d)$, we have that $y \in C^1((0,\infty);\mathbb{R})$ and moreover

$$y'(t) = -2\langle\phi(x(t)), x(t)\rangle + 2\langle\gamma(t), x(t)\rangle.$$

By the Cauchy–Schwarz inequality and (1.5.6), we get

$$\frac{2\langle\phi(x(t)), x(t)\rangle}{||x(t)||} \ge 2\varphi(\|x(t)\|)$$

when $\|x(t)\| \ne 0$, so $-2\langle\phi(x(t)), x(t)\rangle \le -2\varphi(\|x(t)\|)\|x(t)\|$. In the case that $\|x(t)\| = 0$, $2\langle\phi(x(t)), x(t)\rangle = 0$. Therefore, for all $t \ge 0$, $-2\langle\phi(x(t)), x(t)\rangle \le -2\varphi(\|x(t)\|)\|x(t)\|$. Thus $y'(t) \le -2\varphi(\|x(t)\|)\|x(t)\| + 2\|\gamma(t)\|\|x(t)\|$ for $t > 0$ or

$$y'(t) \le -2\varphi(\sqrt{y(t)})\sqrt{y(t)} + 2\|\gamma(t)\|\sqrt{y(t)}, \quad t > 0.$$

Moreover $y_\epsilon(0) = (\|x(0)\| + \epsilon/2)^2 > \|x(0)\|^2 = y(0)$.

Suppose there is $t_2 > 0$ such that $y(t_2) = y_\epsilon(t_2)$ but $y(t) < y_\epsilon(t)$ for $t \in [0, t_2)$. Then as y_ϵ is in $C^1((0,\infty),\mathbb{R})$, we have that $y'(t_2) \geq y'_\epsilon(t_2)$. By construction

$$\begin{aligned} y'_\epsilon(t) &= 2x_{\eta,\epsilon}(t)\{-\varphi(x_{\eta,\epsilon}(t)) + \|\gamma(t)\| + \epsilon/2e^{-t}\} \\ &= -2\sqrt{y_\epsilon(t)}\varphi(\sqrt{y_\epsilon(t)}) + 2\sqrt{y_\epsilon(t)}\|\gamma(t)\| + \epsilon\sqrt{y_\epsilon(t)}e^{-t}. \end{aligned}$$

Thus

$$\begin{aligned} y'_\epsilon(t_2) &= -2\sqrt{y_\epsilon(t_2)}\varphi(\sqrt{y_\epsilon(t_2)}) + 2\sqrt{y_\epsilon(t_2)}\|\gamma(t_2)\| + \epsilon\sqrt{y_\epsilon(t_2)}e^{-t_2} \\ &= -2\sqrt{y(t_2)}\varphi(\sqrt{y(t_2)}) + 2\sqrt{y(t_2)}\|\gamma(t_2)\| + \epsilon\sqrt{y_\epsilon(t_2)}e^{-t_2} \\ &\geq y'(t_2) + \epsilon\sqrt{y_\epsilon(t_2)}e^{-t_2} \\ &\geq y'_\epsilon(t_2) + \epsilon\sqrt{y_\epsilon(t_2)}e^{-t_2}. \end{aligned}$$

or $\sqrt{y_\epsilon(t_2)} \leq 0$. This implies $x_{\eta,\epsilon}(t_2) = 0$. But this is impossible as $x_{\eta,\epsilon}(t) > 0$ for all $t \geq 0$. Therefore $y(t) < y_\epsilon(t)$ for all $t \geq 0$, or $\|x(t)\|^2 < x_{\eta,\epsilon}(t)^2$ for all $t \geq 0$, which proves the result.

Chapter 2

Asymptotic Behaviour of Affine Stochastic Differential Equations

2.1 Introduction

In the last chapter, we examined the asymptotic behaviour of the deterministic differential equation

$$x'(t) = -f(x(t)) + g(t), \quad t \geq 0.$$

We viewed this as an equation with a unique and globally stable equilibrium at zero which is then perturbed by an *external* force g. We view this force as external because it is independent of x. The question naturally arises: what happens if the deterministic external force is replaced by one which is stochastic, and whose intensity is independent of the state x? In that case, we are lead to examine the stochastic differential equation

$$dX(t) = -f(X(t))\,dt + \sigma(t)\,dB(t), \quad t \geq 0,$$

where the function σ is continuous and stochastic and B is a finite dimensional Brownian motion. Specifically, we let

$$\sigma \in C([0,\infty); \mathbb{R}^{d\times r}) \tag{2.1.1}$$

and B be an r–dimensional standard Brownian motion.

In this chapter, we do not address directly the asymptotic behaviour of general nonlinear stochastic differential equations: this is done in the case of scalar stochastic differential equations in Chapter 3 and for finite dimensional equations in Chapter 4. Instead we start by analysing affine stochastic differential equations. This means that the function f is replaced by a linear function, or that $f(x) = Ax$ where A is a $d \times d$ real matrix. Since we are presuming that there is a unique equilibrium at zero, and that it is globally stable, we assume that all the eigenvalues of A have negative real parts. One of the important tasks in this chapter is to classify the asymptotic behaviour of the stochastic differential equation

$$dX(t) = AX(t)\,dt + \sigma(t)\,dB(t)$$

It turns out that this can be achieved by studying the simpler d–dimensional equation with solution Y which is given by

$$dY(t) = -Y(t)dt + \sigma(t)dB(t), \quad t \geq 0; \quad Y(0) = 0. \tag{2.1.2}$$

In fact, we demonstrate that X and Y have equivalent asymptotic behaviour, in the sense that X converges to zero if and only if Y does; is bounded but not convergent if and only if Y is; and is unbounded if and only if Y is.

Therefore, the question of analysing the asymptotic behaviour of the general linear equation reduces to that of studying the special linear equation (2.1.2). If σ is identically zero, it follows that the solution of

$$y'(t) = -y(t), \quad t \geq 0; \quad y(0) = 0.$$

obeys $y(t) = 0$ for all $t \geq 0$ if $y(0) = 0$. The question naturally arises as under what condition on σ does the solution $Y(t)$ obey

$$\lim_{t\to\infty} Y(t) = 0, \quad a.s. \tag{2.1.3}$$

It is shown in [31] that $Y(t)$ obeys (2.1.3) in the one–dimensional case if

$$\lim_{t\to\infty} \sigma^2(t)\log t = 0.$$

Moreover in [31], it is shown that if $t \to \sigma^2(t)$ is decreasing to zero, and $Y(t)$ obeys (2.1.3), then we must have $\lim_{t\to\infty} \sigma^2(t)\log t = 0$. These results are extended to finite–dimensions in [30].

In this chapter, we characterise the convergence, boundedness and unboundedness of solutions of (2.1.2) without imposing monotonicity on $\|\sigma\|_F^2$. Our main results show that Y obeys (2.1.3) if and only if

$$S(\epsilon) = \sum_{n=0}^{\infty} \left\{ 1 - \Phi\left(\frac{\epsilon}{\sqrt{\int_n^{n+1} \|\sigma(s)\|_F^2 ds}} \right) \right\} < +\infty, \quad \text{for every} \quad \epsilon > 0, \tag{2.1.4}$$

where Φ is the distribution function of a standardised normal random variable. We also show that in contrast to (2.1.4), if $S(\epsilon)$ is infinite for all ϵ, then $\limsup_{t\to\infty} \|Y(t)\| = +\infty$; while if the sum is finite for some ϵ and infinite for others, then $c_1 \leq \limsup_{t\to\infty} \|Y(t)\| \leq c_2$ a.s., where $0 < c_1 \leq c_2 < +\infty$ are deterministic. Since $S(\epsilon)$ is monotone in ϵ, it can be seen that we can describe the asymptotic behaviour for every function σ, and that, moreover, the stability, boundedness or unboundedness of the solution depends on σ only through the Frobenius norm $\|\sigma\|_F$. Therefore, as all norms in $\mathbb{R}^{d\times r}$ are equivalent, it follows that the asymptotic behaviour relies only on $\|\sigma\|$, where $\|\cdot\|$ is any norm in $\mathbb{R}^{d\times r}$.

Given that we are dealing with a continuous time equation, it seems appropriate that the conditions which enable us to characterise the asymptotic behaviour should be "continuous" rather than "discrete". The finiteness condition on $S(\epsilon)$, which relies on a particular partition of time, and the convergence of a sum, can certainly be seen as a "discrete" condition, in this sense. Therefore, we develop an integral condition on σ which is equivalent to the summation condition in (2.1.4). More precisely, we define

$$I(\epsilon) = \int_0^\infty \sqrt{\int_t^{t+c} \|\sigma(s)\|_F^2 \, ds} \exp\left(- \frac{\epsilon^2/2}{\int_t^{t+c} \|\sigma(s)\|_F^2} \right) \chi_{(0,\infty)} \left(\int_t^{t+c} \|\sigma(s)\|_F^2 \right) \, ds \quad (2.1.5)$$

for arbitrary $c > 0$. We then show that $I(\epsilon)$ being finite for all ϵ implies that Y tends to 0; if $I(\epsilon)$ is infinite for all ϵ then Y is unbounded; and if $I(\epsilon)$ is finite for some ϵ and infinite for others, then Y is bounded but not convergent to zero.

Although (2.1.4) or $I(\epsilon)$ being finite are necessary and sufficient for Y to obey (2.1.3), these conditions may be hard to apply in practice. For this reason we also deduce sharp sufficient conditions on σ which enable us to determine for which value of ϵ the functions $S(\epsilon)$ or $I(\epsilon)$ are finite. One such condition is the following: if it is known that

$$\lim_{t\to\infty} \int_t^{t+c} \|\sigma(s)\|_F^2 \, ds \log t = L \in [0, \infty],$$

then $L = 0$ implies that Y tends to zero a.s.; if L is positive and finite, then Y is bounded, but does not converge to zero; and if $L = +\infty$, then Y is unbounded. Of course, all these conditions ensure that the solution of the general linear equation possesses the same properties.

One other result of note is established. We ask: is it possible for solutions of the unperturbed ODE $x'(t) = Ax(t)$ to be unstable, but solutions of the SDE to be stable for some nontrivial σ? In other words, can the noise *stabilise* solutions? We prove that it cannot, in the sense that if there are a representative and finite collection of initial conditions ξ for which $X(t,\xi)$ tends to zero with positive probability, then it must be the case that all the eigenvalues of A have negative real parts, and that $S(\epsilon)$ is finite for all $\epsilon > 0$. These conditions are therefore equivalent to $\lim_{t\to\infty} X(t,\xi) = 0$ a.s. for each initial condition ξ.

The next section states and discusses the main results, with proofs and supporting lemmatas in the following section. Then we discuss the sufficient conditions on σ for stability with proofs and supporting lemmatas.

2.2 Main results for linear equation

We first determine necessary and sufficient conditions for the solution of a linear equation defined by (2.1.2) to tend to zero. Note that Y has the representation

$$Y(t) = e^{-t}\int_0^t e^s \sigma(s)\,dB(s), \quad t \ge 0. \tag{2.2.1}$$

Denote by $\Phi : \mathbb{R} \to \mathbb{R}$ the distribution of a standard normal random variable

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-u^2/2}\,du, \quad x \in \mathbb{R}. \tag{2.2.2}$$

We interpret $\Phi(-\infty) = 0$ and $\Phi(\infty) = 1$. Define $\theta_i : \mathbb{N} \to [0,\infty)$ by

$$\theta_i^2(n) = \sum_{l=1}^r \int_n^{n+1} \sigma_{il}^2(s)\,ds, \quad i = 1,\dots,d. \tag{2.2.3}$$

and

$$\theta(n)^2 = \int_n^{n+1} \|\sigma(s)\|_F^2\,ds. \tag{2.2.4}$$

Finally define

$$S(\epsilon) = \sum_{n=0}^{\infty}\left\{1 - \Phi\left(\frac{\epsilon}{\sqrt{\int_n^{n+1}\|\sigma(s)\|_F^2\,ds}}\right)\right\} \tag{2.2.5}$$

Since S is a monotone function of ϵ, it is the case that either (i) $S(\epsilon)$ is finite for all $\epsilon > 0$; (ii) there is $\epsilon' > 0$ such that for all $\epsilon > \epsilon'$ we have $S(\epsilon) < +\infty$ and $S(\epsilon) = +\infty$ for all $\epsilon < \epsilon'$; and (iii) $S(\epsilon) = +\infty$ for all $\epsilon > 0$.

The following theorem characterises the pathwise asymptotic behaviour of solutions of (2.1.2).

Theorem 2.2.1. *Suppose that σ obeys* (2.1.1) *and that Y is the unique continuous adapted process which obeys* (2.1.2)*. Let $S(\cdot)$ be defined by* (2.2.5)*.*

(A) If

$$S(\epsilon) \text{ is finite for all } \epsilon > 0, \tag{2.2.6}$$

then

$$\lim_{t\to\infty} Y(t) = 0, \quad a.s. \tag{2.2.7}$$

(B) If there exists $\epsilon' > 0$ such that

$$S(\epsilon) \text{ is finite for all } \epsilon > \epsilon', \quad S(\epsilon) = +\infty \text{ for all } \epsilon < \epsilon', \tag{2.2.8}$$

then there exists deterministic $0 < c_1 \leq c_2 < +\infty$ such that

$$c_1 \leq \limsup_{t\to\infty} \|Y(t)\| \leq c_2, \quad a.s. \tag{2.2.9}$$

Moreover

$$\liminf_{t\to\infty} \|Y(t)\| = 0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t \|Y(s)\|^2\, ds = 0, \quad a.s.$$

(C) If

$$S(\epsilon) = +\infty \text{ for all } \epsilon > 0, \tag{2.2.10}$$

then

$$\limsup_{t\to\infty} \|Y(t)\| = +\infty, \quad a.s. \tag{2.2.11}$$

The conditions and form of Theorem 2.2.1, as well as other theorems in this section, are inspired by those of [31, Theorem 1] and by [20, Theorem 6, Corollary 7].

Let d be an integer and A be a $d \times d$ matrix with real entries, and consider the deterministic linear differential equation

$$x'(t) = Ax(t), \quad t \geq 0; \quad x(0) = \xi \in \mathbb{R}^d. \tag{2.2.12}$$

Theorem 2.2.1 can be immediately applied to determine necessary and sufficient conditions for the convergence to the unique equilibrium of (2.2.12) of solutions of a stochastically perturbed version of (2.2.12), namely

$$dX(t) = AX(t)\,dt + \sigma(t)\,dB(t), \quad t \geq 0; \quad X(0) = \xi \in \mathbb{R}^d. \tag{2.2.13}$$

Theorem 2.2.2. *Suppose that σ obeys* (2.1.1)*. Let A be a $d \times d$ real matrix for which all eigenvalues have negative real parts. Let X be the solution of* (2.2.13)*. Let θ be defined by* (2.2.4) *and let Φ be given by* (2.2.2)*. Then the following holds:*

(A) If S obeys (2.2.6)*, then $\lim_{t\to\infty} X(t,\xi) = 0$ a.s. for each $\xi \in \mathbb{R}^d$;*

(B) If S obeys (2.2.8)*, then there exist deterministic $0 < c_1 \leq c_2 < \infty$ independent of ξ such that*

$$c_1 \leq \limsup_{t\to\infty} \|X(t,\xi)\| \leq c_2, \quad a.s.$$

Moreover

$$\liminf_{t\to\infty} \|X(t,\xi)\| = 0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t \|X(s,\xi)\|^2\,ds = 0, \quad a.s.$$

(C) If S obeys (2.2.10)*, then $\limsup_{t\to\infty} \|X(t,\xi)\| = +\infty$ a.s. for each $\xi \in \mathbb{R}^d$.*

Suppose that $(t_n)_{n\geq 0}$ is an increasing sequence with $t_0 = 0$ and $\lim_{n\to\infty} t_n = +\infty$. Define

$$S_{t.}(\epsilon) = \sum_{n=0}^{\infty} \left\{ 1 - \Phi\left(\frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2\,ds}} \right) \right\} \tag{2.2.14}$$

This necessary and sufficient condition on $S_{t.}(\epsilon)$ is difficult to evaluate directly, because we do not know Φ in its closed form. However we can show that $S_{t.}(\epsilon)$ is finite or infinite according to whether the sum

$$S'_{t.}(\epsilon) = \sum_{n=0}^{\infty} \sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2\,ds}\, \exp\left(-\frac{\epsilon^2}{2} \frac{1}{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2\,ds} \right) \tag{2.2.15}$$

is finite or infinite, where we interpret the summand to be zero in the case where $\theta'(n) = 0$. Therefore we establish the following Lemmata which enables us to obtain all the above results with $S'_{t.}(\epsilon)$ in place of $S_{t.}(\epsilon)$.

Lemma 2.2.1. *$S_{t.}(\epsilon)$ given by (2.2.14) is finite if and only if $S'_{t.}(\epsilon)$ given by (2.2.15) is finite.*

Proof. We note by e.g., [44, Problem 2.9.22], we have

$$\lim_{x\to\infty} \frac{1-\Phi(x)}{x^{-1}e^{-x^2/2}} = \frac{1}{\sqrt{2\pi}}. \tag{2.2.16}$$

If $S_{t.}(\epsilon)$ is finite, then $1-\Phi(\epsilon/\theta'(n)) \to 0$ as $n\to\infty$, where $\theta'(n)^2 = \int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2\,ds$. This implies $\epsilon/\theta'(n)\to\infty$ as $n\to\infty$. Therefore by (2.2.16), we have

$$\lim_{n\to\infty} \frac{1-\Phi(\epsilon/\theta'(n))}{\theta'(n)/\epsilon\cdot\exp(-\epsilon^2/\{2\theta'^2(n)\})} = \frac{1}{\sqrt{2\pi}}. \tag{2.2.17}$$

Since $(1-\Phi(\epsilon/\theta'(n)))_{n\ge1}$ is summable, it therefore follows that the sequence

$$(\theta'(n)/\epsilon\cdot\exp(-\epsilon^2/\{2\theta'^2(n)\}))_{n\ge1}$$

is summable, so $S'_{t.}(\epsilon)$ is finite, by definition.

On the other hand, if $S'_{t.}(\epsilon)$ is finite, and we define $\phi:[0,\infty)\to\mathbb{R}^d$ by

$$\phi(x) = \begin{cases} x\exp(-1/(2x^2)), & x>0,\\ 0, & x=0,\end{cases}$$

then as we have $\theta'(n)\exp(-\epsilon^2/2\theta'^2(n))$ summable, we have $(\phi(\theta'(n)/\epsilon))_{n\ge1}$ is summable. Therefore $\phi(\theta'(n)/\epsilon)\to0$ as $n\to\infty$. Then, as ϕ is continuous and increasing on $[0,\infty)$, we have that $\theta'(n)/\epsilon\to0$ as $n\to\infty$, or $\epsilon/\theta'(n)\to\infty$ as $n\to\infty$. Therefore (2.2.17) holds, and thus $(1-\Phi(\epsilon/\theta'(n)))_{n\ge1}$ is summable, which implies that $S_{t.}(\epsilon)$ is finite, as required. □

The following theorem then characterises the pathwise asymptotic behaviour of solutions of (2.1.2).

Theorem 2.2.3. *Suppose that σ obeys (2.1.1) and that Y is the unique continuous adapted process which obeys (2.1.2). Let $S_{t.}(\epsilon)$ be defined by (2.2.14) where $t.$ is any ϵ–independent* *sequence obeying*

$$t_0 = 0, \quad 0 < \alpha \le t_{n+1} - t_n \le \beta < +\infty, \quad \lim_{n\to\infty} t_n = +\infty \tag{2.2.18}$$

for some $0 < \alpha \le \beta < +\infty$.

(A) If

$$S_{t.}(\epsilon) \text{ is finite for all } \epsilon > 0, \tag{2.2.19}$$

then

$$\lim_{t\to\infty} Y(t) = 0, \quad a.s.$$

(B) (i) If there exists $\epsilon' > 0$ *such that*

$$S_{t.}(\epsilon) \text{ is finite for all } \epsilon > \epsilon', \tag{2.2.20}$$

then there exists a deterministic $0 < c_2 < +\infty$ *such that*

$$\limsup_{t\to\infty} \|Y(t)\| \le c_2, \quad a.s.$$

(ii) On the other hand, if there exists $\epsilon'' > 0$ *such that*

$$S_{\tau.}(\epsilon) = +\infty \text{ for all } \epsilon < \epsilon'', \tag{2.2.21}$$

where τ *is any* ϵ*–independent sequence obeying* (2.2.18)*, then there exists a deterministic* $0 < c_1 < +\infty$ *such that*

$$\limsup_{t\to\infty} \|Y(t)\| \ge c_1, \quad a.s.$$

(C) If

$$S_{t.}(\epsilon) = +\infty \text{ for all } \epsilon > 0, \tag{2.2.22}$$

then

$$\limsup_{t\to\infty} \|Y(t)\| = +\infty, \quad a.s.$$

We mention a useful corollary which holds when $t_n = nh$ for some $h > 0$. It yields Theorem 2.2.1 in the case $h = 1$. It is also of utility when considering the relationship between the asymptotic behaviour of solutions of stochastic differential equations and the asymptotic behaviour of uniform step–size discretisations.

Corollary 2.2.1. *Suppose that* σ *obeys* (2.1.1) *and* Y *is the unique continuous adapted process which obeys* (2.1.2)*. Suppose that* S_h *is defined by*

$$S_h(\epsilon) = \sum_{n=0}^{\infty} \left\{ 1 - \Phi\left(\frac{\epsilon}{\sqrt{\int_{nh}^{(n+1)h} \|\sigma(s)\|_F^2 \, ds}} \right) \right\}. \tag{2.2.23}$$

(A) If

$$S_h(\epsilon) \text{ is finite for all } \epsilon > 0,$$

then

$$\lim_{t\to\infty} Y(t) = 0, \quad a.s.$$

(B) If there exists $\epsilon' > 0$ *such that*

$$S_h(\epsilon) \text{ is finite for all } \epsilon > \epsilon', \quad S_h(\epsilon) = +\infty \text{ for all } \epsilon < \epsilon',$$

then there exists deterministic $0 < c_1 \le c_2 < +\infty$ *such that*

$$c_1 \le \limsup_{t\to\infty} \|Y(t)\| \le c_2, \quad a.s.$$

Moreover

$$\liminf_{t\to\infty} \|Y(t)\| = 0, \quad \lim_{t\to\infty} \frac{1}{t} \int_0^t \|Y(s)\|^2 \, ds = 0, \quad a.s.$$

(C) If

$$S_h(\epsilon) = +\infty \text{ for all } \epsilon > 0,$$

then

$$\limsup_{t\to\infty} \|Y(t)\| = +\infty, \quad a.s.$$

One important application of Theorem 2.2.3 is to formulate the conditions for asymptotic convergence, boundedness and fluctuation in terms of a continuous integral condition on σ. To this end we introduce for fixed $c > 0$ the ϵ–dependent integral

$$I(\epsilon) = \int_0^\infty \sqrt{\int_t^{t+c} \|\sigma(s)\|_F^2 \, ds} \exp\left(-\frac{\epsilon^2/2}{\int_t^{t+c} \|\sigma(s)\|_F^2 ds}\right) \chi_{(0,\infty)}\left(\int_t^{t+c} \|\sigma(s)\|_F^2\right) ds. \tag{2.2.24}$$

We notice that $\epsilon \mapsto I(\epsilon)$ is a monotone function, and therefore $I(\cdot)$ is either finite for all $\epsilon > 0$; infinite for all $\epsilon > 0$; or finite for all $\epsilon > \epsilon'$ and infinite for all $\epsilon < \epsilon'$. The following theorem is therefore seen to classify the asymptotic behaviour of (2.1.2).

Theorem 2.2.4. *Suppose that σ obeys* (2.1.1) *and that Y is the unique continuous adapted process which obeys* (2.1.2). *Let $I(\cdot)$ be defined by* (2.2.24).

(A) If

$$I(\epsilon) \text{ is finite for all } \epsilon > 0, \tag{2.2.25}$$

then

$$\lim_{t\to\infty} Y(t) = 0, \quad a.s.$$

(B) If there exists $\epsilon' > 0$ such that

$$I(\epsilon) \text{ is finite for all } \epsilon > \epsilon', \quad I(\epsilon) = +\infty \text{ for all } \epsilon < \epsilon', \tag{2.2.26}$$

then there exists deterministic $0 < c_1 \le c_2 < +\infty$ such that

$$c_1 \le \limsup_{t\to\infty} \|Y(t)\| \le c_2, \quad a.s.$$

Moreover,

$$\liminf_{t\to\infty} \|Y(t)\| = 0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t \|Y(s)\|^2 \, ds = 0, \quad a.s.$$

(C) If

$$I(\epsilon) = +\infty \text{ for all } \epsilon > 0, \tag{2.2.27}$$

then

$$\limsup_{t\to\infty} \|Y(t)\| = +\infty, \quad a.s.$$

A similar theorem can of course be formulated for the solution of (2.2.13).

Theorem 2.2.5. *Suppose that σ obeys* (2.1.1) *and that X is the unique continuous adapted process which obeys* (2.2.13). *Suppose all the eigenvalues of A have negative real parts, and let $I(\cdot)$ be defined by* (2.2.24).

(A) If I obeys (2.2.25), *then*

$$\lim_{t\to\infty} X(t) = 0, \quad a.s.$$

(B) If I obeys (2.2.26), *then there exists deterministic $0 < c_1 \le c_2 < +\infty$ such that*

$$c_1 \le \limsup_{t\to\infty} \|X(t)\| \le c_2, \quad a.s.$$

Moreover

$$\liminf_{t\to\infty} \|X(t)\| = 0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t \|X(s)\|^2\, ds = 0, \quad a.s. \tag{2.2.28}$$

(C) If I obeys (2.2.27) *then*

$$\limsup_{t\to\infty} \|X(t)\| = +\infty, \quad a.s.$$

The result of the last theorem shows that $\liminf_{t\to\infty} \|X(t)\| = 0$ a.s. when $I(\epsilon)$ is finite for some $\epsilon > 0$ and infinite for others. In the case when $I(\epsilon) = +\infty$ for every $\epsilon > 0$, we now give an example which shows that no general conclusion can be made about the limit inferior.

Example 2.2.1. Suppose that $d = r \ge 3$, that $A = I_d$ and that $\sigma(t) = \eta(t)I_d$ for $t \ge 0$, where $\eta \in C([0,\infty);(0,\infty))$. Suppose also that

$$\lim_{t\to\infty} \int_0^t e^{2s}\eta^2(s)\, ds = +\infty.$$

Then the i–th component of X obeys

$$X_i(t) = \xi_i e^{-t} + e^{-t} \int_0^t e^s \eta(s)\, dB_i(s), \quad t \geq 0.$$

Hence

$$e^{2t}\|X(t)\|_2^2 = \|\xi\|_2^2 + \sum_{i=1}^d \left(\int_0^t e^s \eta(s)\, dB_i(s) \right)^2, \quad t \geq 0.$$

Define

$$T(t) := \int_0^t e^{2s}\eta^2(s)\, ds, \quad t \geq 0.$$

Then $T : [0,\infty) \to [0,\infty)$ is an increasing and C^1 function with $T(t) \to \infty$ as $t \to \infty$. Define $\tau(t) = T^{-1}(t)$ for $t \geq 0$ and

$$U(t) = \|\xi\|_2^2 + \sum_{i=1}^d \left(\int_0^t e^s \eta(s)\, dB_i(s) \right)^2, \quad t \geq 0.$$

Also define $\tilde{U}(t) = U(\tau(t))$ and

$$B_i^*(t) = \int_0^{\tau(t)} e^s \eta(s)\, dB_i(s), \quad t \geq 0.$$

Let $\mathcal{G}(t) = \mathcal{F}^B(\tau(t))$. Then $\tilde{U}$ and B_i^* are $\mathcal{G}$–adapted and

$$\tilde{U}(t) = \|\xi\|_2^2 + \sum_{i=1}^d B_i^*(t)^2, \quad t \geq 0.$$

We now establish that B_i^* is a $\mathcal{G}$ standard Brownian motion. To do this we must check the conditions of Theorem 0.3.3. First, we see that B_i^* is $\mathcal{F}^B(\tau(t))$ measurable, and therefore $\mathcal{G}(t)$ measurable. Since τ is increasing, $\mathcal{G}$ is a filtration. Also because τ is continuous and $s \mapsto e^s\eta(s)$ is continuous, then $t \mapsto B_i^*(t)$ is continuous. Finally, if we let $I_i(t) = \int_0^t e^s\eta(s) dB_i(s)$, then $\mathbb{E}[I_i(t)^2] = \int_0^t e^{2s}\eta(s)^2 ds = T(t)$. Thus

$$\mathbb{E}[B_i^*(t)^2] = \mathbb{E}[I_i(\tau(t))^2] = T(\tau(t)) = t < +\infty.$$

Therefore, we need only to check that B_i^* obeys the projection property (0.3.1) for martingales. Let $t > s \geq 0$. Then as τ is increasing, we have

$$\begin{aligned}\mathbb{E}[B_i^*(t)|\mathcal{G}(s)] &= \mathbb{E}[I_i(\tau(t))|\mathcal{F}^B(\tau(s))] \\ &= \mathbb{E}[\int_{\tau(s)}^{\tau(t)} e^u\eta(u)dB_i(u) + B_i^*(s)|\mathcal{F}^B(\tau(s))] \\ &= \mathbb{E}[\int_{\tau(s)}^{\tau(t)} e^u\eta(u)dB_i(u)|\mathcal{F}^B(\tau(s))] + B_i^*(s) \\ &= \mathbb{E}[\int_{\tau(s)}^{\tau(t)} e^u\eta(u)dB_i(u)] + B_i^*(s) = B_i^*(s).\end{aligned}$$

Hence B_i^* is a $\mathcal{G}(t)$–martingale. Finally, $\langle B_i^*\rangle(t) = \int_0^{\tau(t)} e^{2s}\eta(s)^2 ds = T(\tau(t)) = t$. Therefore, by Theorem 0.3.3, B_i^* is a $\mathcal{G}$ standard Brownian motion. Also, because the Brownian motions $B_1, \ldots, B_d$ are independent, it follows that $B_1^*, B_2^*, \ldots, B_d^*$ are independent $\mathcal{G}$–adapted standard Brownian motions. Therefore $\tilde{U}$ is a d–dimensional square Bessel process starting at $\|\xi\|_2^2$, and indeed

$$e^{2\tau(t)}\|X(\tau(t))\|_2^2 = \tilde{U}(t), \quad t \geq 0.$$

Thus, $\tilde{U}_2 = \sqrt{\tilde{U}}$ is a d–dimensional Bessel process starting at $\|\xi\|_2$.

Now, if $\xi \neq 0$, it was proven in Appleby and Wu [22] that

$$\liminf_{t\to\infty} \frac{\log \frac{\tilde{U}_2(t)}{\sqrt{t}}}{\log\log t} = -\frac{1}{d-2}, \quad \limsup_{t\to\infty} \frac{\tilde{U}_2(t)}{\sqrt{2t\log\log t}} = 1, \quad \text{a.s.}$$

Hence

$$\liminf_{t\to\infty} \frac{\log \frac{e^{\tau(t)}\|X(\tau(t))\|_2}{\sqrt{t}}}{\log\log t} = -\frac{1}{d-2}, \quad \text{a.s.}$$

which yields

$$\liminf_{t\to\infty} \frac{\log \frac{\|X(t)\|_2}{\sqrt{e^{-2t}T(t)}}}{\log\log T(t)} = -\frac{1}{d-2}, \quad \limsup_{t\to\infty} \frac{\|X(t)\|_2}{\sqrt{2e^{-2t}T(t)\log\log T(t)}} = 1, \quad \text{a.s.} \tag{2.2.29}$$

If we suppose that η is such that $\eta'(t)/\eta(t) \to 0$ as $t \to \infty$, so that η neither decays nor grows at an exponential rate, we have by l'Hôpital's rule that

$$\lim_{t\to\infty} \frac{T(t)}{e^{2t}\eta(t)^2} = \frac{1}{2},$$

and because $\lim_{t\to\infty}\log\eta(t)/t = 0$, we have also that

$$\lim_{t\to\infty}\frac{\log\log T(t)}{\log t} = 1.$$

Therefore, from (2.2.29) we get

$$\liminf_{t\to\infty}\frac{\log\frac{\|X(t)\|_2}{\frac{1}{\sqrt{2}}\eta(t)}}{\log t} = -\frac{1}{d-2}, \quad \limsup_{t\to\infty}\frac{\|X(t)\|_2}{\sqrt{\eta^2(t)\log t}} = 1, \quad \text{a.s.}$$

Now, we suppose that $\eta(t)/t^\alpha \to L \in (0,\infty)$ as $t\to\infty$. If $\alpha \geq 0$, we can show that all the hypotheses hold and that $I(\epsilon) = +\infty$ for all $\epsilon > 0$. Moreover, if $\alpha > 1/(d-2) > 0$, then

$$\lim_{t\to\infty}\|X(t)\|_2 = +\infty, \quad \text{a.s.}$$

while if $0 \leq \alpha < 1/(d-2)$, we have

$$\liminf_{t\to\infty}\|X(t)\|_2 = 0, \quad \limsup_{t\to\infty}\|X(t)\|_2 = +\infty, \quad \text{a.s.}$$

(In the case $\alpha < 0$, we have that $X(t) \to 0$ as $t \to \infty$ a.s. because $I(\epsilon)$ is finite for all $\epsilon > 0$.)

Therefore, it can be seen that without further information on the growth or decay rate of $\|\sigma(t)\|$ as $t\to\infty$, it is impossible to make a general conclusion about the size of $\liminf_{t\to\infty}\|X(t)\|$. In this sense, the overall conclusions of Theorem 2.2.4 cannot be improved upon if $d \geq 3$ without further analysis. However, we will see in the next chapter that when $d = 1$ (in which case we can take $r = 1$ without loss of generality), it can be shown that $I(\epsilon) = +\infty$ for all $\epsilon > 0$ implies

$$\liminf_{t\to\infty}|X(t)| = 0, \quad \limsup_{t\to\infty}|X(t)| = +\infty, \quad \text{a.s.}$$

We now present a result concerning the inability of noise to stabilise the asymptotically stable differential equation $x'(t) = Ax(t)$.

Theorem 2.2.6. *Suppose that σ obeys* (2.1.1) *and that $X(\cdot,\xi)$ is the unique continuous adapted process which obeys* (2.2.13) *with initial condition $X(0)=\xi$. Then the following are equivalent:*

(A) All the eigenvalues of A have negative real parts, and I defined by (2.2.24) *obeys* (2.2.25)*;*

(B) There is a basis $(\xi_i)_{i=1}^d$ of $\mathbb{R}^d$ and an event C with $\mathbb{P}[C]>0$ given by

$$C=\{\omega:\lim_{t\to\infty}X(t,\xi_i,\omega)=0,\ \textit{for } i=1,\dots,d,\ \lim_{t\to\infty}X(t,0,\omega)=0\};$$

(C) For each $\xi\in\mathbb{R}^d$ we have $\lim_{t\to\infty}X(t,\xi)=0$ a.s.

Proof. Theorem 2.2.5 shows that (A) implies (C), and (C) clearly implies (B). It remains to prove that (B) implies (A). Define $\xi_0=0$ and for $i=1,\dots,d$ set $\zeta_i=\xi_i-\xi_{i-1}$. Next, for $\omega\in C$, define $V_i(t,\omega)=X(t,\xi_i,\omega)-X(t,\xi_{i-1},\omega)$ for $i=1,\dots,d$. Therefore by hypothesis we have that $V_i(t,\omega)\to 0$ as $t\to\infty$. Moreover, we see that V_i obeys the differential equation

$$V_i'(t,\omega)=AV_i(t,\omega),\quad t\geq 0,\quad V_i(0,\omega)=\xi_i-\xi_{i-1}=\zeta_i.$$

If $\Psi\in\mathbb{R}^{d\times d}$ is the principal matrix solution given by $\Psi'(t)=A\Psi(t)$ with $\Psi(0)=I_d$, then $V_i(t,\omega)=\Psi(t)\zeta_i$. Therefore we have that $\Psi(t)\zeta_i\to 0$ as $t\to\infty$ for each $i=1,\dots,d$. Since $(\zeta_i)_{i=1}^d$ are linearly independent, we have that $\Psi(t)\to 0$ as $t\to\infty$. Hence it follows that all the eigenvalues of A have negative real parts.

Let Y be the solution of (2.1.2). Writing X as

$$dX(t)=(-X(t)+\{X(t)+AX(t)\})\,dt+\sigma(t)\,dB(t),$$

by variation of constants, we see that

$$X(t)=X(0)e^{-t}+\int_0^t e^{-(t-s)}\{X(s)+AX(s)\}\,ds+Y(t),\quad t\geq 0.$$

Therefore, we see that $Y(t,\omega) \to 0$ as $t \to \infty$ for each $\omega \in C$. Since C is an event of positive probability, we see from Theorem 2.2.4 that $Y(t) \to 0$ as $t \to \infty$ a.s., and that therefore $I(\epsilon)$ is finite for all $\epsilon > 0$. We have therefore shown that (B) implies both conditions in (A), as required. □

2.3 Sufficient conditions on σ for stability

Although the condition on (2.2.5) is necessary and sufficient, it can be quite difficult to check in practice. We supply easily checked sufficient conditions on σ for which the solution of (2.1.2) converge to zero, bounded or unbounded. We deduce some conditions which are more easily verified than (2.2.6). In view of Theorem 4.2.4, in what follows, we therefore concentrate on the case when σ is not in $L^2([0,\infty);\mathbb{R}^{d\times r})$. In this case, there exists a pair of integers $(i,j) \in \{1,\dots,d\}\times\{1,\dots,r\}$ such that $\sigma_{ij} \notin L^2([0,\infty);\mathbb{R})$. Note that Y_i obeys

$$dY_i(t) = -Y_i(t)\,dt + \sum_{j=1}^{r} \sigma_{ij}(t)\,dB_j(t), \quad t \geq 0.$$

Thus there exists a standard Brownian motion $\bar{B}_i$ such that

$$dY_i(t) = -Y_i(t)\,dt + \sqrt{\sum_{l=1}^{r} \sigma_{il}^2(t)\,d\bar{B}_i(t)}, \quad t \geq 0.$$

Define

$$\sigma_i^2(t) = \sum_{l=1}^{r} \sigma_{il}^2(t), \quad t \geq 0. \tag{2.3.1}$$

Then $\sigma_i \notin L^2(0,\infty)$, and it is possible to define a number $T_i > 0$ such that $\int_0^t e^{2s}\sigma_i^2(s)\,ds > e^e$ for $t > T_i$ and so one can define a function $\Sigma_i : [T_i,\infty) \to [0,\infty)$ by

$$\Sigma_i(t) = \left(\int_0^t e^{-2(t-s)}\sigma_i^2(s)\,ds\right)^{1/2} \left(\log\log \int_0^t e^{2s}\sigma_i^2(s)\,ds\right)^{1/2}, \quad t \geq T_i. \tag{2.3.2}$$

The significance of the function Σ_i defined in (2.3.2) is that it characterises the largest possible fluctuations of Y_i.

Lemma 2.3.1. *Suppose that σ is continuous and obeys (2.1.1) and σ_i defined by (2.3.1) is such that $\sigma_i \notin L^2(0,\infty)$. Suppose that Y is the unique continuous process which obeys*

(2.1.2). *If* Σ_i *is defined by* (2.3.2) *then* Y_i *obeys*

$$\limsup_{t\to\infty} \frac{\|Y_i(t)\|}{\Sigma_i(t)} = \sqrt{2}, \quad a.s. \tag{2.3.3}$$

Since Y is given by (2.2.1), Lemma 2.3.1 follows by applying the Law of the iterated logarithm for martingales to $M(t) := \int_0^t e^s \sigma_i(s)\, d\bar{B}_i(s)$. This holds because $\sigma_i \notin L^2([0,\infty);\mathbb{R}^{d\times r})$ implies that $\langle M\rangle(t) = \int_0^t e^{2s}\sigma_i^2(s)\, ds \to \infty$ as $t\to\infty$. Its proof is essentially given in [10].

Note that $\lim_{t\to\infty} \|\sigma(t)\|_F^2 \log t = 0$ implies Σ_i in (2.3.2) goes to zero as $t\to\infty$. Also that $\lim_{t\to\infty} \|\sigma(t)\|_F^2 \log t = +\infty$ implies Σ_i in (2.3.2) goes to ∞ as $t\to\infty$. Finally we have that $\liminf_{t\to\infty} \|\sigma(t)\|_F^2 \log t > 0$ implies $\liminf_{t\to\infty} \Sigma_i(t) > 0$. We state the next result which is discussed in more details in next chapter for one–dimensional nonlinear SDE, and also in Chapter 4.

Theorem 2.3.1. *Suppose* $\sigma \in C([0,\infty);\mathbb{R}^{d\times r})$ *and that* Y *is the unique continuous adapted process which obeys* (2.1.2).

(i) If $\lim_{t\to\infty} \|\sigma(t)\|_F^2 \log t = 0$, *then* Y *obeys* (2.2.7).

(ii) If $\lim_{t\to\infty} \|\sigma(t)\|_F^2 \log t \in (0,\infty)$, *then there are* $c_1, c_2 > 0$ *such that* $0 < c_1 \leq \limsup_{t\to\infty} |Y(t)| \leq c_2$ *a.s.*

(iii) If $\lim_{t\to\infty} \|\sigma(t)\|_F^2 \log t = +\infty$, *then* Y *obeys* (2.2.11).

2.4 Proofs

2.4.1 Proof of Theorem 2.2.4

We start by proving a preliminary lemma.

Lemma 2.4.1. *Suppose* $x \in C([0,\infty);[0,\infty))$.

(i) If $\int_0^\infty x(t)\, dt = +\infty$, *then for every* $h > 0$ *there exists a sequence* $(t_n)_{n\geq 0}$ *obeying*

$$t_0 = 0, \quad h \leq t_{n+1} - t_n \leq 3h, \quad n \geq 0$$

such that

$$\sum_{n=0}^{\infty} x(t_n) = +\infty \tag{2.4.1}$$

(ii) If $\int_0^\infty x(t)\,dt < +\infty$, then for every $h > 0$ there exists a sequence $(t_n)_{n\geq 0}$ obeying

$$t_0 = 0, \quad h \leq t_{n+1} - t_n \leq 3h, \quad n \geq 0$$

such that

$$\sum_{n=0}^{\infty} x(t_n) < +\infty \tag{2.4.2}$$

Proof. We start by proving part (i). Let $s_0 = 0$ and define for $n \geq 1$

$$s_n = \inf\{t \in [nh, (n+1)h] \,:\, x(t) = \max_{s\in[nh,(n+1)h]} x(s)\}. \tag{2.4.3}$$

Clearly $s_n \in [nh, (n+1)h]$. Thus

$$+\infty = \int_0^\infty x(t)\,dt = \int_0^h x(s)\,ds + \sum_{n=1}^{\infty} \int_{nh}^{(n+1)h} x(s)\,ds \leq h \max_{s\in[0,h]} x(s) + \sum_{n=1}^{\infty} hx(s_n).$$

Therefore we have

$$\sum_{n=1}^{\infty} x(s_{2n}) + \sum_{n=0}^{\infty} x(s_{2n+1}) = +\infty.$$

Hence we have that either (I) $\sum_{n=1}^\infty x(s_{2n}) = +\infty$ or (II) $\sum_{n=0}^\infty x(s_{2n+1}) = +\infty$.

If case (I) holds, let $t_n = s_{2n}$ for $n \geq 0$. Then $t_0 = 0$ and $(t_n)_{n\geq 0}$ obeys (2.4.1). Note that $t_1 - t_0 = t_1 = s_2 \in [2h, 3h]$. For $n \geq 1$, we have $t_{n+1} - t_n = s_{2n+2} - s_{2n}$. Hence $t_{n+1} - t_n \leq (2n+3)h - 2nh = 3h$. Also $t_{n+1} - t_n \geq (2n+2)h - (2n+1)h = h$. Therefore t_n obeys all the required properties.

If case (II) holds, let $t_n = s_{2n-1}$ for $n \geq 1$ and $t_0 = 0$. Then $t_0 = 0$ and $(t_n)_{n\geq 0}$ obeys (2.4.1). Note that $t_1 - t_0 = t_1 = s_1 \in [h, 2h]$. Therefore $h \leq t_1 - t_0 \leq 2h < 3h$. For $n \geq 1$, we have $t_{n+1} - t_n = s_{2n+1} - s_{2n-1}$. Hence $t_{n+1} - t_n \leq (2n+2)h - (2n-1)h = 3h$. Also $t_{n+1} - t_n \geq (2n+1)h - (2n-1+1)h = h$. Therefore t_n obeys all the required properties.

We now turn to the proof of part (ii). Construct $(t_n)_{n=0}^{\infty}$ recursively as follows: let $t_0 = 0$, and for $n \in \mathbb{N}$

$$t_{n+1} = \inf\{t \in [t_n + h, t_n + 2h] \,:\, x(t) = \min_{t_n+h \le s \le t_n+2h} x(s)\}. \tag{2.4.4}$$

The existence of such a sequence can be proved by induction on n, taking note that x is continuous on the compact interval $[t_n + h, t_n + 2h]$, and hence attains its minimum. By construction, we have

$$t_{n+1} - t_n \ge h > 0, \tag{2.4.5}$$

and $t_{n+1} - t_n \le 2h$. To prove (2.4.2), note that $x(t_{n+1}) \le x(t)$ for $t_n + h \le t \le t_n + 2h$, so by integrating both sides of this inequality over $[t_n + h, t_n + 2h]$, using the non-negativity of $x(\cdot)$ and $t_n + 2h \le t_{n+1} - h + 2h = t_{n+1} + h$ (which follows from (2.4.5)), we get

$$hx(t_{n+1}) \le \int_{t_n+h}^{t_n+2h} x(t)\,dt \le \int_{t_n+h}^{t_{n+1}+h} x(t)\,dt.$$

Summing both sides of this inequality establishes (2.4.2). □

Lemma 2.4.2. *Suppose that I is defined by* (2.2.24).

(i) Suppose that $I(\epsilon) = +\infty$. Then there exists $(t_n)_{n\ge 0}$ independent of $\epsilon > 0$ such that

$$t_0 = 0, \quad 0 < h \le t_{n+1} - t_n \le 3h < +\infty, \quad n \ge 0,$$

and

$$\sum_{n=0}^{\infty} \left\{1 - \Phi\left(\frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2\,ds}}\right)\right\} = +\infty.$$

(ii) Suppose that $I(\epsilon) < +\infty$. Then there exists $(t_n)_{n\ge 0}$ independent of $\epsilon > 0$ such that

$$t_0 = 0, \quad 0 < h \le t_{n+1} - t_n \le 3h < +\infty, \quad n \ge 0,$$

and

$$\sum_{n=0}^{\infty} \left\{1 - \Phi\left(\frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2\,ds}}\right)\right\} < +\infty.$$

Proof. Define

$$\zeta^2(t) = \int_t^{t+c} \|\sigma(s)\|_F^2 \, ds, \quad t \geq 0, \tag{2.4.6}$$

and $\phi_\epsilon(x) = xe^{-\epsilon^2/(2x^2)}\chi_{(0,\infty)}(x)$ for $x \geq 0$. Therefore for $x \geq 0$ we have

$$\frac{1}{\epsilon}\phi_\epsilon(x) = \frac{1}{\epsilon}xe^{-\epsilon^2/(2x^2)}\chi_{(0,\infty)}(x/\epsilon) = \phi_1(x/\epsilon).$$

Then

$$I(\epsilon)/\epsilon = \int_0^\infty \phi_\epsilon\left(\zeta(t)\right)/\epsilon \, dt = \int_0^\infty \phi_1\left(\zeta(t)/\epsilon\right) \, dt.$$

Let $x_\epsilon(t) = \phi_1(\zeta(t)/\epsilon)$ for $t \geq 0$. Clearly x is a non–negative function on $[0,\infty)$, and as $\lim_{x\to 0^+} \phi_1(x) = 0 = \phi_1(0)$, we have that ϕ_1 is continuous and increasing on $[0,\infty)$. Hence x_ϵ is continuous on $[0,\infty)$. Note therefore that $I(\epsilon)/\epsilon = \int_0^\infty x_\epsilon(t)\, dt$.

We are now in a position to prove part (ii). Suppose that $I(\epsilon) < +\infty$. Let $0 < h \leq c/3$. Then by Lemma 2.4.1 part (ii) there exists $(t_n)_{n\geq 0}$ such that $h \leq t_{n+1} - t_n \leq 3h$ and $\sum_{n=0}^\infty \phi_\epsilon(\zeta(t_n)) < +\infty$. Recall that t_n are defined by (2.4.4) i.e., $t_0 = 0$, and for $n \in \mathbb{N}$ we have

$$t_{n+1} = \inf\{t \in [t_n + h, t_n + 2h] \, : \, x_\epsilon(t) = \min_{t_n+h\leq s\leq t_n+2h} x_\epsilon(s)\}.$$

Since $x_\epsilon(t) = \phi_1(\zeta(t)/\epsilon)$ and ϕ_1 is increasing, it follows that

$$t_{n+1} = \inf\{t \in [t_n + h, t_n + 2h] \, : \, \zeta(t) = \min_{t_n+h\leq s\leq t_n+2h} \zeta(s)\},$$

and since ζ is independent of ϵ, it follows that (t_n) is independent of ϵ.

This is equivalent to

$$\sum_{n=0}^\infty \zeta(t_n) \exp\left(-\frac{\epsilon^2}{2}\frac{1}{\zeta(t_n)^2}\right) < +\infty.$$

This implies that $\zeta(t_n) \to 0$ as $n \to \infty$, and by Mills' estimate that

$$\lim_{n\to\infty} \frac{1 - \Phi(\epsilon/\zeta(t_n))}{\frac{\zeta(t_n)}{\epsilon}\exp\left(-\frac{\epsilon^2}{2}\frac{1}{\zeta^2(t_n)}\right)} = \frac{1}{\sqrt{2\pi}}.$$

Hence we have

$$\sum_{n=0}^{\infty}\left\{1-\Phi\left(\frac{\epsilon}{\sqrt{\int_{t_n}^{t_n+c}\|\sigma(s)\|_F^2\,ds}}\right)\right\}=\sum_{n=0}^{\infty}\left\{1-\Phi\left(\frac{\epsilon}{\zeta(t_n)}\right)\right\}<+\infty. \tag{2.4.7}$$

Since $t_{n+1}\le t_n+3h$, and $3h\le c$, we have

$$\int_{t_n}^{t_n+c}\|\sigma(s)\|_F^2\,ds\ge\int_{t_n}^{t_n+3h}\|\sigma(s)\|_F^2\,ds\ge\int_{t_n}^{t_{n+1}}\|\sigma(s)\|_F^2\,ds.$$

Since Φ is increasing, we have

$$1-\Phi\left(\frac{\epsilon}{\sqrt{\int_{t_n}^{t_n+c}\|\sigma(s)\|_F^2\,ds}}\right)\ge 1-\Phi\left(\frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}}\|\sigma(s)\|_F^2\,ds}}\right).$$

By (2.4.7) we have

$$\sum_{n=0}^{\infty}\left\{1-\Phi\left(\frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}}\|\sigma(s)\|_F^2\,ds}}\right)\right\}\le\sum_{n=0}^{\infty}\left\{1-\Phi\left(\frac{\epsilon}{\sqrt{\int_{t_n}^{t_n+c}\|\sigma(s)\|_F^2\,ds}}\right)\right\}<+\infty,$$

which proves part (ii).

We are now in a position to prove part (i). Suppose that $I(\epsilon)=+\infty$. Let $h\in[c,\infty)$. Then by part (i) of Lemma 2.4.1 there exists $(t_n)_{n\ge 0}$ such that $h\le t_{n+1}-t_n\le 3h$ and $\sum_{n=0}^{\infty}\phi_\epsilon(\zeta(t_n))=+\infty$. We now wish to show that the (t_n) are independent of $\epsilon>0$. Since they depend directly on the sequence (s_n) defined by (2.4.3), we must simply show that the sequence (s_n) is independent of ϵ. By (2.4.3) we have

$$s_n=\inf\{t\in[nh,(n+1)h]\,:\,x_\epsilon(t)=\max_{s\in[nh,(n+1)h]}x_\epsilon(s)\}.$$

Since $x_\epsilon(t)=\phi_1(\zeta(t)/\epsilon)$ and ϕ_1 is increasing, it follows that

$$s_n=\inf\{t\in[nh,(n+1)h]\,:\,\zeta(t)=\max_{s\in[nh,(n+1)h]}\zeta(s)\},$$

and since ζ is independent of ϵ, so are (s_n) and therefore (t_n).

Next, $\sum_{n=0}^{\infty} \phi_\epsilon(\zeta(t_n)) = +\infty$ is equivalent to

$$\sum_{n=0}^{\infty} \zeta(t_n) \exp\left(-\frac{\epsilon^2}{2}\frac{1}{\zeta(t_n)^2}\right) = +\infty.$$

Suppose that

$$\sum_{n=0}^{\infty} \left\{1 - \Phi\left(\frac{\epsilon}{\zeta(t_n)}\right)\right\} < +\infty.$$

Then $\zeta(t_n) \to 0$ as $n \to \infty$, and by Mills' estimate that

$$\lim_{n\to\infty} \frac{1 - \Phi(\epsilon/\zeta(t_n))}{\frac{\zeta(t_n)}{\epsilon}\exp\left(-\frac{\epsilon^2}{2}\frac{1}{\zeta^2(t_n)}\right)} = \frac{1}{\sqrt{2\pi}}.$$

Hence we have that

$$\sum_{n=0}^{\infty} \zeta(t_n) \exp\left(-\frac{\epsilon^2}{2}\frac{1}{\zeta(t_n)^2}\right) < +\infty,$$

a contradiction. Therefore we have

$$\sum_{n=0}^{\infty} \left\{1 - \Phi\left(\frac{\epsilon}{\sqrt{\int_{t_n}^{t_n+c} \|\sigma(s)\|_F^2\, ds}}\right)\right\} = \sum_{n=0}^{\infty} \left\{1 - \Phi\left(\frac{\epsilon}{\zeta(t_n)}\right)\right\} = +\infty. \tag{2.4.8}$$

Next, as $c \le h$ and $t_{n+1} \ge t_n + h$ we have

$$\int_{t_n}^{t_n+c} \|\sigma(s)\|_F^2\, ds \le \int_{t_n}^{t_n+h} \|\sigma(s)\|_F^2\, ds \le \int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2\, ds.$$

Since Φ is increasing, we have

$$1 - \Phi\left(\frac{\epsilon}{\sqrt{\int_{t_n}^{t_n+c} \|\sigma(s)\|_F^2\, ds}}\right) \le 1 - \Phi\left(\frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2\, ds}}\right).$$

By (2.4.8) we have

$$\sum_{n=0}^{\infty} \left\{1 - \Phi\left(\frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2\, ds}}\right)\right\}$$
$$\ge \sum_{n=0}^{\infty} \left\{1 - \Phi\left(\frac{\epsilon}{\sqrt{\int_{t_n}^{t_n+c} \|\sigma(s)\|_F^2\, ds}}\right)\right\} = +\infty,$$

which proves part (i). □

Proof of Theorem 2.2.4. To prove part (A), we have by hypothesis that $I(\epsilon) < +\infty$ for all $\epsilon > 0$. Then, by Lemma 2.4.2 part (ii), for every $h \leq c/3$, there exists $(t_n)_{n\geq 0}$ independent of ϵ for which $h \leq t_{n+1} - t_n \leq 3h$ and

$$\sum_{n=0}^{\infty} \left\{ 1 - \Phi\left(\frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2 \, ds}} \right) \right\} < +\infty.$$

Therefore by Theorem 2.2.3 part (A), it follows that $Y(t) \to 0$ as $t \to \infty$ a.s.

To prove part (C), we have by hypothesis that $I(\epsilon) = +\infty$ for all $\epsilon > 0$. Then, by Lemma 2.4.2 part (i), for every $h \geq c$, there exists $(t_n)_{n\geq 0}$ independent of ϵ for which $h \leq t_{n+1} - t_n \leq 3h$ and

$$\sum_{n=0}^{\infty} \left\{ 1 - \Phi\left(\frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2 \, ds}} \right) \right\} = +\infty.$$

Therefore by Theorem 2.2.3 part (C), it follows that $\limsup_{t\to\infty} |Y(t)| = +\infty$ a.s.

To prove part (B), we have by hypothesis that $I(\epsilon) < +\infty$ for all $\epsilon > \epsilon'$. Then, by Lemma 2.4.2 part (ii), for every $h \leq c/3$, there exists $(t_n)_{n\geq 0}$ independent of ϵ for which $h \leq t_{n+1} - t_n \leq 3h$ and

$$\sum_{n=0}^{\infty} \left\{ 1 - \Phi\left(\frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2 \, ds}} \right) \right\} < +\infty.$$

Therefore by Theorem 2.2.3 part (B), it follows that $\limsup_{t\to\infty} |Y(t)| \leq c_2$ a.s.

On the other hand, we have by hypothesis that $I(\epsilon) = +\infty$ for all $\epsilon < \epsilon'$. Then, by Lemma 2.4.2 part (i), for every $h \geq c$, there exists $(\tau_n)_{n\geq 0}$ independent of ϵ for which $h \leq \tau_{n+1} - \tau_n \leq 3h$ and

$$\sum_{n=0}^{\infty} \left\{ 1 - \Phi\left(\frac{\epsilon}{\sqrt{\int_{\tau_n}^{\tau_{n+1}} \|\sigma(s)\|_F^2 \, ds}} \right) \right\} = +\infty.$$

Therefore by Theorem 2.2.3 part (B), it follows that $\limsup_{t\to\infty} |Y(t)| \geq c_1$ a.s. □

2.4.2 Proof of Theorem 2.2.2

Let $z(t,\omega) = X(t,\omega) - Y(t,\omega)$ for $t \geq 0$. Then $z(0) = \xi$ and

$$z'(t,\omega) = AX(t,\omega) + Y(t,\omega) = Az(t,\omega) + g(t,\omega), \quad t \geq 0$$

where $g(t,\omega) = AY(t,\omega) + Y(t,\omega)$. Let Ψ be the unique continuous $d \times d$–valued matrix solution of

$$\Psi'(t) = A\Psi(t), \quad t \geq 0; \quad \Psi(0) = I_d.$$

Since all eigenvalues of A have negative real parts, there exist $K > 0$ and $\lambda > 0$ such that $|\Psi(t)| \leq Ke^{-\lambda t}$ for all $t \geq 0$. Now by variation of constants, z is given by

$$z(t,\omega) = \Psi(t)\xi + \int_0^t \Psi(t-s)g(s,\omega)\,ds, \quad t \geq 0. \tag{2.4.9}$$

To prove statement (A), note that S obeying (2.2.6) implies by Theorem 2.2.1 that $Y(t,\omega) \to 0$ as $t \to \infty$ for all $\omega \in \Omega^*$ where Ω^* is an a.s. event. We show now that $X(t,\xi,\omega) \to 0$ as $t \to \infty$ for every $\xi \in \mathbb{R}^d$ and every $\omega \in \Omega^*$, which would prove statement (A). Since $Y(t,\omega) \to 0$ as $t \to \infty$ we have $g(t,\omega) \to 0$ as $t \to \infty$. Therefore by (2.4.9), we have $z(t,\omega) \to 0$ as $t \to \infty$. Since $Y(t,\omega) \to 0$ as $t \to \infty$ and $\Psi(t) \to 0$ as $t \to \infty$, it follows that $X(t,\omega) \to 0$ as $t \to \infty$.

To prove statement (B), note that all hypotheses of part (B) of Theorem 4.2.1 hold, so therefore we have that there is a deterministic $c_3 > 0$ such that

$$\limsup_{t\to\infty} \|X(t)\| \geq c_3, \quad \text{a.s.}$$

To prove the upper bound, note that because there is a deterministic $c_2 > 0$ such that $\limsup_{t\to\infty} |Y(t)| \leq c_2$ a.s., we have

$$\limsup_{t\to\infty} \|g(t)\| \leq \|I + A\|c_2, \quad \text{a.s.}$$

Using this estimate, the fact that $\Phi(t) \to 0$ as $t \to \infty$, and (2.4.9) we get

$$\limsup_{t\to\infty} \|z(t)\| \leq \int_0^\infty |\Phi(s)|\,ds \cdot \|I + A\|c_2 =: c_4, \quad \text{a.s.}$$

Hence we have $\limsup_{t\to\infty} \|X(t)\| \leq c_2 + c_4 =: c_5$ a.s., which proves the upper estimate in (B).

To prove statement (C), note that all hypotheses of part (A) of Theorem 4.2.1 hold, so therefore we have that $\limsup_{t\to\infty} \|X(t)\| = +\infty$ a.s. as required.

