Mathematical Transfer by Chemistry Undergraduate Students

Richard Hoban B.Sc. (Hons)

Under the supervision of Dr. Odilla E. Finlayson, School of Chemical Sciences, Dublin City University & Dr. Brien C. Nolan, School of Mathematical Sciences, Dublin City University.

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy is entirely my own work, that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge breach any law of copyright, and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

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Acknowledgements

The only constant is change

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Contents

Abstract

Mathematical Transfer by Chemistry Undergraduate Students

This thesis reports on a study of the transfer of mathematical knowledge by undergraduate chemistry students. Transfer in this research refers to the students' ability to use mathematical concepts, previously experienced within a mathematics course, within chemistry contexts. A pilot study was undertaken with a sample of second-year undergraduate chemistry students in order to determine their ability to transfer mathematical knowledge from a mathematics context to a chemistry context. The results showed that, while certain students could transfer (i.e., answer mathematical items correctly in a mathematics context and then in a chemistry context), many students were unable to transfer due to insufficient mathematical knowledge.

These results motivated the main study, in which students' ability to transfer mathematical concepts was investigated and analysed in two respects. These were the degree to which transfer was present, and the degree to which a particular characteristic, namely students' ability to correctly explain their mathematical reasoning, underpinned successful transfer. It was found that students who evidenced an ability to explain their reasoning in a mathematics context associated with transfer.

An intervention programme was designed which focused on the development of student understanding of mathematical concepts, both in terms of symbolic actions and linking these symbolic actions with mathematical referents/objects. This intervention programme was informed by current mathematics-educational theories. The evaluation of the intervention programme involved determining students' mathematical understanding, their ability to transfer, and their opinions as to its usefulness. While the majority of the students found the intervention programme beneficial, students' competency in respect of linking mathematical actions with referents/objects varied over the different concepts studied. Students' ability to transfer also varied from one concept to another.

The systematic process adopted in this study, of both determining students' ability to transfer and the factors influencing transfer, and using this information together with mathematics-educational theories in developing intervention programmes, is applicable to transfer studies across other disciplines.

Introduction

Anecdotal evidence suggests that chemistry undergraduate students struggle with mathematics in a chemistry context; books that present mathematics in a chemistry context implicitly highlight this [1, 2, 3, 4]. While these books are good in their own right, they do not attempt to answer why chemistry undergraduate students possess mathematical difficulties. A number of reasons are possible:

(1) Students possess insufficient mathematical knowledge;

(2) The discipline-specific chemistry knowledge which is being modelled mathematically, impedes students' ability to apply and interpret the relevant mathematical knowledge; or

(3) Students' have an inability to transfer mathematical knowledge to chemistry.

This research aimed to: 1) investigate the extent to which the problems students have with mathematics in a chemistry context are due to students' inability to transfer mathematical knowledge to chemistry; and 2) improve chemistry undergraduate students' knowledge of mathematics in a chemistry context. More specifically, it was decided to investigate:

> Whether students can transfer mathematical knowledge relevant to chemical kinetics and thermodynamics from a mathematics context to a chemistry context? (hereafter referred to as the *Transfer Question*).

Transfer in the context of this research was defined as getting correct answers to questions using the same mathematical concepts in both a mathematics context and a chemistry context. Such an approach resides, in terms of educational literature, in the domain of the traditional view of the transfer of learning. In addition to looking for evidence of transfer, the significance of the transfer observed (if there so happened to be any transfer observed) was investigated; 'significance' was in terms of whether it could be argued that the transfer observed was not due to chance alone.

Some of the reasons why students can transfer were also explored. In the Pilot-Study aspect of the research, it was decided to investigate whether students could transfer conceptual mathematical knowledge more so than procedural mathematical knowledge (the *Conceptual versus Procedural Question*).This *Conceptual versus Procedural Question* evolved into the following question:

> Do students who evidence an ability to explain in a mathematics context associate with transfer? (Hereafter referred to as the *Explaining and Transfer Question.*)

From a personal perspective, my undergraduate background in the field of science education dictated the manner in which this research was conducted. My undergraduate degree comprised of an emphasis on chemistry, physics, biology and mathematics, all embedded within an educational context. Thus, it could be argued that my background in science is one of being a generalist; this is very much the case.

However, despite my generalist background, I adopted a mathematics-education approach in the undertaking of the research. I adopted this critical stance because of two significant reasons: 1) the findings from my 4th year undergraduate research project; and 2) my own personal experience of learning mathematics.

In investigating chemistry undergraduate students' difficulties in understanding the mole concept for my 4th year undergraduate research project, one of the main findings from such a project was that the difficulties students have with the mole concept are due to students' inability to *transfer/use* necessary mathematical knowledge. I anticipated that this may be the case with students' difficulties in terms of chemical kinetics and thermodynamics.

My own personal experience of learning mathematics was quite haphazard. In terms of my second-level schooling, despite completing the honours leaving certificate curriculum in mathematics, it was a number of years before I became aware of certain mathematical concepts in real-world contexts. For many concepts, such as exponential functions, logarithmic functions and integration, it was largely due to chance that I managed to develop an understanding of these concepts. Such 'chance understanding' materialised during my university studies.

Could it be that this understanding resulted because of: 1) perseverance?; 2) eventually making connections (or transferring) between the world of abstraction and applied contexts?; or 3) the manner in which these concepts were explained during lectures, tutorials, and within textbooks at university? I suspect that it was a combination of these factors, but if so, then why do other students in Dublin City University appear not able to eventually develop this understanding, and thus apply it to mathematical concepts in chemical kinetics and thermodynamics? The research questions investigated in this project aimed to explain such an occurrence.

The research comprised of two phases:

- In Phase 1, the *Transfer Question* was investigated in the form of a Pilot Study; the findings from this Pilot Study grounded the investigation of both the *Transfer Question* and the *Explaining and Transfer Question* in the Main Study. The Main Study was comprised of two studies, Study 1 and Study 2.

- In Phase 2, an Intervention was designed in order to improve students' mathematical understanding. The findings from Phase 1 suggested that doing so would improve students' ability to transfer.

There are four chapters in the thesis. Chapter 1 reviews literature that is relevant to:1) chemistry education; 2) the mathematics problem; 3) learning mathematics; and 4) transfer studies. Such literature informed the *Transfer Question* and the *Explaining and Transfer Question*. In Chapter 2, the research methodology used to investigate the *Transfer Question* and the *Explaining and Transfer Question* in both the Pilot Study and the Main Study is discussed. The results that arose from the Pilot Study are also discussed in this chapter, and in particular how these results grounded the research methodology used in the Main Study.

Chapter 3 presents and discusses the results from the Main Study. How these results grounded the development of an intervention designed to improve students' mathematical understanding is described in Chapter 4. Lastly, the conclusions and implications which arose from: 1) the investigation of both the *Transfer Question* and the *Explaining and Transfer Question;* and 2) the implementation of the Intervention, are detailed.

Chapter 1

A Review of the Literature which Informed the *Transfer Question* and the *Explaining and Transfer Question*

1.1 Chapter Overview

This chapter is composed of two parts. In Part 1, Relevant Literature, the *Transfer Question* and the *Explaining and Transfer Question* are discussed in terms of: 1) chemistry education; 2) the mathematics problem; 3) learning mathematics; and 4) transfer studies. In Part 2, the *Transfer Question* and the *Explaining and Transfer Question*, how such questions can be answered is described. The *Transfer Question* is described in terms of: 1) what is transfer; and 2) how can transfer be assessed.

The precursor to the *Explaining and Transfer Question*—the *Conceptual versus Procedural Question*—is discussed in terms of: 1) the justification for asking such a question; and 2) the reasons why the question evolved into the *Explaining and Transfer Question*.

Lastly, the *Explaining and Transfer Question* is dealt with in terms of the various theories of how students learn (in particular, theories on how students learn mathematics) that were analysed in order to see if a particular theory could be used to categorise the degree to which the students explained in a mathematics context.

1.2 Relevant Literature

1.2.1 Chemistry Education

As has been stated in the Introduction, undergraduate chemistry students struggle with mathematics. Nicoll and Francisco have found that academics agree that mathematics proficiency is the keystone to success in physical chemistry [5]. Such a view may not seem surprising, but the same researchers have also found that it is not mathematics proficiency *per se* which embodies success in physical chemistry, but (more specifically) students' ability to solve word problems and their ability to think logically.

Other researchers claim that while mathematical ability is more important than mathematical exposure as regards success in physical chemistry, the amount of mathematics taken by students cannot be underestimated [6]. The same researchers argue that including a mathematics review session near the beginning of a chemistry course may be an effective way of reminding students of what they have learned, thus helping them to complete the required mathematical elements of the forthcoming chemistry course.

Looking at some of the specific difficulties which chemistry undergraduate students face, De Pierro and Garabala [7] have found that for many students the use of the irrational number 'e' as the base of the natural log (Ln) is deeply mysterious. In a chemistry context, the repercussions of this are obvious—failure to understand the fundamentals of chemical kinetics. Moreover, an inability to translate the symbolic representation of calculus operations is also a difficulty for students. In relation to integration, Bressoud [8] claims that research has always talked about the graphical meaning of the integral as being important in students' development of the concept of integral, yet, it has not been, in many cases, traditional to give students questions examining their understanding of such a graphical depiction.

Certain researchers advocate the teaching of mathematics in a chemistry context as a solution to the mathematical problems which chemistry students possess. For example, Witten [9] argues for adapting the current undergraduate mathematics course to become more contextual, demonstrating the mathematical technique in the applied context. However, there is little evidence that such an approach solves students' mathematical difficulties [10]. Such findings lead to other questions: are the mathematical difficulties experienced by undergraduate students just confined to chemistry students, or do students from other disciplines experience mathematical difficulties?; and, if so, how have such mathematical difficulties been addressed, or attempted to have been addressed? Such questions are hereafter referred to as *The Mathematics Problem*.

1.2.2 The Mathematics Problem

Concerns with 1^{st}-year students' mathematical preparedness for undergraduate courses involving mathematics are nothing new [11,12,13,14,15]. In terms of economics courses which involve a mathematical element, the authors Evensky et al.[16] describe how many students (especially introductory economics students) do not possess the basic graph skills necessary to interpret the relevant economic, contextual information.

In the UK, higher education lecturers reported that undergraduate students struggle with mathematics that is relevant to physics. Moreover, those students who do cope with the mathematics in the mathematics context show relatively little competency in being able to apply this mathematical knowledge to a physics context. An area of calculus which students particularly find difficult is integration.

In order to try and understand these difficulties, a lecturer in mathematics education worked with a number of students at Kings College London [17]. The students were physical science and engineering students. A number of interesting findings emerged. In particular, students' performance on questions concerned with graphs appeared to predict students' success on later mathematics courses. The author, Gill, articulates that teachers at all levels of the education system in the UK tend to overestimate students' abilities to interpret and understand graphs. He poses the question: if there is such a thing as 'graphicacy', should the primary concern of the mathematics educator be to focus on graphing when first introducing a concept?

Gill also states that students see no relationship between their mathematics courses and their subject areas. This suggests that mathematics should be thought in the required context. However, for Gill, the counter-argument to this is that it doesn't work because there are too many contexts, and even if it did work, the mathematics would be tied to that context, leaving, the students no better off. It can be inferred that what Gill means by the use of the words 'no better off' is that students will be unable to transfer if mathematics is taught in only one context.

In a purely Irish context, international studies have indicated that Irish students perform relatively poor on questions that require abstraction or the presentation of mathematical tasks in non-routine formats [18,19]. At the university level, there is widespread agreement that students enter mathematically-laden courses with few of even the most basic mathematical skills necessary to succeed on these courses. Hourighan and O'Donoghoue [20] sought to investigate why this is so. They set about with the aim of trying to identify factors in the pre-testing (or second-level schooling of students) mathematics education of students entering third level that explained students' mathematical under-preparedness for tertiary education in Ireland. A number of interesting findings emerged:

(1) Mathematics tends to be taught in a manner which is divorced from realistic settings;

(2) Students are given little room, if any, to explore mathematical ideas; and

(3) The mode of teaching appears to promote a 'learned helplessness' amongst students.

