Second Order Rossby-Haurwitz Wave Interactions

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy, is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

Signed: Edel Adlaghny

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To my husband, Sean, and my parents

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Abstract

Rossby-Haurwitz waves are large sinuous oscillations in the atmosphere and oceans. These planetary waves owe their existence to the rotation and shape of the earth. They are an important wave type for large-scale meteorological processes as they are dominant in determining the patterns of weather in the middle latitudes.

This thesis concerns the interactions of these Rossby-Haurwitz waves within the framework of the vorticity equation for nondivergent planetary flow at second order. Of particular interest is the potential for generating zonal flow, i.e., large-scale atmospheric flow that occurs in an east-west direction. Examining interactions at first order we distinguish between nonresonant interactions and resonant interactions. Resonant interactions are interactions where two Rossby-Haurwitz waves can create a third Rossby-Haurwitz wave, which over time becomes as strong as the two primary waves. The necessary conditions for resonant interactions to occur are derived. It is also shown that zonal flow waves cannot be produced at this order.

Examining second order interactions it is shown that zonal flow can now be generated by a mechanism that disappears in the $\beta-$plane limit. This is the central result of the thesis. Zonal flow can be generated through the exchange of energy within a triad, and this occurs at second order. The amplitudes of the zonal flow terms are not affected until the second order equation. Detailed numerical results are presented underpinning the theoretical results.

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Chapter 1

Introduction

1.1 The aim of the thesis

This thesis progresses beyond first order interactions to examine second order interactions of Rossby-Haurwitz waves. In particular we want to investigate whether large-scale interacting planetary waves can force zonal flow through a resonant interaction energy transfer mechanism. In this introduction we will discuss the work that will be done in the thesis, from the aims of the thesis through to its conclusions. Previous work done on the area is also discussed in this introduction.

1.2 Rossby-Haurwitz waves and zonal flow

Rossby waves are large sinuous oscillations governing the dynamics of particle flow in the atmosphere and oceans. The existence of these waves in the oceans was first theorised by Carl-Gustav Rossby in a paper in 1939. Since then they have been the subject of many investigations, by many different people.

The solutions of the equations governing the dynamics of the oceans and the atmospheres, in simplified form, are Rossby waves. They are the prime example of the winding large-scale motion of the mid-latitude troposphere. They are transverse waves, waves in which the motion of the medium is perpendicular to the motion of the wave, in the horizontal direction. They travel mainly from east to west, following the parallels of latitude.

These waves can readily be seen in the large-scale meanders of the mid-latitude jet stream that are responsible for prevailing seasonal and day-to-day weather patterns. It is more difficult, though, to spot these waves in the oceans. This is because there is a large scale difference in their horizontal and vertical scales. Their horizontal scale is usually of the order of hundreds of kilometres while the amplitude of the oscillation at the sea surface is just a few centimetres. Therefore it is only in recent times with the advent of the use of radar altimeters that these waves have been observed in the oceans [1].

Rossby waves have a considerable affect on large-scale ocean circulation. This in turn means that they have a large influence on weather and climate in middle latitudes. They play an important part in the process by which signals are transmitted from one side of the ocean basin to the other. The interactions of these waves are important for determining the distribution of energy in the atmosphere and ocean. They also affect the western boundary currents like the Gulf stream. They can intensify such currents and push them off their usual course. They interact with general circulation in the atmosphere and can therefore delay the effect of climate events.

It is the interactions of these waves which we investigate in this thesis. We examine the interactions of two of these waves and then three of these waves at both first and second order. In the course of this investigation we will encounter a phenomenon called zonal flow. This is a wind pattern where the winds are parallel or nearly parallel to the lines of latitudes. This generally west to east flow of the wind is called zonal flow. This type of flow tends to establish a pattern with small temperature contrasts from north to south. This zonal flow and its associated small temperature contrasts contribute to mild weather patterns. As part of this thesis we will examine the interactions of Rossby waves and zonal flow.

Rossby waves also occur in completely different circumstances. As previously mentioned they transport energy and angular momentum and this could be a factor in the kind of banded zonal jets seen on Jupiter, as well as on Earth [25]. Rossby waves have also been discovered in the instabilities of magnetically confined plasmas. Recently, the occurrence of drift waves in plasmas, which are dynamically equivalent to Rossby waves were examined [21].

It is for all the aforementioned reasons that Rossby waves are a very important phenomenon. It is felt, therefore, that it is important to gain a better understanding of both the first and second order interactions of these planetary waves. In this thesis we examine the first and second order interactions of these waves and in the course of this analysis we find the occurrence of zonal flow as a result of these interactions.

1.3 Previous work in the area

The aim of this thesis is to obtain more insight into the question of Rossby-Haurwitz wave interactions. Examining the vorticity equation for nondivergent planetary flow, we show that Rossby-Haurwitz waves are linear solutions of this equation. These waves are also referred to as spherical planetary waves.

Many people have previously worked in the area of examining the interaction of Rossby waves. Carl Gustav Rossby first observed these waves in 1939 and since then Haurwitz [5], Lynch [11] and Pedlosky [16] among many others have continued to examine this interesting phenomenon. The simplest context in which to study the interactions of Rossby waves is in a shallow layer of incompressible fluid on the rotating earth. The geometry of this study is greatly helped by ignoring the effects of sphericity, except for allowing for the change of the vertical component of the earth's rotation with respect to latitude. This method is called the β-plane approximation, and is the preferred method for the majority of work previously done in the area.

Since these waves occur on the spherical Earth we are interested in examining Rossby-Haurwitz waves taking the spherical nature of the Earth into account. It has been shown on the plane that if two specific waves interact with each other then resonance will occur.

This means that the forced wave may build up after a sufficient number of oscillations to be comparable in magnitude with the primary waves. At this stage the perturbation would break down and hence this forced wave would be required to be a leading order wave.

We wish to replicate these results on the sphere. To do this the accepted Cartesian equations are converted into their spherical counterparts and we apply our analysis to this equation instead. We find that quite a few differences occur when this is done. Like Kartashova [9] and Reznik [20], we find that the resonance conditions on the sphere differ quite substantially from those found using the β-plane approximation. The conditions found include both inequalities and equations on the wavenumbers n and m.

In 1969 Newell [14] proposed a mechanism whereby zonal flows could be generated through the resonant interaction of Rossby waves. He derived equations describing the long-time behaviour of a resonantly interacting triad, and found that zonal flow was generated at second order. Newell determined that a quartet mechanism was required for zonal flow to be excited through resonance.

In 1977 Loesch [10] similarly investigated the generation of zonal flows through finite amplitude, discrete Rossby waves. Loesch's analysis found that interacting Rossby waves are capable of generating zonal flow on the required time scale. Analogous to Newell, Loesch also discovered that a quartet of waves is required with which to generate zonal flow. We will show in the course of this thesis that a triad solution is solely required to produce zonal flow when we take full account of the sphericity of the Earth.

More recently there has been a lot of renewed interest in Rossby-Haurwitz waves in general, and specifically in zonal flow. It has been shown that zonal flows in a shallow rotating atmosphere can be excited by finite amplitude Rossby waves [15]. The driving mechanism of this instability is due to the Reynolds stresses. This results in a transfer of spectral energy from short scale Rossby waves to long scale zonal flows in the Earth's atmosphere. Similarly, it has been shown that zonal flows in a nonuniform rotating fluid can be excited by finite amplitude Rossby waves [22]. This was also shown by examining Reynolds stresses of short scale Rossby waves.

In this thesis we examine Newell's mechanism and discuss a method of creating zonal flow which is different from those in [15] and [22]. We put forward the hypothesis and subsequently prove that zonal flow can be created through the energy exchange mechanism of a resonantly interacting triad. This method requires one less wave then those suggested by Newell and Loesch.

1.4 Outline of thesis

In Chapter 2 we begin our discussion by deriving the vorticity equation for nondivergent planetary flow. The solutions to this equation are our Rossby-Haurwitz waves. To derive this equation we derive the mass conservation equation and the equation for momentum in Cartesian coordinates. These equations are then converted into their spherical coordinate equivalents. We nondimensionalise the variables in these equations and examine the orders of the resulting terms. From this analysis we can determine the vorticity equation for nondivergent planetary flow.

In Chapter 3 we begin our study of the solutions to this equation. The similarities between our equation and the Laplace equation suggest that we should consider Legendre functions as our wave solution. In fact, this is the case and we find a single wave solution. We are interested in the interactions of these Rossby-Haurwitz waves. Therefore we examine a dyad solution, which is a two wave solution. We are unable to produce resonant interactions through studying a Rossby-Haurwitz wave dyad solution. We conclude in this chapter that to produce meaningful interactions we require a triad solution. We also conclude in this chapter that it is impossible to produce zonal flow at either first or second order if we commence our studies with a dyad solution.

In Chapter 4 we examine both first and second order resonant interactions in full. We establish the conditions required for triad solutions to exist and we examine the possibility of zonal flow being generated due to these triad interactions. We find that, although zonal flow cannot be created at first order, we can create zonal flow at second order when we consider resonant triad interactions. We have discovered the excitation of zonal flow directly from resonant triad interactions.

Finally in Chapter 5 we apply the theory to several examples. Firstly we determine which waves can form triad solutions. Using some of these particular triads we examine the coupling coefficients for each wave coupling. Inspecting these numbers we can determine which zonal flow terms can be produced, and from this we can establish the conditions on the amplitudes of these zonal flow terms.

Finally, in summary, this thesis is a full and comprehensive study of first and second order Rossby-Haurwitz wave interactions. We study the implications of dyad solutions and the necessity for triad solutions when examining resonant interactions. We establish the required conditions on the wavenumbers for such triad solutions to exist. We find that zonal flow cannot be created through an energy transfer mechanism unless we commence our calculation with a triad solution. Finally we determine that, if we study a triad solution, we can produce zonal flow at second order.

Chapter 2

Derivation of equation

The vorticity equation for nondivergent planetary flow is used to study Rossby wave solutions in the oceans and the atmosphere. This equation is derived from the conservation law of mass and Newton's equation, which describes the non-conservation of momentum due to the presence of forces. In this chapter we state the equations in Cartesian coordinates and then convert them into spherical coordinates.

We reduce the equations in spherical coordinates from three dimensions to the two-dimensional equations which are widely used in oceanography and meteorology. From the equation for the conservation of mass an expression for a stream function ψ can be determined. Reducing the two momentum equations down to one equation by calculating the third component of the curl and nondimensionalising the variables we obtain the main equation which we are examining. This equation is the vorticity equation for nondivergent planetary flow, the solutions of which will be shown to be Rossby-Haurwitz waves.

2.1 Basic equations

2.1.1 Conservation of mass

We will derive the equation for the conservation of mass using a planar coordinate system which is based at the center of the Earth, and is in rotation with the Earth. This law considers the rate of change of mass in a volume V,

$$\frac{d}{dt}\int_V \rho(\vec{x},t)\, d^3x \tag{2.1.1}$$

where ρ is the mass density. In this case, where particles are neither created or destroyed, the only way that mass in a given volume V can change is through particles moving across the boundary ∂V of V. To derive the formula for the flux we consider a small part of the boundary of size $\Delta\sigma$ and a volume of size $\Delta x_n \Delta\sigma$. Here Δx_n is the average distance the fluid particles travel perpendicular to the boundary during a time interval of length Δt. The particles in this volume are those which will leave V during time Δt, and $\rho\frac{\Delta x_n}{\Delta t}\Delta\sigma$ is approximately the mass lost (or gained) per unit time.

Summing over all boundary areas and taking the limit for finer and finer partitions we obtain

$$\int_{\partial V} \rho\frac{d\vec{x}}{dt}\cdot\vec{n}\, d\sigma \tag{2.1.2}$$

for the outward flux, where $\vec{n}$ is the unit outward normal. Combining (2.1.1) and (2.1.2) and using the divergence theorem yields the integral formula for the conservation of mass

$$\begin{aligned}\frac{d}{dt}\int_V \rho\, d^3x &= -\int_{\partial V} \rho \frac{d\vec{x}}{dt}\cdot \vec{n}\, d\sigma \\ &= -\int_V \operatorname{div}\left(\rho \frac{d\vec{x}}{dt}\right) d^3x\end{aligned}$$

Assuming the t derivative can be taken inside the integral we obtain

$$\int_V \left[\frac{\partial \rho}{\partial t} + \operatorname{div}\left(\rho \frac{d\vec{x}}{dt}\right)\right] d^3x = 0$$

Furthermore, we assume that this equation holds true for all volumes V and that the integrand is continuous. Under these assumptions we obtain

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\left(\rho \frac{d\vec{x}}{dt}\right) = 0$$

2.1.2 Momentum equation

We also need to examine the conservation of momentum. We shall consider the momentum in a volume V,

$$\int_V \rho\, \vec{u}\, d^3x$$

The momentum in V changes due to momentum flowing out of a system and forces acting on the system, i.e.

$$\frac{d}{dt}\int_V \rho\, u_i\, d^3x = -\int_{\partial V} \rho\, u_i\, \vec{u}\cdot\vec{n}\, d\sigma + \int_V f_i\, d^3x$$

The first integral in this expression describes the momentum being carried across the boundary of the volume under consideration. The second integral describes the effect of the force per unit volume, denoted by $\vec{f}$. Using both the divergence theorem and our smoothness assumption this becomes

$$\int_V \left[\frac{\partial}{\partial t}(\rho\, u_i) + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\rho\, u_i\, u_j) \right] d^3x = \int_V f_i\, d^3x$$

For this formula to hold for all V and continuous integrands we require

$$\frac{\partial}{\partial t}(\rho\, u_i) + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\rho\, u_i\, u_j) = f_i$$

which, using the mass conservation equation, leads to

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = \frac{1}{\rho}\vec{f} \tag{2.1.3}$$

The forces in this equation which need to be considered are the Coriolis force, gravity and the pressure gradient force. The first of these forces to be considered is the Coriolis force. The Coriolis force is an apparent force on moving objects in a noninertial coordinate system. In meteorology the Coriolis force per unit mass arises solely from the Earth's rotation, with angular velocity $\vec{\Omega}$, and at latitude ϕ is given by

$$\vec{f}_{Coriolis} = 2\rho(\vec{u} \times \vec{\Omega}) = 2\rho\vec{u} \times \begin{pmatrix} 0 \\ \Omega\cos\phi \\ \Omega\sin\phi \end{pmatrix}$$

The second force exerted on the system is gravity. From Newton's law of attraction of mass the gravitational force is given by

$$\vec{f}_{gravity} = -\frac{G\rho M}{(x^2 + y^2 + z^2)^{3/2}}\,\vec{x} \tag{2.1.4}$$

where G is the gravitational constant and M represents the mass of the Earth.

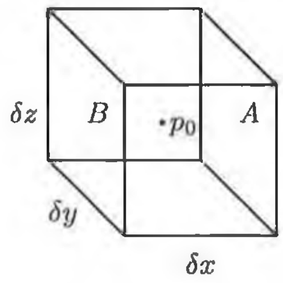

Figure 2.1: The pressure gradient force

The final force which affects the system is the pressure gradient force. If a force at a point p_0 is considered on the wall A in Figure (2.1), we know from a Taylor series expansion that

$$\begin{aligned} p(x_0 + \frac{\delta x}{2}, y, z) &= p(x_0, y, z) + \frac{\partial p}{\partial x}(x_0, y, z)\frac{\delta x}{2} + \cdots \\ &= p_0 + \frac{\partial p}{\partial x}\frac{\delta x}{2} + \cdots \end{aligned}$$

We know that force equals pressure multiplied by the area over which the force is exerted. Therefore,

$$F_A = -\left(p_0 + \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y \delta z$$

and similarly the force exerted on wall B is

$$F_B = \left(p_0 - \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y \delta z$$

Therefore the net force in the x direction is

$$F_x = F_A + F_B = -\frac{\partial p}{\partial x}\delta x \delta y \delta z$$

and hence

$$f_x = -\frac{\partial p}{\partial x} \qquad (2.1.5)$$

Applying similar arguments to the y and z components we see

$$f_y = -\frac{\partial p}{\partial y} \quad \text{and} \quad f_z = -\frac{\partial p}{\partial z} \tag{2.1.6}$$

Including these forces in our equations we see that the equation of non-conservation of momentum which we shall study is

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\frac{GM}{(x^2 + y^2 + z^2)^{3/2}}\,\vec{x} - \frac{1}{\rho}\nabla p + 2\vec{u} \times \vec{\Omega}$$

2.1.3 Conversion to spherical coordinates

From the previous section we see that the equations for mass and momentum which we want to study are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \tag{2.1.7}$$

and

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\frac{GM}{(x^2 + y^2 + z^2)^{3/2}}\,\vec{x} - \frac{1}{\rho}\nabla p + 2\vec{u} \times \vec{\Omega} \tag{2.1.8}$$

We want to examine these equations on the sphere. Therefore it is necessary to convert the above equations into their spherical equivalents. The coordinate system which shall be used is (λ, ϕ, r), where λ represents the longitude $(-\pi \leq \lambda < \pi)$, ϕ represents the latitude $(-\pi/2 \leq \phi \leq \pi/2)$ and r is the radial distance, i.e.

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\lambda\cos\phi \\ r\sin\lambda\cos\phi \\ r\sin\phi \end{pmatrix}$$

Using this coordinate system we see that

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\lambda\cos\phi & -\frac{\sin\lambda}{r\cos\phi} & -\frac{1}{r}\cos\lambda\sin\phi \\ \sin\lambda\cos\phi & \frac{\cos\lambda}{r\cos\phi} & -\frac{1}{r}\sin\lambda\sin\phi \\ \sin\phi & 0 & \frac{1}{r}\cos\phi \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \lambda} \\ \frac{\partial}{\partial \phi} \end{pmatrix} \tag{2.1.9}$$

and

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -\sin\lambda & -\cos\lambda\sin\phi & \cos\lambda\cos\phi \\ \cos\lambda & -\sin\lambda\sin\phi & \sin\lambda\cos\phi \\ 0 & \cos\phi & \sin\phi \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} \tag{2.1.10}$$

where $\tilde{u}$ is the velocity in the λ direction, $\tilde{v}$ represents the velocity in the ϕ direction and $\tilde{w}$ is the velocity in the r direction.

Using these identities we convert equations (2.1.7) and (2.1.8) into their spherical coordinate equivalents. Firstly examining the momentum equation, (2.1.8), we get

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \\ w \end{pmatrix} + \left(\begin{pmatrix} u \\ v \\ w \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \right) \begin{pmatrix} u \\ v \\ w \end{pmatrix} - 2 \begin{pmatrix} u \\ v \\ w \end{pmatrix} \times \begin{pmatrix} 0 \\ \Omega\cos\phi \\ \Omega\sin\phi \end{pmatrix}$$

$$+ \frac{GM}{(x^2+y^2+z^2)^{3/2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{\rho} \begin{pmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{pmatrix} \tag{2.1.11}$$

Substituting (2.1.9) and (2.1.10) into this equation and multiplying the whole equation by

$$\begin{pmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\cos\lambda\sin\phi & -\sin\lambda\sin\phi & \cos\phi \\ \cos\lambda\cos\phi & \sin\lambda\cos\phi & \sin\phi \end{pmatrix}$$

we see that the equations describing the momentum in spherical coordinates are

$$\frac{\partial \tilde{u}}{\partial t}+\frac{1}{r\cos\phi}\tilde{u}\frac{\partial \tilde{u}}{\partial \lambda}-\frac{\tan\phi}{r}\tilde{u}\tilde{v}+\frac{1}{r}\tilde{u}\tilde{w}+\frac{1}{r}\tilde{v}\frac{\partial \tilde{u}}{\partial \phi}+\tilde{w}\frac{\partial \tilde{u}}{\partial r}-2\Omega\tilde{v}\sin\phi$$
$$+2\Omega\tilde{w}\cos\phi=-\frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial \lambda} \tag{2.1.12}$$

$$\frac{\partial \tilde{v}}{\partial t}+\frac{1}{r\cos\phi}\tilde{u}\frac{\partial \tilde{v}}{\partial \lambda}+\frac{\tan\phi}{r}\tilde{u}^2+\frac{1}{r}\tilde{v}\tilde{w}+\frac{1}{r}\tilde{v}\frac{\partial \tilde{v}}{\partial \phi}+\tilde{w}\frac{\partial \tilde{v}}{\partial r}$$
$$+2\Omega\tilde{u}\sin\phi=-\frac{1}{\rho r}\frac{\partial p}{\partial \phi} \tag{2.1.13}$$

$$\frac{\partial \tilde{w}}{\partial t}-\frac{1}{r}\tilde{u}^2+\frac{1}{r\cos\phi}\tilde{u}\frac{\partial \tilde{w}}{\partial \lambda}-\frac{1}{r}\tilde{v}^2+\frac{1}{r}\tilde{v}\frac{\partial \tilde{w}}{\partial \phi}+\tilde{w}\frac{\partial \tilde{w}}{\partial r}-2\Omega\tilde{u}\cos\phi$$
$$+\frac{GM}{r^2}=-\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{2.1.14}$$

Similarly we need to convert the equation of mass conservation into its polar coordinates equivalent. Using the identity,

$$\nabla\cdot(\rho\vec{u})=\rho\nabla\cdot\vec{u}+\vec{u}\cdot\nabla\rho$$

equation (2.1.7) becomes

$$\frac{1}{\rho}\left(\frac{\partial}{\partial t}+\vec{u}\cdot\nabla\right)\rho+\nabla\cdot\vec{u}=0$$

To convert this equation into spherical coordinates we again use equations (2.1.9) and (2.1.10). From these identities we see that

$$\begin{aligned}\nabla\cdot\vec{u}&=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\\&=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tilde{w}\right)+\frac{1}{r\cos\phi}\frac{\partial \tilde{u}}{\partial \lambda}+\frac{1}{r\cos\phi}\frac{\partial}{\partial \phi}\left(\cos\phi\tilde{v}\right)\end{aligned}$$

The next term from the equation which needs to be converted into spherical coordinates is

$$\begin{aligned}\vec{u}\cdot\nabla\rho &= u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}\\ &= \frac{1}{r\cos\phi}\tilde{u}\frac{\partial\rho}{\partial\lambda}+\frac{1}{r}\tilde{v}\frac{\partial\rho}{\partial\phi}+\tilde{w}\frac{\partial\rho}{\partial r}\end{aligned}$$

Therefore we see that the equation of mass conservation which we wish to study is

$$\begin{aligned}\frac{1}{\rho}\frac{\partial\rho}{\partial t}+\frac{1}{\rho}\left(\frac{1}{r\cos\phi}\tilde{u}\frac{\partial\rho}{\partial\lambda}+\frac{1}{r}\tilde{v}\frac{\partial\rho}{\partial\phi}+\tilde{w}\frac{\partial\rho}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tilde{w}\right)\\ +\frac{1}{r\cos\phi}\frac{\partial\tilde{u}}{\partial\lambda}+\frac{1}{r\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\tilde{v}\right)=0\end{aligned}\tag{2.1.15}$$

2.2 Reduction of equations to main equation

We now want to reduce our three-dimensional equations to the two-dimensional equations which are widely used in oceanography and meteorology. A mathematically rigorous justification of this reduction is a very difficult problem and beyond the scope of this thesis. Recent work has been done on this subject including that by Temam and Ziane [24], who have given such a justification for the reduction of the three-dimensional Navier-Stokes equations in the case of incompressible flow in a shell of vanishing thickness. Although the atmosphere is a compressible fluid, we shall consider here a model where ρ is constant. We also set

$$r = a + r', \quad \text{with} \quad \frac{r'}{a} \ll 1$$

where a corresponds to the radius of the Earth and r' is the distance from the surface of the Earth to the phenomenon in question.

Under these conditions the two-dimensional equations under examination are

$$\begin{aligned}
\frac{\partial u}{\partial t} + \frac{1}{a\cos\phi}u\frac{\partial u}{\partial \lambda} - \frac{\tan\phi}{a}uv + \frac{1}{a}v\frac{\partial u}{\partial \phi} - 2\Omega v\sin\phi &= -\frac{1}{\rho a\cos\phi}\frac{\partial p}{\partial \lambda} &\quad (2.2.1)\\
\frac{\partial v}{\partial t} + \frac{1}{a\cos\phi}u\frac{\partial v}{\partial \lambda} + \frac{\tan\phi}{a}u^2 + \frac{1}{a}v\frac{\partial v}{\partial \phi} + 2\Omega u\sin\phi &= -\frac{1}{\rho a}\frac{\partial p}{\partial \phi} &\quad (2.2.2)\\
\frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \phi}(v\cos\phi) &= 0 &\quad (2.2.3)
\end{aligned}$$

when the tildes are dropped. Examining the third equation we note that this equation can be satisfied by introducing a stream function ψ such that

$$\begin{aligned}
u &= -\frac{1}{a}\frac{\partial \psi}{\partial \phi} &\quad (2.2.4)\\
v &= \frac{1}{a\cos\phi}\frac{\partial \psi}{\partial \lambda} &\quad (2.2.5)
\end{aligned}$$

To reduce the equations down to a single equation, and also to reduce the number of variables in the equation by one, we examine the first two equations by calculating the third component of the curl. To do this we calculate

$$\frac{1}{a}\frac{\partial}{\partial \phi}(2.2.1) - \frac{\tan\phi}{a}(2.2.1) - \frac{1}{a\cos\phi}\frac{\partial}{\partial \lambda}(2.2.2)$$

which, using the expressions (2.2.4) and (2.2.5), works down to be

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{1}{\cos^2\phi}\frac{\partial^2\psi}{\partial \lambda^2} - \tan\phi\frac{\partial \psi}{\partial \phi} + \frac{\partial^2\psi}{\partial \phi^2}\right) + 2\Omega\frac{\partial \psi}{\partial \lambda}\\
&+\frac{1}{a^2\cos\phi}\left(\frac{\partial \psi}{\partial \lambda}\frac{\partial}{\partial \phi} - \frac{\partial \psi}{\partial \phi}\frac{\partial}{\partial \lambda}\right)\left(\frac{1}{\cos^2\phi}\frac{\partial^2\psi}{\partial \lambda^2} - \tan\phi\frac{\partial \psi}{\partial \phi} + \frac{\partial^2\psi}{\partial \phi^2}\right) = 0
\end{aligned}$$

We nondimensionalise the variables in this equation to ensure that we can compare the terms in the equation without worrying about the dimensions of each term. The variables which we will introduce for this nondimensionalisation are

$$t = Tt', \quad \psi = \frac{L^2}{T}\psi' \quad \text{and} \quad \Omega = \frac{1}{T}\Omega'$$

These new variables cause our equation to become

$$\frac{\partial}{\partial t'}\left(\frac{1}{\cos^2\phi}\frac{\partial^2\psi'}{\partial\lambda^2}-\tan\phi\frac{\partial\psi'}{\partial\phi}+\frac{\partial^2\psi'}{\partial\phi^2}\right)+2\Omega'\frac{\partial\psi'}{\partial\lambda}$$
$$+\frac{L^2}{a^2\cos\phi}\left(\frac{\partial\psi'}{\partial\lambda}\frac{\partial}{\partial\phi}-\frac{\partial\psi'}{\partial\phi}\frac{\partial}{\partial\lambda}\right)\left(\frac{1}{\cos^2\phi}\frac{\partial^2\psi'}{\partial\lambda^2}-\tan\phi\frac{\partial\psi'}{\partial\phi}+\frac{\partial^2\psi'}{\partial\phi^2}\right)=0$$

We want to examine wave solutions with small amplitudes. Therefore we are assuming that ψ' is small. We indicate this by letting

$$\psi'=\alpha\psi''$$

where ψ'' is order 1. If we substitute this into our equation and drop the resulting primes we have derived the vorticity equation for nondivergent planetary flow, i.e.

$$\frac{\partial}{\partial t}\left(\frac{1}{\cos^2\phi}\frac{\partial^2\psi}{\partial\lambda^2}+\frac{\partial^2\psi}{\partial\phi^2}-\tan\phi\frac{\partial\psi}{\partial\phi}\right)+2\Omega\frac{\partial\psi}{\partial\lambda}$$
$$=\delta\left(\frac{1}{\cos\phi}\frac{\partial\psi}{\partial\phi}\frac{\partial}{\partial\lambda}-\frac{1}{\cos\phi}\frac{\partial\psi}{\partial\lambda}\frac{\partial}{\partial\phi}\right)\left(\frac{1}{\cos^2\phi}\frac{\partial^2\psi}{\partial\lambda^2}+\frac{\partial^2\psi}{\partial\phi^2}-\tan\phi\frac{\partial\psi}{\partial\phi}\right) \tag{2.2.6}$$

Chapter 3

Wave solutions of equation

In this chapter we begin our study of Rossby-Haurwitz wave solutions. The purpose of this section is to study the nature of the nonlinear interactions of these waves when δ is small. Expanding our stream function, ψ, about the small parameter δ, we can find solutions to the vorticity equation for nondivergent planetary flow. Firstly we are interested in examining single wave solutions to this equation. For single wave solutions to occur the necessary condition on the velocity of the wave is derived. Using this as our base we then examine the interactions between these waves.

To examine the interactions of these waves we consider a dyad solution i.e. a two wave solution. When examining a dyad solution we encounter nonlinear combinations of Legendre functions. Using well known properties of the Legendre functions, and also less well known properties which we will derive, we can rewrite these combinations of Legendre functions as sums of single Legendre functions. Using the spectral method we explain how the interactions between waves of a nonresonant Rossby-Haurwitz wave dyad solution only lead to small corrections. The aim of this thesis is to study first and second order resonant interactions of Rossby-Haurwitz waves in general. We find that we must consider a resonantly interacting dyad solution which in turn results in us having to consider a triad solution. The consequences of studying a triad solution are discussed in the next chapter.

