Pricing and Hedging of Asian Options: Quasi-Explicit Solutions via Malliavin Calculus

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Abstract

We use Malliavin calculus and the Clark–Ocone formula to derive the hedging strategy of an arithmetic Asian Call option in general terms. Furthermore we derive an expression for the density of the integral over time of a geometric Brownian motion, which allows us to express hedging strategy and price of the Asian option as an analytic expression. Numerical computations which are based on this expression are provided.

Keywords: Asian options, option pricing, hedging, Malliavin calculus.

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1 Introduction

Asian options are options where the payoff depends on the average of the underlying asset during at least some part of the life time of the option. The average can be taken in several ways, each leading to a different type of Asian option. According to Nelken [16] the name Asian option was coined by employees of Bankers Trust, which sold this type of options to Japanese firms who wanted to hedge their foreign currency exposure. These firms used Asian options because their annual reports were also based on average exchange rates over the year. Average type options are particularly suited to hedge risk at foreign exchange markets and are significantly cheaper than plain vanilla options by reason of the averaging effect. Effectively such options are traded since the mid 1980's and first appeared in the form of commodity linked bonds. Specific examples are the Mexican Petro Bond and the Delaware Gold Index Bond. Asian options are mostly OTC traded, however market and trading volume appear to grow very rapidly. A recent study of CIBC world markets revealed that Asian style options are the most commonly traded exotic options. Similar statements can be found in Nelken [16].

In the Black-Scholes model, the technically easiest case to consider is where the average is a geometric average. Since the product of log-normal distributed random variables is again log-normal distributed, explicit analytical expressions are available for the price of such options as well as their hedging strategies. We therefore focus on the case of arithmetic Asian options. In particular, we study the case of a continuous arithmetic average price call

$$\left(\frac{1}{T}\int_0^T S(t)\,dt - K\right)^+,$$

which is arguable the most important and most popular case. We note, however, that all results can be adapted to cover other cases of Asian options.

Of fundamental importance is the question how these options should be priced. In a standard Black-Scholes framework the unique arbitrage free price is given by the discounted expectation of the payoff under the unique risk-neutral measure. Alternatively a PDE similar to the classical Black-Scholes equation can be found. In contrast to the classical case of a European call, however, it is far more difficult to derive practically useful expressions for the discounted expectation or the solution of the corresponding partial differential equations. In fact, Rogers and Shi [18] pointed out that the existence of a closed form solution to its valuation would be impossible. Consequently, it would also be impossible that a closed form solution for a hedging strategy exists.

However, it is slightly subjective as to what is considered to be a closed form and what not in Mathematical Finance. An analytic formula involving a triple integral and a complicated density expression might be closed form for one but not closed form for another. From a practitioner's point of view, closed form is considered to be something that can easily be implemented and leads to numerically highly accurate results, such as the Black-Scholes formula for example. For this reason, the more complicated looking closed form solutions are often termed quasi-explicit solutions. Geman and Yor (see [7] and [8]) were among the first to derive integral representations for the price of Asian options. More precisely, they derive an explicit expression for the Laplace transform of the price of an Asian option with respect to the duration time T, see also Carr and Schröder [4] for an excellent exposition on this subject. It is not possible to analytically invert this Laplace transformation, but powerful numerical schemes have been developed, see for example Fu et al. [6]. On the other hand, further numerical procedures addressing either Monte Carlo simulation or various approximations of Asian options have been studied by Kemna and Vorst [12], Turnbull and Wakeman [20], Rogers and Shi [18], and Simon, Goovaerts, and Dhaene [19] to mention only a few. For a more comprehensive literature review we refer to the excellent survey article of Boyle and Potapchik [2].

With the exception of Boyle and Potapchik[1] all of the authors above address

[1]Boyle and Potapchik [2] provide a number of Monte Carlo estimators for the Greeks of an Asian option.

exclusively the pricing aspect, but not the hedging aspect. In fact, the aspect of hedging an Asian option does not seem to be very well studied in the literature. To our knowledge, the only references dealing with the hedging strategy of Asian options are Albrecher et al. [1] and Jacques [15].

Albrecher et al. [1] consider an incomplete market model (by considering Lévy processes); thus they cannot get a hedging strategy but only a static superhedging strategy for the payoff structure of the Asian option. They do so by using that the price of a (discretely sampled) Asian option can be approximated from above by a (static) portfolio of European call options with strike prices κ_i satisfying $\kappa_1 + ... + \kappa_n = nK$, where K is the strike price of the Asian option and n is the number of samples in the average. The approximation is very good if the Asian option is deep in–the–money but features a large relative error within the magnitude of 50% and more if the Asian option is out–of–the–money[2][3].

Jacques [15] on the other side considers a discrete arithmetic Asian option on an underlying risky asset where the price dynamics is modeled as a geometric Brownian motion. He uses the Edgeworth expansion introduced by Turnbull and Wakeman [20] (originally to price Asian options) in order to approximate

[2]see Albrecher et al. [1], Exhibit 3 on page 70, last two rows display relative errors of more than 50%.

[3]Intuitively, as Albrecher et al.'s hedge is static, it features low transaction costs. However, in the case of weekly averaging over a one year period, the hedging agent needs to purchase 52 European call options, leading to similar transaction costs as a dynamic hedge in one underlying, which is updated weekly. Additionally, looking at Exhibit 2 in Albrecher et al. [1] (see page 68, last collumn) reveals that the strike prices κ_i for the Europan calls used in the static hedge are all very close to each other. Specifically for the at the money case one has $\kappa_i \in [99.79, 100.13]$ for all $i = 1, ..., n$. Under the realistic assumption that not all strikes are liquid, and e.g. options with strike $95, 100, 105, 110$ are traded, the static hedge proposed by Albrecher et al. would practically consist of holding one European call option only, that is the one with strike 100, i.e. $\kappa_i = 100$ for all $i = 1, ..., n$. In that case the hedging results obtained by Albrecher et al. are worse, ranging from 5% relative error for the in–the–money Asian options to far off (1.7243 instead of actual 0.2342) for the out–of–the–money case (compare Albrecher et al. [1], Exhibit 7, last collumn EC, page 71). On the other side, whether liquid or not, in the Black-Scholes context, the European call options required in Albrecher et al. (at all strikes) can be created artificially using Black-Scholes EC delta hedges (which can be computed explicitly), and a weighted average of these can be used to superhedge the Asian option. The hedge however is then no longer static, but in this form compares well to ours.

the delta and hence the hedging strategy, dynamically updating the fit of the approximating distribution (log-normal and inverse Gaussian). Jacques then studies the performance of the hedging strategy when hedging the approximated option price rather than the actual option price.[4] By comparison, we study a continuous arithmetic Asian option on an underlying risky asset where the price dynamics is modeled as a geometric Brownian motion. Hence, it is possible to compare the approach of Jacques with our approach.

Both approaches, that is Albrecher et al. [1] and Jacques [15], are biased, while ours is not. Our approach is stable, whether the option is in–the–money, at–the–money or out–of–the–money, whether long or short times to maturities, high or low volatilities are present. It is also straightforward to obtain numerical results. Albrecher et al.'s and Jacques's hedging strategy have their own justification and appeal. In practice all three of them could be used with different weights, or using one to control for the other.

For any option, the hedging problem is usually more complex than the pricing problem. In fact, a solution of the hedging problem determines an arbitrage free price via the initial value of the hedging strategy. In this paper, we derive an analytic formula, i.e. a quasi-explicit formula, for the hedging strategy of an arithmetic Asian call option, by using standard techniques from Malliavin calculus and in particular the Clark–Ocone formula. As a byproduct of the hedging strategy we obtain a quasi-explicit expression for the price of an Asian option as indicated above. Note that our derivation via the hedging strategy using Malliavin calculus is conceptually quite different from other approaches – even though our pricing formula is not significantly less complex than those derived by other authors. On the other hand, knowing the actual hedging strategy is of fundamental importance for sellers and buyers of Asian options. In order to demonstrate that our formula is indeed applicable, we include various numerical examples at the end. Finally, we compare our hedging strategy with

[4] Compare Jacques [15], row 1 in Table 1, page 169, Table 3 and Jacques's statement that the error is in relation to the option price 12.68 (4 lines up from the table).

the approach of Jacques [15].

The remainder of this article is structured as follows. In section 2, we set up our model and outline the problem, while in section 3, we provide a brief review of some techniques of Malliavin Calculus. In section 4, we derive an expression for the density function of the time integral of geometric Brownian motion as well as a differential equation which characterizes it. In section 5, we derive our main results, while section 6 gives our numerical examples. The main conclusions are summarized in section 7.