It should be noted that mathematical under-preparedness is not just confined to Ireland, but appears to be a permanent feature of many educational systems in developed countries [21]. Importantly, Hourighan and O'Donoghue [20] state that mathematics-intensive courses need independent learners possessing conceptual and transferrable skills required to solve unfamiliar problems, but that these skills were not fostered in the classrooms which the authors studied. If this is the case, the question of how should mathematics be learned is raised, and is subsequently discussed.

1.2.3 The Learning of Mathematics

Schoenfeld states that human memory, in general, is associative, and that memory contents are organised in chunks [22]. Memory for verbal information as opposed to visual/perceptual information has been found to be different. Anderson [23] in his comments on perception-based knowledge representations states that it has often been found that memory for pictorial material is superior to memory for verbal material. In a particular experiment, called Santa's experiment, it was concluded that visual information such as geometrical objects tends to be stored according to spatial position where as other information such as words tends to be stored according to linear order.

It is obvious that memory is a constituent of mathematical learning, and that without it, learning cannot occur. However, memory is not the only constituent of learning; mathematical learning, according to Romberg and Carpenter [24, p.868] “proceeds through construction not absorption [memory]”. Such a view raises the question: what does it mean to construct knowledge? Bruner [25] provides a partial answer in terms of describing how the active participation of a child in learning may improve their problem-solving ability, thus making material more readily accessible in memory. The very term ‘active participation’ is what Bruner means by the construction of knowledge. This active participation is improved by problem solving.

Schoenfeld [22] also agrees that teaching for understanding should be done through the method of *discovery/mathematical problem solving*. For Schoenfeld, problems should serve as introductions to important mathematical ideas, and good problems in this regard are problems that lead to more problems. It is through problem solving that students will, according to Schoenfeld, come to know a body of mathematical knowledge and see the world through a mathematical lens. Such problem solving is all good and well, but how exactly are mathematics educators supposed to engender it?; more specifically, how can it be engendered for mathematics required in a calculus context?

The authors Rasmussen and King [26] adopted the Realistic Mathematics Education (RME) approach in their design of a course designed to improve students’ understanding of first-order differential equations. The RME approach is used in the Netherlands. It situates mathematical concepts in contexts that are deemed to be experientially real for the students [27]. The approach taken is one of, where possible, guided reinvention. Mathematical concepts are situated in a manner which they would have first confronted mankind. This presentation of the concepts in terms of their historical evolution is designed to improve students’ formal mathematical knowledge.

Rasmussen and King [26] presented students with a rate of change in a realistic setting. Questions were asked in such a way that students were guided (and in the process, expected) to construct an informal Euler method for approximating solution functions to differential equations. The authors found that using a rate-of-change equation to gain information about a quantity of interest was non-trivial for the students. However, the

RME approach is a constantly evolving theory, and the results from experiments, like those carried out by Rasmussen and King contribute to refining the theory.

Perhaps the reasons why RME approaches are not always successful is due to what Hiebert et al. [28] term the distinction between acquiring mathematical knowledge and applying it. Such distinctions have in their view, influenced researchers conceptions of what it means to be able to solve mathematical problems. The authors pose the question: how much emphasis should be placed on acquiring the concepts and skills of mathematical knowledge in a mathematics context and how much emphasis should be placed on being able to apply these skills in realistic settings?

In their view, starting with mathematical problems in realistic settings such as RME does not resolve the difficulties that are inherent in the distinction between acquiring knowledge and applying it. They recommend 'problematising' the acquisition of mathematical knowledge which will in their view influence students' ability to *apply it/transfer it*. Previous studies on students' ability to transfer mathematical knowledge and what has been learned from these transfer studies is now discussed.

1.2.4 Transfer Studies

According to Schoenfeld [29], the issue of transfer in education is important, so important that it deserves attention on its own. For him, the central question concerning transfer is: how do we make sense of the ways in which people use knowledge in circumstances different from the circumstances in which the knowledge was initially acquired? Transfer, in Schoenfeld's eyes is frequent, yet when looked for in educational psychological literature, it appears to vanish. Schoenfeld posits that this is because researchers look for pre-determined transfer; in other words, the researchers do not investigate what students see as similar or dissimilar between the learning context and transfer context.

Detterman [30] adopts a more pessimistic stance in respect of transfer. For him, little transfer occurs, and even when it is claimed that transfer was observed, Detterman feels that this transfer may have been prompted: students may have been told that previous material is useful in the solution of a new problem, and thus artificially transfer. Alternatively, reported transfer may be as a result of the transfer between the learning context and transfer context, being an example of near transfer. Detterman concludes that it would be more worthwhile if educational researchers tried to understand how people acquire knowledge in a particular context, and thus tailor instruction appropriately.

Evans [31] also agrees somewhat with the views of Detterman. They articulate how researchers in education expect the transfer of learning, e.g., from school to everyday situations to be relatively unproblematic. However, for Evans, the notion of the same mathematical task in different contexts is highly problematic. In a similar vein, Lave [32] states that students can use school-type questions but fail to show competency in applying these algorithms when a question becomes more context bound, or more like a question encountered in the real world. Interestingly, Boaler [33] claims that no context can be assumed to enhance or inhibit understanding for all students.

In their study of students' abilities to use knowledge of slope to determine the steepness of a ramp, Lobato [34] found that what researchers considered similar between the learning situation and transfer situation, was, in fact, not so. In particular, Lobato found that students viewed the steepness of the ramp as changing. She underscored the need to provide *instructional treatment/experiences* that enable students to develop the type of general understanding that will help them to make sense of quantitative situations. Such instructional experiences are, in Lobato's opinion, an important area in the field of mathematics education.

Wagner[35] found that student reasoning which appears erratic, when that student is asked to transfer knowledge, can be understood and explained if close attention is paid to the contextual variations deemed important by the student. Their conclusion emerged from a case study analysis of an undergraduate student's attempt to solve a series of problems related to an elementary statistical principle. The student in question, slowly yet eventually, came to identify problems as instances of a single principle. Such a finding, in Wagner's View, challenges the mantra of *situated learning/cognition theories* which claim that transfer results from abstract instruction: on the contrary, Wagner found that abstraction is the consequence of transfer and the result of understanding, not the cause of it.

In terms of transfer studies related to the use of mathematics in a scientific context, Britton et al. [10] designed an instrument to measure students' ability to transfer exponential and logarithms from a mathematics context to a range of scientific contexts. They found that transfer rarely occurs despite (from their perspective) the transfer questions containing enough discipline-specific information to enable students to use mathematical knowledge without any previous knowledge of the particular discipline. Perhaps, from the researcher's perspective, the students needed discipline-specific knowledge of the transfer task in order to transfer. This certainly appeared to be the case in Bassok and Holyoak's study[36] of students' ability to transfer algebra from a mathematics context to a physics context (and vice-versa).

Bassok and Holyoak found that transfer from algebra to physics word problems was impaired if the physics transfer problems were embedded in a discussion of motion concepts. When the discussion of motion concepts in a Physics concept was disregarded, most students were able to transfer the algebra problems to the isomorphic physics context. This was in contrast to students who had learned algebra in the physics context; they almost never exhibited detectable transfer to isomorphic algebra problems. Bassok and Holyoak concluded that content-specific knowledge limits transfer.

Lastly, the authors Potgieter et al. [37] investigated undergraduate students' ability to transfer mathematical knowledge relevant to the Nernst Equation (used in electrochemistry) from a mathematics context to a chemistry context. The authors wanted to see if the mathematically-related difficulties which students exhibit are due to deficiencies in students' mathematical foundations or due to an inability to transfer.

They exposed a group of students to an instrument which contained both algebraic and graphical information relevant to the Nernst Equation in a mathematics context and a chemistry context. A number of interesting findings emerged:

1) Students experienced few problems with algebraic questions in both a mathematics context and chemistry context;
2) Students performed poorly on graphical questions; and
3) The problems students have with mathematics in a chemistry context appears to be a mathematical one, and not due to transfer.

In respect of the second findings, Scaife and Rogers [38] describe how little is known about the cognitive value of graphical representations. They stress that researchers need to:1) address issues such as the nature of the relationship between graphical representations and students' mental, internal representations of such graphs; and 2) consider how graphical representations are used when students solve problems and make inferences. The research undertaken by Potgieter et al. [37] was another reason for investigating undergraduate students' ability to transfer mathematical knowledge, relevant to chemical kinetics and thermodynamics, from a mathematics context to a chemistry context. Potgieter et al. state the extent to which the results from this study are relevant to non-logarithmic functions in chemistry needs to be investigated in the future.

1.3 The Transfer Question

Answering the *Transfer Question* meant that a number of other questions had to be asked, namely: what is transfer?; how can transfer be assessed?; and how has transfer been measured in the past?

1.3.1 What is Transfer?

The transfer of learning (hereafter referred to as *transfer*) for the purpose of this research was defined as the ability to use skills (or knowledge) in a context that is different from the learning context in which the skills (or knowledge) were initially acquired; such a definition agrees with the views of Evans, and Roberts et al. [31,39]. This statement appears to be a straightforward definition of transfer. However, Barnett and Ceci [40] state: "there is a lack of structure in the transfer debate and a failure to specify the various dimensions that may be relevant to determining whether and when transfer occurs" [40, p.614]. The authors provide clarity in answering such a question. They developed a taxonomy which can be used to classify whether transfer occurs. The taxonomy is composed of two factors: 1) the Content Factor; and 2) the Contextual Factor, the latter of which is first discussed.

The Contextual Factor considers the degree to which the learning context and transfer context are different in terms of six dimensions which constitute what a context is. These six dimensions are: 1) the knowledge-domain dimension; 2) the physical dimension; 3) the functional dimension; 4) the temporal dimension; 5) the social dimension; and 6) the modality dimension. Comparing and contrasting each dimension in the learning context and transfer context leads to a degree of difference. The degree to which each dimension is different is what determines whether transfer is near or far along that particular dimension. Looking at the degree to which all of the dimensions are different provides a qualitative view as to whether the transfer in question is near or far. It should be noted that determining the degree of difference for each dimension is subjective.

The knowledge-domain dimension is a measure of the difference between the knowledge in the learning context and in the transfer context. The physical dimension is a measure of the difference between the learning-context environment and the transfer-context environment. The functional dimension is a measure of the difference between the purpose for which the students are required to use a skill/knowledge in the learning context and the transfer context. The temporal dimension is a measure of the time duration between the acquiring of the knowledge/skill in a learning context and its application in a transfer context. The social dimension is a measure of the difference between the degree of social interaction involved in acquiring the knowledge in the learning context and the transfer context. The modality dimension is a measure of the difference between how the knowledge is communicated in the learning context and the transfer context.

The Contextual Factor of Barnett and Ceci's Taxonomy was applied to this research—not to determine if the transfer of mathematical knowledge from a mathematics context to a chemistry context is near or far—but rather to determine if students do transfer when they use mathematical knowledge in a chemistry context. The application of this Contextual Factor, as pertaining to this research, can be seen in Figure 1.1. Because determining the degree to which each dimension is different is subjective, it was decided not to try and classify the degree of difference for each dimension as reflective of either near transfer or far transfer. Rather, Figure 1.1 shows that the six dimensions which make up a context are different in the mathematics context and chemistry context. Whether the degree of difference for each dimension is near or far transfer cannot be determined.