2.5 Proof of Theorem 2.2.3

2.5.1 Preliminary estimates

We start by showing how estimates on the rows of the matrix σ relate to its Frobenius norm. Let $(t_n)_{n\geq 0}$ is an increasing sequence with $t_0 = 0$ and $\lim_{n\to\infty} t_n = +\infty$ and define, by analogy to (2.2.14),

$$S^1_{t\cdot}(\epsilon) = \sum_{n=0}^{\infty}\sum_{i=1}^{d}\left\{1 - \Phi\left(\frac{\epsilon}{\sqrt{\int_{t_n}^{t_{n+1}}\sum_{j=1}^{r}\sigma_{ij}^2(s)\,ds}}\right)\right\}. \tag{2.5.1}$$

Define

$$\theta^2(n) = \int_{t_n}^{t_{n+1}} \|\sigma(s)\|_F^2\,ds, \tag{2.5.2}$$

$$\theta_i^2(n) = \int_{t_n}^{t_{n+1}} \sum_{j=1}^{r}\sigma_{ij}^2(s)\,ds, \quad i = 1,\dots,d. \tag{2.5.3}$$

We can see that as $S^1_{t\cdot}$ is a monotone function of ϵ, it is the case that either (i) $S^1_{t\cdot}(\epsilon)$ is finite for all $\epsilon > 0$; (ii) there is $\epsilon_1' > 0$ such that for all $\epsilon > \epsilon_1'$ we have $S^1_{t\cdot}(\epsilon) < +\infty$ and $S^1_{t\cdot}(\epsilon) = +\infty$ for all $\epsilon < \epsilon_1'$; and (iii) $S^1_{t\cdot}(\epsilon) = +\infty$ for all $\epsilon > 0$. In the next lemma, we show that $S_{t\cdot}$ defined by (2.2.14) is always finite if and only if $S^1_{t\cdot}$ is; that $S_{t\cdot}$ is infinite if and only if $S^1_{t\cdot}$ is; and that $S_{t\cdot}$ and $S^1_{t\cdot}$ are sometimes finite and sometimes infinite only if the other is.

Lemma 2.5.1. *Let $(t_n)_{n\geq 0}$ be an increasing sequence with $t_0 = 0$ and $\lim_{n\to\infty} t_n = +\infty$. Suppose that $S_{t\cdot}$ is defined by (2.2.14) and that $S^1_{t\cdot}$ is defined by (2.5.1).*

(a) The following are equivalent:

(i) $S_{t\cdot}(\epsilon) < +\infty$ for all $\epsilon > 0$;

(ii) $S^1_{t\cdot}(\epsilon) < +\infty$ for all $\epsilon > 0$.

(b) The following are equivalent:

(i) There exists $\epsilon' > 0$ such that for all $\epsilon > \epsilon'$ we have $S_{t\cdot}(\epsilon) < +\infty$ and $S_{t\cdot}(\epsilon) = +\infty$ for all $\epsilon < \epsilon'$;

(ii) *There exists* $\epsilon_1' > 0$ *such that for all* $\epsilon > \epsilon_1'$ *we have* $S^1_{t.}(\epsilon) < +\infty$ *and* $S^1_{t.}(\epsilon) = +\infty$ *for all* $\epsilon < \epsilon_1'$;

(c) *The following are equivalent:*

(i) $S_{t.}(\epsilon) = +\infty$ *for all* $\epsilon > 0$;

(ii) $S^1_{t.}(\epsilon) = +\infty$ *for all* $\epsilon > 0$.

Proof. With θ and θ_i defined by (2.5.2) and (2.5.3), we have $\theta^2(n) \geq \theta_i(n)^2$ for each $i = 1, \dots, d$. Thus

$$\sum_{i=1}^{d} \left\{ 1 - \Phi\left(\frac{\epsilon}{\theta_i(n)}\right) \right\} \leq d\left(1 - \Phi\left(\frac{\epsilon}{\theta(n)}\right)\right). \tag{2.5.4}$$

Suppose, for each n, that $Z_i(n)$ for $i = 1, \dots, d$ are independent standard normal random variables. Define $Z(n) = (Z_1(n), Z_2(n), \dots, Z_d(n))$ and suppose that $(Z(n))_{n\geq 0}$ are a sequence of independent normal vectors. Define finally

$$X_i(n) = \theta_i(n) Z_i(n), \quad X(n) = \sum_{i=1}^{d} X_i(n), \quad n \geq 0.$$

Then we have that X_i is a zero mean normal with variance θ_i^2 and X is a zero mean normal with variance θ^2. Define $Z^*(n) = X(n)/\theta(n)$ is a standard normal random variable. Therefore we have that

$$\mathbb{P}[|X(n)| > \epsilon] = \mathbb{P}[|Z^*(n)| \geq \epsilon/\theta(n)] = 2\mathbb{P}[Z^*(n) \geq \epsilon/\theta(n)] = 2\left(1 - \Phi\left(\frac{\epsilon}{\theta(n)}\right)\right). \tag{2.5.5}$$

With $A_i(n) = \{|X_i(n)| \leq \epsilon/d\}$, $B(n) = \{\sum_{i=1}^{d} |X_i(n)| \leq \epsilon\}$, then $\cap_{i=1}^{d} A_i(n) \subseteq B(n)$, so

$$\mathbb{P}\left[|X(n)| > \epsilon\right] \leq \mathbb{P}[\overline{B}(n)] \leq \mathbb{P}\left[\overline{\cap_{i=1}^{d} A_i(n)}\right] = \mathbb{P}\left[\cup_{i=1}^{d} \overline{A_i(n)}\right] \leq \sum_{i=1}^{d} \mathbb{P}\left[\overline{A_i(n)}\right].$$

Since $X_i = \theta_i Z_i$, we have

$$\mathbb{P}\left[|X(n)| > \epsilon\right] \leq 2\sum_{i=1}^{d} \mathbb{P}\left[X_i(n) \geq \epsilon/d\right] = 2\sum_{i=1}^{d} \left\{1 - \Phi\left(\frac{\epsilon/d}{\theta_i(n)}\right)\right\}. \tag{2.5.6}$$

By (2.5.5) and (2.5.6), we get

$$1 - \Phi\left(\frac{\epsilon}{\theta(n)}\right) \leq \sum_{i=1}^{d} \left\{1 - \Phi\left(\frac{\epsilon/d}{\theta_i(n)}\right)\right\}. \tag{2.5.7}$$

From (2.5.4), we can see that $S_{t.}(\epsilon) < +\infty$ implies that $S^1_{t.}(\epsilon) < +\infty$ and from (2.5.7) that $S^1_{t.}(\epsilon/d) < +\infty$ implies $S_{t.}(\epsilon) < +\infty$. Therefore, we have that part (a) holds. Part (c) holds similarly, because from (2.5.4) we have that $S^1_{t.}(\epsilon) = +\infty$ implies $S_{t.}(\epsilon) = +\infty$, and from (2.5.7) we have that $S^1_{t.}(\epsilon/d) = +\infty$ implies $S_{t.}(\epsilon) = +\infty$. As to the proof of part (b), suppose that (i) holds. Then by (2.5.4), we can see that $S^1_{t.}(\epsilon) \leq S_{t.}(\epsilon) < +\infty$ for all $\epsilon < \epsilon'$, and by (2.5.7) that $S^1_{t.}(\epsilon/d) \geq S(\epsilon) = +\infty$ for all $\epsilon < \epsilon'$. Therefore, there exists $\epsilon'_1 \in [\epsilon', \epsilon'/d]$ such that (ii) holds. The proof that (ii) implies (i) is similar. □

2.5.2 Organisation of the proof of Theorem 2.2.3

The proof is divided into four parts: we first derive estimates and identities common to parts (A)–(C) of Theorem 2.2.3. Second, we prove (2.2.11), which yields (C). Next, we obtain the lower bound on the limit superior in (2.2.9), which is part of (B). Finally, we find the upper bound on the limit superior in (2.2.9), which completes the proof of the limsup in (B). We also prove (2.2.7), which proves (A).

The proof of the liminf in (B) and the ergodic–type result in part (B) are not given at this point. Instead, we prove them for the solution of the general equation (2.2.13). The results for Y are then simply corollaries of this general result, with $A = -I_d$.

2.5.3 Preliminary estimates

Let $V(j) := \int_{t_{j-1}}^{t_j} e^{s-t_j}\sigma(s)\,dB(s)$, $j \geq 1$. Define $V_i(j) = \langle V(j), \mathbf{e}_i\rangle$. Then

$$V_i(j) = \sum_{l=1}^{r} \int_{t_{j-1}}^{t_j} e^{s-t_j}\sigma_{il}(s)\,dB_l(s).$$

For each fixed i, Then $(V_i(j))_{j\geq 1}$ is a sequence of independently and normally distributed random variables with mean zero and variance

$$v_i^2(j-1) := \text{Var}[V_i(j)] = \sum_{l=1}^{r} \int_{t_{j-1}}^{t_j} e^{2s-2t_j}\sigma_{il}^2(s)\,ds \leq \sum_{l=1}^{r} \int_{t_{j-1}}^{t_j} \sigma_{il}^2(s)\,ds = \theta_i^2(j-1).$$

Similarly, $v_i^2(j-1) \geq e^{2(t_{j-1}-t_j)}\theta_i^2(j-1) \geq e^{-2\beta}\theta_i^2(j-1)$, so $v_i(j-1)=0$ if and only if $\theta_i(j-1)=0$. Also, by (2.2.1), we get

$$Y(t_n) = e^{-t_n}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} e^s\sigma(s)\,dB(s) = \sum_{j=1}^{n} e^{-(t_n-t_j)}V(j), \quad n\geq 1. \tag{2.5.8}$$

This also implies that for $n \geq 1$ we have

$$Y(t_{n+1}) = V(n+1) + \sum_{j=1}^{n} e^{-(t_{n+1}-t_j)}V(j) = V(n+1) + e^{-(t_{n+1}-t_n)}Y(t_n). \tag{2.5.9}$$

Next, as $V_i(j)$ is normally distributed, we have $\mathbb{P}[|V_i(j)|>\epsilon] = 2(1-\Phi(\epsilon/v_i(j-1))$. However, as Φ is increasing, and $e^{-\beta}\theta_i(j-1) \leq v_i(j-1) \leq \theta_i(j-1)$, we have $1-\Phi(\epsilon/e^{-\beta}\theta_i(j-1)) \leq 1-\Phi(\epsilon/v_i(j-1)) \leq 1-\Phi(\epsilon/\theta_i(j-1))$, so

$$2\left(1-\Phi(\epsilon/e^{-\beta}\theta_i(j-1))\right) \leq \mathbb{P}[|V_i(j)|>\epsilon] \leq 2\left(1-\Phi(\epsilon/\theta_i(j-1))\right), \quad j\geq 1. \tag{2.5.10}$$

Note that $|V(j)|_1 = \sum_{i=1}^d |V_i(j)|$. Thus, as $|V(j)|_1 \geq |V_i(j)|$, we have that $\mathbb{P}[|V(j)|_1 \geq \epsilon] \geq \mathbb{P}[|V_i(j)|\geq\epsilon]$ for each $i=1,\ldots,d$. Therefore

$$d\mathbb{P}[|V(j)|_1 \geq \epsilon] \geq \sum_{i=1}^{d}\mathbb{P}[|V_i(j)|\geq\epsilon]. \tag{2.5.11}$$

On the other hand, defining $A_i(j) = \{|V_i(j)| \leq \epsilon/d\}$ and $B(j) = \{|V(j)|_1 \leq \epsilon\}$, we see that $\cap_{i=1}^d A_i(j) \subseteq B(j)$. Then

$$\mathbb{P}[|V(j)|_1 \geq \epsilon] = \mathbb{P}[\overline{B(j)}] \leq \mathbb{P}\left[\overline{\cap_{i=1}^d A_i(j)}\right] = \mathbb{P}\left[\cup_{i=1}^d \overline{A_i(j)}\right] \leq \sum_{i=1}^{d}\mathbb{P}\left[|V_i(j)|\geq\epsilon/d\right]. \tag{2.5.12}$$

2.5.4 Proof of part (C)

Suppose $S_{t.}$ obeys (2.2.22). Then by Lemma 2.5.1 we have that $S_{t.}^1(\epsilon)=+\infty$ for every $\epsilon>0$. Therefore by (2.5.10), $\sum_{j=1}^\infty\sum_{i=1}^d \mathbb{P}[|V_i(j)|>\epsilon]=+\infty$ for every $\epsilon>0$. Therefore, by (2.5.11) we have $\sum_{j=1}^\infty \mathbb{P}[|V(j)|_1\geq\epsilon]=+\infty$ for all $\epsilon>0$. Since $(V(j))_{j\geq1}$ are independent, it follows from the Borel–Cantelli Lemma that for every $\epsilon>0$ $\limsup_{n\to\infty}|V(n)|_1>\epsilon$ a.s. Letting $\epsilon\to\infty$ through the integers, we have $\limsup_{n\to\infty}|V(n)|_1=+\infty$ a.s. Thus by (2.5.9), we obtain $\limsup_{n\to\infty}|Y(t_n)|_1=+\infty$ a.s., which implies that $\limsup_{t\to\infty}|Y(t)|_1=+\infty$ a.s.

2.5.5 Proof of lower bound in part (B)

Suppose that $S_{t\cdot}$ obeys (2.2.21). There exists an $\epsilon < \epsilon'$ such that $\sum_{j=1}^{\infty} \{1 - \Phi(\epsilon/\theta(j))\} = +\infty$. Therefore, by Lemma 2.5.1, it follows that there exists $\epsilon_1' > 0$ such that for all $\epsilon/e^{-\beta} < \epsilon_1'$ we have

$$\sum_{j=1}^{\infty}\sum_{i=1}^{d}\left\{1 - \Phi\left(\frac{\epsilon}{e^{-\beta}\theta_i(j)}\right)\right\} = +\infty.$$

By (2.5.10) we therefore have

$$\sum_{j=1}^{\infty}\sum_{i=1}^{d}\mathbb{P}[|V_i(j)| > \epsilon] \geq \sum_{j=1}^{\infty} 2\left\{1 - \Phi\left(\frac{\epsilon e^{-\beta}}{e^{-\beta}\theta(j-1)}\right)\right\} = +\infty.$$

Therefore by (2.5.11) we have

$$\sum_{j=1}^{\infty}\mathbb{P}[|V(j)|_1 > \epsilon] = +\infty.$$

By the independence of $(V(j))$ together with the Borel–Cantelli Lemma, it follows that $\limsup_{n\to\infty} |V(n)|_1 \geq \epsilon$ a.s. Letting $\epsilon \uparrow \epsilon_1' e^{-\beta}$ through the rational numbers gives $\limsup_{n\to\infty} |V(n)|_1 \geq \epsilon_1' e^{-\beta}$ on Ω_1, an a.s. event. By (2.5.9), $V(n+1) = Y(t_{n+1}) - e^{-(t_{n+1}-t_n)}Y(t_n)$, so we have

$$\epsilon_1' e^{-\beta} \leq \limsup_{n\to\infty} |V(n,\omega)|_1 \leq (1 + e^{-\alpha}) \limsup_{n\to\infty} |Y(t_n,\omega)|_1, \quad \text{for } \omega \in \Omega_1.$$

Thus

$$\limsup_{n\to\infty} |Y(t_n)|_1 \geq \epsilon_1' e^{-\beta}/(1 + e^{-\alpha}), \quad \text{a.s.},$$

which implies $\limsup_{t\to\infty} |Y(t)|_1 \geq \epsilon_1' e^{-\beta}/(1 + e^{-\alpha}) =: c_1$, a.s.

2.5.6 Proof of upper bounds in parts (A) and (B)

Suppose that

$$\sum_{j=1}^{\infty}\left\{1 - \Phi\left(\frac{\epsilon}{\theta(j)}\right)\right\} < +\infty. \tag{2.5.13}$$

In part (A), (2.5.13) holds for all $\epsilon > 0$, while in part (B) it holds for all $\epsilon > \epsilon'$. By (2.5.13) and (2.5.4) we have

$$\sum_{j=1}^{\infty}\sum_{i=1}^{d}\left\{1 - \Phi\left(\frac{\epsilon}{\theta_i(j)}\right)\right\} < +\infty,$$

and hence by (2.5.10) we have

$$\sum_{j=1}^{\infty}\sum_{i=1}^{d}\mathbb{P}[|V_i(j)|\geq\epsilon]<+\infty.$$

By the Borel–Cantelli lemma, it follows that $\limsup_{n\to\infty}|V_i(n)|\leq\epsilon$ a.s. Now from (2.5.8), we have that

$$Y_i(t_n)=\sum_{j=1}^{n}e^{-(t_n-t_j)}V_i(j),$$

so therefore, as $t_n-t_j\geq\alpha(n-j)$ for $j=1,\dots,n$, we have that

$$|Y_i(t_n)|\leq\sum_{j=1}^{n}e^{-(t_n-t_j)}|V_i(j)|\leq\sum_{j=1}^{n}e^{-\alpha(n-j)}|V_i(j)|,$$

so

$$\limsup_{n\to\infty}|Y_i(t_n)|\leq\epsilon\sum_{j=0}^{\infty}e^{-\alpha j}=\epsilon\frac{1}{1-e^{-\alpha}},\quad\text{a.s.}\tag{2.5.14}$$

Next let $t\in[t_n,t_{n+1})$. Therefore, from (2.1.2) we have

$$Y_i(t)=Y_i(t_n)e^{-(t-t_n)}+\sum_{l=1}^{r}e^{-t}\int_{t_n}^{t}e^s\sigma_{il}(s)\,dB_l(s),\quad t\in[t_n,t_{n+1}).$$

Therefore

$$\begin{aligned}&\max_{t\in[t_n,t_{n+1}]}|Y_i(t)|\\&\leq|Y_i(t_n)|+\max_{t\in[t_n,t_{n+1}]}e^{-t}\left|\sum_{l=1}^{r}\int_{t_n}^{t}e^s\sigma_{il}(s)\,dB_l(s)\right|\leq|Y_i(t_n)|+Z_i(n),\end{aligned}\tag{2.5.15}$$

where

$$Z_i(n):=e^{-t_n}\max_{t\in[t_n,t_{n+1}]}\left|\sum_{l=1}^{r}\int_{t_n}^{t}e^s\sigma_{il}(s)\,dB_l(s)\right|,\quad n\geq1.$$

Fix $n\in\mathbb{N}$. Now

$$\mathbb{P}[Z_i(n)>\epsilon]=\mathbb{P}\left[\max_{t\in[t_n,t_{n+1}]}\left|\sum_{l=1}^{r}\int_{t_n}^{t}e^s\sigma_{il}(s)\,dB_l(s)\right|>\epsilon e^{t_n}\right]$$

Define $\tau_i(t):=\sum_{l=1}^{r}\int_{t_n}^{t}e^{2s}\sigma_{il}^2(s)\,ds$ for $t\in[n,n+1]$. Consider

$$C_{in}(t)=\sum_{l=1}^{r}\int_{t_n}^{t}e^s\sigma_{il}(s)\,dB_l(s),\quad t\in[t_n,t_{n+1}].$$

Then $C_{in}=\{C_{in}(t):t_n\leq t\leq t_{n+1}\}$ is a continuous martingale with $\langle C_{in}\rangle(t)=\tau_i(t)$ for $t\in[t_n,t_{n+1}]$. Therefore, by the martingale time change theorem [65, Theorem V.1.6], there exists a standard Brownian motion B^*_{in} such that $C_{in}(t) = B^*_{in}(\tau_i(t))$ for $t \in [t_n, t_{n+1}]$, and so we have

$$\begin{aligned}
\mathbb{P}[Z_i(n) > \epsilon] &= \mathbb{P}\left[\max_{t\in[t_n,t_{n+1}]}\left|B^*_{in}\left(\sum_{l=1}^{r}\int_{t_n}^{t} e^{2s}\sigma^2_{il}(s)\,ds\right)\right| > \epsilon e^{t_n}\right] \\
&= \mathbb{P}\left[\max_{u\in[0,\tau_i(n+1)]}|B^*_{in}(u)| > \epsilon e^{t_n}\right] \\
&= \mathbb{P}\left[\left\{\max_{u\in[0,\tau_i(n+1)]} B^*_{in}(u) > \epsilon e^{t_n}\right\} \cup \left\{\max_{u\in[0,\tau_i(n+1)]} -B^*_{in}(u) > \epsilon e^{t_n}\right\}\right] \\
&\le \mathbb{P}\left[\max_{u\in[0,\tau_i(n+1)]} B^*_{in}(u) > \epsilon e^{t_n}\right] + \mathbb{P}\left[\max_{u\in[0,\tau_i(n+1)]} -B^*_{in}(u) > \epsilon e^{t_n}\right] \\
&= \mathbb{P}\left[|B^*_{in}(\tau_i(n+1))| > \epsilon e^{t_n}\right] + \mathbb{P}\left[|B^{**}_{in}(\tau_i(n+1))| > \epsilon e^{t_n}\right],
\end{aligned}$$

where $B^{**}_{in} = -B^*_{in}$ is a standard Brownian motion. Recall that if W is a standard Brownian motion that $\max_{s\in[0,t]} W(s)$ has the same distribution as $|W(t)|$. Therefore, as $B^*_{in}(\tau_i(n+1))$ is normally distributed with zero mean we have

$$\begin{aligned}
\mathbb{P}[Z_i(n) > \epsilon] &\le 2\mathbb{P}\left[|B^*_{in}(\tau_i(n+1))| > \epsilon e^{t_n}\right] = 4\mathbb{P}\left[B^*_{in}(\tau_i(n+1)) > \epsilon e^{t_n}\right] \\
&= 4\left(1-\Phi\left(\frac{\epsilon e^{t_n}}{\sqrt{\tau_i(n+1)}}\right)\right) = 4\left(1-\Phi\left(\frac{\epsilon}{\sqrt{e^{-2t_n}\tau_i(n+1)}}\right)\right).
\end{aligned}$$

If we interpret $\Phi(\infty) = 1$, this formula holds valid in the case when $\tau_i(n+1) = 0$, because in this situation $Z_i(n) = 0$ a.s. Now

$$\begin{aligned}
e^{-2t_n}\tau_i(n+1) &= e^{-2t_n}\sum_{l=1}^{r}\int_{t_n}^{t_{n+1}} e^{2s}\sigma^2_{il}(s)\,ds \\
&\le e^{2(t_{n+1}-t_n)}\int_{n}^{n+1}\sigma^2_{il}(s)\,ds \le e^{2\beta}\theta_i^2(n).
\end{aligned}$$

Since Φ is increasing, we have

$$\mathbb{P}[Z_i(n) > \epsilon] \le 4\left(1-\Phi\left(\frac{\epsilon}{\sqrt{e^{-2t_n}\tau_i(n+1)}}\right)\right) \le 4\left(1-\Phi\left(\frac{\epsilon}{e^{\beta}\theta_i(n)}\right)\right).$$

Therefore we have

$$\mathbb{P}[Z_i(n) > \epsilon e^{\beta}] \le 4\left(1-\Phi\left(\frac{\epsilon}{\theta_i(n)}\right)\right), \quad n \ge 0. \tag{2.5.16}$$

Hence

$$\sum_{n=1}^{\infty}\mathbb{P}[Z_i(n) > \epsilon e^{\beta}] < +\infty,$$

so by the Borel–Cantelli lemma, we have that

$$\limsup_{n\to\infty} Z_i(n) \le \epsilon e^{\beta}, \quad \text{a.s.} \tag{2.5.17}$$

Therefore by (2.5.15), (2.5.14) and (2.5.17), we have that

$$\limsup_{t\to\infty} |Y_i(t)| \le (1/(1-e^{-\alpha}) + e^{\beta})\epsilon, \quad \text{a.s.}$$

and so

$$\limsup_{t\to\infty} |Y(t)|_1 \le d(1/(1-e^{-\alpha}) + e^{\beta})\epsilon, \quad \text{a.s.} \tag{2.5.18}$$

If (2.2.6) holds, (2.5.18) implies that $Y(t) \to 0$ as $t \to \infty$ a.s.

If the first part of (2.2.8) holds, then (2.5.18) holds for every $\epsilon > \epsilon'$. Thus, letting $\epsilon \downarrow \epsilon'$ through the rational numbers we have $\limsup_{t\to\infty} |Y(t)|_1 \le d(1/(1-e^{-\alpha}) + e^{\beta})\epsilon' =: c_2$ a.s., proving (2.2.9).

2.5.7 Proof of (2.2.28) in part (B) of Theorem 2.2.5

We note that $I(\epsilon)$ being finite is equivalent to $S(\epsilon) < +\infty$. Therefore we have that $\int_n^{n+1} \|\sigma(s)\|_F^2\,ds \to 0$ as $n \to \infty$ which implies that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \|\sigma(s)\|_F^2\,ds = 0. \tag{2.5.19}$$

Since all the eigenvalues of A have negative real part, there exists a $d\times d$ positive definite matrices M such that

$$A^T M + MA = -I_d.$$

(see for example Horn and Johnson [42] or Rugh [68]). Define $V(x) = x^T M x$ for all $x \in \mathbb{R}^d$. Notice that

$$\frac{\partial V}{\partial x_i} = [2Mx]_i = \sum_{k=1}^{d} 2M_{ik}x_k.$$

Therefore we have

$$\frac{\partial^2 V}{\partial x_i \partial x_j}(x) = 2M_{ij}.$$

Let $X_i(t) = \langle X(t), e_i\rangle$. Notice that the cross–variation of X_i and X_j obeys

$$d\langle X_i, X_j\rangle(t) = \sum_{k=1}^{r} \sigma_{ik}(t)\sigma_{jk}(t)\,dt.$$

Therefore, as V is a C^2 function, by the multidimensional version of Itô's formula, we have

$$dV(X(t)) = \sum_{i=1}^{d} \frac{\partial V}{\partial x_i}(X(t))dX_i(t) + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial^2 V}{\partial x_i \partial x_j}(X(t))d\langle X_i, X_j\rangle(t).$$

Hence

$$dV(X(t)) = \langle 2MX(t), AX(t)\rangle\, dt + \sum_{i=1}^{d}\sum_{j=1}^{d} M_{ij} \sum_{k=1}^{r} \sigma_{ik}(t)\sigma_{jk}(t)\, dt + \langle 2MX(t), \sigma(t)\, dB(t)\rangle.$$

Next, we note that because $M = M^T$ and $A^T M + MA = -I_d$, we have

$$\begin{aligned}\langle 2Mx, Ax\rangle &= \langle (M + M^T)x, Ax\rangle = \langle Mx, Ax\rangle + \langle Ax, M^T x\rangle \\ &= (Mx)^T Ax + (Ax)^T M^T x = x^T M^T Ax + x^T A^T Mx = -x^T x.\end{aligned}$$

Also, since M is positive definite, there exists a $d \times d$ matrix P such that $M = PP^T$, so we have

$$\begin{aligned}\sum_{i=1}^{d}\sum_{j=1}^{d} M_{ij} \sum_{k=1}^{r} \sigma_{ik}(t)\sigma_{jk}(t) &= \sum_{i=1}^{d}\sum_{k=1}^{r} \left(\sum_{j=1}^{d} M_{ij}\sigma_{jk}(t) \right) \sigma_{ik}(t) \\ &= \sum_{i=1}^{d}\sum_{k=1}^{r} [M\sigma(t)]_{ik}\sigma^T_{ki}(t) = \sum_{i=1}^{d} [M\sigma(t)\sigma(t)^T]_{ii} \\ &= \sum_{i=1}^{d} [PP^T\sigma(t)\sigma(t)^T]_{ii} = \mathrm{tr}(PP^T\sigma(t)\sigma(t)^T) \\ &= \mathrm{tr}(P^T\sigma(t)\sigma(t)^T P) = \|P^T\sigma(t)\|_F^2.\end{aligned}$$

where we have used the fact that $\|C\|_F^2 = \mathrm{tr}(CC^T)$ for any matrix C and that $\mathrm{tr}(CD) = \mathrm{tr}(DC)$ for square matrices C and D. Thus

$$\begin{aligned}V(X(t)) = V(\xi) - \int_0^t X(s)^T X(s)\, ds + \int_0^t \|P^T\sigma(s)\|_F^2\, ds \\ + \sum_{j=1}^{r} \int_0^t \left\{ \sum_{i=1}^{d} [2MX(s)]_i \sigma_{ij}(s) \right\} dB_j(s). \quad (2.5.20)\end{aligned}$$

We consider the third term on the righthand side of (2.5.20). Since $\|P^T\sigma(s)\|_F \le \|P^T\|_F \|\sigma(s)\|_F$, from (2.5.19), we have that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \|P^T\sigma(s)\|_F^2\, ds = 0.$$

Let $K(t)$ be the fourth term on the righthand side of (2.5.20). Then K is a local martingale with

$$\langle K\rangle(t) = \sum_{j=1}^{r}\int_0^t \left\{\sum_{i=1}^{d}[2MX(s)]_i\sigma_{ij}(s)\right\}^2 ds.$$

Hence by the Cauchy–Schwarz inequality, we have

$$\langle K\rangle(t) \le \sum_{j=1}^{r}\int_0^t \sum_{i=1}^{d}[2MX(s)]_i^2 \sum_{i=1}^{d}\sigma_{ij}^2(s)\,ds = 4\int_0^t \|MX(s)\|_2^2\|\sigma(s)\|_F^2\,ds.$$

Since $t\mapsto\|X(t)\|$ is bounded a.s., we may use (2.5.19) to get

$$\lim_{t\to\infty}\frac{1}{t}\langle K\rangle(t) = 0, \quad \text{a.s.}$$

Hence by the strong law of large numbers for martingales, we have that $K(t)/t\to 0$ as $t\to\infty$ a.s. Since $t\mapsto\|X(t)\|$ is bounded a.s. we have that $V(X(t))/t\to 0$ as $t\to\infty$ a.s. Therefore, returning to (2.5.20), we get

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t X(s)^T X(s)\,ds = 0, \quad \text{a.s.} \tag{2.5.21}$$

Suppose now that there is an event A_1 with $\mathbb{P}[A_1]>0$ such that

$$A_1 = \{\omega : \liminf_{t\to\infty}\|X(t,\omega)\| > 0\}.$$

Since $t\mapsto\|X(t)\|$ is bounded, it follows that for each $\omega\in A_1$, there is a positive and finite $\bar{x}(\omega)$ such that

$$\liminf_{t\to\infty}\|X(t,\omega)\|_2 =: \bar{x}(\omega).$$

Therefore for $\omega\in A_1$ we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t X(s,\omega)^T X(s,\omega)\,ds \ge \bar{x}(\omega) > 0.$$

Therefore

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t X(s)^T X(s)\,ds > 0, \quad \text{a.s. on } A_1,$$

which contradicts (2.5.21), because $\mathbb{P}[A_1]>0$. Therefore, it must be the case that $\mathbb{P}[A_1]=0$, which implies that $\mathbb{P}[\overline{A}_1]=1$, or $\liminf_{t\to\infty}\|X(t)\|=0$ a.s. as required.

Chapter 3

Asymptotic Classification of Solutions of Scalar Nonlinear SDEs

3.1 Introduction

In the previous chapter, a complete classification of the asymptotic behaviour was given for an affine stochastic differential equation in the finite dimensional case. In Chapter 1, we saw that if a nonlinear equation is perturbed deterministically, and the mean reverting force is weak as the solution departs far from equilibrium, then solutions may not converge if the maximal size of the perturbation does not decay sufficiently rapidly. Therefore, if we consider scalar nonlinear equations (as in Chapter 1), but perturb them stochastically (as in Chapter 2), it is of interest to ask whether we can perform a classification of the asymptotic behaviour in a manner that equals our success in these earlier problems.

Therefore, in this chapter, we characterise the global asymptotic stability of the unique equilibrium of a scalar deterministic ordinary differential equation when it is subjected to a stochastic perturbation independent of the state. Another major task is to classify the asymptotic behaviour of solutions into convergent, recurrent or bounded under some stronger mean reverting condition on the nonlinearity. What is of special interest is that, in the former case, solutions will be globally convergent under exactly the same conditions on the intensity of the stochastic perturbation σ that apply in the linear case, and indeed, these conditions which ensure stability are *entirely independent* of the type of nonlinear mean reversion: unlike the deterministic case we *do not need* to make any assumption on the strength of the mean–reversion, merely that it is always present. In this sense, by comparing with the results of Chapter 1, we can think of deterministic scalar ODEs as being more robust to exogenous stochastic destabilisation than exogenous deterministic destabilisation.

Furthermore, the classification of the asymptotic behaviour of solutions which can be obtained in the case when there is slightly stronger mean–reversion relies once again on the exactly the same criteria needed to classify the asymptotic behaviour in the affine case, so that the conditions which guarantee bounded or unbounded solutions are once again independent of the type of nonlinear mean reversion. Such results suggest that the affine equation must be of great assistance to their proof, and accordingly we discuss in detail how the results obtained from Chapter 2 (specialised and extended to the scalar case) can assist in analysing the asymptotic behaviour of solutions of the non–linear stochastic differential equation in the scalar case. This analysis will also motivate the extension of the results in this chapter to finite dimensional equations in Chapter 4, and also points the way to explaining how the finite dimensional results from Chapter 2 can be used to achieve this task.

To make our discussion more precise, let us fix a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t\geq 0}, \mathbb{P})$. Let B be a standard one–dimensional Brownian motion which is adapted to $(\mathcal{F}(t))_{t\geq 0}$. We consider the stochastic differential equation

$$dX(t) = -f(X(t))\,dt + \sigma(t)\,dB(t), \quad t \geq 0; \quad X(0) = \xi \in \mathbb{R}. \tag{3.1.1}$$

We suppose that f obeys (1.2.2) and that σ obeys

$$\sigma \in C([0,\infty);\mathbb{R}). \tag{3.1.2}$$

These conditions ensure the existence of a continuous adapted process which obeys (3.1.1) on $[0,\infty)$, and we will refer to any such process as a solution. We do not rule out the existence of more than one process, but part of our analysis will show that all solutions share the same asymptotic properties. Hypotheses such as local Lipschitz continuity or monotonicity can be imposed in order to guarantee that there is a unique such solution. The condition (1.2.2) on f inspire the dissipative condition (4.1.9) in Chapter 4.

In the case when σ is identically zero, it follows under the hypothesis (1.2.2) that any solution x of equation (1.2.5)

$$x'(t) = -f(x(t)), \quad t > 0; \quad x(0) = \xi,$$

obeys equation (1.2.6), that is

$$\lim_{t\to\infty} x(t;\xi) = 0 \text{ for all } \xi \in \mathbb{R}.$$

Clearly $x(t) = 0$ for all $t \geq 0$ if $\xi = 0$. The question naturally arises: if any solution x of (1.2.5) obeys (1.2.6), under what conditions on f and σ does any solution X of (3.1.1) obey

$$\lim_{t\to\infty} X(t,\xi) = 0, \quad \text{a.s. for each } \xi \in \mathbb{R}. \tag{3.1.3}$$

The convergence phenomenon captured in (3.1.3) for any solution of (3.1.1) is often called almost sure *global convergence* (or *global stability* for the solution of (1.2.5)), because solutions of the perturbed equation (3.1.1) converge to the zero equilibrium solution of the underlying unperturbed equation (1.2.5).

Chan and Williams [31] proved the following result:

Theorem 3.1.1. *Suppose*

$$\lim_{x\to\infty} f(x) = \infty, \quad \lim_{x\to-\infty} f(x) = -\infty, \tag{3.1.4}$$

and that σ is a continuous function with $t \mapsto \sigma^2(t)$ non–increasing. Let X be the unique solution of (3.1.1)*. Then the following are equivalent:*

(A)

$$\lim_{n\to\infty} \sigma^2(t)\log(t) = 0; \tag{3.1.5}$$

(B) $X(t) \to 0$ as $t \to \infty$ with positive probability;

(C) $X(t) \to 0$ as $t \to \infty$ a.s.

These results were extended to finite–dimensions by Chan in [30]. The results in [31, 30] are motivated by problems in simulated annealing.

In Appleby, Gleeson and Rodkina [10], the monotonicity condition on f and (3.1.4) were relaxed. It was shown if f is locally Lipschitz continuous and obeys (1.2.2), and in place of (3.1.4) also obeys (1.2.12) then any solution X of (3.1.1) obeys (3.1.3) holds if σ obeys

(3.1.5). The converse of Chan and Williams is also established: if $t \mapsto \sigma^2(t)$ is decreasing, and the solution X of (3.1.1) obeys (3.1.3), then σ must obey (3.1.5). Moreover, it was also shown, without monotonicity on σ, that if σ obeys

$$\lim_{t\to\infty} \sigma^2(t) \log t = +\infty, \tag{3.1.6}$$

then the solution X of (3.1.1) obeys

$$\limsup_{t\to\infty} |X(t,\xi)| = +\infty, \quad \text{a.s. for each } \xi \in \mathbb{R}. \tag{3.1.7}$$

Furthermore, it was shown that the condition (3.1.5) could be replaced by the weaker condition

$$\lim_{t\to\infty} \int_0^t e^{-2(t-s)}\sigma^2(s)\,ds \cdot \log_2 \int_0^t \sigma^2(s)e^{2s}\,ds = 0 \tag{3.1.8}$$

and that (3.1.8) and (3.1.5) are equivalent when $t \mapsto \sigma^2(t)$ is decreasing. In fact, it was even shown that if σ^2 is not monotone decreasing, σ does not have to satisfy (3.1.5) in order for X to obey (3.1.3).

In this chapter, we improve upon the results in [10] and [31, 30] in a number of directions. First, we show that neither the Lipschitz continuity of f nor the condition (1.2.12) is needed in order to guarantee that any solution X of (3.1.1) obeys (3.1.3). Moreover, we give necessary and sufficient conditions for the convergence of solutions which do not require the monotonicity of σ^2. One of our main results shows that if f obeys (1.2.2) and σ is also continuous, then any solution X of (3.1.1) obeys (3.1.3) if and only if

$$S'(\epsilon) := \sum_{n=0}^{\infty} \sqrt{\int_n^{n+1} \sigma^2(s)\,ds}\, \exp\left(-\frac{\epsilon^2}{2}\frac{1}{\int_n^{n+1}\sigma^2(s)\,ds}\right) < +\infty, \quad \text{for every } \epsilon > 0, \tag{3.1.9}$$

and it is even shown that if (3.1.9) does not hold, then $\mathbb{P}[X(t) \to 0 \text{ as } t \to \infty] = 0$ for any $\xi \in \mathbb{R}$ (Theorem 3.5.1). Another significant development from [10] and [31, 30] is a complete classification of the asymptotic behaviour of (3.1.1) in terms of the data, rather than merely satisfactory sufficient conditions. In Theorem 3.4.3, we show that when f obeys (3.1.4), that any solution is either (a) convergent to zero with probability one, (b) bounded but not convergent to zero, with probability one, or (c) recurrent on $\mathbb{R}$ with probability one, according as to whether $S'(\epsilon)$ is always finite, sometimes finite, or never finite, for $\epsilon > 0$. Apart from classifying the asymptotic behaviour, the novel feature here is that bounded but non–convergent solutions are examined.

Of course, the analysis from the last chapter can also be used to replace finiteness conditions on the sum $S'(\epsilon)$ with corresponding integral conditions, although we do not list again in this chapter results which describe the asymptotic behaviour. It suffices to mention that if we fix $c > 0$ and define

$$I(\epsilon) = \int_0^\infty \sqrt{\int_t^{t+c} \sigma(s)^2\, ds} \exp\left(-\frac{\epsilon^2/2}{\int_t^{t+c} \sigma(s)^2 ds}\right) \chi_{(0,\infty)}\left(\int_t^{t+c} \sigma(s)^2\right) ds \qquad (3.1.10)$$

then $I(\epsilon) < +\infty$ for all $\epsilon > 0$ is a necessary and sufficient condition for solutions to obey (3.1.3). Similarly, in the case when f obeys (3.1.4), then any solution is either (a) convergent to zero with probability one, (b) bounded but not convergent to zero, with probability one, or (c) recurrent on $\mathbb{R}$ with probability one, according as to whether $I(\epsilon)$ is always finite, sometimes finite, or never finite, for $\epsilon > 0$.

Although the condition (3.1.9) is necessary and sufficient for X to obey (3.1.3), it may prove to be a little unwieldy for use in some situations. For this reason we deduce some sharp sufficient conditions for X to obey (3.1.3). If f obeys (1.2.2) and σ is continuous and obeys (3.1.8), then any solution X of (3.1.1) obeys (3.1.3) (Theorem 3.4.2). In the spirit of Theorem 3.5.1, we also establish converse results in the case when σ^2 is monotone , and demonstrate that the condition (3.1.8) is hard to relax if we require X to obey (3.1.3). The relationship between the conditions which characterise the asymptotic behaviour, and which involve $S'(\epsilon)$, and sufficient conditions are explored in several results, notably in Proposition 3.3.1. Also, in the case when solutions are bounded, we analyse the relationships between the deterministic bounds on solutions and the drift and diffusion coefficients. In particular, in Propositions 3.4.2,3.10.1 and 3.4.3, we demonstrate the bounds on any solution increase with greater noise intensity, and with weaker mean reversion.

These results are proven by showing that the stability of (3.1.1) is equivalent to the asymptotic stability of a process Y which is the solution of an affine SDE with the same diffusion coefficient σ (Proposition 3.4.1, especially part (A)). A classification of the asymptotic behaviour Y has already been achieved in Chapter 2, and the relevant results of Chapter 2 are listed in Section 3.3. The proof of part (a) of Proposition 3.4.1 is given under the additional condition that $\sigma \notin L^2(0,\infty)$; the case when $\sigma \in L^2(0,\infty)$ is easier, uses different methods, and is dealt with separately. Essentially, in the case when $\sigma \notin L^2(0,\infty)$

the recurrence of one–dimensional standard Brownian motion forces solutions to return to an arbitrarily small neighbourhood of the origin infinitely often. Then, if the noise fades sufficiently quickly so that the affine SDE is convergent to zero, the difference $Z := X - Y$ obeys a perturbed version of the ordinary differential equation (1.2.5) where the perturbation fades to zero asymptotically, and by virtue of the recurrence property, there exist arbitrarily large time when Z is arbitrarily close to zero. By considering an initial value problem for Z starting at such times, deterministic methods can then be used to show that Z tends to zero, and hence that X tends to zero. A similar method is employed in Theorem 3.4.3 to establish an upper bound on $|X|$ when Y is bounded, but does not tend to zero. Establishing that solutions of (3.1.1) is unbounded, or obeys certain lower bounds, is generally achieved by writing a variation of constants formula for X in terms on Y, and then using the known asymptotic behaviour of Y to force a contradiction.

Many parts of the analysis in this chapter apply to finite–dimensional equations, and these questions will be investigated in the next chapter.

Other interesting questions which can be attacked by means of the methods in this chapter include an analysis of local stability, where there are a finite number of equilibria of the underlying deterministic dynamical system (1.2.5). Some work in this direction has been conducted in a discrete–time setting in [2]. Numerical methods under monotonicity assumptions like those of Chan and Williams have been studied and are outlined in Chapter 5. Furthermore, finite dimensional analogues of the results in this chapter are given in Chapter 4, with corresponding numerical results in the multidimensional case concluding the thesis in Chapter 6.

Section 3.2 deals with preliminary results, including the proof that solutions of (3.1.1) exist. Results for an auxiliary affine SDE, proven in Chapter 2, are recapitulated in Section 3.3, along with some new results for the stability of affine equations. Section 3.4 considers general results, including the classification of the almost sure behaviour of solutions under the additional assumption (1.2.12) on f. Section 3.5 considers the characterisation of asymptotic stability using only the assumption (1.2.2). Proofs of many results are deferred to the end of the chapter, and these proofs are presented in Sections 3.6, 3.7, 3.9 and 3.10.

3.2 Preliminaries

3.2.1 Remarks on existence and uniqueness of solutions of (3.1.1)

There is an extensive theory regarding the existence and uniqueness of solutions of stochastic differential equations under a variety of regularity conditions on the drift and diffusion coefficients. Perhaps the most commonly quoted conditions which ensure the existence of a strong local solution are the Lipschitz continuity of the drift and diffusion coefficients. However, in this chapter, we would like to establish our asymptotic results under weaker hypotheses on f. We do not concern ourselves greatly with relaxing conditions on σ, because σ being continuous proves sufficient to ensure the existence of solutions in many cases.

The existence of a unique solution of

$$dX(t) = f(X(t))\,dt + \sigma(t, X(t))\,dB(t) \tag{3.2.1}$$

can be asserted in the case when $|\sigma(t,x)| \geq c > 0$ for some $c > 0$ for all (t,x) and f being bounded, so no continuity assumption is required on f. However, assuming such a lower bound on σ would not natural in the context of this chapter: for asymptotic stability results, we would typically require that $\liminf_{t\to\infty} \sigma^2(t) = 0$. Moreover, f being bounded excludes the important category of strongly mean–reverting functions f that have been investigated for this stability problem in [31] and [10].

One of the aims of this chapter and of [10] is to relax monotonicity assumptions on σ which are required in [31]. Therefore, although we are often interested in functions σ which tend to zero in some sense, we do not want to exclude the cases when $\sigma(t) = 0$ for all t in a given interval (or indeed union of intervals). Our analysis will show that in these cases, the behaviour of σ on the intervals where it is nontrivial can give rise to solutions of (3.1.1) obeying (3.1.3) or (3.1.7). However, on those time intervals I for which σ is zero, the process X obeys the differential equation

$$X'(t) = -f(X(t)), \quad t \in I$$

where $X(\inf I)$ is a random variable. On such an interval, it is conceivable that a lack of regularity in f could give rise to multiple solutions of the ordinary equation (and hence the SDE (3.1.1)), so our most general existence results which make assertions about the existence of solutions (but say nothing about unicity of solutions), and which use the weakest hypotheses on f that we impose in this work, do not appear to be especially conservative.

For these reasons, we prove that there is a continuous and adapted process which obeys (3.1.1) by using very elementary methods, rather than by appealing to a result from the substantial body of sophisticated theory concerning the existence of solutions of (3.2.1).

When f obeys (1.2.2) and σ obeys (3.1.2), we now demonstrate that there exists a continuous and adapted process X which satisfies (3.1.1). The existence of a local solution is ensured by the continuity of f and σ, while the fact that any such solution is well–defined for all time follows from the mean–reverting condition $xf(x) > 0$ for $x \neq 0$ which is part of (1.2.2). In the chapter, the spirit of our approach is to show that *any* solution of (3.1.1) has the stated asymptotic properties, even though multiple solutions exist, without paying particular concern as to whether solutions are unique.

Proposition 3.2.1. *Suppose that f obeys* (1.2.2) *and σ obeys* (3.1.2)*. Then there exists a continuous adapted process X which obeys* (3.1.1) *on* $[0,\infty)$*, a.s.*

The proof is postponed to Section 3.6. In order to ensure that solutions of (3.1.1) are unique, it is often necessary to impose additional regularity properties on f. One common and mild assumption which ensures uniqueness is that (1.2.4). See e.g., [55]. Another assumption which guarantees the uniqueness of the solution is that the drift coefficient $-f$ obeys a one–sided Lipschitz condition. More precisely, imposing such an assumption on f implies

$$\text{There exists } K \in \mathbb{R} \text{ such that } (f(x)-f(y))(x-y) \geq -K|x-y|^2 \quad \text{for all } x, y \in \mathbb{R}. \tag{3.2.2}$$

It is to be noted that if f is non–decreasing, it obeys (3.2.2), because the righthand side is non–negative, and we can choose $K = 0$. Since non–decreasing functions do not have to be Lipschitz continuous, we see that in general (3.2.2) does not imply (1.2.4), so these additional assumptions can be used to cover different situations.

Proposition 3.2.2. *Suppose that f obeys* (1.2.2) *and* (3.2.2) *and that σ obeys* (3.1.2). *Then there exists a unique continuous adapted process X which obeys* (3.1.1) *on* $[0,\infty)$ *a.s.*

Again the proof is deferred to Section 3.6.

In the proof of Proposition 3.2.1, and elsewhere throughout the chapter, it is helpful to introduce the following processes and notation. Consider the affine stochastic differential equation

$$dY(t) = -Y(t)\,dt + \sigma(t)\,dB(t), \quad t \geq 0; \quad Y(0) = 0. \tag{3.2.3}$$

Since σ is continuous, there is a unique continuous adapted process which obeys (3.2.3), and we identify the a.s. event Ω_Y on which this solution is defined:

$$\Omega_Y = \big\{\omega \in \Omega : \text{there is a unique continuous adapted process } Y \text{ for which the realisation } Y(\cdot,\omega) \text{ obeys (3.2.3)}\big\}. \tag{3.2.4}$$

It is also helpful throughout the chapter to identify the event Ω_X on which the continuous adapted process X obeys (3.1.1), so we therefore define

$$\Omega_X = \big\{\omega \in \Omega : \text{the continuous adapted process } X \text{ is such that the realisation } X(\cdot,\omega) \text{ obeys (3.1.1)}\big\}. \tag{3.2.5}$$

By virtue of Proposition 3.2.1, Ω_X is an almost sure event.

3.2.2 Preliminary asymptotic results

We first consider hypotheses on the data i.e., on σ under which any solution X of (3.1.1) obeys (3.1.3). We note that when $\sigma \in L^2(0,\infty)$, we have X obeys (3.1.3). However, we cannot apply directly the semimartingale convergence theorem of Lipster–Shiryaev (see e.g., [51, Theorem 7, p.139]) to the non–negative semimartingale X^2, because it is not guaranteed that $\mathbb{E}[X^2(t)] < +\infty$ for all $t \geq 0$. The proof of the following theorem, which is deferred to the next section, uses the ideas of [51, Theorem 7, p.139] heavily, however.

Theorem 3.2.1. *Suppose that f satisfies* (1.2.2)*. Suppose that σ obeys* (3.1.2) *and $\sigma \in L^2(0,\infty)$. If X is any solution of* (3.1.1)*, then X obeys* (3.1.3)*.*

The proof is relegated to Section 3.7.1. Our next result shows that if, on the contrary, $\sigma \notin L^2(0,\infty)$, we can only guarantee that X visits a neighbourhood of the equilibrium infinitely often.

Theorem 3.2.2. *Suppose that f obeys* (1.2.2)*, and that σ obeys* (3.1.2) *and $\sigma \notin L^2(0,\infty)$. Then any solution X of* (3.1.1) *obeys $\liminf_{t\to\infty} |X(t)| = 0$ a.s.*

Again the proof is postponed to Section 3.7.1.

3.3 Linear Equation

Since scalar linear SDEs have attracted much attention, in this section we explain some of the similarities and differences between our work and that which has appeared in the literature to date. We also restate notation, auxiliary functions and processes in order to state scalar versions of results from Chapter 2 that are relevant to the asymptotic analysis of the nonlinear equation.

3.3.1 Linear equations with time-varying features

In this section, we discuss results from the general asymptotic theory of *linear* stochastic differential equations. A useful nomenclature for classifying various categories of linear equation is given in Mao [55, Chapter 3.1], for it transpires that the asymptotic behaviour of equations—and the corresponding analysis of their asymptotic behaviour— differs across these categories. As we focus in this section on scalar equations, we confine attention now to the most general scalar linear equation. We say that the scalar process X is a solution of a linear stochastic differential equation if it obeys

$$dX(t) = (a_0(t)X(t) + f_0(t))\, dt + \sum_{j=1}^{r}(a_j(t)X(t) + f_j(t))\, dB_j(t) \tag{3.3.1}$$

where $r \geq 1$ is an integer, a_j and f_j for $j = 0, \dots, r$ are appropriately regular functions, and $B = (B_1, \dots, B_r)$ is an r–dimensional standard Brownian motion. To simplify our discussion, we assume the continuity of the f's and a's, which is sufficient to ensure the existence of a unique strong solution of (3.3.1).

The equation (3.3.1) is termed *homogeneous* if $f_j(t) \equiv 0$ for all $t \geq 0$ and all $j = 0, \ldots, r$. For such an equation, if $X(0) = 0$, then the unique solution is $X(t) = 0$ for all $t \geq 0$ a.s., so the presence of the stochastic terms *preserves* the zero equilibrium of the underlying deterministic differential equation

$$x'(t) = a_0(t)x(t). \tag{3.3.2}$$

An extremely comprehensive theory concerning the stability of the zero solution of (3.3.1) exists for homogeneous equations, and is expounded in e.g., Khas'minski [45, Chapter 6], to which we allude presently. For any other *non–homogeneous* equation, $X(0) = 0$ does *not* imply that $X(t) = 0$ for all $t \geq 0$, and it is sometimes said that the non–autonomous perturbations f_j are not *equilibrium–preserving.* For instance, if $a_j(t) \equiv 0$ for all $t \geq 0$ and $j = 1, \ldots, r$, the diffusion coefficient depends only on t (and is thus state–independent) and the equation is termed *linear in the narrow sense* in [55]. These equations are in some sense the simplest in the class of linear equations, as their solutions can be expressed explicitly in terms of the fundamental solution of (3.3.2). It is such *non–homogeneous* equations that are investigated in this section, and discussed also in e.g., [45, Chapter 7.4] and in [30, 31]. For such equations, it can be shown if f_j, $j = 1, \ldots, f$ fade sufficiently rapidly, then the stability of the underlying deterministic equation is preserved. Conditions given in these works, such as the square integrability of the f_j's (cf. e.g., [45, Chapter 7.4, p.255] for the equation studied in this chapter, and generalisations to nonlinear and non–autonomous equations in [45, Theorem 7.4.1]) are covered and improved in this chapter. In [45] and elsewhere, such equations are often referred to as possessing *damped perturbations*, reflecting the fact that the $f(t)$'s are hypothesised to tend to zero in some sense as $t \to \infty$.

In the equation analysed in this section, which is driven by $r = 1$ Brownian motions, a non–autonomous function f_1 appears in the diffusion coefficient (which means that the zero equilibrium is not preserved) but no such perturbation is present in the drift (i.e., $f_0 = 0$). In Appleby and Rodkina [21], a single non–autonomous forcing function also appears, but is present instead in the drift (i.e., f_0 is non–trivial) with the non–autonomous forcing function being absent in the diffusion coefficient (i.e., $f_1 = 0$): accordingly, the equation considered is

$$dX(t) = (a_0 X(t) + f_0(t))\, dt + a_1 X(t)\, dB(t). \tag{3.3.3}$$

As in this work, the results in [21] aim to estimate the critical size of the perturbation at which the stability of zero equilibrium of the autonomous equation is lost. The autonomous equation has all solutions attracted to zero if $a_0 - a_1^2/2 < 0$, and under those conditions it is shown that if

$$\limsup_{t\to\infty} \frac{\log|f_0(t)|}{\log t} < -\frac{a_1^2}{a_1^2 - 2a_0} < 0,$$

then $X(t) \to 0$ as $t \to \infty$ a.s., while if

$$\liminf_{t\to\infty} \frac{\log|f_0(t)|}{\log t} > -\frac{a_1^2}{a_1^2 - 2a_0},$$

then $\limsup_{t\to\infty} |X(t)| = +\infty$ a.s. These conditions for the a.s. asymptotic convergence of $X(t) \to 0$ as $t \to \infty$ are therefore quite sharp. Indeed, the stipulation that $a_0 - a_1^2/2 < 0$ cannot be readily relaxed either, for in the case when $f_0(t) \geq 0$ for all $t \geq 0$ and $X(0) > 0$, the condition $a_0 - a_1^2/2 < 0$ is necessary in order to have $X(t) \to 0$ as $t \to \infty$.

Owing to the deep treatment of the long–time behaviour of linear equations in [45, Chapter 6], the brief synopsis of the diverse results there which impinge on our work, and which was promised earlier, is now given. In particular, we focus on results relating to homogeneous but non–autonomous equations. In [45, Theorem 6.1.1] extremely sharp conditions are given on the time–varying coefficients a_0 and a_1 for the stability and instability of the scalar homogeneous differential equation

$$dX(t) = a_0(t)X(t)\, dt + a_1(t)X(t)\, dB(t).$$

In contrast to our analysis here, which concentrates on pathwise behaviour, [45, Chapter 6.2] is devoted to characterising stability and instability in the mean and mean square sense, with sufficient conditions for stability and instability for general p–th moments being given in [45, Chapter 6.3] The connection between moment stability and instability and stability in probability is developed in [45, Chapter 6.4] for autonomous equations and in [45, Chapter 6.5] for non–autonomous equations. The remainder of the chapter, being largely concerned with results leading to and including the almost sure Lyapunov exponents for *autonomous* equations, is less germane to the results reported in our work.

3.3.2 Specialisation of results from Chapter 2

We can now apply the results from the previous chapter, which concerning the asymptotic behaviour of the related finite–dimensional stochastic differential equation in the scalar case. We start with notation and definitions which parallel our presentation there. Let $\Phi : \mathbb{R} \to [0, 1]$ be the distribution function of a standard normal random variable as defined in (2.2.2).the sequence $\theta : \mathbb{N} \to [0, \infty)$

$$\theta^2(n) = \int_n^{n+1} \sigma^2(s)\, ds. \tag{3.3.4}$$

Let $\epsilon > 0$ and consider the sum

$$S(\epsilon) = \sum_{n=0}^{\infty} \left\{ 1 - \Phi\left(\frac{\epsilon}{\theta(n)}\right)\right\}. \tag{3.3.5}$$

Again, this summation is difficult to evaluate directly, because Φ is not known in closed form. However, it can be shown that $S(\epsilon)$ is finite or infinite according as to whether the sum

$$S'(\epsilon) := \sum_{n=0}^{\infty} \theta(n) \exp\left(-\frac{\epsilon^2}{2}\frac{1}{\theta^2(n)}\right) \tag{3.3.6}$$

is finite or infinite, where we interpret the summand to be zero in the case where $\theta(n) = 0$. This is the result of Mill's estimate, which is proven as Lemma 2.2.1 in Chapter 2.

Moreover, we have the next result immediately as a corollary of Theorem 2.2.1

Corollary 3.3.1. *Suppose that σ obeys* (3.1.2) *and Y is the unique continuous adapted process which obeys* (3.2.3)*. Let θ be defined by* (3.3.4) *and $S'(\cdot)$ by* (3.3.6)*.*

(A) If θ is such that

$$S'(\epsilon) \text{ is finite for all } \epsilon > 0, \tag{3.3.7}$$

then

$$\lim_{t\to\infty} Y(t) = 0, \quad a.s. \tag{3.3.8}$$

(B) If θ is such that there exists $\epsilon' > 0$ such that

$$S'(\epsilon) \text{ is finite for all } \epsilon > \epsilon', \tag{3.3.9a}$$

$$S'(\epsilon) = +\infty \text{ for all } \epsilon < \epsilon', \tag{3.3.9b}$$

then the event Ω_1 defined by

$$\Omega_1 := \{\omega \in \Omega_Y : 0 < \limsup_{t\to\infty} |Y(t,\omega)| < +\infty\}. \tag{3.3.10}$$

is almost sure and there exist deterministic $0 < \underline{Y} \le \overline{Y} < +\infty$ defined by

$$\underline{Y} := \inf_{\omega\in\Omega_1} \limsup_{t\to\infty} |Y(t,\omega)| > 0, \tag{3.3.11}$$

$$\overline{Y} := \sup_{\omega\in\Omega_1} \limsup_{t\to\infty} |Y(t,\omega)| > 0. \tag{3.3.12}$$

Moreover

$$\liminf_{t\to\infty} |Y(t)| = 0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t |Y(s)|^2\,ds = 0, \quad a.s.$$

(C) If θ is such that

$$S'(\epsilon) = +\infty \text{ for all } \epsilon > 0, \tag{3.3.13}$$

then

$$\limsup_{t\to\infty} |Y(t)| = +\infty, \quad \liminf_{t\to\infty} |Y(t)| = 0, \quad a.s.$$

In Corollary 3.3.1, no monotonicity conditions are imposed on σ. The form of Theorem 3.3.1 is inspired by those of [31, Theorem 1] and [20, Theorem 6, Corollary 7].

Remark 3.3.1. The benefit of this result is that it can be used more easily to state the nonlinear result even it's a corollary of Theorem 2.2.1. The existence of deterministic bounds on $|Y|$ in (3.3.11) and (3.3.12) in part (B) was established as part of Theorem 2.2.1 in Chapter 2. Moreover, it was established as part of the proof that explicit bounds on $\overline{Y}$ and $\underline{Y}$ can be given in terms of the critical value of $\epsilon = \epsilon'$ in (2.2.8). The estimates given by the analysis in [4] are

$$\underline{Y} \geq \underline{y} := \frac{e^{-1}}{1+e^{-1}}\epsilon', \quad \overline{Y} \leq \left(\frac{1}{1-e^{-1}} + e\right)\epsilon' =: \overline{y}. \tag{3.3.14}$$

Hence we have $0.2689\epsilon' \leq \limsup_{t\to\infty} |Y(t)| \leq 4.3003\epsilon'$, a.s.