2 Model Setup

We consider the standard Black-Scholes framework, i.e. a model consisting of a risk-free bond $B(\cdot)$ and a risky stock $S(\cdot)$ with dynamics

$$\begin{aligned} dB(t) &= B(t)r\,dt && (1)\\ dS(t) &= S(t)\left[r\,dt + \sigma\,dW(t)\right], && (2) \end{aligned}$$

where $W(\cdot)$ denotes a standard one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, with $(\mathcal{F}(t))_{t\in[0,T]}$ being the canonical Brownian filtration. Here, we assume without loss of generality that the measure $\mathbb{P}$ is in fact the risk-neutral measure. It is well known that the dynamics (2) provides

$$S(t) = s \cdot \exp\left(rt - \frac{\sigma^2}{2}t + \sigma W(t)\right), \quad 0 \le t \le T \tag{3}$$

with $S(0) = s$. Let us denote the investor's wealth and the amount invested in the stock at time t by $V(t)$ and $\pi(t)$, respectively. The amount invested in the bond is given by $V(t) - \pi(t)$. We call $\pi(\cdot)$ an investment strategy. As the wealth process clearly depends on the investment strategy we write $V^{\pi}(\cdot)$ in the following. We assume that the investor follows a self-financing investment strategy with initial wealth $V(0) = v_0$. Assuming this, the wealth process satisfies

$$dV^{\pi}(t) = rV^{\pi}(t)\ dt + \pi(t)\sigma\, dW(t), \quad 0 \leq t \leq T. \tag{4}$$

Due to the stochastic integral on the one hand and to exclude arbitrage on the other hand we must impose the following two conditions on the investment strategy for technical reasons:

$$\int_0^T \pi^2(t)\, dt < \infty \quad \mathbb{P} - a.s. \tag{5}$$

$$\mathbb{P}\left(\{V^{\pi}(t) > 0 \quad \forall t \geq 0\}\right) = 1. \tag{6}$$

In particular, the first condition ensures that the SDE in (4) has a unique strong solution, given by

$$V^{\pi}(t) \cdot \exp(-rt) = v_0 + \int_0^t \exp(-ru)\, \pi(u)\sigma\, dW(u), \quad \text{for } 0 \leq t \leq T, \tag{7}$$

see for example Korn [13], pages 54–55. Now consider the payoff at maturity of the arithmetic Asian option

$$f_T = \left[\frac{1}{T}\int_0^T S(t)\, dt - K\right]^+ \tag{8}$$

with $K > 0$ being the strike. A hedging strategy of this option is a self-financing strategy $\pi(\cdot)$ such that

$$V^{\pi}(T) = f_T$$

holds $\mathbb{P}$-a.s.. It follows from Theorem 1.2.1 in Karatzas [9] that there exists a unique hedging strategy and, moreover, the initial investment that needs to be undertaken to finance this hedge is given by

$$v_0 = \mathbb{E}\left[f_T \cdot \exp(-rT)\right]. \tag{9}$$

The value v_0 is then the only price of the option at time $t = 0$, which does not allow for arbitrage, and is therefore called the fair price of the option f_T at time 0. In the remainder of this article we determine a quasi-explicit formula for the hedging strategy $\pi(\cdot)$ and the fair price of an arithmetic Asian call option, denoted by either $V^\pi(t)$ or simply $V(t)$ with $V^\pi(0) = v_0$.

3 A brief review on Malliavin calculus

Let us review some of the basic concepts of Malliavin calculus. A standard reference for this is e.g. Nualart [17]. We consider the set $\mathcal{S}$ of cylindrical functionals $F : \Omega \to \mathbb{R}$, given by $F = f\left(W(t_1), ..., W(t_l)\right)$ where $f \in C_b^\infty\left(\mathbb{R}^l\right)$ is a smooth function with bounded derivatives of all orders and $(W(t))$ denotes a Brownian motion on Ω.[5] We define the Malliavin derivative operator on $\mathcal{S}$ via

$$D_s F \quad := \quad \sum_{i=1}^{l} \frac{\partial f}{\partial x_i} \left(W(t_1,\omega), ..., W(t_l,\omega)\right) \cdot \mathbf{1}_{[0,t_i]}(s).$$

This operator and the iterated operators D^n are closable and unbounded from $L^p(\Omega)$ into $L^p\left(\Omega \times [0,T]^n\right)$, for all $n \geq 1$. Their respective domains are denoted by $\mathbb{D}^{n,p}$ and obtained as the closure of $\mathcal{S}$ with respect to the norms defined by $\|F\|_{n,p}^p = \|F\|_{L^p(\Omega)}^p + \sum_{k=1}^n \left\|D^k F\right\|_{L^p\left(\Omega\times[0,T]^k\right)}^p$. In the following we concentrate on the Hilbert space $\mathbb{D}^{1,2}$, which will be the relevant space for all of our computations. We need the following version of the chain rule for Malliavin calculus:

Proposition 1. *Let $\varphi \in C^1(\mathbb{R})$ be a continuously differentiable function and let $F \in \mathbb{D}^{1,2}$. Then $\varphi(F) \in \mathbb{D}^{1,2}$ if and only if $\varphi(F) \in L^2(\Omega)$ and $\varphi'(F)DF \in L^2(\Omega \times [0,T])$ and under these hypotheses*

$$D\left[\varphi(F)\right] = \varphi'(F)DF. \tag{10}$$

[5] Within the context of this article, we can assume w.l.o.g. that $\Omega = \mathcal{C}_0([0,T])$ is the space of continuous functions $\omega : [0,T] \to \mathbb{R}$ satisfying $\omega(0) = 0$.

If φ is not C^1 but globally Lipschitz with constant K, then $\varphi(F)$ is still in $\mathbb{D}^{1,2}$ and there exist a random variable G, which is bounded by K, such that

$$D\left[\varphi(F)\right] = GDF. \tag{11}$$

Proof. This is a combination of Lemma 2.1. in Leon et al. [14] and Proposition 1.2.3 in Nualart [17]. $\square$

The following proposition presents the Clark-Ocone formula which is the main link between hedging and Malliavin calculus. It is identical to Proposition 1.3.5 in Nualart [17].

Proposition 2. *Let $F \in \mathbb{D}^{1,2}$, then*

$$F = \mathbb{E}\left[F\right] + \int_0^T \mathbb{E}\left[D_t F \mid \mathcal{F}(t)\right] dW(t). \tag{12}$$

For our application we will later need to compute the Malliavin derivative of the payoff of an arithmetic average Asian option. The following proposition will help us with that.

Proposition 3. *Define $g_T = \frac{1}{T}\int_0^T S(u)\,du - K$ with $S(\cdot)$ given by (3) and $K \in \mathbb{R}$. Then $g_T \in \mathbb{D}^{1,2}$ and*

$$D_t(g_T) = \sigma g_T + \sigma K - \frac{\sigma}{T}\int_0^t S(u)\,du. \tag{13}$$

Proof. It follows from Fournie et al. [5], Property P2 by choosing $X^1(t) = S(t)$ and $X^2(t) = \int_0^t S(u)\,du$ and identifying the first variation process $Y(\cdot)$ of this system as

$$Y(t) = \begin{pmatrix} \frac{S(t)}{S(0)} & 0 \\ \int_0^t \frac{S(u)}{S(0)}\,du & 1 \end{pmatrix}$$

after performing elementary linear algebra operations that, for all $t \in [0,T]$, one has $S(t) \in \mathbb{D}^{1,2}$ and $\int_0^t S(u)\,du \in \mathbb{D}^{1,2}$ with

$$\begin{aligned} D_s S(t) &= \sigma S(t) \cdot \mathbf{1}_{[s \le t]} \\ D_s \left(\int_0^t S(u)\,du \right) &= \left(\int_s^t \sigma S(u)\,du \right) \mathbf{1}_{[s \le t]}. \end{aligned}$$

As the Malliavin derivative is additive and zero on constants, we have that

$$\begin{aligned} D_t g_T &= D_t \left(\frac{1}{T} \int_0^T S(u)\,du - K \right) \\ &= \frac{1}{T} \int_t^T \sigma S(u)\,du \\ &= \sigma g_T - \frac{\sigma}{T} \int_0^t S(u)\,du + \sigma K. \end{aligned}$$

□

4 On the density of the arithmetic Asian average

In this section, we study the density function of the arithmetic average of geometric Brownian motions. We present a rather direct approach resulting in a quasi-explicit formula. In addition to this we derive a PDE for the distribution function. The motivation for this is as follows. As indicated in the introduction, (quasi)-explicit is a relative term, and even though we obtain an integral representation, numerical integration needs to be carried out to compute the expression. This is the same for the formulas derived by Yor et al. [7]. On the other side, powerful methods for the solution of partial differential equations are available, and deriving the density function from this might at least be a good alternative. Yet another alternative is to use a Monte Carlo approach.

Proposition 4. *For $t > 0$, denote by $p(t, x, a, b)$ the probability density of*

$\int_0^t \exp(au + bW(u))\, du$. *Then* $p(t,x,a,b) = 0$ *for* $x \le 0$, *and*

$$p(t,x,a,b) = \Gamma_t(x) \int_0^\infty \Psi_t(v) \left[\int_0^\infty y^{\frac{2a}{b^2}} \exp\left(-\frac{2}{b^2 x} \left[y^2 + 2y\cosh(v) + 1 \right] \right) dy \right] dv$$

for $x > 0$, *where*

$$\begin{aligned} \Gamma_t(x) &= 8 \left(\pi b^3 x^2 \sqrt{2\pi t} \right)^{-1} \exp\left(\frac{4\pi^2 - (at)^2}{2b^2 t} \right) \quad \textit{and} \\ \Psi_t(v) &= \sin\left(\frac{4\pi v}{b^2 t} \right) \cdot \sinh(v) \cdot \exp\left(-\frac{2v^2}{b^2 t} \right). \end{aligned}$$

Proof. Denote by $U_t(x; a, b)$ the probability distribution function

$$U_t(x; a, b) = \mathbb{P}\left(\int_0^t \exp(au + bW(u))\, du \ge x \right).$$

According to Lemma 9.4 in Karatzas et al. [10], one has

$$U_t(x; a, b) = \mathbb{P}\left(\int_0^{\frac{b^2 t}{4}} \exp\left(2 \left(\frac{2au}{b^2} + W(u) \right) \right) du \ge \frac{b^2 x}{4} \right).$$

Obviously,

$$U_t(x; a, b) = \int_\Omega \mathbf{1}_A \, d\mathbb{P},$$

where

$$A = \left\{ \omega \,\middle|\, \int_0^{\frac{b^2 t}{4}} \exp\left(2 \left(\frac{2au}{b^2} + W(u) \right) \right) du \ge \frac{b^2 x}{4} \right\}.$$

Define an equivalent measure $\widetilde{\mathbb{P}}$ such that

$$\left. \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{F}_s} = \exp\left(-\frac{2a^2}{b^4} s - \frac{2a}{b^2} W(s) \right). \tag{14}$$