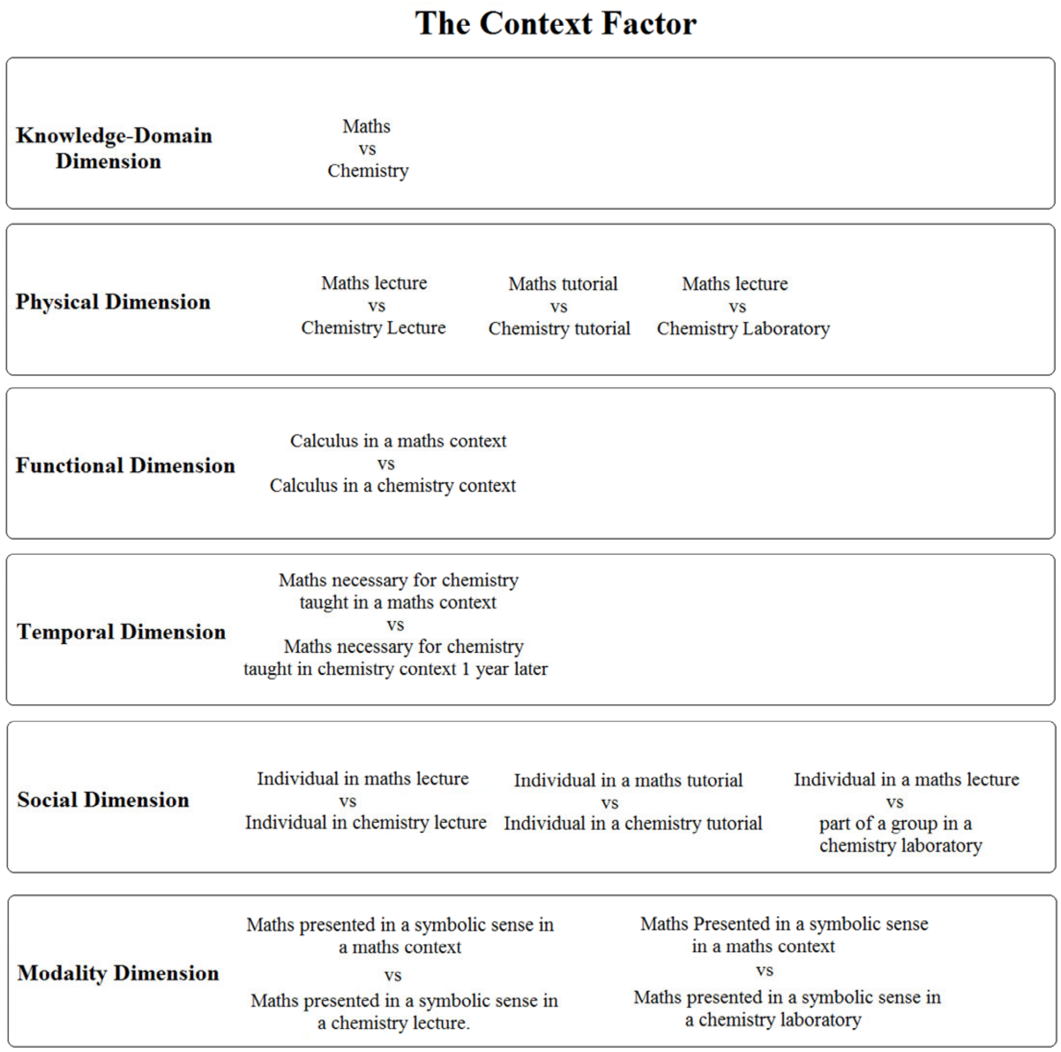


Figure 1.1. The Application of Barnett and Ceci's Contextual Factor as Pertaining to this Research. Adapted from[40].

The Content Factor of Barnett and Ceci's Taxonomy considers what it is that students transfer from the learning context to the transfer context. It comprises of three sub-factors, all of which are shown in Figure 1.2. The Performance-Change Sub-Factor considers how transfer is assessed. It defines whether transfer is assessed in terms of speed, accuracy, or the approach taken by a student when they attempt to transfer. The Memory Demands Sub-Factor considers what it is that students have to remember in order to transfer. For example, do the students have to recall knowledge before transferring it, or are the students allowed to search for the knowledge that they need before attempting to transfer. Lastly, the Learned-Skill Sub-Factor considers what it is that the student transfers, be it a procedure or a more general problem-solving approach.

The Content Factor			
Performance Change	Speed	Accuracy	Approach
Memory Demands	Execute only	Recognise and execute	Recall, recognise and execute
Learned Skill	Procedure	Representation	Principle or heuristic

Figure 1.2. The Application of Barnett and Ceci's Content Factor as Pertaining to the Research. Adapted from [40].

1.3.2 Assessing the Transfer Question

Royer, Mestre and Dufresne [41] describe the different generational approaches that have evolved in terms of the assessment of transfer. They describe the first-generational approach, in psychological terms, as behaviourist. The behaviourist approach is one in which researchers define/identify common elements of similarity between a learning context and transfer context. The researchers seek evidence as to whether students can transfer these elements from the learning context to the transfer context, either entirely or not at all. The main limitation to such an approach is that it misses out on trying to comprehend the mental processes that individuals employ in transferring prior learning—in other words, it misses out on what students transfer (if anything at all) and how students transfer. Such a limitation encouraged the second generational approach to understanding transfer—the cognitive approach.

Essentially, the cognitive approach is concerned with trying to understand the change in learners' conceptual thinking as they transfer. Both the behaviourist approach and cognitive approach are what Lobato [42] terms the traditional view of transfer. Such a view is predominately cognitive in focus and—like the behaviourist approach—considers transfer an all-or-nothing affair.

The third generation approach to assessing transfer stems from the limitations of the cognitive/traditional approach. Various third-generational theories have emerged which seek to investigate the "mediating factors by which individuals activate and apply prior learning, both productively and unproductively, during transfer tasks" [41, p.xvii]. Theories such as 'consequential transitions', 'affordances and constraints', and 'preparation for future learning' [43] are what Lobato classifies as an 'actor-oriented approach' to assessing transfer. This actor-oriented approach is often referred to as 'situative' [44]. The two main features that distinguish the *situative/actor-oriented approach* from the *cognitive/traditional approach* are:1) the cognitive approach considers transfer from the researcher's perspective whereas the situative approach considers transfer from the students' perspective; and 2) the cognitive approach focuses on the complete transfer of knowledge whereas the situative perspective focuses on partial transfer as well as complete transfer. Both approaches, nonetheless, consider how and why transfer occurs.

Royer et al. [41] point out that the different third generation theories [45,46,47,48,49,50] of transfer which come under the umbrella of the *situative/actor-oriented approach* essentially talk about the same thing—transfer in terms of cognition and socio-cultural factors—using different terms. Lobato [42] contrasts the traditional view of transfer with the actor-oriented view, using a number of dimensions as can be seen in Table 1.1.

Dimension	Traditional Transfer	Actor-Oriented Transfer
1. Definition	The application of knowledge learned in one situation to a new situation.	The personal construction of relations of similarity across activities, (i.e., seeing situations as the same).
2. Perspective	Observer's (expert's) perspective.	Actor's (learner's) perspective.
3. Research Method	Researchers look for improved performance between learning and transfer tasks.	Researchers look for the influence of prior activity on current activity and how actors construe situations as similar.
4. Research Questions	Was transfer obtained? What conditions facilitate transfer?	What relations of similarity are created? How are they supported by the environment?
5. Transfer Tasks	Paired learning and transfer tasks share structural features but differ by surface features.	Researchers acknowledge that what experts consider a surface feature may be structurally substantive for a learner.
6. Location of Invariance	Transfer measures a psychological phenomenon.	Transfer is distributed across mental, material, social and cultural planes.
7. Transfer Processes	Transfer occurs if two symbolic mental representations are identical or overlap, or if mapping between them can be constructed.	Multiple processes, such as an attunement to affordances and constraints, assimilation, language use, and 'focusing phenomena', influence transfer.
8. Metaphor	Static application of knowledge.	Dynamic production of 'sameness'.

Table 1.1 The Traditional and Actor-Oriented View of Transfer. Adapted from [42].

Again, looking at Figure 1.2, the second sub-factor of Barnett and Ceci's Taxonomy [40] is the *Memory Demands* of the transfer task. In the context of this research, this factor describes the degree to which students had to remember knowledge in a mathematics context before transferring it. During the investigation of the transfer question in the Pilot Study and Study 1, students were expected to recall the appropriate mathematical knowledge in order to transfer it. In Study 2, the students were reminded of the necessary mathematical knowledge in a mathematics context before they were presented with the transfer task. The results from this approach are presented and discussed in Sections 3.2 and 3.4 respectively in Chapter 3.

1.4 The Explaining and Transfer Question

In Figure 1.2, the third sub-factor that comprises the Content Factor of Barnett & Ceci's Taxonomy [40] is the Learned Skill Sub-Factor. It has been mentioned that this sub-factor refers to the type of skill/knowledge that is transferred. Barnett and Ceci [40] refer to the *learned skill/knowledge* to be transferred as either a specific fact or procedure; a form of representation; or a more general problem-solving heuristic or principle.

Hiebert and Lefevre, in their discussion of conceptual and procedural knowledge in mathematics [51, p.8], state: "rote learning ['rote learning' being the term they use for procedural knowledge] is knowledge that is absent in relationships, resulting in it being tied to the context in which it is learned". Conversely, the same authors, when talking about conceptual knowledge, state: "it has long been recognised that if procedures are understood or learnt in a meaningful way, the procedures ***transfer more easily*** to structurally similar problems" [51, p.13]. In light of this, during the Pilot Study phase of this research, it was decided to investigate whether students could transfer conceptual mathematical knowledge more so than procedural knowledge. This *Conceptual versus Procedural Question* was the precursor to the *Explaining and Transfer Question*.

Investigating the *Conceptual versus Procedural Question* raised a question of its own: what is conceptual and procedural knowledge in mathematics? Hiebert and Lefevre's definition of 'procedural' and 'conceptual understanding' was reviewed; it has been cited frequently [52,53,54,55,56]. The authors define procedural knowledge as made up of two parts. The first part "is composed of the formal language or symbol representation system of mathematics" [51,p.6]. The second part "consists of rules, algorithms, or procedures used to solve mathematical tasks" [51, p.6]. In contrast, the authors define conceptual knowledge as "rich in relationships" which "can be thought of in terms of a connected web of knowledge" [51, p. 3].

Such definitions were considered vague, posing difficulties in terms of: 1) how can it be determined if a student has a procedural or conceptual understanding of a mathematics concept; and 2) how can certain mathematical knowledge be classified as procedural or conceptual in nature. The views in respect of competency with graphs in a mathematics context (hereafter termed *graphicacy*) were considered in order to try and shed light on these difficulties.

1.4.1 Graphicacy

Numerous researchers equate conceptual understanding in a mathematics context with *graphicacy* [17,37,57]. Gill [17] hypothesises that graphicacy is a constituent of conceptual understanding, claiming that understanding graphs and slopes may underlie the ability to understand a number of higher-order concepts in mathematics. In respect of integration, Grundmeier et al. [58] advocate focusing on the verbal definition as well as the graphical aspect of a definite integral; not doing so, will, they claim, mean that students leave with nothing more than 'procedural fluency' in this area. Abboud and Habre [59] articulate that conceptual understanding is comprised of an emphasis not just on the symbolic aspect of a concept but also on the graphical aspect of the concept. Lastly, Potgieter et al. [37] argue that requiring students to *visualise/graphically represent* algebraic thinking is a means of cultivating 'conceptual understanding'.

During the Pilot Study (described in Section 2.3. in Chapter 2), students' ability to transfer mathematical knowledge was investigated. Each distinct piece of mathematical knowledge was termed a mathematical item; these mathematical items are in Appendix A. The algebraic items were classified as procedural while the graphical items were classified as conceptual. One of the aims of the Pilot Study was to see if students could transfer these presupposed conceptual items more so than the presupposed procedural items. This was found not to be the case. More importantly, two of the main findings which emerged from the Pilot Study (the full findings of which can be seen by referring to Section 2.3.4 in Chapter 2), were: 1) mathematical knowledge cannot be objectively classified as either procedural or conceptual in nature; and 2) stating whether students have a conceptual or procedural understanding of a mathematical item is subjective.