It remains an open question as to whether in general the explicit bounds $\overline{y}$ and $\underline{y}$ on $\overline{Y}$ and $\underline{Y}$ can be improved. In part of Theorem 3.3.1 in which case (B) holds, it can be shown by an independent argument that $\underline{y} = \overline{y} = \epsilon'$ and therefore that $\overline{Y} = \underline{Y} = \epsilon'$.

Remark 3.3.2. If σ obeys (3.1.2) and $\sigma \in L^2(0,\infty)$, and Y is the solution of (3.2.3), then Y obeys $\lim_{t\to\infty} Y(t) = 0$ a.s. by Theorem 3.2.1. Moreover, if $\sigma \in L^2(0,\infty)$, then σ obeys (3.3.7). If σ obeys either (2.2.8) or (3.3.13), then $\sigma \notin L^2(0,\infty)$.

The condition that $S'(\epsilon)$ is finite or infinite can be difficult to check. However, in the case when

$$\text{There exists } L \in [0,\infty] \text{ such that } L = \lim_{t\to\infty} \sigma^2(t)\log t, \tag{3.3.15}$$

each of the conditions (3.3.7), (2.2.8) and (3.3.13) is possible according as to whether the limit L is zero, non–zero and finite, or infinite. In this case therefore, the asymptotic behaviour of any solution of (3.1.1) can be classified completely.

Proposition 3.3.1. *Suppose that $\sigma \in C([0,\infty);\mathbb{R})$ obeys* (3.3.15) *and that $S'(\cdot)$ is defined by* (3.3.6)*.*

(A) If $L = 0$, then S' obeys (3.3.7)*.*

(B) If $L \in (0,\infty)$, then S' obeys (2.2.8)*.*

(C) If $L = \infty$, then S' obeys (3.3.13)*.*

Scrutiny of the proof reveals that we can replace the condition (3.3.15) with the weaker condition

$$\text{There exists } L \in [0,\infty] \text{ such that } L = \lim_{n\to\infty} \theta^2(n)\log n, \tag{3.3.16}$$

and still obtain the same trichotomy in Proposition 3.3.1. The proof of Proposition 3.3.1 is postponed to Section 3.8.

The conditions of Corollary 3.3.1 can be quite difficult to check in practice. In Chapter 2, easily–checked sufficient conditions on σ for which Y is bounded, stable or unstable, are developed. These results are extended slightly here, and will also be used to analyse the nonlinear equation (3.1.1). For this reason, they are stated afresh here.

In the case when $\sigma \in L^2(0,\infty)$ we have that Y tends to zero. Therefore, we confine attention to the case where $\sigma \notin L^2(0,\infty)$. In this case, we can define a number $T > 0$ such that $\int_0^t e^{2s}\sigma^2(s)\,ds > e^e$ for $t > T$ and so one can define a function $\Sigma : [T,\infty) \to [0,\infty)$ by

$$\Sigma(t) = \left(\int_0^t e^{-2(t-s)}\sigma^2(s)\,ds\right)^{1/2} (\log t)^{1/2}, \quad t \geq T. \tag{3.3.17}$$

Our main result in this direction can now be stated.

Theorem 3.3.1. *Suppose that* σ *obeys* (3.1.2) *and that* Y *is the unique continuous adapted process which obeys* (3.2.3)*. Let* Σ *be given by* (3.3.17)*.*

(A) If $\lim_{t\to\infty} \Sigma^2(t) = 0$ *then*

$$\lim_{t\to\infty} Y(t) = 0, \quad a.s. \tag{3.3.18}$$

(B) If $\liminf_{t\to\infty} \Sigma^2(t) = L < +\infty$ *then*

$$\limsup_{t\to\infty} |Y(t)| \geq \sqrt{2L}, \quad a.s.$$

(C) If $\limsup_{t\to\infty} \Sigma^2(t) = L < +\infty$ *then*

$$\limsup_{t\to\infty} |Y(t)| \leq \sqrt{2L}, \quad a.s. \tag{3.3.19}$$

(D) If $\lim_{t\to\infty} \Sigma^2(t) = L < +\infty$ *then*

$$\limsup_{t\to\infty} |Y(t)| = \sqrt{2L}, \quad a.s. \tag{3.3.20}$$

(E) If $\lim_{t\to\infty} \Sigma^2(t) = +\infty$ *then*

$$\limsup_{t\to\infty} |Y(t)| = +\infty, \quad a.s.$$

The proof of part (C) uses the methods of [4, Theorem 3.2], so is not given. It is now clear that part (D) is merely a corollary of parts (B) and (C). Parts (A) and (E) may also be thought of as limiting cases of part (D) as $L \to 0$ and $L \to \infty$, respectively. We note that when σ obeys (3.3.15), then $\Sigma^2(t) \to L$ as $t \to \infty$, so that in part (D), we have from the proof of part (B) of Proposition 3.3.1 that S' obeys (2.2.8) with $\epsilon' = \sqrt{2L}$ and by (3.3.20), that $\overline{Y} = \underline{Y} = \sqrt{2L} = \epsilon'$ in (3.3.11) and (3.3.12). This strengthens the general estimates given on $\overline{Y}$ and $\underline{Y}$ in (3.3.14).

3.4 Nonlinear Equation

In this section we explore the asymptotic behaviour of the nonlinear differential equation (3.1.1). In the first part of this section, we establish a connection between the solution of (3.2.3) and solutions of (3.1.1). This enables us to state the main results of the chapter, which appear, together with interpretation and examples, in the second part of this section.

3.4.1 Connection between the linear and nonlinear equation

In our first result, we show that knowledge of the pathwise asymptotic behaviour of $Y(t)$ as $t \to \infty$ enables us to infer a great deal about the asymptotic behaviour of $X(t)$ as $t \to \infty$. Indeed, we show in broad terms that X inherits the asymptotic behaviour exhibited by Y, when f obeys (1.2.2).

Proposition 3.4.1. *Suppose that f satisfies* (1.2.2) *and that σ obeys* (3.1.2)*. Let X be any solution of* (3.1.1)*, and Y the solution of* (3.2.3)*, and suppose that the a.s. events Ω_X and Ω_Y are defined as in* (3.2.5) *and* (3.2.4) *respectively.*

(A) Suppose that there is an a.s. event defined by

$$\{\omega \in \Omega_Y : \lim_{t\to\infty} |Y(t,\omega)| = 0\}.$$

Then $\lim_{t\to\infty} X(t) = 0$ *a.s.*

(B) Suppose that the event Ω_1 defined by (3.3.10) *is almost sure. Then the event*

$$\Omega_2 = \Omega_1 \cap \Omega_X \tag{3.4.1}$$

is almost sure, and there exists a positive and deterministic $\underline{X}$ given by

$$\underline{X} = \inf_{\omega \in \Omega_2} \limsup_{t \to \infty} |X(t,\omega)|. \tag{3.4.2}$$

(C) Suppose that there is an a.s. event defined by

$$\{\omega \in \Omega_Y : \limsup_{t \to \infty} |Y(t,\omega)| = +\infty\}.$$

Then $\limsup_{t\to\infty} |X(t)| = +\infty$ *a.s.*

In the proof of part (B), we can even determine an explicit lower bound for $\underline{X}$. If the event Ω_1 is defined by (3.3.10), we may define as in (3.3.11) and (3.3.12) the deterministic numbers $0 < \underline{Y} \leq \overline{Y} < +\infty$. For any f obeying (1.2.2) it can be shown that there is function $y \mapsto \underline{x}(y) = \underline{x}(f, y)$ which, for $y \geq 0$, obeys

$$2\underline{x} + \max_{|x| \leq \underline{x}} |f(x)| = y. \tag{3.4.3}$$

This leads to the estimate

$$\underline{X} \geq \underline{x}(f, \underline{Y}), \tag{3.4.4}$$

where $\underline{Y}$ is given by (3.3.11). Moreover, as it transpires that $\underline{x}(f,\cdot)$ is an increasing function, by (3.3.14), we can estimate $\underline{X}$ explicitly according to

$$\underline{X} \geq \underline{x}(f, \underline{y}),$$

where $\underline{y}$ is given explicitly by (3.3.14).

An interesting implication of part (C) is that an *arbitrarily strong* mean–reverting force (as measured by f) cannot keep solutions of (3.1.1) within bounded limits if the noise perturbation is so intense that a linear mean–reverting force cannot keep solutions bounded. Therefore, the system will run “out of control” (in the sense of becoming unbounded) however strongly the function f pushes it back towards the equilibrium state.

3.4.2 Main results

Due to Theorem 3.3.1, we can readily use Proposition 3.4.1 to characterise the asymptotic behaviour of solutions of (3.1.1).

Theorem 3.4.1. *Suppose that* σ *obeys* (3.1.2), f *obeys* (1.2.2) *and that* X *is any continuous adapted process which obeys* (3.1.1). *Let* θ *be defined by* (3.3.4) *and* $S'(\cdot)$ *by* (3.3.6).

(A) If θ *is such that* (3.3.7) *holds, then*

$$\lim_{t\to\infty} X(t) = 0, \quad a.s.$$

(B) If θ *is such that* (2.2.8) *holds, then there exists an almost sure event* $\Omega_2 = \Omega_1 \cap \Omega_X$, *and a deterministic* $\underline{X} > 0$ *defined by* (3.4.2) *such that*

$$\underline{X} = \inf_{\omega\in\Omega_2} \limsup_{t\to\infty} |X(t,\omega)| > 0.$$

Moreover, $\underline{X}$ *obeys*

$$\underline{X} \geq \underline{x}(f, \underline{Y}),$$

where $\underline{x}(f,\cdot)$ *is the unique solution of* (3.4.3), *and* $\underline{Y}$ *is defined by* (3.3.11). *Furthermore,*

$$\liminf_{t\to\infty} |X(t)| = 0, \quad a.s.$$

(C) If θ *is such that* (3.3.13) *holds, then*

$$\limsup_{t\to\infty} |X(t)| = +\infty, \quad \liminf_{t\to\infty} |X(t)| = 0, \quad a.s.$$

Proof. If θ is such that (3.3.7) holds, then from Theorem 3.3.1, we have $\lim_{t\to\infty} Y(t) = 0$, a.s. Taking this together with Proposition 3.4.1, part (A) holds. If θ is such that (2.2.8) holds, or if θ is such that (3.3.13) holds, then taken together with Theorem 3.3.1 and Proposition 3.4.1 we have that the first part (B) and of (C) is true. For the second part of (B) and (C), we recall that if (2.2.8) or (3.3.13) hold, Remark 3.3.2 implies that $\sigma \notin L^2(0,\infty)$. In this case, we already know that $\liminf_{t\to\infty} |X(t)| = 0$, a.s. by Theorem 3.2.2.

□

The formula (3.4.3), which is established in the proof of part (B) of Proposition 3.4.1, relates the lower bound on the large fluctuations $\underline{x}$ to the size of the diffusion coefficient σ and the nonlinearity in f. Thus, we may view $\underline{x} = \underline{x}(f, \underline{Y}) = \underline{x}(f, \sigma)$, because $\underline{Y}$ depends on σ but not on f. It is clear that the larger the diffusion coefficient, the larger the value of $\underline{Y}$. We now show for fixed f that $\underline{x}$ is increasing and that $\underline{x}(f, y) \to \infty$ as $y \to \infty$. Moreover, we show for fixed y that $\underline{x}(f_1, y) \geq \underline{x}(f_2, y)$ if

$$|f_2(x)| \geq |f_1(x)|, \quad x \in \mathbb{R}. \tag{3.4.5}$$

These ordering results seem to make intuitive sense, as we would expect weaker mean reversion and a larger diffusion coefficient to lead to larger fluctuations in X.

Proposition 3.4.2. *Suppose that f obeys* (1.2.2)*. Let $\underline{x}$ be the unique solution of* (3.4.3)*. Then*

(i) $y \mapsto \underline{x}(f, y)$ is increasing and $\lim_{y\to\infty} \underline{x}(f, y) = +\infty$, $\lim_{y\to 0^+} \underline{x}(f, y) = 0$.

(ii) If f_1 and f_2 are functions that obey (1.2.2) *and also satisfy* (3.4.5)*, then $\underline{x}(f_1, y) \geq \underline{x}(f_2, y)$.*

Proof. Define $h_f : [0,\infty) \to [0,\infty)$ according to

$$h_f(x) := 2x + \max_{|y|\leq x} |f(y)|, \quad x \geq 0. \tag{3.4.6}$$

Then h_f is increasing and continuous, and obeys the limits $\lim_{x\to\infty} h_f(x) = +\infty$ and $\lim_{x\to 0^+} h_f(x) = 0$. By (3.4.3), $h_f(\underline{x}(f, y)) = y$. Therefore

$$\underline{x}(f, y) = h_f^{-1}(y), \quad y \geq 0. \tag{3.4.7}$$

Hence $y \mapsto \underline{x}(f, y)$ is increasing. Finally, $\lim_{y\to\infty} \underline{x}(f, y) = \infty$ and $\lim_{y\to 0^+} \underline{x}(f, y) = \lim_{y\to 0^+} h_f^{-1}(y)$=0.

To prove part (ii), note by (3.4.5) that

$$h_{f_1}(\underline{x}(f_1,y)) = y = h_{f_2}(\underline{x}(f_2,y)) = 2\underline{x}(f_2,y) + \max_{|u|\le \underline{x}(f_2,y)} |f_2(u)|$$

$$\ge 2\underline{x}(f_2,y) + \max_{|u|\le \underline{x}(f_2,y)} |f_1(u)| = h_{f_1}(\underline{x}(f_2,y)).$$

Since h_{f_1} is an increasing function, we have $\underline{x}(f_1,y) \ge \underline{x}(f_2,y)$ as required. □

Just as the conditions of Theorem 3.3.1 can be quite difficult to check in practice for Y, the same is true for the conditions of Theorem 3.4.1 on θ for X. As in Theorem 3.3.1, and because of Proposition 3.4.1, we can supply easily checked sufficient conditions on σ for which X is bounded, stable or unstable.

In the case when $\sigma \in L^2(0,\infty)$ we have that X tends to zero. Therefore, we confine attention to the case where $\sigma \notin L^2(0,\infty)$. In this case, we can define a number $T > 0$ such that $\int_0^t e^{2s}\sigma^2(s)\,ds > e^e$ for $t > T$ and so one can define, as before, the function $\Sigma : [T,\infty) \to [0,\infty)$ by (3.3.17).

Theorem 3.4.2. *Suppose that f obeys* (1.2.2) *and that σ obeys* (3.1.2). *Let X be any solution of* (3.1.1). *Let Σ be given by* (3.3.17).

(A) If $\lim_{t\to\infty} \Sigma^2(t) = 0$ then $\lim_{t\to\infty} X(t) = 0$ a.s.

(B) If there exists $L \in (0,\infty)$ such that $\liminf_{t\to\infty} \Sigma^2(t) = L$, then there exists an almost sure event $\Omega_2 = \Omega_1 \cap \Omega_X$, and a deterministic $\underline{X} > 0$ defined by (3.4.2) *such that $\underline{X} = \inf_{\omega\in\Omega_2} \limsup_{t\to\infty} |X(t,\omega)| > 0$. Moreover, $\underline{X} \ge \underline{x}(f,\underline{Y})$, where $\underline{x}(f,\cdot)$ is the unique solution of* (3.4.3), *and $\underline{Y}$ is defined by* (3.3.11).

(C) If $\lim_{t\to\infty} \Sigma^2(t) = +\infty$ then $\limsup_{t\to\infty} |X(t)| = +\infty$ a.s.

Proof. If $\lim_{t\to\infty} e^{-2t}\log t \int_0^t e^{2s}\sigma^2(s)ds = 0$ then $\lim_{t\to\infty} Y(t) = 0$ from Theorem 3.3.1. Combining this with Proposition 3.4.1, we get $\lim_{t\to\infty} X(t) = 0$ proving part (A). Similarly, parts (B) and (C) follow from parts (B) and (E) of Theorem 3.3.1 and Proposition 3.4.1. □

We finish this Section by giving a sufficient condition on f for which solutions of (3.1.1) do not tend to zero but are nonetheless bounded. In the case when σ is such that either parts (A) or (C) apply, we have unambiguous information about the asymptotic behaviour of solutions: either almost all sample paths tend to zero, or almost all sample paths exhibit unbounded fluctuations. However, scrutiny of the statement of Proposition 3.4.1 shows that part (B) does not rule out the possibility that $\limsup_{t\to\infty}|X(t)| = +\infty$ with positive probability (or even almost surely). We make a further hypothesis on f, under which this is impossible, and X is forced to be bounded. The hypothesis is

$$\lim_{x\to-\infty} f(x) = -\infty, \quad \lim_{x\to\infty} f(x) = \infty. \tag{3.4.8}$$

An estimate on the lower bound $\underline{X}$ in case (B) is given in (3.4.3), which is found as part of the proof of Proposition 3.4.1. $\underline{X}$ is given in terms of f and σ. Similarly, an estimate can be determined for the upper bound. Towards this end, we introduce functions which are a type of generalised inverse of f by defining the functions f^- and f^+ by

$$f^+(x) = \sup\{z > 0 : f(z) = x\}, \quad x \geq 0, \tag{3.4.9}$$

$$f^-(x) = \inf\{z < 0 : f(z) = x\}, \quad x \leq 0. \tag{3.4.10}$$

These functions are well–defined if f obeys (1.2.2) and (3.4.8). We notice also that if f is increasing, then $f^\pm$ are exactly the inverse of f.

We may therefore define for any f the function $y \mapsto \overline{x}(f,y)$ by

$$\overline{x}(f,y) = 2y + \max(f^+(y), -f^-(-y)), \quad y \geq 0. \tag{3.4.11}$$

The main conclusion of the following theorem is that an explicit upper bound can be found for $\limsup_{t\to\infty}|X(t)|$. In fact, it can be shown that if $\overline{Y}$ obeys (3.3.12), then

$$\limsup_{t\to\infty}|X(t,\omega)| \leq \overline{x}(f,\overline{Y}), \quad \text{for each } \omega \in \Omega_2, \tag{3.4.12}$$

where Ω_2 is given by (3.4.1).

We are finally in a position to state the main result of this section.

Theorem 3.4.3. *Suppose that σ obeys* (3.1.2)*, f obeys* (1.2.2) *and* (3.4.8)*. Suppose that X is any continuous adapted process which obeys* (3.1.1)*. Let θ be defined by* (3.3.4) *and $S'(\cdot)$ by* (3.3.6)*.*

(A) If θ is such that (3.3.7) *holds, then* $\lim_{t\to\infty} X(t) = 0$*, a.s.*

(B) If θ is such that (2.2.8) *holds, then there exists an almost sure event* $\Omega_2 = \Omega_1 \cap \Omega_X$ *where* Ω_1 *defined in* (3.3.10)*, and deterministic* $0 < \underline{X} \le \overline{X} < +\infty$ *such that*

$$\underline{X} = \inf_{\omega\in\Omega_2} \limsup_{t\to\infty} |X(t,\omega)|, \quad \overline{X} = \sup_{\omega\in\Omega_2} \limsup_{t\to\infty} |X(t,\omega)|, \tag{3.4.13}$$

Moreover,

$$\underline{X} \ge \underline{x}(f, \underline{Y}),$$

where $\underline{x}(f,\cdot)$ *is the unique solution of* (3.4.3)*, and* $\underline{Y}$ *is defined by* (3.3.11)*, and*

$$\overline{X} \le \overline{x}(f, \overline{Y}),$$

where $\overline{x}(f,\cdot)$ *is defined by* (3.4.11) *and* $\overline{Y}$ *is defined by* (3.3.12)*. Furthermore,*

$$\liminf_{t\to\infty} |X(t)| = 0, \quad a.s.$$

(C) If θ is such that (3.3.13) *holds, then*

$$\limsup_{t\to\infty} |X(t)| = +\infty, \quad \liminf_{t\to\infty} |X(t)| = 0, \quad a.s.$$

We prove part (B) only, as the results of parts (A) and (C) follow from Theorem 3.4.1. Therefore, under the additional hypothesis that f obeys (3.4.8), it follows from Theorem 3.4.1 and 3.4.3 that either (i) solutions tend to zero with probability one, when σ obeys (3.3.7) (ii) solutions fluctuate within finite bounds with probability one, when σ obeys (2.2.8) or (iii) solutions fluctuate unboundedly with probability one, when σ obeys (3.3.13). In the second case, part (B) of Theorem 3.4.3 can be restated as

$$\underline{x}(f, \underline{Y}) \le \limsup_{t\to\infty} |X(t)| \le \overline{x}(f, \overline{Y}), \quad \text{a.s.},$$

and moreover we have weaker but explicit estimates on these deterministic bounds given by

$$0 < \underline{x}(f, \underline{y}) \le \limsup_{t\to\infty} |X(t)| \le \overline{x}(f, \overline{y}) < +\infty, \quad \text{a.s.},$$

where $\underline{y}$ and $\overline{y}$ are given by (3.3.14).

It is interesting to determine the effect of weaker mean reversion and an increasing diffusion coefficient on the *upper* bound of the large deviations of X, given by $\overline{x}(f,\overline{Y})$, just as we did for the lower bound on the size of the largest fluctuations in Proposition 3.4.2, given by $\underline{x}(f,\underline{Y})$. As before, it can be shown that weaker mean reversion and increasing diffusion coefficients increase the bound $\overline{x}$. Also, if the effect of the diffusion coefficient alone is negligible (so that $\overline{Y} \to 0$), or unboundedly large (so that $\overline{Y} \to \infty$), we see that cases (A) and (C) in Theorem 3.4.3 can be viewed as limiting cases of the asymptotic behaviour described in case (B). These properties of the bounds are established in the following result.

Proposition 3.4.3. *Suppose that f obeys* (1.2.2) *and* (3.4.8)*. Let $\overline{x}$ be given by* (3.4.11)*. Then*

(i) $y \mapsto \overline{x}(f,y)$ *is increasing and* $\lim_{y\to\infty} \overline{x}(f,y) = +\infty$, $\lim_{y\to 0^+} \overline{x}(f,y) = 0$.

(ii) If f_1 and f_2 are functions that obey (1.2.2) *and* (3.4.8)*, and also satisfy* (3.4.5)*, then* $\overline{x}(f_1,y) \geq \overline{x}(f_2,y)$.

The proof is relegated to the final section. We finish the section with an example which shows how estimates of $\underline{X}$ and $\overline{X}$ can be obtained in practice.

Example 3.4.1. We see how these estimates on the fluctuations behave for a specific class of examples. Suppose that $f(x) = x^n$ where n is an odd integer and that $\sigma^2(t)\log t \to L \in (0,\infty)$ as $t \to \infty$. Then by Theorem 3.3.1 it follows that $\limsup_{t\to\infty} |Y(t)| = \sqrt{2L}$ a.s. so we have $\overline{Y} = \underline{Y} = \sqrt{2L}$. Since f is increasing we have for $x \geq 0$ that

$$f^+(x) = f^{-1}(x) = x^{1/n}, \quad f^-(-x) = f^{-1}(-x) = -x^{1/n}, \quad \max_{|y|\leq x} |f(y)| = x^n$$

so that $\underline{x}(L) = \underline{x}(f,\underline{Y})$ and $\overline{x}(L) = \overline{x}(f,\overline{Y})$ satisfy

$$2\underline{x} + \underline{x}^n = \sqrt{2L}, \quad \overline{x} = 2\sqrt{2L} + (\sqrt{2L})^{1/n}.$$

From this, we readily see that

$$\lim_{L\to 0^+} \frac{\underline{x}(L)}{\sqrt{2L}} = \frac{1}{2}, \quad \lim_{L\to 0^+} \frac{\overline{x}(L)}{(\sqrt{2L})^{1/n}} = 1,$$

and that

$$\lim_{L\to \infty} \frac{\underline{x}(L)}{(\sqrt{2L})^{1/n}} = 1, \quad \lim_{L\to \infty} \frac{\overline{x}(L)}{2(\sqrt{2L})} = 1.$$

Notice that $\lim_{L\to 0^+} \overline{x}(L) = 0$ and $\lim_{L\to\infty} \underline{x}(L) = \infty$.

It is clear that these asymptotic bounds are widely spaced, because

$$\lim_{L\to 0^+} \frac{\overline{x}(L)}{\underline{x}(L)} = \lim_{L\to \infty} \frac{\overline{x}(L)}{\underline{x}(L)} = +\infty.$$

It would be an interesting question to determine whether either of these bounds is satisfactory, but we do not pursue this here. We suspect that the upper bound $\overline{x}(L)$ as $L \to \infty$ is very conservative, however, as it does not take into account the strong mean reversion of f.

3.5 Asymptotic Stability

It should be remarked that one consequence of Theorem 3.4.1 is that sample paths of X tend to zero with non–zero probability if and only if θ obeys (3.3.7), in which case almost all sample paths tend to zero. Therefore, we have the following immediate corollary of Theorem 3.4.1.

Theorem 3.5.1. *Suppose f obeys* (1.2.2) *and that σ obeys* (3.1.2)*. Let X be any solution of* (3.1.1)*. Let θ be defined by* (3.3.4) *and let Φ be given by* (2.2.2)*. Then the following are equivalent:*

(A)

$$\sum_{n=1}^{\infty} \theta(n) \exp\left(-\frac{1}{2}\frac{\epsilon^2}{\theta^2(n)}\right) < +\infty, \quad \textit{for every } \epsilon > 0. \tag{3.5.1}$$

(B) $\lim_{t\to\infty} X(t,\xi) = 0$ *with positive probability for some* $\xi \in \mathbb{R}$.

(C) $\lim_{t\to\infty} X(t,\xi) = 0$ *a.s. for each* $\xi \in \mathbb{R}$.

Part (A) refines part of [10, Proposition 3.3]. Also, if $X(t) \to 0$ as $t \to \infty$, it does so a.s., and so θ obeys (3.3.7). Therefore, $Y(t) \to 0$ as $t \to \infty$. This forces $\liminf_{t\to\infty} \Sigma^2(t) = 0$, for else we would have $\limsup_{t\to\infty} |Y(t)| > 0$ a.s, as essentially pointed out by [10, Proposition 3.3].

It should also be noted that no monotonicity conditions are required on σ in order for this result to hold, and that a.s. global stability is independent of the form of f. The conditions and form of Theorem 3.4.1 and 3.5.1 are inspired by those of [31, Theorem 1] and by [20, Theorem 6, Corollary 7].

An interesting fact of Theorem 3.5.1 is that it is unnecessary for $\sigma(t) \to 0$ as $t \to \infty$ in order for X to obey (3.1.3). In fact, we can even have $\limsup_{t\to\infty} |\sigma(t)|^2 = \infty$ and still have $X(t) \to 0$ as $t \to \infty$ a.s. Some examples are supplied in [10].

Note that (3.1.5) implies $\lim_{t\to\infty} \Sigma(t) = 0$, that (3.1.6) implies $\lim_{t\to\infty} \Sigma(t) = \infty$, and finally that $\liminf_{t\to\infty} \sigma^2(t) \log t > 0$ implies that $\liminf_{t\to\infty} \Sigma(t) > 0$. The next result is therefore an easy corollary of Theorem 3.3.1, or of Proposition 3.4.1 and Proposition 3.3.1.

Theorem 3.5.2. *Suppose that f satisfies* (1.2.2)*, and that σ obeys* (3.1.2)*. Let X be any solution of* (3.1.1)*.*

(i) If σ obeys $\lim_{t\to\infty} \sigma^2(t) \log t = 0$, then X obeys (3.1.3)*.*

(ii) If σ obeys $\liminf_{t\to\infty} \sigma^2(t) \log t \in (0,\infty)$, then $\mathbb{P}[\lim_{t\to\infty} X(t) = 0] = 0$

and $\liminf_{t\to\infty} |X(t)| = 0$, a.s..

(iii) If σ obeys $\lim_{t\to\infty} \sigma^2(t) \log t = \infty$, then

$$\limsup_{t\to\infty} |X(t)| = \infty, \quad \liminf_{t\to\infty} |X(t)| = 0, \quad a.s.$$

Part (i) is part of [10, Proposition 3.3(a)]. Part (iii) is [10, Lemma 3.7]. In [31], Chan and Williams have proven in the case when $t \mapsto \sigma^2(t)$ is decreasing, that Y obeys (3.3.18) if and only if σ obeys (3.1.5). Our final result is a corollary of this observation and Theorem 3.5.1, and also of [10, Theorem 3.8]. A stronger result than Theorem 3.5.2 can be stated if the sequence θ in (3.3.4) is decreasing: in this case, $\lim_{n\to\infty}\theta^2(n)\log n = 0$ is equivalent to (3.1.3).

Theorem 3.5.3. *Suppose that f satisfies (1.2.2). Suppose that σ obeys (3.1.2) and $t \mapsto \sigma^2(t)$ is decreasing. Let X be any solution of (3.1.1). Then the following are equivalent:*

(A) σ obeys $\lim_{t\to\infty}\sigma^2(t)\log t = 0$;

(B) $\lim_{t\to\infty} X(t,\xi) = 0$ a.s. for each $\xi \in \mathbb{R}$.

The remark preceding this result points to the importance of the condition $\theta(n)^2\log n \to 0$ as $n \to \infty$.We now supply an example in which $\theta(n)^2\log n \to 0$ as $n\to\infty$, but $t\mapsto\sigma^2(t)$ has "spikes" which prevents it from satisfying the condition $\lim_{t\to\infty}\sigma^2(t)\log t = 0$.

Example 3.5.1. Consider the decomposition of $[0,\infty)$ into a union of disjoint intervals

$$[0,\infty) = \cup_{k=0}^{\infty}\{I_k \cup J_k \cup K_k\},$$

where $\epsilon_k \in (0,1/2)$ for each $k \geq 0$ and

$$I_k = [k, k+\epsilon_k], \quad J_k = (k+\epsilon_k, k+1-\epsilon_k), \quad K_k = [k+1-\epsilon_k, k+1), \quad k\in\mathbb{N}.$$

Let $(l_k)_{k\geq 0}$ and $(q_k)_{k\geq 0}$ be positive sequences and consider the function $\sigma : [0,\infty) \to [0,\infty)$ defined by

$$\sigma^2(t) = \begin{cases} l_k - \frac{l_k - q_k}{\epsilon_k}(t-k), & t\in[k,k+\epsilon_k], \\ q_k, & t\in(k+\epsilon_k, k+1-\epsilon_k), \\ l_{k+1} + \frac{l_{k+1}-q_k}{\epsilon_k}(t-k-1), & t\in[k+1-\epsilon_k, k+1). \end{cases}$$

Then $t\mapsto\sigma^2(t)$ is continuous. If θ is defined by (3.3.4), then

$$\theta^2(k) = q_k(1-\epsilon_k) + \frac{1}{2}\epsilon_k(l_{k+1}+l_k).$$

Notice also that $\sigma^2(k) = l_k$. Suppose $q_k\log k \to 0$, $\epsilon_k(l_k + l_{k+1})\log k \to 0$ but we have that $\limsup_{k\to\infty} l_k\log k > 0$. Then $\theta^2(k)\log k \to 0$ as $k\to\infty$, but

$$\limsup_{t\to\infty}\sigma^2(t)\log t \geq \limsup_{k\to\infty}\sigma^2(k)\log k = \limsup_{k\to\infty} l_k\log k > 0.$$

Concrete examples of sequences for which these conditions hold include

$$q_k = \frac{1}{k+1}, \quad \epsilon_k = \frac{1}{k+3}, \quad l_k = 1,$$

or

$$q_k = \frac{1}{k+1}, \quad \epsilon_k = \frac{1}{(k+3)^2}, \quad l_k = k.$$

3.6 Proof of Existence Results from Section 3.2.1

3.6.1 Proof of Proposition 3.2.1

Consider the affine stochastic differential equation (3.2.3). Since σ is continuous, there is a unique continuous adapted process which obeys (3.2.3). Let Ω_Y be the a.s. event defined by (3.2.4) on which Y is defined. Now, for each $\omega \in \Omega_Y$, define the function

$$\varphi(t, x, \omega) = -f(x + Y(t, \omega)) + Y(t, \omega), \quad t \geq 0.$$

Since f is continuous, and the sample path $t \mapsto Y(t, \omega)$ is continuous, $(t, x) \mapsto \varphi(t, x, \omega)$ is continuous. Consider now the differential equation

$$z'(t, \omega) = \varphi(t, z(t, \omega), \omega), \quad t > 0; \quad z(0, \omega) = \xi.$$

By the continuity of φ in both arguments, by the Peano existence theorem, there exists a continuous local solution $t \mapsto z(t, \omega)$ for each $\omega \in \Omega_Y$ and $0 \leq t < \tau_e(\omega)$. Presently, it will be shown that $\tau_e(\omega) = +\infty$ a.s. on Ω_Y.

Moreover, as Y is adapted to $(\mathcal{F}^B(t))_{t\geq 0}$, z is also adapted to $(\mathcal{F}^B(t))_{t\geq 0}$. Now consider the process X defined on Ω_Y by $X(t) = z(t) + Y(t)$ for $t \in [0, \tau_e)$. By construction it is continuous and adapted. Furthermore, we have for $t \in [0, \tau_e)$

$$
\begin{aligned}
X(t,\omega) &= z(t,\omega) + Y(t,\omega) \\
&= \xi + \int_0^t \varphi(s, z(s,\omega), \omega)\, ds + \int_0^t -Y(s,\omega)\, ds + \left(\int_0^t \sigma(s)\, dB(s)\right)(\omega) \\
&= \xi + \int_0^t \{-f(z(s,\omega) + Y(s,\omega)) + Y(s,\omega)\}\, ds + \int_0^t -Y(s,\omega)\, ds \\
&\qquad + \left(\int_0^t \sigma(s)\, dB(s)\right)(\omega) \\
&= \xi + \int_0^t -f(X(s,\omega))\, ds + \left(\int_0^t \sigma(s)\, dB(s)\right)(\omega) \\
&= \left(\xi + \int_0^t -f(X(s))\, ds + \int_0^t \sigma(s)\, dB(s)\right)(\omega).
\end{aligned}
$$

Hence $X(\cdot,\omega)$ obeys (3.1.1) for each $\omega \in \Omega_Y$ on the interval $[0,\infty)$. The proof that τ_e is infinite a.s. was given in the Appendix of [10].

3.6.2 Proof of Proposition 3.2.2

The proof is inspired by an observation in e.g., [44]. Note first by Proposition 3.2.1 that the continuity of f together with (1.2.2) guarantees the existence of a continuous adapted process which obeys (3.1.1). Suppose therefore that X_1 and X_2 are any two solutions of (3.1.1). Then

$$d(X_1(t) - X_2(t)) = (-f(X_1(t)) + f(X_2(t)))\, dt,$$

and by Itô's rule we have that

$$d(X_1(t) - X_2(t))^2 = -2(X_1(t) - X_2(t))\,(f(X_1(t)) - f(X_2(t)))\, dt, \quad t \geq 0.$$

Since $X_1(0) = X_2(0) = \xi$, we have

$$(X_1(t) - X_2(t))^2 = -2\int_0^t (X_1(s) - X_2(s))\,(f(X_1(s)) - f(X_2(s)))\, ds, \quad t \geq 0.$$

Since f obeys (3.2.2), we have

$$(X_1(t) - X_2(t))^2 \leq 2K \int_0^t (X_1(s) - X_2(s))^2\, ds, \quad t \geq 0.$$

If $K \leq 0$, we can conclude automatically that $X_1(t) = X_2(t)$ for all $t \geq 0$ a.s., and that therefore the solution is unique. If $K > 0$, by applying Gronwall's inequality to the non–negative continuous function $t \mapsto (X_1(t) - X_2(t))^2$, we conclude that $X_1(t) = X_2(t)$ for all $t \geq 0$ a.s., and once again we have uniqueness.

3.7 Proofs of Preliminary Results

3.7.1 Proof of Theorem 3.2.1

By Itô's rule, we have

$$X^2(t) = \xi^2 - \int_0^t 2X(s)f(X(s))\,ds + \int_0^t \sigma^2(s)\,ds + \int_0^t 2X(s)\sigma(s)\,dB(s), \quad t \geq 0. \tag{3.7.1}$$

Since $xf(x) \geq 0$ for all $x \in \mathbb{R}$ and $\sigma \in L^2(0,\infty)$, we have

$$X^2(t) \leq \xi^2 + \int_0^\infty \sigma^2(s)\,ds + 2\int_0^t X(s)\sigma(s)\,dB(s), \quad t \geq 0.$$

Define M to be the local martingale given by $M(t) = \int_0^t 2X(s)\sigma(s)\,dB(s)$ for $t \geq 0$. Let

$$U(t) = \int_0^t 2X(s), f(X(s))ds, \quad A(t) = \int_0^t \sigma(s)^2 ds, \quad t \geq 0.$$

Since $xf(x) \geq 0$ for all $x \in \mathbb{R}$ and $\sigma \in L^2([0,\infty)$, it follows that A and U are continuous adapted increasing processes. Therefore by Theorem 0.3.6, it follows that

$$\lim_{t\to\infty} X(t)^2 = L \in [0,\infty), \quad a.s.$$

and that

$$\lim_{t\to\infty} \int_0^t X(s)f(X(s))ds = I \in [0,\infty), \quad a.s.$$

By continuity this means there is an a.s. event $A = \{\omega : X(t,\omega)^2 \to L \in [0,\infty) \text{ as } t \to \infty\}$ such that $A = A_+ \cup A_- \cup A_0$ where

$$A_+ = \{\omega : X(t,\omega) \to \sqrt{L(\omega)} \in (0,\infty) \text{ as } t \to \infty\},$$

$$A_- = \{\omega : X(t,\omega) \to -\sqrt{L(\omega)} \in (-\infty,0) \text{ as } t \to \infty\},$$

and $A_0 = \{\omega : X(t,\omega) \to 0 \text{ as } t \to \infty\}$. Suppose that $\omega \in A_+$. Then

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X(s,\omega)f(X(s,\omega))\,ds = \sqrt{L(\omega)}f(\sqrt{L(\omega)}) > 0, \tag{3.7.2}$$

by continuity of X, f and the fact that $xf(x) > 0$ for $x \neq 0$. Since the last two terms on the righthand side of (3.7.1) have finite limits as $t \to \infty$, (3.7.2) implies that for $\omega \in A_+$ that

$$0 \leq \lim_{t\to\infty} \frac{X^2(t,\omega)}{t} = -2\sqrt{L(\omega)}f(\sqrt{L(\omega)}) < 0,$$

a contradiction. Therefore $\mathbb{P}[A_+] = 0$. A similar argument yields $\mathbb{P}[A_-] = 0$. Since $\mathbb{P}[A] = 1$, we must have $\mathbb{P}[A_0] = 1$, as required.

3.7.2 Proof of Theorem 3.2.2

Let $A_1 = \{\omega : \liminf_{t\to\infty} X(t,\omega) > 0\}$ and suppose that $\mathbb{P}[A_1] > 0$. In particular, for $\omega \in A_1$, define $(0,\infty] \ni c(\omega) = \liminf_{t\to\infty} X(t,\omega)$ Then there exists $T_1(\omega) > 0$ such that $X(t,\omega) > 0$ for all $t > T_1(\omega)$. Hence for $t \geq T_1(\omega)$, we have

$$\begin{aligned} X(t) &= X(0) - \int_0^{T_1} f(X(s))\,ds - \int_{T_1}^t f(X(s))\,ds + \int_0^t \sigma(s)\,dB(s) \\ &\leq X(0) - \int_0^{T_1} f(X(s))\,ds + \int_0^t \sigma(s)\,dB(s). \end{aligned}$$

Since $\sigma \notin L^2(0,\infty)$ it follows that $\liminf_{t\to\infty} \int_0^t \sigma(s)\,dB(s) = -\infty$ a.s. Therefore a.s. on A_1 we have

$$c(\omega) = \liminf_{t\to\infty} X(t,\omega) \leq X(0) - \int_0^{T_1} f(X(s))\,ds + \liminf_{t\to\infty} \int_0^t \sigma(s)\,dB(s) = -\infty,$$

a contradiction. Hence $\mathbb{P}[A_1] = 0$, therefore $\liminf_{t\to\infty} X(t) \leq 0$ a.s. To prove that $\limsup_{t\to\infty} X(t) \geq 0$ a.s., define $X_-(t) = -X(t)$, $f_-(x) = -f(-x)$, $\sigma_-(t) = -\sigma(t)$. Then

$$dX_-(t) = -f_-(X_-(t))\,dt + \sigma_-(t)\,dB(t), \quad t \geq 0.$$

By the same argument as above, it can be shown that $\liminf_{t\to\infty} X_-(t) \leq 0$ a.s., which yields $\limsup_{t\to\infty} X(t) \geq 0$ a.s. Combining this with $\liminf_{t\to\infty} X(t) \leq 0$ a.s. yields the required result.

3.8 Proofs of Proposition 3.3.1

By (2.2.16) we have

$$\lim_{x\to\infty} \left\{\log(1-\Phi(x)) - \log x^{-1} + x^2/2\right\} = \log\left(1/\sqrt{2\pi}\right),$$

and so

$$\lim_{x\to\infty} \frac{\log(1-\Phi(x))}{x^2/2} = -1.$$

Suppose that $\theta(n) \to 0$ as $n \to \infty$, we have for $\epsilon > 0$ that

$$\lim_{n\to\infty} \frac{\log(1-\Phi(\epsilon/\theta(n)))}{\epsilon^2/(2\theta^2(n))} = -1.$$

Thus

$$\lim_{n\to\infty} \frac{\log(1-\Phi(\epsilon/\theta(n)))}{\log n} = \lim_{n\to\infty} \frac{\log(1-\Phi(\epsilon/\theta(n)))}{\epsilon^2/(2\theta^2(n))} \cdot \frac{\epsilon^2/(2\theta^2(n))}{\log n}$$
$$= -\frac{\epsilon^2}{2} \lim_{n\to\infty} \frac{1}{\theta^2(n)\log n}. \quad (3.8.1)$$

In cases (A) and (B), we have that $\theta^2(n) := \int_n^{n+1} \sigma^2(s)\,ds$ obeys

$$\lim_{n\to\infty} \theta^2(n)\log n = L, \quad (3.8.2)$$

and in each case $\theta(n) \to 0$ as $n \to \infty$. Therefore (3.8.1) holds in both case (A) and case (B). To prove part (A), note that when $L = 0$, from (3.8.2) and (3.8.1), we have

$$\lim_{n\to\infty} \frac{\log(1-\Phi(\epsilon/\theta(n)))}{\log n} = -\infty$$

for every $\epsilon > 0$, so by (3.3.5), we have $S(\epsilon) < +\infty$ for every $\epsilon > 0$. Therefore, by Lemma 2.2.1,S' obeys (3.3.7), as required.

To prove part (B), note that when $L \in (0,\infty)$, from (3.8.2) and (3.8.1), we have

$$\lim_{n\to\infty} \frac{\log(1-\Phi(\epsilon/\theta(n)))}{\log n} = -\frac{\epsilon^2}{2L}.$$

If $\epsilon > \sqrt{2L}$, then by (3.3.5) we have $S(\epsilon) < +\infty$, and thus by Lemma 2.2.1,$S'(\epsilon) < +\infty$. On the other hand, if $\epsilon < \sqrt{2L}$, by (3.3.5) we have that $S(\epsilon) = +\infty$, and so by Lemma 2.2.1,$S'(\epsilon) = +\infty$. Therefore (2.2.8) holds with $\epsilon' = \sqrt{2L}$.

In case (C), suppose that there exists $\epsilon^* > 0$ such that $S'(\epsilon^*) < +\infty$. Then by Lemma 2.2.1,we have that $S(\epsilon^*) < +\infty$. Then we have that $1 - \Phi(\epsilon^*/\theta(n)) \to 0$ as $n \to \infty$. This implies that $\theta(n) \to 0$ as $n \to \infty$. Thus, we have that (3.8.1) holds. Now, because $\sigma^2(t)\log t \to \infty$ as $t \to \infty$, we have that $\theta^2(n)\log n \to \infty$ as $n \to \infty$. Therefore, using this fact and (3.8.1), we have that

$$\lim_{n\to\infty} \frac{\log(1-\Phi(\epsilon^*/\theta(n)))}{\log n} = 0.$$

Therefore, it follows from (3.3.5) that $S(\epsilon^*) = +\infty$, a contradiction. Therefore, we must have that $S'(\epsilon) = +\infty$ for every $\epsilon > 0$, which is (3.3.13), as claimed.

3.9 Proof of Proposition 3.4.1

3.9.1 Proof of Part (A) of Proposition 3.4.1

In the case when $\sigma \in L^2(0,\infty)$, we have that each of the events $\{\omega : \lim_{t\to\infty} Y(t,\omega) = 0\}$ and $\{\omega : \lim_{t\to\infty} X(t,\omega) = 0\}$ are a.s. by Theorem 3.2.1.

Suppose now that $\sigma \notin L^2(0,\infty)$. Define

$$\Omega_e = \Omega_X \cap \Omega_Y, \tag{3.9.1}$$

where Ω_X is given by (3.2.5) and Ω_Y is defined by (3.2.4). Define for each $\omega \in \Omega_e$ the realisation $z(\cdot,\omega)$ by $z(t,\omega) = X(t,\omega) - Y(t,\omega)$ for $t \geq 0$. Then $z(\cdot,\omega)$ is in $C^1(0,\infty)$ and obeys

$$z'(t,\omega) = -f(X(t,\omega)) + Y(t,\omega) = -f(z(t,\omega) + Y(t,\omega)) + Y(t,\omega),\ t \geq 0;\ z(0) = \xi.$$

Define

$$A_2 = \{\omega \in \Omega_e : \lim_{t\to\infty} Y(t,\omega) = 0\}, \quad A_3 = \{\omega \in \Omega_e : \liminf_{t\to\infty} |X(t,\omega)| = 0\}.$$

Therefore A_2 is an a.s. event by hypothesis. Since $\sigma \notin L^2(0,\infty)$, A_3 is an a.s. event by Theorem 3.2.2. Thus the event A_4 defined by $A_4 = A_2 \cap A_3$ is almost sure. Fix $\omega \in A_4$. Since $Y(t,\omega) \to 0$ as $t \to \infty$ and $\liminf_{t\to\infty} |X(t,\omega)| = 0$, it follows that

$$\liminf_{t\to\infty} |z(t,\omega)| \leq \liminf_{t\to\infty} |X(t,\omega)| + |Y(t,\omega)| = \liminf_{t\to\infty} |X(t,\omega)| + \lim_{t\to\infty} |Y(t,\omega)| = 0.$$

Let $\eta \in (0,1)$. We next show that $\limsup_{t\to\infty} |z(t,\omega)| \leq \eta$. Since f is continuous on $\mathbb{R}$, it is uniformly continuous on $[-2,2]$. Therefore, there exists a function $\mu : [0,\infty) \to [0,\infty)$ such that $\mu(0) = 0$, $\mu(\nu) \to 0$ as $\nu \downarrow 0$, and for which for every $\nu \in [0,4]$ is defined by

$$\mu(\nu) = \max_{|x|\vee|y|\leq 2, |x-y|\leq\nu} |f(x) - f(y)|.$$

Thus μ is a modulus of continuity of f on $[-2,2]$. Let $\epsilon > 0$ be so small that

$$\epsilon < \frac{\eta}{4}, \quad \epsilon + \mu(\epsilon) < f(\eta) \wedge |f(-\eta)|.$$

Then for $u \in [\eta-\epsilon, \eta+\epsilon] \subset (0,2)$ we have $|f(u) - f(\eta)| \leq \mu(\epsilon)$, so $f(u) \geq f(\eta) - \mu(\epsilon) > \epsilon$. Therefore

$$\epsilon < \inf_{u\in(\eta-\epsilon,\eta+\epsilon)} f(u). \tag{3.9.2}$$

On the other hand for $u \in [-\eta-\epsilon, -\eta+\epsilon] \subset (-2,0)$ we have $|f(u)-f(-\eta)| \leq \mu(\epsilon)$, so

$$f(u) \leq f(-\eta) + \mu(\epsilon) < -\epsilon.$$

Therefore

$$-\epsilon > \sup_{u \in (\eta-\epsilon, \eta+\epsilon)} f(u). \tag{3.9.3}$$

Since $Y(t,\omega) \to 0$ as $t \to \infty$, there exists $T_1(\epsilon,\omega) > 0$ such that $|Y(t,\omega)| < \epsilon$ for all $t > T_1(\epsilon)$. Suppose that $\limsup_{t\to\infty} |z(t,\omega)| > \eta$. Since $\liminf_{t\to\infty} |z(t,\omega)| = 0$, we may therefore define $T_2(\epsilon,\omega) = \inf\{t > T_1(\epsilon,\omega) : |z(t,\omega)| = \eta/2\}$. Also define $T_3(\epsilon,\omega) = \inf\{t > T_2(\epsilon,\omega) : |z(t,\omega)| = \eta\}$.

In the case when $z(T_3(\epsilon,\omega),\omega) = \eta$, we have $z'(T_3(\epsilon,\omega),\omega) \geq 0$. Since $|Y(T_3(\epsilon,\omega),\omega)| < \epsilon$ we have

$$\begin{aligned} 0 \leq z'(T_3(\epsilon,\omega),\omega) &= -f(z(T_3(\epsilon,\omega)) + Y(T_3(\epsilon,\omega),\omega)) + Y(T_3(\epsilon,\omega),\omega) \\ &= -f(\eta + Y(T_3(\epsilon,\omega),\omega)) + Y(T_3(\epsilon,\omega),\omega) \\ &< -f(\eta + Y(T_3(\epsilon,\omega),\omega)) + \epsilon \leq -\inf_{|u-\eta|<\epsilon} f(u) + \epsilon < 0, \end{aligned}$$

by (3.9.2), a contradiction. On the other hand, in the case when $z(T_3(\epsilon,\omega),\omega)) = -\eta$, we have that $z'(T_3(\epsilon,\omega),\omega) \leq 0$. Since $|Y(T_3(\epsilon,\omega),\omega)| < \epsilon$ we have

$$\begin{aligned} 0 \geq z'(T_3(\epsilon,\omega),\omega) &= -f(z(T_3(\epsilon,\omega),\omega) + Y(T_3(\epsilon,\omega),\omega)) + Y(T_3(\epsilon,\omega)) \\ &= -f(-\eta + Y(T_3(\epsilon,\omega),\omega)) + Y(T_3(\epsilon,\omega),\omega) \\ &> -f(-\eta + Y(T_3(\epsilon,\omega),\omega)) - \epsilon \geq -\sup_{|u+\eta|<\epsilon} f(u) - \epsilon > 0, \end{aligned}$$

by (3.9.3), a contradiction. Hence $T_3(\epsilon,\omega)$ does not exist for any $\omega \in A_4$. Hence $\limsup_{t\to\infty} |z(t,\omega)| \leq \eta$. Since $\eta > 0$ is arbitrary, we make take the limit as $\eta \downarrow 0$ to obtain $\limsup_{t\to\infty} |z(t,\omega)| = 0$. Since $X = Y + z$, and $Y(t,\omega) \to 0$ as $t \to \infty$, we have that $X(t,\omega) \to 0$ as $t \to \infty$, and because this is true for each ω in the a.s. event A_4, the result has been proven.

3.9.2 Proof of Part (C) of Proposition 3.4.1

Let the a.s. event Ω_e be as defined in (3.9.1). Define now the event $\Omega_3 = \{\omega \in \Omega_e : \limsup_{t\to\infty} |Y(t,\omega)| = +\infty\}$ which is a.s. by hypothesis. Define $F(t) = X(t) - f(X(t))$

for $t \geq 0$. Then (3.1.1) can be rewritten as

$$dX(t) = \{-X(t) + F(t)\}\, dt + \sigma(t)\, dB(t), \quad t \geq 0,$$

so by variation of constants we get

$$X(t) = X(0)e^{-t} + \int_0^t e^{-(t-s)} F(s) ds + Y(t), \quad t \geq 0.$$

Rearranging and taking absolute values gives

$$|Y(t)| \leq |X(t)| + |X(0)|e^{-t} + \int_0^t e^{-(t-s)} |F(s)|\, ds, \quad t \geq 0. \tag{3.9.4}$$

Define $A_5 = \left\{\omega \in \Omega_X : \sup_{t\geq 0} |X(t,\omega)| < +\infty\right\}$ and suppose that $\mathbb{P}[A_5] > 0$. Define $A_6 = A_5 \cap \Omega_3$. Then $\mathbb{P}[A_6] = \mathbb{P}[A_5] > 0$. Let $\omega \in A_6$ and define $X_1(\omega) = \sup_{t\geq 0} |X(t,\omega)|$. Then $|X(t,\omega)| \leq X_1(\omega)$ for all $t \geq 0$. Since f is continuous, for all $y \geq 0$, there exists $\overline{f}(y) < +\infty$ such that

$$\max_{|x|\leq y} |f(x)| =: \overline{f}(y). \tag{3.9.5}$$

Therefore $|f(X(t,\omega))| \leq \overline{f}(X_1(\omega))$ for all $t \geq 0$. Hence by (3.9.4), for each $\omega \in A_6$, we have that for all $t \geq 0$

$$\begin{aligned} |Y(t,\omega)| &\leq X_1(\omega) + X_1(\omega) + \int_0^t e^{-(t-s)} (X_1(\omega) + \overline{f}(X_1(\omega)))\, ds \\ &\leq 3X_1(\omega) + \overline{f}(X_1(\omega)). \end{aligned}$$

Since $\limsup_{t\to\infty} |Y(t,\omega)| = +\infty$ for each $\omega \in A_6 \subseteq \Omega_3$, we have a contradiction, so therefore we must have $\mathbb{P}[A_6] = 0$. This, taken together with continuity the continuity of X, gives $\limsup_{t\to\infty} |X(t)| = \infty$ a.s., proving part (C) of Proposition 3.4.1.

3.9.3 Proof of Part (B) of Proposition 3.4.1

Define $\Omega_2 = \Omega_1 \cap \Omega_e$. Then by hypothesis, for every $\omega \in \Omega_2$ we have that there is a finite and positive $Y^*(\omega)$ such that

$$Y^*(\omega) = \limsup_{t\to\infty} |Y(t,\omega)|.$$

By definition $\underline{Y} \leq Y^*(\omega) \leq \overline{Y}$. Define for $\omega \in \Omega_2$

$$X^*(\omega) = \limsup_{t\to\infty} |X(t,\omega)|,$$

where $X^*(\omega) = 0$ and $X^*(\omega) = +\infty$ are admissible values. By (3.9.4), we have

$$Y^*(\omega) \le X^*(\omega) + \limsup_{t\to\infty} \int_0^t e^{-(t-s)} |F(s,\omega)|\, ds \le X^*(\omega) + \limsup_{t\to\infty} |F(t,\omega)|.$$

By the definition of $\overline{f}$, F and X^* we have

$$\limsup_{t\to\infty} |F(t,\omega)| \le X^*(\omega) + \overline{f}(X^*(\omega)).$$

Since $\overline{f}$ is defined by (3.9.5) and h_f by (3.4.6), we obtain

$$Y^*(\omega) \le 2X^*(\omega) + \overline{f}(X^*(\omega)) = h_f(X^*(\omega)).$$

By Proposition 3.4.2, h_f is an increasing function, so we have $X^*(\omega) \ge h_f^{-1}(Y^*(\omega))$. Now by the definition of X^*, $\underline{X}$ and the fact that h_f^{-1} is increasing, we have

$$\underline{X} = \inf_{\omega\in\Omega_2} X^*(\omega) \ge \inf_{\omega\in\Omega_2} h_f^{-1}(Y^*(\omega)) = h_f^{-1}\left(\inf_{\omega\in\Omega_2} Y^*(\omega)\right).$$

Since $\Omega_2 \subseteq \Omega_1$, $\inf_{\omega\in\Omega_2} Y^*(\omega) \ge \inf_{\omega\in\Omega_1} Y^*(\omega) = \underline{Y}$, by the definition of $\underline{Y}$. Thus as h_f^{-1} is increasing,

$$\underline{X} \ge h_f^{-1}\left(\inf_{\omega\in\Omega_2} Y^*(\omega)\right) \ge h_f^{-1}(\underline{Y}) = \underline{x}(f,\underline{Y}),$$

using (3.4.7) at the last step. Notice lastly that part (i) of Proposition 3.4.2 implies that $\underline{x}(f,\underline{Y}) > 0$ because $\underline{Y} > 0$, by hypothesis.

3.10 Proofs of Theorem 3.4.3 and Proposition 3.4.3

3.10.1 Preliminary results

The asymptotic estimate (3.4.12) in Theorem 3.4.3 is shown by first establishing the estimate

$$\limsup_{t\to\infty} |X(t,\omega)| \le \max(x_+(\overline{Y}), x_-(\overline{Y})) + \overline{Y}, \quad \text{for each } \omega \in \Omega_2 \tag{3.10.1}$$

where we define $x_+, x_- : [0,\infty) \to \mathbb{R}$ by

$$x_+(y) = \sup\{x > 0 : \min_{a\in[-y,y]} f(x+a) = y\}, \quad y \ge 0, \tag{3.10.2}$$

$$-x_-(y) = \inf\{x < 0 : \max_{a\in[-y,y]} f(x+a) = -y\}, \quad y \ge 0. \tag{3.10.3}$$

We prefer the estimate in (3.4.12) in part because the estimate on the right hand side of (3.10.1) is difficult to analyse in general, due to the complexity of x_+ and x_-. Moreover, there is no loss of sharpness in the estimate in (3.4.12) relative to (3.10.1) in the case when f is increasing. To see this, first note that when f is increasing on $\mathbb{R}$, it can readily be seen that $x_+(y) = y + f^{-1}(y)$ and $x_-(y) = y - f^{-1}(-y)$. Therefore, if we grant that (3.10.1) holds, it follows that

$$\limsup_{t\to\infty} |X(t,\omega)| \leq 2\overline{Y} + \max(f^{-1}(\overline{Y}), -f^{-1}(\overline{Y})), \quad \text{for each } \omega \in \Omega_2.$$

Therefore, if we define

$$\overline{x}^*(f,y) = 2y + \max(f^{-1}(y), -f^{-1}(-y)), \tag{3.10.4}$$

it can be seen that

$$\limsup_{t\to\infty} |X(t,\omega)| \leq \overline{x}^*(f,\overline{Y}), \quad \text{for each } \omega \in \Omega_2.$$

On the other hand, $\overline{x}^*(f)$ defined in (3.10.4) is equal to $\overline{x}(f)$ defined in (3.4.11) when f is increasing, because $f^-(x) = f^{-1}(x)$ for $x \leq 0$ and $f^+(x) = f^{-1}(x)$ for $x \geq 0$, where f^+ and f^- are defined in (3.4.9) and (3.4.10).

Therefore, the second stage in proving the asymptotic estimate (3.4.12) reduces to showing that

$$y + \max(x_+(y), x_-(y)) \leq \overline{x}(f,y), \quad y \geq 0, \tag{3.10.5}$$

and accordingly, we start the proof of Theorem 3.4.3 by first establishing (3.10.5).

Lemma 3.10.1. *Suppose that f obeys* (1.2.2) *and* (3.4.8)*. Then the functions f^+ and f^- given by* (3.4.9) *and* (3.4.10) *are well–defined and with x_+, x_- and $\overline{x}$ defined by* (3.10.2)*,* (3.10.3) *and* (3.4.11) *respectively, we have* (3.10.5)*.*

Proof. Let $z > x + f^+(x)$. Suppose $u \in [-x,x]$. Then $z + u > f^+(x)$. By the definition of f^+ we have $f(a) > x$ for all $a > f^+(x)$. Therefore, for each $z > x + f^+(x)$, we have $f(z+u) > x$ for all $u \in [-x,x]$. Hence

$$\min_{u\in[-x,x]} f(z+u) > x, \quad \text{for all } z > x + f^+(x).$$

Since $x_+(y) = \sup\{x > 0 : \min_{u\in[-y,y]} f(x+u) = y\}$, we have that

$$y + f^+(y) \geq x_+(y). \tag{3.10.6}$$

Let $x > 0$. Let $z < -x + f^-(-x)$. Suppose $u \in [-x, x]$. Then $z + u < f^-(-x)$. By the definition of f^- we have $f(a) < -x$ for all $a < f^-(-x)$. Therefore, for each $z < -x + f^-(-x)$, we have $f(z+u) < -x$ for all $u \in [-x, x]$. Hence

$$\max_{u\in[-x,x]} f(z+u) < -x, \quad \text{for all } z < -x + f^-(-x).$$

Since $-x_-(y) = \inf\{x > 0 : \max_{u\in[-y,y]} f(x+u) = -y\}$, we have that $-y + f^-(-y) \leq -x_-(y)$, so

$$y - f^-(-y) \geq x_-(y). \tag{3.10.7}$$

Hence by (3.4.11), (3.10.6), (3.10.7), for any $y \geq 0$ we have

$$\begin{aligned}\overline{x}(f, y) &= 2y + \max(f^+(y), -f^-(-y)) \\ &= y + \max(y + f^+(y), y - f^-(-y)) \\ &\geq y + \max(x_+(y), x_-(y)),\end{aligned}$$

which is (3.10.5). □

3.10.2 Proof of Theorem 3.4.3

We start with a lemma.