In this chapter we will also find that if the calculations are commenced with a non-resonant dyad solution it is not possible to produce zonal flow through interactions at either $O(\delta)$ or $O(\delta^2)$.

3.1 Asymptotic expansion of the equation

We wish to examine wave solutions of the vorticity equation for nondivergent planetary flow which was derived in the previous chapter

$$\frac{\partial}{\partial t}\left(\frac{1}{\cos^2\phi}\frac{\partial^2\psi}{\partial\lambda^2}+\frac{\partial^2\psi}{\partial\phi^2}-\tan\phi\frac{\partial\psi}{\partial\phi}\right)+2\frac{\partial\psi}{\partial\lambda}$$
$$=\delta\left(\frac{1}{\cos\phi}\frac{\partial\psi}{\partial\phi}\frac{\partial}{\partial\lambda}-\frac{1}{\cos\phi}\frac{\partial\psi}{\partial\lambda}\frac{\partial}{\partial\phi}\right)\left(\frac{1}{\cos^2\phi}\frac{\partial^2\psi}{\partial\lambda^2}+\frac{\partial^2\psi}{\partial\phi^2}-\tan\phi\frac{\partial\psi}{\partial\phi}\right) \quad (3.1.1)$$

Looking at the terms involved in this equation it can be seen that the equation is nonlinear. The Jacobian term describes this nonlinearity, which results in nonlinear coupling of the waves and energy transfer between the waves. It is important to notice that this nonlinear term is of a lower order than the other linear terms in the equation.

Due to the complicated nature and structure of the equation and the fact that it is nonlinear it would be extremely difficult to try to solve this equation as it is written. Instead we will expand the equation about the small parameter, δ, and examine solutions by looking at the different orders of the equation. Since $\delta \ll 1$ it is reasonable to expand ψ about δ,

$$\psi(\lambda, \phi, t, \delta) = \psi_0(\lambda, \phi, t) + \delta\psi_1(\lambda, \phi, t) + \delta^2\psi_2(\lambda, \phi, t) + \dots \tag{3.1.2}$$

Applying this expansion to our equation generates a sequence of linear problems for ψ_n.

The leading order equation, the highest order equation, which we are studying is

$$\frac{\partial}{\partial t}\nabla^2\psi_0 + 2\frac{\partial\psi_0}{\partial\lambda} = 0 \tag{3.1.3}$$

where

$$\nabla^2 = \frac{1}{\cos^2\phi}\frac{\partial^2}{\partial\lambda^2} + \frac{\partial^2}{\partial\phi^2} - \tan\phi\frac{\partial}{\partial\phi}$$

and the $O(\delta)$ equation is

$$\frac{\partial}{\partial t}\nabla^2\psi_1 + 2\frac{\partial\psi_1}{\partial\lambda} = \left(\frac{1}{\cos\phi}\frac{\partial\psi_0}{\partial\phi}\frac{\partial}{\partial\lambda} - \frac{1}{\cos\phi}\frac{\partial\psi_0}{\partial\lambda}\frac{\partial}{\partial\phi}\right)\nabla^2\psi_0 \tag{3.1.4}$$

It is these equations, along with the corresponding one for $O(\delta^2)$ which we shall study to determine the form of the wave solutions of the equation, and hence we will determine when, if ever, zonal flow can be generated.

3.2 The Legendre differential equation

To solve the full equation (3.1.1), we will firstly examine the leading order equation (3.1.3). The solution to the leading order equation will give an approximate solution for the stream function ψ, i.e. ψ_0. This approximate solution can be improved upon by solving the $O(\delta)$ equation which results in ψ_1. This term is the first correction term to the solution. From here we can continue to find correction terms to the solution by solving the lower order equations, and from this we determine the full solution of the equation.

Firstly we wish to find a Rossby-Haurwitz wave solution for the leading order equation (3.1.3). Examining the Laplacian part of the solution we recognize similarities between this term and the Legendre differential equation. This leads us to consider spherical harmonics as our wave solution.

The Legendre differential equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(n(n+1) - \frac{m^2}{1-x^2}\right)y = 0 \tag{3.2.1}$$

This equation has exactly two linearly independent solutions on $(-1, 1)$, [2]. If we examine this equation we can see that it has two regular singular points, $x = \pm 1$. Therefore the solution to this equation takes the form [3]

$$y_1(x) = (x-1)^{r_1} \sum_{m=0}^{\infty} a_m (x-1)^m \tag{3.2.2}$$

$$y_2(x) = y_1(x)\ln(x-1) + (x-1)^{r_1} \sum_{m=0}^{\infty} b_m (x-1)^m \tag{3.2.3}$$

We can see that the second of these solutions has a logarithmic singularity at the regular singular point. This solution diverges at the boundaries and hence we will not consider it. As a result of this our solutions will take the form of equation (3.2.2).

The solutions to this equation are the associate Legendre functions. If we redefine the variables of this equation by setting $y = P_n^m(\sin\phi)$ and $x = \sin\phi$ the Legendre differential equation becomes

$$\frac{d^2}{d\phi^2}P_n^m(\sin\phi) - \tan\phi\frac{d}{d\phi}P_n^m(\sin\phi) + \left(n(n+1) - \frac{m^2}{\cos^2\phi}\right)P_n^m(\sin\phi) = 0 \qquad (3.2.4)$$

It is clear from examining this equation that there are very strong similarities between this equation and the Laplacian term in the equation we are studying,

$$\frac{1}{\cos^2\phi}\frac{\partial^2\psi}{\partial\lambda^2} + \frac{\partial^2\psi}{\partial\phi^2} - \tan\phi\frac{\partial\psi}{\partial\phi} \qquad (3.2.5)$$

If we can find an eigenfunction of the Laplacian of the form $\psi_0 = e^{i\kappa\lambda}F(\phi)$ then $e^{i(\kappa\lambda-\sigma t)}F(\phi)$ is a solution of the leading order equation if

$$\sigma = \frac{-2\kappa}{\rho}$$

where ρ is the eigenvalue. Therefore we will examine the Laplacian equation in spherical coordinates and examine the solution of this equation. There are many books available on this subject including one by Pinchover and Rubinstein [17].

The Laplace equation in spherical coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\left(\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\cos^2\phi}\frac{\partial^2 u}{\partial\lambda^2}\right) = 0$$

where

$$0 < r < a, \quad -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}, \quad -\pi \leq \lambda \leq \pi$$

Letting $u(r,\lambda,\phi) = R(r)Y(\lambda,\phi)$ the equation separates and reduces to

$$\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial Y}{\partial\phi}\right) + \frac{1}{\cos^2\phi}\frac{\partial^2 Y}{\partial\lambda^2} = -\mu Y \qquad (3.2.6)$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \mu R \qquad (3.2.7)$$

where μ is a constant. We have to impose the conditions

$$Y(-\pi,\phi) = Y(\pi,\phi) \quad \text{and} \quad Y_\lambda(-\pi,\phi) = Y_\lambda(\pi,\phi)$$

on Y and we note that it is also bounded everywhere on the unit sphere.

If we now examine the first equation and apply a second separation of variables

$$Y(\lambda,\phi) = \Lambda(\lambda)\Phi(\phi)$$

we obtain

$$\frac{d^2\Lambda}{d\lambda^2} + \nu\Lambda = 0 \tag{3.2.8}$$

$$\cos\phi\frac{d}{d\phi}\left(\cos\phi\frac{d\Phi}{d\phi}\right) + \left(\mu\cos^2\phi - \nu\right)\Phi = 0 \tag{3.2.9}$$

Examining equation (3.2.8) we see that Λ must be a solution of the form

$$\Lambda = a_m e^{im\lambda} + b_m e^{-im\lambda}$$

where $\nu = m^2$. From the periodic conditions

$$\Lambda(-\pi) = \Lambda(\pi) \quad \text{and} \quad \Lambda_\lambda(-\pi) = \Lambda_\lambda(\pi)$$

we obtain $m \in \mathbb{Z}$. Therefore the equation under consideration is

$$\cos\phi\frac{d}{d\phi}\left(\cos\phi\frac{d\Phi}{d\phi}\right) + (\mu\cos^2\phi - m^2)\Phi = 0 \tag{3.2.10}$$

3.3 Solutions to Legendre differential equation

We want to find solutions to equation (3.2.10). To do this we firstly set $\Phi = P_n^m(\sin\phi)$.

Doing this the equation which we are studying becomes

$$\frac{d^2P_n^m}{d\phi^2} - \tan\phi\frac{dP_n^m}{d\phi} - \frac{m^2}{\cos^2\phi}P_n^m + \mu P_n^m = 0 \tag{3.3.1}$$

If we set

$$L_n^m = \cos^{1/2}\phi P_n^m \tag{3.3.2}$$

the equation becomes

$$\frac{d^2 L_n^m}{d\phi^2} - \frac{m^2 - \frac{1}{4}}{\cos^2\phi} L_n^m + \left(\mu + \tfrac{1}{4}\right) L_n^m = 0 \tag{3.3.3}$$

This is the equation which we shall be studying. Firstly we will prove some general properties of the equation.

3.3.1 General properties of equation

To examine properties of L_n^m we shall define the following, [8]

$$H_m^+ = -(m - \tfrac{1}{2})\tan\phi + \frac{d}{d\phi} \tag{3.3.4}$$

$$H_m^- = -(m - \tfrac{1}{2})\tan\phi - \frac{d}{d\phi} \tag{3.3.5}$$

Using these identities and equation (3.3.3) we can see that

$$H_m^- H_m^+ L_n^m = \left(\mu + \tfrac{1}{4} - (m - \tfrac{1}{2})^2\right) L_n^m \tag{3.3.6}$$

and similarly

$$H_{m+1}^+ H_{m+1}^- L_n^m = \left(\mu + \tfrac{1}{4} - (m + \tfrac{1}{2})^2\right) L_n^m \tag{3.3.7}$$

If we now set $m = m + 1$ in (3.3.6) we get

$$H_{m+1}^- H_{m+1}^+ L_n^{m+1} = \left(\mu + \tfrac{1}{4} - (m + \tfrac{1}{2})^2\right) L_n^{m+1}$$

If we multiply equation (3.3.7) by H^-_{m+1} we obtain

$$H^-_{m+1} H^+_{m+1} H^-_{m+1}\, L^m_n = H^-_{m+1}\, \left(\mu + \tfrac{1}{4} - (m + \tfrac{1}{2})^2\right) L^m_n$$

Examining this equation we can see that $H^-_{m+1}\, L^m_n$ is also a solution to the equation and hence

$$L^{m+1}_n = H^-_{m+1}\, L^m_n \tag{3.3.8}$$

Similarly if we replace m with $m-1$ in equation (3.3.7) and multiply equation (3.3.6) by H^+_m we will find that

$$L^{m-1}_n = H^+_m\, L^m_n \tag{3.3.9}$$

An expression for μ can now be found using the definitions just shown. We require the solutions to the equation under consideration to be square integrable. This means that we require

$$\int_a^b (L^m_n)^2 \, \mathrm{d}\phi = 1$$

By examining this requirement we are able to determine an expression for μ

$$\begin{aligned}
\int_a^b \left(L^{m+1}\right)^2 \, \mathrm{d}\phi &= \int_a^b \left(H^-_{m+1}\, L^m\right)^2 \, \mathrm{d}\phi \\
&= \int_a^b L^m H^+_{m+1} H^-_{m+1}\, L^m \, \mathrm{d}\phi \\
&= \left(\mu + \tfrac{1}{4} - (m + \tfrac{1}{2})^2\right) \int_a^b (L^m)^2 \, \mathrm{d}\phi
\end{aligned}$$

by integration by parts, if we assume that the surface terms vanish. Similarly it can be shown that

$$\int_a^b \left(L^{m+2}\right)^2 \mathrm{d}\phi = \left(\mu + \tfrac{1}{4} - (m + \tfrac{3}{2})^2\right) \int_a^b \left(L^{m+1}\right)^2 \mathrm{d}\phi$$
$$= \left(\mu + \tfrac{1}{4} - (m + \tfrac{3}{2})^2\right)\left(\mu + \tfrac{1}{4} - (m + \tfrac{1}{2})^2\right) \int_a^b \left(L^{m}\right)^2 \mathrm{d}\phi$$

We can continue this argument for higher and higher m. Since m is an increasing function, at some point, say $n+1$ we would arrive at the contradiction

$$\int_a^b \left(L_n^{n+1}\right)^2 \mathrm{d}\phi < 0$$

unless

$$L_n^{n+1} = 0$$

or

$$H_{n+1}^{-} L_n^n = 0$$

If we now apply equation (3.3.7) to this expression we can see that

$$H_{n+1}^{+} H_{n+1}^{-} L_n^n = \left(\mu + \tfrac{1}{4} - (n + \tfrac{1}{2})^2\right) L_n^n$$

$$\Rightarrow \mu = n(n+1) \qquad (3.3.10)$$

Using the expression for μ just found, the equation under examination is

$$\frac{d^2 L_n^m}{d\phi^2} - \frac{m^2 - \frac{1}{4}}{\cos^2\phi} L_n^m + \left(n(n+1) + \tfrac{1}{4}\right) L_n^m = 0 \qquad (3.3.11)$$

3.3.2 Solution to Legendre differential equation

To find our Rossby-Haurwitz wave solutions we must find solutions to equation (3.3.1). To give an indication what these solutions are we firstly examine equation (3.3.11). We notice that

$$L_n^n = \cos^{n+\frac{1}{2}}\phi$$

is a solution to this equation.

We require that these solutions are square integrable, which means that this solution must be normalised

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} L_n^{n2}\,\mathrm{d}\phi = 1 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n+1}\phi\,\mathrm{d}\phi \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n}\phi\cos\phi\,\mathrm{d}\phi \\
&= \sin\phi\cos^{2n}\phi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2n\cos^{2n-1}\phi\sin^2\phi\,\mathrm{d}\phi \\
\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n+1}\phi\,\mathrm{d}\phi &= \frac{2n}{2n+1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n-1}\phi\,\mathrm{d}\phi \\
&= \frac{2n}{2n+1}\frac{2n-2}{2n-1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n-3}\phi\,\mathrm{d}\phi \\
&= \ldots = 2\frac{2n.2n-2\ldots 2}{2n+1.2n-1\ldots 3.1}
\end{aligned}$$

From these results we can see that for L_n^n to be normalised we require that

$$L_n^n = \left(\frac{1.3\ldots 2n-1.2n+1}{2.2.4\ldots 2n-2.2n}\right)^{\frac{1}{2}} \cos^{n+\frac{1}{2}}\phi \tag{3.3.12}$$

Using this solution to the equation and the general properties (3.3.8) and (3.3.9) we find that

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} L_n^{m+1\,2}\,\mathrm{d}\phi &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(H_{m+1}^{-} L_n^m\right)^2\,\mathrm{d}\phi \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} L_n^m H_{m+1}^{+} H_{m+1}^{-} L_n^m\,\mathrm{d}\phi \\
&= (n-m)(n+m+1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} L_n^{m2}\,\mathrm{d}\phi
\end{aligned}$$

$$\Rightarrow L_n^{m+1} = [(n-m)(n+m+1)]^{-\frac{1}{2}} H_{m+1}^{-} L_n^m \tag{3.3.13}$$

It can similarly be shown that

$$L_n^{m-1} = [(n+m)(n-m+1)]^{-\frac{1}{2}} H_m^{+} L_n^m \tag{3.3.14}$$

From these derivations it is now possible to find an expression for all terms L_n^m where $n \geq m$.

3.4 Single wave solution to leading order equation

Using the above identities we shall find a single wave solution to the leading order equation which is called a Rossby-Haurwitz wave after C. G. Rossby and B. Haurwitz who found these solutions. The similar: ies between the Legendre differential equation and our equation leads us to consider Legendre functions as our solution ψ, i.e. we study spherical harmonics as our wave solution [5], [13]

$$\psi_0 = Ae^{i(m\lambda-\sigma t)} P_n^m(\mu) + \bar{A}e^{-i(m\lambda-\sigma t)} P_n^m(\mu), \tag{3.4.1}$$

where

$$P_n^m(\mu) = \cos^{-\frac{1}{2}} \phi L_n^m$$

i.e.

$$\begin{aligned} P_n^n &= \left(\frac{1.3\ldots 2n+1}{2.2.4\ldots 2n}\right)^{\frac{1}{2}} \cos^n \phi \\ P_n^m &= [(n+m+1)(n-m)]^{-\frac{1}{2}} \left(-(m+1)\tan\phi + \frac{d}{d\phi}\right) P_n^{m+1} \\ P_n^m &= [(n+m)(n-m-1)]^{-\frac{1}{2}} \left(-(m-1)\tan\phi - \frac{d}{d\phi}\right) P_n^{m-1} \end{aligned} \tag{3.4.2}$$

Here $P_n^m(\mu)$ represents the associated Legendre polynomial of the first kind. The parameter m represents the zonal wavenumber, i.e. the number of zeroes of the solution along the longitude, while $n-m$ represents the number of zeroes along the latitude. They are called the longitudinal and latitudinal wavenumbers respectively. The total wavenumber is represented by n.

We will use the representation (3.4.2) to derive certain inequalities which are crucial in the context of this thesis. This derivation follows [8] and [23] and is outlined in the appendices. We point out that there is a much better known representation of the solution to the Legendre equation (see, e.g. [17]), which is

$$P_n^m = \frac{1}{2^n n!}\sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}}(1-\mu^2)^{\frac{m}{2}}\frac{d^{n+m}}{d\mu^{n+m}}(\mu^2-1)^n$$

where

$$\mu = \sin\phi$$

To solve the leading order equation the Laplacian term is examined, which produces

$$\begin{aligned}\nabla^2\psi_0 &= \frac{1}{\cos^2\phi}\frac{\partial^2\psi_0}{\partial\lambda^2} + \frac{\partial^2\psi_0}{\partial\phi^2} - \tan\phi\frac{\partial\psi_0}{\partial\phi} \\ &= Ae^{i(m\lambda-\sigma t)}\left(\frac{-m^2}{\cos^2\phi}P_n^m(\mu) + \frac{d^2P_n^m(\mu)}{d\phi^2} - \tan\phi\frac{dP_n^m(\mu)}{d\phi}\right) + \text{c.c.}\end{aligned}$$

Using the Legendre differential equation (3.2.4) this reduces down to

$$\nabla^2\psi_0 = -n(n+1)Ae^{i(m\lambda-\sigma t)}P_n^m(\mu) - n(n+1)\bar{A}e^{-i(m\lambda-\sigma t)}P_n^m(\mu)$$

When this is filled into the leading order equation we find that

$$\left[Ae^{i(m\lambda-\sigma t)}P_n^m(\mu) + \bar{A}e^{-i(m\lambda-\sigma t)}P_n^m(\mu)\right](i\sigma n(n+1) + 2im) = 0$$

This leads to the necessary condition on the velocity, without which the solution we are studying is not a Rossby-Haurwitz wave

$$\sigma = \frac{-2m}{n(n+1)} \tag{3.4.3}$$

It should be noted that $\psi = \psi_0$ is actually a solution to the full equation (3.1.1), since the quadratic term on the right hand side of (3.1.1) vanishes for ψ_0 given in (3.4.1). Since the leading order equation (3.1.3) is linear we notice also that any linear superposition of Rossby-Haurwitz waves is also a solution of (3.1.3).

3.5 Examining a dyad solution

3.5.1 $O(\delta)$ equation

With the condition derived for a single wave solution to occur the next step is to study a dyad solution. We want to firstly discuss nonresonant dyad solutions, whose interaction only leads to small corrections.

The dyad solution which we are going to examine is a superposition of two solutions

$$\begin{aligned}\psi_0 = A_1 e^{i(m_1\lambda - \sigma_1 t)} P_{n_1}^{m_1}(\mu) + \bar{A}_1 e^{-i(m_1\lambda-\sigma_1 t)} P_{n_1}^{m_1}(\mu) \\ + A_2 e^{i(m_2\lambda - \sigma_2 t)} P_{n_2}^{m_2}(\mu) + \bar{A}_2 e^{-i(m_2\lambda-\sigma_2 t)} P_{n_2}^{m_2}(\mu)\end{aligned} \tag{3.5.1}$$

which satisfy the leading order equation (3.1.3) if and only if

$$\sigma_1 = \frac{-2m_1}{n_1(n_1+1)}, \quad \sigma_2 = \frac{-2m_2}{n_2(n_2+1)}$$

To study the interaction of Rossby-Haurwitz waves we have to consider the nonlinear part of the equation (3.1.1). The Jacobian part of the equation represents this nonlinearity, which is of $O(\delta)$. Therefore, by examining the expansion of ψ about δ it is clear that the $O(\delta)$ equation is

$$\frac{\partial}{\partial t}\nabla^2\psi_1 + 2\frac{\partial\psi_1}{\partial\lambda} = J(\psi_0, \nabla^2\psi_0) \tag{3.5.2}$$

where

$$J(a,b) = \frac{\partial a}{\partial\mu}\frac{\partial b}{\partial\lambda} - \frac{\partial b}{\partial\mu}\frac{\partial a}{\partial\lambda}$$

From the previous calculations it is known that

$$\begin{aligned}\nabla^2\psi_0 = &- n_1(n_1+1)A_1 e^{i(m_1\lambda-\sigma_1 t)}P_{n_1}^{m_1}\\ &- n_2(n_2+1)A_2 e^{i(m_2\lambda-\sigma_2 t)}P_{n_2}^{m_2} + \text{c.c.}\end{aligned}$$

Using this we can compute the Jacobian term of the equation

$$\begin{aligned}&J(\psi_0, \nabla^2\psi_0) = \\ &il_{n_1 n_2}\Big\{A_1A_2e^{i[(m_1+m_2)\lambda-(\sigma_1+\sigma_2)t]}\Big(m_2P_{n_2}^{m_2}\frac{dP_{n_1}^{m_1}}{d\mu} - m_1P_{n_1}^{m_1}\frac{dP_{n_2}^{m_2}}{d\mu}\Big)\\ &-A_1\bar{A}_2e^{i[(m_1-m_2)\lambda-(\sigma_1-\sigma_2)t]}\Big(m_2P_{n_2}^{m_2}\frac{dP_{n_1}^{m_1}}{d\mu} + m_1P_{n_1}^{m_1}\frac{dP_{n_2}^{m_2}}{d\mu}\Big)\Big\} + \text{c.c.}\end{aligned} \tag{3.5.3}$$

where

$$l_{n_i n_j} = n_i(n_i+1) - n_j(n_j+1)$$

The main aim of this thesis is to examine first and second order resonant interactions. In particular we want to investigate whether or not zonal flow can be created through Rossby-Haurwitz wave interactions. Zonal flow waves are waves which have no north-south component, they flow in the east-west direction only. In the previous chapter it was derived that the velocity in the north-south direction, v, to leading order, is given by

$$v = \frac{1}{a\cos\phi}\frac{\partial\psi}{\partial\lambda}$$

Therefore for this term to be zero we require ψ to be independent of λ. The only place that the variable λ occurs in this expression is in the exponential term. Therefore for zonal flow to occur we require m to be equal to zero.

Zonal flow terms are thus of the form $e^{i\sigma t}P_n^0$. This type of term is a Rossby-Haurwitz wave if and only if it satisfies the leading order equation (3.1.3)

$$\frac{\partial}{\partial t}\nabla^2\psi_0 + 2\frac{\partial\psi_0}{\partial\lambda} = 0$$
$$\Rightarrow \sigma n(n+1)e^{i\sigma t}P_n^0 = 0$$

This equation is always true if $m = 0$ since in this case σ has to equal zero. Therefore we see that zonal flow terms will always take the form of, and satisfy the conditions required for, Rossby-Haurwitz waves.

If any of the terms on the right hand side of the $O(\delta)$ equation, i.e. equation (3.5.3), satisfy all these criteria then we will have produced zonal flow. So using all this information, the solution for ψ_1, calculated through studying the right hand side of the $O(\delta)$ equation, must be examined to determine if zonal flow could possibly occur. With the right hand side of the equation as it currently stands it is very difficult to tell if either of the terms present take the form of zonal flow. To overcome this problem we shall use the spectral method to rewrite the awkward expressions in terms of single Legendre functions.

We rewrite (3.5.3) in terms of Legendre functions by using the following expansion

[6], [19]

$$\psi(\lambda,\mu,t) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \psi_n^m(t) Y_n^m(\lambda,\mu) \tag{3.5.4}$$

where

$$\psi_n^m(t) = \frac{1}{2\pi}\int_{-1}^{1}\int_{-\pi}^{\pi} \bar{Y}_n^m(\lambda,\mu)\psi(\lambda,\mu,t)\, d\lambda\, d\mu, \tag{3.5.5}$$

and $\mu = \sin\phi$.

As a result of applying this spectral method to our equation, which is outlined in the appendices, each term on the right hand side of (3.5.2) can be written as a finite sum of Legendre polynomials. For example, if

$$\psi = e^{i(m_1+m_2)\lambda}\Big(im_2 P_{n_2}^{m_2}\frac{dP_{n_1}^{m_1}}{d\mu} - im_1 P_{n_1}^{m_1}\frac{dP_{n_2}^{m_2}}{d\mu}\Big) \tag{3.5.6}$$

then

$$\begin{aligned}\psi_n^m &= \frac{1}{2\pi}\int \bar{Y}_n^m \psi\, \delta S \\ &= \frac{1}{2\pi}\int e^{-im\lambda}P_n^m e^{i(m_1+m_2)\lambda}\Big(im_2 P_{n_2}^{m_2}\frac{dP_{n_1}^{m_1}}{d\mu} - im_1 P_{n_1}^{m_1}\frac{dP_{n_2}^{m_2}}{d\mu}\Big)\,\delta S\end{aligned}$$

Firstly examining the exponential part of this expression we know that

$$\int_{-\pi}^{\pi} e^{-im\theta}e^{in\theta}d\theta = \begin{cases} 0 & m \neq n \\ 2\pi & m = n \end{cases}$$

Therefore for a nonzero solution to this equation we require $m = m_1 + m_2$. This reduces our expression down to

$$\psi_n^{m_1+m_2} = \int_{-1}^{1} P_n^{m_1+m_2}\Big(im_2 P_{n_2}^{m_2}\frac{dP_{n_1}^{m_1}}{d\mu} - im_1 P_{n_1}^{m_1}\frac{dP_{n_2}^{m_2}}{d\mu}\Big)\, d\mu$$

This expression will be nonzero within a particular range of n. To determine this range we notice that an explicit formula for the interaction coefficients was derived in [8] and [23]. Through examining this explicit formula the ranges of applicable n for the interaction coefficients become obvious. We see that this derivation results in the following conditions

$$n_1 + n_2 + n \quad \text{is odd} \tag{3.5.7}$$

$$|n_1 - n_2| < n < n_1 + n_2 \tag{3.5.8}$$

The proof detailing how these conditions arise can be found in Appendix B.

If we define this interaction coefficient as follows

$$B^{m_j m_k}_{n_j n_k n} = \int_{-1}^{1} P_n^{m_j+m_k}\left(im_k P^{m_k}_{n_k}\frac{dP^{m_j}_{n_j}}{d\mu} - im_j P^{m_j}_{n_j}\frac{dP^{m_k}_{n_k}}{d\mu}\right) d\mu \tag{3.5.9}$$

we can see that using the spectral method

$$im_2 P^{m_2}_{n_2}\frac{dP^{m_1}_{n_1}}{d\mu} - im_1 P^{m_1}_{n_1}\frac{dP^{m_2}_{n_2}}{d\mu} = \sum_{n=|n_1-n_2|}^{n_1+n_2} P_n^{m_1+m_2} B^{m_1 m_2}_{n_1 n_2 n}$$

Therefore we can write the combinations of Legendre functions determined from the $O(\delta)$ calculations as sums of individual Legendre functions. Using this method our $O(\delta)$ equation (3.5.2) now becomes

$$\frac{\partial}{\partial t}\nabla^2\psi_1 + 2\frac{\partial \psi_1}{\partial \lambda} = l_{n_1 n_2}\Big\{A_1 A_2 e^{i[(m_1+m_2)\lambda-(\sigma_1+\sigma_2)t]}\sum_{n=|n_1-n_2|}^{n_1+n_2} P_n^{m_1+m_2} B^{m_1 m_2}_{n_1 n_2 n}$$
$$+(-1)^{m_2} A_1 \bar{A}_2 e^{i[(m_1-m_2)\lambda-(\sigma_1-\sigma_2)t]}\sum_{n=|n_1-n_2|}^{n_1+n_2} P_n^{m_1-m_2} B^{m_1-m_2}_{n_1 n_2 n}\Big\} + \text{c.c.} \tag{3.5.10}$$

The only way in which either of these terms can represent zonal flow is if $m_1 = m_2$ or $m_1 = -m_2$. These are the only two possibilities that will produce zero for the λ part of the exponential term and hence potential zonal flow. If we examine the case where $m_1 = -m_2$ our equation becomes

$$\frac{\partial}{\partial t}\nabla^2\psi_1 + 2\frac{\partial\psi_1}{\partial\lambda} = l_{n_1n_2}\Big\{A_1A_2e^{-i(\sigma_1+\sigma_2)t}\sum_{n=|n_1-n_2|}^{n_1+n_2} P_n^0 B_{n_1n_2n}^{-m_2m_2}$$
$$+(-1)^{m_2}A_1\bar{A}_2e^{i[(-2m_2)\lambda-(\sigma_1-\sigma_2)t]}\sum_{n=|n_1-n_2|}^{n_1+n_2} P_n^{-2m_2}B_{n_1n_2n}^{-m_2-m_2}\Big\} + \text{c.c.} \qquad (3.5.11)$$

For either of these expressions to represent a Rossby-Haurwitz wave and at the same time to correspond to zonal flow we would require the wave to satisfy the Rossby-Haurwitz wave condition, (3.4.3), along with the condition that the m term is zero in the exponential. If $m_1 = -m_2$ it is only the first expression which can correspond to zonal flow. Examining this term it is clear that the Rossby-Haurwitz wave condition reduces to

$$(\sigma_1+\sigma_2)n(n+1)e^{-i(\sigma_1+\sigma_2)t}P_n^0 = 0$$

i.e.

$$\begin{aligned}\sigma_1+\sigma_2 &= \frac{-2m_1}{n_1(n_1+1)} - \frac{2m_2}{n_2(n_2+1)}\\ &= \frac{2m_2}{n_1(n_1+1)} - \frac{2m_2}{n_2(n_2+1)}\\ &= 0\end{aligned}$$

$$\Leftrightarrow n_1 = n_2$$

But if $n_1 = n_2$ we have a problem because the $l_{n_1n_2}$ term in (3.5.11) is zero and hence we do not get zonal flow.