The Pilot Study findings were bolstered by the views of Anderson [23] who claims that classifying knowledge in mathematics as either procedural or conceptual is not absolute. If this is so, then perhaps knowledge of a particular mathematical concept can be a combination of both procedural and conceptual understanding? Such a question was considered. It was found that numerous researchers support this view, for example, Hiebert and Lefervre state that it is "not easy to imagine conceptual knowledge that is not linked with some procedures" [51, p.8], while "procedures that are learnt with meaning are procedures that are linked to conceptual knowledge" [51, p.8].

Such views re-shaped the *Conceptual versus Procedural Question* into the form of the *Explaining and Transfer Question*. The *Explaining and Transfer Question* sought to investigate whether students who could explain what they are doing in a mathematics context associated with transfer (the 1st aspect of the *Explaining and Transfer Question*). The analysis of such a question was always going to be subjective because classifying students as having or not having evidenced an ability to explain is subjective. Despite this subjective classification, various views on how students learn mathematics were reviewed in order to see if any of these views could be used to determine the degree to which a student explained (the 2nd aspect of the *Explaining and Transfer Question*). It was anticipated that evidencing a certain degree of explanation for each item may associate with transfer.

1.4.2 How Students Learn Mathematics

The *Explaining and Transfer Question* was investigated during the studies subsequent to the Pilot-Study—Study 1 and Study 2. For the sake of brevity, the various views on how students learn mathematics, as detailed in this thesis, are referred to as theories. The theories that were reviewed are: the APOS theory [60]; the van Hiele theory [61] and Tall et al.'s Theory [62] (hereafter referred to as *Tall's Theory*). Piaget's theory of Cognitive Development [63] was also reviewed. While this theory is not a theory on how students learn mathematics, it was felt that such a theory might be capable of being used to categorise the degree to which students explained.

1.4.2.1 Piaget's Theory

In summarising Piaget's Theory of Cognitive Development, the work of Wadsworth [63], in his book, *Piaget's Theory of Cognitive Development,* is drawn upon.

Introduction

Piaget's theory is focused primarily on the description and explanation of the growth and development of intellectual structures and knowledge. Wadsworth points out that the terms "intellectual, cognitive and mental are used interchangeably in Piaget's theory" [63, p.1]. Concepts central to Piaget's work are those of 'assimilation', 'accommodation', 'equilibrium' and 'schemata'. Piaget used these concepts to explain how mental development occurs. In a sense, the interplay between the concepts of assimilation, accommodation and equilibrium dictate the concept of 'schema'.

Schemata

Piaget believed the mind to have 'structures' similar to the manner in which the body does. What constitute these cognitive structures are schemata. In essence, schemata can be defined in terms of the manner by which individuals intellectually adapt to the environment and also to the manner by which individuals intellectually organise the environment.

Schemata evolve during the course of mental development and, in effect, dictate at what stage of cognitive development a person is at. For example, at birth, the schemata of a child are reflexive. Over time, these reflexive schemata become more differentiated and "less sensory" [63, p.12]. Piaget points out that because "schemata are structures of cognitive development that do change, allowance must be made for their growth and development" [63, p.13]. The processes or concepts which are responsible for this development are 'assimilation' and 'accommodation'.

Assimilation

In essence, assimilation is "viewed as the cognitive process of placing new stimulus events into existing schemata" [63, p.15]. It is important to note that this process of assimilation allows for the growth of schemata and does not explain the change or development of schemata. What explains the change of schemata is the concept of 'accommodation'.

Accommodation

For Piaget, when a child is faced with a new stimulus, they will endeavour to assimilate it into an existing schema of theirs. Oftentimes this is not possible, because there are no available schemata into which the stimulus can fit. This absence of an appropriate schemata, gives rise to one of two possible actions on the child's part.

Firstly, "the child can create a new schema into which they can place the stimulus" [63, p.16] or secondly "they can modify an existing schema so that the stimulus will fit into it" [63, p.16]. Both of these options result in a change in or development of cognitive structures (schemata). The last concept, which controls the development of schema, is that of 'equilibrium'.

Equilibrium

Does a child continually assimilate new stimuli?; continually accommodate new stimuli?; or strike a balance (equilibrium) between assimilation and accommodation? For Piaget, the extent to which students strike a balance determines how the child's cognitive structures (schemata) develop. He uses the term 'equilibrium' to describe the balance between assimilation and accommodation.

Piaget's View of Intelligence

For Piaget, intelligence is comprised of three components:

1) ***Content:*** This refers to the content of a student's behaviour as they engage with 'new' material to be learned.

2) ***Function:*** This refers to those characteristics of intellectual activity—assimilation and accommodation—that are stable and continual throughout cognitive development.

3) ***Structure:*** This refers to the inferred, organisational properties (schemata) that explain the content of a student's behaviour as they engage with 'new' material to be learned.

Piaget concerned himself with the ‘structural’ aspect of intelligence and made the claim, according to Wadsworth [63, p.22], that “structure, like content and unlike function, changes with age”. These developmental changes (structural changes) are the major focus of Piaget’s work. For him, these intellectual structures (schemata) are created through the function aspect of intelligence, which, as is already discussed, is the nature by which students assimilate and accommodate (or in essence equilibrate).

Heredity

For Piaget, *structural/cognitive* development is something that is not determined solely by genetic endowment or solely by a child’s experience. He asserts that properties other than neurological structures are inherited that affect cognitive development. He terms these properties ‘functional invariants’.

These ‘functional invariants’ are, in essence, the relative amounts of ‘assimilation’ and ‘accommodation’ inherent in an individual—or put simply, how well an individual equilibrates stimuli in their environment. For Piaget, this equilibration, or ‘mode of functioning’ for a child is fixed or inherited. It does not change. However, possessing this ‘mode of functioning’ does not ensure cognitive development unless the child interacts with the environment. How the child interacts with the environment is made up of a number of factors, each of which is discussed.

Action:

Piaget’s theory stipulates that the child must act in their environment if cognitive development is to proceed. For example, an infant cannot learn to differentiate between a nipple and an edge of their blanket unless they act on both. The child particularly needs to act on their environment in terms of physical and sensorial experience in the early years of life, because they “do not possess the power of symbolic representation [language]” [63, p.24].

As cognitive development progresses, actions on the environment become mediated by “internalised symbols” [63, p.24]. Consequently sensory-motor experience becomes less relevant but may, nonetheless, be still important.

Physical Experience:

Wadsworth [63, p.30] describes how Piaget's theory stipulates that a child "must have experience with objects and stimuli in the environment" in order for them to develop cognitively.

Social Interaction:

Piaget uses this term to mean the interchange of ideas between people. Interestingly, he points out that concepts or schemata that people develop can be classified as one of either two types:

1) Schemata/concepts that have sensory or physical referents in the form of referents that can be seen or heard. An example of such a referent is that for the noun 'tree'.

2) Schemata/concepts which do not have physical referents but instead rely on social interaction for their construction and validation. An example of such a concept would be the noun 'honesty'.

Before discussing the stages of Piaget's Theory of Cognitive Development, it is important to highlight a number of points. According to Wadsworth:

- "Piaget does not suggest that children move from discrete stage to discrete stage in development; rather, cognitive development flows along" [63, p.26].

- The age spans reflective of each stage of Piaget's theory are normative and "only suggest the times during which most children can be expected to display the intellectual behaviours that are characteristic of the particular stage" [63, p.12]. Furthermore, the norms established by Piaget were deduced from the study of children in Geneva.

- The age at which the stages occur "can vary with the nature of both the individual's experience" and their "hereditary potential" [63, p.12]. The implication from this statement is: perhaps students can reach stages of cognitive development more quickly than they normally do by dint of having their environment manipulated.

Piaget's Theory of Cognitive Development comprises of four main stages:1) The Sensory-Motor Stage; 2) The Preoperational Stage; 3) The Concrete-Operational Stage; and 4) The Formal-Operational Stage. The first two of these stages are summarised, with stage three and stage four discussed in comparatively more depth. A more detailed description of the Sensory-Motor Stage and the Preoperational Stage can be seen by referring to Appendix B.

The Sensory-Motor Stage—A Summary

This stage's evolution is as a result of the child acting on the environment. They must do this in order to develop their concept of object and concept of causality. Completion of the stage means that the child's intellectual development begins to take place "primarily in the conceptual-symbolic area rather than the sensory-motor area" [63, p.61]. The development takes place in the sense of language and symbolic development—a stage which Piaget calls the preoperational period. It should be pointed out that in subsequent stages of cognitive development relative to the sensory-motor period, it is not to be construed that "sensory-motor development ends; instead intellectual development is to be dominated by representational and symbolic activity rather than by motor activity" [63, p.63].

The Preoperational Stage—A Summary

The child becomes no longer restricted to immediate perceptual and sensory-motor events. This does not necessarily imply that the child is absolutely free from such events. Their thought gradually becomes more representational and symbolic in nature. Language develops rapidly, and this, in turn, allows the child 'to play out thought' in their head as opposed to having to link it with physical events (sensory-motor experience).

In terms of behaviour, the child moves from largely egocentric and non-social communication to conversations with their peers. This intercommunicative behaviour allows the child to develop cognitively. Despite these developments, the child remains restricted in certain ways. It is not until the end of the stage that they can reverse operations, follow transformations and become less egocentric in the process. Also, their conservational abilities will not have been developed until the end of the stage.

While the preoperational child's behaviour at the start of this stage somewhat resembles the sensory-motor development, by age 7 there is little resemblance. The child now enters the period of concrete operations.

The Concrete-Operational Stage

During this stage, the child develops "logical thought processes (operations) that can be applied to concrete problems" [63, p.90]. This stage of development functions as "a transition between pre-logical (preoperational) thought and the completely logical thought of the older child"; it usually occurs during the ages of 7-11. [63, p.89].

During this stage, when faced with a disparity between thought and perception, as in conservation problems for example, a child's thought process wins out—"they are no longer perception-bound" [63, p.89]. It should be stressed that "if the concrete-operational child is presented with a purely verbal problem, they are typically unable to solve it correctly" [63, p.89]. However, if the same problem is presented "in terms of real objects, the child can apply their logical operations and solve the problem" [63, p.89]. Wadsworth [63] describes how concrete-operational thought differs from preoperational thought in a number of ways:

Egocentrism and Socialisation

The child is able "to look at something from another's viewpoint" [63, p.92], question their reasoning and seek validation from others. All of these acts are considered by Piaget to be acts of accommodation.

Centration

The child's thought becomes decentred and thus allows them to deal with conservation-type problems.

Transformations

The aforementioned change in egocentrism and centration allows the child to deal with and understand the relationships between successive perceptual events.

Reversibility

In contrast to the preoperational child, the child exhibits no difficultly in reversing operations.

Conservation

Wadsworth [63, p.94] states that the “hallmark of preoperational thought is the inability of the child to conserve”. In contrast, concrete-operational children possess this ability due to their related abilities to decentre, follow transformations and to reverse operations.

In addition to the child’s understanding of these aforementioned concepts changing during the concrete-operational stage, the child acquires two other concepts:

Seriation

This concept refers to the ability to mentally arrange elements on a scale according to increasing or decreasing size. It, according to Wadsworth [63, p.95], occurs at different ages for objects such as length, weight and volume. Interestingly, the child invariably seriates length at around age 7, weight at around age 9 and volume at around age 12.

Classification

This involves the child being able to classify something such as beads into two kinds of classes. For example, a student may so happen to be presented with beads of two different colours such as brown and white. For argument sake, all of the beads happen to be wooden. When asked the question: are there more wooden beads than brown beads?, the concrete-operational child is able to answer. In contrast, the preoperational child would not.

Notably, a child’s idea of time and speed develops in terms of a ratio concept of the two dimensions during this period. Prior to this period, the child is unable to understand the relationship.

The Concrete-Operational Period—A Summary

Wadsworth [63, p.100] stresses that the important concept attained during this period is that of reversibility. Piaget considers this an “essential quality in all operations” [63, p.45]. An additional two operations/concepts developed during this period are those of 'seriation' and 'classification'. To re-stress the important point in respect of this stage of cognitive development:

"The concrete-operational child can use logical operations to solve problems involving 'concrete' objects and events. They cannot solve hypothetical problems; problems that are entirely verbal and some problems requiring more complex operations" [63, p.100]. This ability only emerges during the period of Formal Operations.