Lemma 3.10.2. *Let f obey* (1.2.2) *and* (3.4.8)*. Suppose that p is a continuous function such that*

$$\limsup_{t\to\infty} |p(t)| \leq \overline{p}.$$

Suppose that z is any continuous solution of

$$z'(t) = -f(z(t) + p(t)) + p(t), \quad t > 0; \quad z(0) = \xi$$

Then

$$\limsup_{t\to\infty} |z(t)| \le \max(x_+(\overline{p}), x_-(\overline{p})) \le \overline{p} + \max(f^+(\overline{p}), -f^-(-\overline{p}),$$

where x_+ *is defined by* (3.10.2), x_- *by* (3.10.3) *and* $f^\pm$ *by* (3.4.9), (3.4.10). *Moreover, if* $x(t) = z(t) + p(t)$ *for* $t \ge 0$, *and* $\overline{x}$ *is defined by* (3.4.11), *then*

$$\limsup_{t\to\infty} |x(t)| \le \overline{x}(f, \overline{p}).$$

Proof. For every $\eta > 0$, there exists $T(\eta) > 0$ such that for $t \ge T(\eta)$ we have $|p(t)| \le \overline{p}+\eta$.

The bound on p yields the estimate

$$z(t) - \overline{p} - \eta \le z(t) + p(t) \le z(t) + \overline{p} + \eta, \quad t \ge T(\eta).$$

Since $f(x) \to \infty$ as $x \to \infty$, for every $\eta > 0$ there exists $\tilde{x}_+(\eta) > \eta$ such that

$$\min_{a\in[-\overline{p}-\eta,\overline{p}+\eta]} f(x+a) \ge \overline{p} + 2\eta, \quad \text{for all } x \ge \tilde{x}_+(\eta).$$

Note that x_+ defined by (3.10.2) obeys

$$\min_{a\in[-\overline{p},\overline{p}]} f(x+a) \ge \overline{p}, \quad \text{for all } x \ge x_+(\overline{p}). \tag{3.10.8}$$

Also as $f(x) \to -\infty$ as $x \to -\infty$, for every $\eta > 0$ there exists an $\tilde{x}_-(\eta) > \eta$ such that

$$\max_{a\in[-\overline{p}-\eta,\overline{p}+\eta]} f(x+a) \le -\overline{p} - 2\eta, \quad \text{for all } x \le -\tilde{x}_-(\eta).$$

Note that x_- defined by (3.10.2) obeys

$$\max_{a\in[-\overline{p},\overline{p}]} f(x+a) \le -\overline{p}, \quad \text{for all } x \le -x_-(\overline{p}). \tag{3.10.9}$$

Let $x(\eta) = \max(\tilde{x}_+(\eta), \tilde{x}_-(\eta))$.

Suppose that there is $t_1(\eta) > T(\eta)$ such that $z(t_1) > \tilde{x}_+(\eta)$. If not, it follows that

$$z(t) \le \tilde{x}_+(\eta) \text{ for all } t \ge T(\eta)$$

and we have that $\limsup_{t\to\infty} z(t) \le \tilde{x}_+(\eta)$, which implies that $\limsup_{t\to\infty} z(t) \le x_+(\overline{p})$. We will show that there exists a $t_2(\eta) > t_1(\eta)$ such that $z(t_2) = \tilde{x}_+(\eta)$ and moreover for all $t \ge t_2(\eta)$ that $z(t) \le \tilde{x}_+(\eta)$. This implies that $\limsup_{t\to\infty} z(t) \le \tilde{x}_+(\eta)$ or indeed that $\limsup_{t\to\infty} z(t) \le x_+(\overline{p})$.

By the definition of t_1 we have $z(t_1) + p(t_1) > 0$ and

$$z'(t_1) = -f(z(t_1) + p(t_1)) + p(t_1) \le -\min_{a\in[-\overline{p}-\eta,\overline{p}+\eta]} f(z(t_1) + a) + \overline{p} + \eta \le -\eta.$$

Then we have either that $z(t) > \tilde{x}_+(\eta)$ for all $t \ge t_1(\eta)$ or that there is a minimal $t_2(\eta) > t_1(\eta)$ such that $z(t_2) = \tilde{x}_+(\eta)$. In the former case for every $t \ge t_1(\eta)$ we have

$$z'(t) = -f(z(t) + p(t)) + p(t) \le -\min_{a\in[-\overline{p}-\eta,\overline{p}+\eta]} f(z(t) + a) + \overline{p} + \eta \le -\eta.$$

Since $\eta > 0$, we may define $t_3 = (z(t_1) - \tilde{x}_+(\eta))/\eta + t_1 + 1$. Then $z(t_3) \le z(t_1) - \eta(t_3 - t_1) < \tilde{x}_+(\eta)$, a contradiction. Therefore, there exists a $t_2 > t_1$ such that $z(t_2) = \tilde{x}_+(\eta)$. Now

$$z'(t_2) = -f(z(t_2) + p(t_2)) + p(t_2) \le -\min_{a\in[-\overline{p}-\eta,\overline{p}+\eta]} f(\tilde{x}_+(\eta) + a) + \overline{p} + \eta \le -\eta.$$

Then either there exists a minimal $t_3(\eta) > t_2(\eta)$ such that $z(t_3) = \tilde{x}_+(\eta)$ or we have that $z(t) < \tilde{x}_+(\eta)$ for all $t > t_2(\eta)$. In the former case, we must have $z'(t_3) \ge 0$. But once again we have

$$z'(t_3) = -f(z(t_3) + p(t_3)) + p(t_3) \le -\min_{a\in[-\overline{p}-\eta,\overline{p}+\eta]} f(\tilde{x}_+(\eta) + a) + \overline{p} + \eta \le -\eta,$$

a contradiction. Thus we have $z(t) < \tilde{x}_+(\eta)$ for all $t > t_2(\eta)$, which implies that $\limsup_{t\to\infty} z(t) \le x_+(\overline{p})$.

Suppose that there is $t_1(\eta) > T(\eta)$ such that $z(t_1) < -\tilde{x}_-(\eta)$. If not, it follows that

$$z(t) \ge -\tilde{x}_-(\eta) \text{ for all } t \ge T(\eta)$$

and we have that $\liminf_{t\to\infty} z(t) \ge -\tilde{x}_-(\eta)$, which implies that $\liminf_{t\to\infty} z(t) \ge -x_-(\overline{p})$. We will show that there is a $t_2(\eta) > t_1(\eta)$ such that $z(t_2) = -\tilde{x}_-(\eta)$ and moreover that for all $t \geq t_2(\eta)$ that $z(t) \geq -\tilde{x}_-(\eta)$. This will imply that $\liminf_{t\to\infty} z(t) \geq -\tilde{x}_-(\eta)$ or that $\liminf_{t\to\infty} z(t) \geq -x_-(\overline{p})$.

By the definition of t_1 we have $z(t_1) + p(t_1) < 0$ and

$$z'(t_1) = -f(z(t_1) + p(t_1)) + p(t_1) \geq -\max_{a\in[-\overline{p}-\eta,\overline{p}+\eta]} f(z(t_1) + a) - \overline{p} - \eta \geq \eta.$$

Then we have either that $z(t) < -\tilde{x}_-(\eta)$ for all $t \geq t_1(\eta)$ or that there is a minimal $t_2(\eta) > t_1(\eta)$ such that $z(t_2) = -\tilde{x}_-(\eta)$. In the former case for every $t \geq t_1(\eta)$ we have

$$z'(t) = -f(z(t) + p(t)) + p(t) \geq -\max_{a\in[-\overline{p}-\eta,\overline{p}+\eta]} f(z(t) + a) - \overline{p} - \eta \geq \eta.$$

Since $\eta > 0$, we may define $t_3 = (z(t_1)+\tilde{x}_-(\eta))/-\eta+t_1+1$. Then $z(t_3) \geq z(t_1)+\eta(t_3-t_1) > -\tilde{x}_-(\eta)$, a contradiction. Therefore, there exists a $t_2 > t_1$ such that $z(t_2) = -\tilde{x}_-(\eta)$. Now

$$z'(t_2) = -f(z(t_2) + p(t_2)) + p(t_2) \geq -\max_{a\in[-\overline{p}-\eta,\overline{p}+\eta]} f(\tilde{x}_+(\eta) + a) - \overline{p} - \eta \geq \eta.$$

Then either there exists a minimal $t_3(\eta) > t_2(\eta)$ such that $z(t_3) = -\tilde{x}_-(\eta)$ or we have that $z(t) > -\tilde{x}_-(\eta)$ for all $t > t_2(\eta)$. In the former case, we must have $z'(t_3) \leq 0$. But once again we have

$$z'(t_3) = -f(z(t_3) + p(t_3)) + p(t_3) \geq -\max_{a\in[-\overline{p}-\eta,\overline{p}+\eta]} f(\tilde{x}_+(\eta) + a) - \overline{p} - \eta \geq \eta,$$

a contradiction. Thus we have $z(t) > -\tilde{x}_-(\eta)$ for all $t > t_2(\eta)$, which implies that $\liminf_{t\to\infty} z(t) \geq -x_-(\overline{p})$.

We have thus shown that

$$\limsup_{t\to\infty} z(t) \leq x_+(\overline{p}), \quad \liminf_{t\to\infty} z(t) \geq -x_-(\overline{p}),$$

and so $\limsup_{t\to\infty} |z(t)| \leq \max(x_+(\overline{p}), x_-(\overline{p}))$, as required.

Since $\limsup_{t\to\infty} |p(t)| \leq \overline{p}$, it follows that

$$\limsup_{t\to\infty} |x(t)| \leq \overline{p} + \max(x_+(\overline{p}), x_-(\overline{p})).$$

Therefore using Lemma 3.10.1 (specifically (3.10.5)), we have

$$\limsup_{t\to\infty} |x(t)| \le \overline{p} + \max(x_+(\overline{p}), x_-(\overline{p})) \le \overline{x}(f, \overline{p}),$$

which is precisely the final estimate required. □

3.10.3 Proof of Theorem 3.4.3

Let Ω_e be the event defined in (3.9.1). Then for every $\omega \in \Omega_e$ we may define $z(t,\omega) := X(t,\omega) - Y(t,\omega)$ for $t \ge 0$ where Y is the solution of (3.2.3). Then $z(0) = X(0)$ and each sample path of z is in $C^1(0,\infty)$ with

$$z'(t,\omega) = -f(z(t,\omega) + Y(t,\omega)) + Y(t,\omega), \quad t > 0.$$

If θ obeys (3.3.9a) and (3.3.9b), it follows from part (B) of Theorem 3.3.1 that there exists an a.s. event Ω_1, defined by (3.3.10), such that there is a finite, positive and deterministic $\overline{Y}$ satisfying (3.3.12) i.e.

$$\overline{Y} = \sup_{\omega\in\Omega_1} \limsup_{t\to\infty} |Y(t,\omega)|.$$

Let $\Omega_2 = \Omega_1 \cap \Omega_e$. Fix $\omega \in \Omega_2$. Then by Lemma 3.10.2, with $Y(\cdot,\omega)$ in the role of p, and $z(\cdot,\omega)$ in the role of z, we have that

$$\limsup_{t\to\infty} |z(t,\omega)| \le \max(x_+(\overline{Y}), x_-(\overline{Y})).$$

Putting $X(\cdot,\omega)$ in the role of x in Lemma 3.10.2, we can infer from Lemma 3.10.2 that for $\omega \in \Omega_2$

$$\limsup_{t\to\infty} |X(t,\omega)| \le \overline{x}(f, \overline{Y}).$$

Since this estimate holds for all $\omega \in \Omega_2$, we have precisely (3.4.12), as required.

3.10.4 Proof of Proposition 3.4.3

For a given f, f^+ and f^- are non–decreasing functions. We show first that $\lim_{x\to 0} f^+(x) = 0$.

Since $f(x) \to \infty$ as $x \to \infty$, there exists $a > 0$ such that $f(x) \ge 1$ for all $x \ge a$. Let ϵ be any positive number with $\epsilon < a$. Then, as f is continuous and strictly positive on

$[\epsilon, a]$, it follows that there exists $x_\epsilon \in [\epsilon, a]$ such that $0 < f(x_\epsilon) = \min_{\epsilon \le y \le a} f(y)$. Define $\delta(\epsilon) = f(x_\epsilon)$. Then, if $0 < x < \delta(\epsilon)$, we have that $f^+(x) \le \epsilon$. To justify this, suppose to the contrary that $f^+(x') > \epsilon$ for some $x' \in (0, \delta(\epsilon))$. Then $f^+(x') = \sup\{z > 0 : f(z) = x'\} > \epsilon$. Now, for $f^+(x') =: z' > \epsilon$, we have $f(z') \ge f(x_\epsilon) = \delta(\epsilon)$. However, by hypothesis $\delta(\epsilon) > x'$, so $f(z') > x'$. However, $z' = f^+(x') = \sup\{z > 0 : f(z) = x'\}$ implies that $f(z') = x'$, so we have a contradiction. Therefore, for every $\epsilon \in (0, a)$ there exists a $\delta = \delta(\epsilon) > 0$ such that if $0 < x < \delta(\epsilon)$, we have that $f^+(x) \le \epsilon$. Thus, as f^+ is a non–negative function, and $\epsilon \in (0, a)$ is arbitrary, this is precisely $\lim_{x\to 0^+} f^+(x) = 0$. The proof that $\lim_{x\to 0} f^-(x) = 0$ is similar.

Note that $\lim_{x\to\infty} f^+(x) = \lim_{x\to\infty} f^-(x) = \infty$ (by (3.4.8)) so it is clear that $y \mapsto \overline{x}(f, y)$ is increasing, and moreover that $\lim_{y\to\infty} \overline{x}(f, y) = \infty$. Also, as $\lim_{x\to 0^+} f^+(x) = \lim_{x\to 0} f^-(x) = 0$, we have that $\lim_{y\to 0} \overline{x}(f, y) = 0$, which proves part (i).

To prove part (ii), suppose first that there is $x > 0$ such that ${f_1}^+(x) < {f_2}^+(x)$. By definition, $f_1(z) > x$ for all $z > {f_1}^+(x)$. Since ${f_1}^+(x) < {f_2}^+(x)$, we have $f_1({f_2}^+(x)) > x$. But $f_2({f_2}^+(x)) \ge f_1({f_2}^+(x))$ by (3.4.5). Hence $f_2({f_2}^+(x)) > x$. But $f_2({f_2}^+(x)) = x$, by definition, so we have the contradiction $x > x$. Hence

$$ {f_1}^+(x) \ge {f_2}^+(x), \quad x > 0. \tag{3.10.10} $$

Suppose next there is $y < 0$ such that $f_1^-(y) > f_2^-(y)$. By definition, $f_1(z) < y$ for $z < f_1^-(y)$. Since $f_2^-(y) < f_1^-(y)$, it follows that $f_1(f_2^-(y)) < y$. By (3.4.5), we have $-f_2(u) \ge -f_1(u)$ for all $u < 0$. Hence with $u = f_2^-(y)$, we get $-f_2(f_2^-(y)) \ge -f_1(f_2^-(y)) > -y$. But $f_2(f_2^-(y)) = y$, by definition, so we have $-y = -f_2(f_2^-(y)) \ge -f_1(f_2^-(y)) > -y$, a contradiction. Thus we have $f_1^-(y) \le f_2^-(y)$ for all $y < 0$, or

$$ -f_1^-(y) \ge -f_2^-(y), \quad y < 0. \tag{3.10.11} $$

Therefore, it follows from (3.4.11), (3.10.10) and (3.10.11) that

$$ \begin{aligned} \overline{x}(f_2, y) &= 2y + \max({f_2}^+(y), -f_2^-(-y)) \\ &\le 2y + \max({f_1}^+(y), -{f_1}^-(-y)) = \overline{x}(f_1, y), \end{aligned} $$

as required.

Chapter 4

Asymptotic Classification of Finite Dimensional Nonlinear SDEs

4.1 Introduction

4.1.1 Discussion on hypotheses

In the previous chapter, we employed results on the linear equation studied in Chapter 2, to enable us to analyse the asymptotic behaviour of the *scalar* nonlinear SDE

$$dX(t) = -f(X(t))\,dt + \sigma(t)\,dB(t) \tag{4.1.1}$$

where the underlying deterministic ODE has a unique globally stable equilibrium at zero. In this chapter, we seek to extend our results in Chapter 3 to the finite–dimensional case, expecting that the results on finite dimensional affine equations in Chapter 2 can be of help.

Just as in Chapter 2, we will work with a d–dimensional system, so the noise intensity will be a continuous $d \times r$ matrix–valued function and B a r–dimensional standard Brownian motion. f should be a function from $\mathbb{R}^d$ to $\mathbb{R}^d$, and be continuous so that solutions of the SDE can exist. However, it is important to ask how we should capture reasonably the assumption that $x = 0$ is a unique globally stable equilibrium solution of (4.1.1).

As to uniqueness, we must request that $f(x) = 0$ if and only if $x = 0$. Global stability is however more difficult to characterise, and in general even deterministic research has focussed on giving sufficient conditions under which all solutions of

$$x'(t) = -f(x(t)) \tag{4.1.2}$$

obey $x(t) \to 0$ as $t \to \infty$. One popular assumption in the stochastic literature is the so called *dissipative condition*

$$\langle x, f(x)\rangle > 0 \text{ for all } x \neq 0,$$

and it is easy to see that this yields $x(t) \to 0$ as $t \to \infty$, by showing that the Liapunov function $V(x(t)) = \|x(t)\|_2^2$ is decreasing on trajectories. It is also clear that the dissipative condition makes $x = 0$ the unique equilibrium, for if there were another at $x^* \neq 0$, then we have

$$0 = \langle x^*, 0 \rangle = \langle x^*, f(x^*) \rangle > 0$$

a contradiction. We see also that in the one–dimensional case, the condition $xf(x) > 0$ for $x \neq 0$, which characterises the existence of a unique and globally stable equilibrium, is nothing other than the dissipative condition.

The analysis of good sufficient conditions on f which guarantee global stability for the ordinary equation (4.1.2) forms a substantial body of work, and rather than attempting to trace this, we mention the original contributions of Olech and Hartman in a series of papers in the 1960s. In Hartman [35], global stability is assure by

$$[J(x)]_{ij} = \frac{\partial f_i}{\partial x_j}(x) \text{ is such that } H(x) := \frac{1}{2}(J(x) + J(x)^T) \text{ is negative definite} \qquad (4.1.3)$$

In the two–dimensional case, Olech [62] proves that

$$\text{trace} J(x) \leq 0 \text{ and } |f(x)| \geq \phi > 0 \text{ for } |x| \geq x^* \qquad (4.1.4)$$

suffice. The second of these conditions is weakened in Hartman and Olech [36]to

$$|x||f(x)| > K \text{ for all } |x| \geq M, \text{ or } \int_0^\infty \inf_{\|x\|=\rho} |f(x)|\, d\rho = +\infty \qquad (4.1.5)$$

and the first of Olech's assumptions is modified to

$$\alpha(x) \leq 0, \text{ where } \alpha(x) = \max_{1 \leq i < j \leq d} \{\lambda_i(x) + \lambda_j(x)\} \qquad (4.1.6)$$

and the $\lambda(x)$'s are eigenvalues of $H(x)$. The local asymptotic stability of the equilibrium is also assumed. In the 1970's Brock and Scheinkman [29] demonstrated that some of Olech and Hartman's conditions can be deduced from Liapunov considerations. In particular, they show that some of the conditions used in [35] imply the dissipative condition. This is of particular interest to us, as our approach to understanding the stability and boundedness of solutions may be considered a Liapunov–like approach. A more recent paper of Gasull, LLibre and Sotomayor [34] considers the relationships between these conditions and global stability. As the chapter develops, the relationship between these existing conditions and the conditions we will need are drawn out.

Given that our basic assumption which will guarantee the stability of the underlying deterministic equation is the dissipative condition, in this chapter we investigate how the results in Chapter 3 can be extended to the finite–dimensional case. Roughly speaking, we are able to prove analogues of the main results in Chapter 3 concerning a characterisation of asymptotic stability (under weak conditions on f) and a classification of the asymptotic behaviour (under strong mean–reverting conditions far from the equilibrium).

However, in this chapter somewhat stronger assumptions on f are needed in order to achieve this. In Chapter 3, we were able to prove our results requiring only that f be continuous, so that even when solutions might not be unique, we can ensure that all solutions have the same property. In this chapter for our stability and boundedness results, we have imposed a local Lipschitz condition on f. In the case of stability this makes our argument more manageable, and we conjecture that the assumption could be relaxed. However, the proof of boundedness makes use of a comparison argument in which the existence of a unique solution of an equation (whose solution majorises the solution of the SDE) is essential, and the removal of the Lipschitz assumption in this case is more difficult to achieve.

In the case where we prove stability, we have found that it is no longer enough to assume merely the global stability condition that sufficed in the scalar case. Instead, our proof requires that f obey

$$\liminf_{x\to\infty} \inf_{|y|=x} \langle y, f(y)\rangle > 0.$$

It is interesting to notice that this condition implies the first condition in (4.1.5). Moreover, we speculate that in the finite dimensional stochastic case, it may be necessary for the function f to provide some minimal strength of mean reversion at infinity, because the stochastic part of the equation can be transient (in the sense that its norm can grow to infinity as $t \to \infty$). An example of this possibility was given in Example 2.2.1 in Chapter 2. It is reasonable to assign the source of this problem to the *transience* of the stochastic perturbation in the finite dimensional part, because in the scalar case, where no additional condition on f is needed, the perturbation $\int_0^t \sigma(s)\,dB(s)$ being a time–changed one–dimensional Brownian motion, is *recurrent*.

To give some motivation as to why we expect some extra condition on f in the presence of a cumulatively transient perturbation, we recall the deterministic results in Chapter 1, and write the differential equation in the integral form

$$x(t) = \xi - \int_0^t f(x(s))\,ds + \int_0^t g(s)\,ds, \quad t \geq 0. \tag{4.1.7}$$

In the case when $g(t) \to 0$ but $\int_0^t g(s)\,ds = +\infty$, we have shown that unless f has enough strength to counteract the cumulative perturbation $\int_0^t g(s)\,ds$, it is possible that $x(t) \to \infty$ as $t \to \infty$. If one writes the stochastic equation in integral form

$$X(t) = \xi - \int_0^t f(X(s))\,ds + \int_0^t \sigma(s)\,dB(s), \quad t \geq 0,$$

we can guess that when the cumulative perturbation $\int_0^t \sigma(s)\,dB(s)$ is not convergent (which happens when $\sigma \notin L^2([0,\infty);\mathbb{R}^{d\times r})$), some minimal strength in f is needed to keep the solution from escaping to infinity.

There is another reason to believe that the analogy with the deterministic equation here is justified. In the case when g is in $L^1(0,\infty)$ and the cumulative perturbation $\int_0^t g(s)\,ds$ converges, we have shown in Chapter 1 that the solution of (4.1.7) obeys $x(t) \to 0$ as $t \to \infty$ using only the global stability condition $xf(x) > 0$ for $x \neq 0$, which is nothing other than the dissipative condition in one dimension. In this chapter, a direct analogue of this result in the stochastic case is proven. It can be shown that when f obeys only the dissipative condition, and $\sigma \in L^2([0,\infty);\mathbb{R}^{d\times r})$ (so that the cumulative stochastic perturbation $\int_0^t \sigma(s)\,dB(s)$ converges), then $X(t) \to 0$ as $t \to \infty$ a.s.

4.1.2 Set–up of the problem and main results

Given these general considerations, we now summarise the problem to be studied in precise terms, and outline the main results of the chapter. Let d and r be integers. We fix a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t\geq 0}, \mathbb{P})$. Let B be a standard r–dimensional Brownian motion which is adapted to $(\mathcal{F}(t))_{t\geq 0}$. We consider the stochastic differential equation

$$dX(t) = -f(X(t))\,dt + \sigma(t)\,dB(t), \quad t \geq 0; \quad X(0) = \xi \in \mathbb{R}^d. \tag{4.1.8}$$

We suppose that

$$f \in C(\mathbb{R}^d;\mathbb{R}^d); \quad \langle x, f(x)\rangle > 0, \quad x \neq 0; \quad f(0)=0, \tag{4.1.9}$$

and that σ obeys (2.1.1). To simplify the existence and uniqueness of a unique continuous adapted solution of (4.1.8) on $[0,\infty)$, we assume that $f:\mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz continuous. See e.g., [55]. Hereinafter, we refer to this unique continuous and adapted process as the solution of (4.1.8).

In the case when σ is identically zero, it follows under the hypothesis (4.1.9) that the solution x of equation (1.2.5)

$$x'(t) = -f(x(t)), \quad t>0; \quad x(0)=\xi,$$

obeys

$$\lim_{t\to\infty} x(t;\xi) = 0 \text{ for all } \xi \in \mathbb{R}^d. \tag{4.1.10}$$

Clearly $x(t)=0$ for all $t \geq 0$ if $\xi = 0$. The question naturally arises: if the solution x of (1.2.5) obeys (4.1.10), under what conditions on f and σ does the solution X of (4.1.8) obey

$$\lim_{t\to\infty} X(t,\xi) = 0, \quad \text{a.s. for each } \xi \in \mathbb{R}^d. \tag{4.1.11}$$

In Chapter 3, we showed under the scalar version of condition (4.1.9) that the solution X of (4.1.8) obeys (4.1.11) if and only if σ obeys

$$S_{\text{scalar}}(\epsilon)\sum_{n=0}^{\infty}\left\{1-\Phi\left(\frac{\epsilon}{\sqrt{\int_n^{n+1}\sigma^2(s)\,ds}}\right)\right\} < +\infty, \quad \text{for every } \epsilon>0, \tag{4.1.12}$$

where Φ is the distribution function of a standardised normal random variable. Corresponding integral conditions were developed also. In this chapter, we show that a corresponding condition on σ also suffices. In fact, we show in Theorem 3.5.1 that if f obeys (4.1.9) and is locally Lipschitz continuous, and σ is also continuous, then the solution X of (4.1.8) obeys (4.1.11) if and only if the condition (2.1.4) from Chapter 2 holds i.e.,

$$S(\epsilon)=\sum_{n=0}^{\infty}\left\{1-\Phi\left(\frac{\epsilon}{\sqrt{\int_n^{n+1}\|\sigma(s)\|_F^2(s)\,ds}}\right)\right\} < +\infty, \quad \text{for every } \epsilon>0, \tag{4.1.13}$$

provided that f obeys

$$\text{There exists } \phi > 0 \text{ such that } \phi := \liminf_{x\to\infty} \inf_{|y|=x} \langle y, f(y)\rangle, \tag{4.1.14}$$

a condition weaker than, but similar to, (1.2.12). As in the scalar case, therefore, we see that the condition that guarantees the stability of the linear equation when perturbed by σ suffices also for all nonlinear equations for which f obeys (4.1.14)

In the case when (4.1.14) is not assumed, it can still be shown that if (4.1.13) does not hold, then

$$\mathbb{P}[X(t,\xi) \to 0 \text{ as } t \to \infty] = 0 \text{ for each } \xi \in \mathbb{R}^d.$$

Also, if (4.1.13) holds, the only possible limiting behaviour of solutions are that $X(t) \to 0$ as $t \to \infty$ or $\|X(t)\| \to \infty$ as $t \to \infty$. In the case when $\sigma \in L^2(0,\infty)$, X obeys (4.1.11) without any further conditions on f.

The other major result in the chapter (Theorem 4.2.6) gives a complete classification of the asymptotic behaviour of solutions of (4.1.8) under a strengthening of (4.1.14), namely

$$\liminf_{r\to\infty} \inf_{\|x\|=r} \frac{\langle x, f(x)\rangle}{\|x\|} = +\infty, \tag{4.1.15}$$

which is a direct analogue of the condition needed to give a classification of solutions of (4.1.8) in the scalar case. We show that solutions of (4.1.8) are either (a) convergent to zero with probability one (b) bounded, not convergent to zero, but approach zero arbitrarily close infinitely often with probability one or (c) are unbounded with probability one. Possibility (a) occurs when $S(\epsilon)$ is finite for all ϵ; (b) happens when $S(\epsilon)$ is finite for some ϵ, but infinite for others, and (c) occurs when $S(\epsilon)$ is infinite for all ϵ. Therefore, this result is directly analogous to Theorems 2.2.5 which applies to linear stochastic differential equations whose underlying deterministic part is globally stable.

Although the condition (4.1.13) is necessary and sufficient for X to obey (4.1.11), it may prove to be a little unwieldy for use in some situations. For this reason we deduce some sharp sufficient conditions for X to obey (4.1.11). If f obeys (4.1.9) and is locally Lipschitz continuous, and σ is continuous but is not square integrable, because σ_{ij} is not square integrable for $j \in J_i$, then

$$\lim_{t\to\infty} \int_0^t e^{-2(t-s)} \sum_{l\in J_i} \sigma_{il}^2(s)\,ds \cdot \log\log\left(\int_0^t e^{2s} \sum_{l\in J_i} \sigma_{il}^2(s)\,ds\right) = 0, \tag{4.1.16}$$

implies that the solution X of (4.1.8) obeys (4.1.11). In the spirit of Theorem 3.5.1, we also establish converse results in the case when $t \mapsto \|\sigma(t)\|_F^2$ is monotone (Theorem 3.5.3), and demonstrate that the condition (4.1.16) is hard to relax if we require X to obey (4.1.11).

The main results are proven by showing that the stability of (4.1.8) is intimately connected with the the stability of a linear SDE with the same diffusion coefficient (Theorem 4.2.2). The stability of the linear SDE can be characterised by exploiting the fact that an explicit solution for the equation can be written down, and that the solution is a Gaussian process. As to the organisation of the chapter, notation, and statements and discussion about main results are presented in Section 4.2, with the proofs of these results being in the main part deferred to Section 4.3.

4.2 Statement and Discussion of Main Results

We start by showing that solutions of (4.1.8) will become arbitrarily large whenever the diffusion coefficient is such that solutions of the corresponding affine equation (2.1.2) have the same property. Furthermore, if solutions are of (2.1.2) are bounded but not convergent to zero, then solutions of (4.1.8) do not converge to zero.

Theorem 4.2.1. *Suppose that f satisfies* (1.2.4). *Suppose that σ obeys* (2.1.1) *and let S be defined by* (2.2.5). *Let X be the solution of* (4.1.8).

(A) Suppose that S obeys (2.2.10). *Then*

$$\limsup_{t\to\infty} \|X(t)\| = +\infty, \quad a.s.$$

(B) Suppose that S obeys (2.2.8). *Then there is a deterministic $c_3 > 0$ such that*

$$\limsup_{t\to\infty} \|X(t)\| \geq c_3, \quad a.s.$$

We show that its solutions can either tend to zero or their modulus tends to infinity if and only if solutions of a linear equation with the same diffusion tend to zero.

Theorem 4.2.2. *Suppose that f satisfies* (4.1.9) *and* (1.2.4)*. Suppose σ obeys* (2.1.1)*. Let X be the solution of* (4.1.8)*, and Y the solution of* (2.1.2)*. Then there exist a.s. events Ω_1 and Ω_2 such that*

$$\{\omega : \lim_{t\to\infty} X(t,\omega) = 0\} \subseteq \{\omega : \lim_{t\to\infty} Y(t,\omega) = 0\} \cap \Omega_1, \tag{4.2.1}$$

$$\{\omega : \lim_{t\to\infty} Y(t,\omega) = 0\} \subseteq \{\omega : \lim_{t\to\infty} X(t,\omega) = 0\} \cup \{\omega : \lim_{t\to\infty} \|X(t,\omega)\| = \infty\} \cap \Omega_2. \tag{4.2.2}$$

When taken in conjunction with Theorem 2.2.1, we see that the condition (2.2.6) comes close to characterising the convergence of solutions of (4.1.8) to zero, contingent on the possibility that $\|X(t)\| \to \infty$ as $t \to \infty$ being eliminated.

Theorem 4.2.3. *Suppose that f satisfies* (4.1.9) *and* (1.2.4)*. Suppose σ obeys* (2.1.1)*. Let X be the solution of* (4.1.8)*. Let Φ be given by* (2.2.2)*.*

(i) If σ obeys (2.2.6)*, then for each $\xi \in \mathbb{R}^d$,*

$$\{\lim_{t\to\infty} \|X(t,\xi)\| = \infty\} \cup \{\lim_{t\to\infty} \|X(t,\xi)\| = 0\} \quad \textit{is an a.s. event.}$$

(ii) If $X(t,\xi) \to 0$ with positive probability for some $\xi \in \mathbb{R}^d$, then σ obeys (2.2.6)*.*

Proof. To prove part (i), we first note that (2.2.6) and Theorem 2.2.1 implies that $Y(t) \to 0$ as $t \to \infty$ a.s. Theorem 4.2.2 then implies that the event $\{\lim_{t\to\infty} \|X(t,\xi)\| = \infty\} \cup \{\lim_{t\to\infty} X(t,\xi) = 0\}$ is a.s. To show part (ii), by hypothesis and Theorem 4.2.2, we see that $\mathbb{P}[Y(t) \to 0 \text{ as } t \to \infty] > 0$. Therefore, by Theorem 2.2.1, it follows that σ obeys (2.2.6). □

Part (i) of Theorem 4.2.3 is unsatisfactory, as it does not rule out the possibility that $\|X(t)\| \to \infty$ as $t \to \infty$ with positive probability. If further restrictions are imposed on f and σ, however, it is possible to conclude that $X(t,\xi) \to 0$ as $t \to \infty$ a.s. In the scalar case, it was shown in Appleby and Rodkina [6] that no such additional conditions are required.

Our first result in this direction imposes an extra condition on σ, but not on f. We note that when $\sigma \in L^2([0,\infty);\mathbb{R}^{d\times r})$, Y obeys (2.2.7) and that X obeys (4.1.11). However, we cannot apply directly the semimartingale convergence theorem of Lipster–Shiryaev directly (see e.g., [51, Theorem 7, p.139]) to the non–negative semimartingale $\|X\|^2$, because it is not guaranteed that $\mathbb{E}[\|X(t)\|^2] < +\infty$ for all $t \geq 0$. The proof of the following theorem, which is deferred to the next section, uses the ideas of [51, Theorem 7, p.139] heavily, however.

Theorem 4.2.4. *Suppose that f satisfies* (4.1.9) *and* (1.2.4)*. Suppose also that σ obeys* (2.1.1) *and $\sigma \in L^2([0,\infty);\mathbb{R}^{d\times r})$. Let X be the solution of* (4.1.8)*, and Y the solution of* (2.1.2)*. Then X obeys* (4.1.11) *and $\lim_{t\to\infty} Y(t) = 0$ a.s.*

It can be seen from Theorem 4.2.4 that it only remains to prove Theorem 4.2.2 in the case when $\sigma \notin L^2([0,\infty);\mathbb{R}^{d\times r})$. Under an additional restriction on f (but no extra condition on σ) we can give necessary and sufficient conditions in terms of σ for which X tends to zero a.s.

Theorem 4.2.5. *Suppose f obeys* (1.2.4) *and in addition to* (4.1.9)*, obeys*

$$\liminf_{r\to\infty} \inf_{\|x\|=r} \langle x, f(x)\rangle > 0. \tag{4.2.3}$$

Suppose that σ obeys (2.1.1)*. Let X be the solution of* (4.1.8)*. Let θ be defined by* (2.2.4) *and let Φ be given by* (2.2.2)*. Then the following are equivalent:*

(A) S obeys (2.2.6)*;*

(B) $\lim_{t\to\infty} X(t,\xi) = 0$ with positive probability for some $\xi \in \mathbb{R}^d$.

(C) $\lim_{t\to\infty} X(t,\xi) = 0$ a.s. for each $\xi \in \mathbb{R}^d$.

Notice that no monotonicity conditions are required on $\|\sigma\|_F^2$ in order for this result to hold. The condition (4.2.3) was not required to prove an analogous result in the scalar case in [6]. However, the condition is weaker than the condition (1.2.12) which was required in the scalar case to secure the stability of solutions of (4.1.8) in [10].

There is one final result in this section. It gives a complete characterisation of the asymptotic behaviour of solutions of (4.1.8) under a strengthening of (4.2.3), namely

$$\liminf_{r\to\infty} \inf_{\|x\|=r} \frac{\langle x, f(x)\rangle}{\|x\|} = +\infty. \tag{4.2.4}$$

(4.2.4) is a direct analogue of the condition needed to give a classification of solutions of (4.1.8) in the scalar case. The following result is therefore a direct generalisation of a scalar result from [6] to finite dimensions.

Theorem 4.2.6. *Suppose f obeys* (1.2.4), (4.1.9), *and* (4.2.4). *Suppose that σ obeys* (2.1.1). *Let X be the solution of* (4.1.8). *Let θ be defined by* (2.2.4) *and let Φ be given by* (2.2.2). *Then the following are equivalent:*

(A) If S obeys (2.2.6), *then $\lim_{t\to\infty} X(t) = 0$, a.s. for each $\xi \in \mathbb{R}^d$;*

(B) If S obeys (2.2.8), *then there exists deterministic $0 < c_1 \le c_2 < +\infty$ such that*

$$c_1 \le \limsup_{t\to\infty} \|X(t)\| \le c_2, \quad \textit{a.s., for each } \xi \in \mathbb{R}^d;$$

Moreover,

$$\liminf_{t\to\infty} \|X(t)\| = 0, \quad \textit{a.s.}$$

(C) If S obeys (2.2.10), *then $\limsup_{t\to\infty} \|X(t)\| = +\infty$ a.s., for each $\xi \in \mathbb{R}^d$.*

4.3 Sufficient Conditions for Asymptotic Behaviour

Due to Lemma 2.3.1 and Theorem 4.2.2, the functions Σ_i determine the asymptotic behaviour of X. Let $N \subseteq \{1, 2, \dots, d\}$ be defined by

$$N = \{i \in \{1, 2, \dots, d\} : \sigma_i \notin L^2(0, \infty)\}. \tag{4.3.1}$$

where σ_i is defined by (2.3.1). Note that if $i \notin N$, then $\sigma_i \in L^2(0, \infty)$ and we immediately have that $Y_i(t) \to 0$ as $t \to \infty$ a.s.

Theorem 4.3.1. *Suppose that f satisfies* (4.1.9), (1.2.4) *and* (4.2.3). *Suppose that σ obeys* (2.1.1) *and $\sigma \notin L^2([0,\infty);\mathbb{R}^{d\times r})$. Let X be the solution of* (4.1.8). *Let N be the set defined in* (4.3.1) *and Σ_i be defined by* (2.3.2) *for each $i \in N$.*

(i) If $\Sigma_i(t) \to 0$ as $t \to \infty$ for each $i \in N$, then X obeys (4.1.11).

(ii) If X obeys (4.1.11), *then $\liminf_{t\to\infty} \Sigma_i(t) = 0$ for each $i \in N$.*

(iii) If $\liminf_{t\to\infty} \Sigma_i(t) > 0$ for some $i \in N$, then $\mathbb{P}[\lim_{t\to\infty} X(t) = 0] = 0$.

(iv) If $\lim_{t\to\infty} \Sigma_i(t) = \infty$ for some $i \in N$ then $\limsup_{t\to\infty} \|X(t)\| = \infty$ a.s.

An interesting fact of this result is that it is unnecessary for $\sigma(t) \to 0$ as $t \to \infty$ in order for solutions of (4.1.8) to obey (4.1.11). In fact, we can even have $\limsup_{t\to\infty} \|\sigma(t)\|_F^2 = \infty$ and still have $X(t) \to 0$ as $t \to \infty$ a.s. See [10] for examples.

Note that the condition

$$\lim_{t\to\infty} \|\sigma(t)\|_F^2 \log t = 0 \tag{4.3.2}$$

implies that $\Sigma_i(t) \to 0$ as $t \to \infty$ for each $i \in N$, and for $i \notin N$ it still implies that $Y_i(t) \to 0$ as $t \to \infty$. Also note that the condition

$$\lim_{t\to\infty} \sigma_i^2(t) \log t = +\infty \text{ for some } i \in \{1,\dots,d\} \tag{4.3.3}$$

implies that $\Sigma_i(t) \to \infty$ as $t \to \infty$, and finally that the condition

$$\liminf_{t\to\infty} \sigma_i^2(t) \log t > 0$$

implies that $\liminf_{t\to\infty} \Sigma_i(t) > 0$. The next result is therefore an easy corollary of Theorem 4.3.1.

Theorem 4.3.2. *Suppose that f satisfies* (4.1.9), (1.2.4) *and* (4.2.3). *Suppose that σ obeys* (2.1.1). *Let X be the solution of* (4.1.8).

(i) If for all $i \in \{1,\dots,d\}$ σ_i obeys $\lim_{t\to\infty} \sigma_i^2(t) \log t = 0$, then X obeys (4.1.11).

(ii) If there is $i \in \{1, \ldots, d\}$ *such that* σ_i *obeys* $\liminf_{t\to\infty} \sigma_i^2(t) \log t \in (0, \infty)$*, then*

$\mathbb{P}[\lim_{t\to\infty} X(t) = 0] = 0$.

(iii) If there is $i \in \{1, \ldots, d\}$ *such that* σ_i *obeys* $\lim_{t\to\infty} \sigma_i^2(t) \log t = \infty$*, then*

$\limsup_{t\to\infty} \|X(t)\| = \infty$ *a.s.*

In [31], Chan and Williams have proven in the case when $t \mapsto \sigma^2(t)$ is decreasing, that Y obeys (2.2.7) if and only if σ obeys (3.1.5). Therefore, our final result is a corollary of this observation and of Theorem 4.2.2. It can also be deduced from Theorem 4.2.5.

Theorem 4.3.3. *Suppose that* f *satisfies* (4.1.9), (1.2.4) *and* (4.2.3)*. Suppose that* σ *obeys* (2.1.1) *and* $\|\sigma\|_F^2$ *is decreasing. Let* X *be the solution of* (4.1.8)*. Then the following are equivalent:*

(A) σ *obeys* $\lim_{t\to\infty} \|\sigma(t)\|_F^2 \log t = 0$*;*

(B) $\lim_{t\to\infty} X(t, \xi) = 0$ *with positive probability for some* $\xi \in \mathbb{R}^d$*;*

(C) $\lim_{t\to\infty} X(t, \xi) = 0$ *a.s. for each* $\xi \in \mathbb{R}^d$.

Another result in the same direction, but with a slightly weaker monotonicity hypothesis is the following.

Theorem 4.3.4. *Suppose that* f *satisfies* (4.1.9), (1.2.4) *and* (4.2.3)*. Suppose that* σ *obeys* (2.1.1) *and that* $(\int_n^{n+1} \|\sigma(s)\|_F^2 \, ds)_{n\geq 0}$ *is non–increasing. Let* X *be the solution of* (4.1.8)*. Then the following are equivalent:*

(A) σ *obeys* $\lim_{n\to\infty} \int_n^{n+1} \|\sigma(s)\|_F^2 \, ds \cdot \log n = 0$*;*

(B) $\lim_{t\to\infty} X(t, \xi) = 0$ *with positive probability for some* $\xi \in \mathbb{R}^d$*;*

(C) $\lim_{t\to\infty} X(t, \xi) = 0$ *a.s. for each* $\xi \in \mathbb{R}^d$.

4.4 Proof of Results

4.4.1 Proof of Theorem 4.2.4

By Itô's rule, we have

$$\|X(t)\|^2 = \|\xi\|^2 - \int_0^t 2\langle X(s), f(X(s))\rangle\, ds + \int_0^t \|\sigma(s)\|_F^2\, ds \\ + \sum_{j=1}^{r} \int_0^t \sum_{i=1}^{d} 2X_i(s)\sigma_{ij}(s)\, dB_j(s), \quad t \geq 0. \quad (4.4.1)$$

Define M to be the local martingale given by

$$M(t) = \sum_{j=1}^{r} \int_0^t \sum_{i=1}^{d} 2X_i(s)\sigma_{ij}(s)\, dB_j(s), \quad t \geq 0.$$

and let

$$U(t) = \int_0^t 2\langle X(s), f(X(s))\rangle ds, \quad A(t) = \int_0^t \|\sigma(s)\|_F^2 ds, \quad t \geq 0.$$

Since $\langle x, f(x)\rangle \geq 0$ for all $x \in \mathbb{R}^d$ and $\sigma \in L^2([0,\infty); \mathbb{R}^{d\times r})$, it follows that A and U are continuous adapted increasing processes. Therefore by Theorem 0.3.6, it follows that

$$\lim_{t\to\infty} \|X(t)\|^2 = L \in [0,\infty), \quad a.s.$$

and that

$$\lim_{t\to\infty} \int_0^t \langle X(s), f(X(s))\rangle ds = I \in [0,\infty), \quad a.s.$$

By continuity this means that there is an a.s. event $A = \{\omega : \|X(t,\omega)\| \to \sqrt{L(\omega)} \in [0,\infty) \text{ as } t \to \infty\}$. We write $A = A_+ \cup A_0$ where

$$A_+ = \{\omega : \|X(t,\omega)\| \to \sqrt{L(\omega)} \in (0,\infty) \text{ as } t \to \infty\},$$

and $A_0 = \{\omega : X(t,\omega) \to 0 \text{ as } t \to \infty\}$. Suppose that $\omega \in A_+$. Define

$$F(x) = \langle x, f(x)\rangle, \quad x \in \mathbb{R}^d.$$

By (4.1.9), we have that $F(x) = 0$ if and only if $x = 0$. Define for any $r \geq 0$

$$\inf_{|x|=r} F(x) =: \phi(r) \geq 0.$$

Since f is continuous, F is continuous, therefore ϕ is continuous. Hence $\min_{|x|=r} F(x) = \phi(r)$. Suppose there is $r > 0$ such that $\phi(r) = 0$. Then there exists x with $|x| = r$ such that

$F(x) = \phi(r) = 0$. But this implies that $x = 0$, a contradiction. Moreover ϕ is continuous and positive definite. Hence for $\omega \in A_+$ we have

$$\liminf_{t\to\infty} \langle X(t,\omega), f(X(t,\omega)) \geq \phi(\sqrt{L(\omega)}) > 0.$$

Therefore

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \langle X(s,\omega), f(X(s,\omega))\rangle\, ds \geq \phi(\sqrt{L(\omega)}) > 0. \tag{4.4.2}$$

Since the last two terms on the righthand side of (4.4.1) have finite limits as $t \to \infty$, (4.4.2) implies that for $\omega \in A_+$ that

$$0 \leq \lim_{t\to\infty} \frac{X^2(t,\omega)}{t} = -2\phi(\sqrt{L(\omega)}) < 0,$$

a contradiction. Therefore $\mathbb{P}[A_+] = 0$. Since $\mathbb{P}[A] = 1$, we must have $\mathbb{P}[A_0] = 1$, as required.

4.4.2 Proof of Theorem 4.2.1

Define

$$\Omega_X = \big\{\omega \in \Omega : \text{there is a unique continuous adapted process } X \tag{4.4.3}$$
$$\text{for which the realisation } X(\cdot,\omega) \text{ obeys (4.1.8)}\big\}$$

$$\Omega_Y = \big\{\omega \in \Omega : \text{there is a unique continuous adapted process } Y \tag{4.4.4}$$
$$\text{for which the realisation } Y(\cdot,\omega) \text{ obeys (2.1.2)}\big\}.$$

Let

$$\Omega_e = \Omega_X \cap \Omega_Y. \tag{4.4.5}$$

If S obeys (2.2.10), it follows from Theorem 2.2.1 that $\limsup_{t\to\infty} \|Y(t)\| = +\infty$, a.s., and let the event on which this holds be $\Omega_1 \subseteq \Omega_Y$. Suppose that there is an event $A = \{\omega : \limsup_{t\to\infty} \|X(t)\| < \infty\}$ for which $\mathbb{P}[A] > 0$. Define $A_1 = A \cap \Omega_1 \cap \Omega_e$ so that $\mathbb{P}[A_1] > 0$.

Next, rewrite (4.1.8) as

$$dX(t) = (-X(t) + [X(t) - f(X(t))])\, dt + \sigma(t)\, dB(t), \quad t \geq 0; \quad X(0) = \xi.$$

Therefore on Ω_X we obtain

$$X(t) = \xi e^{-t} + \int_0^t e^{-(t-s)}(X(s) - f(X(s)))\,ds + e^{-t}\int_0^t e^s\sigma(s)\,dB(s).$$

Since Y obeys (2.2.1), for $\omega \in \Omega_e$ we have

$$Y(t,\omega) = X(t,\omega) - \xi e^{-t} - \int_0^t e^{-(t-s)}(X(s,\omega) - f(X(s,\omega)))\,ds, \quad t \geq 0. \tag{4.4.6}$$

Define for $\omega \in A_1$

$$X^*(\omega) := \limsup_{t\to\infty} \|X(t)\| < +\infty, \tag{4.4.7}$$

and define $\bar{f}(x) = \sup_{|y|\leq x}|f(y)|$ and $\bar{F}(x) = 2x + \bar{f}(x)$ for $x \geq 0$. Then for each $\omega \in A_1$, it follows from (4.4.6) that

$$\limsup_{t\to\infty} \|Y(t,\omega)\| \leq 2X^*(\omega) + \bar{f}(X^*(\omega)) = \bar{F}(X^*(\omega)),$$

and as $\bar{F}(X^*(\omega)) < +\infty$, a contradiction results.

To prove part (B), first note that $\bar{F}$ is continuous and increasing on $[0,\infty)$ with $\bar{F}(0) = 0$ and $\lim_{x\to\infty}\bar{F}(x) = +\infty$. Therefore, for every $c > 0$ there exists a unique $c' > 0$ such that $\bar{F}(c) = c'$, or $c' = \bar{F}^{-1}(c)$. Suppose that S obeys (2.2.8), so that by Theorem 2.2.1 there is a $c_1 > 0$ such that $\limsup_{t\to\infty}\|Y(t)\| \geq c_1$ a.s. Let the event on which this holds be Ω_2. Suppose now that the event A_2 defined by

$$A_2 = \{\omega \in \Omega_X : \limsup_{t\to\infty}\|X(t,\omega)\| < \bar{F}^{-1}(c_1)\},$$

and suppose that $\mathbb{P}[A_2] > 0$. Define $A_3 = A_2 \cap \Omega_e \cap \Omega_2$. Then $\mathbb{P}[A_3] > 0$. For $\omega \in A_3$, $X^*(\omega)$ as given by (4.4.7) is well–defined and finite, and in fact $X^*(\omega) < \bar{F}^{-1}(c_1)$. As before, from (4.4.6), we deduce that $\limsup_{t\to\infty}\|Y(t,\omega)\| \leq \bar{F}(X^*(\omega))$. But then we have $c_1 \leq \bar{F}(X^*(\omega))$, which implies $\bar{F}^{-1}(c_1) \leq X^*(\omega) < \bar{F}^{-1}(c_1)$, a contradiction. Thus we have that $\mathbb{P}[A_2] = 0$, so $\limsup_{t\to\infty}\|X(t)\| \geq \bar{F}^{-1}(c_1) =: c_3 > 0$ a.s., as required.

4.4.3 Proof of Theorem 4.2.2

In this proof, we implicitly consider the case where $\sigma \notin L^2([0,\infty);\mathbb{R}^{d\times r})$, as Theorem 4.2.4 shows that the result holds in the case where $\sigma \in L^2([0,\infty);\mathbb{R}^{d\times r})$, with each of the events $\{\omega : \lim_{t\to\infty} Y(t,\omega) = 0\}$ and $\{\omega : \lim_{t\to\infty} X(t,\omega) = 0\}$ being a.s.

We prove that $X(t) \to 0$ as $t \to \infty$ implies $Y(t) \to 0$ as $t \to \infty$ i.e., (4.2.1). Since f obeys (4.1.9) it follows from (4.4.6) that for each $\omega \in \{\omega : X(t,\omega) \to 0 \text{ as } t \to \infty\} \cap \Omega_e$ that $Y(t,\omega) \to 0$ as $t \to \infty$, proving (4.2.1).

We now prove that $Y(t) \to 0$ as $t \to \infty$ implies $X(t) \to 0$ as $t \to \infty$ or $\|X(t)\| \to \infty$ as $t \to \infty$, i.e. (4.2.2).

Define $\Omega_2 = \{\omega : \lim_{t\to\infty} Y(t,\omega) = 0\} \cap \Omega_Y$ and

$$A_0 = \{\omega : \liminf_{t\to\infty} |X(t,\omega)| = 0\},$$
$$A_+ = \{\omega : \liminf_{t\to\infty} \|X(t,\omega)\| \in (0,\infty)\},$$
$$A_\infty = \{\omega : \liminf_{t\to\infty} \|X(t,\omega)\| = \infty\}.$$

Also define

$$\Omega_0 = \Omega_2 \cap \Omega_X \cap A_0 = \Omega_2 \cap \Omega_e \cap A_0,$$
$$\Omega_+ = \Omega_2 \cap \Omega_X \cap A_+ = \Omega_2 \cap \Omega_e \cap A_+,$$
$$\Omega_\infty = \Omega_2 \cap \Omega_X \cap A_\infty = \Omega_2 \cap \Omega_e \cap A_\infty.$$

Finally define $A_1 = \{\omega : \lim_{t\to\infty} X(t,\omega) = 0\}$ and $\Omega_1 = \Omega_2 \cap \Omega_X \cap A_1$. Clearly $A_1 \subseteq A_0$ and $\Omega_1 \subseteq \Omega_0$.

Define for each $\omega \in \Omega_e$ the realisation $z(\cdot,\omega)$ by $z(t,\omega) = X(t,\omega) - Y(t,\omega)$ for $t \geq 0$. Then $z(\cdot,\omega)$ is in $C^1(0,\infty)$ and obeys

$$z'(t,\omega) = -f(X(t,\omega)) + Y(t,\omega) = -f(z(t,\omega) + Y(t,\omega)) + Y(t,\omega),\ t \geq 0;\ z(0) = \xi.$$

Let $\omega \in \Omega_0 \cup \Omega_+$. Then $\liminf_{t\to\infty} \|X(t,\omega)\| < +\infty$. Define also

$$g(t,\omega) = f(z(t,\omega)) - f(z(t,\omega) + Y(t,\omega)) + Y(t,\omega), \quad t \geq 0.$$

Since $z(\cdot,\omega)$ is in $C^1(0,\infty)$ we have

$$\begin{aligned}\frac{d}{dt}\|z(t,\omega)\|^2 &= 2\langle z(t,\omega), z'(t,\omega)\rangle \\ &= 2\langle z(t,\omega), -f(z(t,\omega)) + f(z(t,\omega)) - f(z(t,\omega) + Y(t,\omega)) + Y(t,\omega)\rangle \\ &= -2\langle z(t,\omega), f(z(t,\omega))\rangle + 2\langle z(t,\omega), g(t,\omega)\rangle.\end{aligned}$$

Since $Y(t,\omega) \to 0$ as $t \to \infty$ and $\liminf_{t\to\infty} \|X(t,\omega)\| =: L(\omega) < +\infty$, it follows that

$$\begin{aligned}\liminf_{t\to\infty} \|z(t,\omega)\| &\le \liminf_{t\to\infty} \|X(t,\omega)\| + \|Y(t,\omega)\| \\ &= \liminf_{t\to\infty} \|X(t,\omega)\| + \lim_{t\to\infty} \|Y(t,\omega)\| = L(\omega).\end{aligned}$$

Define $\lambda(\omega) := \liminf_{t\to\infty} \|z(t,\omega)\|$. Then $\lambda(\omega) < +\infty$.

STEP A: We now show that $\liminf_{t\to\infty} \|z(t,\omega)\| > 0$ implies

$$\limsup_{t\to\infty} \|z(t,\omega)\| < +\infty.$$

Proof of STEP A: Suppose $\lambda(\omega) > 0$ and $\limsup_{t\to\infty} \|z(t,\omega)\| = +\infty$. Since f is continuous, and $\langle x, f(x)\rangle > 0$ for $x \neq 0$, it follows that there exists $F_\lambda > 0$ such that

$$F_\lambda := \inf_{\|z\|=3\lambda/2} \langle z, f(z)\rangle.$$

Also, as f is locally Lipschitz continuous, there exists $K_{3\lambda} > 0$ such that

$$|f(x) - f(y)| \le K_{3\lambda}|x-y|, \quad \text{for all } |x| \vee |y| \le 3\lambda.$$

Let

$$\epsilon < \frac{3\lambda(\omega)}{2} \vee \frac{2F_{\lambda(\omega)}}{3\lambda(1+K_{3\lambda(\omega)})}.$$

Since $Y(t,\omega) \to 0$ as $t \to \infty$, there exists $T_1(\epsilon,\omega) > 0$ such that $\|Y(t,\omega)\| < \epsilon$ for all $t > T_1(\epsilon,\omega)$. Suppose that

$$\limsup_{t\to\infty} \|z(t,\omega)\| = +\infty.$$

Then there exists $T_2(\epsilon) > T_1(\epsilon)$ such that $T_2(\epsilon) = \inf\{t > T_1(\epsilon) : \|z(t)\| = 3\lambda/2\}$. Define also

$$T_3(\epsilon) = \inf\{t > T_2(\epsilon) : \|z(t)\| = 5\lambda/4\}, \quad T_4(\epsilon) = \inf\{t > T_3(\epsilon) : \|z(t)\| = 3\lambda/2\}.$$

Clearly with $w(t) = \|z(t,\omega)\|^2$, we have $w'(T_3,\omega) \le 0$ and $w'(T_4,\omega) \ge 0$. Since $z(T_4) = 3\lambda/2$ we have $\langle z(T_4), f(z(T_4))\rangle \ge F_\lambda$. Also we have $\|z(T_4)+Y(T_4)\| \le \|z(T_4)\|+\|Y(T_4)\| \le 3\lambda/2 + \epsilon \le 3\lambda$, so

$$\|f(z(T_4)+Y(T_4)) - f(z(T_4))\| \le K_{3\lambda}\|Y(T_4)\| \le K_{3\lambda}\epsilon.$$

Collecting these estimates yields

$$\begin{aligned}
&w'(T_4)\\
&= -2\langle z(T_4), f(z(T_4))\rangle + 2\langle z(T_4), g(T_4)\rangle\\
&= -2\langle z(T_4), f(z(T_4))\rangle + 2\langle z(T_4), f(z(T_4)) - f(z(T_4)+Y(T_4)) + Y(T_4)\rangle\\
&\le -2F_\lambda + 2\cdot\frac{3\lambda}{2}\epsilon + 2\frac{3\lambda}{2}\|f(z(T_4)) - f(z(T_4)+Y(T_4)))\|\\
&\le -2F_\lambda + 3\lambda\epsilon + 3\lambda K_{3\lambda}\epsilon < 0.
\end{aligned}$$

Therefore we have a contradiction, because $w'(T_4) \ge 0$.

STEP B: Next we show that $\liminf_{t\to\infty}\|z(t,\omega)\| = 0$ implies

$$\limsup_{t\to\infty}\|z(t,\omega)\| < +\infty.$$

Proof of STEP B: Suppose to the contrary that $\limsup_{t\to\infty}\|z(t,\omega)\| = +\infty$. Fix $\lambda > 0$ arbitrarily. Proceeding exactly as in STEP A, we can demonstrate that the supposition $\limsup_{t\to\infty}\|z(t,\omega)\| = \infty$ leads to a contradiction. Therefore we have shown that $\liminf_{t\to\infty}\|z(t,\omega)\| \in [0,\infty)$ implies that $\limsup_{t\to\infty}\|z(t,\omega)\| < +\infty$.

STEP C: Next we show that

$$\liminf_{t\to\infty}\|X(t,\omega)\| < +\infty$$

implies that $\liminf_{t\to\infty}|z(t,\omega)| = 0$, $\limsup_{t\to\infty}\|z(t,\omega)\| < +\infty$.