We get the same result when we examine the case for $m_1 = m_2$. Therefore we can see that it is not possible to produce zonal flow at $O(\delta)$ when ψ_0 is a dyad solution where the two waves do not interact resonantly.

3.5.2 The ψ_1 solution

The next possibility to consider is whether zonal flow can be generated at $O(\delta^2)$. The $O(\delta^2)$ equation is

$$\frac{\partial}{\partial t}\nabla^2\psi_2 + 2\frac{\partial\psi_2}{\partial\lambda} = J(\psi_0, \nabla^2\psi_1) + J(\psi_1, \nabla^2\psi_0) \tag{3.5.12}$$

We notice that to solve this equation we must firstly determine an expression for ψ_1. To do this we use the expression derived for the mixed Legendre function terms obtained through applying the spectral method to the equation.

Assuming that none of these Rossby-Haurwitz waves solve the leading order equation (this case will be discussed in the next chapter) we can determine ψ_1 by solving equation (3.5.10). If we consider a solution for ψ_1 to be of the form

$$\psi_1 = Be^{i[(m_1+m_2)\lambda-(\sigma_1+\sigma_2)t]}P_n^{m_1+m_2}$$

then B must satisfy

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^2\psi_1 + 2\frac{\partial\psi_1}{\partial\lambda} &= i(\sigma_1+\sigma_2)n(n+1)Be^{i[(m_1+m_2)\lambda-(\sigma_1+\sigma_2)t]}P_n^{m_1+m_2} \\ &\quad + 2i(m_1+m_2)Be^{i[(m_1+m_2)\lambda-(\sigma_1+\sigma_2)t]}P_n^{m_1+m_2} \\ &= A_1A_2e^{i[(m_1+m_2)\lambda-(\sigma_1+\sigma_2)t]}\sum_{n=|n_1-n_2|}^{n_1+n_2} P_n^{m_1+m_2}B_{n_1n_2n}^{m_1m_2}\end{aligned}$$

$$\Leftrightarrow B = \sum_{n=|n_1-n_2|}^{n_1+n_2}\frac{A_1A_2l_{n_1n_2}B_{n_1n_2n}^{m_1m_2}}{i(\sigma_1+\sigma_2)n(n+1)+2i(m_1+m_2)}$$

Similarly if we let

$$\psi_1 = Ce^{i[(m_1-m_2)\lambda-(\sigma_1-\sigma_2)t]}P_n^{m_1-m_2}$$

we will find that

$$C = \sum_{n=|n_1-n_2|}^{n_1+n_2} \frac{A_1\bar{A}_2 l_{n_1n_2} B^{m_1-m_2}_{n_1n_2n}}{i(\sigma_1-\sigma_2)n(n+1)+2i(m_1-m_2)}$$

Combining these two expressions we can see that ψ_1 is given by

$$\begin{aligned}\psi_1 = &\sum_{n=|n_1-n_2|}^{n_1+n_2} A_1A_2B^{m_1m_2}_{n_1n_2n}b^{m_1m_2}_{n_1n_2n}e^{i[(m_1+m_2)\lambda-(\sigma_1+\sigma_2)t]}P^{m_1+m_2}_n \\ &+(-1)^{m_2}\sum_{n=|n_1-n_2|}^{n_1+n_2} A_1\bar{A}_2B^{m_1-m_2}_{n_1n_2n}b^{m_1-m_2}_{n_1n_2n}e^{i[(m_1-m_2)\lambda-(\sigma_1-\sigma_2)t]}P^{m_1-m_2}_n \\ &+\text{c.c.}\end{aligned} \tag{3.5.13}$$

where

$$b^{m_jm_k}_{n_jn_kn} = \frac{-il_{n_jn_k}}{n(n+1)(\sigma_j+\sigma_k)+2(m_j+m_k)}$$

Since we now know both ψ_0 and ψ_1 we can examine the $O(\delta^2)$ equation (3.5.12) and answer the question of whether or not zonal flow can be created at this order.

3.6 $O(\delta^2)$ equation for a dyad solution

With ψ_0 and ψ_1 determined, the $O(\delta^2)$ equation is considered. Computing (3.5.12) we get

$$\frac{\partial}{\partial t}\nabla^2\psi_2 + 2\frac{\partial \psi_2}{\partial \lambda} = J(\psi_0, \nabla^2\psi_1) + J(\psi_1, \nabla^2\psi_0)$$

$$= \sum_{n=|n_1-n_2|}^{n_1+n_2} l_{nn_1}\Big\{A_1^2 A_2 B_{n_1n_2n}^{m_1m_2} b_{n_1n_2n}^{m_1m_2} e^{i[(2m_1+m_2)\lambda-(2\sigma_1+\sigma_2)t]}\times$$

$$\Big(im_1 P_{n_1}^{m_1}\frac{dP_n^{m_1+m_2}}{d\mu} - i(m_1+m_2)P_n^{m_1+m_2}\frac{dP_{n_1}^{m_1}}{d\mu}\Big)$$

$$+A_1^2\bar{A}_2(-1)^{m_2}B_{n_1n_2n}^{m_1-m_2}b_{n_1n_2n}^{m_1-m_2}e^{i[(2m_1-m_2)\lambda-(2\sigma_1-\sigma_2)t]}\times$$

$$\Big(im_1 P_{n_1}^{m_1}\frac{dP_n^{m_1-m_2}}{d\mu} - i(m_1-m_2)P_n^{m_1-m_2}\frac{dP_{n_1}^{m_1}}{d\mu}\Big)$$

$$+A_1\bar{A}_1A_2B_{n_1n_2n}^{m_1m_2}b_{n_1n_2n}^{m_1m_2}e^{i[m_2\lambda-\sigma_2t]}\times$$

$$\Big(-im_1P_{n_1}^{m_1}\frac{dP_n^{m_1+m_2}}{d\mu} - i(m_1+m_2)P_n^{m_1+m_2}\frac{dP_{n_1}^{m_1}}{d\mu}\Big)$$

$$-A_1\bar{A}_1\bar{A}_2(-1)^{m_2}B_{n_1n_2n}^{m_1-m_2}b_{n_1n_2n}^{m_1-m_2}e^{-i[m_2\lambda-\sigma_2t]}\times$$

$$\Big(i(m_1-m_2)P_n^{m_1-m_2}\frac{dP_{n_1}^{m_1}}{d\mu} + im_1P_{n_1}^{m_1}\frac{dP_n^{m_1-m_2}}{d\mu}\Big)$$

$$+\sum_{n=|n_1-n_2|}^{n_1+n_2} l_{nn_2}A_1A_2^2B_{n_1n_2n}^{m_1m_2}b_{n_1n_2n}^{m_1m_2}e^{i[(m_1+2m_2)\lambda-(\sigma_1+2\sigma_2)t]}\times$$

$$\Big(im_2P_{n_2}^{m_2}\frac{dP_n^{m_1+m_2}}{d\mu} - i(m_1+m_2)P_n^{m_1+m_2}\frac{dP_{n_2}^{m_2}}{d\mu}\Big)$$

$$+A_1A_2\bar{A}_2(-1)^{m_2}B_{n_1n_2n}^{m_1-m_2}b_{n_1n_2n}^{m_1-m_2}e^{i[m_1\lambda-\sigma_1t]}\times$$

$$\Big(im_2P_{n_2}^{m_2}\frac{dP_n^{m_1-m_2}}{d\mu} - i(m_1-m_2)P_n^{m_1-m_2}\frac{dP_{n_2}^{m_2}}{d\mu}\Big)$$

$$-A_1A_2\bar{A}_2B_{n_1n_2n}^{m_1m_2}b_{n_1n_2n}^{m_1m_2}e^{i[m_1\lambda-\sigma_1t]}\times$$

$$\Big(im_2P_{n_2}^{m_2}\frac{dP_n^{m_1+m_2}}{d\mu} + i(m_1+m_2)P_n^{m_1+m_2}\frac{dP_{n_2}^{m_2}}{d\mu}\Big)$$

$$-A_1\bar{A}_2^2(-1)^{m_2}B_{n_1n_2n}^{m_1-m_2}b_{n_1n_2n}^{m_1-m_2}e^{i[(m_1-2m_2)\lambda-(\sigma_1-2\sigma_2t]}\times$$

$$\Big(im_2P_{n_2}^{m_2}\frac{dP_n^{m_1-m_2}}{d\mu} + i(m_1-m_2)P_n^{m_1-m_2}\frac{dP_{n_2}^{m_2}}{d\mu}\Big)\Big\} + \text{c.c.}$$

Examining these terms we see that to produce zonal flow at this order we require one or more of the following to happen

- $2m_1 \pm m_2 = 0$
- $m_2 = 0$
- $m_1 \pm 2m_2 = 0$
- $m_1 = 0$

Any of these conditions will result in the λ part of the exponential term being zero. For zonal flow to occur in the first part of the $O(\delta^2)$ equation it would be required that $2m_1 = -m_2$. The term affected by this requirement is

$$A_1^2 A_2 B_{n_1 n_2 n}^{m_1 m_2} b_{n_1 n_2}^{m_1 m_2} e^{i[(2m_1+m_2)\lambda-(2\sigma_1+\sigma_2)t]} \times \left(i m_1 P_{n_1}^{m_1} \frac{dP_n^{m_1+m_2}}{d\mu} - i(m_1+m_2) P_n^{m_1+m_2} \frac{dP_{n_1}^{m_1}}{d\mu} \right) \tag{3.6.1}$$

The λ term in this expression is zero under this condition and therefore this represents zonal flow. We now need to determine if this zonal flow constitutes a resonant Rossby-Haurwitz wave. For this term to take the form of a Rossby-Haurwitz wave, the wave will have to solve the leading order equation (3.1.3). Letting

$$\psi_0 = e^{i[(2m_1+m_2)\lambda-(2\sigma_1+\sigma_2)t]} P_n^0$$

we see that it solves the leading order equation if and only if

$$(2\sigma_1 + \sigma_2) n(n+1) + 2(2m_1 + m_2) = 0$$

Since $2m_1 + m_2 = 0$ and $n \neq 0$, for the equation to hold true we require

$$\begin{aligned} 2\sigma_1 + \sigma_2 &= 2\left(\frac{-2m_1}{n_1(n_1+1)}\right) - \frac{2m_2}{n_2(n_2+1)} \\ &= \frac{-4m_1}{n_1(n_1+1)} + \frac{4m_1}{n_2(n_2+1)} \\ &= 0 \end{aligned}$$

This equation is true if and only if $n_1 = n_2$. But we encounter a problem with the requirement that $n_1 = n_2$. If this is true then the $b^{m_1 m_2}_{n_1 n_2 n}$ term in (3.6.1) will equal zero. Therefore in the only case for this example where resonant zonal flow is possible, the coefficients are required to be zero. Therefore this expression cannot produce resonant zonal flow.

We will get similar results when the cases $2m_1 = m_2$ and $m_1 = \pm 2m_2$ are examined. For each of these cases it would be required that $n_1 = n_2$ and this results in the corresponding $b^{m_j m_k}_{n_j n_k n}$ expression equalling zero. For the other cases, $m_1 = 0$ and $m_2 = 0$, it can also be shown that the coefficients in front of these expressions will also equal zero. Therefore it can be concluded that it is impossible to produce zonal flow at $O(\delta)$ or at $O(\delta^2)$ if we allow ψ_0 to be a nonresonantly interacting dyad solution.

3.7 Considering resonant interactions

Upon discovering that zonal flow cannot be created when we commence our calculations with a nonresonantly interacting dyad solution the next possibility to consider is starting out with two resonantly interacting Rossby-Haurwitz waves. Starting out with the superposition (3.5.1), which this time consists of two resonantly interacting waves, we examine the $O(\delta)$ equation

$$\frac{\partial}{\partial t}\nabla^2\psi_1 + 2\frac{\partial\psi_1}{\partial\lambda} = l_{n_1n_2}\Big(\sum_{n=|n_1-n_2|}^{n_1+n_2} A_1A_2B^{m_1m_2}_{n_1n_2n}e^{i[(m_1+m_2)\lambda-(\sigma_1+\sigma_2)t]}P_n^{m_1+m_2}$$
$$+(-1)^{m_2}\sum_{n=|n_1-n_2|}^{n_1+n_2} A_1\bar{A}_2B^{m_1-m_2}_{n_1n_2n}e^{i[(m_1-m_2)\lambda-(\sigma_1-\sigma_2)t]}P_n^{m_1-m_2} + \text{c.c.}\Big) \tag{3.7.1}$$

It is important now to examine if any of the terms on the right hand side solve (3.1.3). If, for example, the first term (for some n) solves the leading order equation then this would result in the solution ψ_1 of (3.7.1) containing the term

$$tB^{m_1m_2}_{n_1n_2n}b^{m_1m_2}_{n_1n_2n}e^{i[(m_1+m_2)\lambda-(\sigma_1+\sigma_2)t]}P_n^{m_1+m_2} \tag{3.7.2}$$

Any other term on the right hand side of (3.7.1) leads to a term in ψ_1 which is a constant times

$$B^{m_1m_2}_{n_1n_2n}b^{m_1m_2}_{n_1n_2n}e^{i[(m_1+m_2)\lambda-(\sigma_1+\sigma_2)t]}P_n^{m_1+m_2} \tag{3.7.3}$$

The term (3.7.2) grows linearly in time and indicates resonance. This means that two waves combine to force a third wave to be formed which, over time, becomes as strong as the original two waves. This type of interaction is called a resonant interaction and is of great interest to us. Nonresonant interactions (3.7.3) will merely create small background noise, but for resonant interactions we get a strong time dependent solution for our correction term of the form (3.7.2).

We insisted that the expansion (3.1.2) is valid for times of $O(1/\delta)$, in the sense that the $O(\delta)$ term in (3.1.2) is much smaller than the leading order term, for time t of order $1/\delta$. This means that we cannot tolerate terms of the form (3.7.2) in ψ_1. If any such term was produced it would have to be included in the leading order solution, ψ_0, to ensure that the approximation remains valid for all times t of order $1/\delta$. As a result all potential resonant terms must be checked and a corresponding condition will be derived.

Taking each term on the right hand side of equation (3.7.1) in turn, we must check to see if it satisfies the leading order equation. The equation under consideration is

$$\frac{\partial}{\partial t}\left(\frac{1}{\cos^2\phi}\frac{\partial^2\psi}{\partial\lambda^2}+\frac{\partial^2\psi}{\partial\phi^2}-\tan\phi\frac{\partial\psi}{\partial\phi}\right)+2\frac{\partial\psi}{\partial\lambda}=0 \tag{3.7.4}$$

To determine which terms could cause problems we need to examine the two possible terms

$$e^{i[(m_1+m_2)\lambda-(\sigma_1+\sigma_2)t]}P_n^{m_1+m_2} \tag{3.7.5}$$

and

$$e^{i[(m_1-m_2)\lambda-(\sigma_1-\sigma_2)t]}P_n^{m_1-m_2} \tag{3.7.6}$$

The first example considered is (3.7.5). Studying this term it is clear that when this solution is filled into the leading order equation (3.7.4) it is satisfied if and only if

$$\big(i(\sigma_1+\sigma_2)n(n+1)+2i(m_1+m_2)\big)e^{i[(m_1+m_2)\lambda-(\sigma_1+\sigma_2)t]}P_n^{m_1+m_2}=0$$

We find that this equation can be true if and only if

$$i(\sigma_1+\sigma_2)n(n+1)+2i(m_1+m_2)=0$$

Using the expressions determined for σ_1 and σ_2 from the leading order equation this reduces down to

$$\frac{m_1}{n_1(n_1+1)}+\frac{m_2}{n_2(n_2+1)}=\frac{m_1+m_2}{n(n+1)}$$

So we find that if this condition is satisfied then we have resonance. This would mean that the expansion will not remain valid under the time scale we are considering. To get around this problem we must include this third wave into our leading order solution to ensure that the approximation remains valid. We do this by letting the corresponding n be called n_3 and set $m_1 + m_2 = m_3$.

Examining the second possibility (3.7.6) it can be found that none of the terms in this expression will cause the aforementioned problem. Therefore none of these terms will cause problems with resonance. Without loss of generality we have assumed that the Rossby-Haurwitz wave is one of the terms in the first sum; if it is not, we simply replace m_2 by $-m_2$ in (3.5.1).

The result of this analysis is that we must include this Rossby-Haurwitz wave into the leading order solution. Therefore we study a solution of the form

$$\begin{aligned}\psi_0 =& A_1 e^{i(m_1\lambda-\sigma_1 t)} P_{n_1}^{m_1}(\mu) + \bar{A}_1 e^{-i(m_1\lambda-\sigma_1 t)} P_{n_1}^{m_1}(\mu)\\ &+A_2 e^{i(m_2\lambda-\sigma_2 t)} P_{n_2}^{m_2}(\mu) + \bar{A}_2 e^{-i(m_2\lambda-\sigma_2 t)} P_{n_2}^{m_2}(\mu)\\ &+A_3 e^{i(m_3\lambda-\sigma_3 t)} P_{n_3}^{m_3}(\mu) + \bar{A}_3 e^{-i(m_3\lambda-\sigma_3 t)} P_{n_3}^{m_3}(\mu),\end{aligned} \tag{3.7.7}$$

where

$$m_3 = m_1 + m_2, \quad \sigma_3 = \frac{-2m_3}{n_3(n_3+1)}$$

and

$$\frac{m_1}{n_1(n_1+1)} + \frac{m_2}{n_2(n_2+1)} = \frac{m_3}{n_3(n_3+1)} \tag{3.7.8}$$

Chapter 4

Triad solutions and Zonal flow

The aim of this thesis is to study first and second order resonant interactions of Rossby-Haurwitz waves in general. In particular, we want to investigate in detail whether large-scale interacting planetary waves with finite amplitudes can force zonal flow through a resonant interaction energy transfer mechanism. It has been proved in the previous chapter that it is not possible to produce zonal flow at either $O(\delta)$ or $O(\delta^2)$ if we consider a nonresonantly interacting Rossby-Haurwitz wave dyad solution.

It has also been shown that resonant interactions between Rossby-Haurwitz waves lead to a triad configuration. The longitudinal wavenumber and frequency of the third Rossby-Haurwitz wave equal the sums of the longitudinal wavenumbers and frequencies of the other two Rossby-Haurwitz waves. Also the interaction of this Rossby-Haurwitz wave with each of the original two Rossby-Haurwitz waves generates the other. These three waves transfer energy between themselves and do not produce any further waves at first order. We are interested in studying this triad configuration to determine if any new waves can in fact be produced at higher order.

We wish to try to generate zonal flow through these Rossby-Haurwitz wave triad interactions. In this chapter we examine the excitation of a zonal flow through a direct triad resonance mechanism. As was previously stated a triad configuration seemingly does not produce new waves at $O(\delta)$. We examine this theory and then we also examine whether zonal flow can be produced at $O(\delta^2)$.

4.1 Time Dependence of Amplitudes

The triad solution we are considering which was determined in Chapter 3 will, of course, interact to produce, once again, the unwanted terms similar to (3.7.2) in ψ_1. To avoid this problem it is assumed that the amplitudes A_1, A_2 and A_3 are slowly varying in time, such that their time derivatives are of order δ. We implement this idea by introducing a slow time $\tau_1 = \delta t$ into our calculations. As a bookkeeping device the time derivative shall be rewritten as

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \delta \frac{\partial}{\partial \tau_1}$$

Using these new slowly time dependent amplitudes, our main equation, (3.1.1) becomes

$$\frac{\partial}{\partial t}\nabla^2\psi + 2\frac{\partial\psi}{\partial\lambda} = \delta J(\psi, \nabla^2\psi) - \delta\frac{\partial}{\partial\tau_1}\nabla^2\psi \tag{4.1.1}$$

Once again we shall expand ψ about δ

$$\psi(\lambda,\phi,t,\delta) = \psi_0(\lambda,\phi,t) + \delta\psi_1(\lambda,\phi,t) + \ldots$$

We see that our leading order equation remains unchanged

$$\frac{\partial}{\partial t}\nabla^2\psi_0 + 2\frac{\partial\psi_0}{\partial\lambda} = 0 \tag{4.1.2}$$

but the $O(\delta)$ equation becomes

$$\frac{\partial}{\partial t}\nabla^2\psi_1 + 2\frac{\partial\psi_1}{\partial\lambda} = J(\psi_0,\nabla^2\psi_0) - \frac{\partial}{\partial\tau_1}\nabla^2\psi_0 \tag{4.1.3}$$

From the previous chapter we have

$$\begin{aligned}\psi_0 = {} & A_1(\tau_1)e^{i(m_1\lambda-\sigma_1 t)}P_{n_1}^{m_1}(\mu) + \bar{A}_1(\tau_1)e^{-i(m_1\lambda-\sigma_1 t)}P_{n_1}^{m_1}(\mu)\\ & +A_2(\tau_1)e^{i(m_2\lambda-\sigma_2 t)}P_{n_2}^{m_2}(\mu) + \bar{A}_2(\tau_1)e^{-i(m_2\lambda-\sigma_2 t)}P_{n_2}^{m_2}(\mu)\\ & +A_3(\tau_1)e^{i(m_3\lambda-\sigma_3 t)}P_{n_3}^{m_3}(\mu) + \bar{A}_3(\tau_1)e^{-i(m_3\lambda-\sigma_3 t)}P_{n_3}^{m_3}(\mu)\end{aligned} \tag{4.1.4}$$

where

$$m_1 + m_2 = m_3$$

and the following conditions on the velocities

$$\sigma_1 = \frac{-2m_1}{n_1(n_1+1)}, \quad \sigma_2 = \frac{-2m_2}{n_2(n_2+1)} \quad \text{and} \quad \sigma_3 = \frac{-2m_3}{n_3(n_3+1)}$$

To examine interactions of these Rossby-Haurwitz waves we must study the $O(\delta)$ equation. Through substituting (4.1.4) into this equation, (4.1.3), we see that we must examine the following

$$\frac{\partial}{\partial t}\Big(\frac{1}{\cos^2\phi}\frac{\partial^2\psi_1}{\partial\lambda^2}+\frac{\partial^2\psi_1}{\partial\phi^2}-\tan\phi\frac{\partial\psi_1}{\partial\phi}\Big)+2\frac{\partial\psi_1}{\partial\lambda}=$$

$$\sum_{j=1}^{3}\sum_{k=j+1}^{3} l_{n_jn_k}\Big\{A_jA_ke^{i[(m_j+m_k)\lambda-(\sigma_j+\sigma_k)t]}\sum_{n=|n_j-n_k|}^{n_j+n_k}P_n^{m_j+m_k}B_{n_jn_kn}^{m_jm_k}$$

$$+(-1)^{m_k}A_j\bar{A}_ke^{i[(m_j-m_k)\lambda-(\sigma_j-\sigma_k)t]}\sum_{n=|n_j-n_k|}^{n_j+n_k}P_n^{m_j-m_k}B_{n_jn_kn}^{m_j-m_k}\Big\}$$

$$+\sum_{j=1}^{3}n_j(n_j+1)\frac{dA_j}{d\tau_1}e^{i(m_j\lambda-\sigma_jt)}P_{n_j}^{m_j}+\text{c.c.} \tag{4.1.5}$$

If any term on the right hand side of this equation satisfies the leading order equation (4.1.2) then a problem with resonance will be encountered as before. Therefore we must examine these equations to determine whether or not this problem occurs.

Examining the first term

$$e^{i[(m_j+m_k)\lambda-(\sigma_j+\sigma_k)t]}P_n^{m_j+m_k}$$

we see that this term solves the leading order equation if and only if

$$i(\sigma_j+\sigma_k)n(n+1)+2i(m_j+m_k)=0$$

Using the conditions derived from the leading order equation, the left hand side of this equation reduces to

$$\Big(\frac{-2m_j}{n_j(n_j+1)}-\frac{2m_k}{n_k(n_k+1)}\Big)n(n+1)+2(m_j+m_k)=0$$

Examining this term we can see that it will equal zero if and only if $j=1$ and $k=2$, i.e. this will represent the third part of the triad, the (m_3,n_3) Rossby-Haurwitz wave. So taking these values into account the problematic term is

$$l_{n_1 n_2} A_1 A_2 e^{i[(m_1+m_2)\lambda-(\sigma_1+\sigma_2)t]} P_{n_3}^{m_1+m_2} B_{n_1 n_2 n}^{m_1 m_2}$$

There are no other cases for this example for which this term will cause resonance.

Next examining the other possibility we study

$$e^{i[(m_j-m_k)\lambda-(\sigma_j-\sigma_k)t]} P_n^{m_j-m_k}$$

This term solves the leading order equation if and only if

$$i(\sigma_j - \sigma_k)n(n+1) + 2i(m_j - m_k) = 0$$

Using the conditions derived from the leading order equation the left hand side of this equation reduces to

$$\left(\frac{-2m_j}{n_j(n_j+1)} + \frac{2m_k}{n_k(n_k+1)}\right) n(n+1) + 2(m_j - m_k) = 0$$

Examining this term we see that there are two cases for which this term can equal zero. The first case is if $j = 2$ and $k = 3$. This represents the conjugate of the first Rossby-Haurwitz wave of our leading order solution when $n = n_1$ to give the wave (m_1, n_1) i.e.,

$$(-1)^{m_3} l_{n_2 n_3} A_2 \bar{A}_3 e^{i[(m_2-m_3)\lambda-(\sigma_2-\sigma_3)t]} P_{n_1}^{m_2-m_3} B_{n_2 n_3 n}^{m_2-m_3}$$

The other possibility for this case which causes problems is $j = 1$ and $k = 3$, which corresponds to the conjugate of the second Rossby-Haurwitz wave when $n = n_2$, (m_2, n_2), i.e.

$$(-1)^{m_3} l_{n_1 n_3} A_1 \bar{A}_3 e^{i[(m_1-m_3)\lambda-(\sigma_1-\sigma_3)t]} P_{n_2}^{m_1-m_3} B_{n_1 n_3 n}^{m_1-m_3}$$

Examining the τ_1 derivative terms we also see that we have expressions akin to each term in the leading order solution which will also cause problems with resonance. Therefore, combining all this, we note that the following terms will prove to be problematic if they are left in the equation

- $l_{n_1n_2}A_1A_2e^{i[(m_1+m_2)\lambda-(\sigma_1+\sigma_2)t]}P_{n_3}^{m_1+m_2}B_{n_1n_2n}^{m_1m_2}$
- $(-1)^{m_3}l_{n_2n_3}\bar{A}_2A_3e^{-i[(m_2-m_3)\lambda-(\sigma_2-\sigma_3)t]}P_{n_1}^{m_2-m_3}(-1)B_{n_2n_3n}^{m_2-m_3}$
- $(-1)^{m_3}l_{n_1n_3}\bar{A}_1A_3e^{-i[(m_1-m_3)\lambda-(\sigma_1-\sigma_3)t]}P_{n_2}^{m_1-m_3}(-1)B_{n_1n_3n}^{m_1-m_3}$
- $n_1(n_1+1)\frac{dA_1}{d\tau_1}e^{i(m_1\lambda-\sigma_1t)}P_{n_1}^{m_1}$
- $n_2(n_2+1)\frac{dA_2}{d\tau_1}e^{i(m_2\lambda-\sigma_2t)}P_{n_2}^{m_2}$
- $n_3(n_3+1)\frac{dA_3}{d\tau_1}e^{i(m_3\lambda-\sigma_3t)}P_{n_3}^{m_3}$

These terms will cause problems with resonance if they are left in the equations as they are. The complex conjugates of all these terms also need to be included to ensure that we have encompassed all terms of this format.