The Formal-Operational Stage

When a child (now an adolescent) reaches this stage of thought, (usually between the age of 11- 15) they typically possess "the cognitive structural equipment to think as well as adults'" [63, p.101]. From a Piagetian perspective, formal thought and concrete thought are the same, as they "both employ logical operations" [63, p.44].

However, "concrete thought is limited to solving tangible problems of the present" [63, p.102]. This is in contrast to formal-operational thought, where the child/adolescent can "deal with all classes of problem: the present, past, future, the hypothetical and the verbal" [63, p.102]. Moreover, the child/adolescent becomes free from the 'content of problems'. What this means is best explained by example. If a logical argument happens to be "prefixed by the statement: 'suppose coal is white. . .', the concrete-operational child when asked to solve the problem, declares that coal is black and they cannot answer the question" [63, p.104]. The formal-operational child/adolescent would not declare this.

The concept/schemata of proportion develops at all stages of cognitive development, but it is not until the formal-operational stage, that the child will have a fully-fledged understanding of it. Likewise, during the formal-operational period, the child/adolescent is able to deal with the concept of conservation of movement more fully.

The distinction between a child's formal-operational thought and an adult's formal-operational thought is: an adult is less egocentric. The adult—if they have reached the formal-operational stage—is able to separate reality from idealism. A child/adolescent is not; they are idealists.

1.4.2.2 The APOS Theory

APOS is an acronym for the four stages involved in Dubinskey et al.'s [60] theory, namely Action, Process, Object and Schema. Each of these stages is discussed.

Action:

The 'action stage' is concerned with the "transformation of objects perceived by the individual as essentially external and as requiring, either explicitly or from memory, step-by-step instructions on how to perform an operation [the transformation of objects]" [60, p.2]. After reflecting and repeating the 'action stage' for a length of time (the exact duration of which is not stated), students reach a 'process stage'.

Process:

Once in the 'process stage', students can "perform the same kind of action but no longer with the need of external stimuli; they can think of performing the action without actually doing it, and can reverse and compose the action with other processes" [60, p.3]. Reflecting on and repeating the 'process stage', students reach the 'object phase'.

Object:

Once students become aware of a process as a totality, such as realising transformations acting on it (the object), the students are considered to be at an 'object stage' of APOS Development [60].

Schema:

Lastly, when a student integrates actions, processes and objects with some general principles to form a framework which "may be brought to bear upon a problem situation involving that concept" [60, p.3], from an APOS perspective, the student is deemed to have reached the Schema Stage of APOS theory.

An example where the APOS theory has been applied may make it more clear. The authors Briendenbach et al. [64] used the theory to categorise students' understanding of function. They articulate that the distinction between a student at an 'action stage' and a student at a 'process stage' is not clear-cut. Notwithstanding this, the authors claim that an indicator of an 'action conception' of function is one where students require an explicit formula for calculating a value of the dependent variable, given a value of the independent variable. On the other hand, students who possess a 'process conception' of function are, when presented with a graph, able to see the graph as a function even though there may not be an explicit formula defining the function.

Interestingly, drawing inspiration from the APOS theory, Oehrtmann et al. [65] define what they consider to be an 'action view' and 'process view' of function. This is shown in Table 1.2.

Tall et al. [66] discuss the APOS theory in the context of the distinction between 'process' and 'object'. They articulate that it is problematic to explain precisely what is meant by the term 'object'. Nonetheless, Tall et al. [66] use the development of number concept as an example to demonstrate the 'object aspect' of the APOS theory.

They articulate that different ways of counting a set of objects which number '5' is very much indicative of the 'process stage' a person goes through when developing their concept of number. This 'process stage' aids the development of a 'cognitive structure' which in turn allows the individual to use the symbol '5' as if it refers to an 'object'. The fact that there may be no absolute object corresponding to the number '5' is irrelevant.

In effect, when a student reaches the 'object phase' of APOS, it does not matter what the 'object' is, but what a student can do with the object. By 'acting upon' such an object, new processes are generated, which, in turn, generate new objects, which, in turn, generate new schema. Thus, mathematical schemas become more advanced.

Action View	Process View
A function is tied to a specific rule, formula, or computation and requires the completion of specific computations and/or steps.	A function is a generalised input-output process that defines a mapping of a set of input values to a set of output values.
A student must perform or imagine *each action*.	A student can imagine the entire process without having to perform each action.
The "answer" depends on the formula.	The process is independent of the formula.
A student can only imagine a single value at a time as input or output (e.g., x stands for a specific number).	A student can imagine all input at once or "run through" a continuum of inputs. A function is a transformation of entire spaces.
Composition is substituting a formula or expression for x.	Composition is a coordination of two input-output processes; input is processed by one function and its output is processed by a second function.
Inverse is about algebra (switch y and x, then solve) or geometry (reflect across y = x).	Inverse is the reversal of a process that defines a mapping from a set of output values to a set of input values.
Finding domain and range is conceived at most as an algebra problem (e.g., the denominator cannot be zero, and the radicand cannot be negative).	Domain and range are produced by operating and reflecting on the set of all possible inputs and outputs.
Functions are conceived as static.	Functions are conceived as dynamic.
A function's graph is a geometric figure.	A function's graph defines a specific mapping of a set of input values to a set of output values.

Table 1.2 Action and Process Views of Functions. Adapted from [65]

1.4.2.3 The Van Hiele Theory

Van Hiele's Theory [61] of mathematics education is concerned with how children (or students) learn to reason in geometry. Van Hiele postulates that there are five levels which describe how students develop their understanding of geometry. In order to move between these levels, a 'crisis of thinking' is necessary. The memorisation of structures in each level avoids this 'crisis'. There are multiple levels in van Hiele's theory, the first three of which are briefly discussed. These three levels were deemed most applicable to the *Explaining and Transfer Question* because the levels beyond level three become more concerned with mathematical proof, a type of understanding that was not investigated in any of the mathematical items administered to the students. The three levels are discussed in the context of how students might develop an understanding of the theorem of Pythagoras.

Level 1

This level is concerned with what confronts students in a visual sense. In respect of Pythagoras's theorem, a student might be presented with the image of a right-angled triangle superimposed on a grid, as shown in Figure 1.3. As can be seen in Figure 1.3, the square on each side of the triangle is drawn.

Level 2

Students describe what they see in terms of formulating and manipulating symbols. The students must be aware of the properties attributed to these symbols, and that the symbols represent different content in different contexts. In the second level, attention is called to shape.

Relating Level 2 to Pythagoras's theorem, students' attention could be called to the area in terms of square units that is encompassed by each of the squares on the right-angled triangle in Figure 1.3. Students could be asked to count the square units encompassed by each of the squares on the right-angled triangle, thus producing the following image, as shown in Figure 1.4.

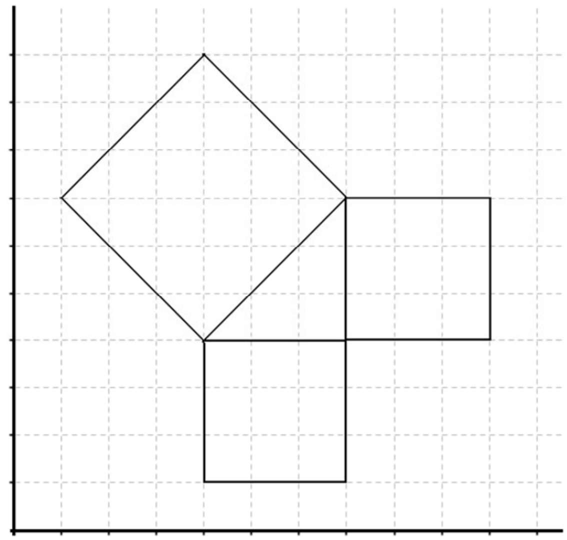


Figure 1.3 A Right-Angled Triangle Superimposed on a Grid.

Level 3

Level 3 is concerned with students' ability to mentally manipulate symbols in order to construct mathematical theories. Students can accomplish this without necessarily having to be aware of what it is the symbols refer to. In essence, what van Hiele terms 'deductive coherence' becomes present. Simply put, van Hiele defines deductive coherence as an ability to use symbolic expressions to formulate theories both from an algebraic and geometric perspective.

Relating this level to Pythagoras's theorem, the students would be expected to deduce and symbolise the relationship that they observed (or should have observed) in Figure 1.4, namely, the area of the square on the hypotenuse of a right-angled triangle is equal to the sum of the area of the squares on the opposite two sides. Furthermore the students operating in this level would be able to determine the length of one of the sides of a right-angled triangle, if they knew the length of the other two sides. Going beyond this level, students would be able to use the relationship in an algebraic sense for more advanced mathematics such as producing the equation of a circle.

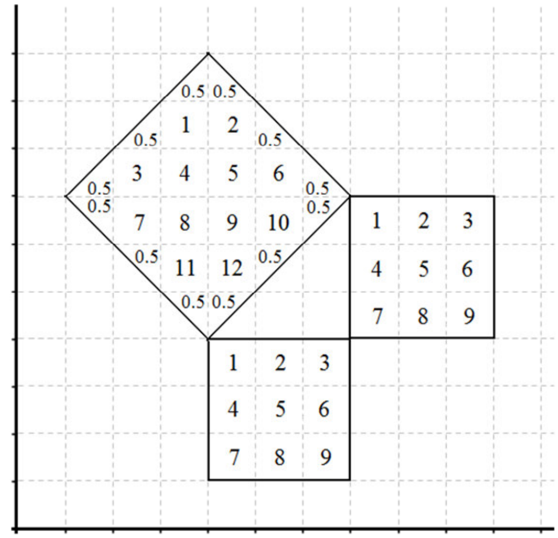


Figure 1.4 A Right-Angled Triangle Superimposed on a Grid.

For educators, not being cognisant of van Hiele's levels may mean that mathematical information is presented to students at too high a level. Van Hiele articulates that the first and second levels are as important as the more advanced levels. He stresses that starting at the first level gives students a better introduction to the deduction method (the third level and beyond).

1.4.2.4 Tall's Theory

To understand Tall's theory, it is necessary to understand the distinction between embodied mathematical objects and mathematical objects—terms used in the theory.

For Tall and Gray [66], there are two types of mathematical object, the distinction between the two being best illustrated by way of example. A triangle or the graph of a function can be called an *embodied mathematical object* because these are objects which begin with perception using the fundamental senses such as sight and which become more mentally based over time. Embodied mathematical objects manifest themselves in the geometrical aspect of mathematics or the graphical aspect of mathematics.

In contrast, *mathematical objects* manifest themselves in the aspect of mathematics concerned with symbols. For example, the symbol for the number 5 can be called an object, whereas a mental image of 5 fingers can be considered to be an embodied mathematical object that embodies the idea/concept of 'five-ness'.

Tall articulates that there are three aspects to mathematics [67], namely:

- Geometric
- Symbolic
- Axiomatic

Other terms that Tall uses for these aspects are: The 1st World; the 2nd World and the 3rd World respectively.

The 1st World

This is what Tall also terms the Conceptual-Embodied World, which consists of a student's thinking about things that they perceive and sense (embodied mathematical objects) not only in the Physical World but in their own mental world of meaning; it is, in essence, concerned with a students' visual-spatial imagery.

The 2nd World

In this World, the student works with symbols used for calculations and manipulation. A student begins with actions such as pointing and counting that are encapsulated as mathematical objects in the form of symbols. The symbols allow the student to switch effortlessly from a procedure or process-to-do in mathematics to a procedure or process-to-think-about. For Tall et al. [62, p. 7], a procedure is a "specific sequence of steps carried out a step at a time", while a process is "any number of procedures which essentially have the same effect"—supposedly the same effect as a single procedure.

The 3rd World

Students work not with familiar objects of experience (embodied mathematical objects or mathematical objects) but with axioms. The axioms then act as a bridge to building theorems.