Girsanov's theorem shows that $(\tilde{W}(t))$ defined by $\tilde{W}(t) = W(t) + \frac{2at}{b^2}$ is a Brownian motion under $\tilde{\mathbb{P}}$. Since

$$A = \left\{ \omega \,\middle|\, \int_0^{\frac{b^2 t}{4}} \exp\left(2\tilde{W}_u\right)\, du \geq \frac{b^2 x}{4} \right\},$$

setting $s = b^2 t/4$ in equation (14) gives

$$U_t(x; a, b) = \int_\Omega \mathbf{1}_A \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}\, d\tilde{\mathbb{P}} = \int_\Omega \mathbf{1}_A \exp\left(\frac{2a}{b^2} \tilde{W}_{\frac{b^2 t}{4}} - \frac{2a^2}{b^4} \left(\frac{b^2 t}{4} \right) \right) d\tilde{\mathbb{P}}.$$

Let $f_t(x, y)$ denote the joint density function of $\left(\int_0^t \exp(2\tilde{W}_u)\, du, \tilde{W}_t \right)$. Then

$$U_t(x; a, b) = \exp\left(-\frac{a^2 t}{2b^2} \right) \int_{\frac{b^2 x}{4}}^\infty \int_{-\infty}^\infty \exp\left(\frac{2ay}{b^2} \right) f_{\frac{b^2 t}{4}}(v, y)\, dy\, dv.$$

The joint density $f_t(x, y)$ has a closed form solution: It is a direct consequence of equation (6.c) in Yor [21] that $f_t(x, y) = 0$ for $x \leq 0$, and

$$f_t(x, y) = \rho_t(x, y) \int_0^\infty \exp\left(-\frac{z^2}{2t} - \frac{\exp(y)}{x} \cosh(z) \right) \sinh(z) \sin\left(\frac{\pi z}{t} \right) dz \quad (15)$$

for $x > 0$, where

$$\rho_t(x, y) = \left(x^2 \sqrt{2\pi^3 t} \right)^{-1} \exp\left(\frac{2xyt + \pi^2 x - t - t\exp(2y)}{2xt} \right).$$

Since

$$p(t, x, a, b) = -\frac{\partial U_t(x; a, b)}{\partial x},$$

we obtain

$$p(t, x, a, b) = \exp\left(-\frac{a^2 t}{2b^2} \right) \left(\frac{b^2}{4} \right) \int_{-\infty}^\infty \exp\left(\frac{2ay}{b^2} \right) f_{\frac{b^2 t}{4}}\left(\frac{b^2 x}{4}, y \right) dy.$$

The expression for $p(t, x, a, b)$ stated in the Lemma is obtained by inserting the representation for $f_{\frac{b^2 t}{4}}(b^2 x/4, y)$ given in (15), substituting $\ln(y)$ for y and, finally, rearranging terms. $\square$

Let us now define

$$\begin{aligned} U(t,x) &\equiv \mathbb{P}\left[\int_0^t \exp\left(ru - \frac{\sigma^2}{2}u + \sigma W(u)\right) du > x\right] \\ &= U_t\left(x, r - \frac{\sigma^2}{2}, \sigma\right). \end{aligned} \tag{16}$$

Then it follows directly from (16) and the definition of $U_t(x,a,b)$ that

$$U(t,x) = \int_x^{+\infty} p\left(t, u, r - \frac{\sigma^2}{2}, \sigma\right) du \quad \text{for } t>0,\ x>0. \tag{17}$$

As indicated before this expression can be evaluated by running numerical integration. Additionally, we derive a partial differential equation characterizing this expression in the following proposition.

Proposition 5. *The function $U(\cdot,\cdot)$ is the unique solution to the following partial differential equation*

$$\frac{1}{2}\sigma^2 x^2 U_{xx} + \left[\left(\sigma^2 - r\right)x - 1\right] U_x - U_t = 0, \quad \textit{for } t>0,\ x>0, \tag{18}$$

with boundary conditions

$$\begin{aligned} U(0+,x) &= 0 \quad \textit{for } x>0 \\ U(0+,0) &= 1 \\ U(t,x) &= 1 \quad \textit{for } t \geq 0,\ x \leq 0\,. \end{aligned}$$

Proof. Using the notation

$$X(t) = \exp\left(-rt + \frac{\sigma^2}{2}t - \sigma W(t)\right) \cdot \left[x - \int_0^t \exp\left(ru - \frac{\sigma^2}{2}u + \sigma W(u)\right) du\right], \tag{19}$$

we find that

$$\int_0^t \exp\left(ru - \frac{\sigma^2}{2}u + \sigma W(u)\right)\, du > x \quad \text{if and only if } X(t) < 0,$$

and therefore that

$$U(t,x) = \mathbb{P}\left[X(t) < 0 \,|\, X(0) = x\right]. \tag{20}$$

On the other hand, (19) is the unique strong solution to the stochastic differential equation

$$dX(t) = \left[\left(\sigma^2 - r\right) X(t) - 1\right]\, dt - \sigma X(t)\, dW(t), \quad \text{for } X(0) = x > 0.$$

and therefore our statement follows from the Kolmogorov backward equation (see e.g. Equation 5.7 in Karlin and Taylor [11]). The initial and boundary value conditions are obvious. That the function $U(t,x)$ defined in (20) is sufficiently smooth follows from Theorem 2.3.3 in Nualart [17] (see page 133). □

5 Quasi-explicit hedge and price of the Asian option

Theorem 1. *The hedging strategy of the arithmetic Asian option f_T consists of investing the amount of*

$$\pi(t) = V^{\pi}(t) + \frac{S(t)}{T} \cdot G(t) \cdot \exp\left(-r(T-t)\right) \cdot U(T-t, G(t)) \tag{21}$$

at time $t \in [0,T]$ in the stock, with $U(\cdot,\cdot)$ given by (17) or (18), and

$$G(t) = \frac{1}{S(t)}\left[K \cdot T - \int_0^t S(u)\, du\right] = -\frac{T}{S(t)}\left[\frac{1}{T} \cdot \int_0^t S(u)\, du - K\right]. \tag{22}$$

The amount invested in the bond is

$$-\frac{S(t)}{T} \cdot G(t) \cdot \exp\left(-r(T-t)\right) \cdot U(T-t, G(t)). \tag{23}$$

Proof. It follows from Proposition 1 and Proposition 3 that $f_T \in \mathbb{D}^{1,2}$ and therefore that

$$f_T \exp\left(-rT\right) \in \mathbb{D}^{1,2}.$$

Furthermore,

$$\begin{aligned} D_t\left[f_T \exp\left(-rT\right)\right] &= \exp\left(-rT\right) D_t(f_T) &(24)\\ &= \left[f_T + \mathbf{1}_{(g_T>0)}(\omega)\left\{K - \frac{1}{T}\int_0^t S(u)\, du\right\}\right] \sigma \exp\left(-rT\right). \end{aligned}$$

We now conclude from Proposition 2 that

$$f_T \exp\left(-rT\right) = \mathbb{E}[f_T \exp\left(-rT\right)] + \int_0^T \mathbb{E}[D_t(f_T \exp\left(-rT\right)) \,|\, \mathcal{F}(t)]\, dW(t). \tag{25}$$

By definition of a hedging strategy and equation (7) the amount invested in the stock $\pi(\cdot)$ satisfies

$$f_T \exp\left(-rT\right) = \mathbb{E}[f_T \exp\left(-rT\right)] + \int_0^T \exp\left(-rt\right) \sigma\pi(t)\, dW(t). \tag{26}$$

Note, that we know from an arbitrage argument alone that the initial value of the hedging strategy must satisfy $v_0 = \mathbb{E}[f_T \exp\left(-rT\right)]$. The representation theorem of Wiener functionals (see for example Korn [13], page 71) states that the integrand in the representations (24) and (25) is unique and hence that

$$\pi(t) = \frac{1}{\sigma} \exp\left(rt\right) \cdot \mathbb{E}\left[D_t(f_T \exp\left(-rT\right)) \,|\, \mathcal{F}(t)\right]. \tag{27}$$

It follows automatically from the discussion above that this strategy is tame and a hedge in the sense defined in section 2. Furthermore (0.2.18), (0.2.19) in

Karatzas [9] together with (9) imply that $V^\pi(t)\exp(-rt)$ is a martingale and hence that

$$V^\pi(t)\exp(-rt) = \mathbb{E}[f_T \exp(-rT) \mid \mathcal{F}(t)] \quad \mathbb{P}-a.s.. \tag{28}$$

Hence by means of equation (24) and equation (28), we obtain

$$\pi(t) = V^\pi(t) + \exp(-r(T-t)) \cdot \left[K - \frac{1}{T}\int_0^t S(u)\,du\right] \cdot \mathbb{P}\left(g_T > 0 \,|\, \mathcal{F}(t)\right). \tag{29}$$

Using the properties of Brownian motion, it is not difficult to derive that

$$\begin{aligned} \mathbb{P}\left(g_T > 0 \,|\, \mathcal{F}(t)\right) &= \mathbb{P}\left(\int_t^T \exp\left(\left[r - \frac{\sigma^2}{2}\right]\cdot[u-t] + \sigma W_{u-t}\right) du > G(t) \,\middle|\, \mathcal{F}(t)\right) \\ &= U(T-t, G(t)). \end{aligned} \tag{30}$$

Finally, equation (21) follows from equation (29) and equation (30). Equation (23) follows from the self–financing condition. □

We have already indicated, that – for general reasons of arbitrage – $V^\pi(t)$ in Theorem 1 is the fair price of the arithmetic Asian option at time t. We provide the following two formulas (one of them quasi-explicit), which can be used to compute these values.