In order to avoid this problem we can equate all these problematic terms and cancel them from the right hand side of the $O(\delta)$ equation (4.1.5). Remembering that $m_1+m_2=m_3$ we get

$$-l_{n_2n_3}(-1)^{m_1}(-1)^{m_3}\bar{A}_2A_3e^{i(m_1\lambda-\sigma_1t)}P_{n_1}^{m_1}B_{n_2n_3n}^{m_2-m_3}+n_1(n_1+1)\frac{dA_1}{d\tau_1}e^{i(m_1\lambda-\sigma_1t)}P_{n_1}^{m_1}=0$$

$$-l_{n_1n_3}(-1)^{m_2}(-1)^{m_3}\bar{A}_1A_3e^{i(m_2\lambda-\sigma_2t)}P_{n_2}^{m_2}B_{n_1n_3n}^{m_1-m_3}+n_2(n_2+1)\frac{dA_2}{d\tau_1}e^{i(m_2\lambda-\sigma_2t)}P_{n_2}^{m_2}=0$$

$$l_{n_1n_2}A_1A_2e^{i(m_3\lambda-\sigma_3t)}P_{n_3}^{m_3}B_{n_1n_2n}^{m_1m_2}+n_3(n_3+1)\frac{dA_3}{d\tau_1}e^{i(m_3\lambda-\sigma_3t)}P_{n_3}^{m_3}=0$$

along with their complex conjugates. Tidying this up a little we see that we get

$$\frac{dA_1}{d\tau_1} = \frac{(-1)^{m_1}(-1)^{m_3} l_{n_2n_3} B^{m_2-m_3}_{n_2n_3n_1}}{n_1(n_1+1)} \bar{A}_2 A_3 \tag{4.1.6}$$

$$\frac{dA_2}{d\tau_1} = \frac{(-1)^{m_2}(-1)^{m_3} l_{n_1n_3} B^{m_1-m_3}_{n_1n_3n_2}}{n_2(n_2+1)} \bar{A}_1 A_3 \tag{4.1.7}$$

$$\frac{dA_3}{d\tau_1} = \frac{l_{n_2n_1} B^{m_1m_2}_{n_1n_2n_3}}{n_3(n_3+1)} A_1 A_2 \tag{4.1.8}$$

Next we examine the $B^{m_jm_k}_{n_jn_kn}$ terms from the amplitude equations to see if they can be written in a nicer form. Firstly examining $B^{m_1-m_3}_{n_1n_3n}$ we see that

$$\begin{aligned}
B^{m_1-m_3}_{n_1n_3n_2} &= \int_{-1}^{1} P^{m_1-m_3}_{n_2}\left(-im_3P^{-m_3}_{n_3}\frac{dP^{m_1}_{n_1}}{d\mu} - im_1P^{m_1}_{n_1}\frac{dP^{-m_3}_{n_3}}{d\mu}\right)d\mu \\
&= -(-1)^{m_2}(-1)^{m_3}\int_{-1}^{1} P^{m_2}_{n_2}\left(im_3P^{m_3}_{n_3}\frac{dP^{m_1}_{n_1}}{d\mu} + im_1P^{m_1}_{n_1}\frac{dP^{m_3}_{n_3}}{d\mu}\right)d\mu \\
&= -(-1)^{m_2}(-1)^{m_3}\int_{-1}^{1} P^{m_3}_{n_3}\left(im_2P^{m_2}_{n_2}\frac{dP^{m_1}_{n_1}}{d\mu} - im_1P^{m_1}_{n_1}\frac{dP^{m_2}_{n_2}}{d\mu}\right)d\mu \\
&= -(-1)^{m_2}(-1)^{m_3}B^{m_1m_2}_{n_1n_2n_3}
\end{aligned}$$

Applying similar calculations to $B^{m_2-m_3}_{n_2n_3n_1}$ we can see that to ensure that we do not encounter problems with these resonant terms we insist that the amplitudes must satisfy the following differential equations

$$\frac{dA_1}{d\tau_1} = \frac{l_{n_2n_3}}{n_1(n_1+1)} B^{m_1m_2}_{n_1n_2n_3} \bar{A}_2 A_3 \tag{4.1.9}$$

$$\frac{dA_2}{d\tau_1} = \frac{l_{n_3n_1}}{n_2(n_2+1)} B^{m_1m_2}_{n_1n_2n_3} \bar{A}_1 A_3 \tag{4.1.10}$$

$$\frac{dA_3}{d\tau_1} = \frac{l_{n_2n_1}}{n_3(n_3+1)} B^{m_1m_2}_{n_1n_2n_3} A_1 A_2 \tag{4.1.11}$$

If a change of variable is applied to these equations it should be noticed that they represent the three-wave equations which model the nonlinear dynamics of the amplitudes of three waves in fluids or plasmas. There are well known conservation quantities connected to the three-wave equations which will also apply to our equations above. Examples of studies on these conservation properties are in [7] and [11].

4.2 Checking for occurrence of zonal flow at $O(\delta)$ and $O(\delta^2)$

The conditions on the amplitudes must now be taken into account when examining the $O(\delta)$ equation and hence the ψ_1 solution. Applying these conditions we see that the our ψ_1 solution is now

$$\begin{aligned}\psi_1 = \sum_{j=1}^{3}\sum_{k=j+1}^{3}\Big(&\sum_{\substack{n=|n_j-n_k|\\ n\neq n_{jk}}}^{n_j+n_k} A_j A_k B_{n_j n_k n}^{m_j m_k} b_{n_j n_k n}^{m_j m_k} e^{i[(m_j+m_k)\lambda-(\sigma_j+\sigma_k)t]} P_n^{m_j+m_k} \\ &+ (-1)^{m_k} \sum_{\substack{n=|n_j-n_k|\\ n\neq \tilde{n}_{jk}}}^{n_j+n_k} A_j \bar{A}_k B_{n_j n_k n}^{m_j -m_k} b_{n_j n_k n}^{m_j -m_k} e^{i[(m_j-m_k)\lambda-(\sigma_j-\sigma_k)t]} P_n^{m_j-m_k}\Big) \\ &+ \text{c.c.}\end{aligned} \tag{4.2.1}$$

where

$$n_{jk} = \begin{cases} n_3, & j=1, k=2 \\ n_j+n_k+1, & \text{otherwise}\end{cases} \qquad \tilde{n}_{jk} = \begin{cases} n_1, & j=2, k=3 \\ n_2, & j=1, k=3 \\ n_j+n_k+1, & \text{otherwise}\end{cases}$$

All that the conditions $n \neq n_{jk}$ and $n \neq \tilde{n}_{jk}$ do is to take out the resonant term.

We must check all these terms now to see if any of them will correspond to zonal flow. As explained before, zonal flow terms are λ independent wave solutions. Examining these terms we see that the λ part of the exponential term will be zero if and only if $m_j = m_k$

or $m_j = -m_k$. Firstly examining the case for $m_j = -m_k$ the expression for ψ_1 reduces to

$$\psi_1 = \sum_{j=1}^{3}\sum_{k=j+1}^{3}\Bigg(\sum_{\substack{n=|n_j-n_k| \\ n\neq n_{jk}}}^{n_j+n_k} A_j A_k B_{n_j n_k n}^{-m_k m_k} b_{n_j n_k n}^{-m_k m_k} e^{-i(\sigma_j+\sigma_k)t} P_n^0$$
$$+ (-1)^{m_k} \sum_{\substack{n=|n_j-n_k| \\ n\neq \bar{n}_{jk}}}^{n_j+n_k} A_j \bar{A}_k B_{n_j n_k n}^{-m_k -m_k} b_{n_j n_k n}^{-m_k -m_k} e^{i[(-2m_k)\lambda-(\sigma_j-\sigma_k)t]} P_n^{-2m_k}\Bigg)$$
$$+ \text{ c.c.} \tag{4.2.2}$$

Only the first of these two terms can represent zonal flow. For this zonal flow term to be of the form of a Rossby-Haurwitz wave it must satisfy the condition

$$-i(\sigma_j + \sigma_k)n(n+1) = 0$$

But like in the previous chapter we see that this is true if and only if $n_j = n_k$. If this holds true then $b_{n_j n_k n}^{-m_k m_k}$ is zero. Similar results are determined for the case where $m_j = m_k$. Therefore we see once again that it is not possible to produce zonal flow at $O(\delta)$ even when we start with a resonantly interacting triad solution.

The next logical step to take is to check for the occurrence of zonal flow at $O(\delta^2)$. The $O(\delta^2)$ equation is

$$\frac{\partial}{\partial t}\nabla^2\psi_2 + 2\frac{\partial\psi_2}{\partial\lambda} = J(\psi_0, \nabla^2\psi_1) + J(\psi_1, \nabla^2\psi_0)$$
$$-\frac{\partial}{\partial\tau_1}\nabla^2\psi_1 \tag{4.2.3}$$

When we substitute in our expressions for ψ_0 and ψ_1, the $O(\delta^2)$ equation is

$$\frac{\partial}{\partial t}\nabla^2\psi_2 + 2\frac{\partial\psi_2}{\partial\lambda} = -\frac{\partial}{\partial\tau_1}\nabla^2\psi_1$$

$$+\sum_{l=1}^{3}\sum_{j=1}^{3}\sum_{k=j+1}^{3}\left\{\sum_{\substack{n=|n_j-n_k|\\ n\neq n_{jk}}}^{n_j+n_k}(n_l(n_l+1)-n(n+1))\times\right.$$

$$B^{m_jm_k}_{n_jn_kn}b^{m_jm_k}_{n_jn_kn}\Big(A_lA_jA_ke^{i[(m_l+m_j+m_k)\lambda-(\sigma_l+\sigma_j+\sigma_k)t]}$$

$$\Big(i(m_j+m_k)P_n^{m_j+m_k}\frac{dP^{m_l}_{n_l}}{d\mu} - im_lP^{m_l}_{n_l}\frac{dP_n^{m_j+m_k}}{d\mu}\Big)$$

$$-A_l\bar{A}_j\bar{A}_ke^{i[(m_l-m_j-m_k)\lambda-(\sigma_l-\sigma_j-\sigma_k)t]}$$

$$\Big(i(m_j+m_k)P_n^{m_j+m_k}\frac{dP^{m_l}_{n_l}}{d\mu} + im_lP^{m_l}_{n_l}\frac{dP_n^{m_j+m_k}}{d\mu}\Big)\Big)$$

$$+\sum_{\substack{n=|n_j-n_k|\\ n\neq \bar{n}_{jk}}}^{n_j+n_k}(n_l(n_l+1)-n(n+1))\times$$

$$(-1)^{m_k}B^{m_j-m_k}_{n_jn_kn}b^{m_j-m_k}_{n_jn_kn}\Big(A_lA_j\bar{A}_ke^{i[(m_l+m_j-m_k)\lambda-(\sigma_l+\sigma_j-\sigma_k)t]}$$

$$\Big(i(m_j-m_k)P_n^{m_j-m_k}\frac{dP^{m_l}_{n_l}}{d\mu} - im_lP^{m_l}_{n_l}\frac{dP_n^{m_j-m_k}}{d\mu}\Big)$$

$$-A_l\bar{A}_jA_ke^{i[(m_l-m_j+m_k)\lambda-(\sigma_l-\sigma_j+\sigma_k)t]}$$

$$\left.\Big(i(m_j-m_k)P_n^{m_j-m_k}\frac{dP^{m_l}_{n_l}}{d\mu} + im_lP^{m_l}_{n_l}\frac{dP_n^{m_j-m_k}}{d\mu}\Big)\Big)\right\} \tag{4.2.4}$$

We want to examine all these terms to see if zonal flow can be produced. If we take, for example, $j=1$, $k=2$ and $l=3$ then the second term will correspond to zonal flow. This term is

$$\sum_{\substack{n=|n_j-n_k|\\ n\neq n_{jk}}}^{n_j+n_k}(n_3(n_3+1)-n(n+1))\,\bar{A}_1\bar{A}_2A_3e^{-i[(m_1+m_2-m_3)\lambda-(\sigma_1+\sigma_2-\sigma_3)t]}$$

$$\times\Big(i(m_1+m_2)P_n^{m_1+m_2}\frac{dP^{m_3}_{n_3}}{d\mu} + im_3P^{m_3}_{n_3}\frac{dP_n^{m_1+m_2}}{d\mu}\Big)\Big)$$

Examining this term we see that the λ term is in fact zero and hence it represents zonal flow. To see if it represents a Rossby-Haurwitz wave it must satisfy the Rossby-Haurwitz wave condition, i.e.

$$\frac{m_1}{n_1(n_1+1)} + \frac{m_2}{n_2(n_2+1)} = \frac{m_3}{n_3(n_3+1)}$$

This condition was one of the conditions of our triad solution. The σ terms in the above expression equate to

$$\begin{aligned}\sigma_1 + \sigma_2 - \sigma_3 &= -\frac{2m_1}{n_1(n_1+1)} - \frac{2m_2}{n_2(n_2+1)} + \frac{2m_3}{n_3(n_3+1)} \\ &= 0\end{aligned}$$

Therefore we can conclude from these workings that nonresonantly interacting Rossby-Haurwitz waves are incapable of generating zonal flow for either the $O(\delta)$ equation or the $O(\delta^2)$ equation. By examining resonantly interacting Rossby-Haurwitz waves we have shown that it is possible to generate zonal flow. Zonal flow first occurs at the $O(\delta^2)$ equation and it is only possible to generate such waves when a triad interaction is considered. Thus the preceding calculations illustrate the capacity of Rossby-Haurwitz waves to generate zonal flow through the resonant mechanism for a triad solution.

4.3 Zonal flow

The consequences of the previous section will now be discussed in detail. We proved that resonant zonal flow terms are generated which will cause the asymptotic expansion of ψ to become invalid for certain time, when we consider a resonantly interacting Rossby-

Haurwitz wave triad solution up to $O(\delta^2)$. As a result of this we include these particular zonal flow terms in our leading order solution to ensure that this doesn't happen.

One should note, however, that at this stage this does not rule out the possibility that the zonal flow terms are $O(\delta)$. According to our asymptotic scheme, at each order, certain equations are imposed on the amplitudes to avoid resonances. Solving these equations at each order updates the amplitudes, which are given as expressions in δ. It could well turn out that the equations we will later impose on the zonal flow amplitudes lead to amplitudes of $O(\delta)$.

In the previous section we saw that we can produce one example of resonant zonal flow when examining the $O(\delta^2)$ equation for the triad solution. Therefore at this order we have produced zonal flow terms which are also Rossby-Haurwitz wave solutions. Similar terms can also be produced for the other parts of the $O(\delta^2)$ equation which will similarly correspond to nonzero resonant zonal flow. Therefore these terms will cause resonance and are problematic. To overcome this problem these terms are included in the leading order solution from the outset. Therefore we study

$$\psi_0 = \sum_{j=1}^{N} \psi_0^j \tag{4.3.1}$$

where

$$\psi_0^j = (A_j(\tau_1)e^{i(m_j\lambda-\sigma_j t)} + \bar{A}_j(\tau_1)e^{-i(m_j\lambda-\sigma_j t)})P_{n_j}^{m_j}(\mu),$$

$$\sigma_j = \frac{-2m_j}{n_j(n_j+1)},$$

$$m_j = 0, \quad A_j = \bar{A}_j, \quad j = 4, 5, \cdots, N$$

Using this new solution for ψ_0 we once again have to make sure that resonant Rossby-Haurwitz waves do not occur on the right hand side of the $O(\delta)$ equation. To ensure that this does not happen we must enforce the following conditions on the amplitudes

$$\begin{aligned}
\frac{dA_1}{d\tau_1} &= \frac{l_{n_2n_3}B^{m_1m_2}_{n_1n_2n_3}}{n_1(n_1+1)}\bar{A}_2A_3 + 2\sum_{k=4}^{N}\frac{l_{n_kn_1}B^{m_10}_{n_1n_kn_1}}{n_1(n_1+1)}A_1A_k \\
\frac{dA_2}{d\tau_1} &= \frac{l_{n_3n_1}B^{m_1m_2}_{n_1n_2n_3}}{n_2(n_2+1)}\bar{A}_1A_3 + 2\sum_{k=4}^{N}\frac{l_{n_kn_2}B^{m_20}_{n_2n_kn_2}}{n_2(n_2+1)}A_2A_k \\
\frac{dA_3}{d\tau_1} &= \frac{l_{n_2n_1}B^{m_1m_2}_{n_1n_2n_3}}{n_3(n_3+1)}A_1A_2 + 2\sum_{k=4}^{N}\frac{l_{n_kn_3}B^{m_30}_{n_3n_kn_3}}{n_3(n_3+1)}A_3A_k \\
\frac{dA_k}{d\tau_1} &= 0 \quad \text{for } k = 4, 5, \cdots, N
\end{aligned} \tag{4.3.2}$$

We see that the amplitudes of the zonal flow terms do not gain or lose energy at this time scale. At this order the zonal flow actually acts as a catalyst, helping the other waves to exchange energy between them. The next order of the equation must be studied to establish conditions on variations for the zonal flow amplitudes.

Before continuing to study the next order equation we should notice that the zonal flow terms at this order only affect the phase of the amplitudes, A_i, $i = 1, 2, 3$. If we take the derivative of the modulus squared of A_i, $i = 1, 2, 3$, in each case we will see that the terms containing the zonal flow amplitudes will reduce to zero. The other terms will be nonzero after applying the same calculation to them. Therefore we can infer that the first part of the amplitude conditions affect the wave itself while the zonal flow terms affect the phase only.

Examining these equations we notice that it is possible, in theory, to solve these equations. Solving these equations we can determine expressions for A_i. The result of this is that A_i will be dependent on δ, t and N constants of integration, i.e. the amplitudes are functions of the form $A(\delta, t, c_1, c_2, \ldots c_N)$. At the $O(\delta^2)$ equation the problem with resonance will once again crop up. To overcome this problem of resonance we now make the integration constants t dependent. These terms will be slowly time dependent and hence will first appear in the $O(\delta^2)$ equation. This will cause terms of the form

$$\frac{\partial A_j}{\partial c_1}\frac{dc_1}{dt} + \ldots + \frac{\partial A_j}{\partial c_N}\frac{dc_N}{dt}$$

to appear in the $O(\delta^2)$ equation. We symbolically write

$$\frac{\partial A_j}{\partial \tau_2} = \frac{\partial A_j}{\partial c_1}\frac{dc_1}{dt} + \ldots + \frac{\partial A_j}{\partial c_N}\frac{dc_N}{dt}$$

It is important to note that in our asymptotic scheme τ_1 and τ_2 are not independent variables. We will explain later that treating τ_1 and τ_2 as independent variables leads to inconsistent equations.

To study the next order of the equation an expression is required for ψ_1. Taking into account the above conditions, it can be deduced that

$$\begin{aligned}\psi_1 = \sum_{j=1}^{3}\sum_{k=j+1}^{N}\Bigg(&\sum_{\substack{n=|n_j-n_k| \\ n\neq n_{jk}}}^{n_j+n_k} A_j A_k B_{n_j n_k n}^{m_j m_k} b_{n_j n_k n}^{m_j m_k} e^{i[(m_j+m_k)\lambda-(\sigma_j+\sigma_k)t]} P_n^{m_j+m_k} \\ &+ (-1)^{m_k}\sum_{\substack{n=|n_j-n_k| \\ n\neq \bar{n}_{jk}}}^{n_j+n_k} A_j \bar{A}_k B_{n_j n_k n}^{m_j-m_k} b_{n_j n_k n}^{m_j-m_k} e^{i[(m_j-m_k)\lambda-(\sigma_j-\sigma_k)t]} P_n^{m_j-m_k}\Bigg) \\ &+ \text{c.c.}\end{aligned} \tag{4.3.3}$$

where

$$n_{jk} = \begin{cases} n_3, & j = 1, k = 2 \\ n_j, & j = 1, 2, 3 \text{ and } k = 4, 5, \cdots, N \\ n_j + n_k + 1, & \text{otherwise} \end{cases}$$

and

$$\tilde{n}_{jk} = \begin{cases} n_1, & j = 2, k = 3 \\ n_2, & j = 1, k = 3 \\ n_j, & j = 1, 2, 3 \text{ and } k = 4, 5, \cdots, N \\ n_j + n_k + 1, & \text{otherwise} \end{cases}$$

When we examine the $O(\delta^2)$ equation for these values of ψ_0 and ψ_1 we find that resonating terms will be produced which will cause problems with the approximation. We overcome these resonance problems by making the c_i slowly time dependent variables. Using this and the symbolic notation in terms of partial τ_2 derivatives, the next set of amplitude equations are given below. Note that this is a system of N ordinary differential equations for $c_1, \ldots, c_N$.

$$
\begin{aligned}
\frac{\partial A_1}{\partial \tau_2} &= \frac{1}{n_1(n_1+1)}\Big[- A_1 A_3 \bar{A}_3 \Big(\sum_{\substack{n=|n_3-n_1| \\ n\neq n_2}}^{n_1+n_3} l_{nn_3} (B^{m_1-m_3}_{n_1n_3n})^2 b^{m_1-m_3}_{n_1n_3n} \\
&\quad + \sum_{n=|n_3-n_1|}^{n_1+n_3} l_{nn_3} (B^{m_1m_3}_{n_1n_3n})^2 b^{m_1m_3}_{n_1n_3n} \Big) \\
&\quad - A_1 A_2 \bar{A}_2 \Big(\sum_{n=|n_2-n_1|}^{n_1+n_2} l_{nn_2} (B^{m_1-m_2}_{n_1n_2n})^2 b^{m_1-m_2}_{n_1n_2n} + \sum_{\substack{n=|n_2-n_1| \\ n\neq n_3}}^{n_1+n_2} l_{nn_2} (B^{m_1m_2}_{n_1n_2n})^2 b^{m_1m_2}_{n_1n_2n} \Big) \\
&\quad - \sum_{k=4}^{N} 2\bar{A}_2 A_3 A_k \Big(\sum_{\substack{n=|n_1-n_3| \\ n\neq n_2}}^{n_1+n_3} l_{nn_3} (-1)^{m_3} (-1)^{m_2} B^{m_1-m_3}_{n_1n_3n} B^{m_20}_{n_2n_kn} b^{m_20}_{n_2n_kn} \\
&\quad + \sum_{\substack{n=|n_2-n_1| \\ n\neq n_3}}^{n_1+n_2} l_{nn_2} B^{m_1m_2}_{n_1n_2n} B^{m_30}_{n_3n_kn} b^{m_30}_{n_3n_kn} \\
&\quad + \sum_{\substack{n=|n_2-n_3| \\ n\neq n_1}}^{n_2+n_3} l_{nn_k} (-1)^{m_1} (-1)^{m_3} B^{m_10}_{n_1n_kn} B^{m_2-m_3}_{n_2n_3n} b^{m_2-m_3}_{n_2n_3n} \Big) \\
&\quad - \sum_{k=4}^{N} \sum_{l=4}^{N} 4 A_1 A_k A_l \sum_{\substack{n=|n_1-n_k| \\ n\neq n_l}}^{n_1+n_k} l_{nn_l} B^{m_10}_{n_1n_ln} B^{m_10}_{n_1n_kn} b^{m_10}_{n_1n_kn} \Big]
\end{aligned}
\tag{4.3.4}
$$

$$\begin{aligned}
\frac{\partial A_2}{\partial \tau_2} &= \frac{1}{n_2(n_2+1)}\Big[A_1\bar{A}_1A_2\Big(\sum_{\substack{n=|n_2-n_1|\\ n\neq n_3}}^{n_1+n_2} l_{nn_1}(B^{m_1m_2}_{n_1n_2n})^2 b^{m_1m_2}_{n_1n_2n}\\
&- \sum_{n=|n_2-n_1|}^{n_1+n_2} l_{nn_1}(B^{m_1-m_2}_{n_1n_2n})^2 b^{m_1-m_2}_{n_1n_2n}\Big)\\
&- A_2A_3\bar{A}_3\Big(\sum_{n=|n_2-n_3|}^{n_2+n_3} l_{nn_3}(B^{m_2m_3}_{n_2n_3n})^2 b^{m_2m_3}_{n_2n_3n} + \sum_{\substack{n=|n_2-n_3|\\ n\neq n_1}}^{n_2+n_3} l_{nn_3}(B^{m_2-m_3}_{n_2n_3n})^2 b^{m_2-m_3}_{n_2n_3n}\Big)\\
&- \sum_{k=4}^{N} 2\bar{A}_1A_3A_k\Big(\sum_{\substack{n=|n_2-n_3|\\ n\neq n_1}}^{n_2+n_3} l_{nn_3}(-1)^{m_3}(-1)^{m_1}B^{m_2-m_3}_{n_2n_3n}B^{m_10}_{n_1n_kn}b^{m_10}_{n_1n_kn}\\
&+ \sum_{\substack{n=|n_3-n_1|\\ n\neq n_2}}^{n_1+n_3} l_{nn_k}(-1)^{m_2}(-1)^{m_3}B^{m_1-m_3}_{n_1n_3n}b^{m_1-m_3}_{n_1n_3n}B^{m_20}_{n_2n_kn}\\
&- \sum_{\substack{n=|n_2-n_1|\\ n\neq n_3}}^{n_1+n_2} l_{nn_1}B^{m_1m_2}_{n_1n_2n}B^{m_30}_{n_3n_kn}b^{m_30}_{n_3n_kn}\Big)\\
&- \sum_{k=4}^{N}\sum_{l=4}^{N} 4A_2A_kA_l \sum_{\substack{n=|n_2-n_k|\\ n\neq n_2}}^{n_2+n_k} l_{nn_l}B^{m_20}_{n_2n_ln}B^{m_20}_{n_2n_kn}b^{m_20}_{n_2n_kn}\Big] \qquad (4.3.5)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial A_3}{\partial \tau_2} &= \frac{1}{n_3(n_3+1)}\Big[A_1\bar{A}_1A_3\Big(\sum_{n=|n_3-n_1|}^{n_1+n_3} l_{nn_1}(B^{m_1m_3}_{n_1n_3n})^2 b^{m_1m_3}_{n_1n_3n} \\
&- \sum_{\substack{n=|n_3-n_1|\\ n\neq n_2}}^{n_1+n_3} l_{nn_1}(B^{m_1-m_3}_{n_1n_3n})^2 b^{m_1-m_3}_{n_1n_3n}\Big) \\
&+ A_2\bar{A}_2A_3\Big(\sum_{n=|n_2-n_3|}^{n_2+n_3} l_{nn_2}(B^{m_2m_3}_{n_2n_3n})^2 b^{m_2m_3}_{n_2n_3n} - \sum_{\substack{n=|n_2-n_3|\\ n\neq n_1}}^{n_2+n_3} l_{nn_2}(B^{m_2-m_3}_{n_2n_3n})^2 b^{m_2-m_3}_{n_2n_3n}\Big) \\
&- \sum_{k=4}^{N} 2A_1A_2A_k\Big(\sum_{\substack{n=|n_1-n_2|\\ n\neq n_3}}^{n_1+n_2} l_{nn_k}B^{m_1m_2}_{n_1n_2n}b^{m_1m_2}_{n_1n_2n}B^{m_30}_{n_3n_kn} \\
&+ \sum_{\substack{n=|n_2-n_3|\\ n\neq n_1}}^{n_2+n_3} l_{nn_2}(-1)^{m_1}(-1)^{m_3}B^{m_2-m_3}_{n_2n_3n}B^{m_10}_{n_1n_kn}b^{m_10}_{n_1n_kn} \\
&+ \sum_{\substack{n=|n_1-n_3|\\ n\neq n_2}}^{n_1+n_3} l_{nn_1}(-1)^{m_2}(-1)^{m_3}B^{m_1-m_3}_{n_1n_3n}B^{m_20}_{n_2n_kn}b^{m_20}_{n_2n_kn}\Big) \\
&- \sum_{k=4}^{N}\sum_{l=4}^{N} 4A_3A_kA_l \sum_{\substack{n=|n_3-n_k|\\ n\neq n_3}}^{n_3+n_k} l_{nn_l}B^{m_30}_{n_3n_ln}B^{m_30}_{n_3n_kn}b^{m_30}_{n_3n_kn}\Big] \qquad (4.3.6)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial A_k}{\partial \tau_2} &= \left(A_1A_2\bar{A}_3 - \bar{A}_1\bar{A}_2A_3\right)\frac{1}{n_k(n_k+1)} \\
&\Big(-\sum_{\substack{n=|n_2-n_3|\\ n\neq n_1}}^{n_2+n_3} l_{nn_1}(-1)^{m_3}(-1)^{m_1}B^{m_10}_{n_1n_kn}B^{m_2-m_3}_{n_2n_3n}b^{m_2-m_3}_{n_2n_3n} \\
&- \sum_{\substack{n=|n_1-n_3|\\ n\neq n_2}}^{n_1+n_3} l_{nn_2}(-1)^{m_3}(-1)^{m_2}B^{m_20}_{n_2n_kn}B^{m_1-m_3}_{n_1n_3n}b^{m_1-m_3}_{n_1n_3n} \\
&+ \sum_{\substack{n=|n_1-n_2|\\ n\neq n_3}}^{n_1+n_2} l_{nn_3}B^{m_30}_{n_3n_kn}B^{m_1m_2}_{n_1n_2n}b^{m_1m_2}_{n_1n_2n}\Big) \qquad k=4,\ldots,N \qquad (4.3.7)
\end{aligned}$$

At this stage we point out that treating τ_1 and τ_2 as independent variables, as is done in other asymptotic schemes, simplifies the equations, of course, but is inconsistent here. Such an approach turns the equations (4.3.2) into partial differential equations. In the equations (4.3.4) to (4.3.7), the τ_2 derivative is now also a partial derivative, not just a shorthand notation. Differentiating the last equation in (4.3.2) with respect to τ_2 and (4.3.7) with respect to τ_1, however, leads to a different result and therefore to a contradiction.