Each of these mathematical aspects/worlds is associated with a cognitive development. Tall et al. [62, p.1] term this cognitive development "how people build from activities in the environment to developing highly subtle abstract concepts". They claim that this comprises various combinations of 'perception', 'action' and 'reflection'. Tall et al. [62] represent diagrammatically these activities as shown in Figure 1.5.

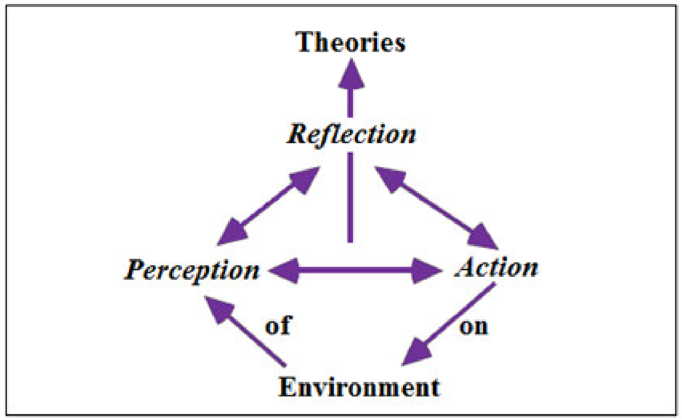


Figure 1.5 Reflection, Perception and Action. Adapted from [62]

A focus on one or more of these activities more so than the others leads to the construction (from the students' perspective) of different aspects of mathematics, as seen in Figure 1.6. Looking at Figure 1.6, the Space and Shape mathematics is similar to Tall's 1st World or Conceptual Embodied World of mathematics [67]. The Symbolic Type Mathematics is reflective of Tall's 2nd World of mathematics, also known as the Proceptual Symbolic World [67]. Lastly, the Axiomatic Mathematics is reflective of Tall's 3rd World of mathematics [67]. Figure 1.7 represents these similarities; the figure also shows examples of mathematical concepts which are encompassed by each of these three *mathematical aspects/worlds*.

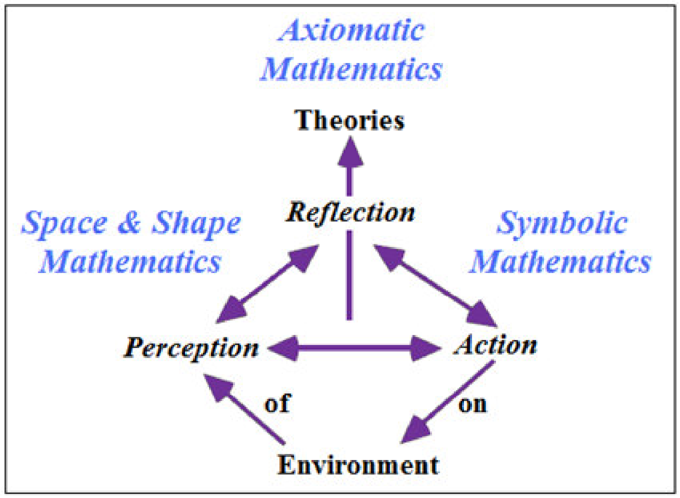


Figure 1.6 Different Aspects of Mathematics. Adapted from [62].

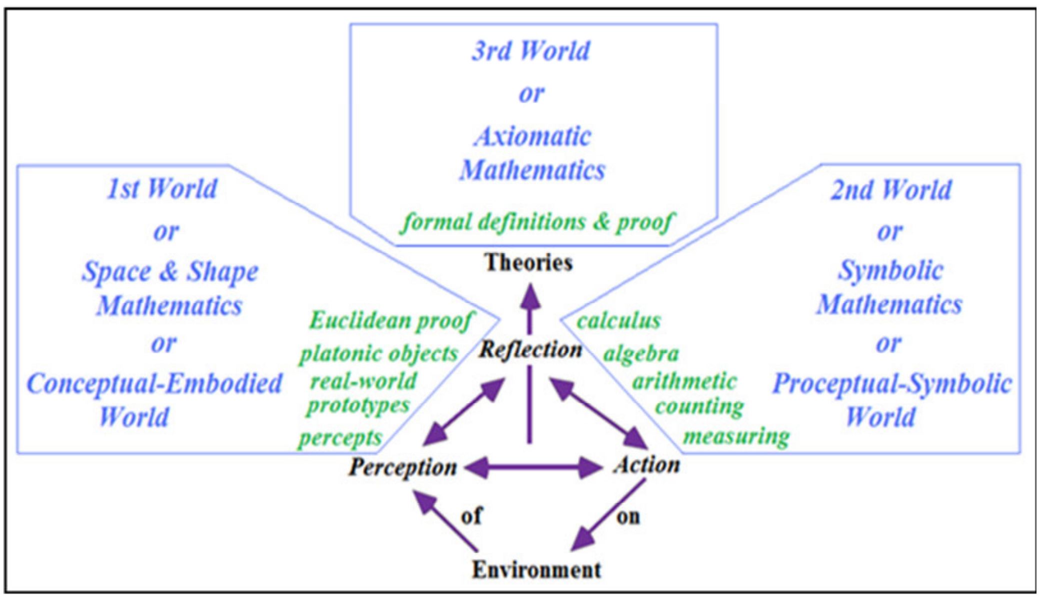


Figure 1.7 Mathematical Concepts Encompassed by Each of the Three Aspects of Mathematics. Adapted from [62].

It is useful to note the distinction that Tall and his fellow researchers make between the terms ‘mathematical concept’ and ‘mathematical object’. According to Tall, the two terms are used in different contexts to “express appropriate ideas” [67, p.9]. The term ‘mathematical concept’ is used colloquially. In contrast, the term ‘mathematical object’ is used in formal mathematics.

Tall [62] postulates that each of the three aspects of mathematics is accompanied by a different type of cognitive development. This cognitive development is highlighted in the purple arrows in Figure 1.7. For example, acting on mathematical objects (symbols) and perceiving these actions would be deemed to be indicative of the cognitive development associated with symbolic mathematics. Furthermore, reflection on these symbolic actions would be considered a precursor to formal mathematics.

1.5 Chapter Summary

Four key strands were identified in the literature as being relevant to this research, namely: 1) chemistry education; 2) the mathematics problem; 3) learning mathematics; and 4) transfer studies. Each of these strands informed the asking of the *Transfer Question* and the *Explaining and Transfer Question*. It was found that transfer is considered, in its basic sense, to be the transfer of skills/knowledge from a learning context to a transfer context. A number of approaches can be taken in assessing the transfer of learning, namely a behaviourist and cognitive approach (referred to as the traditional view of transfer) or the 'situative' approach which is referred to as the actor-oriented view of transfer.

The precursor to the *Explaining and Transfer Question*—the *Conceptual versus Procedural Question*—stemmed from research which claims that conceptual mathematical knowledge transfers more easily than procedural mathematical knowledge. It was felt that mathematical knowledge cannot be classified objectively as either procedural or conceptual in nature. Consequently, the *Conceptual versus Procedural Question* evolved into the form of the *Explaining and Transfer Question*. Various theories on how students learn mathematics—the APOS theory, the van Hiele theory and Tall's theory—were reviewed in order to determine if one of these theories could be used to categorise the degree to which students explained in a mathematics context. Piaget's theory of cognitive development was also reviewed. The research methodology used to answer both the *Transfer Question* and the *Explaining and Transfer Question* is the subject of the next chapter.

Chapter 2

The Research Methodology Used to Investigate the *Transfer Question* and the *Explaining and Transfer Question*

2.1 Chapter Overview

This chapter discusses the research methodology used to answer the *Research Questions* as outlined in Chapter 1. The *Transfer Question* and the *Explaining and Transfer Question* (undertaken in the Main Study) were *informed/grounded* in results which emanated from a Pilot Study.

The theoretical framework chosen to investigate the *Transfer Question* in both the Pilot Study and Main Study is discussed in terms of its strengths and weaknesses. The rationale behind the *Conceptual Versus Procedural Question* (the precursor to the *Explaining and Transfer Question*) is also articulated. The theoretical framework which supported the investigation of the *Explaining and Transfer Question* is also discussed in terms of the framework's strengths and weaknesses. The findings which resulted from the Pilot Study, and in particular, how these findings grounded the evolution of the *Conceptual versus Procedural Question* into the *Explaining and Transfer Question* are detailed.

Lastly, the validity and reliability of the instruments used in the Main Study, the sample profile in both the Pilot Study and the Main Study, and the ethical and implementation issues are all described.

2.2 Overall Research Methodology

The question of which approach was the most appropriate to use to answer the *Transfer Question* during the Pilot Study, Study 1 and Study 2 was raised. Lobato [68; p.187] states: "there is no one best approach to conceive of the transfer of learning". Nonetheless, it was decided that the traditional view of transfer would be the best approach to adopt in order to investigate the *Transfer Question*. The reasons for using such an approach were:

1) The aim of the *Transfer Question* was to determine if students could transfer (apply mathematics knowledge/skills in a chemistry context)—as opposed to investigating more general questions of transfer which the actor-oriented view allows.

2) The *Transfer Question* was investigated using relatively large numbers of students (30 and 45 students in the thermodynamics, and kinetics aspects respectively of the Pilot Study; 30 students in Study 1 and 24 students in Study 2) Time-wise; using an actor-oriented approach would have been impractical, and is more suitable for clinical studies as opposed to group studies.

3) The methods used in the traditional approach are well established [68], whereas "the methods used to document most of the alternative transfer perspectives are emerging" [68 p. 168].

The downside to using the traditional view of transfer lies in the fact that such an approach misses out on what students do transfer to learning situations (if anything at all) when predetermined transfer has not been observed by the researcher.

It can be seen from the 'Research Question dimension' in row four of Table 1.1 that both the *Transfer Question* and the *Explaining and Transfer Question* are respectively reflective of the traditional-view-of-transfer questions, namely: was transfer obtained?; and what conditions facilitate transfer?

During the Pilot Study, it wanted to be determined if students could transfer conceptual mathematical items more so than procedural mathematical items—*The Conceptual versus Procedural Question*. The reason for this stemmed from previous research (as discussed in Section 1.4 in Chapter 1) which suggests that conceptual knowledge transfers more easily than procedural. This raised the question: how is conceptual knowledge distinguished from procedural knowledge?

There is some debate in the literature as to what exactly this distinction is. Anderson [23] claims that the distinction is not absolute. Nonetheless, in this study, it was decided to accept the argument (discussed in Sections 1.4 and 1.4.1 in Chapter 1) that procedural knowledge is found in the symbolic aspect of mathematics while conceptual knowledge is found in the geometric aspect. Consequently, the Pilot-Study Items were classified as procedural or conceptual, depending on whether they were symbolic or graphical in nature; this is shown in Table 2.1.

Item Number	**Mathematical Item**	**Item Type**
1	Calculation of Slope.	Procedural
2	Determining which Line has the Greatest Rate of Change.	Conceptual
3	Differentiation.	Procedural
4	Graphical Interpretation of the Meaning of Derivative.	Conceptual
5	Multiplication of Fractions.	Procedural
6	Usage of Exponent Laws.	Procedural
7	Graphing a Function.	Conceptual
8	Evaluation of an Integral.	Procedural
9	Graphing an Integral.	Conceptual

Table 2.1 The Pilot-Study Mathematical Items and their Classification as either Procedural or Conceptual in Nature.

During the Main Study, to determine the degree to which students explained in the *Explaining and Transfer Question,* Tall's theory was chosen for three reasons, the last of which was considered most important.

1) Van Hiele's theory is only directly relevant to geometry. In contrast, Tall's theory encompasses both the symbolic and geometrical aspect of mathematics, as well as the axiomatic.

2) The APOS theory does not make it explicit how exactly a researcher can determine if a student is at an action, process, object or schema stage for any mathematical concept.

3) In Tall's theory, there is a cognitive development which is explicitly described. This cognitive development is shown in Figure 1.7. Being aware of this cognitive development, it was felt that probing students' explanations could unlock the degree to which student's explained. How this was accomplished is described in Section 2.4.3.2 in Chapter 2.