Proof of STEP C: First, we note that $\liminf_{t\to\infty}\|X(t,\omega)\| < +\infty$ implies that $\liminf_{t\to\infty}\|z(t,\omega)\| < +\infty$. By STEPs A and B, implies $\limsup_{t\to\infty}\|z(t,\omega)\| < +\infty$. Define

$$\limsup_{t\to\infty}\|z(t,\omega)\| =: \Lambda'(\omega) \in [0,\infty).$$

Suppose that $\liminf_{t\to\infty}\|z(t,\omega)\| = \lambda(\omega) > 0$. Then $\Lambda' \ge \lambda > 0$. By the continuity of f, the fact that $\Lambda' \ge \lambda > 0$, and the fact that f obeys $\langle x, f(x)\rangle > 0$ for all $x \ne 0$, there exists an $F_{\lambda,\Lambda'} > 0$ defined by

$$F_{\lambda(\omega),\Lambda'(\omega)} := \min_{\lambda(\omega)/2\le|x|\le\Lambda'(\omega)-\lambda(\omega)/2}\langle x, f(x)\rangle.$$

For every $\beta > 0$ there exists a $K_\beta > 0$ such that

$$|f(x) - f(y)| \le K_\beta|x-y|, \quad \text{for all } |x|\vee|y| \le \beta.$$

Suppose now that $\epsilon > 0$ is so small that

$$0 < \epsilon < \frac{\lambda(\omega)}{2} \wedge \frac{F_{\lambda(\omega),\Lambda'(\omega)}}{2(1+K_{\Lambda'(\omega)+\lambda(\omega)})(\Lambda'(\omega)+\lambda(\omega)/2)}.$$

Then there exists $T_1(\epsilon,\omega) > 0$ such that $|Y(t,\omega)| < \epsilon$ for all $t > T_1(\epsilon,\omega)$. Also, there exists $T_2(\omega) > 0$ such that $|z(t,\omega)| \leq \Lambda'(\omega) + \lambda(\omega)/2$ for all $t \geq T_2(\omega)$. Define $\Lambda = \Lambda' + \lambda$. Now there exists a $K_\Lambda > 0$ such that

$$|f(x) - f(y)| \leq K_\Lambda |x - y|, \quad \text{for all } |x| \vee |y| \leq \Lambda.$$

Now let $T_3(\epsilon,\omega) = 1 + T_1(\epsilon,\omega) \vee T_2(\omega)$. Then for $t \geq T_3(\epsilon,\omega)$ we have $\|z(t,\omega) + Y(t,\omega)\| \leq \Lambda'(\omega) + \lambda(\omega)/2 + \epsilon < \Lambda'(\omega) + \lambda(\omega) = \Lambda(\omega)$ and $\|z(t,\omega)\| \leq \Lambda(\omega)$. Therefore for $t \geq T_3(\epsilon,\omega)$ we have

$$\begin{aligned}
|\langle g(t,\omega), z(t,\omega)\rangle| &\leq \|z(t,\omega)\| \|f(z(t,\omega)+Y(t,\omega)) - f(z(t,\omega))\| + \langle z(t,\omega), Y(t,\omega)\rangle \\
&\leq K_\Lambda \|Y(t,\omega)\| \|z(t,\omega)\| + \|z(t,\omega)\| \|Y(t,\omega)\| \\
&\leq (1+K_\Lambda)\epsilon(\Lambda' + \lambda/2) = (1+K_{\Lambda'+\lambda})(\Lambda' + \lambda/2)\epsilon.
\end{aligned}$$

Since $\liminf_{t\to\infty} \|z(t,\omega)\| = \lambda(\omega) > 0$ there exists $T_4(\omega) > 0$ such that $\|z(t,\omega)\| > \lambda(\omega)/2$ for all $t \geq T_4(\omega)$. Define $T_5(\epsilon,\omega) = 1 + T_4(\omega) \vee T_3(\epsilon,\omega)$. Then for $t \geq T_5(\epsilon,\omega)$ we have $0 < \lambda(\omega)/2 < \|z(t,\omega)\| \leq \Lambda'(\omega) + \lambda(\omega)/2$, which implies that

$$\langle z(t,\omega), f(z(t,\omega))\rangle \geq F_{\lambda,\Lambda'} > 0.$$

Therefore for $t \geq T_5(\epsilon,\omega)$ we have

$$\begin{aligned}
\frac{d}{dt}\|z(t,\omega)\|^2 &= -2\langle z(t,\omega), f(z(t,\omega))\rangle + 2\langle g(t,\omega), z(t,\omega)\rangle \\
&\leq -2\langle z(t,\omega), f(z(t,\omega))\rangle + 2(1+K_{\Lambda'+\lambda})(\Lambda' + \lambda/2)\epsilon \\
&\leq -2F_{\lambda,\Lambda'} + 2(1+K_{\Lambda'+\lambda})(\Lambda' + \lambda/2)\epsilon \\
&< -F_{\lambda,\Lambda'}.
\end{aligned}$$

Therefore for $t \geq T_5(\epsilon,\omega)$ we have

$$\|z(t,\omega)\|^2 \leq \|z(T_5)\|^2 - F_{\lambda,\Lambda'}(t - T_5).$$

Hence we have that $\|z(t,\omega)\|^2 \to -\infty$ as $t \to \infty$, which is a contradiction. Thus $\liminf_{t\to\infty} \|z(t,\omega)\| = 0$, as required.

STEP D: Suppose that

$$\liminf_{t\to\infty} \|X(t,\omega)\| < +\infty.$$

Then $\lim_{t\to\infty} X(t,\omega) = 0$.

Proof of STEP D: By STEP C, $\liminf_{t\to\infty} \|X(t,\omega)\| < +\infty$, this implies that $\liminf_{t\to\infty} \|z(t,\omega)\| = 0$ and $\limsup_{t\to\infty} \|z(t,\omega)\| < +\infty$. If we can show that

$$\lim_{t\to\infty} \|z(t,\omega)\| = 0,$$

we are done because $X(t,\omega) = z(t,\omega) + Y(t,\omega)$ and $Y(t,\omega) \to 0$ as $t \to \infty$. Let $\eta > 0$. We next show that $\limsup_{t\to\infty} \|z(t,\omega)\| \le \eta$. Since f is locally Lipschitz there exists $K_2 > 0$ such that $|f(x) - f(y)| \le K_2|x - y|$ for $|x| \vee |y| \le 2\eta$. There also exists $F_\eta > 0$ such that

$$F_\eta := \min_{|x|=\eta} \langle x, f(x)\rangle.$$

Let $\epsilon > 0$ be so small that

$$\epsilon < \frac{\eta}{2} \wedge \frac{F_\eta}{\eta(1+K_2)}.$$

Since $Y(t,\omega) \to 0$ as $t \to \infty$, there exists $T_1(\epsilon,\omega) > 0$ such that $\|Y(t,\omega)\| < \epsilon$ for all $t > T_1(\epsilon)$. Suppose that $\limsup_{t\to\infty} \|z(t,\omega)\| > \eta$. Since $\liminf_{t\to\infty} \|z(t,\omega)\| = 0$, we may therefore define

$$T_2(\epsilon,\omega) = \inf\{t > T_1(\epsilon,\omega) : \|z(t,\omega)\| = \eta/2\},$$

$$T_3(\epsilon,\omega) = \inf\{t > T_2(\epsilon,\omega) : \|z(t,\omega)\| = \eta\}.$$

Therefore, with $w(t) = \|z(t,\omega)\|^2$ we have that $w'(T_3(\epsilon,\omega)) \ge 0$. Furthermore, for $t \in [T_2(\epsilon,\omega), T_3(\epsilon,\omega)]$ we have $\|z(t,\omega)\| \le \eta$ and $\|z(t,\omega) + Y(t,\omega)\| \le \eta + \epsilon < 2\eta$ so

$$\|g(t,\omega)\| \le \|f(z(t,\omega)) - f(z(t,\omega) + Y(t,\omega))\| + \|Y(t,\omega)\| \le K_2\|Y(t,\omega)\| + \epsilon \le (1+K_2)\epsilon.$$

Thus as $\|z(T_3)\| = \eta$, we have

$$|\langle z(T_3), g(T_3)\rangle| \le \|z(T_3)\|\|g(T_3)\| = \eta\|g(T_3)\| \le \eta(1+K_2)\epsilon.$$

Since $\|z(T_3)\| = \eta$, we have $\langle z(T_3), f(z(T_3)) \ge F_\eta$ so therefore we have the estimate

$$w'(T_3(\epsilon,\omega)) = -2\langle z(T_3), f(z(T_3))\rangle + 2\langle z(T_3), g(T_3)\rangle$$

$$\le -2F_\eta + 2(1+K_2)\eta\epsilon < 0,$$

a contradiction. Hence $T_3(\epsilon,\omega)$ does not exist for any $\omega \in \Omega_0 \cup \Omega_+$. Therefore we have $\limsup_{t\to\infty} \|z(t,\omega)\| \le \eta$. Since $\eta > 0$ is arbitrary, we make take the limit as $\eta \downarrow 0$ to obtain $\limsup_{t\to\infty} \|z(t,\omega)\| = 0$. Since $X = Y + z$, and $Y(t,\omega) \to 0$ as $t \to \infty$, we have that $X(t,\omega) \to 0$ as $t \to \infty$.

4.4.4 Proof of Theorem 4.2.5

Let Y be the solution of (2.1.2). We prove first that (2.2.6) implies (4.1.11). First, from Theorem 2.2.1, we have that (2.2.6) implies $Y(t) \to 0$ as $t \to \infty$ a.s. Moreover, if (2.2.6) holds it follows that

$$\sum_{n=0}^{\infty} 1 - \Phi(\epsilon/\theta_i(n)) < +\infty \quad \text{for each } \epsilon > 0$$

for each $i \in \{1,\dots,d\}$. Therefore, we have that $\Phi(\epsilon/\theta_i(n)) \to 1$ as $n \to \infty$. Hence we have $\theta_i(n) \to 0$ as $n \to \infty$. Define $\Sigma_i(t)^2 := \sum_{j=1}^r \sigma_{ij}^2(t)$ for $t \ge 0$. Then with $a_i(n) := \int_n^{n+1} \Sigma_i^2(s)\,ds$ we have $\lim_{n\to\infty} a_i(n) = 0$, and so with $a(n) := \sum_{i=1}^d a_i(n)$, we have $\lim_{n\to\infty} a(n) = 0$. Hence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} a(j) = 0.$$

Note that $\|\sigma(t)\|_F^2 = \sum_{i=1}^d \Sigma_i(t)^2$. For every $t > 0$ there is $n \in \mathbb{N}_0$ such that $t \in [n, n+1]$. Now

$$\begin{aligned}\int_0^t \|\sigma(s)\|_F^2\,ds &\le \int_0^{n+1} \|\sigma(s)\|_F^2\,ds = \int_0^{n+1} \sum_{i=1}^d \Sigma_i(s)^2\,ds \\ &= \sum_{i=1}^d \int_0^{n+1} \Sigma_i(s)^2\,ds = \sum_{i=1}^d \sum_{l=0}^n \int_l^{l+1} \Sigma_i(s)^2\,ds \\ &= \sum_{i=1}^d \sum_{l=0}^n a_i(l) = \sum_{l=0}^n a(l).\end{aligned}$$

Therefore we have

$$\frac{1}{t}\int_0^t \|\sigma(s)\|_F^2\,ds \le \frac{n}{t}\cdot\frac{1}{n}\sum_{l=0}^n a(l) \le \frac{1}{n}\sum_{l=0}^n a(l),$$

and so

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \|\sigma(s)\|_F^2\,ds = 0. \tag{4.4.8}$$

Define the event $A = \{\omega : \|X(t,\omega)\| \to \infty \text{ as } t \to \infty\}$. We prove that $\mathbb{P}[A] = 0$. Suppose to the contrary that $\mathbb{P}[A] > 0$. Define $\Omega_3 = \Omega_2 \cap \Omega_X \cap A$. Then by assumption $\mathbb{P}[\Omega_3] > 0$. By (4.4.1) we have

$$\|X(t)\|^2 = \|\xi\|^2 - \int_0^t 2\langle X(s), f(X(s))\rangle\, ds + \int_0^t \|\sigma(s)\|_F^2\, ds + 2M(t), \quad t \geq 0. \tag{4.4.9}$$

where M to be the local (scalar) martingale given by

$$M(t) = \sum_{j=1}^{r} \int_0^t \sum_{i=1}^{d} X_i(s)\sigma_{ij}(s)\, dB_j(s), \quad t \geq 0. \tag{4.4.10}$$

Since f obeys (4.2.3), i.e.,

$$\liminf_{r\to\infty} \inf_{|x|=r} \langle x, f(x)\rangle =: \lambda > 0,$$

for $\omega \in \Omega_3$ we have that

$$\liminf_{s\to\infty} \langle X(s,\omega), f(X(s,\omega)) \geq \lambda,$$

so

$$\liminf_{t\to\infty} \frac{2}{t}\int_0^t \langle X(s,\omega), f(X(s,\omega))\rangle\, ds \geq 2\lambda,$$

so for each $\epsilon < \lambda/3$, there exists $T_1(\epsilon,\omega) > 0$ such that

$$\frac{2}{t}\int_0^t \langle X(s,\omega), f(X(s,\omega))\rangle\, ds \geq 2\lambda - \epsilon, \quad t \geq T_1(\epsilon,\omega).$$

By (4.4.8), for every $\epsilon > 0$ there is $T_2(\epsilon) > 0$ such that

$$\frac{\|\xi\|^2}{t} < \epsilon, \quad \frac{1}{t}\int_0^t \|\sigma(s)\|_F^2\, ds < \epsilon, \quad t > T_2(\epsilon).$$

Let $T(\epsilon,\omega) = 1 + T_1(\epsilon,\omega) \vee T_2(\epsilon)$.

Suppose there is a subevent A' of A with $\mathbb{P}[A'] > 0$ such that $\langle M\rangle(t,\omega) \to \infty$ as $t \to \infty$ for each $\omega \in A'$. Then $\liminf_{t\to\infty} M(t,\omega) = -\infty$ and $\limsup_{t\to\infty} M(t,\omega) = +\infty$ for each $\omega \in A'$. Then by the continuity of M there exists $\tau(\omega) > T(\epsilon,\omega)$ such that $M(\tau(\omega)) = 0$. Let $t \geq T(\epsilon,\omega)$. Then

$$\begin{aligned}\frac{\|X(t,\omega)\|^2}{t} &= \frac{\|\xi\|^2}{t} - 2\frac{1}{t}\int_0^t \langle X(s,\omega), f(X(s,\omega))\rangle\, ds + \frac{\int_0^t \|\sigma(s)\|_F^2\, ds}{t} + \frac{2M(t,\omega)}{t} \\ &\leq \epsilon - 2\lambda + \epsilon + \epsilon + 2\frac{M(t,\omega)}{t} \\ &= -2\lambda + 3\epsilon + 2\frac{M(t,\omega)}{t} < -\lambda + \frac{M(t,\omega)}{t}.\end{aligned}$$

Hence

$$0 \leq \frac{\|X(\tau(\omega))\|^2}{\tau(\omega)} < -\lambda + 2\frac{M(\tau(\omega))}{\tau(\omega)} = -\lambda < 0,$$

a contradiction. Therefore we have that $\lim_{t\to\infty}\langle M\rangle(t) < +\infty$ a.s. on A. Hence $M(t)$ tends to a limit as $t \to \infty$ a.s. on A and so $M(t)/t \to 0$ as $t \to \infty$ a.s. on A. Therefore,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{\|X(t,\omega)\|^2}{t} \\
&= \limsup_{t\to\infty} \frac{\|\xi\|^2}{t} - \frac{2}{t}\int_0^t \langle X(s,\omega), f(X(s,\omega))\rangle\, ds + \frac{1}{t}\int_0^t \|\sigma(s)\|_F^2\, ds + \frac{2M(t,\omega)}{t} \\
&= \limsup_{t\to\infty} -2\frac{1}{t}\int_0^t \langle X(s,\omega), f(X(s,\omega))\rangle\, ds \\
&= -2\liminf_{t\to\infty} \frac{1}{t}\int_0^t \langle X(s,\omega), f(X(s,\omega))\rangle\, ds \leq -2\lambda < 0,
\end{aligned}$$

a contradiction. Therefore, we must have $\mathbb{P}[A] = 0$. Thus by Theorem 4.2.2, it follows that $X(t) \to 0$ as $t \to \infty$ a.s. We have shown that statement (A) and (C) are equivalent.

Statement (C) implies statement (B). It remains to show that statement (B) implies statement (A). By Theorem 4.2.2, it follows that $\mathbb{P}[Y(t) \to 0 \text{ as } t \to \infty] > 0$. Therefore by Theorem 2.2.1 it follows that (2.2.6) (or statement (A)) holds. Thus (C) implies (B) implies (A).

4.5 Proof of Theorem 4.2.6

We start by noticing that parts (A) and (C) of the theorem have already been proven; part (A) is a consequence of Theorem 4.2.5, while part (C) is part (A) of Theorem 4.2.1. The lower bound in part (B) is a result of part (B) from Theorem 4.2.1.

Therefore, it remains to establish the upper bound in part (B). However, the proof of this result is technical, and relies on a number of subsidiary results. The main step is a comparison theorem, in which $\|X\|$ is bounded by the above by the positive solution of Z of a scalar stochastic differential equation.

4.5.1 Auxiliary functions and processes

We start by introducing some functions and processes and deducing some of their important properties. Let $\phi : [0, \infty) \to \mathbb{R}$ be defined by

$$\phi(x) = \inf_{|y|=x} \frac{\langle y, f(y)\rangle}{|y|}, \quad x > 0; \qquad \phi(0) = 0. \tag{4.5.1}$$

Since f obeys (4.1.9) it follows that $\phi : [0, \infty) \to [0, \infty)$. We start by proving that ϕ is locally Lipschitz continuous.

Lemma 4.5.1. *Suppose that f obeys* (4.1.9) *and* (1.2.4). *Then ϕ defined by* (4.5.1) *is locally Lipschitz continuous and if $\phi_2(x) := \sqrt{x}\phi(\sqrt{x})$ then $\phi_2 : [0, \infty) \to \mathbb{R}$ is continuous.*

Proof. For $x > 0$ we have $\phi(x) = \inf_{|z|=1}\langle z, f(xz)\rangle$ and as $\phi(0) = 0$, the same formula holds for $x = 0$. Then for any $x, y \geq 0$ we have

$$\begin{aligned}
|\phi(x) - \phi(y)| &= \left| \inf_{|z|=1} \langle z, f(xz)\rangle - \inf_{|z|=1} \langle z, f(yz)\rangle \right| \\
&= \left| \sup_{|z|=1} \langle z, -f(xz)\rangle - \sup_{|z|=1} \langle z, -f(yz)\rangle \right| \\
&\leq \sup_{|z|=1} |\langle z, -f(xz)\rangle - \langle z, -f(yz)\rangle| \\
&= \sup_{|z|=1} |\langle z, f(yz) - f(xz)\rangle| .
\end{aligned}$$

By the Cauchy–Schwartz inequality, $|\phi(x) - \phi(y)| \leq \sup_{|z|=1} |f(yz) - f(xz)|$. Now, since f is locally Lipschitz continuous we have for every $n \in \mathbb{N}$ that there is $K_n > 0$ such that $|f(u) - f(v)| \leq K_n\|u - v\|$ for all $\|u\| \vee \|v\| \leq n$. Now suppose that $x \vee y \leq n$. Therefore, we have

$$|\phi(x) - \phi(y)| \leq \sup_{|z|=1} |f(yz) - f(xz)| \leq \sup_{|z|=1} K_n|y - x| = K_n|y - x|,$$

so ϕ is locally Lipschitz continuous. Notice also that $|\phi(x)| \leq K_n|x|$ for all $x \leq n$.

To prove that ϕ_2 is locally Lipschitz continuous, suppose that $x, y \in [0, n]$ and suppose without loss of generality that $0 \leq y \leq x \leq n$. Hence $0 \leq \sqrt{y} \leq \sqrt{x} \leq \sqrt{n}$. Write

$$\phi_2(x) - \phi_2(y) = \sqrt{x}(\phi(\sqrt{x}) - \phi(\sqrt{y})) + \phi(\sqrt{y})(\sqrt{x} - \sqrt{y}),$$

so because ϕ is non–negative and $\sqrt{x} \geq \sqrt{y}$ we have

$$|\phi_2(x) - \phi_2(y)| \leq \sqrt{x}|\phi(\sqrt{x}) - \phi(\sqrt{y})| + \phi(\sqrt{y})(\sqrt{x} - \sqrt{y}).$$

Therefore, using the Lipschitz continuity of ϕ and the estimate $|\phi(y)| \leq K_{\sqrt{n}}\sqrt{y}$ for all $y \leq n$ we have

$$\begin{aligned}|\phi_2(x) - \phi_2(y)| &\leq \sqrt{x}K_{\sqrt{n}}|\sqrt{x} - \sqrt{y}| + K_{\sqrt{n}}\sqrt{y}(\sqrt{x} - \sqrt{y}) \\ &= \sqrt{x}K_{\sqrt{n}}(\sqrt{x} - \sqrt{y}) + K_{\sqrt{n}}\sqrt{y}(\sqrt{x} - \sqrt{y}) = K_{\sqrt{n}}(x - y),\end{aligned}$$

so that $|\phi_2(x) - \phi_2(y)| \leq K_{\sqrt{n}}|x - y|$ for $0 \leq y \leq x \leq n$. Hence ϕ_2 is also locally Lipschitz continuous. □

Let X be the unique continuous solution of (4.1.8) and define the r scalar processes $\bar{\sigma}_j : [0, \infty) \to \mathbb{R}$ by

$$\bar{\sigma}_j(t) = \begin{cases} \sum_{i=1}^d \frac{\langle X(t), \mathbf{e}_i\rangle}{\|X(t)\|}\sigma_{ij}(t), & X(t) \neq 0, \\ \frac{1}{\sqrt{d}}\sum_{i=1}^d |\sigma_{ij}(t)|, & X(t) = 0. \end{cases} \tag{4.5.2}$$

We define $\bar{\sigma}(t) \geq 0$ by

$$\bar{\sigma}^2(t) := \sum_{j=1}^r \bar{\sigma}_j^2(t), \quad t \geq 0. \tag{4.5.3}$$

Hence $\bar{\sigma}_j$ for $j = 1, \ldots, r$ and $\bar{\sigma}$ are adapted processes. Therefore using the Cauchy–Schwartz inequality and (4.5.2) we get

$$\bar{\sigma}_j^2(t) \leq \sum_{i=1}^d \sigma_{ij}^2(t), \quad t \geq 0,$$

and so $\bar{\sigma}^2(t) \leq \|\sigma(t)\|_F^2$ for all $t \geq 0$. Hence $\bar{\sigma}$ and $\bar{\sigma}_j$ for $j = 1, \ldots, r$ are bounded functions on any compact interval. Therefore, the process $\tilde{Y}_0$ given by

$$\tilde{Y}_0(t) = \sum_{j=1}^r \int_0^t e^s \bar{\sigma}_j(s)\, dB_j(s), \quad t \geq 0$$

is well–defined and is moreover a continuous square integrable martingale. Therefore the process Y_0 defined by

$$Y_0(t) = e^{-t}\tilde{Y}_0(t), \quad t \geq 0 \tag{4.5.4}$$

is a continuous semimartingale and obeys

$$dY_0(t) = -Y_0(t)\,dt + \sum_{j=1}^{r} \bar{\sigma}_j(t)\,dB_j(t), \quad t \geq 0. \tag{4.5.5}$$

Next define $W(0) = 1 + \|\xi\| > 0$ and

$$W'(t) = -\phi(W(t) + Y_0(t)) + \frac{\|\sigma(t)\|_F^2 + e^{-t}}{W(t) + Y_0(t)} + Y_0(t), \quad t \geq 0. \tag{4.5.6}$$

Since ϕ is locally Lipschitz continuous, $\|\sigma\|_F^2$ is continuous and the paths of Y_0 are continuous, there is a unique continuous solution of (4.5.6) on the interval $[0, \tau)$ where

$$\tau = \inf\{t > 0 : Z(t) \notin (0, \infty)\}$$

and $Z(t) = W(t) + Y_0(t)$ for $t \in [0, \tau)$. Therefore, as W is the unique continuous solution of (4.5.6) on $[0, \tau)$, it follows that on $[0, \tau)$ that Z so defined is the unique solution of the stochastic differential equation

$$dZ(t) = \left(-\phi(Z(t)) + \frac{\|\sigma(t)\|_F^2 + e^{-t}}{Z(t)} \right) dt + \sum_{j=1}^{r} \bar{\sigma}_j(t)\,dB_j(t), \tag{4.5.7}$$

with initial condition $Z(0) = |\xi| + 1 > 0$. The adaptedness of Y_0 ensures that the process W is adapted, and therefore so is Z.

The first step is to show that $\tau = +\infty$ a.s., which means that $Z(t)$ is well–defined and strictly positive for all $t \geq 0$, a.s.

Lemma 4.5.2. *Suppose that f obeys* (4.1.9) *and* (1.2.4), *and that σ obeys* (2.1.1). *Let Z be the unique continuous adapted solution of* (4.5.7). *Then $\tau = +\infty$ a.s.*

Proof. Let $\zeta = |\xi| + 1 > 0$ and define $k^* \in \mathbb{N}$ such that $k^* > \zeta$. Define for each $k \geq k^*$ the stopping time $\tau_k^\zeta = \inf\{t > 0 : Z(t) = k \text{ or } 1/k\}$. We see that τ_k^ζ is an increasing sequence of times and so $\tau_\infty^\zeta := \lim_{k\to\infty} \tau_k^\zeta$. Suppose, in contradiction to the desired claim, that $\tau_\infty^\zeta < +\infty$ with positive probability for some ζ. Then, there exists $T > 0$, $\epsilon > 0$ and $k_0 \in \mathbb{N}$ such that

$$\mathbb{P}[\tau_k^\zeta \leq T] \geq \epsilon, \quad k \geq k_0 > k^*.$$

Therefore, by Itô's rule we have that

$$Z(T\wedge\tau_k^\zeta)+\frac{1}{Z(T\wedge\tau_k^\zeta)}=\zeta+\frac{1}{\zeta}$$
$$+\int_0^{T\wedge\tau_k^\zeta}\left\{-\phi(Z(s))+\frac{\phi(Z(s))}{Z(s)}\frac{1}{Z(s)}-\frac{e^{-s}}{Z(s)^3}+\frac{\|\sigma(s)\|_F^2+e^{-s}}{Z(s)}\right\}ds$$
$$+\sum_{j=1}^r\int_0^{T\wedge\tau_k^\zeta}(1-Z(s)^{-2})\bar{\sigma}_j(s)\,dB_j(s).$$

We remove the non–autonomous terms in the first integral by noting that $\|\sigma(s)\|_F^2\leq\sigma_T^2<+\infty$ for all $s\in[0,T]$, so we arrive at

$$Z(T\wedge\tau_k^\zeta)+\frac{1}{Z(T\wedge\tau_k^\zeta)}=\zeta+\frac{1}{\zeta}+\int_0^{T\wedge\tau_k^\zeta}b_T(Z(s))\,ds+M(T)$$

where we have defined

$$b_T(z)=-\phi(z)+\frac{\phi(z)}{z}\frac{1}{z}-\frac{e^{-T}}{z^3}+\frac{1+\sigma_T^2}{z},\quad z>0, \tag{4.5.8}$$

and $M=\{M(t):t\in[0,T]\}$ is the martingale defined by

$$M(t)=\sum_{j=1}^r\int_0^{t\wedge\tau_k^\zeta}(1-Z(s)^{-2})\bar{\sigma}_j(s)\,dB_j(s),\quad t\in[0,T].$$

For $z\geq 1$, since ϕ is non–negative we have

$$b_T(z)=-\phi(z)(1-z^{-2})-\frac{e^{-T}}{z^3}+\frac{1+\sigma_T^2}{z}\leq\frac{1+\sigma_T^2}{z}\leq 1+\sigma_T^2.$$

For $z\in(0,1]$, the Lipschitz continuity of ϕ guarantees that $|\phi(z)|\leq K_1z$ for some $K_1>0$. Therefore we have

$$b_T(z)\leq\frac{K_1+1+\sigma_T^2}{z}-\frac{e^{-T}}{z^3},$$

and so we can readily show that there is $K_2(T)>0$ such that $b_T(z)\leq K_2(T)$ for all $z\in(0,1]$. Define $K_3(T)=\max(K_2(T),1+\sigma_T^2)$. Therefore we have $b_T(z)\leq K_3(T)$ for all $z>0$. Since $Z(s)\in(0,\infty)$ for all $s\in[0,T\wedge\tau_k^\zeta]$ we have that

$$Z(T\wedge\tau_k^\zeta)+\frac{1}{Z(T\wedge\tau_k^\zeta)}\leq\zeta+\frac{1}{\zeta}+\int_0^{T\wedge\tau_k^\zeta}K_3(T)+M(T)\leq\zeta+\frac{1}{\zeta}+TK_3(T)+M(T).$$

By the optional sampling theorem, we have that

$$\mathbb{E}\left[Z(T\wedge\tau_k^\zeta)+\frac{1}{Z(T\wedge\tau_k^\zeta)}\right]\le\zeta+\frac{1}{\zeta}+TK_3(T)=:K(T,\zeta)<+\infty.$$

Define next the event $C_k=\{\tau_k^\zeta\le T\}$. Then for $k\ge k_0$ we have $\mathbb{P}[C_k]\ge\epsilon$. If $\omega\in C_k$, we have that $\tau_k^\zeta\le T$, so $Z(T\wedge\tau_k^\zeta)=k$ or $Z(T\wedge\tau_k^\zeta)=1/k$. Hence $Z(T\wedge\tau_k^\zeta)+1/Z(T\wedge\tau_k^\zeta)=k+1/k$ for $\omega\in C_k$. Hence

$$\begin{aligned}K(T,\zeta)&\ge\mathbb{E}\left[Z(T\wedge\tau_k^\zeta)+\frac{1}{Z(T\wedge\tau_k^\zeta)}\right]\\&\ge\mathbb{E}\left[\left(Z(T\wedge\tau_k^\zeta)+\frac{1}{Z(T\wedge\tau_k^\zeta)}\right)1_{C_k}\right]\\&=(k+1/k)\mathbb{P}[C_k]\ge(k+1/k)\epsilon.\end{aligned}$$

Therefore, we have that $K(T,\zeta)\ge(k+1/k)\epsilon$ for all $k\ge k_0$. Letting $k\to\infty$ gives a contradiction. □

Given that Z is positive and well–defined for all $t\ge 0$, we are now in a position to formulate and prove a comparison result, which shows that $\|X(t)\|\le Z(t)$ for all $t\ge0$ a.s. Once this result is proven, the main theorem will be established if we show that the solution Z of (4.5.7) is bounded.

Lemma 4.5.3. *Suppose that f obeys* (4.1.9) *and* (1.2.4)*, and that σ obeys* (2.1.1)*. Suppose that Z is the unique continuous adapted solution of* (4.5.7) *and that X is the unique continuous adapted solution of* (4.1.8)*. Then $\|X(t)\|\le Z(t)$ for all $t\ge0$ a.s.*

Proof. Define $Y_2(t)=\|X(t)\|^2$ for $t\ge0$. Then by the definition of $\bar\sigma_j$ for $j=1,\dots,r$ from (4.5.2), we have

$$2\sum_{i=1}^d X_i(t)\sigma_{ij}(t)=2\sqrt{Y_2(t)}\bar\sigma_j(t),\quad t\ge0.$$

By Itô's rule, we have

$$dY_2(t)=\left(-2\langle X(t),f(X(t))\rangle+\|\sigma(t)\|_F^2\right)\,dt+2\sum_{j=1}^r\sum_{i=1}^d X_i(t)\sigma_{ij}(t)\,dB_j(t),\quad t\ge0.$$

Using this semimartingale decomposition and the previous identity, we get

$$dY_2(t) = \left(-2\langle X(t), f(X(t))\rangle + \|\sigma(t)\|_F^2\right)\, dt + 2\sqrt{Y_2(t)}\sum_{j=1}^{r}\bar{\sigma}_j(t)\, dB_j(t). \tag{4.5.9}$$

Let ϕ be the function defined by (4.5.1), $\bar{\sigma}$ the process defined by (4.5.3), and define the processes η_1 and η_2 by

$$\eta_1(t) = \|\sigma(t)\|_F^2 + 2e^{-t} + \bar{\sigma}(t)^2, \quad t \geq 0,$$

$$\eta_2(t) = 2\sqrt{Y_2(t)}\phi(\sqrt{Y_2(t)}) - 2\langle X(t), f(X(t))\rangle, \quad t \geq 0,$$

and the processes β_1 and β_2 by

$$\beta_1(t) = b(Z_2(t), t) + \eta_1(t), \quad t \geq 0, \tag{4.5.10}$$

$$\beta_2(t) = b(Y_2(t), t) + \eta_2(t), \quad t \geq 0, \tag{4.5.11}$$

where we have defined $b : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$b(x,t) = -2\phi_2(x) + \|\sigma(t)\|_F^2, \quad x \geq 0, t \geq 0, \tag{4.5.12}$$

where ϕ_2 is defined in Lemma 4.5.1.

Granted these definitions, we can rewrite (4.5.9) as

$$dY_2(t) = \beta_2(t)\, dt + 2\sqrt{Y_2(t)}\sum_{j=1}^{r}\bar{\sigma}_j(t)\, dB_j(t). \tag{4.5.13}$$

Next, by virtue of Lemma 4.5.2 it follows that there is a positive and process $Z_2 = \{Z_2(t) : t \geq 0\}$ define by $Z_2(t) = Z(t)^2$ for all $t \geq 0$. Therefore, applying Itô's rule to (4.5.7), and using the definition (4.5.3), we have

$$dZ_2(t) = \left(2Z(t)\left\{-\phi(Z(t)) + \frac{e^{-t} + \|\sigma(t)\|_F^2}{Z(t)}\right\} + \bar{\sigma}^2(t)\right)\, dt + 2Z(t)\sum_{j=1}^{r}\bar{\sigma}_j(t)\, dB_j(t).$$

Hence by the definition of ϕ_2, (4.5.10) and Z_2 we have

$$dZ_2(t) = \beta_1(t)\,dt + 2\sqrt{Z_2(t)}\sum_{j=1}^{r}\bar{\sigma}_j(t)\,dB_j(t). \tag{4.5.14}$$

Notice also that $Y_2(0) = |\xi|^2 < 1 + |\xi|^2 = Z_2(0)$.

Our proof now involves comparing Y_2 and Z_2, viewed as solutions of (4.5.13) and (4.5.14) respectively. Proving that $Y_2(t) \le Z_2(t)$ for all $t \ge 0$ a.s. suffices. The proof is an adaptation of standard comparison proofs. Extant results can not be applied immediately, because we must carefully deal with the fact that the state–dependence in the drift in both (4.5.13) and (4.5.14) is merely *locally* Lipschitz continuous, and that the diffusion coefficients are non–autonomous through the presence of a *process* rather than simple deterministic dependence of time.

To prove that Y_2 is dominated by Z_2, we first show that $\eta_1(t) > 0 \ge \eta_2(t)$ for $t \ge 0$. The first inequality is immediate. To show that $\eta_2(t) \le 0$ for all $t \ge 0$, first note that if $X(t) = 0$, then $\eta_2(t) = 0$. If $\|X(t)\| > 0$, by (4.5.1) and the definition of Y_2, we have that

$$\frac{\langle X(t), f(X(t))\rangle}{\|X(t)\|} \ge \inf_{\|x\|=\|X(t)\|}\frac{\langle x, f(x)\rangle}{\|x\|} = \phi(\sqrt{Y_2(t)}).$$

Hence $\langle X(t), f(X(t))\rangle \ge \|X(t)\|\phi(\sqrt{Y_2(t)}) = \sqrt{Y_2(t)}\phi(\sqrt{Y_2(t)})$, so $\eta_2(t) \le 0$. Therefore, because $\eta_2 \le 0$ and $\eta_1 > 0$, we have

$$\beta_2(t) \le b(Y_2(t), t), \quad \beta_1(t) > b(Z_2(t), t), \quad t \ge 0. \tag{4.5.15}$$

By Lemma 4.5.1, ϕ_2 is locally Lipschitz continuous, so for every $n \ge 0$ there is a $\kappa_n > 0$ such that

$$|b(x,t) - b(y,t)| = |2\phi_2(x) - 2\phi_2(y)| \le \kappa_n|x - y| \quad \text{for all } x, y \in [0, n]. \tag{4.5.16}$$

Now define $\Delta(t) := Y_2(t) - Z_2(t)$ for $t \ge 0$. Let $\rho(x) = 4x$ for $x \ge 0$. Then ρ is increasing and $\int_{0+} 1/\rho(x)\,dx = +\infty$. Now by (4.5.3)

$$d[\Delta](t) = 4\left(\sqrt{Y_2(t)} - \sqrt{Z_2(t)}\right)^2\sum_{j=1}^{r}\bar{\sigma}_j^2(t)\,dt = 4\left(\sqrt{Y_2(t)} - \sqrt{Z_2(t)}\right)^2\bar{\sigma}^2(t)\,dt.$$

If

$$\int_0^t \rho(\Delta(s))^{-1} I_{\{\Delta(s)>0\}}\, d[\Delta](s) < +\infty, \text{a.s.} \tag{4.5.17}$$

then $\Lambda_t^0(\Delta) = 0$ a.s., where $\Lambda_\cdot^0(\Delta)$ is the local time of Δ in zero (see [67, Proposition V.39.3]).

If $y \geq x \geq 0$, we have that $(\sqrt{y}-\sqrt{x})^2 \leq y-x$. Define $J = \{s \in [0,t] : \Delta(s) > 0\}$. Therefore, $s \in J$ we have $Y_2(s) > Z_2(s) > 0$ and so

$$\left(2\sqrt{Y_2(t)} - 2\sqrt{Z_2(t)}\right)^2 \leq 4(Y_2(s) - Z_2(s)) = 4\Delta(s) = \rho(\Delta(s)).$$

Thus

$$\begin{aligned}
&\int_0^t \rho(\Delta(s))^{-1} I_{\{\Delta(s)>0\}}\, d[\Delta](s) \\
&= \int_J \rho(\Delta(s))^{-1} I_{\{\Delta(s)>0\}}\, d[\Delta](s) + \int_{[0,t]\setminus J} \rho(\Delta(s))^{-1} I_{\{\Delta(s)>0\}}\, d[\Delta](s) \\
&= \int_J \rho(\Delta(s))^{-1} \cdot 4\left(\sqrt{Y_2(s)} - \sqrt{Z_2(s)}\right)^2 \bar{\sigma}^2(s)\, ds \\
&\leq \int_J \bar{\sigma}^2(s)\, ds \leq \int_0^t \bar{\sigma}^2(s)\, ds \leq \int_0^t \|\sigma(s)\|_F^2\, ds < +\infty,
\end{aligned}$$

as required.

Next, let

$$\tau_n = \inf\{t > 0 : Y_2(t) = n \text{ or } Z(t) = n\}, \quad n \geq \lceil (1+|\zeta|^2) \rceil.$$

By Lemma 4.5.2, Z does not explode in finite time, so neither does Z_2. Also, as $\|X\|$ does not explode in finite time, we have that $\tau_n \to \infty$ as $n \to \infty$. Using the fact that $\Lambda_t^0(\Delta) = 0$ a.s., together with (4.5.13) and (4.5.14) we get

$$\Delta(t\wedge\tau_n)^+ = \Delta(0)^+ + \int_0^{t\wedge\tau_n} I_{\{\Delta(s)>0\}}(\beta_2(s) - \beta_1(s))\, ds + M(t). \tag{4.5.18}$$

where we have defined

$$M(t) = \int_0^{t\wedge\tau_n} I_{\{\Delta(s)>0\}} 2\left(\sqrt{Y_2(s)} - \sqrt{Z_2(s)}\right) \sum_{j=1}^r \bar{\sigma}_j(s)\, dB_j(s).$$

Therefore by (4.5.3), and the fact that $\sqrt{Y_2(s)} \vee \sqrt{Z_2(s)} \le \sqrt{n}$ for $s \in [0, t \wedge \tau_n]$

$$\begin{aligned}\langle M\rangle(t) &= 4\int_0^{t\wedge\tau_n} I_{\{\Delta(s)>0\}} \left(\sqrt{Y_2(s)} - \sqrt{Z_2(s)}\right)^2 \bar{\sigma}^2(s)\,ds \\ &\le 4\int_0^{t\wedge\tau_n} I_{\{\Delta(s)>0\}} \left(\sqrt{Y_2(s)} - \sqrt{Z_2(s)}\right)^2 \|\sigma(s)\|_F^2\,ds \\ &\le 4n\int_0^{t\wedge\tau_n} \|\sigma(s)\|_F^2\,ds \le 4n\int_0^t \|\sigma(s)\|_F^2\,ds.\end{aligned}$$

Now $\Delta(0) = Y_2(0) - Z_2(0) < 0$, so by the optional sampling theorem, we deduce from (4.5.18) that

$$0 \le \mathbb{E}[\Delta(t\wedge\tau_n)^+] = \mathbb{E}\left[\int_0^{t\wedge\tau_n} I_{\{\Delta(s)>0\}}(\beta_2(s) - \beta_1(s))\,ds\right]. \tag{4.5.19}$$

We now estimate the integrand on the right–hand side. If $\Delta(s) > 0$, we have $\Delta(s) = Y_2(s) - Z_2(s) > 0$. Thus for $s \in [0, t\wedge\tau_n]$, because $Y_2(s) \vee Z_2(s) \le n$, we may use (4.5.15) and then (4.5.16) to get

$$\begin{aligned}I_{\{\Delta(s)>0\}}(\beta_2(s) - \beta_1(s)) &= \beta_2(s) - \beta_1(s) \le b(Y_2(s), s) - b(Z_2(s), s) \\ &\le |b(Y_2(s), s) - b(Z_2(s), s)| \le \kappa_n |Y_2(s) - Z_2(s)|.\end{aligned}$$

Since $Y_2(s) - Z_2(s) > 0$, this gives $I_{\{\Delta(s)>0\}}(\beta_2(s) - \beta_1(s)) \le \kappa_n(Y_2(s) - Z_2(s)) = \kappa_n \Delta(s)^+$. In the case when $\Delta(s) \le 0$, we have $I_{\{\Delta(s)>0\}}(\beta_2(s) - \beta_1(s)) = 0 \le \kappa_n\Delta(s)^+$. Thus, the estimate $I_{\{\Delta(s)>0\}}(\beta_2(s) - \beta_1(s)) = 0 \le \kappa_n\Delta(s)^+$ holds for all $s \in [0, t\wedge\tau_n]$, so inserting this bound into (4.5.19), we get

$$0 \le \mathbb{E}[\Delta(t\wedge\tau_n)^+] \le \mathbb{E}\left[\int_0^{t\wedge\tau_n} \kappa_n\Delta(s)^+\,ds\right] = \kappa_n\mathbb{E}\int_0^{t\wedge\tau_n} \Delta(s)^+\,ds. \tag{4.5.20}$$

As to the term on the righthand side, by considering the cases when (a) $\tau_n \le t$ and (b) $\tau_n > t$, we can show that

$$\int_0^{t\wedge\tau_n} \Delta(s)^+\,ds \le \int_0^t \Delta(s\wedge\tau_n)^+\,ds.$$

Putting this estimate into (4.5.20) gives

$$0 \le \mathbb{E}[\Delta(t\wedge\tau_n)^+] \le \kappa_n \int_0^t \mathbb{E}[\Delta(s\wedge\tau_n)^+]\,ds, \quad t \ge 0. \tag{4.5.21}$$

Since $t \mapsto \Delta(t)$ has a.s. continuous sample paths, so does $t \mapsto \Delta(t \wedge \tau_n)$, and therefore $\delta_n : [0,\infty) \to \mathbb{R}$ defined by $\delta_n(t) = \mathbb{E}[\Delta(t \wedge \tau_n)]$ for $t \geq 0$ is a non–negative and continuous function obeying $\delta_n(t) \leq \kappa_n \int_0^t \delta_n(s)\, ds$ for all $t \geq 0$. By Gronwall's inequality, $\delta_n(t) = 0$ for all $t \geq 0$. Therefore we have $Y_2(t \wedge \tau_n) - Z_2(t \wedge \tau_n) \leq 0$ for all $t \geq 0$ a.s. and for each $n \in \mathbb{N}$. Since $\tau_n \to \infty$ as $n \to \infty$, it follows that $Y_2(t) - Z_2(t) \leq 0$ for all $t \geq 0$ a.s., as required. □

In the next lemma, we show that Y_0 defined by (4.5.4) is bounded.

Lemma 4.5.4. *Suppose that S obeys* (2.2.8)*. If Y_0 is defined by* (4.5.4)*, then there is $c_1 > 0$ such that*

$$\limsup_{t\to\infty} |Y_0(t)| \leq c_1, \quad a.s.$$

Proof. Let $V_0(n) := \int_{n-1}^n e^{s-n} \sum_{j=1}^r \bar{\sigma}_j(s)\, dB(s)$, $n \geq 1$. Then by (4.5.4) we get

$$Y_0(n) = e^{-n} \sum_{l=1}^n \int_{l-1}^l e^s \sum_{j=1}^r \bar{\sigma}_j(s)\, dB_j(s) = \sum_{l=1}^n e^{-(n-l)} V_0(l), \quad n \geq 1. \tag{4.5.22}$$

Define

$$\tilde{Y}_{n-1}(t) = \int_{n-1}^t e^s \sum_{j=1}^r \bar{\sigma}_j(s)\, dB_j(s), \quad t \in [n-1, n].$$

Clearly $\tilde{Y}_{n-1}$ is a continuous $\mathcal{F}^B$ martingale, and by (4.5.3) we have

$$\langle \tilde{Y}_{n-1} \rangle(t) = \int_{n-1}^t e^{2s} \bar{\sigma}^2(s)\, ds, \quad t \in [n-1, n].$$

Therefore there is an extension $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ of $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined a one–dimensional Brownian motion $\bar{B}_n = \{\bar{B}_n(t) : n-1 \leq t \leq n; \mathcal{F}_n\}$ such that

$$\tilde{Y}_{n-1}(t) = \int_{n-1}^t e^s \bar{\sigma}(s)\, d\bar{B}_n(s), \quad t \in [n-1, n].$$

(cf. [44, Theorem 3.4.2]). Now define

$$\bar{Y}_{n-1}(t) = \int_{n-1}^t e^s \|\sigma(s)\|_F\, d\bar{B}(s), \quad t \in [n-1, n].$$

Since $\bar{\sigma}(t) \leq \|\sigma(t)\|_F$ for all $t \geq 0$, by applying a result of Hajek (cf. e.g., [44, Exercise 3.4.24]) we have that

$$\mathbb{P}[V_0(n) > \epsilon] = \mathbb{P}[\tilde{Y}_{n-1}(n) > \epsilon e^n] \leq 2\mathbb{P}[\bar{Y}_{n-1}(n) \geq \epsilon e^n]. \tag{4.5.23}$$

Noting that $-\tilde{Y}_{n-1}$ is also a continuous martingale, by applying Hajek's result once more, we have that

$$\mathbb{P}[V_0(n) \leq -\epsilon] = \mathbb{P}[-\tilde{Y}_{n-1}(n) \geq \epsilon e^n] \leq 2\mathbb{P}[\bar{Y}_{n-1}(n) \geq \epsilon e^n].$$

Combining this estimate with (4.5.23), we get

$$\mathbb{P}[|V_0(n)| > \epsilon] \leq 4\mathbb{P}[\bar{Y}_{n-1}(n) \geq \epsilon e^n]. \tag{4.5.24}$$

Now, we notice that $\bar{Y}_{n-1}(n)$ is a normally distributed random variable with mean zero and variance

$$\bar{v}(n)^2 := \int_{n-1}^{n} e^{2s}\|\sigma(s)\|_F^2\, ds.$$

Notice that $e^{-2}\theta(n)^2 \leq e^{-2n}\bar{v}^2(n) \leq \theta(n)^2$. Since Φ is increasing, we have

$$\begin{aligned}\mathbb{P}[|V_0(n)| > \epsilon] &\leq 4\left(1 - \Phi\left(\frac{\epsilon e^n}{\bar{v}(n)}\right)\right) = 4\left(1 - \Phi\left(\frac{\epsilon}{e^{-n}\bar{v}(n)}\right)\right)\\ &\leq 4\left(1 - \Phi\left(\frac{\epsilon}{\theta(n)}\right)\right).\end{aligned}$$

Therefore, for every $\epsilon > \epsilon'$, by (2.2.10) it follows that

$$\sum_{n=1}^{\infty}\mathbb{P}[|V_0(n)| \geq \epsilon] < +\infty.$$

Thus by the Borel–Cantelli lemma, it follows that $\limsup_{n\to\infty}|V_0(n)| \leq \epsilon$ a.s. for every $\epsilon > \epsilon'$. Hence by (4.5.22), we have that

$$\limsup_{n\to\infty}|Y_0(n)| \leq \epsilon \cdot \sum_{k=0}^{\infty} e^{-k} = \epsilon\frac{1}{1-e^{-1}}, \quad \text{a.s.} \tag{4.5.25}$$

Next let $t \in [n, n+1)$. Therefore, from (4.5.4) we have

$$Y_0(t) = Y_0(n)e^{-(t-n)} + e^{-t}\int_n^t e^s \sum_{j=1}^r \bar{\sigma}_j(s)\, dB_j(s), \quad t \in [n, n+1).$$

With $Z_0(n) := e^{-n} \max_{t\in[n,n+1]} \left|\int_n^t e^s \sum_{j=1}^r \bar{\sigma}_j(s)\, dB_j(s)\right|$ for $n \geq 1$, we have

$$\max_{t\in[n,n+1]} |Y_0(t)| \leq |Y_0(n)| + \max_{t\in[n,n+1]} e^{-t} \left|\int_n^t e^s \sum_{j=1}^r \bar{\sigma}_j(s)\, dB_j(s)\right| \leq |Y_0(n)| + Z_0(n). \quad (4.5.26)$$

Next we estimate $\mathbb{P}[Z_0(n) > \epsilon e]$. Fix $n \in \mathbb{N}$. Now

$$\mathbb{P}[Z_0(n) > \epsilon e] = \mathbb{P}\left[\max_{t\in[n,n+1]} |\bar{Y}_n(t)| > \epsilon e e^n\right].$$

Define $\tau(t) := \int_n^t e^{2s}\bar{\sigma}^2(s)\, ds$ for $t \in [n, n+1]$. Therefore, by the martingale time change theorem [65, Theorem V.1.6], there exists a standard Brownian motion B_n^* such that

$$\mathbb{P}[Z_0(n) > \epsilon e] = \mathbb{P}\left[\max_{t\in[n,n+1]} |B_n^*(\tau(t))| > \epsilon e e^n\right] = \mathbb{P}\left[\max_{u\in[0,\tau(n+1)]} |B_n^*(u)| > \epsilon e e^n\right].$$

Notice now that $\tau(t) \leq \int_n^t e^{2s}\|\sigma(s)\|_F^2\, ds$, so

$$\begin{aligned}
\mathbb{P}[Z_0(n) > \epsilon e] &\leq \mathbb{P}\left[\max_{u\in[0,\int_n^{n+1} e^{2s}\|\sigma(s)\|_F^2\, ds]} |B_n^*(u)| > \epsilon e e^n\right] \\
&= \mathbb{P}\left[\max_{u\in[0,\bar{v}^2(n+1)]} |B_n^*(u)| > \epsilon e e^n\right] \\
&\leq \mathbb{P}\left[\max_{u\in[0,\bar{v}^2(n+1)]} B_n^*(u) > \epsilon e^n e\right] + \mathbb{P}\left[\max_{u\in[0,\bar{v}^2(n+1)]} -B_n^*(u) > \epsilon e^n e\right] \\
&= \mathbb{P}\left[|B_n^*(\bar{v}^2(n+1))| > \epsilon e^n e\right] + \mathbb{P}\left[|B_n^{**}(\bar{v}^2(n+1))| > \epsilon e^n e\right],
\end{aligned}$$

where $B_n^{**} = -B_n^*$ is a standard Brownian motion. Recall that if W is a standard Brownian motion that $\max_{s\in[0,t]} W(s)$ has the same distribution as $|W(t)|$. Therefore, as $B_n^*(\bar{v}(n+1))$ is normally distributed with zero mean we have

$$\begin{aligned}
\mathbb{P}[Z_0(n) > \epsilon e] &= 2\mathbb{P}\left[|B_n^*(\bar{v}^2(n+1))| > \epsilon e e^n\right] = 4\mathbb{P}\left[B_n^*(\bar{v}^2(n+1)) > \epsilon e e^n\right] \\
&= 4\left(1 - \Phi\left(\frac{\epsilon e e^n}{\bar{v}(n+1)}\right)\right) = 4\left(1 - \Phi\left(\frac{\epsilon e}{\sqrt{e^{-2n}\bar{v}^2(n+1)}}\right)\right).
\end{aligned}$$

If we interpret $\Phi(\infty) = 1$, this formula holds valid in the case when $\bar{v}(n+1) = 0$, because in this case $Z_0(n) = 0$ a.s. Now $e^{-2n}\bar{v}^2(n+1) = e^{-2n}\int_n^{n+1} e^{2s}\|\sigma(s)\|_F^2\,ds \le e^2\theta^2(n)$. Since Φ is increasing, we have

$$\mathbb{P}[Z(n) > \epsilon e] = 4\left(1 - \Phi\left(\frac{\epsilon e}{\sqrt{e^{-2n}\tau(n+1)}}\right)\right) \le 4\left(1 - \Phi\left(\frac{\epsilon e}{e\theta(n)}\right)\right),$$

so

$$\mathbb{P}[Z_0(n) > \epsilon e] \le 4\left(1 - \Phi\left(\frac{\epsilon}{\theta(n)}\right)\right). \tag{4.5.27}$$

Therefore by (2.2.8) and (4.5.27) we have $\sum_{n=1}^{\infty}\mathbb{P}[Z(n) > \epsilon e] < +\infty$ for all $\epsilon > \epsilon'$. Therefore by the Borel–Cantelli Lemma, we have that

$$\limsup_{n\to\infty} Z_0(n) \le \epsilon e, \quad \text{a.s.} \tag{4.5.28}$$

By (4.5.25), (4.5.26) and (4.5.28) we have

$$\limsup_{n\to\infty}\max_{t\in[n,n+1]}|Y_0(t)| \le \limsup_{n\to\infty}|Y_0(n)| + \limsup_{n\to\infty} Z_0(n) \le \frac{1}{1-e^{-1}}\epsilon + e\epsilon,$$

Therefore, letting $\epsilon \downarrow \epsilon'$ through the rational numbers we have

$$\limsup_{t\to\infty}\|Y(t)\| \le (1/(1-e^{-1}) + e)\epsilon' =: c_1, \quad \text{a.s.},$$

proving the result. □

Before proceeding with the final supporting lemma, we show that whenever $S(\epsilon)$ is finite, we must have

$$\lim_{t\to\infty}\int_t^{t+1}\|\sigma(s)\|_F^2\,ds = 0. \tag{4.5.29}$$

Lemma 4.5.5. *Suppose that S obeys* (2.2.8). *Then σ obeys* (4.5.29).

Proof. By (2.2.8), there exists $\epsilon > 0$ such that $\sum_{n=1}^{\infty}\{1 - \Phi(\epsilon/\theta(n))\} < +\infty$. Therefore, it follows that $\theta(n) \to 0$ as $n \to \infty$. For every $t > 0$, there exists $n(t) \in \mathbb{N}$ such that

$n(t) \leq t < n(t) + 1$. Hence

$$\begin{aligned}\int_t^{t+1} \|\sigma(s)\|_F^2\,ds \leq \int_{n(t)}^{t+1} \|\sigma(s)\|_F^2\,ds &= \int_{n(t)}^{n(t)+1} \|\sigma(s)\|_F^2\,ds + \int_{n(t)+1}^{t+1} \|\sigma(s)\|_F^2\,ds \\ &\leq \int_{n(t)}^{n(t)+1} \|\sigma(s)\|_F^2\,ds + \int_{n(t)+1}^{n(t)+2} \|\sigma(s)\|_F^2\,ds \\ &= \theta(n(t))^2 + \theta(n(t+1))^2.\end{aligned}$$

Since $n(t) \to \infty$ as $t \to \infty$ and $\theta(n) \to 0$ as $n \to \infty$, taking limits yields (4.5.29). □

Before we can show that W is bounded, we must first prove that

$$\liminf_{t\to\infty} Z(t) < +\infty, \quad \text{a.s.} \tag{4.5.30}$$

Lemma 4.5.6. *Suppose that f obeys* (4.1.9), (1.2.4) *and* (4.2.4)*. Suppose that σ obeys* (2.1.1) *and that S obeys* (2.2.8)*. Then the solution Z of* (4.5.7) *obeys* (4.5.30).

Proof. Note that if f obeys (4.2.4), then ϕ given by (4.5.1) satisfies $\lim_{x\to\infty}\phi(x) = +\infty$. Using (4.5.7), we have

$$\frac{Z(t)}{t} = \frac{1+\|\xi\|}{t} - \frac{1}{t}\int_0^t \phi(Z(s))\,ds + \frac{1}{t}\int_0^t \frac{\|\sigma(s)\|_F^2 + e^{-s}}{Z(s)}\,ds + \frac{M_2(t)}{t}, \tag{4.5.31}$$

where M_2 is the continuous martingale given by

$$M_2(t) = \sum_{j=1}^r \int_0^t \bar{\sigma}_j(s)\,dB_j(s), \quad \text{a.s.}$$

Using (4.5.3) we get

$$\langle M_2\rangle(t) = \int_0^t \bar{\sigma}^2(s)\,ds \leq \int_0^t \|\sigma(s)\|_F^2\,ds,$$

and in the case when $S(\epsilon)$ is finite, we may appeal to the proof of Theorem 4.2.5, which shows that (4.4.8) holds. On the event A for which $\langle M_2\rangle(t)$ tends to a finite limit as $t \to \infty$, we have that $M_2(t)$ converges to a finite limit, in which case $M_2(t)/t \to 0$ as $t \to \infty$ on A. On $\bar{A}$, we have that $\langle M_2\rangle(t) \to \infty$ as $t \to \infty$, so by the strong law of large numbers for martingales, we have

$$\begin{aligned}\limsup_{t\to\infty}\frac{|M_2(t)|}{t} &\leq \limsup_{t\to\infty}\frac{M_2(t)}{\langle M_2\rangle(t)}\limsup_{t\to\infty}\frac{\langle M_2\rangle(t)}{t}\\ &= \limsup_{t\to\infty}\frac{M_2(t)}{\langle M_2\rangle(t)}\limsup_{t\to\infty}\frac{1}{t}\int_0^t \|\sigma(s)\|_F^2\,ds = 0,\end{aligned}$$

so a.s. we have

$$\lim_{t\to\infty}\frac{M_2(t)}{t} = 0, \quad \text{a.s.} \tag{4.5.32}$$

Now define the event A_1 by $A_1 := \{\omega : \lim_{t\to\infty} Z(t,\omega) = \infty\}$ and suppose that $\mathbb{P}[A_1] > 0$. By Lemma 4.5.2 we note that there is an a.s. event $\Omega_3 = \{\omega : Z(t,\omega) > 0 \text{ for all } t \geq 0\}$. Let $A_2 = A_1 \cap \Omega_1 \cap \Omega_2$, where Ω_1 is the a.s. event in (4.5.32). Thus $\mathbb{P}[A_2] > 0$. Then for each $\omega \in A_2$, we have that $\lim_{t\to\infty}\phi(Z(t,\omega)) = +\infty$, and so

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \phi(Z(s))\,ds = +\infty, \quad \text{on } A_2. \tag{4.5.33}$$

For each $\omega \in A_2$, there is a $T^*(\omega) > 0$ such that $Z(t,\omega) \geq 1$ for all $t \geq T^*(\omega)$. Therefore, for $t \geq T^*(\omega)$, we have the bound

$$\frac{1}{t}\int_0^t \frac{\|\sigma(s)\|_F^2 + e^{-s}}{Z(s)}\,ds \leq \frac{1}{t}\int_0^{T^*} \frac{\|\sigma(s)\|_F^2 + e^{-s}}{Z(s)}\,ds + \frac{1}{t}\int_{T^*}^t \{\|\sigma(s)\|_F^2 + e^{-s}\}\,ds.$$

Since $t \mapsto e^{-t}$ is integrable, and σ obeys (4.4.8), it follows that the second term on the right–hand side has a zero limit as $t \to \infty$. To deal with the first term, note that the continuity of Z on the compact interval $[0, T^*]$ and the positivity of Z implies there is a $T_1^* \in [0, T^*]$ such that $\inf_{t\in[0,T^*]} Z(t) = Z(T_1^*) > 0$, and so the first term also tends to zero as $t \to \infty$. Thus the third term on the righthand side of (4.5.31) tends to zero as $t \to \infty$ on A_2. Noting this zero limit, we take the limit as $t \to \infty$ in (4.5.31), and using (4.5.33) and (4.5.32), arrive at

$$\lim_{t\to\infty}\frac{Z(t,\omega)}{t} = -\infty, \text{for each } \omega \in A_2.$$

which implies that $Z(t,\omega) \to -\infty$ as $t \to \infty$ for each $\omega \in A_2$. But since $Z(t,\omega) > 0$ for all $t \geq 0$ for each $\omega \in A_2$, we have a contradiction, proving the result. □

Lemma 4.5.7. *Suppose that f obeys* (4.1.9), (1.2.4) *and* (4.2.4). *Suppose that σ obeys* (2.1.1) *and that S obeys* (2.2.8). *Then the solution W of* (4.5.6) *obeys*

$$\limsup_{t\to\infty} \|W(t)\| \leq c_2, \quad a.s.$$

for some deterministic $c_2 > 0$.