Assuming that the zonal flow amplitudes are of $O(\delta)$, allows us to remove the corresponding terms from the right hand side in (4.3.2). These terms will reappear in equations (4.3.4) to (4.3.6), while the terms with the zonal flow amplitudes can be removed. Cross-differentiation shows that the two systems of partial differential equations for A_1, A_2 and A_3 are inconsistent. In our asymptotic scheme, the integration constants from the previous order are made time-dependent at the next order, and a set of ordinary differential equations are imposed on them. This does not lead to any inconsistency.

To examine the equations (4.3.4) to (4.3.7) further we will take the first example

$$\begin{aligned}\frac{\partial A_1}{\partial \tau_2} &= iA_1A_2\bar{A}_2\alpha + iA_1A_3\bar{A}_3\beta \\ &\quad + i\sum_{k=4}^{N} 2\bar{A}_2A_3A_k\gamma_k + i\sum_{k=4}^{N}\sum_{l=4}^{N} 4A_1A_kA_l\delta_{kl}\end{aligned}$$

where α, β, γ and δ are the constants given in equation (4.3.4). The imaginary part of these constants has been taken out explicitly.

Taking the derivative of the modulus we see that

$$\begin{aligned}
\frac{\partial |A_1|^2}{\partial \tau_2} &= A_1 \frac{\partial \bar{A}_1}{\partial \tau_2} + \bar{A}_1 \frac{\partial A_1}{\partial \tau_2} \\
&= A_1\big(-i\alpha \bar{A}_1 A_2 \bar{A}_2 - i\beta \bar{A}_1 A_3 \bar{A}_3 - i \sum_{k=4}^{N} \gamma_k A_2 \bar{A}_3 A_k \\
&\qquad - i \sum_{k=4}^{N} \sum_{l=4}^{N} \delta_{kl} \bar{A}_1 A_k A_l \big) \\
&\quad + \bar{A}_1 \big(i\alpha A_1 A_2 \bar{A}_2 + i\beta A_1 A_3 \bar{A}_3 + i \sum_{k=4}^{N} \gamma_k \bar{A}_2 A_3 A_k + i \sum_{k=4}^{N} \sum_{l=4}^{N} \delta_{kl} A_1 A_k A_l \big) \\
&= -i \sum_{k=4}^{N} \gamma_k \left(A_1 A_2 \bar{A}_3 A_k - \bar{A}_1 \bar{A}_2 A_3 A_k \right)
\end{aligned}$$

From this equation we can see that the zonal flow terms can in fact affect the amplitude of the wave, and not just its phase. Similar results will be found for the other two amplitude equations, A_2 and A_3. The final thing to check is how the zonal flow amplitudes are affected by these equations. Examining the last equation, (4.3.7) we rewrite it as

$$\frac{\partial A_k}{\partial \tau_2} = i\kappa_k \left(A_1 A_2 \bar{A}_3 - \bar{A}_1 \bar{A}_2 A_3 \right) \qquad k = 4, \ldots, N \tag{4.3.8}$$

Taking the derivative of the modulus squared we obtain

$$\frac{\partial |A_k|^2}{\partial \tau_2} = 2i\kappa_k A_k \left(A_1 A_2 \bar{A}_3 - \bar{A}_1 \bar{A}_2 A_3 \right)$$

Since this term is, in general, nonzero we can see that the amplitudes of the zonal flow terms are indeed affected at the $O(\delta^2)$ equation. These amplitudes are affected in a meaningful way as the wave itself is altered, not just the phase of the wave.

In this chapter we have shown that zonal flow is created at $O(\delta^2)$ if we commence our calculations with a resonant triad solution. This zonal flow only affects the phase of the amplitudes at $O(\delta)$. At $O(\delta^2)$ though it affects both the phase and the amplitudes of the waves.

Chapter 5

Numerical Calculations

In this chapter we will now apply all the previous theory to some examples. By doing this we will illustrate the mechanism of this work. Firstly we will go through all the necessary conditions that a triad solution must satisfy. Applying all these conditions to a small set of numbers we determine the first thirteen triads. We shall examine some of these triad examples to further our understanding of the complicated method explained in the previous chapters.

Using these triad solutions we firstly examine the numbers involved for our ψ_1 solution. To do this we must determine the range of non zero solutions applicable for $B^{m_j m_k}_{n_j n_k n}$ and $b^{m_j m_k}_{n_j n_k n}$ for each wave under consideration. From this we can examine the $O(\delta^2)$ equation and determine how many zonal flow terms, if any, are created. Due to the aforementioned problem with resonance, all these terms must form part of the leading order solution. Using this leading order solution the required amplitude conditions are computed for these numbers.

5.1 Triads

To examine the described method numerically we firstly need to determine which waves can actually form a triad. From examining the complicated list of conditions which need to be satisfied it becomes clear that it is only a few particular waves which satisfy these requirements. The list of conditions which a triad would have to satisfy are

$$\sigma_1 + \sigma_2 = \sigma_3$$

$$m_1 + m_2 = m_3$$

$$m_i \leq n_i \quad \forall \quad i = 1, 2, 3, \ldots$$

$$|n_1 - n_2| < n_3 < n_1 + n_2$$

$$n_1 + n_2 + n_3 \quad \text{odd}$$

$$n_1 \neq n_2 \neq n_3$$

as we have seen in Chapter 3 and Appendix B. Clearly not all waves can participate in resonant interactions.

Applying all the required conditions necessary for the occurrence of a Rossby-Haurwitz wave triad, the first thirteen triads for the lowest wavenumbers are given in Table 5.1. In order to show the workings of this theory an example is explicitly studied to generate nu-

(m_1, n_1)	(m_2, n_2)	(m_3, n_3)
(1, 6)	(2,14)	(3, 9)
(1, 6)	(11, 20)	(12, 15)
(2, 6)	(3, 8)	(5, 7)
(2, 6)	(4, 14)	(6, 9)
(2, 7)	(11, 20)	(13, 14)
(2, 14)	(17, 20)	(19, 19)
(3, 6)	(6, 14)	(9, 9)
(3, 9)	(8, 20)	(11, 14)
(3, 14)	(1, 20)	(4, 15)
(4, 12)	(5, 14)	(9, 13)
(6, 14)	(2, 20)	(8, 15)
(6, i8)	(7, 20)	(13, 19)
(9, 14)	(3, 20)	(12, 15)

Table 5.1: The first 13 triads

merical results. We shall study the first triad, namely, $(1,6)$, $(2,14)$ and $(3,9)$. Therefore for the rest of this section we have

$$(m_1, n_1) = (1,6), \quad (m_2, n_2) = (2,14) \quad \text{and} \quad (m_3, n_3) = (3,9)$$

Examining the first part of this triad we can see that when $m = 1, n = 6$ the wave in question is $e^{i(\lambda + \frac{1}{21}t)}P_6^1$. The plot of the real part of this wave is given in Figure (5.1).

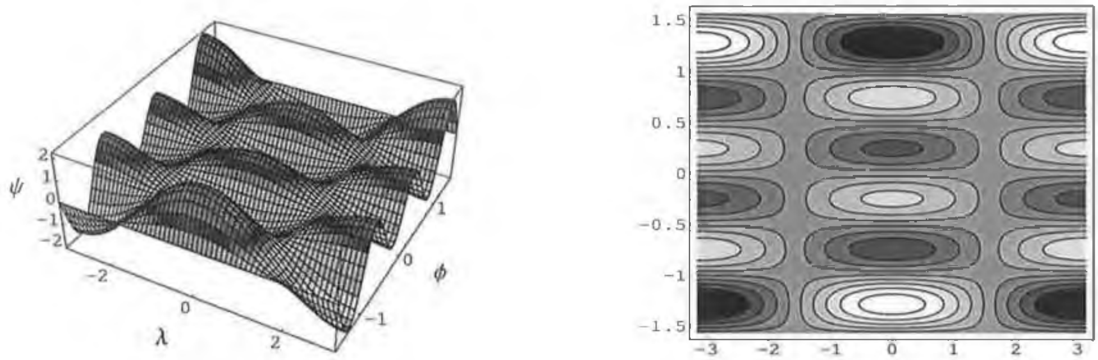

Figure 5.1: Rossby-Haurwitz wave $e^{i(\lambda+\frac{1}{21}t)}P_6^1(\sin\phi)$ and the corresponding contour plot

Similarly we see that the other two waves $m = 2, n = 14$ and $m = 3, n = 9$ produce the waves $e^{i(2\lambda+\frac{2}{105}t)}P_{14}^2$ and $e^{i(3\lambda+\frac{1}{15}t)}P_9^3$ respectively. The plot of the real parts of these waves are given in Figures (5.2) and (5.3)

If we consider all three of these waves together the triad which we are examining is

$$\psi = A_1e^{i(\lambda+\frac{1}{21}t)}P_6^1 + A_2e^{i(2\lambda+\frac{2}{105}t)}P_{14}^2$$
$$+ A_3e^{i(3\lambda+\frac{1}{15}t)}P_9^3 + \text{ c.c.} \tag{5.1.1}$$

The plot of the real part of this triad is shown in Figure (5.4)

From these contour plots it can be seen that the zonal wavenumber is given by m, and $n - m$ is the wavenumber in the north-south direction. Therefore we can see that n represents the total wavenumber.

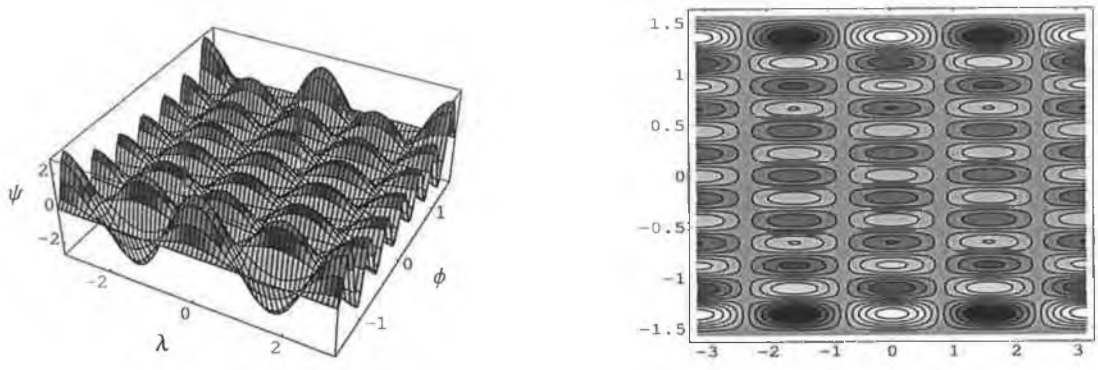

Figure 5.2: Rossby-Haurwitz wave $e^{i(2\lambda+\frac{2}{105}t)}P_{14}^{2}(\sin\phi)$ and the corresponding contour plot

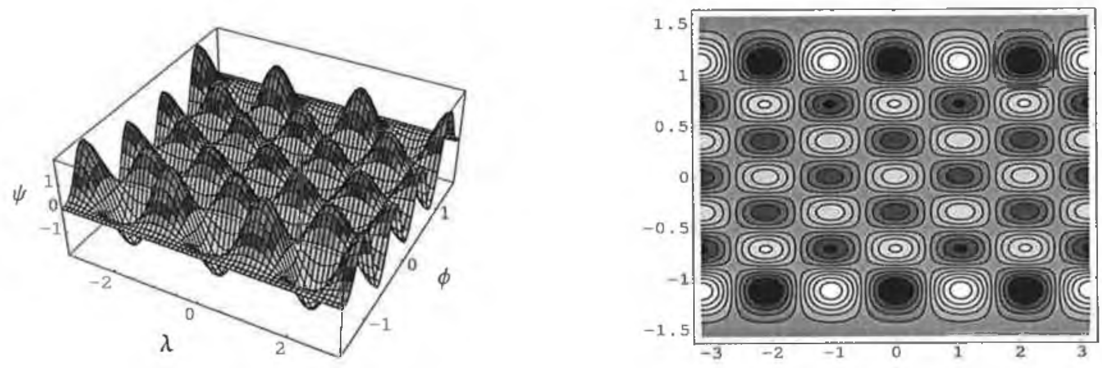

Figure 5.3: Rossby-Haurwitz wave $e^{i(3\lambda+\frac{1}{15}t)}P_{9}^{3}(\sin\phi)$ and the corresponding contour plot

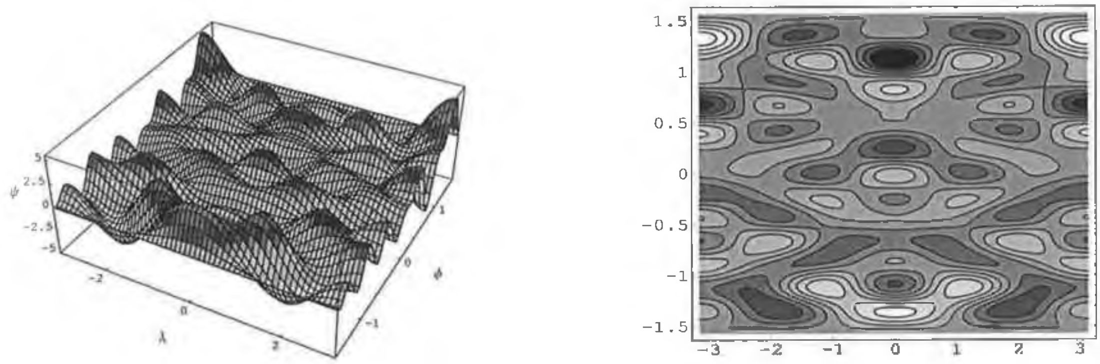

Figure 5.4: Rossby-Haurwitz wave triad solution, equation (5.1.1) and the corresponding contour plot

5.2 The generation of Zonal flow terms

5.2.1 $O(\delta)$ interactions

Using this triad solution (5.1.1) we firstly examine the $O(\delta)$ interactions. Examining this equation we will be able to derive the ψ_1 solution. The $O(\delta)$ equation is stated in Chapter 4, equation (4.1.3). We examine this $O(\delta)$ equation for two reasons. Firstly, we want to see if any zonal flow terms are generated. For zonal flow to be generated the m part of the exponential term must be zero. The m parts of this expression equates to

$$m_1 \pm m_2, \quad m_1 \pm m_3, \quad m_2 \pm m_3$$

From the numbers which we are using all the m_i are unique and hence none of these expressions can equate to zonal flow. Therefore zonal flow cannot be generated at $O(\delta)$ of this equation even when we start out our calculations using a triad solution.

Secondly we must examine this equation to determine if any resonant terms are present. If any terms on the right hand side of this equation solve the leading order equation then we have a problem. We find that there are indeed terms which cause resonance at this order. To counteract this problem we apply amplitude conditions on these offending terms. These conditions were derived in Chapter 4, equations (4.1.9), (4.1.10) and (4.1.11). Filling in the values for m_i and n_i which we are considering we obtain

$$\frac{dA_1}{d\tau_1} = 58.5582i\bar{A}_2 A_3$$
$$\frac{dA_2}{d\tau_1} = 4.68466i\bar{A}_1 A_3$$
$$\frac{dA_3}{d\tau_1} = 38.258iA_1 A_2$$

Implementing these conditions the resonance problem can now be solved. We have shown that we cannot generate zonal flow at this order and as a result we next examine the $O(\delta^2)$ equation.

5.2.2 Deriving the ψ_1 solution

Taking the amplitude conditions into account in our $O(\delta)$ equation we try to determine the ψ_1 solution. It has been shown that the solution for ψ_1 reduces to equation (4.2.1). Examining the terms in this expression it is clear that we need to determine values for $B^{m_j m_k}_{n_j n_k n}$ and $b^{m_j m_k}_{n_j n_k n}$. By running programs in mathematica we get the following tables of values. Table 5.2 has the values for $B^{m_1 m_2}_{n_1 n_2 n}$ and $b^{m_1 m_2}_{n_1 n_2 n}$, Table 5.3 contains values for $B^{m_1 - m_2}_{n_1 n_2 n}$ and $b^{m_1 - m_2}_{n_1 n_2 n}$, Table 5.4 represents $B^{m_1 m_3}_{n_1 n_3 n}$ and $b^{m_1 m_3}_{n_1 n_3 n}$, Table 5.5 contains values for $B^{m_1-m_3}_{n_1n_3n}$ and $b^{m_1-m_3}_{n_1n_3n}$, Table 5.6 represents $B^{m_2m_3}_{n_2n_3n}$ and $b^{m_2m_3}_{n_2n_3n}$ and finally Table 5.7 represents $B^{m_2-m_3}_{n_2n_3n}$ and $b^{m_2-m_3}_{n_2n_3n}$.

n	$B^{m_1m_2}_{n_1n_2n}$	$b^{m_1m_2}_{n_1n_2n}$
11	17.2301i	-60i
13	12.4548i	-27.3913i
15	8.29596i	-16.8i
17	4.96375i	-11.6667i
19	2.34338i	-8.68966i

Table 5.2: Coupling constants $B^{m_1m_2}_{n_1n_2n}$ and $b^{m_1m_2}_{n_1n_2n}$

n	$B^{m_1-m_2}_{n_1n_2n}$	$b^{m_1-m_2}_{n_1n_2n}$
9	3.49628i	-36.75i
11	8.84906i	-29.1089i
13	15.2673i	-23.3333i
15	21.6655i	-18.9677i
17	26.1733i	-15.6383i
19	24.7929i	-13.0667i

Table 5.3: Coupling constants $B^{m_1-m_2}_{n_1n_2n}$ and $b^{m_1-m_2}_{n_1n_2n}$

If we look at these tables of values we can see that $B^{m_1\pm m_2}_{n_1n_2n}$ is only nonzero for n between 9 and 19, with n odd. Similarly we can see that $B^{m_1\pm m_3}_{n_1n_3n}$ and $B^{m_2\pm m_3}_{n_2n_3n}$ are only

n	$B^{m_1 m_3}_{n_1 n_3 n}$	$b^{m_1 m_3}_{n_1 n_3 n}$
4	$5.85239i$	$8.4i$
6	$-8.42563i$	$15i$
8	$-11.9572i$	$-210i$
10	$-13.2368i$	$-10.5i$
12	$-12.9962i$	$-4.88372i$
14	$-10.4018i$	$-3i$

Table 5.4: Coupling constants $B^{m_1 m_3}_{n_1 n_3 n}$ and $b^{m_1 m_3}_{n_1 n_3 n}$

n	$B^{m_1 - m_3}_{n_1 n_3 n}$	$b^{m_1 - m_3}_{n_1 n_3 n}$
4	$-8.84799i$	$-13.2632i$
6	$-9.22981i$	$-15i$
8	$-6.00544i$	$-18.2609i$
10	$1.35831i$	$-25.2i$
12	$11.9378i$	$-46.6667i$

Table 5.5: Coupling constants $B^{m_1 - m_3}_{n_1 n_3 n}$ and $b^{m_1 - m_3}_{n_1 n_3 n}$

nonzero for n even with n between 4 and 14 and n between 6 and 22 respectively. It is clear that $b^{m_j m_k}_{n_j n_k n}$ is not as restricted as the values for $B^{m_j m_k}_{n_j n_k n}$ but we restrict our tables to these same ranges of values because these ranges for n are the intervals about which we will apply the summations.

$B^{m_1 m_2}_{n_1 n_2 n}$ is the coupling coefficient which causes resonance for $n = 9$. Examining the

n	$B^{m_2m_3}_{n_2n_3n}$	$b^{m_2m_3}_{n_2n_3n}$
6	-5.88655i	-18.75i
8	26.6356i	-31.3433i
10	12.4948i	-210i
12	-4.82547i	35.5932i
14	-18.6664i	15i
16	-28.0914i	9.01288i
18	-32.9823i	6.21302i
20	-32.8872i	4.61538i
22	-25.9314i	3.59589i

Table 5.6: Coupling constants $B^{m_2m_3}_{n_2n_3n}$ and $b^{m_2m_3}_{n_2n_3n}$

tables of numbers for this coefficient we can see that for $n = 11$ the corresponding $b^{m_1m_2}_{n_1n_2n}$ result is the largest by some margin. This indicates that this term is near resonance, i.e. for this term, the resonance condition is approximately equal to zero. Similarly, $B^{m_1-m_3}_{n_1n_3n}$ is resonant for $n = 14$. Once again for this term we can see that the $b^{m_1-m_3}_{n_1n_3n}$ term for $n = 12$ the nearest integer under investigation is also the largest, also suggesting near resonance. Lastly $B^{m_2-m_3}_{n_2n_3n}$ is resonant for $n = 6$ and, once again, we see that for the nearest integer to this, $n = 8$, the corresponding term in $b^{m_2-m_3}_{n_2n_3n}$ is the largest by some margin.

It is through using the numbers above we have determined the ψ_1 solution. Using this solution we can now consider the $O(\delta^2)$ equation and investigate if nonzero zonal flow

n	$B^{m_2-m_3}_{n_2n_3n}$	$b^{m_2-m_3}_{n_2n_3n}$
8	14.5228i	-84i
10	0.162338i	-37.0588i
12	-15.9295i	-22.1053i
14	-27.7451i	-15i
16	-29.5888i	-10.9565i
18	-17.2662i	-8.4i
20	9.12981i	-6.66667i
22	37.8582i	-5.43103i

Table 5.7: Coupling constants $B^{m_2-m_3}_{n_2n_3n}$ and $b^{m_2-m_3}_{n_2n_3n}$

is produced.

5.2.3 $O(\delta^2)$ interactions

With the coupling constants for ψ_1 for the triad solution determined we must examine the $O(\delta^2)$ equation to see whether zonal flow terms can be created at this order. We now substitute the expression found for ψ_1 into the $O(\delta^2)$ equation and determine which terms, if any, cause resonance. If any of these terms that cause resonance are zonal flow terms then we have created zonal flow at this order. The $O(\delta^2)$ equation is given in Chapter 4, equation (4.2.4).

Looking at this we can see that when $m_1 = 1$, $m_2 = 2$ and $m_3 = 3$ we do indeed get some zero λ terms in the exponential term. Therefore we have the possibility of zonal flow being generated. We must ensure that the coefficients in front of these potential zonal flow terms are not zero. Once that is ensured then we have generated zonal flow.

If $l = 3$, $j = 1$ and $k = 2$ then the second exponential term has a zero exponential part. This term is

$$\sum_{\substack{n=|n_1-n_2| \\ n\neq n_3}}^{n_1+n_2} (n_3(n_3+1) - n(n+1))A_3\bar{A}_1\bar{A}_2 B^{m_1m_2}_{n_1n_2n} b^{m_1m_2}_{n_1n_2n}$$
$$\times \left(i(m_1+m_2)P_n^{m_1+m_2}\frac{dP^{m_3}_{n_3}}{d\mu} + im_3P^{m_3}_{n_3}\frac{dP^{m_1+m_2}_n}{d\mu}\right)$$

Applying the spectral method to this expression we see that this equates to

$$\sum_{\substack{\tilde{n}=|n_3-n| \\ \tilde{n}\neq n_3}}^{n_3+n} \bar{A}_1\bar{A}_2A_3P^0_{\tilde{n}}(n(n+1) - n_3(n_3+1))(-1)^{m_3}B^{m_1m_2}_{n_1n_2n}b^{m_1m_2}_{n_1n_2n}B^{m_3-m_3}_{n_3n\tilde{n}}$$

where the bounds for n are $|n_1 - n_2| < n < n_1 + n_2$ from the summation in equation (4.2.4). Examining these bounds and the interaction coefficients connected to this term we can see that there is zonal flow generated for $\tilde{n} = 3 \to 27$ where $\tilde{n}$ is odd.

The next possibility for which zonal flow can be generated is if $l = 1$, $j = 2$ and $k = 3$. For this example the third exponential term from equation (4.2.4) is zero. Therefore this term potentially generates zonal flow. Taking out that term and applying the spectral method we get

$$\sum_{\substack{\tilde{n}=|n_1-n| \\ \tilde{n}\neq n_1}}^{n_1+n} A_1A_2\bar{A}_3P^0_{\tilde{n}}(n_1(n_1+1) - n(n+1))(-1)^{m_3}B^{m_2-m_3}_{n_2n_3n}b^{m_2-m_3}_{n_2n_3n}B^{m_1-m_1}_{n_1n\tilde{n}}$$

where the bounds for n are $|n_2 - n_3| < n < n_2 + n_3$ from equation (4.2.4). Once again we see that this means that nonzero zonal flow is generated for $\tilde{n} = 3 \to 27$ where $\tilde{n}$ is odd.

The last possibility to be considered is when $l = 2$, $j = 1$ and $k = 3$. For this example the third exponential term from equation (4.2.4) is again zero and after applying the spectral method to this term it becomes

$$\sum_{\substack{\bar{n}=|n_2-n| \\ \bar{n}\neq n_2}}^{n_2+n} A_1 A_2 \bar{A}_3 P_{\bar{n}}^0 (n_2(n_2+1) - n(n+1))(-1)^{m_3} B_{n_1 n_3 n}^{m_1-m_3} b_{n_1 n_3 n}^{m_1-m_3} B_{n_2 n \bar{n}}^{m_2-m_2}$$

where the bounds for n are $|n_1 - n_3| < n < n_1 + n_3$ from equation (4.2.4). Once again, nonzero zonal flow is generated for $\bar{n} = 3 \to 27$ where $\bar{n}$ is odd.

We will also get equivalent results for the complex conjugates of these three equations. Therefore we see that we have produced many zonal flow terms at the $O(\delta^2)$ calculations. It can readily be shown that these terms are in fact resonant Rossby-Haurwitz waves and hence they should be included in the ψ_0 solution from the outset. This will ensure that the solution will remain valid for all time under consideration.

5.3 Leading order solution with zonal flow

As a result of examining a triad solution of the vorticity equation for nondivergent planetary flow up to $O(\delta^2)$ the leading order solution which we should consider is

$$\psi_0 = \sum_{j=1}^{16} \psi_0^j$$

where

$$\psi_0^j = (A_j(\tau_1)e^{i(m_j\lambda - \sigma_j t)} + \bar{A}_j(\tau_1)e^{-i(m_j\lambda-\sigma_j t)})P_{n_j}^{m_j}(\mu),$$

$$\sigma_j = \frac{-2m_j}{n_j(n_j+1)},$$

$$m_j = 0, \quad A_j = \bar{A}_j, \quad j = 4, 5, \cdots, 16$$

and

$$n_4 = 3, \quad n_5 = 5, \quad n_6 = 7, \quad \ldots \quad n_{16} = 27$$

Using this as our leading order solution the $O(\delta)$ equation is now slightly different. In the equation given by (4.1.5) the sum over k now must go up to 16 and similarly the second sum which incorporates the τ_1 derivatives must have an upper limit of 16. As a result of this we need to produce tables for further $B^{m_j m_k}_{n_j n_k n}$ expressions. These new expressions are required for examining the ψ_1 solution. These numbers are provided in Tables 5.8, 5.10 and 5.12 for $B^{m_1 0}_{n_1 n_k n}$, $B^{m_2 0}_{n_2 n_k n}$ and $B^{m_3 0}_{n_3 n_k n}$ respectively. The numbers produced for $b^{m_1 0}_{n_1 n_k n}$, $b^{m_2 0}_{n_2 n_k n}$ and $b^{m_3 0}_{n_3 n_k n}$ are given in Tables 5.9, 5.11 and 5.13 respectively.