Theorem 2. *At any time t with $0 \le t \le T$, the fair price of the option f_T can be computed by either*

$$V(t) = \exp(-r(T-t))\,\frac{S(t)}{T}\int_{G(t)}^{+\infty} U(T-t, x)\,dx \tag{31}$$

or

$$V(t) = \exp(-r(T-t))\,\frac{S(t)}{T}\int_{G(t)}^{+\infty} (x - G(t)) \cdot p\left(T-t, x, r - \frac{\sigma^2}{2}, \sigma\right) dx, \tag{32}$$

where the function $U(\cdot,\cdot)$ is calculated by employing Proposition 5 or (17) to-

gether with Proposition 4, respectively.[6] *The expression $G(t)$ is defined in Theorem 1. In particular, the fair price of the arithmetic Asian option f_T at time $t = 0$ is given by*

$$V(0) = \exp(-rT)\,\frac{S(0)}{T}\int_{\frac{K\cdot T}{S(0)}}^{+\infty}\left(x - \frac{K\cdot T}{S(0)}\right)\cdot p\left(T, x, r - \frac{\sigma^2}{2}, \sigma\right)dx. \tag{33}$$

Proof. According to equation (28), we obtain

$$V(t) = \exp(-r(T-t))\,\mathbb{E}[f_T \,|\, \mathcal{F}(t)] \quad \mathbb{P} - a.s.. \tag{34}$$

Furthermore, it is obvious that

$$f_T = \frac{S(t)}{T}\left[\int_t^T \exp\left(\left[r - \frac{\sigma^2}{2}\right][u-t] + \sigma\left[W(u) - W(t)\right]\right)du - G(t)\right]^+. \tag{35}$$

Facilitating the properties of the Brownian motion, it follows from equation (35) and the definition of function $U(\cdot,\cdot)$ that

$$\mathbb{E}[f_T \,|\, \mathcal{F}(t)] = -\int_{G(t)}^{+\infty} S(t)\frac{-G(t)+x}{T}\,dU(T-t,x). \tag{36}$$

Therefore, we obtain equation (32) from equations (34), (36) and (18). For $t = 0$ we have equation (33). Finally, it is not difficult to verify that

$$\lim_{x\to+\infty} U(T-t,x) = 0, \qquad \text{and} \qquad \lim_{x\to+\infty} x\cdot U(T-t,x) = 0.$$

Thus, by means of the integration by parts formula, we derive that

$$\mathbb{E}[f_T \,|\, \mathcal{F}(t)] = \frac{S(t)}{T}\int_{G(t)}^{+\infty} U(T-t,x)\,dx$$

[6] We write $V = V^\pi$ here, as the emphasis is on the option value, rather than the value of the hedge.

and consequently, equation (31) also follows from equation (34). □

The following Corollary is directly implied by equation (31).

Corollary 1. *The amount invested in the stock is always non–negative. Especially, when $\int_0^t S(u)du > K \cdot T$, i.e. the Asian option is in the money, we have that*

$$\pi(t) = \exp\left(-r(T-t)\right)\frac{S(t)}{T}\int_{G(t)}^{+\infty} x \cdot p\left(T-t, x, r-\frac{\sigma^2}{2}, \sigma\right)\,dx. \tag{37}$$

Remark 1. *By means of equation (21) and equation (31), it is easy to verify that*

$$\pi(t) = S(t)\frac{\partial V(t)}{\partial S(t)}, \tag{38}$$

which means that the Asian option is Delta-hedged. This result can also be obtained by following the usual Black-Scholes like argument of creating a risk-free portfolio consisting of one arithmetic average Asian call option and Δ shares of the underlying stock or risky asset.

From the put-call-parity for arithmetic average Asian options, we obtain the following corollary.

Corollary 2. *We obtain the fair price at time $t \in [0,T]$ for an arithmetic average Asian put option via*

$$\bar{V}(t) = \exp\left(-r(T-t)\right)\frac{S(t)}{T}\int_0^{G(t)} (G(t)-x) \cdot p\left(T-t, x, r-\frac{\sigma^2}{2}, \sigma\right)\,dx. \tag{39}$$

The hedging strategy is given as follows. The amount invested in the stock is

$$\bar{\pi}(t) = \bar{V}(t) + G(t)\exp\left(-r(T-t)\right)\frac{S(t)}{T}[1-U(T-t, G(t))], \tag{40}$$

and the amount invested in the bond is $\bar{V}(t) - \bar{\pi}(t)$.

We also obtain the following expressions for the Greeks delta and gamma. These Greeks may not only be important for hedging, but also for a general analysis of risk associated to Asian options. The derivation is straightforward from the formulas presented here.

Proposition 6. *For the Greeks delta and gamma associated to an arithmetic average Asian call option we have*

$$
\begin{aligned}
\Delta &\equiv \frac{\partial V(t)}{\partial S(t)} \\
&= \frac{1}{S(t)}\left[V(t) + \frac{S(t)}{T}\cdot G(t)\cdot U(T-t, G(t))\cdot \exp\left(-r(T-t)\right)\right], \quad (41)\\
\Gamma &\equiv \frac{\partial^2 V(t)}{\partial^2 S(t)} \\
&= \frac{(G(t))^2}{TS(t)} p\left(T-t, G(t); r-\frac{\sigma^2}{2}, \sigma\right)\cdot \exp\left(-r(T-t)\right), \quad (42)\\
\frac{\partial V(t)}{\partial K} &= -U(T-t, G(t))\cdot \exp\left(-r(T-t)\right). \quad (43)
\end{aligned}
$$

We do not provide an expression for the vega of an Asian option here, but refer to Carr, Ewald, and Xiao [3] instead for an interesting result on the volatility dependence of Asian option prices.

6 Numerical Analysis

6.1 Estimating $U(t, x)$

Theoretically, there are three possibilities to calculate $U(t, x)$. First, using equation (17) to calculate $U(t, x)$ as the solution of a triple integral. Second, getting $U(t, x)$ as the solution of the partial differential equation (18) with the corresponding boundary conditions. The third possibility is using a Monte Carlo method to calculate $U(t, x)$ via equation (16).

Let us consider the first possibility. Even though there exists a closed form solution for U in form of equation (17), it entails calculating a triple integral. Additionally, even though the underlying function of $p\left(t, x, r - \frac{\sigma^2}{2}, \sigma\right)$ (see Proposition 4) is smooth, it has various singularities, numerically speaking.

In order to be more precise, let us define

$$f(t,x,y,v,a,b) = \Gamma_t(x) \cdot \Psi_t(v) \cdot y^{\frac{2a}{b^2}} \cdot \exp\left(-\frac{2}{b^2 x}\left[y^2 + 2y\cosh(v) + 1\right]\right),$$

where Γ_t and Ψ_t have been defined in Proposition 4. With this notation, we have

$$p(t,x,a,b) = \int_0^\infty \int_0^\infty f(t,x,y,v,a,b)\,dy\,dv \tag{44}$$

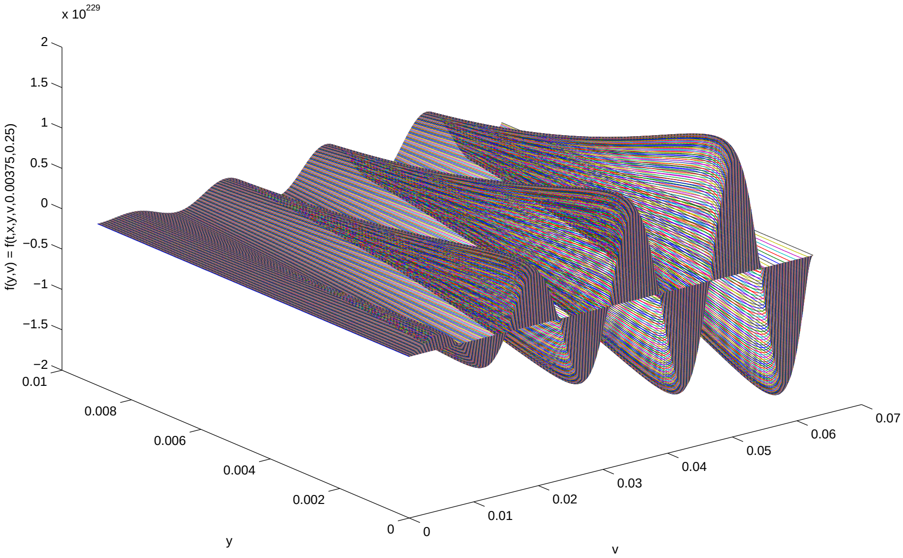
and

$$\begin{aligned} U(t,x) &= \int_x^\infty p\left(t,u,r-\frac{\sigma^2}{2},\sigma\right)du \quad \text{for } t>0, x>0 \\ &= \int_x^\infty \int_0^\infty \int_0^\infty f\left(t,u,y,v,r-\frac{\sigma^2}{2},\sigma\right)dy\,dv\,du\,. \end{aligned} \tag{45}$$

In this section, let us choose $r = 0.035$ and $\sigma = 0.25$, that is $r - \frac{\sigma^2}{2} = 0.00375$. Choosing $r = 0.03$ would lead to $r - \frac{\sigma^2}{2} = -0.00125$, which causes f to converge only slowly to zero in the y-variable for $y \to \infty$. The larger $r - \frac{\sigma^2}{2}$ is, the faster is the convergence.