Proof. We have by Lemma 4.5.4 that $\limsup_{t\to\infty} |Y_0(t)| \leq c_1$, a.s. From this fact and (4.5.29), it follows that for every $\epsilon > 0$ there exists a $T(\omega, \epsilon) > 0$ such that

$$|Y_0(t,\omega)| \leq c_1 + 1 := \bar{Y}, \quad \int_{t-1}^{t} \left\{\|\sigma(s)\|_F^2 + e^{-2}\right\} ds < 1, \quad t \geq T(\epsilon, \omega). \tag{4.5.34}$$

Suppose this holds on the a.s. event Ω_1. By (4.2.4) and (4.5.1) we have that $\phi(x) \to \infty$ as $x \to \infty$. Therefore, we can choose $M > 0$ so large that

$$\frac{M}{2} \geq 2\bar{Y} + 1, \quad \inf_{x \geq M/2 - \bar{Y}} \phi(x) > \frac{1}{\bar{Y}+1} + \bar{Y} + 1. \tag{4.5.35}$$

By (4.2.4) and (4.5.1) we have that $\phi(x) \to \infty$ as $x \to \infty$.

By Lemma 4.5.6, there is an a.s. event Ω_2 such that $\Omega_2 = \{\omega : \liminf_{t\to\infty} Z(t,\omega) < +\infty\}$. Since $|Y_0|$ has a finite limsup on Ω_2, if follows that $\liminf_{t\to\infty} \|W(t,\omega)\| < +\infty$ on $\Omega_1 \cap \Omega_2$. Next suppose there is an event $A_3 = \{\omega : \limsup_{t\to\infty} W(t,\omega) > M\}$ for which $\mathbb{P}[A_3] > 0$. Let $A_4 = A_3 \cap \Omega_2 \cap \Omega_3$. Notice that $\liminf_{t\to\infty} W(t) = \liminf_{t\to\infty} Z(t) + Y_0(t) \geq \liminf_{t\to\infty} Y_0(t) \geq -c_1$, so we do not need to consider the absolute value of W in the definition of A_3. Suppose that $\omega \in A_4$. It then follows that there exists $t_1 > T(\epsilon)$ such that $t_1 = \inf\{t > T(\epsilon) : W(t) = M/2\}$ and a $t_2 > t_1$ such that $t_2 = \inf\{t > t_1 : W(t) = M\}$. It also follows that there is $t_1' \in [t_1, t_2)$ such that $t_1' = \sup\{t > t_1 : W(t) = M/2\}$.

Suppose first that $t_2 - t_1' \geq 1$. Then $t_2 - 1 \geq t_1' \geq t_1 > T(\epsilon)$. Define $t_3 = t_2 - 1$. Then $M > W(t_3) > M/2$. Hence

$$\begin{aligned} M - W(t_3) &= W(t_2) - W(t_3) \\ &= -\int_{t_2-1}^{t_2} \phi(W(s) + Y_0(s))\, ds + \int_{t_2-1}^{t_2} \left\{ \frac{e^{-s} + \|\sigma(s)\|_F^2}{W(s) + Y_0(s)} + Y_0(s) \right\} ds. \end{aligned}$$

Since $W(t) > M/2$ and $|Y_0(t)| \leq \bar{Y}$ for all $t \in [t_2 - 1, t_2]$, we have that $W(t) + Y_0(t) \geq M/2 - \bar{Y} > 0$. Thus $\phi(W(t) + Y(t)) \geq \inf_{x \geq M/2 - \bar{Y}} \phi(x)$. Using these estimates leads to

$$\begin{aligned} M - W(t_3) &\leq -\int_{t_2-1}^{t_2} \inf_{x \geq M/2-\bar{Y}} \phi(x)\, ds + \int_{t_2-1}^{t_2} \left\{ \frac{e^{-s} + \|\sigma(s)\|_F^2}{M/2 - \bar{Y}} + \bar{Y} \right\} ds \\ &= -\inf_{x \geq M/2-\bar{Y}} \phi(x) + \frac{1}{M/2 - \bar{Y}} \int_{t_2-1}^{t_2} \left\{ e^{-s} + \|\sigma(s)\|_F^2 \right\} ds + \bar{Y}. \end{aligned}$$

Using the fact that $t_2 - 1 > T(\epsilon)$, we may use the second condition in (4.5.34), the first condition in (4.5.35) and then the last condition in (4.5.35) to get

$$\begin{aligned} 0 < M - W(t_3) &\leq -\inf_{x \geq M/2-\bar{Y}} \phi(x) + \frac{1}{M/2 - \bar{Y}} + \bar{Y} \\ &\leq -\inf_{x \geq M/2-\bar{Y}} \phi(x) + \frac{1}{\bar{Y} + 1} + \bar{Y} < 0, \end{aligned}$$

a contradiction.

Suppose on the other hand that $t_2 - t_1' < 1$. Once again, for all $t \in (t_1', t_2)$ we have $M/2 < W(t) < M$ with $W(t_1') = M/2$ and $W(t_2) = M$. Then, as $\phi(x) \geq 0$ for all $x \geq 0$, we have

$$\begin{aligned} M/2 &= W(t_2) - W(t_1') \\ &= -\int_{t_1'}^{t_2} \phi(Z(s))\, ds + \int_{t_1'}^{t_2} \frac{e^{-s} + \|\sigma(s)\|_F^2}{W(s) + Y_0(s)}\, ds + \int_{t_1'}^{t_2} Y_0(s)\, ds \\ &\leq \int_{t_1'}^{t_2} \frac{e^{-s} + \|\sigma(s)\|_F^2}{W(s) + Y_0(s)}\, ds + \int_{t_1'}^{t_2} |Y_0(s)|\, ds. \end{aligned}$$

Now, for all $t \in [t_1', t_2]$ we have that $W(t) \geq M/2$ and $|Y_0(t)| \leq \bar{Y}$, so $W(t) + Y_0(t) \geq M/2 - \bar{Y} > 0$. Using these estimates, and then the assumption that $t_2 - t_1' < 1$, we get

$$\begin{aligned} M/2 &\leq \int_{t_1'}^{t_2} \frac{e^{-s} + \|\sigma(s)\|_F^2}{W(s) + Y_0(s)}\, ds + \int_{t_1'}^{t_2} |Y_0(s)|\, ds \\ &\leq \frac{1}{M/2 - \bar{Y}} \int_{t_1'}^{t_2} \{e^{-s} + \|\sigma(s)\|_F^2\}\, ds + \int_{t_1'}^{t_2} \bar{Y}\, ds \\ &\leq \frac{1}{M/2 - \bar{Y}} \int_{t_2-1}^{t_2} \{e^{-s} + \|\sigma(s)\|_F^2\}\, ds + \bar{Y}. \end{aligned}$$

Finally, we notice that $t_2 > t_1 > T(\epsilon)$, so we may use the second estimate in (4.5.34) to get $M/2 \leq 1/(M/2 - \bar{Y}) + \bar{Y}$. Since $M/2 > \bar{Y}$, this rearranges to give $(M/2 - \bar{Y})^2 \leq 1$ or $M/2 - \bar{Y} \leq 1$. This is $M/2 \leq \bar{Y} + 1$. But as $\bar{Y} > 0$, this contradicts the second condition in (4.5.35), i.e, $M/2 \geq 2\bar{Y} + 1$. □

The proof of the main result is now immediate. We have from Lemma 4.5.3 that

$$\|X(t)\| \leq Z(t) = W(t) + Y_0(t), \quad t \geq 0.$$

where W and Y_0 are given by (4.5.6) and (4.5.4) respectively. By Lemma 4.5.4, we have that $\limsup_{t\to\infty} \|Y_0(t)\| \leq c_1$ a.s. By Lemma 4.5.7, we have that $\limsup_{t\to\infty} \|W(t)\| \leq c_2$ a.s. Notice that both c_1 and c_2 are deterministic bounds. Therefore, it follows that

$$\limsup_{t\to\infty} \|X(t)\| \leq c_1 + c_2, \quad \text{a.s.},$$

as required.

4.5.2 Proof that limit inferior is zero

It remains to prove the second part of (B), namely that

$$\liminf_{t\to\infty} \|X(t)\| = 0, \quad \text{a.s.}$$

We have already shown that $t \mapsto \|X(t)\|$ is bounded. Furthermore, since $S(\epsilon) < +\infty$ for all $\epsilon > \epsilon'$, we can prove as in the proof of Theorem 4.2.5 that (4.4.8) holds i.e.,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \|\sigma(s)\|_F^2 \, ds = 0.$$

Recall from (4.4.9) that we have the representation

$$|X(t)|^2 = |\xi|^2 - \int_0^t 2\langle X(s), f(X(s))\rangle \, ds + \int_0^t \|\sigma(s)\|_F^2 \, ds + 2M(t), \quad t \geq 0.$$

where M is the local (scalar) martingale given by (4.4.10) i.e.,

$$M(t) = \sum_{j=1}^{r} \int_0^t \sum_{i=1}^{d} X_i(s)\sigma_{ij}(s) \, dB_j(s), \quad t \geq 0.$$

The quadratic variation of M is given by

$$\langle M\rangle(t) = \sum_{j=1}^{r}\int_0^t \left(\sum_{i=1}^{d} X_i(s)\sigma_{ij}(s)\right)^2 ds,$$

and so by the Cauchy–Schwarz inequality, we have

$$\langle M\rangle(t) \le \sum_{j=1}^{r}\int_0^t \sum_{i=1}^{d} X_i^2(s)\sum_{i=1}^{d}\sigma_{ij}^2(s)\,ds \le \int_0^t \|X(s)\|_2^2\|\sigma(s)\|_F^2\,ds.$$

Therefore, as $t\mapsto \|X(t)\|$ is a.s. bounded, we have

$$\lim_{t\to\infty}\frac{1}{t}\langle M\rangle(t) = 0, \quad \text{a.s.}$$

In the case that $\langle M\rangle$ converges, we have that M tends to a finite limit and so

$$\lim_{t\to\infty}\frac{1}{t}M(t) = 0.$$

If, on the other hand $\langle M\rangle(t)\to\infty$ as $t\to\infty$, by the strong law of large numbers for martingales, we have

$$\lim_{t\to\infty}\frac{1}{t}M(t) = \lim_{t\to\infty}\frac{M(t)}{\langle M\rangle(t)}\cdot\frac{\langle M\rangle(t)}{t} = 0.$$

Using the fact that $t\mapsto\|X(t)\|$ is bounded, we have $\|X(t)\|^2/t\to 0$ as $t\to\infty$. Therefore, by rearranging (4.4.9), dividing by t and letting $t\to\infty$, we get

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \langle X(s), f(X(s))\rangle\,ds = 0, \quad \text{a.s.} \tag{4.5.36}$$

Now we suppose that there is an event A_1 of positive probability such that

$$A_1 = \{\omega : \liminf_{t\to\infty}\|X(t,\omega)\| > 0\}.$$

Since X is bounded, it follows that for a.a. $\omega\in A_1$, there are $\bar{X}(\omega),\bar{x}(\omega)\in(0,\infty)$ such that

$$\liminf_{t\to\infty}\|X(t,\omega)\| = \bar{x}(\omega), \quad \limsup_{t\to\infty}\|X(t,\omega)\| = \bar{X}(\omega).$$

Thus, there exists $T(\omega)>0$ such that

$$\frac{\bar{x}(\omega)}{2} \le \|X(t,\omega)\| \le 2\bar{X}(\omega), \quad t\ge T(\omega).$$

By the continuity of f and the fact that $\langle x, f(x)\rangle > 0$ for all $x \neq 0$, it follows that for any $0 < a \leq b < +\infty$

$$\inf_{\|x\|\in[a,b]} \langle x, f(x)\rangle = L(a,b) > 0.$$

Hence for $t \geq T(\omega)$ we have

$$\langle X(t,\omega), f(X(t,\omega))\rangle \geq L\left(\frac{\bar{x}(\omega)}{2}, 2\bar{X}(\omega)\right) =: \lambda(\omega) > 0.$$

Hence for $t \geq T(\omega)$ we have

$$\frac{1}{t}\int_0^t \langle X(s,\omega), f(X(s,\omega))\rangle\, ds \geq \frac{1}{t}\int_{T(\omega)}^t \langle X(s,\omega), f(X(s,\omega))\rangle\, ds \geq \frac{t - T(\omega)}{t} \cdot \lambda(\omega).$$

Hence for a.a. $\omega \in A_1$ we have

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \langle X(s,\omega), f(X(s,\omega))\rangle\, ds \geq \lambda(\omega) > 0,$$

which implies that

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \langle X(s), f(X(s))\rangle\, ds > 0, \quad \text{a.s. on } A_1.$$

This limit, taken together with the fact that A_1 is an event of positive probability, contradicts (4.5.36). Hence, it must follow that $\mathbb{P}[A_1] = 0$. This implies that $\mathbb{P}[\overline{A_1}] = 1$, or that $\liminf_{t\to\infty} \|X(t)\| = 0$ a.s. as required.

Chapter 5

Discretisation of scalar nonlinear stochastic differential equations

5.1 Introduction

The results in the previous chapter deal with a very wide class of nonlinear Stochastic Differential Equations(in the sense that f can exhibit either very strong or very weak reversion to equilibrium). It presents a challenge therefore to devise a single numerical method which will reproduce the asymptotic behaviour of solution for all f in this class.

In this chapter we show that the asymptotic stability of a special implicit Euler scheme for discretising a scalar stochastic differential equation is equivalent to the asymptotic stability of the differential equation. The reason we use implicit scheme rather than the explicit scheme is because if we want to use explicit scheme, we need to require some kind of global linear bound on f, for example, a condition like $|f(x)| \leq K|x|$ for all x, where K is a constant. At the same time we also need the step size to be sufficiently small. It is known that explicit discretisation is not always effective for preserving stability of scalar ODEs(never mind about SDEs), if the step size is too large. Consider the discretisation of

$$x'(t) = -ax(t), \quad t > 0, \quad x(0) = 1,$$

where $a > 0$. Under a standard Euler explicit method, with step size h, we have

$$x_h(n+1) - x_h(n) = -ahx_h(n), \quad n \geq 0, \quad x_h(0) = 1.$$

or

$$x_h(n+1) = (1 - ah)x_h(n), \quad n \geq 0, \quad x_h(0) = 1.$$

Thus, if $1 - ah < -1$(or $h > \frac{2}{a}$), we have that $|x_h(n)| \to \infty$ as $n \to \infty$ and x_h osullates unboundedly.

On the other hand, an implicit Euler method yields

$$x_h(n+1) - x_h(n) = -ahx_h(n+1), \quad n \geq 0, \quad x_h(0) = 1,$$

or

$$x_h(n+1) = \frac{1}{1+ah}x_h(n), \quad n \geq 0, \quad x_h(0) = 1.$$

Here, we have that $x_h(n) \to 0$ monotonically as $n \to \infty$ for every step size $h > 0$, just as the solution of the ODE does. This is also for nonlinear equation: indeed it can be shown in the same manner that an explicit discretisation of $x' = -f(x)$ will have unbounded and osullating solutions if the global linear bound is violated, and the initial condition is sufficiently large. Even though the standard theory says we can use an explicit scheme(but with very small step size)to characterise the behaviour of solutions in a *finite* interval, as the interval gets larger(or indeed in our case where we are interested in the long run behaviour of the solution), with the explicit scheme, we would require smaller and smaller step size. Therefore the cost of reducing the step size becomes considerable as the length of the interval increases. It would be very nice to be able to consider the long run behaviour of the solution without worrying about the step size. Therefore, it seems like a good idea to apply an implicit scheme to the discretisation of the nonlinear SDE. In this work, we have employed so-called Split-Step Backward Euler method, which we now discuss in detail.

Let B be a standard one–dimensional Brownian motion. Suppose that

$$f(0) = 0; \quad f \text{ is continuous and non-decreasing}; \quad xf(x) > 0 \text{ for all } x \neq 0, \tag{5.1.1}$$

and let f be locally Lipschitz continuous on $\mathbb{R}$. Let $\sigma : [0, \infty) \to \mathbb{R}$ be continuous and $\xi \in \mathbb{R}$ be deterministic. In this chapter we show that the asymptotic behaviour of the equation (3.1.1) as $t \to \infty$ can be mimicked by a uniform implicit discretisation for every uniform step size $h > 0$. The method is known as the split–step backward Euler method (SSBE), and is given by

$$X_h^*(n) = X_h(n) - hf(X_h^*(n)), \quad n \geq 0, \tag{5.1.2}$$

$$X_h(n+1) = X_h^*(n) + \sqrt{h}\sigma(nh)\xi(n+1), \quad n \geq 0. \tag{5.1.3}$$

The SSBE method was introduced in Higham et al.,[39, 60]. Deterministic versions of the split–step method can be found in [33, 37]. The preservation of boundedness and asymptotic stability of deterministic equations under the SSBE method is given in Stuart and Humphries [76].

The asymptotic behaviour of solutions of SDEs and their split step discretisations were studied in Higham et al., [41]. Another important work in which different implicit methods are used to analyse pathwise asymptotic behaviour of SDEs is Schurz [70]. The preservation of asymptotic stability of solutions of SDEs for explicit Euler methods as well as corresponding rates of decay has been considered recently in [11, 19, 13, 17, 66]. The stochastic differential equation in question is equation (3.1.1). In this chapter we will focus on one–dimensional equation, where we refine our attention on finite–dimensional problems in the next chapter. We show that the asymptotic stability of the SSBE method for discretising a scalar stochastic differential equation is equivalent to the asymptotic stability of the differential equation. Most of the results in this chapter are overseen by the next chapter. Even though, it is a great help in studying the finite-dimensional discrete equation. The results in this chapter was published in the Proceedings of Neural, Parallel and Scientific Computations, Volume 4, 2010. The continuous result in question here was proven in [31], which was motivated by simulated annealing problem.

In our main result we show that if $t \mapsto \sigma^2(t)$ is non–increasing then for any $h > 0$ $\lim_{t\to\infty} X(t) = 0$ a.s. if and only if $\lim_{n\to\infty} X_h(n) = 0$ a.s., and that both processes tend to zero a.s. if and only if $\sigma^2(t)\log t \to 0$ as $t \to \infty$. Therefore the split–step method is a.s. asymptotically stable if and only if the SDE (3.1.1) is a.s. asymptotically stable. In this chapter, other results are developed for stochastic difference equations and for the discretisation of an ordinary differential equation. Results are stated and discussed in Section 2; proofs are mainly postponed to Section 3. For standard results from probability theory, we refer the reader to Shiryaev [69].

This chapter and the next, are the last in the thesis, and both are devoted to analysing the long run behaviour of numerical schemes for the SDEs studied in Chapter 3 and 4. Essentially, in these remaining chapters we show that the SSBE method recovers in all cases the type of asymptotic behaviour exhibited by the underlying continuous SDE.

Moreover, these results which preserve the behaviour make no restriction on the uniform size of the step, $h > 0$.

5.2 Statements and Discussions of Main Results

5.2.1 Deterministic Equation

Suppose that f obeys (5.1.1) and consider the differential equation

$$x'(t) = -f(x(t)) + \gamma(t), \quad t \geq 0; \quad x(0) = \zeta. \tag{5.2.1}$$

Part (A) of the following result is essentially proven in [10], with the weaker hypothesis $\liminf_{|x|\to\infty} |f(x)| > 0$ in place of the monotonicity of f. The result concerns the asymptotic convergence of solutions of (5.2.1) to the zero equilibrium of the unperturbed equation $y'(t) = -f(y(t))$.

Theorem 5.2.1. *Suppose f obeys* (1.2.4) *and* (5.1.1). *Let $\gamma : [0,\infty) \to \mathbb{R}$ be continuous. Let $x(0) = \zeta \in \mathbb{R}$. Then there is a unique continuous solution of (5.2.1) and the following hold:*

(A) If $\gamma(t) \to 0$ as $t \to \infty$, then $x(t) \to 0$ as $t \to \infty$;

(B) If $x(t) \to 0$ as $t \to \infty$, then $\int_t^{t+h} \gamma(s)\,ds \to 0$ as $t \to \infty$ for any $h > 0$, and hence $\liminf_{t\to\infty} \gamma(t) = 0$.

Proof. To prove part (B), note for any $h > 0$ we have the identity

$$x(t+h) - x(t) = -\int_t^{t+h} f(x(s))\,ds + \int_t^{t+h} \gamma(s)\,ds.$$

Since $x(t) \to 0$ as $t \to \infty$, x is continuous and f obeys (5.1.1), we have the desired result on taking the limit as $t \to \infty$ on both sides of the identity. □

We remark that $x(t) \to 0$ as $t \to \infty$ does *not* imply $\gamma(t) \to 0$ as $t \to \infty$. If γ is positive and integrable it can be shown under the hypotheses of Theorem 5.2.1 that $x(t) \to 0$ as

$t \to \infty$. However, there exist continuous, positive and integrable functions γ for which $\limsup_{t\to\infty} \gamma(t) > 0$.

Let $h > 0$ and $x(0) = \zeta \in \mathbb{R}$. We consider the following difference equation

$$x^*(n) = x(n) - hf(x^*(n)), \quad n \geq 0, \tag{5.2.2}$$

$$x(n+1) = x^*(n) + hg(n), \quad n \geq 0. \tag{5.2.3}$$

This can be thought of as the split–step discretisation of the ordinary differential equation (5.2.1) with $g(n) = \gamma(nh)$. The following result therefore parallels Theorem 5.2.1; its proof is given in Section 3.

Theorem 5.2.2. *Let $h > 0$, $(g(n))_{n\geq 0}$ be a real sequence. Let $x(0) = \zeta \in \mathbb{R}$. Suppose f obeys* (1.2.4) *and* (5.1.1). *Then there is a unique solution of (5.2.2) and (5.2.3) and the following are equivalent:*

(A) $g(n) \to 0$ as $n \to \infty$;

(B) $x(n) \to 0$ as $n \to \infty$.

5.2.2 Stochastic Equation

We fix a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t\geq 0}, \mathbb{P})$. Let B be a standard one–dimensional Brownian motion which is adapted to $(\mathcal{F}(t))_{t\geq 0}$. If σ is continuous, then there is a unique continuous process, adapted to $(\mathcal{F}(t))_{t\geq 0}$, which obeys the stochastic differential equation (3.1.1). This process will be referred to hereinafter as the solution of (3.1.1). Chan and Williams [31] have proved Theorem 3.1.1 This Theorem has been extended in [10] to deal with non–monotone f and σ.

Can we reproduce this asymptotic behaviour in discrete time using the SSBE method (5.1.2)–(5.1.3)? To answer this question, we first suppose that

$(\xi(n))_{n\geq 0}$

is a sequence of iid standard normal variables with common distribution function Φ. (5.2.4)

Here $\Phi(x) = (\sqrt{2\pi})^{-1}\int_{-\infty}^{x} \exp(-u^2/2)\,du$ for $x \in \mathbb{R}$ and we define $\Phi(\infty) = 1$, $\Phi(-\infty) = 0$. We consider the difference equation for (X, X^*) given by (5.1.2)–(5.1.3), as this system may be thought of as the split–step Euler–Maruyama method of Higham, Mao and Stuart [39] applied to (3.1.1).

Our first result on the asymptotic behaviour of (5.1.2)–(5.1.3) does not impose monotonicity on σ.

Theorem 5.2.3. *Let $h > 0$. Suppose ξ obeys (5.2.4). Let $\zeta \in \mathbb{R}$ be deterministic and $X_h(0) = \zeta$. If f obeys* (1.2.4) *and* (5.1.1)*. Then there is a unique solution (X_h, X_h^*) of (5.1.2) and (5.1.3) and the following are equivalent:*

(A) $\sum_{n=1}^{\infty} 1 - \Phi(\epsilon/|\sigma(nh)|) < +\infty$ *for every* $\epsilon > 0$*;*

(B) $\lim_{n\to\infty} X_h(n) = 0$ *with positive probability for some* $\zeta \in \mathbb{R}$*;*

(C) $\lim_{n\to\infty} X_h(n) = 0$ *a.s. for all* $\zeta \in \mathbb{R}$.

The proof of Theorem 5.2.3 is postponed. We state next an application of Theorem 5.2.3 in the case when σ is decreasing.

Theorem 5.2.4. *Let $h > 0$. Suppose ξ obeys (5.2.4). Let $\zeta \in \mathbb{R}$ be deterministic and $X(0) = \zeta$. Suppose that $n \mapsto \sigma^2(nh)$ is decreasing. If f obeys* (1.2.4) *and* (5.1.1)*. Then there is a unique solution (X_h, X_h^*) of (5.1.2) and (5.1.3) and the following are equivalent:*

(A) $\lim_{n\to\infty} \sigma^2(nh)\log(nh) = 0$*;*

(B) $\lim_{n\to\infty} X_h(n) = 0$ *with positive probability for some* $\zeta \in \mathbb{R}$*;*

(C) $\lim_{n\to\infty} X_h(n) = 0$ *a.s for all* $\zeta \in \mathbb{R}$.

Proof. We use Theorem 5.2.3 to prove Theorem 5.2.4. It was shown in Appleby, Riedle and Rodkina [20] that when σ_n^2 is non–increasing, and $(\xi(n))_{n\geq 0}$ are a sequence of independent standard normal random variables, the following statements are equivalent:

(i) $\sum_{n=1}^{\infty} |\sigma_n| \exp\left(-2^{-1}\varepsilon^2\sigma_n^{-2}\right) < \infty$ for all $\varepsilon \in \mathbb{Q}^+$;

(ii) $\lim_{n\to\infty} \sigma_n^2 \log n = 0$;

(iii) $\lim_{n\to\infty} \sigma_n \xi(n+1) = 0$ a.s.

Let $\sigma_n := \sigma(nh)$. We show that (A) implies (C) implies (B) implies (A). Statement (i) in this setting is equivalent to statement (A) in Theorem 5.2.3. If statement (A) in Theorem 5.2.4 holds, then statement (ii) holds and therefore statement (i). Statement (i) in this setting is equivalent to statement (A) in Theorem 5.2.3. Thus, by Theorem 5.2.3 we have (C) in Theorem 5.2.4. Hence (A) implies (C). Obviously (C) in Theorem 5.2.4 implies (B) in Theorem 5.2.4. Thus (C) implies (B). Since statement (B) in Theorem 5.2.4 is equivalent to (B) in Theorem 5.2.3, by Theorem 5.2.3, we have statement (A) in Theorem 5.2.3. But this is equivalent to statement (i) above, and therefore to statement (ii). But (ii) in this context is nothing other than statement (A) in Theorem 5.2.4, so (B) implies (A). Hence (A)–(C) are equivalent, ending the proof. □

Theorem 5.2.4 captures exactly the behaviour of Theorem 3.1.1 in discrete time. The monotonicity of σ^2 ensures that statement (A) in Theorems 3.1.1 and 5.2.4 are equivalent. Theorem 5.2.4 can be contrasted with results in [2, 11] in which step–size restrictions are needed to mimic the asymptotic behaviour of SDEs using explicit Euler–Maruyama schemes.

In Figures 1 and 2 overleaf we show representative simulated sample paths of the equation

$$dX(t) = -2X(t)^3dt + (1+t)^{-2}dB(t) \quad with \quad X(0) = 20.$$

The first figure shows a path for the SSBE scheme (5.1.2)–(5.1.3), while the second shows a path for the explicit Euler scheme

$$X_h(n+1) = X_h(n) - 2hX_h(n)^3 + \sqrt{h}(1+nh)^{-2}\xi(n+1).$$

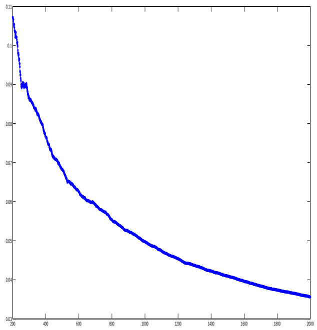

Figure 5.1: SSBE method: $h = 0.1$, $X(0) = 20$.

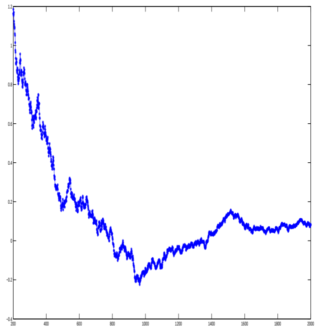

Figure 5.2: Explicit E-M: $h = 0.001$, $X(0) = 20$.

It can be seen that the SSBE scheme gives the regular convergence to zero predicted by continuous–time theory (see [12, 19] and references), while the explicit scheme performs less well, even with a smaller step size. Indeed, for some larger values of h, we see solutions which oscillate unboundedly. Note that the small time behaviour has been cropped from each figure below.

5.3 Proof of Theorem 5.2.2

5.3.1 Supporting Lemmata

Lemma 5.3.1. *Suppose f obeys* (1.2.4) *and* (5.1.1)*. There exists a unique continuous function $x \mapsto F_h(x)$ such that*

$$F_h(x) = x - hf(F_h(x)), \ x \in \mathbb{R}, \tag{5.3.1}$$

where $|F_h(x)| < |x|$ for $x \neq 0$ and $F_h(0) = 0$.

Proof. We show that, for every $x \in \mathbb{R}$, there is a unique $y = F_h(x) \in \mathbb{R}$ such that $y = x - hf(y)$, and $x \mapsto F_h(x)$ is continuous. For every $x \in \mathbb{R}/\{0\}$, by the continuity of f and the intermediate value theorem, there is a y between 0 and x such that $\Delta_h(x, y) = 0$ where

$$\Delta_h(x, y) = y + hf(y) - x.$$

Clearly $\Delta_h(0,0) = 0$, so $y = x - hf(y)$ has a solution for each $x \in \mathbb{R}$. Let $y_1 \neq y_2$. Since f is non-decreasing, we have

$$(y_2 - y_1)(\Delta_h(x, y_2) - \Delta_h(x, y_1)) = (y_2 - y_1)^2 + h(y_2 - y_1)(f(y_2) - f(y_1)) \geq (y_2 - y_1)^2 > 0.$$

Therefore for each fixed x, $y \mapsto \Delta_h(x, y)$ is increasing. Moreover $(x, y) \mapsto \Delta_h(x, y)$ is continuous, as f is continuous. Therefore by a variant of the implicit function theorem (see e.g., Kudryavtsev [47]), there is a unique continuous function $x \mapsto F_h(x)$ such that $\Delta_h(x, F_h(x)) = 0$ for all $x \in \mathbb{R}$. Hence (5.3.1) holds and $|F_h(x)| < |x|$ for $x \neq 0$, and $F_h(0) = 0$, as required. □

Proposition 5.3.1. *Let $x(0) = \zeta \in \mathbb{R}$. Suppose f obeys* (1.2.4) *and* (5.1.1). *Then there is a unique solution to (5.2.2), (5.2.3).*

Proof. A consequence of Lemma 5.3.1 is that (5.2.2) is equivalent to $x^*(n) = F_h(x(n))$, $n \geq 0$ and that therefore (5.2.2), (5.2.3) is equivalent to

$$x(n+1) = F_h(x(n)) + hg(n), \ n \geq 0. \tag{5.3.2}$$

Clearly this equation has a unique solution, as required. □

We introduce another auxiliary function, related to F_h, and deduce some of its salient properties.

Lemma 5.3.2. *Suppose f obeys* (1.2.4) *and* (5.1.1) *and F_h is given by (5.3.1). Define*

$$F_h^+(x) = \max(F_h(x), -F_h(-x)), \ x \geq 0. \tag{5.3.3}$$

Then $F_h^+(x) < x$ for all $x > 0$, $F_h^+(0) = 0$, and $F_h^+(x) > F_h^+(y)$ for all $x > y \geq 0$.

Proof. Define

$$G_h(x) = x + hf(x), \quad x \in \mathbb{R}. \tag{5.3.4}$$

Then G_h is increasing, therefore G_h^{-1} exists. Hence we have that

$$x - hf(G_h^{-1}(x)) = G_h^{-1}(x),$$

therefore by Lemma 5.3.1 and (5.3.1), we have that $F_h = G_h^{-1}$. Evidently, we have that $0 < F_h(x) < x$ for $x > 0$ and $x < F_h(x) < 0$ for $x < 0$. First we show that $x > y$ implies $F_h(x) > F_h(y)$. Now

$$G_h(F_h(x)) = x > y = G_h(F_h(y)).$$

Since G_h is increasing, we have $F_h(x) > F_h(y)$. For $x \geq 0$, $F_h^+(x) = \max[F_h(x), -F_h(-x)]$. Since $F_h(x) < x$ for $x > 0$ and $-F_h(-x) < x$, we have

$$F_h^+(x) - x = \max[F_h(x) - x, -F_h(-x) - x] < 0, \quad F_h^+(0) = 0$$

because $F_h(0) = 0$. Let $x > y \geq 0$. Then $F_h(x) > F_h(y)$, $F_h(-x) < F_h(-y)$, therefore

$$F_h^+(x) = \max[F_h(x), -F_h(-x)] > \max[F_h(y), -F_h(-y)] = F_h^+(y),$$

which completes the proof. □

Since f is non–decreasing and positive, we define for all $\eta > 0$ sufficiently small

$$x_1(\eta) = \min\{x > 0 : f(x) = \eta/h\}, \quad x_2(\eta) = \min\{x > 0 : f(-x) = -\eta/h\}.$$

Lemma 5.3.3. *Suppose f obeys* (1.2.4) *and* (5.1.1). *Let $\eta > 0$ and $x_1(\eta)$, $x_2(\eta)$ be defined as above. If $x_3(\eta) = \max[\eta + x_1(\eta), \eta + x_2(\eta)]$, and F_h^+ is defined by (5.3.3), then*

$$F_h^+(x) - x \leq -\eta \quad \textit{for} \quad \textit{all} \quad x \geq x_3(\eta) \quad \textit{and} \lim_{\eta \to 0^+} x_3(\eta) = 0.$$

Proof. Let $x \geq \eta + x_1(\eta)$. Thus as f is nondecreasing we have

$$f(x - \eta) \geq f(x_1(\eta)) = \eta/h.$$

Hence $hf(x - \eta) \geq \eta$. Thus $x - \eta + hf(x - \eta) \geq x$, or $G_h(x - \eta) \geq x$ for $x \geq \eta + x_1(\eta)$. Hence $x - \eta \geq F_h(x)$ for $x \geq \eta + x_1(\eta)$ because $F_h = G_h^{-1}$.

Define $y_2(\eta) := -x_2(\eta) - \eta$ and suppose $y \leq y_2(\eta)$. Thus $y + \eta \leq -x_2(\eta)$. Therefore

$$f(y+\eta) \leq f(-x_2(\eta)) = -\eta/h,$$

so $hf(y+\eta) \leq -\eta$. Hence $y+\eta+hf(y+\eta) \leq y$, or $G_h(y+\eta) \leq y$ for all $y \leq y_2(\eta)$. Hence $y + \eta \leq F_h(y)$ for all $y \leq y_2(\eta)$. Thus $-x + \eta \leq F_h(-x)$ and $-x \leq -(x_2(\eta) + \eta)$. Hence $-F_h(-x) - x \leq -\eta$ for $x \geq x_2(\eta) + \eta$ and $F_h(x) - x \leq -\eta$ for $x \geq x_1(\eta) + \eta$. Now let

$$x \geq x_3(\eta) = \max[x_1(\eta) + \eta, x_2(\eta) + \eta].$$

Then $-F_h(-x) - x \leq -\eta$ and $F_h(x) - x \leq -\eta$. Hence for $x \geq x_3(\eta)$ we have

$$F_h^+(x) - x = \max[F_h(x) - x, -F_h(-x) - x] \leq -\eta$$

as required. To see that $x_3(\eta) \to 0$ as $\eta \to 0$, note that $x_1(\eta)$ and $x_2(\eta)$ are non-decreasing and positive on $(0, \infty)$, and $\lim_{\eta \to 0^+} x_1(\eta) = \lim_{\eta \to 0^+} x_2(\eta) = 0$. Therefore $x_3(\eta) \to 0$ as $\eta \to 0$, and the proof is complete. $\square$

5.3.2 Proof of Theorem 5.2.2

We first prove (A) implies (B). Since $g(n) \to 0$ as $n \to \infty$, for every $\epsilon > 0$, there is $N(\epsilon) \in \mathbb{N}$ such that $|hg(n)| \leq \epsilon$ for all $n \geq N(\epsilon)$. Let F_h^+ be defined by (5.3.3). Then F_h^+ is continuous and $|F_h(x)| \leq F_h^+(|x|)$ for $x \geq 0$. We define the sequence $(x^+(n))_{n \geq 0}$ by

$$x^+(n+1) = F_h^+(x^+(n)) + |hg(n)|, \quad n \geq 0; \qquad x^+(0) = |x(0)| + 1.$$

Since F_h^+ is increasing, we have $x^+(n) > 0$ for all $n \geq 0$ and $|x(n)| < x^+(n)$ for $n \geq 0$. Let $n \geq N(\epsilon)$, and suppose $x^+(n) \geq x_3(2\epsilon)$ for all $n \geq N(\epsilon)$. By Lemma 5.3.3, $F_h^+(x) - x \leq -2\epsilon$ for $x \geq x_3(2\epsilon)$, therefore we have

$$x^+(n+1) \leq \epsilon + x^+(n) + [F_h^+(x^+(n)) - x^+(n)] \leq x^+(n) - \epsilon.$$

Hence $(x^+(n))_{n \geq N(\epsilon)}$ is decreasing, so $x^+(n) \to L \geq x_3(2\epsilon)$ as $n \to \infty$. Therefore

$$F_h^+(L) = \lim_{n \to \infty} F_h^+(x^+(n)) = \lim_{n \to \infty} [x^+(n+1) - |hg(n)|] = L$$

which by Lemma 5.3.2 implies $L = 0$, a contradiction.

Therefore, we must have that there exists $n' = n'(\epsilon) > N(\epsilon)$ such that $x^+(n') \geq x_3(2\epsilon)$ and $x^+(n'+1) < x_3(2\epsilon)$. Hence, as $n' > N(\epsilon)$, $|g(n')| \leq \epsilon$ and F_h^+ is increasing, we have

$$x^+(n'+2) \leq F_h^+(x_3(2\epsilon)) + \epsilon \leq -2\epsilon + x_3(2\epsilon) + \epsilon < x_3(2\epsilon).$$

Continuing by induction in this manner, we have $x^+(n) < x_3(2\epsilon)$ for all $n \geq n'(\epsilon)+1$. We can therefore summarise the behaviour of x^+ as follows:

(i) If $x^+(n) \geq x_3(2\epsilon)$ for some $n \geq N(\epsilon)$, there exists $n'(\epsilon) > N(\epsilon)$ such that $x^+(n) < x_3(2\epsilon)$ for all $n \geq n'(\epsilon)+1$;

(ii) otherwise $x^+(n) < x_3(2\epsilon)$ for all $n \geq N(\epsilon)$.

Hence in either event, for every $\epsilon > 0$, there is $N_2(\epsilon) > 0$ such that $x^+(n) < x_3(2\epsilon)$ for all $n \geq N_2(\epsilon)$. Hence $|x(n)| < x_3(2\epsilon)$ for all $n \geq N_2(\epsilon)$. Thus $\limsup_{n\to\infty} |x(n)| \leq x_3(2\epsilon)$. Since $x_3(2\epsilon) \to 0$ as $\epsilon \to 0$, we have $|x(n)| \to 0$ as $n \to \infty$.

To prove that (B) implies (A), we first note that

$$x(n+1) = F_h(x(n)) + hg(n) \quad \text{for all} \quad n \geq 0.$$

Since F_h is continuous and $x(n) \to 0$ as $n \to \infty$ by hypothesis, we have

$$\lim_{n\to\infty} F_h(x(n)) = F_h(0) = 0.$$

Hence

$$hg(n) = x(n+1) - F_h(x(n)) \to 0 \quad as \quad n \to \infty,$$

as required.

5.3.3 Proof of Theorem 5.2.3

Under the hypotheses, we have by Theorem 5.2.2 that (5.1.2) and (5.1.3) has a unique solution, and moreover, it can be written in the form (5.2.2) and (5.2.3) where $hg(n) = \sqrt{h}\sigma(nh)\xi(n+1)$. By the independence of sequence $(\xi(n))_{n\geq 1}$ and the fact that $\sigma(nh)$ is deterministic, we observe that (5.2.4) implies

$$\begin{aligned}\mathbb{P}[|\sqrt{h}\sigma(nh)\xi(n+1)| \leq \epsilon\sqrt{h}] &= \mathbb{P}[-\epsilon/|\sigma(nh)| \leq \xi(n+1) \leq \epsilon/|\sigma(nh)|] \\ &= \Phi(\epsilon/|\sigma(nh)|) - \Phi(-\epsilon/|\sigma(nh)|).\end{aligned}$$

Since $\Phi(-x) = 1 - \Phi(x)$ for all $x \in \mathbb{R}$, statement (A) is equivalent to

$$\sum_{n=1}^{\infty} \mathbb{P}[|\sqrt{h}\sigma(nh)\xi(n+1)| > \epsilon\sqrt{h}] < +\infty \quad \text{for all} \quad \epsilon > 0.$$

By the Borel–Cantelli lemma, statement (A) is equivalent to

$$\limsup_{n\to\infty} |\sqrt{h}\sigma(nh)\xi(n+1)| \leq \epsilon\sqrt{h}$$

on an almost sure event Ω_ϵ for every $\epsilon > 0$. Hence statement (A) is equivalent to $\sigma(nh)\xi(n+1) \to 0$ as $n \to \infty$ almost surely. Suppose the almost sure event on which this holds is Ω^*. Therefore, by Theorem 5.2.2, for every $\omega \in \Omega^*$, we have that $X_h(n,\omega) \to 0$ as $n \to \infty$. Hence (A) implies (C), and clearly (C) implies (B). To see that (B) implies (A), suppose that $X_h(n) \to 0$ as $n \to \infty$ on the event A which has non–zero probability. Then by Theorem 5.2.2, it follows that for each $\omega \in A$ we have

$$\sigma(nh)\xi(n+1,\omega) \to 0 \quad as \quad n \to \infty.$$

However, by the independence of the random variables $\xi(n+1)$ and the Kolmogorov Zero–One Law, it follows that we must have $\mathbb{P}[A] = 1$, which implies statement (A), by the Borel–Cantelli lemma. Hence (A)–(C) are equivalent.

Chapter 6

Discretisation of finite dimensional affine and nonlinear stochastic differential equations

6.1 Introduction

In this chapter the asymptotic behaviour of certain discretisations of perturbed nonlinear ordinary and stochastic differential equations is considered. We consider the perturbed stochastic differential equation (3.1.1)

$$dX(t) = -f(X(t))\,dt + \sigma(t)\,dB(t), \quad t \geq 0.$$

The equation is finite–dimensional, with $f : \mathbb{R}^d \to \mathbb{R}^d$, $: [0,\infty) \to \mathbb{R}^{d\times r}$ and B being an r–dimensional standard Brownian motion. We presume that f and σ are sufficiently smooth to ensure the existence of unique solutions. The appropriate conditions are that f is locally Lipschitz continuous and that σ is continuous. Throughout we assume that the unperturbed differential equation

$$y'(t) = -f(y(t)), \quad t \geq 0 \tag{6.1.1}$$

has a unique equilibrium which is translated to zero:

$$f(x) = 0 \quad \text{if and only if } x = 0. \tag{6.1.2}$$

This equilibrium is globally stable by imposing the dissipative condition

$$\langle x, f(x)\rangle > 0, \quad \text{for all } x \neq 0. \tag{6.1.3}$$

Existence of a continuous solution of (6.1.1) is guaranteed by assuming that

$$f \in C(\mathbb{R}^d;\mathbb{R}^d) \tag{6.1.4}$$

The assumptions (6.1.2), (6.1.3) and (6.1.4) imply that all continuous solutions y of (6.1.1) obey $y(t) \to 0$ as $t \to \infty$.

The purpose of this chapter is to examine whether we can mimic the asymptotic behaviour of the solutions of (4.1.8) under discretisation.

This should be achieved using only the conditions required to ensure stability, boundedness or unboundedness in the continuous–time case. A particular challenge is to perform a successful discretisation even in the case when the function f is not globally linearly bounded, and with a uniform mesh size $h > 0$ if possible. As already discussed in the previous chapter, it is known that for such highly nonlinear equations that explicit methods are unlikely to preserve the long run behaviour of solutions; see examples in [60] and [41]. It has been shown in the deterministic case by Stuart Humphries and for stochastic differential equations that implicit methods are very useful for achieving such results. For this reason, we have adopted the split–step backward Euler method (SSBE) developed in [39, 60]. This method reduces to the standard backward Euler method for deterministic differential equations [33, 37]. In this work, we demonstrate that the split step backward Euler method for SDEs, which was introduced by Mao, Higham and Stuart, and by Mattingly, Stuart and Higham achieves these ends.

The results in this chapter extend and improve those presented in [5], in which a scalar equation with a monotone increasing f was considered. A classification of the solutions of scalar linear stochastic differential equations in continuous time was presented in [4].

6.2 The Equation

6.2.1 Set–up of the problem

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space. Suppose that ξ is a stochastic sequence in $\mathbb{R}^r$ with the following property:

Assumption 6.2.1. $\xi = \{\xi(n) : n \geq 1\}$ *is a sequence of r–dimensional independent and identically distributed Gaussian vectors. Moreover, with the notation $\xi^{(j)}(n) = \langle \xi(n), e_j \rangle$ for $j = 1, \ldots, r$, we assume each of the Gaussian random variables $\xi^j(n)$ has zero mean and unit variance, and that $\xi^{(j)}(n)$, $j = 1, \ldots, r$ are mutually independent for each n.*

This sequence generates a natural filtration $\mathcal{F}(n) := \sigma\{\xi(j) : 1 \le j \le n\}$. In what follows we denote by $\Phi : \mathbb{R} \to \mathbb{R}$ the distribution of a standard normal random variable as in (2.2.2) We interpret $\Phi(-\infty) = 0$ and $\Phi(\infty) = 1$.

Remark 6.2.1. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ obey (4.1.9). This condition on f guarantees that the equilibrium at zero of the unperturbed equation is unique. Suppose to the contrary that there is $x^* \neq 0$ such that $f(x^*) = 0$. Then $0 < \langle x^*, f(x^*)\rangle = \langle x^*, 0\rangle = 0$, a contradiction.

Suppose also that

$$\Sigma \in C([0,\infty); \mathbb{R}^{d\times r}). \tag{6.2.1}$$

We consider uniform discretisation of the stochastic differential equation

$$dX(t) = -f(X(t))\,dt + \Sigma(t)\,dB(t), \quad t \ge 0; \quad X(0) = \zeta \in \mathbb{R}^d. \tag{6.2.2}$$

If, for example, we wish to guarantee the existence of a unique strong solution of (6.2.2), we may assume that f is locally Lipschitz continuous on $\mathbb{R}^d$ or satisfies a global one–sided Lipschitz condition.

However, if one wants only to assure the existence of a solution, the continuity of f and σ guarantee the existence of a local solution. Moreover, the second part of condition (4.1.9) guarantees that any such continuous solution does not explode in finite time almost surely, so we have global existence of the solution. Local existence and uniqueness is standard from e.g., [55]; a proof of non–explosion and global existence is given in [10].

6.2.2 Construction of the discretisation and existence and uniqueness of its solutions

We propose to discretise the strong solution X of (6.2.2) as follows. Let $h > 0$, and let $\sigma_h : \mathbb{N}_0 \to \mathbb{R}^{d\times r}$ be a $d \times r$–matrix valued sequence with real entries. Let ξ be the sequence defined by Assumption 6.2.1. Consider the system of stochastic difference equations described by

$$X_h(0) = \zeta; \tag{6.2.3a}$$

$$X_h^\star(n) = X_h(n) - hf(X_h^\star(n)), \quad n \geq 0; \tag{6.2.3b}$$

$$X_h(n+1) = X_h^\star(n) + \sqrt{h}\sigma_h(n)\xi(n+1), \quad n \geq 0. \tag{6.2.3c}$$

(6.2.3b), (6.2.3c) with the initial condition (6.2.3a) is the so–called *split–step method* for discretising the stochastic differential equation (6.2.2). This makes sense if we presume that $\sigma_h(n) = \Sigma(nh)$ for $n \geq 0$, where Σ is the diffusion coefficient in (6.2.2).

6.2.3 Existence and uniqueness of solutions of split–step scheme

We *assume at first* that (6.2.3) has at least one well–defined solution This is assured by the following deterministic— and potentially h–dependent— condition on f.

Assumption 6.2.2. *For every $x \in \mathbb{R}^d$ there exists $x^\star \in \mathbb{R}^d$ such that*

$$x^\star = x - hf(x^\star). \tag{6.2.4}$$

In this situation, we say that (6.2.3) has a solution if there is a pair of processes $(X_h, X_h^\star)$ which obey (6.2.3). Such a solution will automatically be global (i.e, defined for all $n \geq 0$): there is no possibility of finite time explosion, because each member of the sequence ξ is a.s. finite. Such a solution will be adapted to the natural filtration generated by ξ.

Remark 6.2.2. In the scalar case ($d = 1$), and f obeys (4.1.9), then Assumption 6.2.2 is satisfied.

Proof. Consider for each $x \in \mathbb{R}$ the function $G_x : \mathbb{R} \to \mathbb{R}$

$$G_x(y) = y - x + hf(y), \quad y \in \mathbb{R}.$$

Notice that the continuity of f ensures that G_x is continuous. Then $G_x(0) = -x$ and $G_x(x) = hf(x)$. Therefore by (4.1.9), $G_x(0)G_x(x) = -hxf(x) < 0$ for $x \neq 0$, so that there is a solution $x^\star$ of (6.2.4) between 0 and x for every $x \neq 0$. In the case when $x = 0$, we have $yG_0(y) = y^2 + yhf(y) > 0$ for $y \neq 0$ and $G_0(0) = 0$. Thus 0 is the only solution of (6.2.4) in the case when $x = 0$. $\square$

Conditions can be imposed on f which guarantee that there is a unique solution of (6.2.3). These include f obeying the so–called one–sided (global) Lipschitz condition

$$(f(x) - f(y))(x - y) \leq \mu(x - y)^2, \quad \text{for all } x, y \in \mathbb{R}$$

and some $\mu \in \mathbb{R}$. This condition guarantees the existence of a unique solution of (6.2.4) provided the step size h is chosen to be sufficiently small. Although this is weaker than requesting that f satisfy a global Lipschitz condition, it places a restriction on f on all $\mathbb{R}$, and still excludes some functions f which grow faster than polynomially as $|x| \to \infty$.

In this chapter, we do not worry about the uniqueness of the solution of (6.2.3). Instead, we show that *all* solutions of the equation will have the correct asymptotic behaviour. This is in the spirit of generalised dynamical systems considered by Stuart and Humphries [76]. This enables us to impose a weaker regularity condition on f and to therefore consider a wider class of functions f than are covered by the one–sided Lipschitz condition. But if uniqueness of the solution of (6.2.3) is required, we are still free to impose extra conditions on f.

6.2.4 Mean reversion of split–step method under (4.1.9)

Before proceeding, it is worthwhile to note that the first, "deterministic" equation in the split–step method (namely (6.2.3b)) forces the intermediate estimate $X_h^\star$ to always be closer to the equilibrium than X_h.

Lemma 6.2.1. *Suppose* $(X_h, X_h^\star)$ *is a solution of* (6.2.3) *and that* f *obeys* (4.1.9). *Then for each* $n \in \mathbb{N}$,

$$0 < \|X_h^\star(n)\| < \|X_h(n)\| \text{ if } \|X_h(n)\| > 0, \text{ and } X_h^\star(n) = 0 \text{ if and only if } X_h(n) = 0.$$

Proof. To prove part (a), suppose first that $\|X_h(n)\| > 0$. Notice from (6.2.3b) that $X_h^\star(n) = 0$ implies that $X_h(n) = 0$, so we have $\|X_h^\star(n)\| > 0$. By taking the innerproduct with $X_h^\star(n)$ on each side of (6.2.3b), and using the second statement in (4.1.9) we get

$$\|X_h^\star(n)\|^2 = \langle X_h(n), X_h^\star(n)\rangle - h\langle f(X_h^\star(n)), X_h^\star(n)\rangle < \langle X_h(n), X_h^\star(n)\rangle.$$

Applying the Cauchy–Schwartz inequality to the rightmost inequality, this implies that $\|X_h^\star(n)\|^2 < \|X_h(n)\|\|X_h^\star(n)\|$, as required.

We have already seen that $X_h^\star(n) = 0$ implies that $X_h(n) = 0$. To prove the converse, let $X_h(n) = 0$ and suppose that $\|X_h^\star(n)\| > 0$. From (6.2.3b) we have $X_h^\star(n) = -hf(X_h^\star(n))$, so taking the innerproduct as before and using (4.1.9) yields $0 < \|X_h^\star(n)\|^2 = -h\langle f(X_h^\star(n)), X_h^\star(n)\rangle < 0$, a contradiction. □

6.3 Statement and Discussion of Main Results

6.3.1 Affine equations

Before discussing the asymptotic behaviour of solutions of (6.2.3), it is fruitful to first understand the asymptotic behaviour of the d–dimensional sequence $U_h = \{U_h(n) : n \geq 1\}$ defined by

$$U_h(n+1) = \sqrt{h}\sigma_h(n)\xi(n+1), \quad n \geq 0 \tag{6.3.1}$$

Define

$$S_h(\epsilon) = \sum_{n=0}^{\infty} \left\{1 - \Phi\left(\frac{\epsilon}{\|\sigma_h(n)\|_F}\right)\right\}. \tag{6.3.2}$$

Notice that $S_h(\epsilon)$ is monotone in $\epsilon > 0$. Therefore, there are only three possible types of behaviour for S, for a given σ_h, namely: (i) $S_h(\epsilon) < +\infty$ for all $\epsilon > 0$; (ii) $S_h(\epsilon) = +\infty$ for all $\epsilon > 0$; and (iii) $S_h(\epsilon) < +\infty$ for all $\epsilon > \epsilon' > 0$ and $S_h(\epsilon) = +\infty$ for all $\epsilon < \epsilon'$. Due to this trichotomy, it can be seen that the following result enables the long–run pathwise behaviour of $U_h(n)$ to be classified in terms of S_h.

Lemma 6.3.1. *Let $\xi = \{\xi(n) \in \mathbb{R}^r : n \in \mathbb{N}\}$ be a sequence of random vectors obeying Assumption 6.2.1. Let U_h be given by* (6.3.1)*, and $S_h(\epsilon)$ be defined by* (6.3.2).

(A) If $S_h(\epsilon) < \infty$ *for all* $\epsilon > 0$*, then*

$$\lim_{n\to\infty} U_h(n) = 0, \quad a.s. \tag{6.3.3}$$

(B) If $S_h(\epsilon) = +\infty$ *for all* $\epsilon > 0$*, then*

$$\limsup_{n\to\infty} \|U_h(n)\| = +\infty, \quad a.s. \tag{6.3.4}$$

(C) If $S_h(\epsilon) < +\infty$ *for all* $\epsilon > \epsilon'$*, and* $S_h(\epsilon) = +\infty$ *for all* $\epsilon < \epsilon'$*, then there exist deterministic* $0 < c_1 \leq c_2 < +\infty$ *such that*

$$c_1 \leq \limsup_{n\to\infty} \|U_h(n)\| \leq c_2 < +\infty, \quad a.s. \tag{6.3.5}$$

This result enables us to classify the asymptotic behaviour of the discretisation of the d–dimensional affine stochastic differential equation

$$dY(t) = AY(t)\,dt + \Sigma(t)\,dB(t), \quad t \geq 0; \quad X(0) = \zeta, \tag{6.3.6}$$

where A is a $d \times d$ matrix with real entries. We assume that all solutions of the underlying deterministic differential equation

$$y'(t) = Ay(t), \quad t > 0, \quad x(0) = \zeta \tag{6.3.7}$$

obey $y(t) \to 0$ as $t \to \infty$. This means that

$$\text{Re}(\lambda) < 0 \text{ for all eigenvalues } \lambda \text{ of } A. \tag{6.3.8}$$

Let c_A be the characteristic polynomial of A, so that $c_A(\lambda) = \det(\lambda I_d - A)$. By (6.3.8), it follows that there are no positive real solutions of the characteristic equation $c_A(\lambda) = 0$. In particular, $c_A(1/h) \neq 0$ for every $h > 0$, so we have that $\det(I - hA) \neq 0$ and therefore the matrix $C(h)$ given by

$$C(h) = (I - hA)^{-1} \tag{6.3.9}$$

is well–defined. Therefore, there is a unique solution of the split–step scheme

$$Y_h(0) = \zeta, \tag{6.3.10a}$$

$$Y_h^\star(n) = Y_h(n) + hAY_h^\star(n), \quad n \geq 0, \tag{6.3.10b}$$

$$Y_h(n+1) = Y_h^\star(n) + \sqrt{h}\sigma_h(n)\xi(n+1), \quad n \geq 0. \tag{6.3.10c}$$

which is equivalent to

$$Y_h(n+1) = C(h)Y_h(n) + \sqrt{h}\sigma_h(n)\xi(n+1), \quad n \geq 0; \quad Y_h(0) = \zeta.$$

The asymptotic behaviour of Y_h can now be given.

Theorem 6.3.1. *Suppose that* $A \in \mathbb{R}^{d\times d}$ *obeys* (6.3.8). *Let* $\xi = \{\xi(n) \in \mathbb{R}^r : n \in \mathbb{N}\}$ *be a sequence of random vectors obeying Assumption 6.2.1. Let* $S_h(\epsilon)$ *be defined by* (6.3.2), *and* $(Y_h, Y_h^\star)$ *be the unique solution of* (6.3.10).

(A) If $S_h(\epsilon) < +\infty$ *for every* $\epsilon > 0$, *then* $Y_h(n) \to 0$ *as* $n \to \infty$ *a.s.*

(B) If there exists $\epsilon' > 0$ *such that* $S_h(\epsilon) < +\infty$ *for all* $\epsilon > \epsilon'$ *and* $S_h(\epsilon) = +\infty$ *for all* $\epsilon < \epsilon'$, *then there exist deterministic* $0 < c_3 \leq c_4 < +\infty$ *such that*

$$c_3 \leq \limsup_{n\to\infty} \|Y_h(n)\| \leq c_4, \quad a.s.$$

and

$$\liminf_{n\to\infty} \|Y_h(n)\| = 0, \quad \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \|Y_h(j)\|^2 = 0, \quad a.s.$$

(C) If $S_h(\epsilon) = +\infty$ *for all* $\epsilon > 0$, *then* $\limsup_{n\to\infty} \|Y_h(n)\| = +\infty$ *a.s.*

6.3.2 Nonlinear equation

We now discuss the asymptotic behaviour of solutions of (6.2.3). We first show that X_h has a zero limit in the case when σ_h is square summable, without placing any condition on f stronger than (4.1.9). This is a direct analogue of results available in continuous time.

Theorem 6.3.2. *Suppose that* $(X_h, X_h^\star)$ *is a solution of* (6.2.3). *Suppose* f *obeys* (4.1.9), $\sigma_h \in \ell^2(\mathbb{N}, \mathbb{R})$, *and that the sequence* ξ *obeys Assumption 6.2.1. Then* $\lim_{n\to\infty} X_h(n) = 0$, *a.s.*

We show that when U_h is unbounded, so is $\|X_h\|$, and also that if U_h is bounded, $\|X_h\|$ is bounded away from zero by a deterministic constant.

Theorem 6.3.3. *Suppose that* $(X_h, X_h^\star)$ *is a solution of* (6.2.3). *Suppose that* f *obeys* (4.1.9) *and that the sequence* ξ *obeys Assumption 6.2.1. Let* $S_h(\epsilon)$ *be defined by* (6.3.2).

(A) If $S_h(\epsilon) = +\infty$ *for every* $\epsilon > 0$, *then*

$$\limsup_{n\to\infty} \|X_h(n)\| = +\infty, \quad a.s.$$

(B) If $S_h(\epsilon) < +\infty$ *for all* $\epsilon > \epsilon'$ *and* $S_h(\epsilon) = +\infty$ *for all* $\epsilon < \epsilon'$, *then*

$$\limsup_{n\to\infty} \|X_h(n)\| \geq \frac{c_1}{2}, \quad a.s.,$$

where c_1 *is defined by* (6.3.5).

Theorem 6.3.4. *Suppose that* $(X_h, X_h^\star)$ *is a solution of* (6.2.3). *Suppose that* f *obeys* (4.1.9) *and that the sequence* ξ *obeys Assumption 6.2.1. Let* $S_h(\epsilon)$ *be defined by* (6.3.2).

(i) If $S_h(\epsilon) < +\infty$ *for every* $\epsilon > 0$, *then*

$$\{\lim_{n\to\infty} \|X_h(n)\| = 0\} \cup \{\lim_{n\to\infty} \|X_h(n)\| = +\infty\} \text{ is an a.s. event.}$$

(ii) If $\lim_{n\to\infty} X_h(n) = 0$ *with positive probability, then* $S_h(\epsilon) < +\infty$ *for every* $\epsilon > 0$.

Under an additional mean–reverting condition on f, we can characterise the conditions on σ_h under which solutions of (6.2.3) tend to zero.