With these changes to the $O(\delta)$ equation the τ_1 amplitude equations will also be different. This is due to the fact that more terms will be generated which will cause resonance. The analytic solution for these new amplitude equations is given in Chapter 4,

k	n							
	8	10	12	14	16	18	20	22
4	$3.46018i$	0	0	0	0	0	0	0
5	$7.26387i$	$5.64521i$	0	0	0	0	0	0
6	$11.1079i$	$10.3182i$	$7.8108i$	0	0	0	0	0
7	$14.3495i$	$14.5375i$	$13.3193i$	$9.96593i$	0	0	0	0
8	$16.009i$	$17.7719i$	$17.897i$	$16.292i$	$12.1147i$	0	0	0
9	$14.1332i$	$19.0478i$	$21.1284i$	$21.2179i$	$19.2472i$	$14.2592i$	0	0
10	0	$16.3086i$	$22.0426i$	$24.4479i$	$24.5152i$	$22.1911i$	$16.4008i$	0
11	0	0	$18.4696i$	$25.012i$	$27.7443i$	$27.7968i$	$25.1271i$	$18.5403i$
12	0	0	0	$20.6221i$	$27.9654i$	$31.0254i$	$31.0675i$	$28.0573i$
13	0	0	0	0	$22.7692i$	$30.908i$	$34.2958i$	$34.3303i$
14	0	0	0	0	0	$24.9127i$	$33.8432i$	$37.5584i$
15	0	0	0	0	0	0	$27.0535i$	$36.7728i$
16	0	0	0	0	0	0	0	$29.1924i$

Table 5.8: Coupling constant $B^{m_1 0}_{n_1 n_k n}$

k	n							
	8	10	12	14	16	18	20	22
4	21i	9.26471i	5.52632i	3.75i	2.73913i	2.1i	1.66667i	1.35776i
5	8.4i	3.70588i	2.21053i	1.5i	1.09565i	0.84i	0.666667i	0.543103i
6	-9.8i	-4.32353i	-2.57895i	-1.75i	-1.27826i	-0.98i	-0.777778i	-0.633621i
7	-33.6i	-14.8235i	-8.84211i	-6i	-4.38261i	-3.36i	-2.66667i	-2.17241i
8	-63i	-27.7941i	-16.5789i	-11.25i	-8.21739i	-6.3i	-5i	-4.07328i
9	-98i	-43.2353i	-25.7895i	-17.5i	-12.7826i	-9.8i	-7.77778i	-6.33621i
10	-138.6i	-61.1471i	-36.4737i	-24.75i	-18.0783i	-13.86i	-11i	-8.96121i
11	-184.8i	-81.5294i	-48.6316i	-33i	-24.1043i	-18.48i	-14.6667i	-11.9483i
12	-236.6i	-104.382i	-62.2632i	-42.25i	-30.8609i	-23.66i	-18.7778i	-15.2974i
13	-294i	-129.706i	-77.3684i	-52.5i	-38.3478i	-29.4i	-23.3333i	-19.0086i
14	-357i	-157.5i	-93.9474i	-63.75i	-46.5652i	-35.7i	-28.3333i	-23.0819i
15	-425.6i	-187.765i	-112i	-76i	-55.513i	-42.56i	-33.7778i	-27.5172i
16	-499.8i	-220.5i	-131.526i	-89.25i	-65.1913i	-49.98i	-39.6667i	-32.3147i

Table 5.9: Coupling constant $b^{m_1 0}_{n_1 n_k n}$

k	n				
	4	6	8	10	12
4	0	0	0	0	$6.87045i$
5	0	0	0	$10.981i$	$14.1207i$
6	0	0	$14.6386i$	$18.8231i$	$20.9821i$
7	0	$17.4814i$	$21.6379i$	$24.2082i$	$26.2708i$
8	$18.5777i$	$21.0763i$	$23.4235i$	$26.0741i$	$28.8314i$
9	$13.3824i$	$15.8179i$	$19.351i$	$23.4075i$	$27.6474i$
10	$-4.01468i$	$2.68028i$	$9.12658i$	$15.5735i$	$21.9684i$
11	$-23.0355i$	$-15.3559i$	$-6.37266i$	$2.63688i$	$11.4867i$
12	0	$-29.391i$	$-23.9272i$	$-14.1288i$	$-3.39074i$
13	0	0	$-34.7066i$	$-31.1868i$	$-21.0662i$
14	0	0	0	$-39.5423i$	$-37.7237i$
15	0	0	0	0	$-44.116i$
16	0	0	0	0	0

Table 5.10: Coupling constant $B^{m_2 0}_{n_2 n_k n}$

	n				
k	4	6	8	10	12
4	-54.7105i	-61.875i	-75.3261i	-103.95i	-192.5i
5	-49.7368i	-56.25i	-68.4783i	-94.5i	-175i
6	-42.5526i	-48.125i	-58.587i	-80.85i	-149.722i
7	-33.1579i	-37.5i	-45.6522i	-63i	-116.667i
8	-21.5526i	-24.375i	-29.6739i	-40.95i	-75.8333i
9	-7.73684i	-8.75i	-10.6522i	-14.7i	-27.2222i
10	8.28947i	9.375i	11.413i	15.75i	29.1667i
11	26.5263i	30i	36.5217i	50.4i	93.3333i
12	46.9737i	53.125i	64.6739i	89.25i	165.278i
13	69.6316i	78.75i	95.8696i	132.3i	245i
14	94.5i	106.875i	130.109i	179.55i	332.5i
15	121.579i	137.5i	167.391i	231i	427.778i
16	150.868i	170.625i	207.717i	286.65i	530.833i

Table 5.11: Coupling constant $b^{m_2 0}_{n_2 n_k n}$

k	n				
	11	13	15	17	19
4	9.68024i	0	0	0	0
5	15.9894i	15.0291i	0	0	0
6	15.6375i	20.0493i	20.0394i	0	0
7	6.97454i	15.8943i	23.4323i	24.8466i	0
8	-7.92121i	3.19092i	15.6298i	26.4245i	29.5218i
9	-22.422i	-14.0255i	-0.611462i	15.0951i	29.17i
10	-26.0408i	-27.9296i	-19.6557i	-4.36602i	14.4062i
11	-8.30671i	-28.1223i	-32.8054i	-24.9489i	-8.05991i
12	28.6664i	-5.85785i	-29.8404i	-37.2756i	-29.9979i
13	0	33.4367i	-3.47309i	-31.3437i	-41.4713i
14	0	[illegible]	38.0873i	-1.14539i	-32.7115i
15	0	0	0	42.6545i	1.13458i
16	0	0	0	0	47.1613i

Table 5.12: Coupling constant $B^{m_3 0}_{n_3 n_k n}$

	n				
k	11	13	15	17	19
4	$27.8571i$	$12.7174i$	$7.8i$	$5.41667i$	$4.03448i$
5	$21.4286i$	$9.78261i$	$6i$	$4.16667i$	$3.10345i$
6	$12.1429i$	$5.54348i$	$3.4i$	$2.36111i$	$1.75862i$
7	0	0	0	0	0
8	$-15i$	$-6.84783i$	$-4.2i$	$-2.91667i$	$-2.17241i$
9	$-32.8571i$	$-15i$	$-9.2i$	$-6.38889i$	$-4.75862i$
10	$-53.5714i$	$-24.4565i$	$-15i$	$-10.4167i$	$-7.75862i$
11	$-77.1429i$	$-35.2174i$	$-21.6i$	$-15i$	$-11.1724i$
12	$-103.571i$	$-47.2826i$	$-29i$	$-20.1389i$	$-15i$
13	$-132.857i$	$-60.6522i$	$-37.2i$	$-25.8333i$	$-19.2414i$
14	$-165i$	$-75.3261i$	$-46.2i$	$-32.0833i$	$-23.8966i$
15	$-200i$	$-91.3043i$	$-56i$	$-38.8889i$	$-28.9655i$
16	$-237.857i$	$-108.587i$	$-66.6i$	$-46.25i$	$-34.4483i$

Table 5.13: Coupling constant $b^{m_3 0}_{n_3 n_k n}$

equations (4.3.2). These changes cause our first set of amplitude equations to become

$$\frac{dA_1}{d\tau_1} = 58.5582i\bar{A}_2A_3 + iA_1\big(- 5.83115A_4 - 4.29985A_5 + 7.19403A_6$$
$$+29.4452A_7 + 51.1306A_8\big)$$
$$\frac{dA_2}{d\tau_1} = 4.68466i\bar{A}_1A_3 - iA_2\big(15.4011A_4 + 25.9221A_5 + 32.3569A_6$$
$$+31.8289A_7 + 23.3353A_8 + 8.47426A_9 - 8.04488A_{10}$$
$$-18.3342A_{11} - 12.0286A_{12} + 21.679A_{13} + 89.4804A_{14}$$
$$+183.628A_{15} + 252.359A_{16}\big)$$
$$\frac{dA_3}{d\tau_1} = 38.258iA_1A_2 - iA_3\big(18.3121A_4 + 18.9536A_5 + 7.91261A_6$$
$$+14.8045A_8 + 47.6088A_9 + 36.0399A_{10} - 113.833A_{11}\big)$$

Examining the $O(\delta^2)$ equation with our new ψ_0 and ψ_1 solutions we see that the τ_2

amplitude equations are

$$\frac{\partial A_1}{\partial \tau_2} = -19976.5iA_1A_2\bar{A}_2 + 1071.72iA_1A_3\bar{A}_3 + \bar{A}_2A_3i\big(20303A_4 + 51835A_5$$
$$+ 113671A_6 + 189242A_7 + 213132A_8 + 116086A_9 - 148447A_{10} - 196615A_{11}$$
$$+ 122340A_{12} + 181811A_{13} + 468326A_{14} + 739663A_{15} + 40635A_{16}\big)$$
$$+ A_1i\big(1182A_4^2 + 3752A_4A_5 + 3626A_5^2 + 2609A_4A_6 - 650A_5A_6$$
$$- 11885A_6^2 - 3861A_4A_7 - 22031A_5A_7 - 51289A_6A_7 - 39310A_7^2$$
$$- 26589A_4A_8 - 56911A_5A_8 - 54560A_6A_8 + 5492A_7A_8 + 60256A_8^2$$
$$- 38145A_4A_9 - 87431A_5A_9 - 84680A_6A_9 + 25669A_7A_9 + 252976A_8A_9$$
$$+ 249185A_9^2 - 47116A_5A_{10} - 100164A_6A_{10} - 95384A_7A_{10} + 16849A_8A_{10}$$
$$+ 237934A_9A_{10} + 235255A_{10}^2 - 61501A_6A_{11} - 124460A_7A_{11} - 115516A_8A_{11}$$
$$+ 15814A_9A_{11} + 263517A_{10}A_{11} + 258100A_{11}^2 - 79271A_7A_{12} - 155023A_8A_{12}$$
$$- 140744A_9A_{12} + 16808A_{10}A_{12} + 303524A_{11}A_{12} + 294062A_{12}^2 - 100023A_8A_{13}$$
$$- 190765A_9A_{13} - 170121A_{10}A_{13} + 18753A_{11}A_{13} + 352638A_{12}A_{13}$$
$$+ 338217A_{13}^2 - 123622A_9A_{14} - 231320A_{10}A_{14} - 203316A_{11}A_{14}$$
$$+ 21307A_{12}A_{14} + 409073A_{13}A_{14} + 388881A_{14}^2 - 150012A_{10}A_{15}$$
$$- 276531A_{11}A_{15} - 240181A_{12}A_{15} + 24329A_{13}A_{15} + 472062A_{14}A_{15}$$
$$+ 445320A_{15}^2 - 179167A_{11}A_{16} - 326320A_{12}A_{16} - 280641A_{13}A_{16}$$
$$+ 27751A_{14}A_{16} + 541222A_{15}A_{16} + 507164A_{16}^2\big)$$

Similar expressions were also determined for A_2 and A_3. In these expressions none of the coefficients were zero.

Lastly we examined the numbers involved for the A_k derivatives. These derivatives will tell us how the zonal flow terms react and interact. The conditions on the zonal flow amplitudes take the form

$$\frac{\partial A_k}{\partial \tau_2} = \Lambda_k(A_1 A_2 \bar{A}_3 - \bar{A}_1 \bar{A}_2 A_3), \qquad k = 4, 5, \dots, 16 \tag{5.3.1}$$

where Λ_k, $k = 4, 5, \dots, 16$ are given in Table 5.14.

From all these calculations we can clearly see which waves will interact and which waves cannot do so. It can also be seen that only certain zonal flow terms can be generated starting out with this triad example and that within these zonal flow terms some will interact more strongly.

5.4 Examining a second example

We will take another example to show that these results are not unique for the first example. Taking the third triad from the table of the list of triads, Table 5.1, we now examine $(2, 6)$, $(3, 8)$ and $(5, 7)$. Therefore for the rest of this chapter we shall assume that

$$(m_1, n_1) = (2, 6), \quad (m_2, n_2) = (3, 8) \quad \text{and} \quad (m_3, n_3) = (5, 7)$$

Examining the first part of this triad we can see that when $m = 2, n = 6$ the wave in question is $e^{i(2\lambda + \frac{2}{21}t)} P_6^2$. The plot of the real part of this wave is given in Figure (5.5). We also see that the other two waves $m = 3, n = 8$ and $m = 5, n = 7$ produce the waves $e^{i(3\lambda + \frac{1}{12}t)} P_8^3$ and $e^{i(5\lambda + \frac{5}{28}t)} P_7^5$ respectively. The plot of the real part of these waves are given in Figures (5.6) and (5.7)

k	Λ_k
4	$7249.72i$
5	$14514.5i$
6	$20640.9i$
7	$24040.2i$
8	$23165.6i$
9	$15626.6i$
10	$8386.19i$
11	$-618.13i$
12	$1009.79i$
13	$-1009.82i$
14	$-2570.45i$
15	$-3481.92i$
16	$-3315.51i$

Table 5.14: The τ_2 derivative values for the zonal flow amplitudes

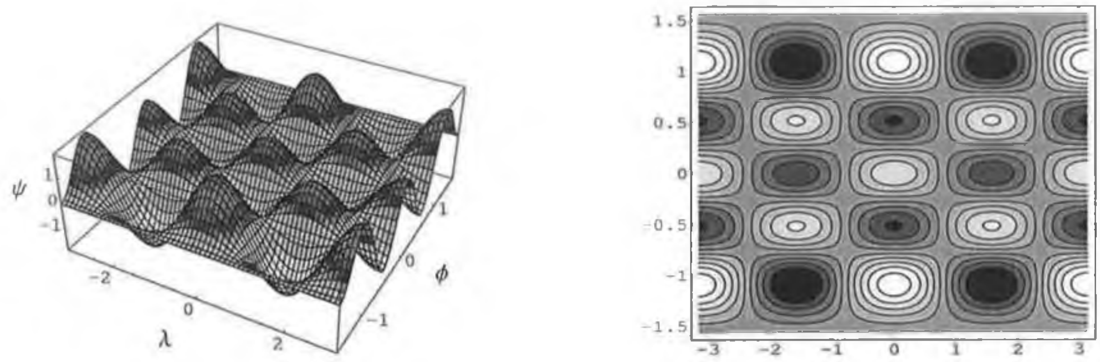

Figure 5.5: Rossby-Haurwitz wave $e^{i(2\lambda+\frac{2}{21}t)}P_6^2(\sin\phi)$ and the corresponding contour plot

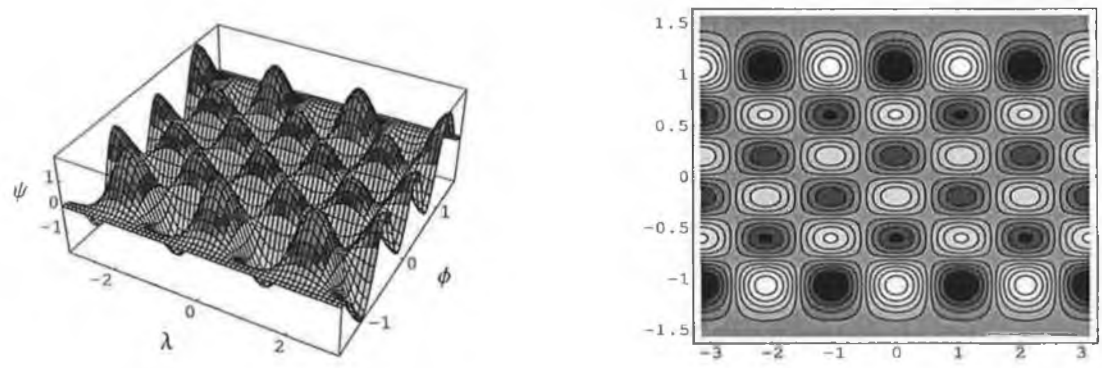

Figure 5.6: Rossby-Haurwitz wave $e^{i(3\lambda+\frac{1}{12}t)}P_8^3(\sin\phi)$ and the corresponding contour plot

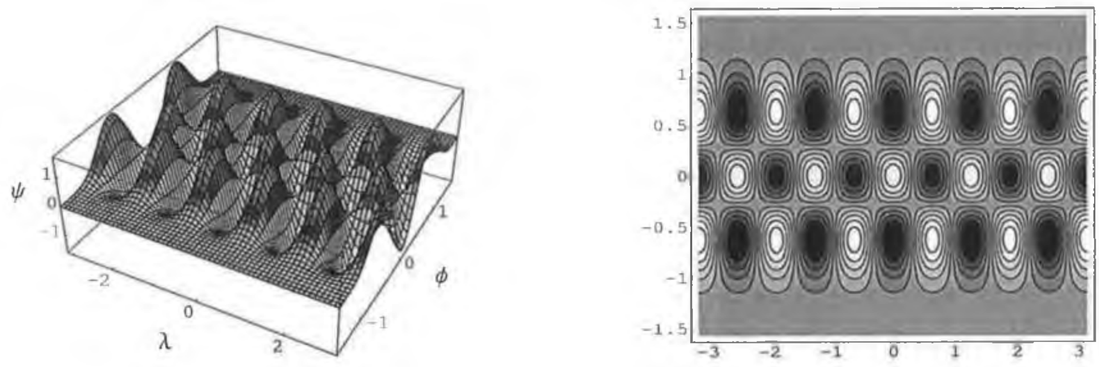

Figure 5.7: Rossby-Haurwitz wave $e^{i(5\lambda+\frac{5}{28}t)}P_7^5(\sin\phi)$ and the corresponding contour plot

Considering these three waves together the triad which we are examining is

$$\psi = A_1 e^{i(2\lambda+\frac{2}{21}t)}P_6^2 + A_2 e^{i(3\lambda+\frac{1}{12}t)}P_8^3$$
$$+ A_3 e^{i(5\lambda+\frac{5}{28}t)}P_7^5 + \text{ c.c.} \tag{5.4.1}$$

The plot of the real part of this triad is shown in Figure (5.8)

5.5 The generation of Zonal flow terms

5.5.1 $O(\delta)$ interactions

Once again we examine the $O(\delta)$ interactions using the triad solution (5.4.1). Since all the m_i terms are unique in our example none of the terms generated can produce zonal flow at this order. Again we find resonant terms present. To get rid of these terms we insist

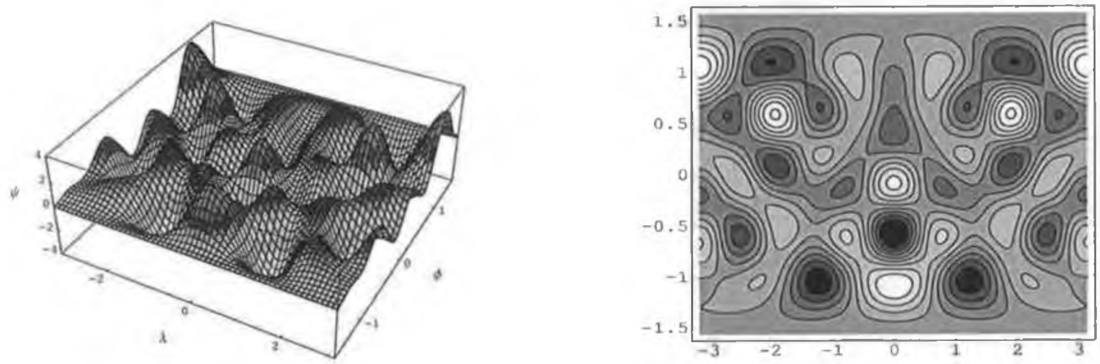

Figure 5.8: Rossby-Haurwitz wave triad solution, equation (5.4.1) and the corresponding contour plot

that

$$\frac{dA_1}{d\tau_1} = 0.846313i\bar{A}_2 A_3$$
$$\frac{dA_2}{d\tau_1} = 0.431972i\bar{A}_1 A_3$$
$$\frac{dA_3}{d\tau_1} = 1.19013iA_1 A_2$$

With the resonant problem now solved and since we cannot generate zonal flow at this order we next examine the $O(\delta^2)$ equation.

5.5.2 Deriving the ψ_1 solution

The ψ_1 solution is now determined, taking into account the amplitude conditions just derived. To do this we need to determine values for $B^{m_j m_k}_{n_j n_k n}$ and $b^{m_j m_k}_{n_j n_k n}$. By running programs in mathematica we get the following tables of values. Table 5.15 has the values for $B^{m_1 m_2}_{n_1 n_2 n}$ and $b^{m_1 m_2}_{n_1 n_2 n}$, Table 5.16 contains values for $B^{m_1 - m_2}_{n_1 n_2 n}$ and $b^{m_1 - m_2}_{n_1 n_2 n}$, Table 5.17

represents $B^{m_1 m_3}_{n_1 n_3 n}$ and $b^{m_1 m_3}_{n_1 n_3 n}$, Table 5.18 contains values for $B^{m_1 -m_3}_{n_1 n_3 n}$ and $b^{m_1 -m_3}_{n_1 n_3 n}$, Table 5.19 represents $B^{m_2 m_3}_{n_2 n_3 n}$ and $b^{m_2 m_3}_{n_2 n_3 n}$ and Table 5.20 represents $B^{m_2 -m_3}_{n_2 n_3 n}$ and $b^{m_2 -m_3}_{n_2 n_3 n}$.

n	$B^{m_1 m_2}_{n_1 n_2 n}$	$b^{m_1 m_2}_{n_1 n_2 n}$
5	11.8024i	6.46154i
9	-1.16121i	-4.94118i
11	-2.55439i	-2.21053i
13	-2.54484i	-1.33333i

Table 5.15: Coupling constants $B^{m_1 m_2}_{n_1 n_2 n}$ and $b^{m_1 m_2}_{n_1 n_2 n}$

n	$B^{m_1 -m_2}_{n_1 n_2 n}$	$b^{m_1 -m_2}_{n_1 n_2 n}$
5	-6.10832i	-12.7273i
7	-12.762i	-11.25i
9	-13.7224i	-9.76744i
11	-3.11169i	-8.4i
13	17.4001i	-7.2i

Table 5.16: Coupling constants $B^{m_1 -m_2}_{n_1 n_2 n}$ and $b^{m_1 -m_2}_{n_1 n_2 n}$

It is from these numbers that we can now determine the ψ_1 solution. Using this solution we can now consider the $O(\delta^2)$ equation and investigate if nonzero zonal flow is produced.

n	$B^{m_1 m_3}_{n_1 n_3 n}$	$b^{m_1 m_3}_{n_1 n_3 n}$
8	-8.73962i	-2.45i
10	-15.048i	-0.868538i
12	-17.6126i	-0.487562i

Table 5.17: Coupling constants $B^{m_1 m_3}_{n_1 n_3 n}$ and $b^{m_1 m_3}_{n_1 n_3 n}$

n	$B^{m_1 - m_3}_{n_1 n_3 n}$	$b^{m_1 - m_3}_{n_1 n_3 n}$
4	9.52663i	-3.23077i
6	11.5972i	-5.6i
10	-17.6525i	4.42105i
12	8.73274i	2i

Table 5.18: Coupling constants $B^{m_1 - m_3}_{n_1 n_3 n}$ and $b^{m_1 - m_3}_{n_1 n_3 n}$

n	$B^{m_2 m_3}_{n_2 n_3 n}$	$b^{m_2 m_3}_{n_2 n_3 n}$
8	17.1202i	5.6i
10	-6.11945i	1.24907i
12	-18.3689i	0.643678i
14	-21.4695i	0.410256i

Table 5.19: Coupling constants $B^{m_2 m_3}_{n_2 n_3 n}$ and $b^{m_2 m_3}_{n_2 n_3 n}$

5.5.3 $O(\delta^2)$ interactions

Next we must investigate whether zonal flow can be generated at this order using these numbers. We now substitute the expression found for ψ_1 into the $O(\delta^2)$ equation and

n	$B^{m_2-m_3}_{n_2n_3n}$	$b^{m_2-m_3}_{n_2n_3n}$
2	$3.25396i$	$4.66667i$
4	$-13.28i$	$7.63636i$
8	$16.0704i$	$-5.6i$
10	$8.96317i$	$-2.47059i$
12	$-21.6333i$	$-1.47368i$
14	$7.67055i$	$-1i$

Table 5.20: Coupling constants $B^{m_2-m_3}_{n_2n_3n}$ and $b^{m_2-m_3}_{n_2n_3n}$

determine which terms, if any, cause resonance. If any of the terms that cause resonance are zonal flow terms then we have created zonal flow at this order. We will see that for this example we can produce resonant zonal flow for $\tilde{n} = 3 \to 19$. Therefore we must now reexamine our previous leading order solution and include these zonal flow terms in it. With these changes made to the leading order solution the amplitude equations will also be changed.

5.6 Leading order solution with zonal flow

As a result of the calculations with this second example the leading order solution which we will consider is

$$\psi_0 = \sum_{j=1}^{12} \psi_0^j$$

where

$$\psi_0^j = (A_j(\tau_1)e^{i(m_j\lambda-\sigma_j t)} + \bar{A}_j(\tau_1)e^{-i(m_j\lambda-\sigma_j t)})P_{n_j}^{m_j}(\mu),$$

$$\sigma_j = \frac{-2m_j}{n_j(n_j+1)},$$

$$m_j = 0, \quad A_j = \bar{A}_j, \quad j = 4, 5, \cdots, 12$$

and

$$n_4 = 3, \quad n_5 = 5, \quad n_6 = 7, \quad \ldots, \quad n_{12} = 19$$

Using this as our leading order solution the $O(\delta)$ equation is now slightly different and we need to produce tables for further $B_{n_j n_k n}^{m_j m_k}$ expressions. These new expressions are required to examine the ψ_1 solution. These numbers are provided in Tables 5.21, 5.23 and 5.25 for $B_{n_1 n_k n}^{m_1 0}$, $B_{n_2 n_k n}^{m_2 0}$ and $B_{n_3 n_k n}^{m_3 0}$ respectively. The numbers produced for $b_{n_1 n_k n}^{m_1 0}$, $b_{n_2 n_k n}^{m_2 0}$ and $b_{n_3 n_k n}^{m_3 0}$ are given in Tables 5.22, 5.24 and 5.26 respectively.

Once again we find that, when applying these changes to the $O(\delta)$ equation the τ_1 amplitude equations will be altered due to the generation of more resonating terms. Due to these changes the first set of amplitude equations are now

$$\frac{dA_1}{d\tau_1} = 0.846313i\bar{A}_2A_3 - iA_1(10.2045A_4 + 5.5898A_5 - 4.67612A_6$$
$$+5.88905A_7 + 63.9133A_8)$$
$$\frac{dA_2}{d\tau_1} = 0.431972i\bar{A}_1A_3 - iA_2(16.739A_4 + 13.7912A_5 + 2.24964A_6$$
$$+4.22128A_7 + 32.2665A_8 + 39.1953A_9 - 87.6338A_{10})$$
$$\frac{dA_3}{d\tau_1} = 1.19013iA_1A_2 - iA_3(8.38064A_4 - 10.3854A_5 - 23.5688A_7$$
$$+30.1748A_8 - 9.65395A_9)$$

	n					
k	2	4	8	10	12	14
4	0	$6.11398i$	$6.53913i$	0	0	0
5	$6.63747i$	$7.66428i$	$11.2565i$	$10.3067i$	0	0
6	$-5.68399i$	$1.42146i$	$11.7555i$	$14.7567i$	$13.9326i$	0
7	0	$-10.8057i$	$5.96598i$	$13.2708i$	$17.9886i$	$17.4814i$
8	0	0	$-6.05081i$	$4.86703i$	$14.5938i$	$21.0763i$
9	0	0	$-18.6964i$	$-9.27371i$	$3.76881i$	$15.8179i$
10	0	0	0	$-22.3314i$	$-12.3573i$	$2.68028i$
11	0	0	0	0	$-25.8857i$	$-15.3559i$
12	0	0	0	0	0	$-29.391i$

Table 5.21: Coupling constant $B^{m_1 0}_{n_1 n_k n}$

k	n					
	2	4	8	10	12	14
4	-8.75i	-14.3182i	10.5i	4.63235i	2.76316i	1.875i
5	-3.5i	-5.72727i	4.2i	1.85294i	1.10526i	0.75i
6	4.08333i	6.68182i	-4.9i	-2.16176i	-1.28947i	-0.875i
7	14i	22.9091i	-16.8i	-7.41176i	-4.42105i	-3i
8	26.25i	42.9545i	-31.5i	-13.8971i	-8.28947i	-5.625i
9	40.8333i	66.8182i	-49i	-21.6176i	-12.8947i	-8.75i
10	57.75i	94.5i	-69.3i	-30.5735i	-18.2368i	-12.375i
11	77i	126i	-92.4i	-40.7647i	-24.3158i	-16.5i
12	98.5833i	161.318i	-118.3i	-52.1912i	-31.1316i	-21.125i

Table 5.22: Coupling constant $b^{m_1 0}_{n_1 n_k n}$

	n			
k	4	6	10	12
4	0	$8.85402i$	$9.49108i$	0
5	$10.179i$	$9.66525i$	$14.4842i$	$14.5744i$
6	$-2.75265i$	0	$11.5014i$	$17.5655i$
7	$-14.3836i$	$-13.6191i$	0	$10.3152i$
8	$8.25674i$	$-14.1513i$	$-15.1383i$	$-4.83894i$
9	0	$13.8082i$	$-23.274i$	$-20.8429i$
10	0	0	$-11.2906i$	$-26.3563i$
11	0	0	$23.4454i$	$-9.72156i$
12	0	0	0	$27.9322i$

Table 5.23: Coupling constant $B^{m_2 0}_{n_2 n_k n}$

k	n			
	4	6	10	12
4	-13.8462i	-24i	18.9474i	8.57143i
5	-9.69231i	-16.8i	13.2632i	6i
6	-3.69231i	-6.4i	5.05263i	2.28571i
7	4.15385i	7.2i	-5.68421i	-2.57143i
8	13.8462i	24i	-18.9474i	-8.57143i
9	25.3846i	44i	-34.7368i	-15.7143i
10	38.7692i	67.2i	-53.0526i	-24i
11	54i	93.6i	-73.8947i	-33.4286i
12	71.0769i	123.2i	-97.2632i	-44i

Table 5.24: Coupling constant $b^{m_2 0}_{n_2 n_k n}$

	n			
k	5	9	11	13
4	6.82625i	11.4241i	0	0
5	-14.0941i	-2.2032i	14.5323i	0
6	11.6941i	-17.1146i	-8.96107i	16.9968i
7	-4.59601i	0	-19.6084i	-15.0047i
8	0.714535i	23.1088i	6.94335i	-20.7633i
9	0	-17.0207i	25.3212i	13.1776i
10	0	3.76867i	-22.5168i	26.8368i
11	0	0	5.48876i	-27.736i
12	0	0	0	7.26142i

Table 5.25: Coupling constant $B^{m_3 0}_{n_3 n_k n}$

k	n			
	5	9	11	13
4	-9.47692i	7.24706i	3.24211i	1.95556i
5	-5.6i	4.28235i	1.91579i	1.15556i
6	0	0	0	0
7	7.32308i	-5.6i	-2.50526i	-1.51111i
8	16.3692i	-12.5176i	-5.6i	-3.37778i
9	27.1385i	-20.7529i	-9.28421i	-5.6i
10	39.6308i	-30.3059i	-13.5579i	-8.17778i
11	53.8462i	-41.1765i	-18.4211i	-11.1111i
12	69.7846i	-53.3647i	-23.8737i	-14.4i

Table 5.26: Coupling constant $b^{m_3 0}_{n_3 n_k n}$

Using these new ψ_0 and ψ_1 solutions we then find the τ_2 amplitude equations from the $O(\delta^2)$ equation. We determine expressions for these derivatives. They are once again similar in structure to those derived in the first example and we find once again that the coefficients are nonzero.