Clearly, f is periodic in the v–variable with changing amplitude (see Figures 1 to 2). The period length is $\frac{\sigma^2 \cdot t}{2}$, that is, the smaller t – the time to maturity, the smaller the period length. The amplitude might increase at first, however, eventually it will decrease (compare with Figure 2). It is difficult to integrate f along v, because the values for which the local maximum and the local minimum in each period are attained is different (see Figure 2). This implies that the integral has to be calculated anew for each period. Moreover, if the amplitude is first increasing, the integral which is taken over the first periods is negative. Integrating over all periods the integral will be non–negative, however, in practice, one can only integrate over a finite number of periods which might lead to

Figure 1: The Function $f(0.5, 0.3, y, v, 0.00375, 0.25)$



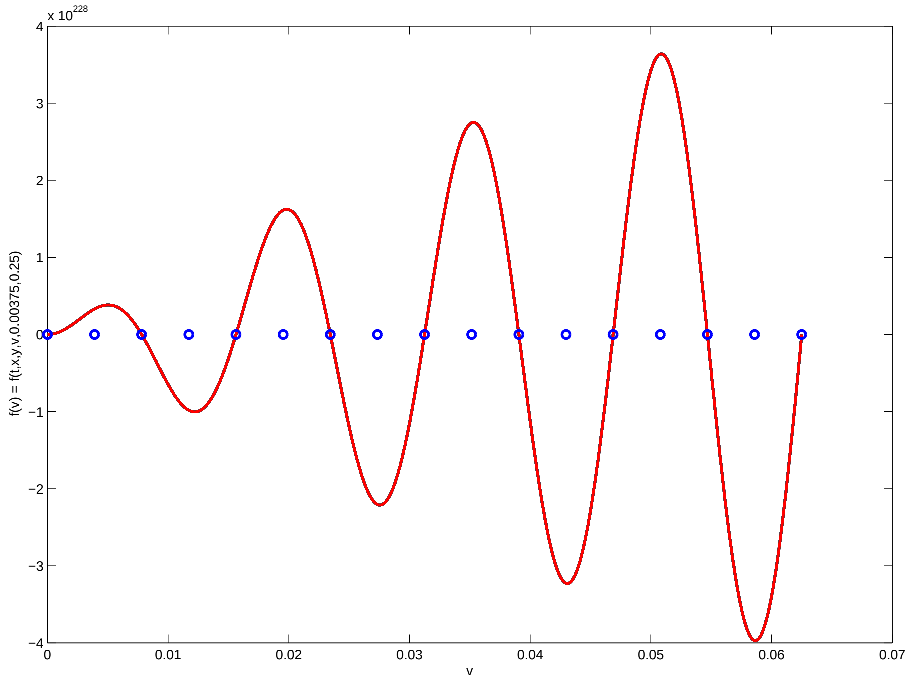
This figure shows the function f for $t = 0.5$, $u = 0.3$, $r = 0.035$, and $\sigma = 0.25$. Please be aware of the scales.

the situation that the integral is still negative (what is not possible, since it is a density). Using advanced numerical methods such as the Levin transformation did not remedy the situation.

Except for the large values, f is not too difficult to integrate over the y–variable. Note also that, the smaller the values are for u and t, the larger are the values of f. The chosen values for u and t in the figures ($u = 0.3$ and $t = 0.5$) are *by no means* the smallest values needed later. However, f gets already very large which makes it very difficult to get an accurate estimate for the integral. Altogether, we think that it is very difficult, not to say impossible, to get an accurate estimate for p using equation (44), let alone for U using equation (45).

Remember that the boundary condition for U is given by $U(0, x) = \mathbb{1}_{\{x<0\}}(x)$, which is discontinuous at zero in the x variable. Since any PDE solver needs

Figure 2: The Function $f(0.5, 0.3, 0.0089774, v, 0.00375, 0.25)$

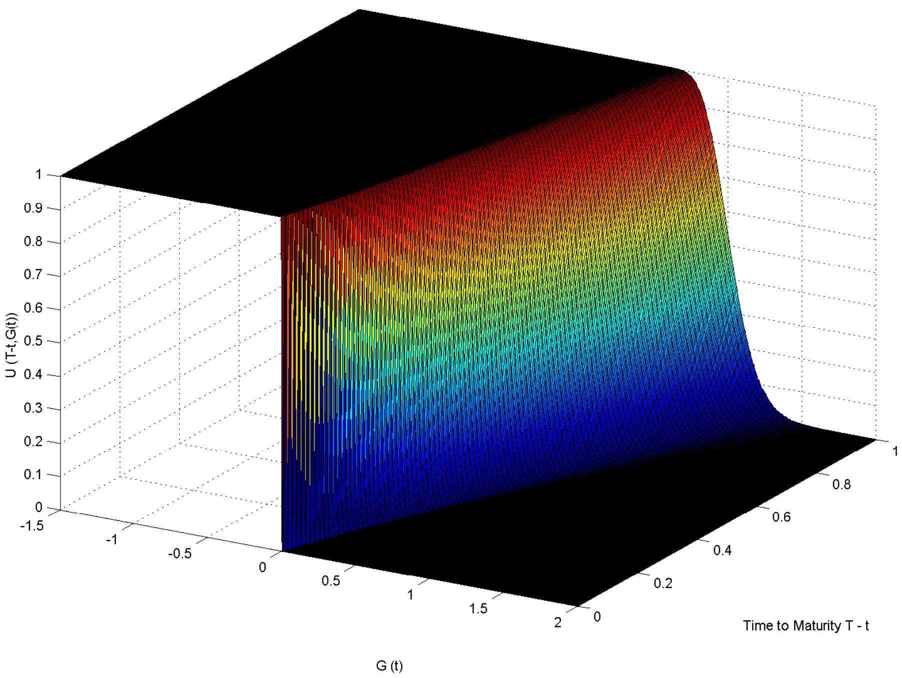


This figure shows the function f for $t = 0.5$, $u = 0.3$, $y = 0.0089774$, $r = 0.035$, and $\sigma = 0.25$. Please note the scales.

continuity in x for a good approximation, the discontinuity is the major source of error made by any PDE solver. This error can be reduced by using an upwind scheme together with an adapted mesh for the PDE solver. However, this error is intrinsic to the PDE solver and cannot be eliminated with any PDE solver. On the other hand, using a Monte Carlo method to estimate $U(t, x)$ via equation (16) is promising since it does not need any structure in the x variable. Note that equation (16) implies only a dependence structure in the t variable. However, since the random source is a Brownian motion, even the t dependency is non–existent. This means, that it is possible to estimate $U(t, x)$ for each t and x on its own (– although we will not use this property for our purposes, that is we will use the t dependency for stability reasons). Therefore, we used the Monte Carlo approach in the following, that is equation (16), to calculate $U(t,x)$ (see Figure 3) and $\int_{G(t)}^{\infty} U(T-t,x)\,dx$ (see Figure 4). In particular, note that it is easier to get a good estimate of the integral using the Monte Carlo approach then a PDE solver due to the properties discussed above.

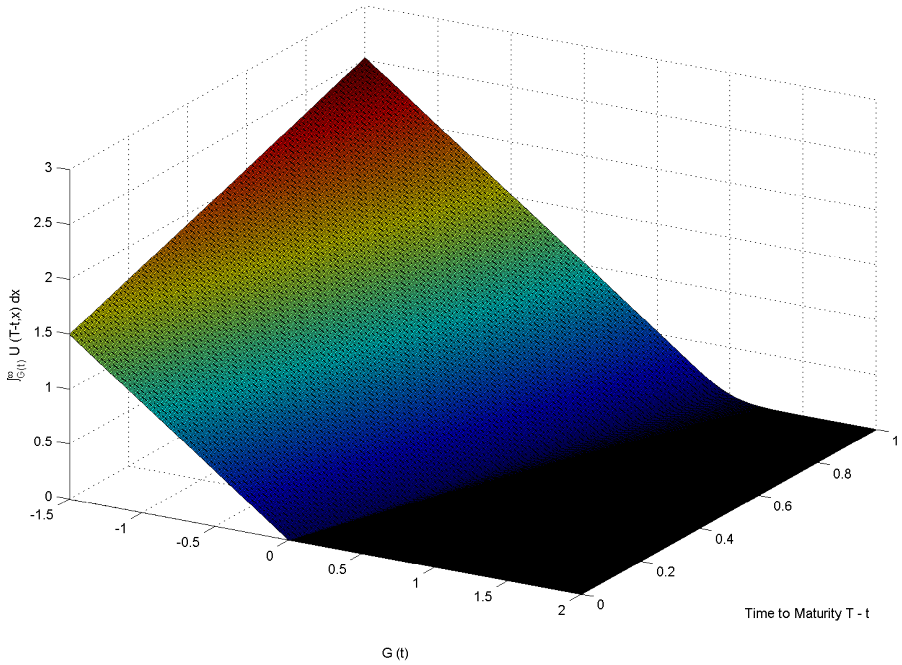
All figures in this paper are generated using Matlab. In particular, the random number generator used for the Monte Carlo method was the Mersenne Twister which is the default random number generator of Matlab.

Figure 3: The Function $U(T-t, G(t))$



This figure shows the function $U(T-t, G(t))$ for $r = 0.035$ and $\sigma = 0.25$. The function U has been approximated by using $n = 10,000$ simulations for each t and x.

Figure 4: The Function $\int_{G(t)}^{\infty} U(T-t,x)\,dx$



This figure shows the function $\int_{G(t)}^{\infty} U(T-t,x)\,dx$ for $r = 0.035$ and $\sigma = 0.25$. The function U has been approximated by using $n = 10,000$ simulations for each t and x.

Before we analyse the solutions further, let us recall that

$$\pi(t) = e^{-r(T-t)} \frac{S(t)}{T} \left[\int_{G(t)}^{+\infty} U(T-t,x)\,dx + G(t) \cdot U(T-t,G(t)) \right] \quad (46)$$

$$V^{\pi}(t) = e^{-r(T-t)} \frac{S(t)}{T} \int_{G(t)}^{+\infty} U(T-t,x)\,dx \,. \quad (47)$$

In particular, one gets for $t = T$

$$\begin{aligned} \pi(T) &= \frac{S(T)}{T}\left[\int_{G(T)}^{+\infty} U(0,x)\,dx + G(t)\cdot U(0,G(T))\right] \\ &= \frac{S(T)}{T}\left[\int_{G(T)}^{0-} \mathbb{1}_{\{x<0\}}\,dx + G(t)\cdot \mathbb{1}_{\{G(T)<0\}}\right] \\ &= 0\,,\ \text{and} \qquad (48) \\ V^{\pi}(T) &= \frac{S(T)}{T}\int_{G(T)}^{+\infty} U(0,x)\,dx \\ &= -\frac{S(T)}{T}G(t)\cdot \mathbb{1}_{\{G(T)<0\}}\,, \qquad (49) \end{aligned}$$

which is greater or equal to zero. Since U is a probability, one has that $U(t,x) \in [0,1]$ for all t, x. Most important is the area where U is strictly within the corresponding open interval, which is depicted in Figure 5. The area between the two curves is approximately the area where U is strictly within the open interval.