The validity of the mathematical items used in the Main Study, as measuring students' ability to carry out particular mathematical tasks in a mathematics context, (e.g. Item 14 dealing with the evaluation of an integral), could not be determined by means of an average inter-item correlation [69] because there was only one item in this regard in a mathematics context. Likewise, there was only one corresponding item in the chemistry context. Despite not being able to measure an average inter-item correlation for any of the items (this not being the focus of the research), it was determined if the items in a mathematics context were appropriate in terms of preparing students to answer a similar item, which they would encounter in a chemistry context. Such a determination involved input from chemistry lecturers and mathematics lecturers from the School of Chemical Sciences and the School of Mathematical Sciences respectively.

It was not possible to measure the internal reliability of each item across both studies. To do so would have involved using a two-sample t-test for performance on each item across both studies in either a mathematics context or chemistry context. This would have required multiple measures of the students' performance in Study 1 on the same item (via different questions in respect of this item) in both the mathematics context and chemistry context; the results of which would then have to be compared for statistically-significant difference between the students' performance in Study 2 (using measures similar to those used in Study 1) in both the mathematics context and chemistry context.

Instead, a more qualitative approach was taken towards assessing the reliability of the items across both studies.

The sample of students involved in both the Pilot Study and Main Study were second-year undergraduate chemistry students, coming from a variety of science undergraduate degrees. In terms of ethics, all of the students were given an opportunity to participate, on a voluntary basis. The students were informed as to the purpose of the research, namely improving chemistry undergraduate students' understanding of mathematics. Also, all of the students were allowed to stop participating in the research at any time. Lastly, the students were informed that information used from such research would be stored in accordance with relevant Data Protection Acts. The research was implemented in Dublin City University lecture halls.

2.3 The Pilot Study

A Pilot Study was undertaken during the 07/08 academic year. The *Transfer Question* was: can students transfer mathematical knowledge from a mathematics context to a chemistry context? The *Conceptual versus Procedural Question* was: do students transfer conceptual mathematical knowledge more so than procedural mathematical knowledge?

2.3.1 The Pilot-Study Sample

The study was conducted amongst a sample of second-year university students. All of the participants were completing a core module in chemical kinetics and thermodynamics as part of their science degree. The students were drawn from degree programmes in the field of Chemical and Pharmaceutical Science, Analytical Science, Environmental Science and Health, and Science Education. All of the students had completed and passed a calculus module during the first-year of their studies. The calculus module was taught by lecturers from the School of Mathematical Sciences. All of the calculus concepts were presented in an abstract manner, using mathematical notation in terms of x and y. This module was designed to equip the students with basic calculus concepts that they could use in later parts of their studies such as in chemical kinetics and thermodynamics.

2.3.2 Diagnostic Tools

To determine the students' mathematical knowledge and transfer ability, diagnostic tools were developed. There were four Diagnostic Tools. Two of the tools were based on knowledge of mathematical items in a mathematics context, and two were based on these mathematical items in a chemistry context. The mathematical items contained in each of the tools are shown in Tables 2.2 and 2.3. The items were chosen on the basis of their importance in chemical kinetics and thermodynamics.

Diagnostic Tool 1 contained an array of mathematical items relevant to chemical kinetics asked within a mathematics context, while Diagnostic Tool 2 replicated these items, but asked them within a chemistry context. Diagnostic Tool 3 contained mathematical items relevant to thermodynamics asked within a mathematics context, while Diagnostic Tool 4 contained the same mathematical items, but asked within a chemistry context. The items in the Diagnostic Tools are given in Appendix A.

A mathematics context was defined in terms of containing a mathematical concept represented in an abstract sense, using the symbolic notation x and y: for example, finding the derivative of the expression: $y = x^{-1}$. A chemistry context was defined in terms of containing the same mathematical concept in chemistry-notational form. A backdrop as to the origin of the chemistry concept, which the mathematical concept in chemistry-notational form represented, was also present e.g. in the form of an explanatory sentence. However, despite the backdrop, it was deemed that students did not necessarily have to understand it in order to answer the mathematical concept in chemistry-notational form. For example: given that the volume (V) of a gas is inversely proportional to the pressure (P) of the gas, find the derivative of the expression: $P = V^{-1}$: this question does not require an understanding of the gas laws to answer.

An example of an item (Item 9) used in the Pilot Study is shown in Figure 2.1. All of the items contained a Part A and a Part B both in the mathematics context and chemistry context. The Part A gave data to address the *Transfer Question* and the *Conceptual versus Procedural Question*.

The Kinetics Mathematical Items
Tools
Diagnostic Tool 1: Mathematical Items in a Mathematics Context
Diagnostic Tool 2: Mathematical Items in a Chemistry Context
Items:
▪Item 1: Calculation of slope.
▪Item 2: Determining which Line has the Greatest Rate of Change.
▪Item 3: Differentiation.
▪Item 4: Graphical Interpretation of the Meaning of Derivative.
▪Item 5: Multiplication of Fractions.
▪Item 6: Usage of Exponent Laws.

Table 2.2 The Kinetics Mathematical Items Used in the Pilot Study.

The Thermodynamics Mathematical Items
Tools
Diagnostic Tool 3: Mathematical Items in a Mathematics Context
Diagnostic Tool 4: Mathematical Items in a Chemistry Context
Items:
▪Item 7: Graphing a Function.
▪Item 8: Evaluation of an Integral.
▪Item 9: Graphing an Integral.

Table 2.3 The Thermodynamics Mathematical Items Used in the Pilot Study.

Item 9: Graphing an Integral

Mathematics Context

(A) Draw a diagram (in Figure 1) that represents the area corresponding to the integral:

$$\int_1^3 \frac{1}{x}dx$$

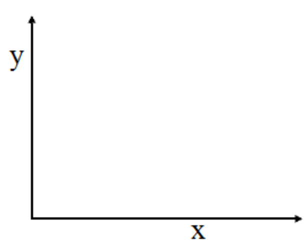


Figure 1

Chemistry Context

(A) The relationship:

$$P = \frac{1}{V},$$

where P is the pressure of a gas and V is its volume represents the ideal gas law applied to an isothermal system. Indicate in Figure 1, the area corresponding to the integral:

$$w = -\int_{V_1}^{V_2} \frac{1}{V}dV$$

which represents the work done by the system (the gas) in expanding from an initial volume:
($V_1 = 1m^3$) to a final volume ($V_2 = 3m^3$), for a reversible isothermal gas expansion.

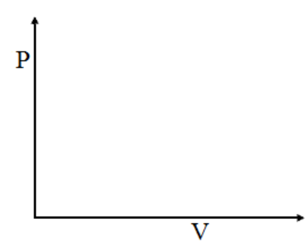


Figure 1

Figure 2.1 The Pilot Study Mathematical Item for the Graphing of an Integral.

The students were administered all of the Diagnostic Tools separately. Each tool, in a mathematics context, was spaced a week apart from its matching chemistry-context tool, so as to avoid, for example, the possibility of Diagnostic Tool 1 helping students to answer the same items in Diagnostic Tool 2 due to a training or recognising-of-patterns effect [10]. All of the students were allowed approximately half an hour to complete each tool. The tools were administered separately to all of the students during a lecture. The number of students attending lectures varied: because of this, if a student was present when Diagnostic Tool 1 was administered, but not when the corresponding Diagnostic Tool 2 was administered, their information was not able to be used to answer the *Transfer Question* and the *Conceptual versus Procedural Question*. Thus, for the chemical kinetics mathematical items (Items 1-6), there were 45 students who completed both Diagnostics Tool 1 and Diagnostic Tool 2. For the thermodynamics items (Items 7-9), there were 30 students who completed Diagnostic Tool 3 and Diagnostic Tool 4.

It was determined if each item in a mathematics context was appropriate, in terms of preparing students to answer the similar, matching item in a chemistry context. Such a determination involved input from chemistry lecturers and mathematics lecturers from the School of Chemical Sciences and the School of Mathematical Sciences respectively.

2.3.3 Data Analysis

2.3.3.1 The Transfer Question

Students' responses to Part A for each item in each tool were marked as correct or incorrect. The students were considered to have transferred a mathematical item if they answered that mathematical item correctly in the mathematics context and correctly in the corresponding chemistry context.

For each item, categorical-statistical tests were used to determine if there was statistically-significant transfer (that is, an association between answering correctly in a mathematics context and correctly in a chemistry context). Depending on the data set, either the Chi-Squared Test or Fisher's Exact Test was used. For example, considering Item 7—Graphing a Function—the manner in which students responded determined which cell they were positioned within, as shown in Table 2.4.

Chemistry Context	Mathematics Context		
	Correct	***Incorrect***	***Total***
Correct	6(3)	4(7)	10
Incorrect	3(6)	17(14)	20
Total	9	21	30
			$p = 3.0\times10^{-2}$

Table 2.4. How the Significance of Observed Transfer was Investigated for Item 7 in the Pilot Study: The frequencies of students falling into each cell were the observed frequencies. The values adjacent to these (in parentheses) were the expected frequencies.

From Table 2.4, of the nine students who answered Item 7 correctly in a mathematics context, six of these answered correctly in the corresponding chemistry context. Of the ten students who answered correctly in the chemistry context, six of these students answered correctly in the corresponding mathematics context. The basic laws of probability dictate that the number of students that would be expected to answer correctly in both the mathematics context and chemistry context due to chance alone is determined from multiplying the probability of being correct in a mathematics context by the probability of being correct in a chemistry context and then multiplying this value by the sample size. The value obtained (shown in parentheses in Table 2.4) was three—the expected number of students answering correctly in both contexts due to chance alone. The actual value observed was six which raised the question of whether this value was significant. Using the relevant categorical-statistical test—in this case, Fisher's Exact Test— for the two-by-two contingency table, it was found that six was statistically significant at a confidence level of 95%. Thus, the conclusion was: if a student answers the item correctly in a mathematics context, the likelihood of them answering the same item in a chemistry context correctly not due to chance alone is strong. A more detailed discussion of the theory behind Fisher's Exact Test and the Chi-Squared Test can be found in Appendix C.

2.3.3.2 The Conceptual versus Procedural Question

For the mathematical items classified as procedural and conceptual during the Pilot Study (as shown in Table 2.1) it was investigated if instances of the transfer of conceptual items were more commonplace than instances of the transfer of procedural items.

2.3.4 Results

2.3.4.1 The Transfer Question

The observed and expected values in the contingency tables that were used to probe whether the transfer observed for each mathematical item was significant are given (in row form) in Table 2.5. In reading the table, it is clear, for example, for Item 1, the calculation of slope, that 22 students transferred the item (answered the item correctly in both a mathematics context and chemistry context).

The number adjacent to twenty-two (in parentheses) is twenty which is the number of students that would have been expected to answer correctly in both the mathematics context and chemistry context due to chance alone. In column five of the table, the p-value for the observed number of students who transferred is equal to 0.11. Thus, the transfer observed for this item was deemed not to be statistically significant. In other words, because the p-value was not less than or equal to 0.05, it could not be concluded that answering this item correctly in a mathematics context meant a student is likely to associate with answering it correctly in a chemistry context. Nonetheless, transfer was observed for the item because 22 students answered the item correctly in both contexts. As can be seen from column three, transfer was observed for all Items 1-7, but not for Items 8-9.

In terms of the significance of the transfer observed, for Item 7—the graphical representation of a function— the p-value for the transfer observed in column three was found to be 0.03, less than a p-value of 0.05, thus suggesting that the transfer observed was significant. In other words if a student answers this item correctly in a mathematics context, they are likely to associate with answering the item in a chemistry context, not due to chance alone.

2.3.4.2 The Conceptual versus Procedural Question

The procedural items in Table 2.5 were Items 1, 3, 5, 6 and 8. Forty-five students answered Items 1, 3, 5 and 6 in both the mathematics context and chemistry context. Of these students, 22 transferred Item 1; 4 transferred Item 3; 25 transferred Item 5 and 3 transferred Item 6. Thirty students answered Item 8 in both the mathematics context and chemistry context. Of these students, none transferred Item 8.