Finally using all these previous calculations we can determine the numbers for the amplitude conditions for the zonal flow terms. The conditions on the zonal flow amplitudes will take the form

$$\frac{\partial A_k}{\partial \tau_2} = \Lambda_k(A_1 A_2 \bar{A}_3 - \bar{A}_1 \bar{A}_2 A_3), \qquad k = 4, 5, \ldots, 12 \tag{5.6.1}$$

where Λ_k, $k = 4, 5, \ldots, 12$ are given in Table 5.27.

k	Λ_k
4	$368.67i$
5	$407.9i$
6	$132.21i$
7	$-126.19i$
8	$-65.47i$
9	$-30.7i$
10	$29.2i$
11	0
12	0

Table 5.27: The τ_2 derivative values for the zonal flow amplitudes

Chapter 6

Conclusions

In this chapter we will discuss the main findings of this thesis. In the first chapter we derived the vorticity equation for nondivergent planetary flow, the solutions of which are shown to be Rossby-Haurwitz waves. In the derivation of this equation we clearly stated all the assumptions we made. This makes it possible to decide if, and under what conditions, the particular phenomenon under consideration can occur.

Using our assumptions we determined the vorticity equation for nondivergent planetary flow. We found that one class of solutions to this equation are the Legendre polynomials. We are interested in studying the interaction of Rossby-Haurwitz waves, so firstly we study dyad interactions. Taking a dyad solution as our leading order solution, where the two waves under consideration do not produce a third wave that grows in time, we find that these two waves cannot produce zonal flow at either $O(\delta)$ or $O(\delta^2)$. This is because any terms produced in the interactions of the two waves that have the possibility of generating zonal flow, will have a zero coupling coefficient.

We next consider the case where two waves interact resonantly. We find that if two waves interact with each other resonantly then one of the waves that should form part of the correction term will actually grow in time. After a certain time it will grow to be as large, and eventually larger, then the leading order terms. This would be a big problem. To overcome this problem this term must form part of the leading order solution and hence we conclude that we must commence our calculations with a triad solution. As a result of this analysis we determine that to consider a resonantly interacting solution we must consider a triad solution. The conditions on the wavenumbers, which Rossby-Haurwitz waves must satisfy to form part of a triad solution, were also determined.

Next we examined the repercussions of starting our calculations using a triad solution. We found that at $O(\delta)$ the problematic, resonance causing terms are still present. In order to avoid this problem we determined conditions on the amplitudes of the Rossby-Haurwitz waves, without which the asymptotic solution will not be valid. With these conditions applied to the equations we see that zonal flow cannot be created at $O(\delta)$. We find that we cannot generate zonal flow at $O(\delta)$ if we commence our calculations with either a dyad or triad solution. This leads us to examine the $O(\delta^2)$ interactions.

When we examined the $O(\delta^2)$ interactions we found that we can in fact generate zonal flow. We have proved that examining a triad solution up to $O(\delta^2)$ will produce zonal flow. In the previous cases where the generation of zonal flow was examined through this resonance interaction mechanism it was found, by both Loesch [10] and Newell [14], that a quartet of Rossby waves was required to produce zonal flow. Therefore, significantly, in the spherical case one less wave is required to produce this phenomenon.

To preserve the expansion of ψ we must include these zonal flow terms in the leading order solution. As a result we find that if we were to examine the vorticity equation for nondivergent planetary flow up to $O(\delta^2)$ we must examine at least a quartet (including zonal flow terms) of Rossby-Haurwitz waves. Using our new leading order solution we derive different amplitude equations required at $O(\delta)$. It was found that zonal flow terms form part of the amplitude equations. We see that the amplitudes of the zonal flow terms themselves are not affected at this order, and that they do not gain or lose energy at this order. Instead we find that they act as catalysts, helping the other waves to exchange energy between them. At this order the zonal flow terms affect the phase of the amplitudes of the Rossby-Haurwitz wave triad solution.

With these amplitude conditions taken into account the ψ_1 solution is determined. Using this ψ_1 solution the $O(\delta^2)$ interactions are examined. Again we find that we need to determine conditions on the amplitudes of the waves to ensure that resonance does not occur. We discovered that the amplitudes of the zonal flows themselves are affected at this order. We see that both the phase and the amplitudes of the waves of the triad solution are affected by zonal flow, and that the amplitude of the zonal flow terms themselves are altered. Therefore the generation and subsequent interactions of the zonal flow terms are significant.

As previously mentioned, Loesch examined the capability of simple discrete Rossby waves to generate zonal flows through nonlinear resonant interaction [10]. Loesch concludes that it is felt that the resonant interaction mechanism may be one of the important mechanisms in strengthening and maintaining the mid-latitude jet against frictional dissipation. He also suggests that by altering the structure of the mean flow, the interaction mechanism affects the stability properties of the atmosphere, thereby influencing cyclogenesis and shorter term climate changes. As a result it is deemed that the generation of zonal flow is important. We have found a different mechanism to generate zonal flow and we have proved that one less wave is required to generate it, compared to the plane case.

In the last chapter we examined the above analysis numerically. We determined the first few Rossby-Haurwitz triad solutions. The list of potential triad solutions shows that very few waves can satisfy the conditions imposed on the wavenumbers. Taking two different examples we showed both the resonant and nonresonant waves which were generated. Using these waves we were able to determine which zonal flow waves were consequently created. Unfortunately in the numerics there is no obvious pattern of the formation of dominant terms but it clear that zonal flow can be created and that these zonal flow waves do significantly affect the amplitude conditions on the triad solution.

This thesis is a comprehensive study of second order interactions of Rossby-Haurwitz waves. We established the necessary conditions on the wavenumbers n and m for resonant interactions to occur. Using a triad solution, we determined that we can create zonal flow purely through an energy exchange mechanism. Hence, on the sphere, a triad solution is solely required to generate zonal flow in comparison to the plane case where a quartet solution is required.

Appendix A

Properties of Legendre functions

In this appendix we derive some of the Legendre function properties which are used over the course of the thesis. Following mainly Infeld [8], we derive a number of lesser known facts about Legendre functions. These results, and in particular those in Appendix B, are of crucial importance for the main part of this thesis. That is why we discuss them in detail, rather than just referring to the literature.

In Chapter 3 we defined a function L_n^m. We want to find an association between this function and the Legendre differential equation. This association is defined to be

$$\cos^{-\frac{1}{2}} \phi L_n^m = P_n^m = \alpha_n^m \bar{P}_n^m \qquad (A.0.1)$$

In this appendix we shall look at some properties of these three functions and derive some of the frequently used identities.

A.1 Identities for $\bar{P}_n^m$

We shall firstly look at some properties of $\bar{P}_n^m$. We will define $\bar{P}$ as the unnormalised Legendre function. The expression P is defined as the fully normalised solution.

To determine this set of solutions we examine equation (3.2.10). If we set

$$z = \log\tan\left(\frac{\phi + \frac{\pi}{2}}{2}\right) \tag{A.1.1}$$

and fill this into the aforementioned equation we obtain

$$\frac{d^2\bar{P}_n^m}{dz^2} + \frac{n(n+1)}{\cosh^2 z}\bar{P}_n^m - m^2\bar{P}_n^m = 0 \tag{A.1.2}$$

Applying calculations similar to those which where done for L_n^m we notice that

$$\left((n+1)\tanh z + \frac{d}{dz}\right)\left((n+1)\tanh z - \frac{d}{dz}\right)\bar{P}_n^m = \left[(n+1)^2 - m^2\right]\bar{P}_n^m \tag{A.1.3}$$

$$\left(n\tanh z - \frac{d}{dz}\right)\left(n\tanh z + \frac{d}{dz}\right)\bar{P}_n^m = \left[n^2 - m^2\right]\bar{P}_n^m \tag{A.1.4}$$

If we multiply equation (A.1.4) by

$$n\tanh z + \frac{d}{dz}$$

and set $n = n - 1$ in equation (A.1.3) we will find that

$$\bar{P}_{n-1}^m = \left(n\tanh z + \frac{d}{dz}\right)\bar{P}_n^m \tag{A.1.5}$$

Similarly it can be shown that

$$\bar{P}_{n+1}^m = \left((n+1)\tanh z - \frac{d}{dz}\right)\bar{P}_n^m \tag{A.1.6}$$

We wish to ensure that $\bar{P}$ satisfies the same conditions as L. Therefore we require $\bar{P}$ to be square integrable. This means that we want

$$\int_a^b \left(\bar{P}_n^m\right)^2 \mathrm{d}\mu = 1 \tag{A.1.7}$$

Examining this requirement we will find that

$$\begin{aligned}\int_a^b \left(\bar{P}_n^m\right)^2 \mathrm{d}\phi &= \int_a^b \left(\left(n\tanh z - \frac{d}{dz}\right)\bar{P}_{n-1}^m\right)^2 \mathrm{d}\mu \\ &= \int_a^b \bar{P}_{n-1}^m \left(n\tanh z + \frac{d}{dz}\right)\left(n\tanh z - \frac{d}{dz}\right)\bar{P}_{n-1}^m \,\mathrm{d}\mu \\ &= \left(n^2 - m^2\right)\int_a^b \left(\bar{P}_{n-1}^m\right)^2 \mathrm{d}\mu\end{aligned}$$

Therefore we can see that to ensure that these functions are square integrable we require

$$\bar{P}_n^m = \left[(n-m)(n+m)\right]^{-\frac{1}{2}}\left(n\tanh z - \frac{d}{dz}\right)\bar{P}_{n-1}^m \tag{A.1.8}$$

Using the definition for z in equation (A.1.1) it can be shown that

$$\frac{d}{dz} = \cos\phi\frac{d}{d\phi} \tag{A.1.9}$$

$$\tanh z = \sin\phi \tag{A.1.10}$$

Filling these into equation (A.1.8) we have

$$\bar{P}_n^m = \left[(n-m)(n+m)\right]^{-\frac{1}{2}}\left(n\sin\phi - \cos\phi\frac{d}{d\phi}\right)\bar{P}_{n-1}^m \tag{A.1.11}$$

If we also multiply equation (A.1.8) across by

$$n\tanh z + \frac{d}{dz}$$

and apply the same substitutions we will find that

$$\bar{P}_{n-1}^m = \left[(n-m)(n+m)\right]^{-\frac{1}{2}}\left(n\sin\phi + \cos\phi\frac{d}{d\phi}\right)\bar{P}_n^m \tag{A.1.12}$$

If we could find a solution to this equation (A.1.2), then using both this solution and the equations above will ensure that we will be able to determine any function $\bar{P}$ for any given n and m. It can be shown that

$$\bar{P}_m^m = \cosh^{-m} z$$

is in fact a solution to the equation. We must now force similar conditions to $\bar{P}$ as those which were applied to L. This means that we require

$$\begin{aligned}
&\int_{-\infty}^{\infty} (\bar{P}_m^m)^2 \,\mathrm{d}z = 1 \\
\Rightarrow \int_{-\infty}^{\infty} \cosh^{-2m} z \,\mathrm{d}z &= \frac{\sinh z \cosh^{-2m+1} z}{2m-1}\Big|_{-\infty}^{\infty} + \frac{2m-2}{2m-1}\int_{-\infty}^{\infty} \cosh^{-2m+2} z \,\mathrm{d}z \\
&= \frac{2m-2}{2m-1}\int_{-\infty}^{\infty} \cosh^{-2m+2} z \,\mathrm{d}z \\
&= \frac{2m-2}{2m-1}\frac{2m-4}{2m-3}\int_{-\infty}^{\infty} \cosh^{-2m+4} z \,\mathrm{d}z \\
&= \ldots = \left(\frac{2}{1}\frac{2.4\ldots 2m-4.2m-2}{3.5\ldots 2m-3.2m-1}\right)
\end{aligned}$$

Applying this to our solution we find that a solution to the equation is

$$\bar{P}_m^m = \left(\frac{1}{2}\frac{3.5\ldots 2m-3.2m-1}{2.4\ldots 2m-4.2m-2}\right)^{\frac{1}{2}} \cosh^{-m} z$$

Using both this single solution when n and m are equal and the definition above, all valid solutions for $\bar{P}_n^m$ can be determined.

A.2 Identities for P_n^m

In this section we shall examine the properties of P_n^m. From Chapter 3 we know that

$$L_n^m = \cos^{\frac{1}{2}} \phi P_n^m \tag{A.2.1}$$

It was also derived in this chapter that

$$L_n^m = [(n+m+1)(n-m)]^{-\frac{1}{2}} \left(-(m+\tfrac{1}{2}) \tan\phi + \frac{d}{d\phi} \right) L_n^{m+1} \tag{A.2.2}$$

So applying equation (A.2.1) to this equation we see that

$$\begin{aligned}
\cos^{\frac{1}{2}}\phi P_n^m &= [(n+m+1)(n-m)]^{-\frac{1}{2}} \left(-(m+\tfrac{1}{2}) \tan\phi + \frac{d}{d\phi} \right) \cos^{\frac{1}{2}}\phi P_n^{m+1} \\
&= [(n+m+1)(n-m)]^{-\frac{1}{2}} \left(-(m+1) \frac{\sin\phi}{\cos^{\frac{1}{2}}\phi} P_n^{m+1} + \cos^{\frac{1}{2}}\phi \frac{dP_n^{m+1}}{d\phi} \right) \\
\Rightarrow P_n^m &= [(n+m+1)(n-m)]^{-\frac{1}{2}} \left(-(m+1) \tan\phi + \frac{d}{d\phi} \right) P_n^{m+1} \qquad \text{(A.2.3)}
\end{aligned}$$

Similarly it can be shown that

$$P_n^m = [(n+m)(n-m-1)]^{-\frac{1}{2}} \left(-(m-1) \tan\phi - \frac{d}{d\phi} \right) P_n^{m-1} \tag{A.2.4}$$

It can also be shown that

$$P_n^n = \left(\frac{1.3 \ldots 2n+1}{2.2.4 \ldots 2n} \right)^{\frac{1}{2}} \cos^n\phi$$

It follows from this definition that P_n^n is square integrable, i.e.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (P_n^n)^2 \, \mathrm{d}\mu = 1$$

Using the definition for P_n^n and the definitions for P_n^{m-1} and P_n^{m+1} we can now determine an expression for P for any given n and m.

A.3 Identities for L_n^m

Finally in this appendix we shall examine some of the properties of L_n^m. It is assumed that the functions L_n^m are orthogonal. To show this we must show that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} L_n^m L_{n'}^m \,\mathrm{d}\phi = \begin{cases} 0 & n \neq n' \\ \text{nonzero} & n = n' \end{cases} \tag{A.3.1}$$

This can be proved in the following way

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} L_n^m L_{n'}^m \,\mathrm{d}\phi &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} H_m^- L_n^{m-1} \, H_m^- L_{n'}^{m-1} \,\mathrm{d}\phi \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} L_n^{m-1} \, H_m^+ H_m^- L_{n'}^{m-1} \,\mathrm{d}\phi \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(n'(n'+1) + \tfrac{1}{4} - (m - \tfrac{1}{2})^2\right) L_n^{m-1} L_{n'}^{m-1} \,\mathrm{d}\phi \end{aligned}$$

Similarly we see that

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} L_n^m L_{n'}^m \,\mathrm{d}\phi &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} H_m^- H_m^+ L_n^{m-1} L_{n'}^{m-1} \,\mathrm{d}\phi \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(n(n+1) + \tfrac{1}{4} - (m - \tfrac{1}{2})^2\right) L_n^{m-1} L_{n'}^{m-1} \,\mathrm{d}\phi \end{aligned}$$

These two equations must be equal. We see that the integrals are equal if and only if $n = n'$. If $n \neq n'$ then the integral itself must be zero. Hence we have proved equation (A.3.1).

In the previous section it is assumed that P_n^m is the fully normalised form of $\bar{P}_n^m$. Therefore if we assume that α_n^m is the correction to the normalisation of $\bar{P}_n^m$ then we have the identity

$$\cos^{-\frac{1}{2}} \phi L_n^m = P_n^m = \alpha_n^m \bar{P}_n^m \tag{A.3.2}$$

If we can determine an expression for α_n^m then we will be able to derive expressions for $L_{n\pm1}^m$.

Using the relationship between L_n^m and $\bar{P}_n^m$, equation (A.1.12) and equation (A.3.2) we find that

$$\begin{aligned}\frac{1}{\alpha_{n-1}^m}\cos^{-\frac{1}{2}}\phi L_{n-1}^m &= [(n-m)(n+m)]^{-\frac{1}{2}}\left(n\sin\phi+\cos\phi\frac{d}{d\phi}\right)\frac{1}{\alpha_n^m}\cos^{-\frac{1}{2}}\phi L_n^m \\ &= \frac{[(n-m)(n+m)]^{-\frac{1}{2}}}{\alpha_n^m}\left((n+\tfrac{1}{2})\sin\phi\cos^{-\frac{1}{2}}\phi+\cos^{\frac{1}{2}}\phi\frac{d}{d\phi}\right)L_n^m\end{aligned}$$

Rearranging this equation we see that

$$L_{n-1}^m = \frac{\alpha_{n-1}^m}{\alpha_n^m}[(n-m)(n+m)]^{-\frac{1}{2}}\left((n+\tfrac{1}{2})\sin\phi+\cos\phi\frac{d}{d\phi}\right)L_n^m \tag{A.3.3}$$

If we next examine equation (A.1.11) and apply similar calculations to it we will find that

$$L_{n+1}^m = \frac{\alpha_{n+1}^m}{\alpha_n^m}[(n-m+1)(n+m+1)]^{-\frac{1}{2}}\left((n+\tfrac{1}{2})\sin\phi-\cos\phi\frac{d}{d\phi}\right)L_n^m \tag{A.3.4}$$

If we use these two expressions we can write L_{n+1}^{m+1} in terms of L_n^m by two different methods. Firstly we see that

$$L_{n+1}^{m+1} = \frac{\alpha_{n+1}^{m+1}}{\alpha_n^{m+1}}[(n-m)(n+m+2)]^{-\frac{1}{2}}\left((n+\tfrac{1}{2})\sin\phi-\cos\phi\frac{d}{d\phi}\right)L_n^{m+1} \tag{A.3.5}$$

$$\begin{aligned}&= \frac{\alpha_{n+1}^{m+1}}{\alpha_n^{m+1}}[(n-m)(n+m+2)(n-m)(n+m+1)]^{-\frac{1}{2}} \\ &\quad\left((n+\tfrac{1}{2})\sin\phi-\cos\phi\frac{d}{d\phi}\right)\left(-(m+\tfrac{1}{2})\tan\phi-\frac{d}{d\phi}\right)L_n^m\end{aligned} \tag{A.3.6}$$

If we now apply this formula the other way around we will find that

$$\begin{aligned}L_{n+1}^{m+1} &= \frac{\alpha_{n+1}^m}{\alpha_n^m}[(n-m+1)(n+m+2)(n-m+1)(n+m+1)]^{-\frac{1}{2}} \\ &\quad\left(-(m+\tfrac{1}{2})\tan\phi-\frac{d}{d\phi}\right)\left((n+\tfrac{1}{2})\sin\phi-\cos\phi\frac{d}{d\phi}\right)L_n^m\end{aligned} \tag{A.3.7}$$

If these two equation are worked out and simplified down it can be shown that

$$\begin{aligned} L_{n+1}^{m+1} &= \frac{\alpha_{n+1}^{m+1}}{\alpha_n^{m+1}} \left[(n+m+2)(n+m+1)\right]^{-\frac{1}{2}} \\ &\quad \times \left(-\sin\phi \frac{d}{d\phi} - (m+\tfrac{1}{2})\frac{1}{\cos\phi} - (n+\tfrac{1}{2})\cos\phi \right) L_n^m \\ &= \frac{\alpha_{n+1}^{m}}{\alpha_n^{m}} \left[(n+m+2)(n+m+1)\right]^{-\frac{1}{2}} \\ &\quad \times \left(-\sin\phi \frac{d}{d\phi} - (m+\tfrac{1}{2})\frac{1}{\cos\phi} - (n+\tfrac{1}{2})\cos\phi \right) L_n^m \end{aligned} \tag{A.3.8}$$

We can see therefore that α is independent of m. Setting $n = m$ in equation (3.3.13) we get

$$L_{n+1}^{n+1} = (2n+2)^{-\frac{1}{2}} \left(-(n+\tfrac{1}{2})\tan\phi - \frac{d}{d\phi} \right) L_{n+1}^{n} \tag{A.3.9}$$

If we multiply this equation across by

$$\left(-(n+\tfrac{1}{2})\tan\phi + \frac{d}{d\phi} \right)$$

and apply the identity (3.3.7) we determine

$$\left(-(n+\tfrac{1}{2})\tan\phi + \frac{d}{d\phi} \right) L_{n+1}^{n+1} = (2n+2)^{\frac{1}{2}} L_{n+1}^{n}$$

If we next set $m+1 = n$ in equation (A.3.5) we see that

$$L_{n+1}^{n} = \frac{\alpha_{n+1}}{\alpha_n} (2n+1)^{-\frac{1}{2}} \left((n+\tfrac{1}{2})\sin\phi - \cos\phi \frac{d}{d\phi} \right) L_n^n \tag{A.3.10}$$

Using the identities for L_n^n and L_{n+1}^{n+1}, and comparing these equations we see that

$$\frac{\alpha_{n+1}}{\alpha_n} = -\left(\frac{2n+3}{2n+1} \right)^{\frac{1}{2}}$$

It is from this expression that we can now determine expression for L_{n+1}^m and L_{n-1}^m, i.e.

$$\left((n+\tfrac{1}{2})\sin\phi + \cos\phi\frac{d}{d\phi}\right) L_n^m = -\left(\frac{(n-m)(n+m)(2n+1)}{2n-1}\right)^{\frac{1}{2}} L_{n-1}^m \quad \text{(A.3.11)}$$

$$\left((n+\tfrac{1}{2})\sin\phi - \cos\phi\frac{d}{d\phi}\right) L_n^m = - \left(\frac{(n-m+1)(n+m+1)(2n+1)}{2n+3}\right)^{\frac{1}{2}} L_{n+1}^m \quad \text{(A.3.12)}$$

If we now add equations (A.3.11) and (A.3.12) together we find that

$$\sin\phi L_n^m = -\left(\frac{(n-m+1)(n+m+1)}{(2n+1)(2n+3)}\right)^{\frac{1}{2}} L_{n+1}^m - \left(\frac{(n-m)(n+m)}{(2n+1)(2n-1)}\right)^{\frac{1}{2}} L_{n-1}^m \quad \text{(A.3.13)}$$

Multiplying equation (3.3.14) across by $\cos\phi$ gives

$$\cos\phi L_n^m = -\left(\frac{(2n+1)(n-m-1)}{(2n-1)(n-m)}\right)^{\frac{1}{2}} L_{n-1}^{m+1} - \left(\frac{n+m+1}{n-m}\right)^{\frac{1}{2}} \sin\phi L_n^{m+1}$$

which, when examined in relation to equation (A.3.13) gives

$$\cos\phi L_n^m = \left(\frac{(n+m+1)(n+m+2)}{(2n+1)(2n+3)}\right)^{\frac{1}{2}} L_{n+1}^{m+1} - \left(\frac{(n-m-1)(n-m)}{(2n-1)(2n+1)}\right)^{\frac{1}{2}} L_{n-1}^{m+1} \quad \text{(A.3.14)}$$

We can also determine an expression for $\cos\phi L_n^m$ in terms of L_{n-1}^{m-1} and L_{n+1}^{m-1}. To do this we examine equation (3.3.13) and set $m = m-1$. Multiplying the resulting expression by $\cos\phi$ we obtain

$$
\begin{aligned}
[(n-m+1)(n+m)]^{\frac{1}{2}} \cos\phi\, L_n^m &= \left(-(m-\tfrac{1}{2})\sin\phi - \cos\phi\frac{d}{d\phi}\right) L_n^{m-1} \\
&= \left((n+\tfrac{1}{2})\sin\phi - \cos\phi\frac{d}{d\phi}\right) L_n^{m-1} - (n+m)\sin\phi L_n^{m-1} \\
&= -\left(\frac{(2n+1)(n+m+2)(n+m)}{(2n+3)}\right)^{\frac{1}{2}} L_{n+1}^{m-1} - (n+m)\sin\phi L_n^{m-1} \\
\Rightarrow \cos\phi L_n^m &= \left(\frac{(n+m)(n+m-1)}{(2n+1)(2n-1)}\right)^{\frac{1}{2}} L_{n-1}^{m-1} \\
&\quad - \left(\frac{(n-m+2)(n-m+1)}{(2n+3)(2n+1)}\right)^{\frac{1}{2}} L_{n+1}^{m-1} \qquad \text{(A.3.15)}
\end{aligned}
$$

Lastly to derive an expression for $\cos^l\phi$ we will again use

$$
L_n^m = [(n+m+1)(n-m)]^{-\frac{1}{2}} \left(-(m+\tfrac{1}{2})\tan\phi + \frac{d}{d\phi}\right) L_n^{m+1}
$$

Multiplying this equation across by $\cos^l\phi$ gives

$$
\begin{aligned}
\cos^l\phi L_n^m &= \cos^l\phi\,[(n+m+1)(n-m)]^{-\frac{1}{2}} \left(-(m+\tfrac{1}{2})\tan\phi + \frac{d}{d\phi}\right) L_n^{m+1} \\
&= [(n+m+1)(n-m)]^{-\frac{1}{2}} \left(-(m+\tfrac{1}{2})\sin\phi\cos^{l-1}\phi + \cos^l\phi\frac{d}{d\phi}\right) L_n^{m+1} \\
&= [(n+m+1)(n-m)]^{-\frac{1}{2}} \left(-(m+\tfrac{1}{2}-l)\tan\phi + \frac{d}{d\phi}\right) \cos^l\phi L_n^{m+1} \quad \text{(A.3.16)}
\end{aligned}
$$

These expressions will be used in the derivation of the conditions on n and m which are necessary when examining triad interactions.

Appendix B

Derivation of conditions on n

In this appendix we shall examine the following integral

$$B^{m_1 m_2}_{n_1 n_2 n_3} = \int_{-1}^{1} P^{m_1+m_2}_{n_3} \left(i m_2 P^{m_2}_{n_2} \frac{dP^{m_1}_{n_1}}{d\mu} - i m_1 P^{m_1}_{n_1} \frac{dP^{m_2}_{n_2}}{d\mu} \right) \mathrm{d}\mu \tag{B.0.1}$$

Following [8] and [23], we derive the necessary conditions on the wavenumbers n and m without which the integral above would be zero. In deriving these conditions we will also obtain an alternative expression for the above equation which will make the numerical calculations in Chapter 5 a little easier to manage.

Examining this integral we know that $\mu = \sin\phi$ and $m_1 + m_2 = m_3$ so applying these to the expression we are examining we get

$$\tilde{B}^{m_1 m_2 m_3}_{n_1 n_2 n_3} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P^{m_3}_{n_3} \left(i m_2 P^{m_2}_{n_2} \frac{dP^{m_1}_{n_1}}{d\phi} - i m_1 P^{m_1}_{n_1} \frac{dP^{m_2}_{n_2}}{d\phi} \right) \mathrm{d}\phi \tag{B.0.2}$$

It can be seen immediately from this expression that

$$\tilde{B}^{m_1 m_2 m_3}_{n_1 n_2 n_3} = -\tilde{B}^{-m_1 -m_2 -m_3}_{n_1 n_2 n_3} = -\tilde{B}^{m_2 m_1 m_3}_{n_2 n_1 n_3}$$

We next want to derive a redundancy formula for this equation which shall be used in the proof.