Theorem 6.3.5. *Suppose that* $(X_h, X_h^\star)$ *is a solution of* (6.2.3). *Suppose that* f *obeys* (4.1.9) *and*

$$\liminf_{y\to\infty} \inf_{\|x\|=y} \langle x, f(x)\rangle =: \phi > 0. \tag{6.3.11}$$

and that the sequence ξ *obeys Assumption 6.2.1.*

(A) $S_h(\epsilon)$ *defined by* (6.3.2) *obeys* $S_h(\epsilon) < +\infty$ *for every* $\epsilon > 0$;

(B) $\lim_{n\to\infty} X_h(n) = 0$ *a.s. for all* $\zeta \in \mathbb{R}^d$;

(C) $\lim_{n\to\infty} X_h(n) = 0$ *with positive probability for some* $\zeta \in \mathbb{R}^d$*;*

Furthermore, in the scalar case, we can characterise the stability of the equilibrium without requiring to assume (6.3.11). In fact, it suffices to just assume that f obeys (4.1.9).

Theorem 6.3.6. *Suppose that* $(X_h, X_h^\star)$ *is a solution of* (5.1.2) *and* (5.1.3)*. Suppose that* f *obeys* (4.1.9) *and* $S_h(\epsilon) = \sum_{n=0}^{\infty}\{1-\Phi(\epsilon/|\sigma_h(n)|)\} < +\infty$ *for all* $\epsilon > 0$*. Then*

$$\lim_{n\to\infty} X_h(n,\zeta) = 0 \quad a.s. \text{ for all } \zeta \in \mathbb{R}.$$

The next result enables us to completely classify the asymptotic behaviour of the solutions of (6.2.3). In order to do so, we must strengthen once again the mean–reverting hypothesis on f.

Theorem 6.3.7. *Suppose that* $(X_h, X_h^\star)$ *is a solution of* (6.2.3)*. Suppose that* f *obeys* (4.1.9) *and*

$$\liminf_{y\to\infty} \inf_{\|x\|=y} \frac{\langle x, f(x)\rangle}{\|x\|} = +\infty, \tag{6.3.12}$$

and that the sequence ξ *obeys Assumption 6.2.1. Let* $S_h(\epsilon)$ *be defined by* (6.3.2)*.*

(A) If $S_h(\epsilon) < +\infty$ *for all* $\epsilon > 0$*, then* $\lim_{n\to\infty} X_h(n) = 0$ *a.s.*

(B) If $S_h(\epsilon) < +\infty$ *for all* $\epsilon > \epsilon'$ *and* $S_h(\epsilon) = +\infty$ *for all* $\epsilon < \epsilon'$*, then there exists deterministic* $0 < c_3 \le c_4 < +\infty$ *such that*

$$c_3 < \limsup_{n\to\infty} \|X_h(n)\| \le c_4, \quad a.s.$$

and

$$\liminf_{n\to\infty} \|X_h(n)\| = 0, \quad a.s.$$

(C) If $S_h(\epsilon) = +\infty$ *for all* $\epsilon > 0$*, then* $\limsup_{n\to\infty} \|X_h(n)\| = +\infty$ *a.s.*

This necessary and sufficient condition on $S_h(\epsilon)$ is difficult to evaluate directly, because we do not know Φ in its closed form. However we can show that $S_h(\epsilon)$ is finite or infinite according as to whether the sum

$$S'_h(\epsilon) = \sum_{n=0}^{\infty} \|\sigma_h(n)\|_F \exp\left(-\frac{\epsilon^2}{2}\frac{1}{\|\sigma_h(n)\|_F^2}\right) \tag{6.3.13}$$

is finite or infinite, we interpret the summand to be zero in the case where $\|\sigma_h(n)\|_F = 0$. Therefore we establish the following Lemmata which enables us to obtain all the above results with $S'_h(\epsilon)$ in place of $S_h(\epsilon)$.

Lemma 6.3.2. *$S_h(\epsilon)$ given by (6.3.2) is finite if and only if $S'_h(\epsilon)$ given by (6.3.13) is finite.*

Proof. We note by e.g., [44, Problem 2.9.22], we have (2.2.16). If $S_h(\epsilon)$ is finite, then $1-\Phi(\epsilon/\|\sigma_h(n)\|_F) \to 0$ as $n\to\infty$. This implies $\epsilon/\|\sigma_h(n)\|_F \to \infty$ as $n \to \infty$. Therefore by (2.2.16), we have

$$\lim_{n\to\infty} \frac{1-\Phi(\epsilon/\|\sigma_h(n)\|_F)}{\|\sigma_h(n)\|_F/\epsilon \cdot \exp(-\epsilon^2/\{2\|\sigma_h(n)\|_F^2\})} = \frac{1}{\sqrt{2\pi}}. \tag{6.3.14}$$

Since $(1-\Phi(\epsilon/\|\sigma_h(n)\|_F))_{n\geq 1}$ is summable, it therefore follows that the sequence

$$\left(\|\sigma_h(n)\|_F/\epsilon \cdot \exp(-\epsilon^2/\{2\|\sigma_h(n)\|_F^2(n)\})\right)_{n\geq 1}$$

is summable, so $S'_h(\epsilon)$ is finite, by definition.

On the other hand, if $S'(\epsilon)$ is finite, and we define $\phi : [0,\infty) \to \mathbb{R}^d$ by

$$\phi(x) = \begin{cases} x\exp(-1/(2x^2)), & x > 0, \\ 0, & x = 0, \end{cases}$$

then we have $\|\sigma_h(n)\|_F \exp(-\epsilon^2/2\|\sigma_h(n)\|_F^2(n))$ is summable, hence $(\phi(\|\sigma_h(n)\|_F/\epsilon))_{n\geq 1}$ is summable. Therefore $\phi(\|\sigma_h(n)\|_F/\epsilon) \to 0$ as $n\to\infty$. Then, as ϕ is continuous and increasing on $[0,\infty)$, we have that $\|\sigma_h(n)\|_F/\epsilon \to 0$ as $n\to\infty$, or $\epsilon/\|\sigma_h(n)\|_F \to \infty$ as $n\to\infty$. Therefore (6.3.14) holds, and thus $(1-\Phi(\epsilon/\|\sigma_h(n)\|_F))_{n\geq 1}$ is summable, which implies that $S_h(\epsilon)$ is finite, as required. □

6.3.3 Connection with continuous results

To see how these results mimic the asymptotic behaviour of (6.2.2) and (6.3.6), we record corresponding result for solutions of these equations. To this end, we define

$$S(\epsilon) = \sum_{n=0}^{\infty} \left\{ 1 - \Phi\left(\frac{\epsilon}{\sqrt{\int_n^{n+1} \|\Sigma(t)\|_F^2 \, dt}} \right) \right\} \tag{6.3.15}$$

and for $h > 0$

$$S_h^{(c)}(\epsilon) = \sum_{n=0}^{\infty} \left\{ 1 - \Phi\left(\frac{\epsilon}{\sqrt{\frac{1}{h}\int_{nh}^{(n+1)h} \|\Sigma(t)\|_F^2 \, dt}} \right) \right\}. \tag{6.3.16}$$

Perusal of the proof of Theorem 2.2.3 in Chapter 2 shows that$S(\cdot)$ above can be replaced by S_h^c. The result therefore is

Theorem 6.3.8. *Suppose that $A \in \mathbb{R}^{d\times d}$ obeys (6.3.8). Let $h > 0$ and suppose that $S_h^{(c)}(\epsilon)$ be defined by (6.3.16). Let Y be the unique solution of (6.3.6).*

(A) If $S_h^{(c)}(\epsilon) < +\infty$ for every $\epsilon > 0$, then $Y(t) \to 0$ as $t \to \infty$ a.s.

(B) If there exists $\epsilon' > 0$ such that $S_h^{(c)}(\epsilon) < +\infty$ for all $\epsilon > \epsilon'$ and $S_h^{(c)}(\epsilon) = +\infty$ for all $\epsilon < \epsilon'$, then there exist deterministic $0 < c_1 \le c_2 < +\infty$ such that

$$c_1 \le \limsup_{t\to\infty} \|Y(t)\| \le c_2, \quad a.s.$$

and

$$\liminf_{t\to\infty} \|Y(t)\| = 0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t \|Y(s)\|^2 \, ds = 0, \quad a.s.$$

(C) If $S_h^{(c)}(\epsilon) = +\infty$ for all $\epsilon > 0$, then $\limsup_{t\to\infty} \|Y(t)\| = +\infty$ a.s.

Similarly, we may replace S by S_h^c in Theorem 4.2.6 in Chapter 4 to get

Theorem 6.3.9. *Suppose that f obeys (4.1.9) and (6.3.12). Suppose that X is a solution of (6.2.2). Let $h > 0$ and suppose that $S_h^{(c)}(\epsilon)$ be defined by (6.3.16).*

(A) If $S_h^{(c)}(\epsilon) < +\infty$ for all $\epsilon > 0$, then $\lim_{n\to\infty} X(t) = 0$ a.s.

(B) If $S_h^{(c)}(\epsilon) < +\infty$ *for all* $\epsilon > \epsilon'$ *and* $S_h^{(c)}(\epsilon) = +\infty$ *for all* $\epsilon < \epsilon'$*, then there exist deterministic* $0 < c_3 \leq c_4 < +\infty$ *such that*

$$c_3 < \limsup_{t\to\infty} \|X(t)\| < c_4, \quad a.s.$$

and

$$\liminf_{t\to\infty} \|X(t)\| = 0, \quad a.s.$$

(C) If $S_h^{(c)}(\epsilon) = +\infty$ *for all* $\epsilon > 0$*, then* $\limsup_{n\to\infty} \|X(t)\| = \infty$ *a.s.*

If we take a uniform step size $h > 0$ in a forward Euler–discretisation of (6.2.2), this is tantamount to setting

$$\sigma_h(n) = \Sigma(nh), \quad n \geq 0 \tag{6.3.17}$$

in (6.2.3). In this case, the continuity of Σ ensures for each fixed n that

$$\lim_{h\to 0} \left\{ \frac{1}{h} \int_{nh}^{(n+1)h} \|\Sigma(s)\|_F^2 \, ds - \|\sigma_h(n)\|_F^2 \right\} = 0,$$

so it can be seen that the conditions classifying the finiteness S_h and S_h^c are in some sense "close". We now give some examples where S_h and S_h^c share the same finiteness properties, and therefore, the asymptotic behaviour of solutions of (6.2.2) and (6.2.3) coincide.

In the case when the integral $\int_a^b \Sigma_{ij}^2(s)\, ds$ can be computed explicitly for any $0 \leq a < b < +\infty$ and $(i,j) \in \{1,\ldots,d\} \times \{1,\ldots,r\}$, it is reasonable to approximate the stochastic integral

$$\int_{nh}^{(n+1)h} \Sigma_{ij}(s)\, dB_j(s) \text{ by } \left(\int_{nh}^{(n+1)h} \Sigma_{ij}^2(s)\, ds \right)^{1/2} \xi_j(n+1)$$

where ξ obeys Assumption 6.2.1. This is because the two random variables displayed above have the same distribution. In terms of (6.2.3) (particularly (6.2.3c)) this amounts to choosing σ_h according to

$$[\sigma_h(n)]_{ij} = \frac{1}{\sqrt{h}} \left(\int_{nh}^{(n+1)h} \Sigma_{ij}^2(s)\, ds \right)^{1/2}, \quad n \geq 0, \quad (i,j) \in \{1,\ldots,d\} \times \{1,\ldots,r\}. \tag{6.3.18}$$

In this case, it is seen that $S_h(\epsilon) = S_h^c(\epsilon)$. Applying Theorems 6.3.7 and 6.3.9, we immediately have the following result.

Theorem 6.3.10. *Suppose that f obeys* (4.1.9) *and* (6.3.12) *and suppose that Σ obeys* (6.2.1). *Assume that the sequence ξ obeys Assumption 6.2.1, and for $h > 0$ that f obeys Assumption 6.2.2. Let X be a solution of* (6.2.2) *and $(X_h, X_h^\star)$ is a solution of* (6.2.3). *Then exactly one of the events*

$$\{\omega : \lim_{t\to\infty} X(t,\omega) = 0\}, \quad \{\omega : 0 < \limsup_{t\to\infty} \|X(t,\omega)\| < +\infty, \liminf_{t\to\infty} \|X(t,\omega)\| = 0\},$$
$$\text{and } \{\omega : \limsup_{t\to\infty} \|X(t,\omega)\| = +\infty\}$$

is almost sure, and exactly one of the events

$$\{\omega : \lim_{n\to\infty} X_h(n,\omega) = 0\}, \quad \{\omega : 0 < \limsup_{n\to\infty} \|X_h(n,\omega)\| < +\infty, \liminf_{n\to\infty} \|X_h(n,\omega)\| = 0\},$$
$$\text{and } \{\omega : \limsup_{n\to\infty} \|X_h(n,\omega)\| = +\infty\}$$

is almost sure.

If σ_h is given by (6.3.18), *and $n \mapsto \int_{nh}^{(n+1)h} \Sigma_{ij}^2(s)\,ds$ can be computed exactly for all $(i,j) \in \{1,\dots,d\} \times \{1,\dots,r\}$ and all $n \in \mathbb{N}$, we have the following equivalences:*

(i) $\lim_{t\to\infty} X(t) = 0$ *a.s., if and only if* $\lim_{n\to\infty} X_h(n) = 0$ *a.s.*

(ii) $\limsup_{t\to\infty} \|X(t)\| \in (0,\infty)$ *a.s., if and only if* $\limsup_{n\to\infty} \|X_h(n)\| \in (0,\infty)$ *a.s.*

(iii) $\limsup_{t\to\infty} \|X(t)\| = +\infty$ *a.s., if and only if* $\limsup_{n\to\infty} \|X_h(n)\| = +\infty$ *a.s.*

We next consider a situation where finiteness conditions on $S_h(\epsilon)$ and $S_h^c(\epsilon)$ also coincide, but in which we do not need to have a closed–form expression for $\int_a^b \Sigma_{ij}^2(s)\,ds$. This is the case when $t \mapsto \|\Sigma(t)\|_F^2$ is decreasing and $\sigma_h(n) = \Sigma(nh)$.

Theorem 6.3.11. *Suppose that f obeys* (4.1.9) *and* (6.3.12) *and suppose that Σ obeys* (6.2.1). *Assume that the sequence ξ obeys Assumption 6.2.1, and for $h > 0$ that f obeys Assumption 6.2.2. Let X be a solution of* (6.2.2) *and $(X_h, X_h^\star)$ is a solution of* (6.2.3).

Then exactly one of the events

$$\{\omega : \lim_{t\to\infty} X(t,\omega) = 0\}, \quad \{\omega : 0 < \limsup_{t\to\infty} \|X(t,\omega)\| < +\infty, \liminf_{t\to\infty} \|X(t,\omega)\| = 0\},$$

$$\text{and } \{\omega : \limsup_{t\to\infty} \|X(t,\omega)\| = +\infty\}$$

is almost sure, and exactly one of the events

$$\{\omega : \lim_{n\to\infty} X_h(n,\omega) = 0\}, \quad \{\omega : 0 < \limsup_{n\to\infty} \|X_h(n,\omega)\| < +\infty, \liminf_{n\to\infty} \|X_h(n,\omega)\| = 0\},$$

$$\text{and } \{\omega : \limsup_{n\to\infty} \|X_h(n,\omega)\| = +\infty\}$$

is almost sure.

If we further suppose that $t \mapsto \|\Sigma(t)\|_F^2$ *is non–increasing, and* $\sigma_h(n)$ *is given by* (6.3.17), *we have the following equivalences:*

(i) $\lim_{t\to\infty} X(t) = 0$ *a.s., if and only if* $\lim_{n\to\infty} X_h(n) = 0$ *a.s.*

(ii) $\limsup_{t\to\infty} \|X(t)\| \in (0,\infty)$ *a.s., if and only if* $\limsup_{n\to\infty} \|X_h(n)\| \in (0,\infty)$ *a.s.*

(iii) $\limsup_{t\to\infty} \|X(t)\| = +\infty$ *a.s., if and only if* $\limsup_{n\to\infty} \|X_h(n)\| = +\infty$ *a.s.*

Proof. Define $\vartheta_h(n)^2 = \int_{nh}^{(n+1)h} \|\Sigma(t)\|_F^2\, dt/h$. Since $t \mapsto \|\Sigma(t)\|_F^2$ is non–increasing, for $t \in [nh,(n+1)h]$ we have $\|\Sigma((n+1)h)\|_F^2 \le \|\Sigma(t)\|_F^2 \le \|\Sigma(nh)\|_F^2$. Therefore integrating over $[nh,(n+1)h]$ and using (6.3.17) we get $\|\sigma_h(n+1)\|_F \le \vartheta_h(n) \le \|\sigma_h(n)\|_F$. For $\epsilon > 0$, as Φ is increasing, we have

$$1 - \Phi\left(\frac{\epsilon}{\|\sigma_h(n+1)\|_F}\right) \le 1 - \Phi\left(\frac{\epsilon}{\|\vartheta_h(n)\|_F}\right) \le 1 - \Phi\left(\frac{\epsilon}{\|\sigma_h(n)\|_F}\right).$$

Summing across this inequality and using the definitions (6.3.2) and (6.3.16) we get

$$S_h(\epsilon) - \left\{1 - \Phi\left(\frac{\epsilon}{\|\sigma_h(0)\|_F}\right)\right\} \le S_h^{(c)}(\epsilon) \le S_h(\epsilon).$$

Therefore, for any $\epsilon > 0$, $S_h(\epsilon)$ is finite if and only if $S_h^{(c)}(\epsilon)$ is finite.

We now prove the equivalence (i). Suppose that $\lim_{t\to\infty} X(t) = 0$ a.s. Then, as $S_h^{(c)}(\epsilon)$ must be (i) finite for all $\epsilon > 0$; (ii) infinite for all $\epsilon > 0$; or (iii) finite for all $\epsilon > \epsilon'$ and infinite for all $\epsilon < \epsilon'$ for some $\epsilon' > 0$, it follows from Theorem 6.3.9 that $S_h^{(c)}(\epsilon) < +\infty$ for all $\epsilon > 0$. Therefore, we have that $S_h(\epsilon) < +\infty$ for all $\epsilon > 0$. Theorem 6.3.7 now implies that $X_h(n) \to 0$ as $n \to \infty$ a.s.

Conversely, suppose that $X_h(n) \to 0$ as $n \to \infty$ a.s. Since $S_h(\epsilon)$ must be (i) finite for all $\epsilon > 0$; (ii) infinite for all $\epsilon > 0$; or (iii) finite for all $\epsilon > \epsilon'$ and infinite for all $\epsilon < \epsilon'$ for some $\epsilon' > 0$, it follows from Theorem 6.3.7 that $S_h(\epsilon) < +\infty$ for all $\epsilon > 0$. Therefore, we have that $S_h^{(c)}(\epsilon) < +\infty$ for all $\epsilon > 0$, and hence by Theorem 6.3.9, $X(t) \to 0$ as $t \to \infty$ a.s., completing the proof of (i).

The proof of the equivalences (ii) and (iii) are similar, and hence omitted. □

The condition that $S_h'(\epsilon)$ is finite or infinite can be difficult to check. However we can provide a sufficient condition on which each case of $S_h'(\epsilon)$ being finite all the time, sometime finite sometime infinite and infinite all the time is possible according to whether $\lim_{t\to+\infty} \|\sigma_h(n)\|_F^2 \log n$ being zero, non-zero and finite, or infinite. Therefore the asymptotic behaviour of the solution of (3.1.1) can be classified completely.

Lemma 6.3.3. *Define* $\lim_{n\to\infty} \|\sigma_h(n)\|_F^2 \log n = L \in [0,\infty]$*, then we have the following:*

(A) If $L = 0$*, then* $S_h'(\epsilon) < +\infty$ *for all* $\epsilon > 0$*;*

(B) If $L \in (0,+\infty)$*, then* $S_h'(\epsilon) < +\infty$ *for all* $\epsilon > \epsilon'$ *and* $S_h'(\epsilon) = +\infty$ *for all* $\epsilon < \epsilon'$*;*

(C) If $L = +\infty$*, then* $S_h'(\epsilon) = +\infty$ *for all* $\epsilon > 0$

Proof. Notice from e.g., [44, Problem 2.9.22], $\lim_{x\to\infty}(1-\Phi(x))/(x^{-1}e^{-x^2/2}) = 1/\sqrt{2\pi}$. Therefore we have

$$\lim_{x\to\infty} \log(1-\Phi(x)) + \log x + x^2/2 = \log(1/\sqrt{2\pi}),$$

hence

$$\lim_{x\to\infty} \frac{\log(1-\Phi(x))}{x^2/2} = -1.$$

Let $x = \epsilon/\|\sigma_h(n)\|_F \to \infty$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{\log(1-\Phi(\epsilon/\|\sigma_h(n)\|_F))}{\epsilon^2/2\|\sigma_h(n)\|_F} = -1.$$

Moreover,

$$\begin{aligned}\lim_{n\to\infty} \frac{\log(1-\Phi(\epsilon/\|\sigma_h(n)\|_F))}{\log n} &= \lim_{n\to\infty} \frac{\log(1-\Phi(\epsilon/\|\sigma_h(n)\|_F))}{\epsilon^2/2\|\sigma_h(n)\|_F} \cdot \frac{\epsilon^2/2\|\sigma_h(n)\|_F}{\log n} \\ &= -\frac{\epsilon^2}{2} \lim_{n\to\infty} \frac{1}{\|\sigma_h(n)\|_F \log n}\end{aligned}$$

If $L = 0$, then

$$\lim_{n\to\infty} \frac{\log(1-\Phi(\epsilon/\|\sigma_h(n)\|_F))}{\log n} \to -\infty.$$

Therefore there exists an $N(\epsilon)$, such that for $n > N(\epsilon)$

$$\log(1-\Phi(\epsilon/\|\sigma_h(n)\|_F)) < -2\log n$$

$$1-\Phi(\epsilon/\|\sigma_h(n)\|_F) \le n^{-2} \to 0 \quad \text{as } n \to \infty$$

This implies that $S_h(\epsilon) < +\infty$, which implies $S'_h(\epsilon) < +\infty$ by Lemma 6.3.2 proving part (A).

If $L \in (0, +\infty)$, we have

$$\lim_{n\to\infty} \frac{\log(1-\Phi(\epsilon/\|\sigma_h(n)\|_F))}{\log n} = \frac{-\epsilon^2}{2L}.$$

Therefore either $\epsilon > \sqrt{2L}$, in which case $\lim_{n\to\infty} 1-\Phi(\epsilon/\|\sigma_h(n)\|_F) = 0$, hence $S_h(\epsilon) < +\infty$, and $S'_h(\epsilon) < +\infty$. Or $\epsilon < \sqrt{2L}$, in which case $1-\Phi(\epsilon/\|\sigma_h(n)\|_F)$ is not going to zero, hence not summable, therefore $S_h(\epsilon) = +\infty$ which implies $S'_h(\epsilon) = +\infty$.

Finally, if $L = +\infty$, suppose that $S_h(\epsilon) < +\infty$, then

$$\lim_{n\to\infty} \frac{\log(1-\Phi(\epsilon/\|\sigma_h(n)\|_F))}{\log n} = 0.$$

Then for all $\epsilon > 0$, there exists an $N(\epsilon) > 0$ such that

$$\frac{\log(1 - \Phi(\epsilon/\|\sigma_h(n)\|_F))}{\log n} > -1/2$$

$$\log(1 - \Phi(\epsilon/\|\sigma_h(n)\|_F)) > -1/2 \log n = \log n^{-1/2}$$

$$1 - \Phi(\epsilon/\|\sigma_h(n)\|_F) > n^{-1/2} \quad \text{for all } n \geq N(\epsilon)$$

This implies $S_h(\epsilon) = +\infty$, which is a contradiction, hence the required result, completing the proof. □

6.4 Preliminary Results

In this section, we deduce some simple preliminary facts about (6.2.3) contingent on a solution $(X_h, X_h^\star)$ existing. We also present some results on the asymptotic behaviour of martingales that will be of utility in the sequel.

6.4.1 Estimates and representation

In our next result, we obtain a representation for $\|X_h(n)\|^2$.

Lemma 6.4.1. *Suppose $(X_h, X_h^\star)$ is a solution of (6.2.3). Then*

$$\|X_h(n)\|^2 = \|X_h(0)\|^2 - 2\sum_{i=1}^{n} h\langle f(X_h^\star(i-1)), X_h^\star(i-1)\rangle + \sum_{i=1}^{n} h\|\sigma_h(i-1)\xi(i)\|^2 - \sum_{i=1}^{n} h^2\|f(X_h^\star(i-1))\|^2 + M(n), \quad n \geq 1, \quad (6.4.1)$$

where

$$Y^{(j)}(n) = 2\sqrt{h}\sum_{k=1}^{d}[X_h^\star(n)]_k[\sigma_h(n)]_{kj}, \quad j = 1, \ldots, r, \quad n \geq 1, \quad (6.4.2)$$

$$M(n) = \sum_{i=1}^{n}\sum_{j=1}^{r} Y^{(j)}(i-1)\xi^{(j)}(i), \quad n \geq 1. \quad (6.4.3)$$

Proof. Notice that with $Y^{(j)}$ as defined in (6.4.2) and M as defined in (6.4.3), we have

$$\begin{aligned}M(n) &= \sum_{i=1}^{n}\sum_{j=1}^{r}\left(2\sqrt{h}\sum_{k=1}^{d}[X_h^\star(i-1)]_k[\sigma_h(i-1)]_{kj}\right)\xi^{(j)}(i)\\ &= \sum_{i=1}^{n}\sum_{k=1}^{d}2\sqrt{h}[X_h^\star(i-1)]_k\sum_{j=1}^{r}[\sigma_h(i-1)]_{kj}\xi^{(j)}(i)\\ &= \sum_{i=1}^{n}\sum_{k=1}^{d}2\sqrt{h}[X_h^\star(i-1)]_k[\sigma_h(i-1)\xi(i)]_k,\end{aligned}$$

so that M defined by (6.4.3) obeys

$$M(n) = 2\sqrt{h}\sum_{i=1}^{n}\langle X_h^\star(i-1), \sigma_h(i-1)\xi(i)\rangle, \quad n \geq 1. \tag{6.4.4}$$

Next, we rewrite (6.2.3b) according to $X_h(n) = X_h^\star(n) + hf(X_h^\star(n))$. Then

$$\|X_h(n)\|^2 = \|X_h^\star(n)\|^2 + 2h\langle f(X_h^\star(n)), X_h^\star(n)\rangle + h^2\|f(X_h^\star(n))\|^2. \tag{6.4.5}$$

From (6.2.3c), for $n \geq 0$ we get

$$\|X_h(n+1)\|^2 = \|X_h^\star(n)\|^2 + h\|\sigma_h(n)\xi(n+1)\|^2 + 2\sqrt{h}\langle X_h^\star(n), \sigma_h(n)\xi(n+1)\rangle,$$

so by using (6.4.5) we get

$$\begin{aligned}\|X_h(n+1)\|^2 = \|X_h(n)\|^2 &- 2h\langle f(X_h^\star(n)), X_h^\star(n)\rangle - h^2\|f(X_h^\star(n))\|^2\\ &+ h\|\sigma_h(n)\xi(n+1)\|^2 + 2\sqrt{h}\langle X_h^\star(n), \sigma_h(n)\xi(n+1)\rangle. \end{aligned}\tag{6.4.6}$$

Therefore for $n \geq 1$, by summing on both sides, and using (6.4.4) we have

$$\begin{aligned}\|X_h(n)\|^2 = \|X_h(0)\|^2 &+ \sum_{i=1}^{n} h\left\{-2\langle f(X_h^\star(i-1)), X_h^\star(i-1)\rangle + \|\sigma_h(i-1)\xi(i)\|^2\right\}\\ &- \sum_{i=1}^{n} h^2\|f(X_h^\star(i-1))\|^2 + M(n),\end{aligned}$$

where M is defined in (6.4.3), as claimed. □

6.4.2 A result on the asymptotic behaviour of martingales

We prove now a useful lemma on the asymptotic behaviour of a martingale built from ξ and sequences adapted to its natural filtration. It is based on a result of Bramson, Questel and Rosenthal [28, Theorem 1.1].

Lemma 6.4.2. *Let $M = \{M(n) : n \geq 1\}$ be a martingale with respect to the filtration $(\mathcal{F}(n))_{n\geq 0}$ of σ–fields on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that*

$$M(n) = \sum_{i=1}^{n} Y(i), \quad n \geq 1.$$

If there exists a constant $K \in [1, \infty)$ such that

$$\mathbb{E}[Y(n)^2|\mathcal{F}(n-1)] \leq K\mathbb{E}[|Y(n)||\mathcal{F}(n-1)]^2, \quad \textit{a.s. for all } n \geq 1, \tag{6.4.7}$$

then

$$\{\omega : \lim_{n\to\infty} M(n,\omega) \textit{ exists and is finite}\} \\ \cup \{\omega : \liminf_{n\to\infty} M(n,\omega) = -\infty, \quad \limsup_{n\to\infty} M(n,\omega) = +\infty\} \textit{ is an a.s. event} \tag{6.4.8}$$

We now prove a consequence of Lemma 6.4.2.

Lemma 6.4.3. *Suppose that ξ obeys Assumption 6.2.1. Suppose that $Y^{(j)} = \{Y^{(j)}(n) : n \geq 0\}$ for $j = 1, \ldots, r$ are sequences of $\mathcal{F}^\xi(n)$–measurable random variables. Define $M = \{M(n) : n \geq 1\}$*

$$M(n) = \sum_{i=1}^{n} \sum_{j=1}^{r} Y^{(j)}(i-1)\xi^{(j)}(i), \quad n \geq 1. \tag{6.4.9}$$

Then M obeys (6.4.8).

Proof of Lemma 6.4.3. Define

$$Y(n) = \sum_{j=1}^{r} Y^{(j)}(n-1)\xi^{(j)}(n), \quad n \geq 1.$$

Since $Y^{(j)}(n-1)$ is $\mathcal{F}^\xi(n-1)$ measurable, and ξ obeys Assumption 6.2.1, it follows that

$$\mathbb{E}\left[Y(n)^2|\mathcal{F}^\xi(n-1)\right] = \sum_{j=1}^{r} Y^{(j)}(n-1)^2 =: \varsigma^2(n).$$

Next, we recall that if Z is a normal random variable with mean zero and variance c^2, then

$$\mathbb{E}[|Z|]^2 = \frac{1}{2\pi}c^2.$$

Since $\xi^{(j)}(n)$ for $j = 1, \ldots, r$ are independent standard normal random variables, and $Y^{(j)}(n-1)$ is $\mathcal{F}^\xi(n-1)$ measurable, it follows that, conditional on $\mathcal{F}^\xi(n-1)$, $Y(n)$ is normally distributed with zero mean and variance $\varsigma^2(n)$. Therefore

$$\mathbb{E}\left[|Y(n)||\mathcal{F}^\xi(n-1)\right]^2 = \frac{1}{2\pi}\varsigma^2(n) = \frac{1}{2\pi}\mathbb{E}\left[Y(n)^2|\mathcal{F}^\xi(n-1)\right],$$

so (6.4.7) holds with $K = 2\pi$. Therefore all the hypotheses of Lemma 6.4.2 apply to M, and so we have the claimed conclusion (6.4.8). □

We employ one other result from the convergence theory of discrete process. It appears as Lemma 2 in [18].

Lemma 6.4.4. *Let $\{Z(n)\}_{n\in\mathbb{N}}$ be a non-negative $\mathcal{F}(n)$-measurable process, $\mathbb{E}|Z(n)| < \infty$ for all $n \in \mathbb{N}$ and*

$$Z(n+1) \le Z(n) + W(n) - V(n) + \nu(n+1), \quad n = 0, 1, 2, \ldots, \tag{6.4.10}$$

where $\{\nu(n)\}_{n\in\mathbb{N}}$ is an $\mathcal{F}(n)$-martingale–difference, $\{W(n)\}_{n\in\mathbb{N}}$, $\{V(n)\}_{n\in\mathbb{N}}$ are nonnegative $\mathcal{F}(n)$–measurable processes, $\mathbb{E}|W(n)| < +\infty$, $\mathbb{E}|V(n)| < +\infty$ for all $n \in \mathbb{N}$. Then

$$\left\{\omega : \sum_{n=1}^{\infty} W(n) < +\infty\right\} \subseteq \left\{\omega : \sum_{n=1}^{\infty} V(n) < +\infty\right\} \bigcap \{Z(n) \to\},$$

where $\{Z(n) \to\}$ denotes the set of all $\omega \in \Omega$ for which $\lim_{n\to\infty} Z(n,\omega)$ exists and is finite.

6.4.3 Proof of Lemma 6.3.1

For $n \geq 1$, we have $[U_h(n)]_i = \sqrt{h}\sum_{j=1}^{r}[\sigma_h(n-1)]_{ij}\xi_j(n+1)$. Hence $[U_h(n)]_i$ is normally distributed with mean zero and variance $\theta_i(n)^2 := h\sum_{j=1}^{r}[\sigma_h(n-1)]_{ij}^2$. Therefore,

$$\mathbb{P}[|[U_h(n)]_i| \geq \epsilon] = 1 - \Phi\left(\frac{\epsilon}{\theta_i(n)}\right). \tag{6.4.11}$$

Define $\theta^2(n) = \sum_{i=1}^{d}\theta_i(n)^2 = h\|\sigma_h(n-1)\|_F^2$. Since $\theta^2(n) \geq \theta_i(n)^2$ for each $i = 1,\ldots,d$, we have

$$\sum_{i=1}^{d}\left\{1 - \Phi\left(\frac{\epsilon}{\theta_i(n)}\right)\right\} \leq d\left(1 - \Phi\left(\frac{\epsilon}{\theta(n)}\right)\right).$$

Suppose, for each n, that $Z_i(n)$ for $i = 1,\ldots,d$ are independent standard normal random variables. Define $Z(n) = (Z_1(n), Z_2(n),\ldots,Z_d(n))$ and suppose that $(Z(n))_{n\geq 0}$ are a sequence of independent normal vectors. Define finally

$$X_i(n) = \theta_i(n)Z_i(n), \quad X(n) = \sum_{i=1}^{d}X_i(n), \quad n \geq 0.$$

Then we have that X_i is a zero mean normal with variance θ_i^2 and X is a zero mean normal with variance θ^2. Define $Z^*(n) = X(n)/\theta(n)$ is a standard normal random variable. Therefore we have that

$$\mathbb{P}[|X(n)| > \epsilon] = \mathbb{P}[|Z^*(n)| \geq \epsilon/\theta(n)] = 2\mathbb{P}[Z^*(n) \geq \epsilon/\theta(n)] = 2\left(1 - \Phi\left(\frac{\epsilon}{\theta(n)}\right)\right). \tag{6.4.12}$$

With $A_i(n) = \{|X_i(n)| \leq \epsilon/d\}$, $B(n) = \{\sum_{i=1}^{d}|X_i(n)| \leq \epsilon\}$, then $\cap_{i=1}^{d}A_i(n) \subseteq B(n)$, so

$$\mathbb{P}\left[|X(n)| > \epsilon\right] \leq \mathbb{P}[\overline{B}(n)] \leq \mathbb{P}\left[\overline{\cap_{i=1}^{d}A_i(n)}\right] = \mathbb{P}\left[\cup_{i=1}^{d}\overline{A_i(n)}\right] \leq \sum_{i=1}^{d}\mathbb{P}\left[\overline{A_i(n)}\right].$$

Since $X_i = \theta_i Z_i$, we have

$$\mathbb{P}\left[|X(n)| > \epsilon\right] \leq 2\sum_{i=1}^{d}\mathbb{P}\left[X_i(n) \geq \epsilon/d\right] = 2\sum_{i=1}^{d}\left\{1 - \Phi\left(\frac{\epsilon/d}{\theta_i(n)}\right)\right\}. \tag{6.4.13}$$

By (6.4.12) and (6.4.13), we get (2.5.7), i.e.,

$$1 - \Phi\left(\frac{\epsilon}{\theta(n)}\right) \leq \sum_{i=1}^{d}\left\{1 - \Phi\left(\frac{\epsilon/d}{\theta_i(n)}\right)\right\}.$$

Define $\|U_h(n)\|_1 = \sum_{i=1}^{d}|[U_h(n)]_i|$ for $n \geq 1$. Therefore, as $\|U_h(n)\|_1 \geq |[U_h(n)]_i|$, we have that $\mathbb{P}[\|U_h(n)\|_1 \geq \epsilon] \geq \mathbb{P}[|[U_h(n)]_i| \geq \epsilon]$ for each $i = 1,\ldots,d$. Therefore by (6.4.11) and

(2.5.7), we have

$$d\mathbb{P}[\|U_h(n)\|_1 \geq \epsilon] \geq \sum_{i=1}^{d} \mathbb{P}[|[U_h(n)]_i| \geq \epsilon] = \sum_{i=1}^{d} \left\{1 - \Phi\left(\frac{\epsilon}{\theta_i(n)}\right)\right\} \geq 1 - \Phi\left(\frac{d\epsilon}{\theta(n)}\right). \tag{6.4.14}$$

On the other hand, defining $A_i(j) = \{|[U_h(n)]_i| \leq \epsilon/d\}$ and $B(j) = \{\|U_h(n)\|_1 \leq \epsilon\}$, we see that $\cap_{i=1}^d A_i(n) \subseteq B(n)$. Then

$$\mathbb{P}[\|U_h(n)\|_1 \geq \epsilon] = \mathbb{P}[\overline{B(n)}] \leq \mathbb{P}\left[\overline{\cap_{i=1}^d A_i(n)}\right]$$
$$= \mathbb{P}\left[\cup_{i=1}^d \overline{A_i(n)}\right] \leq \sum_{i=1}^{d} \mathbb{P}\left[|[U_h(n)]_i| \geq \epsilon/d\right].$$

Hence by (6.4.11) and (2.5.4) we get

$$\mathbb{P}[\|U_h(n)\|_1 \geq \epsilon] \leq \sum_{i=1}^{d} \mathbb{P}\left[|[U_h(n)]_i| \geq \epsilon/d\right] = \sum_{i=1}^{d} \left\{1 - \Phi\left(\frac{\epsilon/d}{\theta_i(n)}\right)\right\}$$
$$\leq d\left(1 - \Phi\left(\frac{\epsilon/d}{\theta(n)}\right)\right). \tag{6.4.15}$$

Part (A). Suppose $S_h(\epsilon) < +\infty$ for all $\epsilon > 0$. Then, by (6.4.15) we have that

$$\mathbb{P}[\|U_h(n)\|_1 \geq \epsilon] < +\infty$$

and so by the Borel–Cantelli lemma, $\limsup_{n\to\infty} \|U_h(n)\|_1 \leq \epsilon$ a.s. for each $\epsilon > 0$. Letting $\epsilon \downarrow 0$ through the rational numbers gives $\lim_{n\to\infty} U_h(n) = 0$ a.s.

Part (B). Suppose $S_h(\epsilon) = +\infty$ for all $\epsilon > 0$. Then, by (6.4.14) we have that

$$\mathbb{P}[\|U_h(n)\|_1 \geq \epsilon] = +\infty$$

Since $(\|U_h(n)\|_1)_{n\geq 1}$ is a sequence of independent random variables, by the Borel–Cantelli lemma we have that $\limsup_{n\to\infty} \|U_h(n)\|_1 \geq \epsilon$ a.s. for each $\epsilon > 0$. Letting $\epsilon \to \infty$ through the integers gives $\limsup_{n\to\infty} \|U_h(n)\| = +\infty$ a.s.

Part (C). Suppose $S_h(\epsilon) < +\infty$ for all $\epsilon > \epsilon'$. If $\epsilon > \epsilon'$, then by (6.4.15) we have

$$\sum_{n=1}^{\infty} \mathbb{P}[\|U_h(n)\|_1 \geq dh\epsilon] \leq \sum_{n=0}^{\infty} d\left(1 - \Phi\left(\frac{\epsilon}{\|\sigma_h(n)\|_F}\right)\right) < +\infty,$$

and so $\limsup_{n\to\infty} \|U_h(n)\|_1 \leq dh\epsilon' =: c_2$, a.s. On the other hand, if $\epsilon < \epsilon'$, by (6.4.14) we get

$$\sum_{n=1}^{\infty} \mathbb{P}[\|U_h(n)\|_1 \geq h\epsilon/d] \geq \sum_{n=0}^{\infty} \frac{1}{d}\left\{1 - \Phi\left(\frac{\epsilon}{\|\sigma_h(n)\|_F}\right)\right\} = +\infty.$$

Therefore, using the Borel–Cantelli lemma and independence of $\|U_h(n)\|_1$, we have that $\limsup_{n\to\infty} \|U_h(n)\|_1 \geq h\epsilon'/d =: c_2$, a.s.

6.5 Proof of Theorem 6.3.2

Recall from Lemma 6.4.1 that X_h obeys (6.4.1) with M given by (6.4.4). Since f obeys (4.1.9), this implies that

$$\|X_h(n)^2\| - \|X_h(0)\|^2 \leq \sum_{i=1}^{n} h\|\sigma_h(i-1)\xi(i)\|^2 + M(n), \quad n \geq 1.$$

We want to prove that $\limsup_{n\to\infty} \|X_h(n)\| < +\infty$, therefore we need to prove that $\limsup_{n\to\infty} \sum_{i=1}^{n} h\|\sigma_h(i-1)\xi(i)\|^2 < +\infty$ and $\limsup_{n\to\infty} M(n) < +\infty$. Define $P(n) = \sum_{i=1}^{n} h\|\sigma_h(i-1)\xi(i)\|^2$. Since $(P(n))_{n\geq 1}$ is a non–decreasing sequence, we have that $P_\infty = \lim_{n\to\infty} P(n)$ exists a.s. We wish to show that P_∞ must be finite a.s. Suppose to the contrary that there is an event $A = \{\omega : P_\infty(\omega) = \infty\}$ with $\mathbb{P}[A] > 0$. Then as P_∞ is a non–negative random variable, we have that $\mathbb{E}[P_\infty] = +\infty$. However by Fubini's Theorem we have

$$\mathbb{E}[P_\infty] = \mathbb{E}\sum_{i=1}^{\infty} \|\sigma_h(i-1)\xi(i)\|^2 = \sum_{i=1}^{\infty} \|\sigma_h(i-1)\|_F^2 < +\infty,$$

which is a contradiction. Therefore it must be that $\lim_{n\to\infty} P(n) = P_\infty$ exists and is finite a.s. From (6.4.6) and (4.1.9) we have

$$\|X_h(n+1)\|^2 - \|X_h(n)\|^2 \leq h\|\sigma_h(n)\xi(n+1)\|^2 + 2\sqrt{h}\langle X_h^\star(n), \sigma_h(n)\xi(n+1)\rangle. \quad (6.5.1)$$

We know that $\mathbb{E}[\|X_h(0)\|^2] < +\infty$. We wish to prove that $\mathbb{E}[\|X_h(n)\|^2] < +\infty$ for each $n \in \mathbb{N}$, which we prove by induction. Suppose that $\mathbb{E}[\|X_h(n)\|^2] < +\infty$. Then, we get

$$\mathbb{E}[\|X_h(n+1)\|^2] \leq \mathbb{E}[\|X_h(n)\|^2] + \mathbb{E}[h\|\sigma_h(n)\xi(n+1)\|^2] + 2\sqrt{h}\mathbb{E}[\langle X_h^\star(n), \sigma_h(n)\xi(n+1)\rangle].$$

We now compute the second term on the right–hand side. Because $X_h^\star(n)$ depends on $X_h(n)$ and is $\mathcal{F}(n)$–measurable, and $\xi(n+1)$ is $\mathcal{F}(n+1)$–measurable and independent of $\mathcal{F}(n)$, therefore $\xi(n+1)$ is independent of $X_h^\star(n)$. Moreover $\mathbb{E}[\|X_h^\star(n)\|] \leq \mathbb{E}[\|X_h(n)\|] < \infty$

and similarly $\mathbb{E}[\|\xi(n+1)\|^2]$ is finite. We get

$$\begin{aligned}\mathbb{E}[\langle X_h^\star(n), \sigma_h(n)\xi(n+1)\rangle] &= \mathbb{E}\left[\sum_{i=1}^d [X_h^\star(n)]_i[\sigma_h(n)\xi(n+1)]_i\right] \\ &= \mathbb{E}\left[\sum_{i=1}^d [X_h^\star(n)]_i \sum_{j=1}^r [\sigma_h(n)]_{ij}\xi_j(n+1)\right] \\ &= \mathbb{E}\left[\sum_{j=1}^r \left(\sum_{i=1}^d [X_h^\star(n)]_i[\sigma_h(n)]_{ij}\right)\xi_j(n+1)\right] \\ &= \sum_{j=1}^r \mathbb{E}\left[\left(\sum_{i=1}^d [X_h^\star(n)]_i[\sigma_h(n)]_{ij}\right)\xi_j(n+1)\right].\end{aligned}$$

Since $\mathbb{E}[\|X_h^\star(n)\|^2] < +\infty$ and $\mathbb{E}[\|\xi(n+1)\|^2] < +\infty$ and σ_h is deterministic, it follows from independence and the fact that $\mathbb{E}[\xi_j(n+1)] = 0$ for all n and j, that

$$\mathbb{E}\left[\left(\sum_{i=1}^d [X_h^\star(n)]_i[\sigma_h(n)]_{ij}\right)\xi_j(n+1)\right] = \mathbb{E}\left[\sum_{i=1}^d [X_h^\star(n)]_i[\sigma_h(n)]_{ij}\right]\mathbb{E}[\xi_j(n+1)] = 0.$$

Hence

$$\mathbb{E}[\langle X_h^\star(n), \sigma_h(n)\xi(n+1)\rangle] = 0.$$

Next, we return to $P(n)$ to get

$$\begin{aligned}\mathbb{E}[\|\sigma_h(n)\xi(n+1)\|^2] &= \mathbb{E}\sum_{i=1}^d [\sigma_h(n)\xi_i(n+1)]_i^2 = \mathbb{E}\sum_{i=1}^d \left(\sum_{j=1}^r [\sigma_h(n)]_{ij}\xi_j(n+1)\right)^2 \\ &= \mathbb{E}\sum_{i=1}^d \left\{\sum_{j=1}^r [\sigma_h(n)]_{ij}^2\xi_j^2(n+1) + \sum_j\sum_{k\neq j}[\sigma_h(n)]_{ij}\sigma_{ik}(n)\xi_j(n+1)\xi_k(n+1)\right\}.\end{aligned}$$

By the independence of $\xi_j(n+1)$, $\xi_i(n+1)$ for $i \neq j$, we have

$$\mathbb{E}[\|\sigma_h(n)\xi(n+1)\|^2] = \sum_{i=1}^d\sum_{j=1}^r [\sigma_h(n)]_{ij}^2 = \|\sigma_h(n)\|_F^2. \tag{6.5.2}$$

Therefore

$$\mathbb{E}[\|X_h(n+1)\|^2] \leq \mathbb{E}[\|X_h(n)\|^2] + h\|\sigma(n)\|_F^2 < +\infty.$$

Thus by induction we have $\mathbb{E}[\|X_h(n+1)\|^2] < +\infty$ for all $n \in \mathbb{N}$. Now by (6.5.1) we get

$$\begin{aligned}\|X_h(n)\|^2 - \|X_h(0)\|^2 &= \sum_{j=0}^{n-1}\{\|X_h(j+1)\|^2 - \|X_h(j)\|^2\} \\ &\leq h\sum_{j=0}^{n-1}\|\sigma_h(j)\xi(j+1)\|^2 + 2\sqrt{h}\sum_{j=0}^{n-1}\langle X_h^\star(j), \sigma_h(j)\xi(j+1)\rangle \\ &= hP(n) + 2\sqrt{h}\sum_{j=0}^{n-1}\langle X_h^\star(j), \sigma_h(j)\xi(j+1)\rangle.\end{aligned}$$

Because $\mathbb{E}[\|X_h(n)\|^2] < +\infty$ and $\mathbb{E}[\|X_h^\star(n)\|^2] \leq \mathbb{E}[\|X_h(n)\|^2]$, thus $\mathbb{E}[\|X_h^\star(n)\|^2] < +\infty$ for all. Therefore

$$M(n) = \sum_{j=0}^{n-1} 2\sqrt{h}\langle X_h^\star(j), \sigma_h(j)\xi(j+1)\rangle$$

is a martingale. Next we compute the quadratic variation of M. To this end, we may write M according to

$$M(n) = 2\sqrt{h}\sum_{j=0}^{n-1}\sum_{l=1}^{r} Q_l(j)\xi_l(j+1),$$

where $Q_l(j) = \sum_{i=1}^d [X_h^\star(j)]_i[\sigma_h(j)]_{il}$. Thus $M(j+1) - M(j) = 2\sqrt{h}\sum_{l=1}^r Q_l(j)\xi_l(j+1)$. Hence the quadratic variation of M is given by

$$\begin{aligned}
\langle M\rangle(n) &= 4h\sum_{j=0}^{n-1}\mathbb{E}\left[\left(\sum_{l=1}^{r} Q_l(j)\xi_l(j+1)\right)^2 \middle| \mathcal{F}_j\right] \\
&= 4h\sum_{j=0}^{n-1}\mathbb{E}[\sum_{l=1}^{r} Q_l(j)^2\xi_l(j+1)^2 \\
&\quad + \sum_{m=1}^{r}\sum_{l\neq m} Q_l(j)Q_m(j)\xi_l(j+1)\xi_m(j+1)|\mathcal{F}_j] \\
&= 4h\sum_{l=1}^{r} Q_l(j)^2\mathbb{E}[\xi_l(j+1)^2|\mathcal{F}_j] \\
&\quad + \sum_{m=1}^{r}\sum_{l\neq m} Q_l(j)Q_m(j)\mathbb{E}[\xi_l(j+1)\xi_m(j+1)|\mathcal{F}_j] \\
&= 4h\sum_{j=0}^{n-1}\sum_{l=1}^{r} Q_l^2(j).
\end{aligned}$$

Therefore, by using the Cauchy–Schwartz inequality, we obtain the estimate

$$\begin{aligned}
\langle M\rangle(n) &\leq 4h\sum_{j=0}^{n-1}\sum_{l=1}^{r}\left\{\sum_{i=1}^{d} X_h^\star(j)_i^2\sum_{i=1}^{d}\sigma_{il}^2(j)\right\} \\
&= 4h\sum_{j=0}^{n-1}\left(\sum_{i=1}^{d} X_i^\star(j)^2\right)\cdot\sum_{l=1}^{r}\sum_{i=1}^{d}\sigma_{il}^2(j) = 4h\sum_{j=0}^{n-1}\|X_h^\star(j)\|^2\|\sigma_h(j)\|_F^2. \quad (6.5.3)
\end{aligned}$$

Define the events

$$A_1 = \{\omega : \lim_{n\to\infty} P(n,\omega) = P_\infty \in (0,\infty)\}, \quad A_2 = \{\omega : \lim_{n\to\infty}\langle M\rangle(n) = +\infty\}.$$

Suppose that $P[A_2] > 0$. Let $A_3 = A_1 \cap A_2$, so that $\mathbb{P}[A_3] > 0$. Then a.s. on A_3 we have

$$\lim_{n\to\infty}\frac{M(n)}{\langle M\rangle(n)} = 0.$$

Next suppose that $\epsilon \in (0,1)$ is so small that

$$4\epsilon h \sum_{n=1}^{\infty} \|\sigma_h(n)\|_F^2 < \frac{1}{2}. \tag{6.5.4}$$

Thus for every $\omega \in A_3$ and for every $\epsilon < 1$, there is an $N(\omega,\epsilon) > 1$ such that $|M(n,\omega)| \leq \epsilon\langle M\rangle(n,\omega)$ for all $n \geq N(\omega,\epsilon)$. Therefore for $n \geq N(\omega,\epsilon)$ we have

$$\begin{aligned}\|X_h(n,\omega)\|^2 &\leq \|X_h(0,\omega)\|^2 + hP(n,\omega) + M(n,\omega)\\ &\leq \|X_h(0,\omega)\|^2 + hP_\infty(\omega) + \epsilon\langle M\rangle(n,\omega).\end{aligned}$$

Since $\|X_h(n,\omega)\|^2 \leq \max_{0\leq j\leq N(\omega,\epsilon)} \|X_h(j,\omega)\|^2 =: X_h^{\star\star}(\epsilon,\omega) < +\infty$ for $0 \leq n \leq N(\omega,\epsilon)$. Define $C_1(\epsilon,\omega) := \|X_h(0,\omega)\|^2 + hP_\infty(\omega) + X_h^{\star\star}(\epsilon,\omega)$ which is finite. Therefore

$$\|X_h(n,\omega)\|^2 \leq C_1(\epsilon,\omega) + \epsilon\langle M\rangle(n,\omega), \quad n \geq 1.$$

We drop the ω–dependence temporarily. Define $y(n) = \|\sigma_h(n)\|_F^2\|X_h(n)\|^2$ for $n \geq 0$. Hence by the last inequality and (6.5.3), we have

$$y(n) = \|\sigma_h(n)\|_F^2\|X_h(n)\|^2 \leq C_1(\epsilon)\|\sigma_h(n)\|_F^2 + 4\epsilon h\|\sigma_h(n)\|_F^2 \sum_{j=0}^{n-1} y(j), \quad n \geq 1,$$

where we have used the fact that $\|X_h^\star(j)\| \leq \|X_h(j)\|^2$ for all $j \geq 0$. Thus for $m \geq 1$ we have

$$\begin{aligned}\sum_{n=1}^{m} y(n) &\leq C_1(\epsilon)\sum_{n=1}^{m}\|\sigma_h(n)\|_F^2 + 4\epsilon h\sum_{n=1}^{m}\|\sigma_h(n)\|_F^2\sum_{j=0}^{n-1} y(j)\\ &\leq C_1(\epsilon)\sum_{n=1}^{m}\|\sigma_h(n)\|_F^2 + 4\epsilon h\sum_{n=1}^{m}\|\sigma_h(n)\|_F^2\sum_{j=0}^{m} y(j)\\ &\leq C_1(\epsilon)\sum_{n=1}^{\infty}\|\sigma_h(n)\|_F^2 + 4\epsilon h\sum_{n=1}^{\infty}\|\sigma_h(n)\|_F^2\left(\sum_{j=1}^{m} y(j) + y(0)\right)\\ &\leq C(\epsilon)\sum_{n=1}^{\infty}\|\sigma_h(n)\|_F^2 + \frac{1}{2}\sum_{j=1}^{m} y(j)\end{aligned}$$

where $C(\epsilon) = C_1(\epsilon) + 4\epsilon h\|X_h(0)\|^2\|\sigma_h(0)\|^2$, condition (6.5.4) was used at the last step, the non-negativity and definition of y was used throughout. Therefore $\sum_{j=1}^{m} y(j) \leq 2C(\epsilon)\sum_{n=1}^{\infty}\|\sigma_h(n)\|^2$ for all $m \geq 1$. Thus

$$\sum_{n=1}^{\infty}\|\sigma_h(n)\|^2\|X_h(n,\omega)\|^2 < +\infty \quad \text{for each } \omega \in A_3.$$

This implies that $\lim_{n\to\infty}\langle M\rangle(n,\omega) < +\infty$ for each $\omega \in A_3$, which is a contradiction. Therefore we have that $\mathbb{P}[A_2] = 0$. Thus we have that

$$\lim_{n\to\infty}\langle M\rangle(n) \quad \text{exists and is a.s. finite}$$

This implies $\lim_{n\to\infty} M(n)$ exists and is finite a.s. Thus we have $\limsup_{n\to\infty}\|X_h(n)\| < +\infty$ a.s.

Next we show that $\lim_{n\to\infty}\|X_h(n)\|^2 =: L \in [0,+\infty)$ a.s. To do this we apply Lemma 6.4.4 with $Z(n+1) := \|X_h(n+1)\|^2$, $Z(n) := \|X_h(n)\|^2$, $V(n) := 0$, $W(n) := h\|\sigma_h(n)\xi(n+1)\|^2$, $\nu(n+1) := 2\sqrt{h}\langle X_h^*(n), \sigma_h(n)\xi(n+1)\rangle$. Therefore, by (6.5.2) we get $\mathbb{E}[\sum_{n=1}^{\infty} W(n)] = \sum_{n=1}^{\infty} h\|\sigma_h(n)\|_F^2 < +\infty$, which implies that $\sum_{n=1}^{\infty} W(n) < +\infty$ a.s. Therefore, $\lim_{n\to\infty}\|X_h(n)\|^2 =: L \in [0,\infty)$ a.s. Moreover, as $W(n) \geq 0$ it also follows that $\lim_{n\to\infty} W(n) = 0$ a.s., so $\lim_{n\to\infty} U_h(n) = 0$ a.s.

We are now in a position to prove that $X_h(n) \to 0$ as $n \to \infty$ a.s. Recall from (6.7.3) and that $U_h(n) \to 0$ as $n \to \infty$. Since $\|X_h(n)\| \to \sqrt{L}$ as $n \to \infty$, it follows that $\|X_h^*(n)\| = \|X_h(n+1) - U_h(n+1)\| \to \sqrt{L}$ as $n \to \infty$. Hence $|\langle X_h^*(n), U_h(n+1)\rangle| \leq \|X_h^*(n)\|\|U_h(n+1)\| \to 0$ as $n \to \infty$. Therefore, rearranging (6.7.3) gives

$$\begin{aligned}2h\langle f(X_h^*(n)), X_h^*(n)\rangle &+ h^2\|f(X_h^*(n)\|^2 \\ &= \|X_h(n)\|^2 - \|X_h(n+1)\|^2 + \|U_h(n+1)\|^2 + 2\langle X_h^*(n), U_h(n+1)\rangle\end{aligned}$$

which goes to 0 as $n \to \infty$. Thus $\lim_{n\to\infty}\left\{2\langle f(X_h^*(n)), X_h^*(n)\rangle + h\|f(X_h^*(n)\|^2\right\} = 0$. Next define $R : \mathbb{R}^d \to \mathbb{R}$ by

$$R(x) = 2\langle x, f(x)\rangle + h\|f(x)\|^2, \quad x \in \mathbb{R}^d. \tag{6.5.5}$$

Then we have $R(0) = 0$, $x \mapsto R(x)$ is continuous, $R(x) > 0$ for all $x \neq 0$. Therefore we have $\lim_{n\to\infty} R(X_h^*(n)) = 0$ and $\lim_{n\to\infty}\|X_h(n)\| = \sqrt{L}$. Thus

$$R(X_h^*(n)) \geq \inf_{\|x\|=\|X_h^*(n)\|} R(x) \geq 0.$$

Hence $0 = \limsup_{n\to\infty} R(X_h^*(n)) \geq \limsup_{n\to\infty} \inf_{\|x\|=\|X_h^*(n)\|} R(x) \geq 0$. Therefore

$$\lim_{n\to\infty} \inf_{\|x\|=\|X_h^*(n)\|} R(x) = 0.$$

Now define $R^* : \mathbb{R}^+ \to \mathbb{R}$ by $R^*(y) = \inf_{\|x\|=y} R(x)$. Since R is continuous, R^* is continuous. Thus, because $\lim_{n\to\infty} R^*(\|X_h^*(n)\|) = 0$ and $\|X_h^*(n)\| \to \sqrt{L}$ as $n \to \infty$, we have that

$$0 = \lim_{n\to\infty} R^*(\|X_h^*(n)\|) = R^*\left(\lim_{n\to\infty} \|X_h^*(n)\|\right) = R^*(\sqrt{L}).$$

Thus $\inf_{\|x\|=\sqrt{L}} R(x) = 0$. Since R is continuous, there exists X^* with $\|X^*\| = \sqrt{L}$ such that $R(x^*) = 0$, but since $R(0) = 0$ and $R(x) > 0$ for all $x \neq 0$, this forces $x^* = 0$, so $L = 0$. Hence, $\lim_{n\to\infty} \|X_h(n)\|^2 = 0$, a.s., as required.