6.2 Calculating and Analysing the Hedge Ratio $\frac{\pi(t,G(t),S(t))}{V(t,G(t),S(t))}$

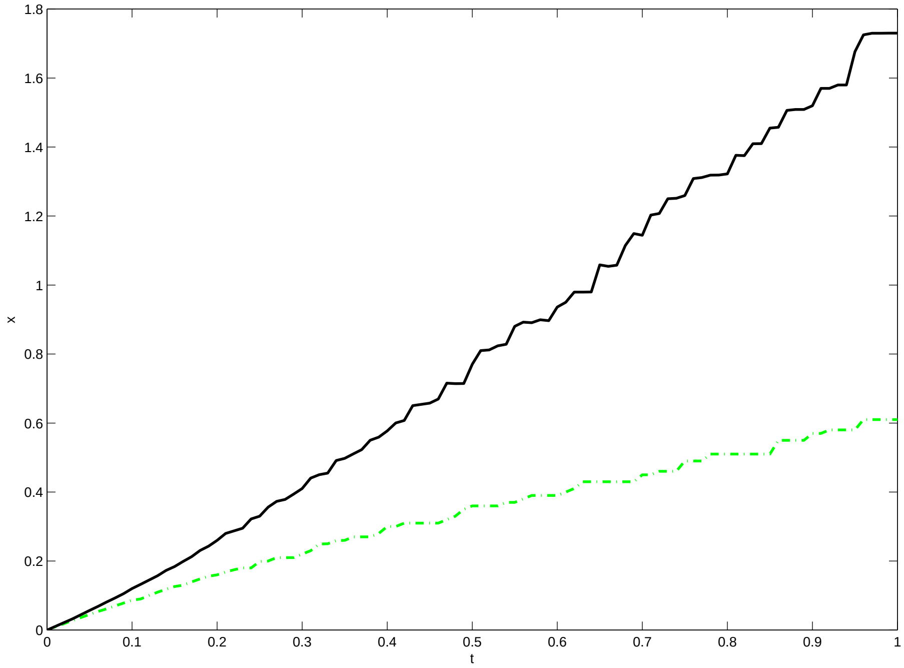
Observe that the hedging strategy $\pi(t, G(t), S(t))$ as well as the fair price of the Asian option $V(t, G(t), S(t))$ depends on t, $G(t)$, and $S(t)$, with $G(t)$ defined in (22). However, if the hedging strategy is written as the fraction invested in the risky asset, one arrives at

$$\frac{\pi(t,G(t),S(t))}{V(t,G(t),S(t))} = 1 + \frac{G(t)\cdot U(T-t,G(t))}{\int_{G(t)}^{\infty} U(T-t,x)\,dx}\,,$$

which depends only on t and $G(t)$. Hence, we will concentrate on this expression in the following. Apparently, the fraction invested in the risk–free asset is given by

$$1 - \frac{\pi(t,G(t),S(t))}{V(t,G(t),S(t))} = -\frac{G(t)\cdot U(T-t,G(t))}{\int_{G(t)}^{\infty} U(T-t,x)\,dx}\,.$$

Figure 5: Two Level Lines of $U(t, x)$



This figure approximates the level lines for $U(t, x) \geq 1 - 10^{-\varepsilon}$ (green dash–dotted line) and for $U(t, x) \leq 10^{-\varepsilon}$ (black solid line), where $\varepsilon = 10$, $r = 0.035$, and $\sigma = 0.25$. The level lines have been approximated by using $n = 10,000$ simulations for each t and x. The approximations have been inproved by using the bisection method.

Notice that $\frac{\pi(t,G(t),S(t))}{V(t,G(t),S(t))}$ is not only independent of $S(t)$ (and thus $S(0)$), but is also independent of K and T. The fraction depends only on r and σ which are the parameters of the market (risk–free interest rate and volatility of the underlying risky asset), but not on the parameters of the actual option contract. The parameters of the actual option contract enter the fraction only via the random variable $G(t)$. Hence $\frac{\pi(t,G(t),S(t))}{V(t,G(t),S(t))}$ holds for any Asian call option contract which is based on an underlying risky asset that has volatility σ and is written in a market with risk–free interest rate r.

Observe that $U(T - t, x)$ is non–negative (it corresponds to a probability measure), hence so is the integral $\int_{G(t)}^{\infty} U(T-t,x)\,dx$. Moreover, $\int_0^{\infty} U(T-t,x)\,dx > 0$. Altogether, this gives

$$\frac{\pi(t,G(t),S(t))}{V(t,G(t),S(t))} \left\{ \begin{array}{ll} >1 & \text{if } G(t)>0 \\ =1 & \text{if } G(t)=0 \\ <1 & \text{if } G(t)<0 \end{array} \right\}.$$

This behavior can be observed in Figure 6. Moreover, Figure 6 shows clearly that

$$\frac{\pi(t,G(t),S(t))}{V(t,G(t),S(t))} \longrightarrow 0 \qquad \text{if } T-t \longrightarrow 0 \text{ and } G(t)<0\,.$$

Note also that the numerical solution is unstable for $T-t$ close to 0 and for $G(t)$ large, that is, the option is out–of–the–money. In this case, both π and V converge to zero as the maturity of the option is approached. The instability of the approximation in this case implies that V converges faster to zero then π.

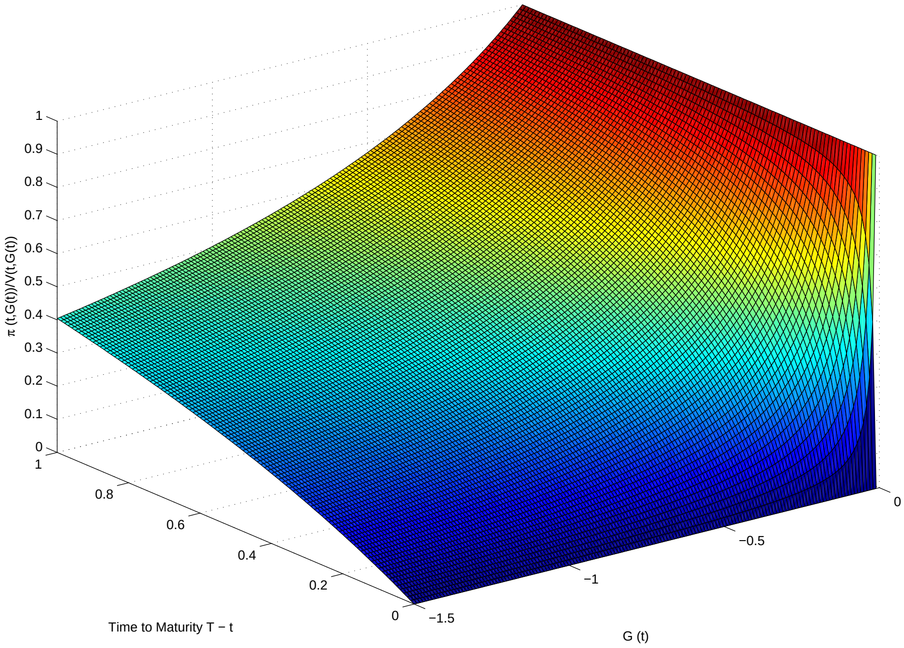
Observe that π and V are not that unstable (and they are important to practitioners), only $\frac{\pi}{V}$ is unstable in that way that its scale is not very reliable for $G(t)>0$. However, $\frac{\pi}{V}$ has the advantage of being independent of $S(t)$ (thus one dimension less) and it depends only on σ and r. The interesting result which can be established here, is that the higher the stock price is (that is the higher the present value of the underlying Asian option is [or the more is the Asian option in–the–money]), the lower is the fraction invested in the stock within the hedging strategy.

6.3 Calculating the Delta and Comparing it to the Approach of Jacques

The Asian option's delta is given in equation (41), Proposition 6. Together with equation (47), this gives

$$\frac{\partial V(t)}{\partial S(t)} = \frac{\pi(t)}{S(t)} = \frac{e^{-r(T-t)}}{T}\left[\int_{G(t)}^{+\infty} U(T-t,x)\,dx + G(t)\cdot U(T-t,G(t))\right]$$

Figure 6: The fraction invested in the risky asset, $\frac{\pi(t,G(t),S(t))}{V(t,G(t),S(t))}$

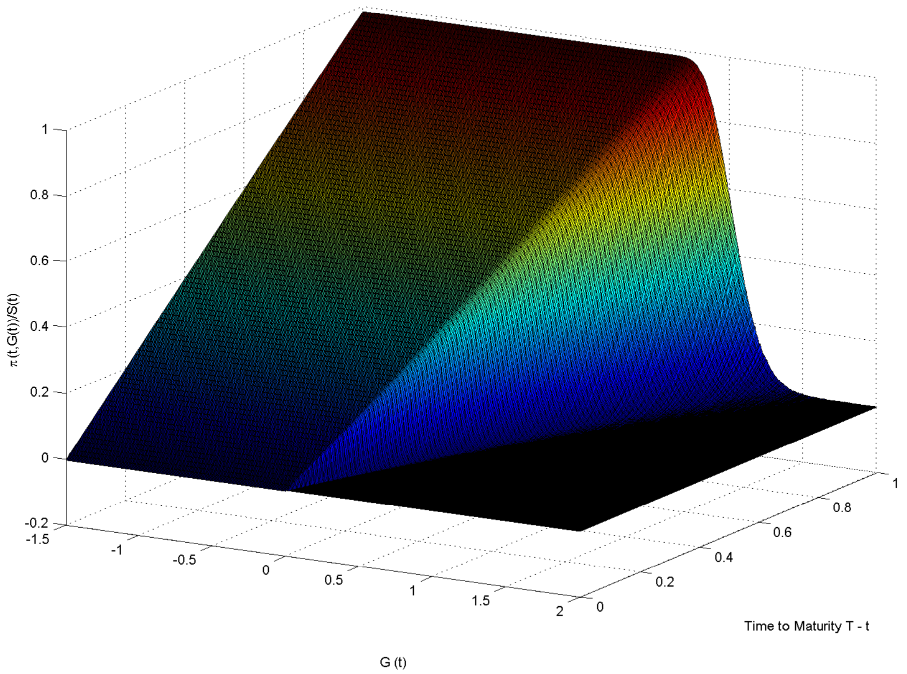


This figure shows the function $\frac{\pi(t,G(t),S(t))}{V(t,G(t),S(t))}$ for $r = 0.035$, $\sigma = 0.25$, and $T = 1$. The fraction has been approximated by using $n = 10,000$ simulations for each t and x.