B.1 Redundancy Formula

It is claimed that [18]

$$\tilde{B}^{m_1 m_2 m_3}_{n_1 n_2 n_3} = (-1)^{m_2} \tilde{B}^{m_3 -m_2 m_1}_{n_3 n_2 n_1} = (-1)^{m_1} \tilde{B}^{-m_1 m_3 m_2}_{n_1 n_3 n_2} \tag{B.1.1}$$

We can show that this redundancy formula holds through integration by parts. Firstly to prove that

$$\tilde{B}^{m_1 m_2 m_3}_{n_1 n_2 n_3} = (-1)^{m_2} \tilde{B}^{m_3 -m_2 m_1}_{n_3 n_2 n_1} \tag{B.1.2}$$

we want to show that

$$\begin{aligned} &\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P^{m_3}_{n_3} \left(im_2 P^{m_2}_{n_2} \frac{dP^{m_1}_{n_1}}{d\phi} - im_1 P^{m_1}_{n_1} \frac{dP^{m_2}_{n_2}}{d\phi} \right) d\phi \\ &\quad = -(-1)^{m_2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P^{m_1}_{n_1} \left(im_2 P^{-m_2}_{n_2} \frac{dP^{m_3}_{n_3}}{d\phi} + im_3 P^{m_3}_{n_3} \frac{dP^{-m_2}_{n_2}}{d\phi} \right) d\phi \end{aligned} \tag{B.1.3}$$

Noting that $m_1 + m_2 = m_3$ we see that the expression that we are examining is

$$im_2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(P^{m_3}_{n_3} P^{m_2}_{n_2} \frac{dP^{m_1}_{n_1}}{d\phi} + P^{m_1}_{n_1} P^{m_2}_{n_2} \frac{dP^{m_3}_{n_3}}{d\phi} + P^{m_1}_{n_1} P^{m_3}_{n_3} \frac{dP^{m_2}_{n_2}}{d\phi} \right) d\phi \tag{B.1.4}$$

We now apply integration by parts to the first term in this expression.

Letting

$$u = P_{n_2}^{m_2} P_{n_3}^{m_3} \quad \Rightarrow \quad du = \left(P_{n_2}^{m_2} \frac{dP_{n_3}^{m_3}}{d\phi} + P_{n_3}^{m_3} \frac{dP_{n_2}^{m_2}}{d\phi} \right) d\phi$$

and

$$dv = \frac{dP_{n_1}^{m_1}}{d\phi} d\phi \quad \Rightarrow \quad v = P_{n_1}^{m_1}$$

results in the term under consideration becoming

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_2}^{m_2} P_{n_3}^{m_3} \frac{dP_{n_1}^{m_1}}{d\phi} d\phi = \left[P_{n_1}^{m_1} P_{n_2}^{m_2} P_{n_3}^{m_3} \right]_{\phi=-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_1}^{m_1} \left(P_{n_3}^{m_3} \frac{dP_{n_2}^{m_2}}{d\phi} + P_{n_2}^{m_2} \frac{dP_{n_3}^{m_3}}{d\phi} \right) d\phi$$

From our definition of Legendre functions we know that

$$P_n^m(\pm\tfrac{\pi}{2}) = 0$$

Therefore the expression which we are examining is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -P_{n_1}^{m_1} P_{n_2}^{m_2} \frac{dP_{n_3}^{m_3}}{d\phi} - P_{n_1}^{m_1} P_{n_3}^{m_3} \frac{dP_{n_2}^{m_2}}{d\phi} + P_{n_1}^{m_1} P_{n_2}^{m_2} \frac{dP_{n_3}^{m_3}}{d\phi} + P_{n_1}^{m_1} P_{n_3}^{m_3} \frac{dP_{n_2}^{m_2}}{d\phi} \, \mathrm{d}\phi = 0$$

as required.

Similarly it can be shown that

$$\bar{B}_{n_1 n_2 n_3}^{m_1 m_2 m_3} = (-1)^{m_1} \bar{B}_{n_1 n_3 n_2}^{-m_1 m_3 m_2}$$

To prove this identity we must show that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_3}^{m_3} \left(i m_2 P_{n_2}^{m_2} \frac{dP_{n_1}^{m_1}}{d\phi} - i m_1 P_{n_1}^{m_1} \frac{dP_{n_2}^{m_2}}{d\phi} \right) d\phi$$
$$= -(-1)^{m_1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_2}^{m_2} \left(i m_3 P_{n_3}^{m_3} \frac{dP_{n_1}^{-m_1}}{d\phi} + i m_1 P_{n_1}^{-m_1} \frac{dP_{n_3}^{m_3}}{d\phi} \right) d\phi$$

Once again setting $m_1 + m_2 = m_3$ this reduces to examining

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} im_1 \left(P_{n_1}^{m_1} P_{n_3}^{m_3} \frac{dP_{n_2}^{m_2}}{d\phi} + P_{n_2}^{m_2} P_{n_3}^{m_3} \frac{dP_{n_1}^{m_1}}{d\phi} + P_{n_1}^{m_1} P_{n_2}^{m_2} \frac{dP_{n_3}^{m_3}}{d\phi} \right) d\phi \tag{B.1.5}$$

This expression is the same as before and hence is also zero. Therefore we have derived the desired result (B.1.1).

B.2 Reformulation of the equation

We are interested in deriving a more manageable formula for equation (B.0.2), i.e. $\tilde{B}_{n_1 n_2 n_3}^{m_1 m_2 m_3}$.

If we replace L_n^m in equation (3.3.13) with an expression for P we have

$$\begin{aligned}
\cos^{\frac{1}{2}} \phi P_n^{m+1} &= \left[(n+m+1)(n-m)\right]^{-\frac{1}{2}} \left(-(m+\tfrac{1}{2}) \tan\phi - \frac{d}{d\phi} \right) \cos^{\frac{1}{2}} \phi P_n^{m+1} \\
\Rightarrow \frac{dP_n^m}{d\phi} &= -\left[(n+m+1)(n-m)\right]^{\frac{1}{2}} P_n^{m+1} - m \tan\phi P_n^m
\end{aligned}$$

Filling this into (B.0.2) we see that

$$\begin{aligned}
&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_3}^{m_3} \left(im_2 P_{n_2}^{m_2} \frac{dP_{n_1}^{m_1}}{d\phi} - im_1 P_{n_1}^{m_1} \frac{dP_{n_2}^{m_2}}{d\phi} \right) d\phi \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} im_2 P_{n_2}^{m_2} P_{n_3}^{m_3} \left(-\left[(n_1+m_1+1)(n_1-m_1)\right]^{\frac{1}{2}} P_{n_1}^{m_1+1} - m_1 \tan\phi P_{n_1}^{m_1} \right) \\
&\quad - im_1 P_{n_1}^{m_1} P_{n_3}^{m_3} \left(-\left[(n_2+m_2+1)(n_2-m_2)\right]^{\frac{1}{2}} P_{n_2}^{m_2+1} - m_2 \tan\phi P_{n_2}^{m_2} \right) d\phi \\
&= \left[(n_2+m_2+1)(n_2-m_2)\right]^{\frac{1}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} im_1 P_{n_1}^{m_1} P_{n_3}^{m_3} P_{n_2}^{m_2+1} \, d\phi \\
&\quad - \left[(n_1+m_1+1)(n_1-m_1)\right]^{\frac{1}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} im_2 P_{n_2}^{m_2} P_{n_3}^{m_3} P_{n_1}^{m_1+1} \, d\phi
\end{aligned}$$

We know from our definition of P_n^m that

$$P_n^m = \left(\frac{(n-m)!}{(n+m)!} \right)^{\frac{1}{2}} (1-\mu^2)^{\frac{m}{2}} \frac{d^m}{d\mu^m} P_n^0 \tag{B.2.1}$$

From MacRobert [12] we know that

$$\frac{d}{d\mu}P_n^0 = (2n+1)^{\frac{1}{2}}\sum_q (2q+1)^{\frac{1}{2}}P_q^0 \tag{B.2.2}$$

where $q = n-1, n-3, \dots 1 \quad \text{or} \quad 0$.

Using these identities we can find an expression for P_n^{m+1}

$$P_n^{m+1} = \left(\frac{(n-m-1)!(2n+1)}{(n+m+1)!}\right)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}\sum_q\left(\frac{(q+m)!(2q+1)}{(q-m)!}\right)^{\frac{1}{2}}P_q^m$$

Applying all this to our expression we see that the equation which we want to examine is

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_3}^{m_3}&\left(im_2P_{n_2}^{m_2}\frac{dP_{n_1}^{m_1}}{d\phi} - im_1P_{n_1}^{m_1}\frac{dP_{n_2}^{m_2}}{d\phi}\right)\mathrm{d}\phi \\
&= im_1\left(\frac{(n_2-m_2)!(2n_2+1)}{(n_2+m_2)!}\right)^{\frac{1}{2}}\sum_q\left(\frac{(2q+1)(q+m_2)!}{(q-m_2)!}\right)^{\frac{1}{2}} \\
&\qquad\qquad\times\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_1}^{m_1}P_{n_3}^{m_3}P_q^{m_2}\cos\phi\,\mathrm{d}\phi \\
&\quad -im_2\left(\frac{(n_1-m_1)!(2n_1+1)}{(n_1+m_1)!}\right)^{\frac{1}{2}}\sum_p\left(\frac{(2p+1)(p+m_1)!}{(p-m_1)!}\right)^{\frac{1}{2}} \\
&\qquad\qquad\times\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_2}^{m_2}P_{n_3}^{m_3}P_p^{m_1}\cos\phi\,\mathrm{d}\phi
\end{aligned}$$

where

$$q = n_2-1, n_2-3, \dots, m_2+1 \quad \text{or} \quad m_2$$

$$p = n_1-1, n_1-3, \dots, m_1+1 \quad \text{or} \quad m_1$$

We shall examine one of the integrals above and determine conditions on n and m in the process, i.e. we will examine

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_1}^{m_1}P_q^{m_2}P_{n_3}^{m_3}\cos\phi\,\mathrm{d}\phi \tag{B.2.3}$$

B.3 Manipulation of equation

We wish to find an alternative formulation for equation (B.2.3). We do this with the help of the identities proved in Appendix A. Using these properties we will be able to rewrite the expression as a sum of factors and as a result, it will be easier to study this expression and determine conditions on this integral [4], [8].

Using the fact that

$$L_n^m = \cos^{\frac{1}{2}}\phi P_n^m \tag{B.3.1}$$

the equation under consideration, equation (B.2.3) can be rewritten as

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_3}^{m_3} L_{n_1}^{m_1} L_q^{m_2}\, \mathrm{d}\phi \tag{B.3.2}$$

Using equation (A.2.3) this becomes

$$\begin{aligned}
&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_3}^{m_3} L_{n_1}^{m_1} L_q^{m_2}\, \mathrm{d}\phi \\
&= [(n_3+m_3+1)(n_3-m_3)]^{-\frac{1}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\left(-(m_3+1)\tan\phi + \frac{d}{d\phi}\right) P_{n_3}^{m_3+1}\right] L_{n_1}^{m_1} L_q^{m_2}\, \mathrm{d}\phi \\
&= [(n_3+m_3+1)(n_3-m_3)]^{-\frac{1}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_3}^{m_3+1} \left(-(m_3+1)\tan\phi - \frac{d}{d\phi}\right) L_{n_1}^{m_1} L_q^{m_2}\, \mathrm{d}\phi
\end{aligned}$$

Noting that $m_1 + m_2 = m_3$ this becomes

$$\begin{aligned}
&[(n_3+m_3+1)(n_3-m_3)]^{-\frac{1}{2}} \\
&\quad \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_3}^{m_3+1} \left(-(m_1+\tfrac{1}{2})\tan\phi - (m_2+\tfrac{1}{2})\tan\phi - \frac{d}{d\phi}\right) L_{n_1}^{m_1} L_q^{m_2}\, \mathrm{d}\phi \\
&= [(n_3+m_3+1)(n_3-m_3)]^{-\frac{1}{2}} \\
&\quad \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_3}^{m_3+1} \left[L_q^{m_2}\left(-(m_1+\tfrac{1}{2})\tan\phi - \frac{d}{d\phi}\right) L_{n_1}^{m_1} \right. \\
&\quad \left. + L_{n_1}^{m_1}\left(-(m_2+\tfrac{1}{2})\tan\phi - \frac{d}{d\phi}\right) L_q^{m_2}\right] \mathrm{d}\phi
\end{aligned}$$

By substituting equation (3.3.13) derived in Chapter 3 into this expression this equation becomes

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_3}^{m_3} L_{n_1}^{m_1} L_q^{m_2}\,\mathrm{d}\phi = [(n_3+m_3+1)(n_3-m_3)]^{-\frac{1}{2}}$$
$$\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_3}^{m_3+1} \Big([(q+m_2+1)(q-m_2)]^{-\frac{1}{2}} L_q^{m_2+1} L_{n_1}^{m_1}$$
$$+ [(n_1+m_1+1)(n_1-m_1)]^{-\frac{1}{2}} L_{n_1}^{m_1+1} L_q^{m_2} \Big)\,\mathrm{d}\phi \quad \text{(B.3.3)}$$

We can now repeat these calculations. We notice that we can write $P_{n_3}^{m_3+1}$ as

$$P_{n_3}^{m_3+1} = [(n_3-m_3-1)(n_3+m_3+2)]^{-\frac{1}{2}} \left(-(m_3+2)\tan\phi + \frac{d}{d\phi}\right) P_{n_3}^{m_3+2}$$

Using this we can repeat the above calculations. This method can be applied $n_3 - m_3$ times which reduces the equation to

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_3}^{m_3}\, L_q^{m_2}\, L_{n_1}^{m_1}\,\mathrm{d}\phi = \left(\frac{(n_3+m_3)!(q-m_2)!(n_1-m_1)!}{(n_3-m_3)!(2n_3)!(q+m_2)!(n_1+m_1)!}\right)^{\frac{1}{2}}$$
$$\times \sum_{i=0}^{n_3-m_3} \binom{n_3-m_3}{i} \left(\frac{(q+n_3+m_2-m_3-i)!(n_1+m_1+i)!}{(q-n_3-m_2+m_3+i)!(n_1-m_1-i)!}\right)^{\frac{1}{2}}$$
$$\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_3}^{n_3} L_q^{m_2+n_3-m_3-i} L_{n_1}^{m_1+i}\,\mathrm{d}\phi \quad \text{(B.3.4)}$$

We know that the superscript of the Legendre function must be less than or equal to the subscript. Therefore we require

$$m_1 + i \leq n_1$$

and

$$m_2 + n_3 - m_3 - i \leq q$$

If we take $i = 0$ this condition corresponds to requiring

$$m_2 + n_3 - m_3 \leq q$$

$$\Rightarrow n_3 - m_1 \leq q$$

$$\Rightarrow n_3 - q \leq m_1 \leq n_1$$

If we start with $i = n_3 - m_3$ we will arrive at the same condition. Therefore from this equation we can deduce that we require

$$n_3 \leq n_1 + q \tag{B.3.5}$$

We know that $q = n_2 - 1, n_2 - 3, \ldots m_2 + 1$ or m_2. Therefore the condition derived is

$$n_3 < n_1 + n_2 \tag{B.3.6}$$

To further reduce equation (B.3.4) we shall examine

$$P_{n_3}^{n_3} L_q^{m_2+n_3-m_3-i}$$

Using equations (3.3.2) and (3.3.12) derived in Chapter 3, and equation (A.3.15) in Appendix A we can rewrite this expression as

$$\begin{aligned} P_{n_3}^{n_3} L_q^{m_2+n_3-m_3-i} &= \left(\frac{1.3\ldots 2n_3+1}{2.2.4\ldots 2n_3}\right)^{\frac{1}{2}} \cos^{n_3}\phi L_q^{m_2+n_3-m_3-i} \\ &= c_1 \cos^{n_3-1}\phi \cos\phi L_q^{n_3-m_1-i} \\ &= c_1 \cos^{n_3-1}\phi \left(c_2 L_{q+1}^{n_3-m_1-i-1} + c_3 L_{q-1}^{n_3-m_1-i-1}\right) \\ &= c_1 \cos^{n_3-2}\phi \left(c_2 L_{q+2}^{n_3-m_1-i-2} + c_3 L_q^{n_3-m_1-i-2} + c_4 L_{q-2}^{n_3-m_1-i-2}\right) \end{aligned}$$

where c_i are constants. If we repeat this calculation n_3 times and use the identity

$$L_n^{-m} = (-1)^m L_n^m$$

the equation reduces to

$$P_{n_3}^{n_3} L_q^{m_2+n_3-m_3-i} = c_1 L_{q+n_3}^{m_1+i} + c_2 L_{q+n_3-2}^{m_1+i} + \ldots + c_n L_{q-n_3}^{m_1+i}$$

Filling this expansion into our equation we can see that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_3}^{n_3} \, L_q^{m_2+n_3-m_3-i} L_{n_1}^{m_1+i} \, \mathrm{d}\phi = c_1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} L_{q+n_3}^{m_1+i} L_{n_1}^{m_1+i} \, \mathrm{d}\phi$$
$$+ c_2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} L_{q+n_3-2}^{m_1+i} L_{n_1}^{m_1+i} \, \mathrm{d}\phi + \ldots + c_n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} L_{q-n_3}^{m_1+i} L_{n_1}^{m_1+i} \, \mathrm{d}\phi$$

For equation (B.0.1) to be nonzero, we require that one of the above integrals is nonzero. From Appendix A we know that L_n^m are orthogonal functions. Therefore to ensure that this expression is nonzero we require

$$q + n_3 - n = n_1$$

where n is an even number. We also require that one of the following expressions are true

$$q + n_3 = n_1$$
$$q + n_3 - 2 = n_1$$
$$\vdots$$
$$q - n_3 = n_1$$

Tidying up these two expression we see that for the integral under consideration to be nonzero we require

$$n_1 + q + n_3 \quad \text{is even} \tag{B.3.7}$$

Comparing q and n_2 we can see that this condition reduces down to the sum of the n's being odd. As a result the conditions required for the integral to be nonzero are

$$n_1 + n_2 + n_3 \quad \text{is odd} \tag{B.3.8}$$

$$n_2 - n_1 < n_3 < n_1 + n_2 \tag{B.3.9}$$

With these set of conditions on the wavenumbers determined we return to the main equation to continue the derivation.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_3}^{n_3} L_q^{m_2+n_3-m_3-i} L_{n_1}^{m_1+i}\, \mathrm{d}\phi = \underbrace{\left(\frac{1.3\ldots 2n_3+1}{2.2.4\ldots 2n_3}\right)^{\frac{1}{2}}}_{\eta} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n_3}\phi L_q^{m_2+n_3-m_3-i} L_{n_1}^{m_1+i}\, \mathrm{d}\phi$$
$$= \eta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n_3}\phi L_q^{\alpha} L_{n_1}^{\beta}\, \mathrm{d}\phi$$

where $\alpha + \beta = n_3$. Applying equation (A.3.16) to this integral we get

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n_3}\phi L_q^{\alpha} L_{n_1}^{\beta}\, \mathrm{d}\phi &= [(q+\alpha+1)(q-\alpha)]^{-\frac{1}{2}} \\ &\quad\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(-(\alpha+\tfrac{1}{2}-n_3)\tan\phi + \frac{d}{d\phi}\right)\cos^{n_3}\phi L_q^{\alpha+1} L_{n_1}^{\beta}\, \mathrm{d}\phi \\ &= -[(q+\alpha+1)(q-\alpha)]^{-\frac{1}{2}} \\ &\quad\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n_3}\phi L_q^{\alpha+1}\left(-(\beta-\tfrac{1}{2})\tan\phi + \frac{d}{d\phi}\right) L_{n_1}^{\beta}\, \mathrm{d}\phi \\ &= -[(q+\alpha+1)(q-\alpha)]^{-\frac{1}{2}} \\ &\quad\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n_3}\phi L_q^{\alpha+1}\,[(n_1+\beta)(n_1-\beta+1)]^{-\frac{1}{2}}\, L_{n_1}^{\beta-1}\, \mathrm{d}\phi\end{aligned} \tag{B.3.10}$$

If we apply this calculation $q - \alpha$ times we will find that

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_3}^{m_3} L_q^{\alpha} L_{n_1}^{\beta}\, \mathrm{d}\phi &= (-1)^{q-\alpha}\eta\left(\frac{(n_1+\beta)!(n_1+q-n_3)!(q+\alpha)!}{(n_1-q+n_3)!(n_1-\beta)!(q-\alpha)!(2q)!}\right)^{\frac{1}{2}} \\ &\quad\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n_3}\phi L_q^{q} L_{n_1}^{n_3-q}\, \mathrm{d}\phi\end{aligned}$$

Substituting the known expression for L_n^n into this equation it becomes

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_3}^{m_3} L_q^{\alpha} L_{n_1}^{\beta}\, \mathrm{d}\phi = (-1)^{q-\alpha}\eta \left(\frac{(n_1+\beta)!(n_1+q-n_3)!(q+\alpha)!}{(n_1-q+n_3)!(n_1-\beta)!(q-\alpha)!(2q)!}\right)^{\frac{1}{2}}$$
$$\times \left(\frac{1.3.5\ldots 2q+1}{2.2.4\ldots 2q}\right)^{\frac{1}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n_3+q+\frac{1}{2}}\phi L_{n_1}^{n_3-q}\, \mathrm{d}\phi \quad \text{(B.3.11)}$$

To examine this integral we now re-examine equation (3.3.3). Multiplying this equation across by $\cos^l\phi$ and examining the first term gives

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^l\phi \frac{d^2L_n^m}{d\phi^2}\, \mathrm{d}\phi = \cos^l\phi \frac{dL_n^m}{d\phi}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + l\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{l-1}\phi \sin\phi \frac{dL_n^m}{d\phi}\, \mathrm{d}\phi$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} l(l-1)\cos^{l-2}\phi L_n^m\, \mathrm{d}\phi - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} l^2\cos^l\phi L_n^m\, \mathrm{d}\phi$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{l-2}\phi(m^2-\tfrac{1}{4})L_n^m - \left[n(n+1)+\tfrac{1}{4}\right]\cos^l\phi L_n^m\, \mathrm{d}\phi$$

From this equation we can see that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{l-2}\phi L_n^m\, \mathrm{d}\phi = \frac{(l+n+\frac{1}{2})(l-n-\frac{1}{2})}{(l+m-\frac{1}{2})(l-m-\frac{1}{2})}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^l\phi L_n^m\, \mathrm{d}\phi \quad \text{(B.3.12)}$$

If we now examine

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^l\phi L_n^m\, \mathrm{d}\phi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^l\phi\,[(n+m+1)(n-m)]^{-\frac{1}{2}}\left(-(m+\tfrac{1}{2})\tan\phi + \frac{d}{d\phi}\right)L_n^{m+1}\, \mathrm{d}\phi$$
$$= [(n+m+1)(n-m)]^{-\frac{1}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\left(-(m+\tfrac{1}{2})\tan\phi - \frac{d}{d\phi}\right)\cos^l\phi\right]L_n^{m+1}\, \mathrm{d}\phi$$
$$= [(n+m+1)(n-m)]^{-\frac{1}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -(m-l+\tfrac{1}{2})\sin\phi\cos^{l-1}\phi L_n^{m+1}\, \mathrm{d}\phi$$
$$= [(n+m+1)(n-m)(n+m+2)(n-m-1)]^{-\frac{1}{2}}(m-l+\tfrac{1}{2})$$
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[(m-l+\tfrac{5}{2})\cos^{l-2}\phi + (l-m-\tfrac{3}{2})\cos^l\phi\right]L_n^{m+2}\, \mathrm{d}\phi$$

Applying equation (B.3.12) to this we get

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^l \phi L_n^m \,\mathrm{d}\phi = \left(\frac{m-l+\frac{1}{2}}{l+m+\frac{3}{2}}\right)\left(\frac{(n+m+2)(n-m-1)}{(n+m+1)(n-m)}\right)^{\frac{1}{2}} \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^l \phi L_n^{m+2}\,\mathrm{d}\phi \qquad \text{(B.3.13)}$$

If we now apply this $\frac{q-n_3+n_1}{2}$ times to equation (B.3.11) we find that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{q+n_3+\frac{1}{2}} \phi L_{n_1}^{n_3-q}\,\mathrm{d}\phi = (-1)^{\frac{n_1-n_3+q}{2}}\left(\frac{(2q)!!(2n_3)!!}{(n_3+q-n_1)!!(q+n_1+n_3)!!}\right) \times \left(\frac{(2n_1)!!(n_1-n_3+q-1)!!(n_1+n_3-q-1)!!}{(n_1+n_3-q)!!(n_1-n_3+q)!!(2n_1-1)!!}\right)^{\frac{1}{2}} \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n_3+q+\frac{1}{2}} L_{n_1}^{n_1}\,\mathrm{d}\phi \qquad \text{(B.3.14)}$$

where $n!! = n(n-2)(n-4)\ldots 2$ or $1 \quad 0!! = (-1)!! = 1$

Noting that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n_3+q+\frac{1}{2}} \phi L_{n_1}^{n_1}\,\mathrm{d}\phi = \left(\frac{1.3\ldots 2n_1+1}{2.2.4\ldots 2n_1}\right)^{\frac{1}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n_1+q+n_3+1}\phi\,\mathrm{d}\phi$$
$$= \left(\frac{1.3\ldots 2n_1+1}{2.2.4\ldots 2n_1}\right)^{\frac{1}{2}} \frac{(n_1+q+n_3)(n_1+q+n_3-2)\ldots 2.2}{(n_1+q+n_3+1)(n_1+q+n_3-1)\ldots 3.1}$$

gives a final expression for equation (B.3.14). Using this we can now see that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n_3} \phi L_q^{\alpha} L_{n_1}^{\beta} = (-1)^{\frac{n_1-n_3-q}{2}+\alpha}\left((2q+1)(2n_1+1)\frac{(n_1+\beta)!(q+\alpha)!}{(n_1-\beta)!(q-\alpha)!}\right)^{\frac{1}{2}} \times \frac{(2n_3)!!(n_1+q-n_3-1)!!}{(n_3+q-n_1)!!(n_1+q+n_3+1)!!(n_1+n_3-q)!!} \qquad \text{(B.3.15)}$$

Filling this into the main equation which we were examining (B.3.4), we find that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_1}^{m_1} P_q^{m_2} P_{n_3}^{m_3} \cos\phi\,\mathrm{d}\phi = \frac{(n_1+q-n_3-1)!!\left[\frac{1}{2}(2n_1+1)(2q+1)(2n_3+1)\right]^{\frac{1}{2}}}{(n_3+q-n_1)!!(n_1+q+n_3+1)!!(n_1+n_3-q)!!} \times \left(\frac{(n_3+m_3)!(n_1-m_1)!(q-m_2)!(n_3-m_3)!}{(n_1+m_1)!(q+m_2)!}\right)^{\frac{1}{2}} \times \sum_{i=0}^{n_3-m_3} \frac{(-1)^{\frac{q-n_3-n_1}{2}+m_1+i}(n_1+m_1+i)!(q+n_3-m_1-i)!}{i!(n_3-m_3-i)!(q-n_3+m_1+i)!(n_1-m_1-i)!} \qquad \text{(B.3.16)}$$

Finally to bring this all together we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_3}^{m_3}\left(im_2 P_{n_2}^{m_2}\frac{dP_{n_1}^{m_1}}{d\phi} - im_1 P_{n_1}^{m_1}\frac{dP_{n_2}^{m_2}}{d\phi}\right)\mathrm{d}\phi$$

$$\begin{aligned}
&= im_1\left(\frac{(n_2-m_2)!(2n_2+1)}{(n_2+m_2)!}\right)^{\frac{1}{2}}\sum_q\left(\frac{(2q+1)(q+m_2)!}{(q-m_2)!}\right)^{\frac{1}{2}} \\
&\quad\times\frac{(q+n_1-n_3-1)!!\,[(2n_1+1)(2q+1)(2n_3+1)]^{\frac{1}{2}}}{(n_1+n_3-q)!!(n_1+q+n_3+1)!!(q+n_3-n_1)!!} \\
&\quad\times\left(\frac{(n_3+m_3)!(n_1-m_1)!(q-m_2)!(n_3-m_3)!}{2(n_1+m_1)!(q+m_2)!}\right)^{\frac{1}{2}} \\
&\quad\times\sum_{i=0}^{n_3-m_3}\frac{(-1)^{\frac{q-n_1-n_3}{2}+m_1+i}(n_1+m_1+i)!(q+n_3-m_1-i)!}{i!(n_3-m_3-i)!(q-n_3+m_1+i)!(n_1-m_1-i)!} \\
&- im_2\left(\frac{(n_1-m_1)!(2n_1+1)}{(n_1+m_1)!}\right)^{\frac{1}{2}}\sum_p\left(\frac{(2p+1)(p+m_1)!}{(p-m_1)!}\right)^{\frac{1}{2}} \\
&\quad\times\frac{(n_2+p-n_3-1)!!\,[(2p+1)(2n_2+1)(2n_3+1)]^{\frac{1}{2}}}{(n_3+p-n_2)!!(p+n_2+n_3+1)!!(n_2+n_3-p)!!} \\
&\quad\times\left(\frac{(n_3+m_3)!(p-m_1)!(n_2-m_2)!(n_3-m_3)!}{2(p+m_1)!(n_2+m_2)!}\right)^{\frac{1}{2}} \\
&\quad\times\sum_{i=0}^{n_3-m_3}\frac{(-1)^{\frac{p-n_2-n_3}{2}+m_1+i}(n_2+m_2+i)!(p+n_3-m_2-i)!}{i!(n_3-m_3-i)!(p-n_3+m_2+i)!(n_2-m_2-i)!}
\end{aligned} \qquad \text{(B.3.17)}$$

where

$$q = n_2-1, n_2-3, \ldots, m_2+1 \quad \text{or} \quad m_2$$

$$p = n_1-1, n_1-3, \ldots, m_1+1 \quad \text{or} \quad m_1$$

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