6.6 Proof of Theorems 6.3.3

We start by proving part (A). Suppose that $A := \{\omega : \limsup_{n\to\infty} \|X_h(n,\omega)\| < +\infty\}$ is an event with $\mathbb{P}[A] > 0$. Define for $\omega \in A$ the quantity $L(\omega) \in [0,\infty)$ such that $L(\omega) = \limsup_{n\to\infty} \|X_h(n,\omega)\|$. By Lemma 6.2.1, we have $\|X_h^\star(n)\| \leq \|X_h(n)\|$ for all $n \geq 0$. Therefore, for every $\omega \in A$, we have $\limsup_{n\to\infty} \|X_h^\star(n,\omega)\| \leq L(\omega)$. By (6.2.3c), we have $U_h(n+1,\omega) = X_h(n+1,\omega) - X_h^\star(n,\omega)$. Since $S_h(\epsilon) = +\infty$ for every $\epsilon > 0$, by Lemma 6.3.1 the process U_h given by (6.3.1) obeys $\limsup_{n\to\infty} \|U_h(n)\| = +\infty$ a.s. Suppose Ω_4 is the a.s. event such that $\Omega_4 = \{\omega : \limsup_{n\to\infty} \|U_h(n,\omega)\| = +\infty\}$. Then $A_1 = A \cap \Omega_4$ is an event with $\mathbb{P}[A_1] > 0$. Therefore for $\omega \in A_1$ we have

$$\begin{aligned}+\infty &= \limsup_{n\to\infty} \|U_h(n+1,\omega)\| = \limsup_{n\to\infty} \|X_h(n+1,\omega) - X_h^\star(n,\omega)\| \\ &\leq \limsup_{n\to\infty} \|X_h(n+1,\omega)\| + \limsup_{n\to\infty} \|X_h^\star(n,\omega)\| \leq 2L(\omega),\end{aligned}$$

a contradiction. Therefore we have that $\mathbb{P}[A] = 0$, which proves part (A).

For the proof of part (B), because $S_h(\epsilon) < +\infty$ for all $\epsilon > \epsilon'$ and $S_h(\epsilon) = +\infty$ for all $\epsilon < \epsilon'$, Lemma 6.3.1 implies that the process U_h defined by (6.3.1) obeys $0 < c_1 \leq \limsup_{n\to\infty} \|U_h(n)\| \leq c_2 < +\infty$ a.s. for some deterministic c_1 and c_2. In fact

$$U_h^*(\omega) := \limsup_{n\to\infty} \|U_h(n,\omega)\| \in [c_1, c_2].$$

Therefore, we know that $\limsup_{n\to\infty} \|X_n(n,\omega)\| > 0$ for all $\omega \in \Omega_1$ where Ω_1 is an almost sure event.

Let $\omega \in \Omega_1$. We have that

$$0 < c'(\omega) := \limsup_{n\to\infty} \|X_h(n,\omega)\|.$$

Clearly $c''(\omega) := \limsup_{n\to\infty} \|X_h^\star(n,\omega)\| \le c'(\omega)$, where the latter inequality holds by Lemma 6.2.1. We have that $c''(\omega) > 0$, because if $X_h^\star(n,\omega) \to 0$ as $n \to \infty$, and f obeys (4.1.9), we have

$$\lim_{n\to\infty} X_h(n,\omega) = \lim_{n\to\infty} X_h^\star(n,\omega) + f(X_h^\star(n,\omega)) = 0.$$

By (6.2.3c), since $c'(\omega) \ge c''(\omega)$, we get

$$\begin{aligned} U_h^*(\omega) &= \limsup_{n\to\infty} \|U_h(n+1,\omega)\| \le \limsup_{n\to\infty} \|X_h(n+1,\omega)\| + \|X_h^\star(n,\omega)\| \\ &= c'(\omega) + c''(\omega) \le 2c'(\omega). \end{aligned}$$

Therefore $c'(\omega) \ge U_h^*(\omega)/2 \ge c_1/2$, as required.

6.7 Proof of Theorems 6.3.4, 6.3.5, and 6.3.6

6.7.1 Properties of the data

Before we turn to the proof of Theorem 6.3.4 we first require some auxiliary results concerning the function f.

Lemma 6.7.1. *Suppose that $f \in C(\mathbb{R}^d);\mathbb{R}^d)$. Suppose Assumption 6.2.2 holds. If $K > 0$ and $\|x\| > K > 0$, then every solution $x^\star$ of (6.2.4) obeys $\|x^\star\| > F_h^{-1}(K) > 0$, where*

$$F_h(x) := x + h \sup_{\|u\|\le x} \|f(u)\|, \quad x \ge 0. \tag{6.7.1}$$

Proof. Since $F_h : [0,\infty) \to [0,\infty)$ is increasing, F_h^{-1} is increasing. Let $K > 0$ and define $M = F_h^{-1}(K) > 0$. Since $\|x\| > K = F_h(M)$, and $x^\star$ obeys $x = x^\star + hf(x^\star)$, we get

$$\begin{aligned} K < \|x\| = \|x^\star + hf(x^\star)\| &\le \|x^\star\| + h\|f(x^\star)\| \\ &\le \|x^\star\| + h \sup_{\|u\|\le\|x^\star\|} \|f(u)\| = F_h(\|x^\star\|). \end{aligned}$$

Thus $K < F_h(\|x^\star\|)$, therefore $F_h^{-1}(K) < \|x^\star\|$, as required. □

Lemma 6.7.2. *Suppose that f obeys* (4.1.9). *Define $\bar{f} : [0,\infty) \to \mathbb{R}$ by*

$$\bar{f}(y) := \inf_{\|x\|=y} \langle x, f(x)\rangle, \tag{6.7.2}$$

and $\varphi : [0,\infty) \to \mathbb{R}$ by

$$\varphi(y) = \inf_{x\in[F_h^{-1}(\frac{3y}{4}),\frac{5y}{4}]} \bar{f}(x).$$

where F_h is defined by (6.7.1). *Then $\bar{f}(x) > 0$ for all $x > 0$ and $\varphi(x) > 0$ for all $x > 0$.*

Proof. Since f is continuous, it follows that $\bar{f}$ is continuous. Also, as F_h is continuous and invertible, F_h^{-1} exists and is continuous, and therefore φ is continuous also. Notice that the continuity of f and the dissipative condition in (4.1.9) implies that $\bar{f}(y) > 0$ for all $y > 0$. We show also that $\varphi(y) > 0$ for $y > 0$. Suppose to the contrary that $\varphi(y) = 0$ for some $y > 0$. Then, as $\bar{f}$ is continuous, there exists $x \in [F_h^{-1}(\frac{3y}{4}), \frac{5y}{4}]$ such that $\bar{f}(x) = 0$. However, as $y > 0$, we have that $F_h^{-1}(3y/4) > 0$, and so this implies that there is $x > 0$ for which $\bar{f}(x) = 0$. □

6.7.2 Asymptotic results

We are now ready to prove the first step of the main result of this section, which is namely to establish that $\liminf_{n\to\infty} \|X_h(n)\| = 0$.

Lemma 6.7.3. *Suppose that $(X_h, X_h^\star)$ is a solution of* (6.2.3). *Suppose that f obeys* (4.1.9), *and that the sequence ξ obeys Assumption 6.2.1. If $S_h(\epsilon)$ defined by* (6.3.2) *obeys $S_h(\epsilon) < +\infty$ for all $\epsilon > 0$, then*

$$\{\liminf_{n\to\infty} \|X_h(n)\| = 0\} \cup \{\lim_{n\to\infty} \|X_h(n)\| = +\infty\} \text{ is an a.s. event.}$$

Proof. Using (6.4.6) together with (6.3.1) we get

$$\|X_h(n+1)\|^2 = \|X_h(n)\|^2 - 2h\langle X_h^\star(n), f(X_h^\star(n))\rangle - h^2\|f(X_h^\star(n))\|^2 + 2\langle X_h^\star(n), U_h(n+1)\rangle + \|U_h(n+1)\|^2, \tag{6.7.3}$$

and therefore

$$\|X_h(n+1)\|^2 \le \|X_h(n)\|^2 - 2h\langle X_h^\star(n), f(X_h^\star(n))\rangle$$
$$+ 2\|X_h^\star(n)\|\|U_h(n+1)\| + \|U_h(n+1)\|^2. \quad (6.7.4)$$

Suppose that Ω_5 is the a.s. event such that $\Omega_5 = \{\omega : \lim_{n\to\infty} \|U_h(n,\omega)\| = 0\}$. Clearly, we have that either the liminf of $\|X_h(n)\|$ is finite or not. Suppose that there exists a nontrivial event Ω_6 such that

$$\Omega_6 = \{\omega : \liminf_{n\to\infty} \|X_h(n,\omega)\| < +\infty\}.$$

In order to prove the result, it suffices to show that Ω_6 is a.s. the same event as $\{\omega : \liminf_{n\to\infty} \|X_h(n,\omega)\| = +\infty\}$.

In order to do this, we suppose to the contrary that there exists an event $A = \{\omega \in \Omega_6 : \liminf_{n\to\infty} \|X_h(n,\omega)\| = l(\omega) \in (0,\infty)\}$ for which $\mathbb{P}[A] > 0$. The finiteness of the liminf is a consequence of A being a subset of Ω_6. Let $A_1 = A\cap\Omega_5$: then $\mathbb{P}[A_1] = \mathbb{P}[A] > 0$. Fix $\omega \in A_1$. Suppose that $\liminf_{n\to\infty} \|X_h^\star(n,\omega)\| = 0$. Then, because $\|X_h^\star(n,\omega)\| \le \|X_h(n,\omega)\|$ we have that $\liminf_{n\to\infty} \|X_h(n,\omega)\| = 0$, a contradiction. Hence, for every $\omega \in A_1$ there exists $l^\star(\omega) > 0$ such that $\liminf_{n\to\infty} \|X_h^\star(n,\omega)\| = l^\star(\omega) > 0$.

Since $l(\omega) > 0$, we note that $\varphi(l(\omega)) > 0$. Because for each $\omega \in A_1$ we have $U_h(n+1,\omega) \to 0$ as $n \to \infty$, it follows that for every $\omega \in A_1$ and for every

$$\epsilon \in \left(0, 1 \wedge \frac{5l(\omega)}{2} \wedge h\frac{\varphi(l(\omega))}{5l(\omega)}\right),$$

there is $N_1(\epsilon,\omega) \in \mathbb{N}$ such that $\|U_h(n+1,\omega)\| < \epsilon$ for all $n > N_1(\epsilon,\omega)$. There also exists $N_2(\omega) \in \mathbb{N}$ such that $\|X_h(n,\omega)\| > 3l(\omega)/4$ for all $n \ge N_2(\omega)$.

Now let $N_3(\epsilon,\omega) = \max(N_1(\epsilon,\omega), N_2(\omega))$. By the definition of the event $A \supseteq A_1$, it follows for each $\omega \in A_1$ that there is a finite $N_4(\epsilon,\omega)$ such that $N_4(\epsilon,\omega) = \inf\{n > N_3(\epsilon,\omega) : \|X_h(n,\omega)\| < 5l(\omega)/4\}$. Therefore $3l(\omega)/4 < \|X_h(N_4,\omega)\| < 5l(\omega)/4$.

We now show by induction that our supposition leads us to conclude that $3l(\omega)/4 < \|X_h(n,\omega)\| < 5l(\omega)/4$ for all $n \geq N_4(\epsilon,\omega)$. This is certainly true for $n = N_4(\epsilon,\omega)$. Suppose that it is true for a general $n \geq N_4(\epsilon,\omega)$. Clearly, as $n \geq N_4(\epsilon,\omega) > N_3(\epsilon,\omega) \geq N_2(\omega)$, we have $3l(\omega)/4 < \|X_h(n{+}1,\omega)\|$, so it remains to establish the upper bound $\|X_h(n{+}1,\omega)\| < 5l(\omega)/4$.

Since F_h is increasing, by using Lemmas 6.2.1 and 6.7.1, we get

$$F_h^{-1}(3l(\omega)/4) < \|X_h^\star(n,\omega)\| \leq \|X_h(n,\omega)\| < \frac{5l(\omega)}{4}.$$

Hence

$$0 < F_h^{-1}(3l(\omega)/4) < \|X_h^\star(n,\omega)\| < \frac{5l(\omega)}{4}.$$

Since $\bar{f}$ is continuous, for all $y_2 > y_1 > 0$, we have

$$\inf_{y_1\leq\|x\|\leq y_2}\langle x, f(x)\rangle = \inf_{y\in[y_1,y_2]}\bar{f}(y) > 0.$$

Thus

$$\langle X_h^\star(n,\omega), f(X_h^\star(n,\omega))\rangle \geq \min_{y\in[F_h^{-1}(3l(\omega)/4),5l(\omega)/4]}\bar{f}(y) = \varphi(l(\omega)) > 0.$$

We now return to (6.7.4) to estimate the terms on the righthand side. For $\|X_h^\star(n,\omega)\| < 5l(\omega)/4$, we have

$$\begin{aligned}&2\|X_h^\star(n,\omega)\|\|U_h(n+1,\omega)\| + \|U_h(n+1,\omega)\|^2\\ &\qquad< 2\frac{5l(\omega)}{4}\epsilon + \epsilon^2 < \frac{5l(\omega)}{2}\epsilon + \frac{5l(\omega)}{2}\epsilon = 5l(\omega)\epsilon.\end{aligned}$$

Therefore

$$\begin{aligned}&-2h\langle X_h^\star(n,\omega), f(X_h^\star(n,\omega))\rangle + 2\|X_h^\star(N_4,\omega)\|\|U_h(n+1,\omega)\| + \|U_h(n+1,\omega)\|^2\\ &\qquad\leq -2h\varphi(l(\omega)) + 5l(\omega)\epsilon < -2h\varphi(l(\omega)) + 5l(\omega)h\frac{\varphi(l(\omega))}{5l(\omega)} = -h\varphi(l(\omega)).\end{aligned}$$

Therefore, by (6.7.4), we obtain $\|X_h(n+1,\omega)\|^2 \leq \|X_h(n,\omega)\|^2 - h\varphi(l(\omega))$ and since by hypothesis we assume $\|X_h(n,\omega)\| < 5l(\omega)/4$, we have $\|X_h(n+1,\omega)\| < 5l(\omega)/4$, as required. Moreover, scrutiny of the above argument shows that one can equally prove that

$$\|X_h(n+1,\omega)\|^2 \leq \|X_h(n,\omega)\|^2 - h\varphi(l(\omega)), \quad \text{for all } n \geq N_4(\epsilon,\omega).$$

Therefore for any $N \in \mathbb{N}$ we have

$$\|X_h(N_4+N,\omega)\|^2 \leq \|X_h(N_4,\omega)\|^2 - Nh\varphi(l(\omega)).$$

In particular, let N be any integer satisfying

$$N > \frac{2}{h\varphi(l(\omega))}\left\{\left(\frac{5l(\omega)}{4}\right)^2 - \left(\frac{l(\omega)}{4}\right)^2\right\}.$$

Since $3l(\omega)/4 < X_h(n,\omega) < 5l(\omega)/4$ for all $n \geq N_4$, we get

$$\begin{aligned}\left(\frac{3l(\omega)}{4}\right)^2 \leq \|X_h(N_4+N,\omega)\|^2 &\leq \|X_h(N_4,\omega)\|^2 - Nh\varphi(l(\omega)) \\ &< \left(\frac{5l(\omega)}{4}\right)^2 - Nh\varphi(l(\omega)) \\ &< \left(\frac{l(\omega)}{4}\right)^2,\end{aligned}$$

which contradicts the original supposition. This proves the desired result. □

We are finally in a position to provide a proof of Theorem 6.3.4.

6.7.3 Proof of Theorem 6.3.4

To prove part (i), by virtue Lemma 6.7.3, it suffices to show on the event Ω_7 defined by $\Omega_7 = \{\omega : \liminf_{n\to\infty} \|X_h(n,\omega)\| = 0\}$ (modulo some null event), we have $X_h(n) \to 0$ as $n \to \infty$. We can assume, without loss of generality, that Ω_7 is an event of positive probability, because, if it is not, Lemma 6.7.3 implies the event $\{\lim_{n\to\infty} \|X_h(n)\| = +\infty\}$ is a.s., and our claim is trivially true.

Recall also the a.s. event Ω_5 defined in Lemma 6.7.3, viz.,

$$\Omega_5 = \{\omega : \lim_{n\to\infty} U_h(n,\omega) = 0\}.$$

By Lemma 6.7.2, it follows that the function $\bar{f}$ defined in (6.7.2) obeys $\bar{f}(y) > 0$ for all $y > 0$ and by the continuity of f, $\bar{f}$ is also continuous on $[0, \infty)$. Therefore, for any $l > 0$ we have that

$$\min_{\frac{l}{32} \le y \le \frac{l}{16}} \bar{f}(y) > 0. \tag{6.7.5}$$

Hence, we may choose an $\epsilon = \epsilon(l) > 0$ so small that

$$2\epsilon(l) = 1 \wedge \frac{l}{32} \wedge \left\{ \frac{32}{10l} 2h \min_{\frac{l}{32} \le y \le \frac{l}{16}} \bar{f}(y) \right\}. \tag{6.7.6}$$

Let $\omega \in \Omega_8 := \Omega_5 \cap \Omega_7$. Therefore, there exists $N_1(l, \omega) \in \mathbb{N}$ such that $\|U_h(n+1, \omega)\| < \epsilon(l)$ for all $n > N_1(l, \omega)$. Moreover, as $\liminf_{n \to \infty} \|X_h(n, \omega)\| = 0$, it follows that there exists an integer $N_2 = N_2(l, \omega) > N_1(l, \omega)$ such that $\|X_h(N_2, \omega)\| < l/16$.

Suppose that there exists an integer $N_3 > N_2$ such that $\|X_h(n, \omega)\| < l/16$ for $n = N_2, N_2 + 1, \ldots, N_3$, but $\|X_h(N_3 + 1, \omega)\| \ge l/16$. By (6.2.3c) and (6.3.1) we have $X_h(N_3 + 1, \omega) = X_h^\star(N_3, \omega) + U_h(N_3 + 1, \omega)$, and since $N_3 > N_1$, we obtain

$$\|X_h^\star(N_3, \omega)\| \ge \|X_h(N_3 + 1, \omega)\| - \|U_h(N_3 + 1, \omega)\| > \frac{l}{16} - \epsilon > \frac{l}{32},$$

where (6.7.6) is used at the last step. Now using Lemma 6.2.1, we get $\|X_h^\star(N_3)\| \le \|X_h(N_3)\| < l/16$, and so $l/32 < \|X_h^\star(N_3)\| < l/16$. Therefore by the definition of $\bar{f}$, we have

$$\langle X_h^\star(N_3), f(X_h^\star(N_3)) \rangle \ge \min_{\frac{l}{32} \le y \le \frac{l}{16}} \bar{f}(y) > 0,$$

where the last inequality is a consequence of (6.7.5).

We now insert these estimates into (6.7.4) to get

$$\begin{aligned}\|X_h(N_3+1,\omega)\|^2 &\leq \|X_h(N_3,\omega)\|^2 - 2h\langle X_h^\star(N_3,\omega), f(X_h^\star(N_3,\omega))\rangle \\ &\qquad + 2\|X_h^\star(N_3,\omega)\|\|U_h(N_3+1,\omega)\| + \|U_h(N_3+1,\omega)\|^2 \\ &\leq \|X_h(N_3,\omega)\|^2 - 2h\langle X_h^\star(N_3,\omega), f(X_h^\star(N_3,\omega))\rangle \\ &\qquad + 2\|X_h^\star(N_3,\omega)\|\epsilon(l) + \epsilon(l)^2 \\ &\leq \left(\frac{l}{16}\right)^2 - 2h\min_{\frac{l}{32}\leq y\leq\frac{l}{16}} \bar{f}(y) + 2\frac{l}{16}\epsilon(l) + \epsilon(l)^2 \\ &< \left(\frac{l}{16}\right)^2 - 2h\min_{\frac{l}{32}\leq y\leq\frac{l}{16}} \bar{f}(y) + 2\frac{l}{16}\epsilon(l) + \epsilon(l)\frac{l}{32} \\ &= \left(\frac{l}{16}\right)^2 - 2h\min_{\frac{l}{32}\leq y\leq\frac{l}{16}} \bar{f}(y) + \frac{5l}{32}\epsilon(l) \\ &< \left(\frac{l}{16}\right)^2,\end{aligned}$$

where once again (6.7.5) is used at the last step, and (6.7.6) has been used throughout. Therefore, by hypothesis we have

$$\left(\frac{l}{16}\right)^2 \leq \|X_h(N_3+1,\omega)\|^2 < \left(\frac{l}{16}\right)^2,$$

a contradiction. Therefore, it must follow for each $\omega \in \Omega_8$ that for every $l > 0$ there exists an integer $N_2 = N_2(l,\omega)$ such that $\|X_h(n,\omega)\| < l/16$ for all $n \geq N_2(l,\omega)$. Therefore, we have that $X_h(n,\omega) \to 0$ as $n \to \infty$ for all $\omega \in \Omega_8$, and as Ω_8 is a.s., the first part of the result has been proven.

To prove part (ii), define $A = \{\omega : \lim_{n\to\infty} X(n,\omega) = 0\}$. Then $\mathbb{P}[A] > 0$ by hypothesis. By Lemma 6.2.1, we have that $\|X_h^\star(n)\| \leq \|X_h(n)\|$ for all $n \geq 0$. Therefore, for $\omega \in A$, we have $X_h^\star(n,\omega) \to 0$ as $n \to \infty$. By (6.2.3c), we have that

$$\lim_{n\to\infty} U_h(n+1,\omega) = \lim_{n\to\infty} \{X_h(n+1,\omega) - X_h^\star(n,\omega)\} = 0.$$

Therefore $U_h(n) \to 0$ on a set of positive probability. By Lemma 6.3.1, it follows that $S_h(\epsilon) < +\infty$ for all $\epsilon > 0$.

6.7.4 Proof of Theorem 6.3.5

Scrutiny of Theorem 6.3.4 shows that we can establish Theorem 6.3.5 provided that the condition (6.3.11) together with $S_h(\epsilon)$ always being finite implies $\liminf_{n\to\infty} \|X_h(n)\| <$

$+\infty$ a.s. This is the subject of the next result.

Lemma 6.7.4. *Suppose that* $(X_h, X_h^\star)$ *is a solution of* (6.2.3). *Suppose that* f *obeys* (4.1.9) *and* (6.3.11) *and that the sequence* ξ *obeys Assumption 6.2.1. If* $S_h(\epsilon)$ *defined by* (6.3.2) *obeys* $S_h(\epsilon) < +\infty$ *for every* $\epsilon > 0$, *then*

$$\liminf_{n\to\infty} \|X_h(n)\| < +\infty, \quad a.s.$$

Proof. Suppose to the contrary that

$$A = \{\omega : \liminf_{n\to\infty} \|X(n,\omega)\| = +\infty\}$$

is an event with $\mathbb{P}[A] > 0$. Since $\Omega_1 = \{\omega : U_h(n,\omega) \to 0 \text{ as } n \to \infty\}$ is an a.s. event, we have that $A_1 = A \cap \Omega_1$ obeys $\mathbb{P}[A_1] > 0$. Therefore by (6.3.11) for each $\omega \in A_1$, there is an $N(\omega) \in \mathbb{N}$ such that

$$\langle X_h^\star(n-1,\omega), f(X_h^\star(n-1,\omega))\rangle \geq \frac{\phi}{2}, \quad n \geq N_1(\omega).$$

On the other hand, as $U_h(n,\omega) \to 0$ as $n \to \infty$ for each $\omega \in A_1$, we have that there is $N_2(\omega)$ such that

$$\|U_h(n,\omega)\|^2 < h\frac{\phi}{4}, \quad n \geq N_2(\omega).$$

Suppose $N_3(\omega) = \max(N_1(\omega), N_2(\omega))$. Then by Lemma 6.4.1, we have that $\|X_h\|^2$ obeys

$$\|X_h(n)\|^2 = \|X_h(N_3)\|^2 - \sum_{i=N_3+1}^{n} h\left\{2\langle f(X_h^\star(i-1)), X_h^\star(i-1)\rangle - \frac{1}{h}\|U_h(i)\|^2\right\}$$
$$- \sum_{i=N_3+1}^{n} h^2\|f(X_h^\star(i-1))\|^2 + M(n) - M(N_3), \quad n \geq N_3+1.$$

Since for $n \geq N_3(\omega)$ we have

$$2\langle X_h^\star(n-1,\omega), f(X_h^\star(n-1,\omega))\rangle - \frac{1}{h}\|U_h(n,\omega)\|^2 > \frac{3\phi}{4},$$

we get

$$\|X_h(n,\omega)\|^2 \le \|X_h(N_3(\omega),\omega)\|^2 - \frac{3\phi h}{4}(n - N_3(\omega)) + M(n,\omega) - M(N_3(\omega),\omega),$$
$$n \ge N_3(\omega)+1, \quad \omega \in A_1. \quad (6.7.7)$$

Now, recall that M is defined by (6.4.3) where $Y^{(j)}$ is given by (6.4.2) for $j = 1,\dots,r$. Notice by (6.4.2) that $Y^{(j)}(n)$ is an $\mathcal{F}^\xi(n)$–measurable random variable. Since ξ obeys Assumption 6.2.1, it follows that all the conditions of Lemma 6.4.3 hold, and that the martingale M is in the form of (6.4.9) in Lemma 6.4.3. Therefore, it follows that M obeys (6.4.8), so that, if we define

$$\Omega_l = \{\omega : \lim_{n\to\infty} M(n,\omega) \text{ exists and is finite}\}$$

and

$$\Omega_\infty = \{\omega : \liminf_{n\to\infty} M(n,\omega) = -\infty, \quad \limsup_{n\to\infty} M(n,\omega) = +\infty\}$$

then $\Omega_l \cup \Omega_\infty =: \Omega_2$ is an a.s. event. Since Ω_2 is a.s., it follows that either (or both) of $A_2 := A_1 \cap \Omega_l$ and $A_3 := A_1 \cap \Omega_\infty$ are events of positive probability.

Suppose that $\mathbb{P}[A_2] > 0$. Then, for each $\omega \in A_2$ we have that $M(n,\omega)$ has a finite limit (say $L(\omega)$) as $n \to \infty$, and that $\|X_h(n,\omega)\| \to \infty$ as $n \to \infty$. Taking the liminf as $n \to \infty$ on both sides of (6.7.7) gives

$$\begin{aligned}
+\infty &= \liminf_{n\to\infty} \|X_h(n,\omega)\|^2 \\
&\le \|X_h(N_3(\omega),\omega)\|^2 - M(N_3(\omega),\omega) + \liminf_{n\to\infty}\left\{-\frac{3\phi h}{4}(n - N_3(\omega)) + M(n,\omega)\right\} \\
&= -\infty,
\end{aligned}$$

a contradiction. Therefore, we have $\mathbb{P}[A_2] = 0$.

Suppose now that $\mathbb{P}[A_3] > 0$. Then, for each $\omega \in A_3$ it follows from the definition of A_3 that $\liminf_{n\to\infty} M(n,\omega) = -\infty$, and that $|X_h(n,\omega)| \to \infty$ as $n \to \infty$. Taking the liminf as $n \to \infty$ on both sides of (6.7.7) gives

$$\begin{aligned} +\infty &= \liminf_{n\to\infty} \|X_h(n,\omega)\|^2 \\ &\leq \|X_h(N_3(\omega),\omega)\|^2 - M(N_3(\omega),\omega) + \liminf_{n\to\infty}\left\{-\frac{3\phi h}{4}(n-N_3(\omega)) + M(n,\omega)\right\} \\ &= -\infty, \end{aligned}$$

a contradiction. Therefore, we have $\mathbb{P}[A_3]=0$. Therefore, we have that $0=\mathbb{P}[A_2\cup A_3] = \mathbb{P}[A_1\cap\Omega_2]>0$, a contradiction. Hence $\mathbb{P}[A_1]=0$, and so $\mathbb{P}[A]=0$, which proves the result. □

6.7.5 Proof of Theorem 6.3.6

To prove this, we first consider the case when $\sigma_h \in \ell^2(\mathbb{N})$. In this case, Theorem 6.3.2 implies that $\lim_{n\to\infty} X_h(n)=0$, a.s. Therefore, we concentrate next on the case when $\sigma_h \notin \ell^2$. An important step to achieve this is to prove the following lemma.

Lemma 6.7.5. *Suppose that f obeys* (4.1.9) *and (X_h, X_h^*) is a solution of* (5.1.2) *and* (5.1.3). *Suppose also that $\sigma_h \notin l^2(\mathbb{N})$. Then*

$$\liminf_{n\to\infty} X_h(n) \leq 0 \leq \limsup_{n\to\infty} X_h(n), \quad a.s.$$

Proof. Suppose $\liminf_{n\to\infty} X_h(n)>0$ with positive probability. Then there exists an event A with $\mathbb{P}[A]>0$, such that

$$A=\{\omega : \liminf_{n\to\infty} X_h(n,\omega)=\underline{X}(\omega)>0\}.$$

For $\omega\in A$ define $\underline{X}(\omega) := \liminf_{n\to\infty} X_h(n,\omega)>0$. Suppose $\liminf_{n\to\infty} X_h^\star(n,\omega)=0$, so that there exists a sequence $(n_j(\omega))_{j=1}^\infty$ such that $n_j(\omega)\uparrow\infty$ as $j\to\infty$ such that $\lim_{j\to\infty} X_h^\star(n_j(\omega),\omega)=0$. Therefore, as $X_h(n,\omega)=X_h^\star(n,\omega)+hf(X_h^\star(n,\omega))$, we have that

$$0 < \underline{X}(\omega) = \liminf_{n\to\infty} X(n,\omega) \le \lim_{j\to\infty} X_h(n_j(\omega),\omega)$$

$$= \lim_{j\to\infty} \{X_h^\star(n_j(\omega),\omega) + hf(X_h^\star(n_j(\omega),\omega))\} = 0,$$

a contradiction. Hence for each $\omega \in A$ we have that

$$\liminf_{n\to\infty} X_h^\star(n,\omega) =: \underline{X}^*(\omega) > 0.$$

Therefore, for each $\omega \in A$, there is $N^*(\omega) \in \mathbb{N}$ such that $X_h(n,\omega) \ge \underline{X}(\omega)/2$ and $X_h^\star(n,\omega) \ge \underline{X}^*(\omega)/2$ for all $n \ge N^*(\omega)$. Let $n \ge N^*(\omega)$. Since

$$X_h(n+1) = X_h^\star(n) + \sqrt{h}\sigma_h(n)\xi(n+1) = X_h(n) - hf(X_h^\star(n)) + \sqrt{h}\sigma_h(n)\xi(n+1),$$

we have

$$\begin{aligned}
X_h(n+1,\omega) &= X_h(N^*(\omega),\omega) + \sum_{j=N^*(\omega)}^{n} \{X_h(j+1,\omega) - X_h(j,\omega)\} \\
&= X_h(N^*(\omega),\omega) + \sum_{j=N^*(\omega)}^{n} -hf(X_h^\star(j,\omega)) + \sum_{j=N^*(\omega)}^{n} \sqrt{h}\sigma_h(j)\xi(j+1,\omega) \\
&= X_h(N^*(\omega),\omega) - h\sum_{j=N^*(\omega)}^{n} f((X_h^*(j,\omega)) + \sum_{j=0}^{n} \sqrt{h}\sigma_h(j)\xi(j+1,\omega) \\
&\quad - \sum_{j=0}^{N^*(\omega)-1} \sqrt{h}\sigma_h(j)\xi(j+1,\omega) \\
&\le X_h(N^*(\omega),\omega) - \sum_{j=0}^{N^*(\omega)-1} \sqrt{h}\sigma_h(j)\xi(j+1,\omega) + M_h(n+1),
\end{aligned}$$

where we have defined the martingale M_h by

$$M_h(n+1) = \sum_{j=0}^{n} \sqrt{h}\sigma_h(j)\xi(j+1), \quad n \ge 0.$$

Since $\sigma_h \notin \ell^2(\mathbb{N})$, we have that for a.a. $\omega \in A$, $\liminf_{n\to\infty} M_h(n+1,\omega) = -\infty$. Therefore, we have

$$0 < \liminf_{n\to\infty} X_h(n+1,\omega) \le -\infty \quad \text{for a.a. } \omega \in A,$$

a contradiction. Therefore $\mathbb{P}[A] = 0$, so $\liminf_{n\to\infty} X_h(n) \le 0$, a.s. One can proceed analogously to prove that $\limsup_{n\to\infty} X_h(n) \ge 0$ a.s. □

Proof of Theorem 6.3.6. Define

$$A_1 = \{\omega : \lim_{n\to\infty} |X_h(n,\omega)| = +\infty\}, \quad A_0 = \{\omega : \lim_{n\to\infty} X_h(n,\omega) = 0\}.$$

Note that Theorem 6.3.4 and the hypothesis $S_h(\epsilon) < +\infty$ implies that $\Omega^* = A_1 \cup A_0$ is an a.s. event. Suppose A_1 is an event with positive probability. Let

$$\Omega_1 = \{\omega : \liminf_{n\to\infty} X_h(n,\omega) \le 0, \quad \limsup_{n\to\infty} X_h(n,\omega)\} \ge 0\}$$

and $\Omega_2 = \{\omega : \lim_{n\to\infty} \sqrt{h}\sigma_h(n)\xi(n+1,\omega) = 0\}$. By Lemma 6.7.5, Ω_1 is an a.s. event, and $S_h(\epsilon) < +\infty$ for all $\epsilon > 0$ implies that Ω_2 is an a.s. event. Define $A_2 = A_1 \cup \Omega_1 \cup \Omega_2$. Then $\mathbb{P}[A_2] = \mathbb{P}[A_1] > 0$.

Next, let $\epsilon \in (0, 1/2)$. Then for every $\omega \in A_2$, there exists an $N_0(\omega,\epsilon)$ such that for all $n \ge N_0(\omega,\epsilon)$ we have $|\sqrt{h}\sigma_h(n)\xi(n+1,\omega)| < \epsilon$ and $|X_n(n,\omega)| > 1/\epsilon$. Since $\lim_{n\to\infty} |X_h(n,\omega)| = +\infty$, $\liminf_{n\to\infty} X_h(n,\omega) \le 0$ and $\limsup_{n\to\infty} X(n,\omega) \ge 0$, we must have

$$\liminf_{n\to\infty} X_h(n,\omega) = -\infty, \quad \limsup_{n\to\infty} X_h(n,\omega) = +\infty.$$

Therefore as $\lim_{n\to\infty} |X_h(n,\omega)| = +\infty$, it follows that there exists $N^*(\omega,\epsilon) > N_0(\omega,\epsilon)$ such that

$$X_h(N^*(\omega,\epsilon),\omega) < -\frac{1}{\epsilon}, \quad X_h(N^*(\omega,\epsilon)+1,\omega) > \frac{1}{\epsilon}.$$

Therefore

$$\begin{aligned}\frac{1}{\epsilon} < X_h(N^*(\omega,\epsilon)+1,\omega) &= X_h^*(N(\omega,\epsilon),\omega) + \sqrt{h}\sigma_h(n)\xi(N(\omega,\epsilon),\omega)\\ &\le X_h^*(N(\omega,\epsilon),\omega) + \epsilon.\end{aligned}$$

Finally, because $X_h(N^*(\omega,\epsilon),\omega) < -1/\epsilon < 0$, therefore, we have that $X_h(N^*(\omega,\epsilon),\omega) \leq X_h^\star(N^*(\omega,\epsilon),\omega) \leq 0$. Therefore

$$\frac{1}{\epsilon} \leq X_h^\star(N(\omega,\epsilon),\omega) + \epsilon \leq \epsilon.$$

Hence $\epsilon^2 \geq 1$. But $\epsilon \in (0,1/2)$, which is a contradiction. Therefore $\mathbb{P}[A_1] = 0$ and so as A_0 and A_1 are disjoint events we have

$$1 = \mathbb{P}[\Omega^*] = \mathbb{P}[A_1 \cup A_0] = \mathbb{P}[A_1] + \mathbb{P}[A_0] = \mathbb{P}[A_0].$$

Thus $A_0 = \{\omega : \lim_{n\to\infty} X_h(n,\omega) = 0\}$ is an a.s. event, which finishes the proof. □

6.8 Proof of Theorem 6.3.7

6.8.1 Proof of parts (C), (A), and limsup in part (B)

Part (C) of the Theorem follows from part (A) of Theorem 6.3.3. Part (A) is a consequence of Theorem 6.3.5, because the condition (6.3.11) on f is implied by (6.3.12). The lower bound in part (B) is a consequence of part (B) of Theorem 6.3.3. Hence the result holds if we can establish the upper bound in part (B).

To do this, notice first by part (B) of Lemma 6.3.1 that there exists an a.s. event Ω_1 given by $\Omega_1 = \{\omega : \limsup_{n\to\infty} \|U_h(n,\omega)\|_1 \leq c_2\}$, where c_2 is given by (6.3.5). Therefore, there is a deterministic $B_0 > c_2$ such that for each $\omega \in \Omega_1$ there is an $N = N(\omega) \in \mathbb{N}$ such that $\|U_h(n+1,\omega)\|_2 \leq \|U_h(n+1,\omega)\|_1 \leq B_0$ for all $n \geq N$. Since f obeys (6.3.12), we may define

$$M(B_0) = \sup\{y > 0 : \inf_{\|x\|_2 \geq y} \frac{\langle x, f(x)\rangle}{\|x\|_2} \leq \frac{2B_0}{h}\}.$$

Define $C(B_0) = B_0 + M(B_0)$. Now suppose that $\|X_h(n,\omega)\|_2 > C(B_0)$ for all $n \geq N(\omega)$. Let $n \geq N(\omega)$. By (6.2.3c) and (6.3.1), we have $\|X_h^\star(n,\omega)\|_2 \geq \|X_h(n+1,\omega)\|_2 - \|U_h(n+1,\omega)\|_2 \geq C(B_0) - B_0 = M(B_0)$. Hence by the definition of $M(B_0)$ we have

$$\frac{\langle X_h^\star(n,\omega), f(X_h^\star(n,\omega))\rangle}{\|X_h^\star(n,\omega)\|_2} \geq \frac{2B_0}{h}.$$

Therefore by (6.2.3b) we get

$$\langle X_h^\star(n,\omega), X_h(n,\omega)\rangle = \|X_h^\star(n,\omega)\|_2^2 + h\langle f(X_h^\star(n,\omega)), X_h^\star(n,\omega)\rangle$$
$$\geq \|X_h^\star(n,\omega)\|_2^2 + h\frac{2B_0}{h}\|X_h^\star(n,\omega)\|_2.$$

By the Cauchy–Schwartz inequality,

$$\|X_h^\star(n,\omega)\|_2\|X_h(n,\omega)\|_2 \geq \|X_h^\star(n,\omega)\|_2^2 + 2B_0\|X_h^\star(n,\omega)\|_2.$$

Since $\|X_h^\star(n,\omega)\| > 0$, we have $\|X_h(n,\omega)\|_2 \geq \|X_h^\star(n,\omega)\|_2 + 2B_0$, or $\|X_h^\star(n,\omega)\|_2 \leq \|X_h(n,\omega)\|_2 - 2B_0$. Therefore, for $n \geq N$ by (6.2.3c) we have

$$\|X_h(n+1,\omega)\|_2 \leq \|X_h^\star(n,\omega)\| + B_0 \leq \|X_h(n,\omega)\|_2 - B_0.$$

Therefore, we have

$$C(B_0) \leq \|X_h(N+n,\omega)\|_2 \leq \|X_h(N,\omega)\|_2 - B_0 n, \quad n \geq 0,$$

which is a contradiction. Thus, there exists $N_1 = N_1(\omega) \geq N(\omega)$ such that $\|X_h(N_1)\|_2 \leq C(B_0)$.

We prove by induction that $\|X_h(n)\|_2 \leq C(B_0)$ for all $n \geq N_1$. Suppose that this is true at level n. Suppose that $\|X_h^\star(n,\omega)\| > C(B_0) - B_0$. Now by (6.2.3b) we get

$$\langle X_h^\star(n,\omega), X_h(n,\omega)\rangle = \|X_h^\star(n,\omega)\|^2 + h\langle f(X_h^\star(n,\omega)), X_h^\star(n,\omega)\rangle$$
$$\geq \|X_h^\star(n,\omega)\|^2 + h\frac{2B_0}{h}\|X_h^\star(n,\omega)\|.$$

By the Cauchy–Schwartz inequality,

$$\|X_h^\star(n,\omega)\|_2\|X_h(n,\omega)\|_2 \geq \|X_h^\star(n,\omega)\|_2^2 + 2B_0\|X_h^\star(n,\omega)\|_2.$$

Since $\|X_h^\star(n,\omega)\| > 0$, we have $\|X_h(n,\omega)\|_2 \geq \|X_h^\star(n,\omega)\|_2 + 2B_0 > C(B_0) + B_0$. But $C(B_0) \geq \|X_h(n,\omega)\|_2 > C(B_0) + B_0$, a contradiction. Hence $\|X_h^\star(n,\omega)\| \leq C(B_0) - B_0$. Therefore by (6.2.3c), we have

$$\|X_h(n+1,\omega)\|_2 \leq \|X_h^\star(n,\omega)\|_2 + B_0 \leq C(B_0),$$

which proves the claim at level $n+1$. Therefore we have $\|X_h(n,\omega)\|_2 \leq C(B_0)$ for all $n \geq N_1(\omega)$ and all $\omega \in \Omega_1$, which is an a.s. event. Hence $\limsup_{n\to\infty} \|X_h(n,\omega)\|_2 \leq C(B_0)$ for each $\omega \in \Omega_1$. Therefore, we have $\limsup_{n\to\infty} \|X_h(n)\|_2 \leq c_4$ a.s., where $c_4 := C(B_0)$ is deterministic.

6.8.2 Proof of liminf in part (B)

It remains to prove in the following result.

Lemma 6.8.1. *Suppose that $S_h(\epsilon) < +\infty$ for all $\epsilon > \epsilon'$ and $S_h(\epsilon) = +\infty$ for all $\epsilon < \epsilon'$. Then*

$$\liminf_{n\to\infty} \|X_h(n)\| = 0, \quad a.s.$$

In order to do this we need first a technical lemma.

Lemma 6.8.2. *$S_h(\epsilon) < +\infty$ for all $\epsilon > \epsilon'$ and $S_h(\epsilon) = +\infty$ for all $\epsilon < \epsilon'$. Then*

$$\lim_{n\to\infty} \|\sigma_h(n)\|_F = 0, \tag{6.8.1}$$

and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \|\sigma_h(j-1)\xi(j)\|^2 = 0, \quad a.s. \tag{6.8.2}$$

Proof. First, we note that if $S_h(\epsilon) < +\infty$ for some $\epsilon > 0$, it follows that

$$1 - \Phi\left(\frac{\epsilon}{h\|\sigma_h(n)\|_F}\right) \to 0, \text{ as } n \to \infty.$$

and therefore (6.8.1) holds. Define

$$\beta(n) = \|\sigma_h(n-1)\xi(n)\|^2, \quad n \geq 1.$$

Notice that the independence of $\xi(n)$ imply that $(\beta(n))_{n\geq 1}$ is a sequence of independent random variables. Using (6.5.2), we have that

$$\mathbb{E}[\beta(n)] = \mathbb{E}[\|\sigma_h(n-1)\xi(n)\|^2] = \|\sigma_h(n-1)\|_F^2, \quad n \geq 1.$$

Notice from (6.8.1) that $\mathbb{E}[\beta(n)] \to 0$ as $n \to \infty$. Define $\tilde{\beta}(n) = \beta(n) - \mathbb{E}[\beta(n)]$ for $n \geq 1$. Then $(\tilde{\beta}(n))_{n\geq 1}$ is a sequence of independent zero mean random variables. We will presently show that

$$\lim_{n\to\infty} \mathbb{E}[\beta(n)^4] = 0. \tag{6.8.3}$$

Taken together with $\mathbb{E}[\beta(n)] \to 0$ as $n \to \infty$, we see that $\lim_{n\to\infty} \mathbb{E}[\tilde{\beta}(n)^4] = 0$, so that there exists a constant $K > 0$ for which $\mathbb{E}[\tilde{\beta}(n)^4] \le K$ for all $n \ge 0$. Therefore, by this estimate, and the fact that $(\tilde{\beta}(n))_{n\ge1}$ is a sequence of independent zero mean random variables, the version of the strong law of large numbers appearing in Theorem 7.2 in [79], enables us to conclude that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \tilde{\beta}(j) = 0, \quad \text{a.s.}$$

Since $\mathbb{E}[\beta(n)] \to 0$ as $n \to \infty$, we have that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \beta(j) = 0, \quad \text{a.s.}$$

which is precisely (6.8.2).

It remains to prove (6.8.3). Since $\|Ax\|_2 \le \|A\|_F\|x\|_2$ for any $x \in \mathbb{R}^r$ and $A \in \mathbb{R}^{d\times r}$, we have that

$$\begin{aligned}\mathbb{E}[\beta(n)^4] &= \mathbb{E}[\|\sigma_h(n-1)\xi(n)\|_2^8] \le \mathbb{E}[\|\sigma_h(n-1)\|_F^8\|\xi(n)\|_2^8] \\ &= \|\sigma_h(n-1)\|_F^8\mathbb{E}[\|\xi(n)\|_2^8].\end{aligned}$$

Since $(\xi(n))_{n\ge1}$ are identically and distributed Gaussian vectors with independent entries (each of which is a standard normal random variable), we have that there is $K_1 := \mathbb{E}[\|\xi(n)\|_2^8]$ for all $n \ge 1$. Hence $\mathbb{E}[\beta(n)^4] \le K_1\|\sigma_h(n-1)\|_F^8$ for $n \ge 1$. Since (6.8.1) holds, we have that $\mathbb{E}[\beta(n)^4] \to 0$ as $n \to \infty$, as claimed. □

6.8.3 Proof of Lemma 6.8.1

Recall the representation of $\|X_h\|^2$ in (6.4.1) i.e.,

$$\begin{aligned}\|X_h(n)\|^2 = \|X_h(0)\|^2 - 2\sum_{i=1}^{n} h\langle f(X_h^\star(i-1)), X_h^\star(i-1)\rangle + \sum_{i=1}^{n} h\|\sigma_h(i-1)\xi(i)\|^2 \\ - \sum_{i=1}^{n} h^2\|f(X_h^\star(i-1))\|^2 + M(n), \quad n \ge 1, \quad (6.8.4)\end{aligned}$$

where the martingale M defined by (6.4.2) and (6.4.3) i.e.,

$$Y^{(j)}(n) = 2\sqrt{h}\sum_{k=1}^{d}[X_h^\star(n)]_k[\sigma_h(n)]_{kj}, \quad j=1,\dots,r, \quad n\geq 1,$$

$$M(n) = \sum_{i=1}^{n}\sum_{j=1}^{r} Y^{(j)}(i-1)\xi^{(j)}(i), \quad n \geq 1.$$

Then M has quadratic variation estimated by (6.5.3) i.e.,

$$\langle M\rangle(n) \leq 4h\sum_{j=0}^{n-1}\|X_h^\star(j)\|^2\|\sigma_h(j)\|_F^2.$$

Since $\|X_h^*(n)\|$ is a bounded sequence, and $\|\sigma_h(n)\|_F \to 0$ as $n\to\infty$, we have that

$$\lim_{n\to\infty}\frac{1}{n}\langle M\rangle(n) = 0, \quad \text{a.s.}$$

Suppose that $A_1 = \{\omega : \lim_{n\to\infty}\langle M\rangle(n,\omega) = +\infty\}$. Then by the Law of Large numbers for martingales, we have

$$\lim_{n\to\infty}\frac{1}{n}M(n,\omega) = \lim_{n\to\infty}\frac{M(n,\omega)}{\langle M\rangle(n,\omega)}\cdot\frac{\langle M\rangle(n,\omega)}{n} = 0,$$

for a.a. $\omega \in A_1$. Suppose that $A_2 = \{\omega : \lim_{n\to\infty}\langle M\rangle(n,\omega) < +\infty\}$. Then by the martingale convergence theorem we have that $\lim_{n\to\infty} M(n,\omega)$ is finite for a.a. $\omega\in A_2$, so we automatically have $\lim_{n\to\infty} M(n,\omega)/n = 0$ for a.a. $\omega \in A_2$. Therefore we have that

$$\lim_{n\to\infty}\frac{M(n)}{n} = 0, \quad \text{a.s.} \tag{6.8.5}$$

By Lemma 6.8.1 we have that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} h\|\sigma_h(i-1)\xi(i)\|^2 = 0, \quad \text{a.s.}$$

Recalling that $n\mapsto \|X_h(n)\|$ is a.s. bounded, we can use the last limit, (6.8.5) and (6.8.4) to obtain

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} hR(X_h^\star(i-1)) = 0, \quad \text{a.s.} \tag{6.8.6}$$

recalling the definition of R from (6.5.5).

Next, we suppose that A defined by

$$A = \{\omega : \liminf_{n\to\infty}\|X_h(n,\omega)\| > 0\}.$$

is such that $\mathbb{P}[A] > 0$. Let $\Omega_1 = \{\omega : \limsup_{n\to\infty} \|X_h(n,\omega)\| < +\infty\}$ and $A_1 = A \cap \Omega_1$. Then $\mathbb{P}[A_1] = \mathbb{P}[A] > 0$. Then for each $\omega \in A_1$ we have $\liminf_{n\to\infty} \|X_h^\star(n,\omega)\| > 0$. Therefore, using the fact that $\|X_h^\star(n)\| \le \|X_h(n)\|$, we see that $\|X_h^\star(n,\omega)\|$ is bounded for $\omega \in A_1$ and therefore, for every $\omega \in A_1$ there is an $N(\omega) \in \mathbb{N}$ and $0 < \underline{X}_h(\omega) \le \overline{X}_h(\omega) < +\infty$ such that

$$\frac{1}{2}\underline{X}_h(\omega) \le \|X_h^\star(n,\omega)\| \le 2\overline{X}_h(\omega), \quad n \ge N(\omega).$$

Now, we recall that $R : \mathbb{R}^d \to \mathbb{R}$ defined by (6.5.5) is continuous and obeys $R(x) > 0$ for all $x \neq 0$ and $R(0) = 0$. Therefore, for any $0 < a \le b < +\infty$, we have

$$\inf_{a\le\|x\|\le b} R(x) =: L_h(a,b) > 0.$$

Therefore, for all $n \ge N(\omega)$ we have

$$R(\|X_h^\star(n,\omega)\|) \ge L_h\left(\frac{1}{2}\underline{X}_h(\omega), 2\overline{X}_h(\omega)\right) =: \lambda_h(\omega) > 0.$$

Hence, as $R(x) \ge 0$ for all $x \ge 0$, we have for $n \ge N(\omega)+1$ that

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} hR(X_h^\star(i-1,\omega)) &\ge \frac{1}{n}\sum_{i=N(\omega)+1}^{n} hR(X_h^\star(i-1,\omega)) \\ &\ge \frac{1}{n}\sum_{i=N(\omega)+1}^{n} h\lambda_h(\omega) = \frac{1}{n}(n-N(\omega))h\lambda_h(\omega).\end{aligned}$$

Therefore, we have for each $\omega \in A_1$

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} hR(X_h^\star(i-1,\omega)) \ge h\lambda_h(\omega) > 0,$$

or

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} hR(X_h^\star(i-1)) > 0, \quad \text{on } A_1.$$

Since $\mathbb{P}[A_1] > 0$ this contradicts (6.8.6), and so we must have $\mathbb{P}[A_1] = 0$. Hence we have that $\liminf_{n\to\infty} \|X_h(n)\| = 0$ a.s. as claimed.

6.9 Proof of Theorem 6.3.1

We prove the result in two parts. First, we prove everything apart from the limit inferior in part (B), and then show that

$$\liminf_{n\to\infty} \|Y_h(n)\| = 0, \quad \text{a.s.}$$

in case (B), when the solution has already been shown to be bounded.

6.9.1 Proof of Theorem 6.3.1 apart from liminf in part (B)

Part (C) is a direct consequence of part (A) of Theorem 6.3.3. The lower bound in part (B) is an automatic consequence of part (B) of Theorem 6.3.3.

It remains to prove part (A) and the upper bound in part (B). We start by determining the eigenvalues of $C(h)$. If $c_{C(h)}$ be the characteristic polynomial of $C(h)$, then we have $c_{C(h)}(0) = (-1)^d \det(C(h)) \neq 0$ and

$$c_{C(h)}(\lambda) = \frac{1}{\det(I - Ah)}(\lambda h)^d c_A\left(\frac{\lambda - 1}{\lambda h}\right), \quad \lambda \neq 0.$$

Therefore, λ_A is an eigenvalue of A if and only if $\lambda_h = 1/(1 - \lambda_A h)$ is an eigenvalue of $C(h)$. Since (6.3.8) holds, 0 is not an eigenvalue of A, and for every $h > 0$,

$$\text{Re}(\lambda_A) < 0 < \frac{h}{2}|\lambda_A|^2.$$

This implies that $|1 - h\lambda_A| < 1$, and hence that $|\lambda_h| < 1$ for each eigenvalue of $C(h)$. Y_h obeys

$$Y_h(n) = C(h)^n \zeta + \sum_{j=1}^{n} C(h)^{n-j} U_h(j), \quad n \geq 0.$$

For part (A), if $S_h(\epsilon) < +\infty$ for every $\epsilon > 0$, by Lemma 6.3.1, we have that $U_h(n) \to 0$ as $n \to \infty$. Since all eigenvalues of $C(h)$ are less than unity in modulus, it follows that $\sum_{j=1}^{n} C(h)^{n-j} U_h(j) \to 0$ as $n \to \infty$, proving the result. To prove the upper bound in part (B), we note that for every $\epsilon \in (0, (1 - \rho(C(h)))/2)$, there is a norm $\|\cdot\|_N$ such that

$$\|C(h)^k x\|_N \leq \|C(h)^k\|_N \|x\|_N \leq (\rho(C(h)) + \epsilon)^k \|x\|_N \text{ for all } k \geq 0 \text{ and all } x \in \mathbb{R}^d.$$

Hence we have

$$\|Y_h(n)\|_N \leq (\rho(C(h)) + \epsilon)^n \|\zeta\|_N + \sum_{j=1}^{n} (\rho(C(h)) + \epsilon)^{n-j} \|U_h(j)\|_N.$$

Therefore taking limits and using the fact that there is a $c > 0$ such that $\|x\|_N \leq c\|x\|_1$ for all $x \in \mathbb{R}^d$, we obtain

$$\limsup_{n\to\infty} \|Y_h(n)\|_N \leq \frac{1}{1 - (\rho(C(h)) + \epsilon)} c \limsup_{n\to\infty} \|U_h(n)\|_1.$$

By part (C) of Lemma 6.3.1, the righthand side is deterministic and finite, so the upper bound in part (B) has been established.

6.9.2 Proof of zero liminf and average in case (B)

We start by recalling a result of which may be found in e.g., Rugh [68].

Lemma 6.9.1. *Let C be a $d \times d$ real matrix. If all the eigenvalues of C lie within the unit disc in the complex plane, then there exists a positive definite $d \times d$ real matrix M such that*

$$C^T MC - M = -I_d.$$

Conversely, the existence of a positive definite M implies that all the eigenvalues of C lie inside the unit disc in the complex plane.

We will have achieved our goal once we have shown the following result.

Lemma 6.9.2. *Suppose that the matrix A obeys* (6.3.8) *and that there exists $\epsilon' > 0$ such that $S_h(\epsilon)$ defined by* (6.3.2) *obeys $S_h(\epsilon) < +\infty$ for all $\epsilon > \epsilon'$ and $S_h(\epsilon) = +\infty$ for all $\epsilon < \epsilon'$. Then*

$$\liminf_{n\to\infty} \|Y_h(n)\| = 0, \quad \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \|Y_h(j)\|^2 = 0, \quad a.s.$$

Proof. It has been shown above that all the eigenvalues of the matrix $C = C(h)$ lie inside the unit disc in the complex plane. Therefore, by Lemma 6.9.1 there exists a positive definite matrix $M = M(h)$ such that

$$C(h)^T M(h)C(h) - M(h) = -I_d.$$

Hereinafter, we write $M = M(h)$ and $C = C(h)$.

Define the function $V : \mathbb{R}^d \to \mathbb{R}$ by $V(x) = x^T Mx$ for $x \in \mathbb{R}^d$. We have that $Y_h(n+1) = CY_h(n) + U_h(n+1)$ for $n \geq 0$ with $Y_h(0) = \zeta$. Therefore, we have

$$\begin{aligned}V(Y_n(n+1)) - V(Y_h(n)) &= -Y^T(n)Y(n) + Y_h(n)^T C^T MU_h(n+1)\\ &\quad + U_h(n+1)^T MCY_h(n) + U_h(n+1)^T MU_h(n+1), \quad n \geq 0.\end{aligned}$$

using $C^TMC - M = -I_d$ to simplify the first term on the right hand side. We now simplify the other terms on the right hand side.

Since M is a positive definite matrix, there exists a matrix P such that $M = PP^T$. Then

$$U_h(n+1)^T M U_h(n+1) = U_h(n+1)^T PP^T U_h(n+1)$$
$$= (P^T U_h(n+1))^T P^T U_h(n+1) = \|P^T U_h(n+1)\|_2^2.$$

Define $k(n+1) = Y_h(n)^T C^T M U_h(n+1) + U_h(n+1)^T M C Y_h(n)$ for $n \geq 0$. Then using the fact that M is symmetric and the definition of U_h, we get

$$\begin{aligned}
k(n+1) &= (M^T C Y_h(n))^T U_h(n+1) + U_h(n+1)^T M C Y_h(n) \\
&= (MCY_h(n))^T U_h(n+1) + U_h(n+1)^T M C Y_h(n) \\
&= 2\langle MCY_h(n), U_h(n+1)\rangle \\
&= 2\sqrt{h}\langle MCY_h(n), \sigma_h(n)\xi(n+1)\rangle.
\end{aligned}$$

Therefore

$$k(n+1) = 2\sqrt{h}\sum_{j=1}^{r}\left(\sum_{i=1}^{d}[MCY_h(n)]_i[\sigma_h(n)]_{ij}\right)\xi_j(n+1), \quad n \geq 0. \tag{6.9.1}$$

Hence we have

$$V(Y_h(n+1)) - V(Y_h(n)) = -Y_h^T(n)Y_h(n) + k(n+1) + \|P^T U_h(n+1)\|_2^2, \quad n \geq 0,$$

so if we define

$$K(n) = \sum_{l=1}^{n} k(l) = \sum_{j=1}^{r}\sum_{l=1}^{n}\left(\sum_{i=1}^{d} 2\sqrt{h}[MCY_h(l-1)]_i[\sigma_h(l-1)]_{ij}\right)\xi_j(l), \quad n \geq 1,$$

then K is a martingale and

$$V(Y_h(n)) - V(\zeta) = -\sum_{l=0}^{n-1}\|Y(l)\|_2^2 + K(n) + \sum_{l=0}^{n-1}\|P^T U_h(l+1)\|_2^2, \quad n \geq 1. \tag{6.9.2}$$

We now estimate the asymptotic behaviour of the last two terms on the righthand side of (6.9.2). The quadratic variation of K is given by

$$\langle K\rangle(n) = \sum_{l=1}^{n}\left(\sum_{i=1}^{d} 2\sqrt{h}[MCY_h(l-1)]_i[\sigma_h(l-1)]_{ij}\right)^2.$$

By the Cauchy–Schwartz inequality, we have

$$\begin{aligned}\langle K\rangle(n) &\leq \sum_{l=1}^{n} 4h\left(\sum_{i=1}^{d}[MCY_h(l-1)]_i^2\sum_{i=1}^{d}[\sigma_h(l-1)]_{ij}^2\right)\\ &\leq \sum_{l=1}^{n} 4h\|MCY_h(l-1)\|^2\|\sigma_h(l-1)\|_F^2.\end{aligned}$$

Since $\|Y_h(n)\|$ is bounded and $\|\sigma_h(n) \to 0$ as $n \to \infty$ (by Lemma 6.8.2) we have that

$$\lim_{n\to\infty}\frac{1}{n}\langle K\rangle(n) = 0, \quad \text{a.s.}$$

Arguing as in the proof of Lemma 6.8.1, we see that

$$\lim_{n\to\infty}\frac{1}{n}K(n) = 0, \quad \text{a.s.} \tag{6.9.3}$$

As for the last term on the right hand side of (6.9.2)

$$0 \leq \limsup_{n\to\infty}\frac{1}{n}\sum_{l=0}^{n-1}\|P^T U_h(l+1)\|_2^2 \leq \|P^T\|_2^2 \limsup_{n\to\infty}\frac{1}{n}\sum_{l=0}^{n-1}\|U_h(l+1)\|_2^2 = 0, \quad \text{a.s.}$$

by (6.8.2) in Lemma 6.8.2. Hence

$$\lim_{n\to\infty}\frac{1}{n}\sum_{l=0}^{n-1}\|P^T U_h(l+1)\|_2^2 = 0, \quad \text{a.s.} \tag{6.9.4}$$

Since $\|Y_h(n)\|$ is a.s. bounded, we have $V(Y_h(n))/n \to 0$ as $n \to \infty$ a.s. Therefore, using this limit and (6.9.4) and (6.9.3) in (6.9.2) we get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{l=0}^{n-1}\|Y_h(l)\|_2^2 = 0, \quad \text{a.s.}$$

This proves the second statement required.

Moreover, it also implies that $\liminf_{n\to\infty}\|Y_h(n)\|_2 = 0$ a.s. for otherwise we would have

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{l=0}^{n-1}\|Y_h(l)\|_2^2 > 0 \quad \text{with positive probability,}$$

a contradiction. □

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