The delta represents the number of shares purchased to setup the hedge.

Considering in–the–money Asian options, the delta suggests to buy almost one share in the underlying risky asset at the starting point of the averaging process (see Figure 7). The position in the risky asset will be dissolved in what seems to be a linear way over the life time of the averaging process for as long as the option is in–the–money. For out–of–the–money options, the delta is significantly smaller then one and for deep out–of–the–money options, it is close to one. Moreover, for out–of–the–money options, the position in the risky asset is usually dissolved faster and in a non–linear way, that is, the delta is already zero (or close to zero) long before the option expires. These results

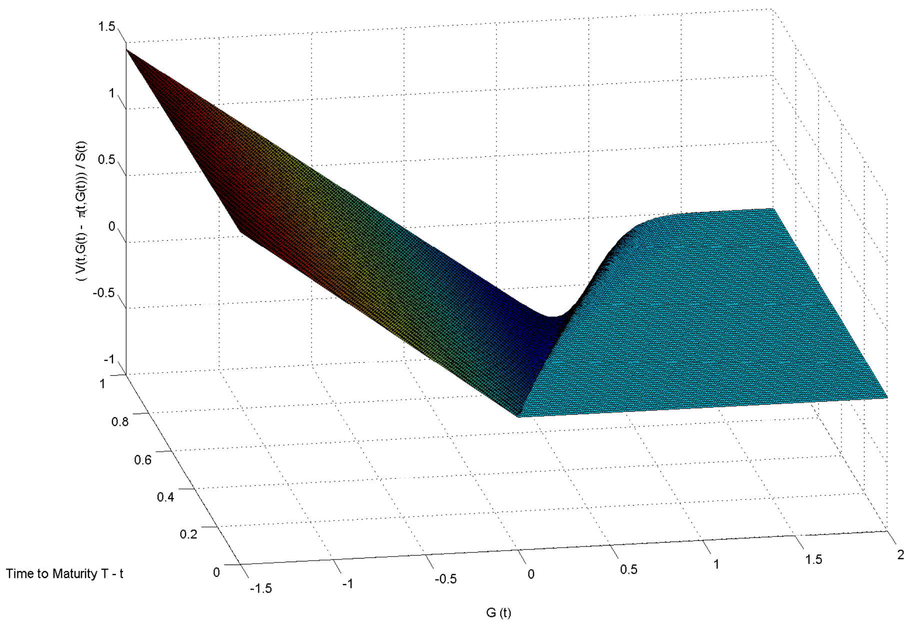
Figure 7: The Delta of an Asian Call Option, $\frac{\pi(t)}{S(t)}$



This figure shows the delta of an Asian call option $\frac{\pi(t)}{S(t)}$ over $G(t)$ and the time to maturity $T - t$ for $r = 0.035$, $\sigma = 0.25$, and $T = 1$.

are qualitatively the same as the results of Jacques [15]. However, our results are such that the delta does not always start with one at the beginning of the averaging process. It actually depends on the market parameters r and σ as well as the maturity T of the option where the delta starts for in–the–money options. If, for instance, in the example of Figure 7, r and T are increased, the initial delta will decrease. We are convinced that this result carries over to the approach of Jacques [15]. This is, because Jacques uses a very short averaging period of 30 days (we use one year and increased it to five years). Nevertheless, the seemingly linear decrease of the delta for in–the–money options is not affected by this.

Figure 8: The fraction invested in the risk–free asset, $\frac{V(t,G(t),S(t))-\pi(t,G(t),S(t))}{S(t)}$



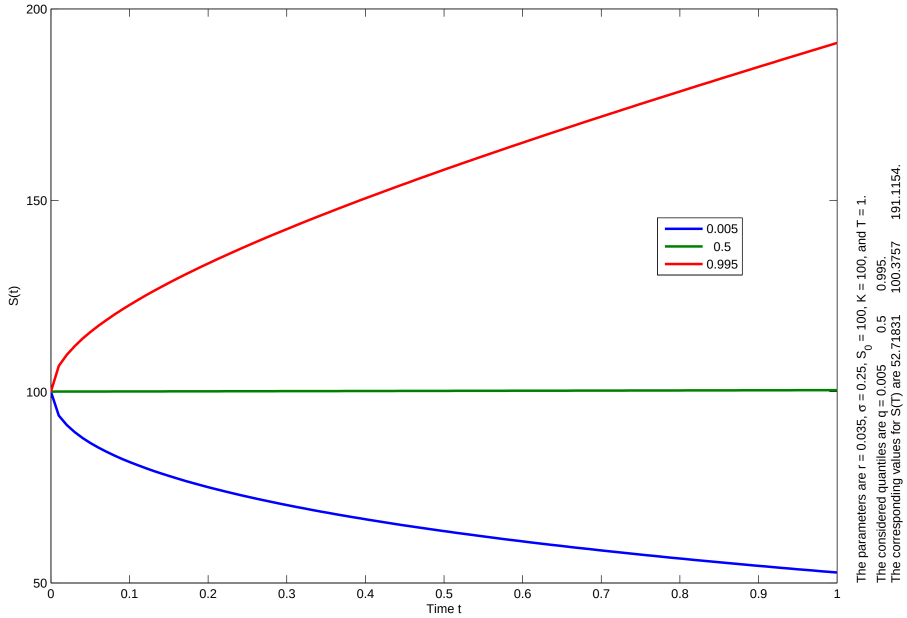
This figure shows the function $\frac{V(t,G(t),S(t))-\pi(t,G(t),S(t))}{S(t)}$ for $r = 0.035$, $\sigma = 0.25$, and $T = 1$.

Figure 8 depicts the fraction invested in the risk–free asset, where the fraction is taken with respect to the price of the risky underlying $S(t)$. Notice that the fraction invested in the risk–free asset is positive only if $G(t) < 0$. However, for $t = 0$ (that is, at the beginning of the lifetime of the Asian option), $G(0) < 0$ implies that $K < 0$ which is usually not feasible. Thus, initially, the hedging strategy consists of going short in the risk–free asset.

As already indicated, it depends on the specification of the contract of the Asian call option, at which $G(0)$ this specific option will start on the various surfaces of $\frac{\pi(t,G(t),S(t))}{V(t,G(t),S(t))}$ or $\frac{\pi(t,G(t),S(t))}{S(t)}$ at time $t = 0$. For convenience, let us assume that $r = 0.035$, $\sigma = 0.25$, and $T = 1$ as above. Moreover, assume that $S(0) = 100$, $K = 100$. Hence, it is straightforward to calculate that $G(0) =$

1. Hence, in this section, the Asian option always starts at–the–money and possibly move quickly in–the–money or out–of–the–money while Jacques [15] starts either in–the–money or out–of–the–money. Instead of considering specific sample paths of $S(t)$ and calculating the corresponding $G(t)$ of this, let us do a more general analysis. Figure 9 depicts the 0.5%, the 50% (the median), and the 99.5% quantiles of $S(t)$ for $t \in [0, 1]$, which are straightforward to calculate. Together, the upper and the lower curve constitute the 99% confidence interval (see Figure 9).

Figure 9: Quantiles of $S(t)$

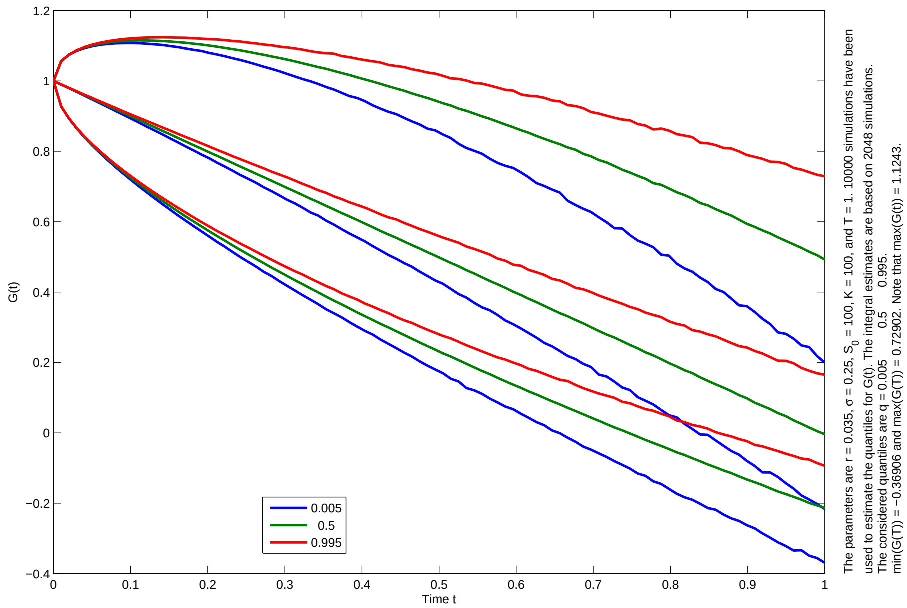


This figure shows the quantiles of $S(t)$ where the parameters are $\mu = 0.035$, $\sigma = 0.25$, and $t \in [0, 1]$.

Figure 10 shows the 0.5%, the 50% (the median), and the 99.5% quantiles of $G(t)$ for $t \in [0, 1]$ given that $S(t)$ follows one of the three lines given in Figure 9. The lower three depicted quantiles of $G(t)$ are calculated on the basis that $S(t)$ moves along the 99.5% quantile, the upper three plotted quantiles of $G(t)$

are calculated on the basis that $S(t)$ is the 0.5% quantile, and the remaining quantiles of $G(t)$ are based on the assumption that $S(t)$ is the median. The quantiles of $G(t)$ have been calculated by simulating $10,000$ paths between $S(0)$ and $S(t)$ and computing the corresponding $G(t)$ for each path. The quantiles of $G(t)$ have been estimated of these $10,000$ values using order statistics. This procedure has been done for every $t \in [0,1]$ independently. Figure 10 gives us a good idea about the possible range of $G(t)$. Observe that $G(t)$ will most likely be in the range of $[-0.4, 1.2]$ for $t \in [0,1]$. The range is much smaller for fixed $t \in [0,1]$.

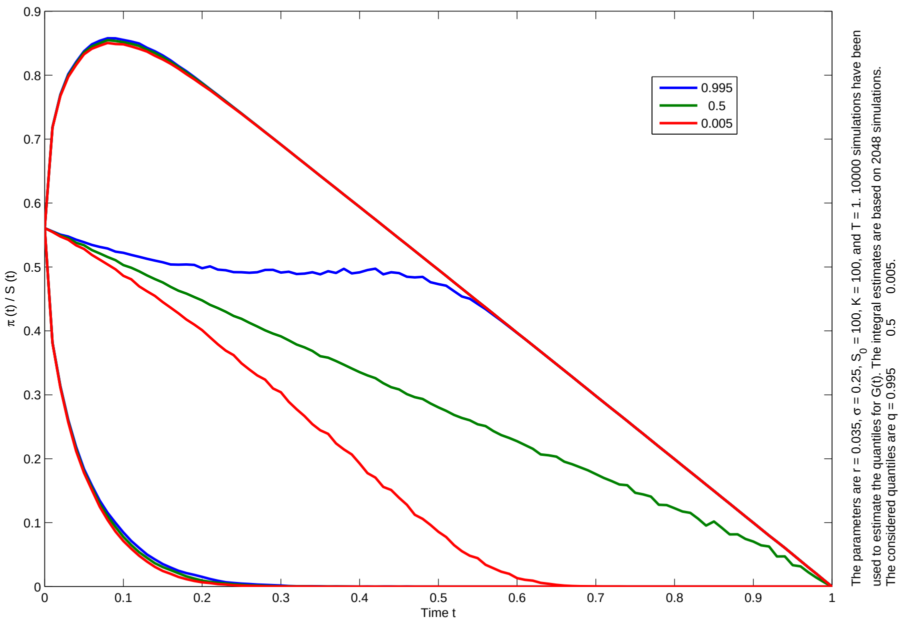
Figure 10: Quantiles of $G(t)$, given $S(t)$



This figure shows the quantiles of $G(t)$ given that $S(t)$ is one of the three values given in Figure 9. The parameters are set to $r = 0.035$, $\sigma = 0.25$, and $T = 1$.

Figure 11 shows the most likely range for the fraction of $\frac{\pi(t,G(t),S(t))}{S(t)}$ given the quantiles of $G(t)$ which are based on the assumption that $S(t)$ moves along its 0.5% quantile (lower (undistinguishable) three lines), its median (the three clearly distinguishable lines in the middle), and its 99.5% quantile (upper (undistinguishable) three lines). Figure 12 is the corresponding figure of $\frac{V(t,G(t),S(t))-\pi(t,G(t),S(t))}{S(t)}$ given the quantiles of $G(t)$ which are based on on the assumption that $S(t)$ moves along its 0.5% quantile (initially the upper (undistinguishable) three lines), its median (the three clearly distinguishable lines which are initially in the middle), and its 99.5% quantile (initially the lower three lines), respectively.

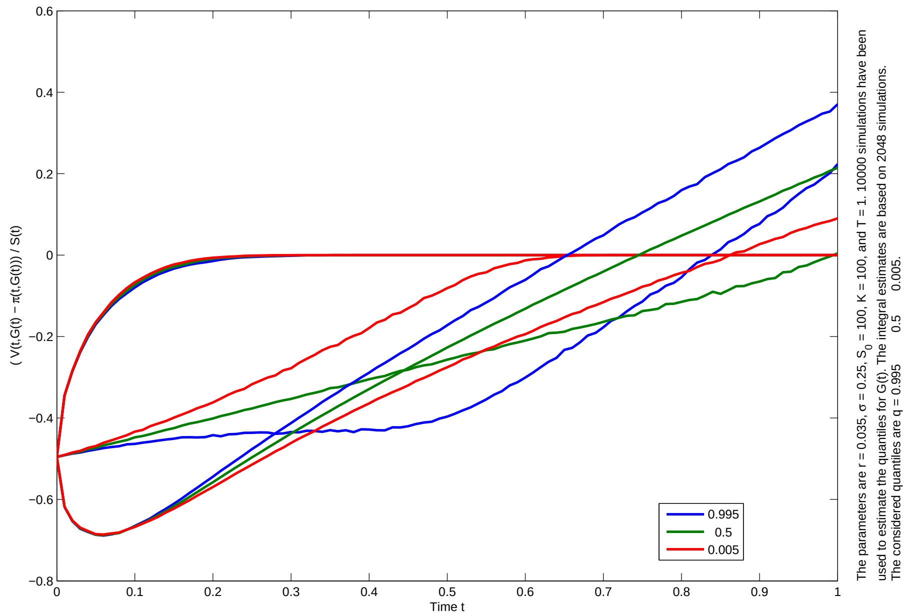
Figure 11: Evolution of $\frac{\pi(t,G(t),S(t))}{S(t)}$ given the various quantiles of $G(t)$



This figure shows the evolution of $\frac{\pi(t,G(t),S(t))}{S(t)}$ given the quantiles of $G(t)$ which are based on the assumption that $S(t)$ moves along its 0.5% quantile (lower (undistinguishable) three lines), its median (the three clearly distinguishable lines in the middle), and its 99.5% quantile (upper (undistinguishable) three lines). The parameters are set to $r = 0.035$, $\sigma = 0.25$, and $T = 1$.

If $S(t)$ moves along the 0.5% quantile, the Asian option is deep out–of–the–money and the hedging strategy consists of dissolving the hedging portfolio in

Figure 12: Evolution of $\frac{V(t,G(t),S(t))-\pi(t,G(t),S(t))}{S(t)}$ given the various quantiles of $G(t)$



This figure shows the evolution of $\frac{V(t,G(t),S(t))-\pi(t,G(t),S(t))}{S(t)}$ given the quantiles of $G(t)$ which are based on the assumption that $S(t)$ moves along its 0.5% quantile (initially the upper (undistinguishable) three lines), its median (the three clearly distinguishable lines which are initially in the middle), and its 99.5% quantile (initially the lower three lines). The parameters are set to $r = 0.035$, $\sigma = 0.25$, and $T = 1$.

the risky underlying (lower (undistinguishable) three lines in Figure 11) and in the risk–free asset (upper (undistinguishable) three lines in Figure 12). In this situation, the option has value zero at maturity which is reflected by the fact that the hedging portfolio has also value zero.

If $S(t)$ moves along the 99.5% quantile, the Asian option is deep in–the–money and the hedging strategy consists of first increasing both the long position in the risky asset and the short position in the risk–free asset. Between $t = 0.05$ and $t = 0.1$ this strategy changes. That is, over the remaining lifetime of the option, the long position in the risky asset will be liquidated in an seemingly linear way (see upper (undistinguishable) three lines in Figure 11). This is in line with the observation of Jacques [15]. At the same time, the short position in the risk–free asset (lower (initially undistinguishable) three lines in Figure 12) is turned into a long position (in all three cases). In this situation, the option has a positive value in all three scenarios at maturity which is reflected by the fact that the hedging portfolio has also positive values.

If $S(t)$ moves along its median, the hedging portfolio will be liquidated almost linearly over the lifetime of the option in both positions, the risky asset (see Figure 11, the green line in the middle) and the risk–free asset (see Figure 12, the green line which is initially in the middle). The value of the option (and hence of the hedge) at maturity is about zero. Most interesting is the confidence interval in this case. If the option is out–of–the–money (that is the risky asset moves along the 0.5% quantile), both positions in the hedging portfolio are closed faster. In particular, the hedge is closed before the maturity of the option, that is between $t = 0.6$ and $t = 0.7$ (and the maturity is at $T = 1$). This is the scenario where the corresponding lines in Figures 11 and 12 join the deep out-of–the–money scenario between $t = 0.6$ and $t = 0.7$. The terminal value of both option and hedge is zero. However, considering the 99.5% quantile, the corresponding hedging strategy is to decrease the long position at most slightly up to the time between $t = 0.5$ and $t = 0.6$ at which point it meets the lines for the deep in–the–money case and it follows that line from thereon. That is, from that time on the long position in the stock is liquidated in a linear way over the remaining lifetime of the option. Similarly, the corresponding short position in the risk–free asset is in this scenario reduced at most slightly up to between $t = 0.4$ and $t = 0.5$ at which point the short position in the risk–free asset is turned in an almost linear way into a long position at maturity. That is, the option matures in–the–money. This case is not considered by Jacques [15]. He considers only an in–the–mone scenario and an out–of–the–money scenario, but not an at–the–money scenario. Nevertheless, we also observe a liquidation of the long position in the risky asset in the at–the–money case which is also seemingly linear.

7 Conclusions

We have derived quasi analytic expressions for price and hedge of an arithmetic Asian call option using Malliavin calculus as well as PDE techniques. We have tested their performance and presented various numerical results using the Monte Carlo method as this gave the most reliable results. Moreover, we provided a comparative analysis of the hedge and, in particular, observed that the lower the current value of the underlying Asian option is, the higher the fraction invested in the stock within the hedging strategy. Finally, we compared our results to those obtained in Jacques [15].

It remains to future research to verify that the seemingly linear decrease of the delta for in–the–money options is indeed linear. Moreover, it would be interesting and it would improve the accuracy of the calculation of the integral of U (see Figure 4) if an explicit expression (even in the form of an approximation, e.g. using the approximation of Jacques [15]) for the level lines (depicted in Figure 5) is derived. Finally, it remains for future research to get a more explicit expression for the relationship between the delta and its parameters, that is r, σ, and T, especially for in–the–money options.

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