The conceptual items in Table 2.5 were Items 2, 4, 7 and 9. Forty-five students answered Items 2 and 4 in both the mathematics context and chemistry context. Of these students, 15 transferred Item 2 and 9 transferred Item 4. Thirty students answered Items 7 and 9 in both the mathematics context and chemistry context. Of these students, 6 transferred Item 7 but none transferred Item 9. The percentage of students who answered the items correctly in a mathematics context is shown in Table 2.6.

Because conceptual items did not appear to be transferred by students any more so than procedural items, it was concluded that:

1) conceptual knowledge may be no more transferrable than procedural knowledge; ***or***
2) the view that conceptual mathematical knowledge is graphical in nature, while procedural mathematical knowledge is symbolic in nature, is not correct.

Item Number	Mathematical Item	Observed Frequencies and Expected Frequencies (in parentheses) in the Contingency Tables.				Transfer (p-values)
		Correct in MC* and CC**	**Correct in MC and Incorrect in CC**	**Incorrect in MC and Correct in CC**	**Incorrect in MC and CC**	
1	Calculation of Slope.	22(20.0)	0(2.0)	19(21.0)	4(2.0)	0.11
2	Determining which Line has the Greatest Rate of Change.	15(16.8)	21(19.2)	6(4.2)	3(4.8)	0.27
3	Differentiation.	4(3.1)	0(0.9)	31(31.9)	10(9.1)	0.56
4	Graphical Interpretation of the Meaning of Derivative.	9(9.8)	8(7.2)	17(16.2)	11(11.8)	0.61
5	Multiplication of Fractions.	25(23.2)	4(5.8)	11(12.8)	5(3.2)	0.24
6	Use of Exponent Laws.	3(1.2)	1(2.8)	10(11.8)	31(29.2)	0.06
7	Graphical Representation of a Function.	6(3.0)	3(6.0)	4(7.0)	17(4.0)	0.03
8	Evaluation of an Integral.	0(0.0)	5(5.0)	0(0.0)	25(25.0)	1.00
9	Graphical Representation of an Integral.	0(0.2)	1(0.8)	5(4.8)	24(24.2)	1.00

MC* *- Mathematics Context;* ***CC***** *- Chemistry Context.*

Table 2.5 Results from the Pilot-Study Contingency Tables that Were Used to Investigate the Significance of Observed Transfer for Each Item.

Item Number	Mathematical Item	Item Type	%* of Correct Students in a Mathematics Context
1	Calculation of Slope.	Procedural	85%
2	Determining which Line has the Greatest Rate of Change.	Conceptual	53%
3	Differentiation.	Procedural	74%
4	Graphical Interpretation of the Meaning of Derivative.	Conceptual	54%
5	Multiplication of Fractions.	Procedural	76%
6	Use of Exponent Laws.	Procedural	29%
7	Graphical Representation of a Function.	Conceptual	31%
8	Evaluation of an Integral.	Procedural	16%
9	Graphical Representation of an Integral.	Conceptual	9%

* The % figures are rounded to the nearest whole number.

Table 2.6 The Percentage of Students who Answered Each Item Correctly in a Mathematics Context in the Pilot Study.

2.3.5 Implications for Main Study

The main findings from the Pilot Study are:

- In terms of the *Transfer Question*, transfer was observed for Items 1-7, but not for Items 8-9. In respect of statistically significant transfer, it was observed for one item only, namely Item 7. A possible reason for the lack of statistically significant transfer observed may be that for certain items, low percentages of students answered correctly in a mathematics context, as can be seen from Table 2.6.

- Looking at Table 2.6, the percentage of students who could answer the items correctly in a mathematics context ranged from 9% to 85%. This suggested that the problems students have with mathematics in a chemistry context may not be due to transfer, but instead due to an absence of mathematical knowledge in a mathematics context. The low percentages of students who answered correctly in a mathematics context for certain items (especially Items 8 and 9) also suggested that students cannot be expected to transfer mathematical knowledge to a chemistry context if they do not possess the knowledge in a mathematics context.

Furthermore, the performance of students in the mathematics context for the majority of items indicated the need for mathematical interventions in order to improve students' mathematical ability, with a view that doing so, would improve students' ability to transfer. The design and impact of such mathematical interventions are discussed in Chapter 4.

- In terms of the *Conceptual versus Procedural Question*, it was found that, based on the presupposition that procedural knowledge is symbolic in nature and conceptual knowledge is graphical in nature, conceptual knowledge does not transfer any more so than procedural knowledge. This suggested:1) the view that conceptual knowledge transfers more easily than procedural knowledge may be wrong; or 2) classifying conceptual knowledge as graphical in nature, and procedural knowledge as symbolic in nature, may not be correct.

  Thus, conceptual mathematical knowledge may be transferred more easily by students than procedural mathematical knowledge if the definition as to what constitutes conceptual mathematical knowledge is re-defined. However, as Anderson [23] articulates: the difference between conceptual and procedural knowledge is not absolute. Consequently, it was decided not to investigate the *Conceptual versus Procedural Question* further.

  Rather, in the Main Study it was decided to investigate:1) whether students who explained their mathematical reasoning in a mathematics context associated with transfer (the 1st aspect of the *Explaining and Transfer Question*); and 2) whether students who evidenced a certain degree of explanation in terms of Tall's theory [62] associated with transfer more so than other students (the 2nd aspect of the *Explaining and Transfer Question*).

The main findings from the Pilot Study informed the development of the Main Study as follows:

- The *Transfer Question* remained the same;

- The *Conceptual versus Procedural Question* evolved into the *Explaining and Transfer Question* because of the difficulty in classifying mathematical items as reflective of procedural knowledge or conceptual knowledge*; and*

- The mathematical items in the Diagnostic Tools for the Main Study (given in Appendix D) were modified to be more realistic in terms of how the mathematics necessary for the chemistry context is replicated in the corresponding mathematics context.

2.4 The Main Study

Informed by the Pilot Study, the Main Study comprised of two studies—Study 1 and Study 2. Study 1 and Study 2 were conducted with two samples of second-year undergraduate students during the academic years 08/09 and 09/10 respectively. The *Transfer Question* was: can students transfer mathematical knowledge from a mathematics context to a chemistry context? In Study 2, students were reminded of the mathematical knowledge (in a mathematics context) that they needed to be able to transfer. The effect of this reminder was investigated to determine if there were: 1) improved instances of transfer; and 2) if it had an effect on the statistical significance of the transfer observed. There were two aspects to the *Explaining and Transfer Question*: 1) do students who evidence an ability to explain their reasoning in a mathematics context associate with transfer?; and 2) do students who evidence a certain degree of explanation for a particular item in a mathematics context in terms of Tall's theory [62] associate with transfer more so than other students.

2.4.1 The Samples

Thirty students participated in Study 1, while 24 students participated in Study 2. All of the participants were volunteers. All of the participants were completing a core module in chemical kinetics and thermodynamics as part of their science degree at the time the studies were undertaken. Like the students in the Pilot Study, the students were drawn from degree programmes in the field of Chemical and Pharmaceutical Science, Analytical Science, Environmental Science and Health and Science Education. The participants had completed and passed a calculus-based mathematics module during the first year of their studies. This mathematics module was designed to equip the students with basic calculus concepts applicable to their courses. In order to answer the *Transfer Question* and the *Explaining and Transfer Question*, the Diagnostic Tools used in the Pilot Study were re-designed, as already referred to in Section 2.3.5.

2.4.2 The Diagnostic Tools

There were four Diagnostic Tools. Two of the tools were based on determining knowledge of mathematical items in a mathematics context, and two were based on these mathematical items in a chemistry context. The mathematical items contained in each of the tools are shown in Tables 2.7 and 2.8. Diagnostic Tool 1 contained twelve mathematical items relevant to chemical kinetics asked within a mathematics context, while Diagnostic Tool 2 replicated these items, but within a chemistry context. Diagnostic Tool 3 contained three mathematical items relevant to thermodynamics asked within a mathematics context, while Diagnostic Tool 4 contained the same mathematical items, but within a chemistry context. The items in the diagnostic tools are given in Appendix D.

The Thermodynamics Mathematical Items
Tools
Diagnostic Tool 3: Mathematical Items in a Mathematics Context
Diagnostic Tool 4: Mathematical Items in a Chemistry Context
Items:
▪Item 13: Graphing a Function.
▪Item 14: Evaluation of an Integral.
▪Item 15: Graphing an Integral.

Table 2.7 The Thermodynamics Mathematical Items Used in the Main Study.

The Kinetics Mathematical Items
Tools
Diagnostic Tool 1: Mathematical Items in a Mathematics Context
Diagnostic Tool 2: Mathematical Items in a Chemistry Context
Items:
▪Item 1: Calculating Slope.
▪Item 2: Sketching a Line with Positive Slope.
▪Item 3: Sketching a Line with Positive Slope.
▪Item 4: Sketching a Line with Negative Slope.
▪Item 5: Generating an Expression for Slope.
▪Item 6: Generating an Expression for Derivative.
▪Item 7: Interpreting Derivative.
▪Item 8: Usage of Exponentials.
▪Item 9: Usage of Natural Logarithms.
▪Item 10: Proportionality.
▪Item 11: Graphing an Exponential Function.
▪Item 12: Graphing a Natural Logarithmic Expression.

Table 2.8 The Kinetics Mathematical Items Used in the Main Study.

An example of one of the items used in the Main Study is shown in Figure 2.2. Students need to be able to calculate rate of change in a chemistry context in order to understand chemical kinetics. Calculating the rate of change is similar to calculating the slope of a line in a mathematics context. It can also be seen that the item in Figure 2.2 contains a Part A and a Part B. The Part A allowed the answering of the *Transfer Question.* The Part B in the mathematics context was used to answer the *Explaining and Transfer Question* in terms of: 1) whether students who explained their reasoning in a mathematics context associated with transfer; and 2) whether a certain degree of explanation given by students in terms of Tall’s theory [62] in a mathematics context, associated with students being able to transfer.

Item 1: Calculating Slope

Mathematics Context

(A) Calculate the slope of the straight line from the two points given in Figure 1.

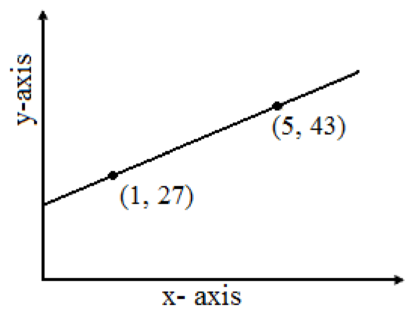


Figure 1

(B) Explain what this number means.

Chemistry Context

(A) Calculate the rate of change of the concentration of the reactant with respect to time over the time interval (Δt) from the two points given in Figure 1.

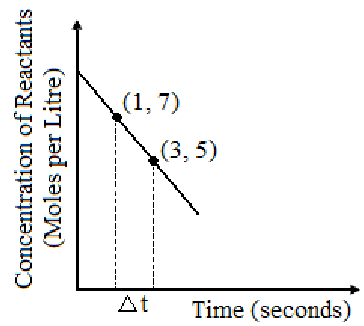


Figure 2

(B) Explain what this value means.

Figure 2.2 Item 1 Used in the Main Study.

It can be seen in Appendix D that each item (except Item 6) contained a Part B. Item 6 did not contain a Part B because the item did not require a calculation or a graphical representation to be performed by the student, but instead only an explanation. Thus, looking for evidence of transfer for this item also represented looking for evidence of students' ability to explain in a mathematics context and transfer. It should be noted that during Study 1, Items 2-5 and Item 11 did not contain a Part B. In the case of Items 2-5, it was felt that such questions would lead to a significant amount of repetition in student answers. It was thought that the Part B data from Items 2-5 would be similar to the Part B data from Item 1.

The chemistry items in both Study 1 and Study 2 were analogous to the items in the mathematics context, containing both Part A and Part B questions. While information in the form of Part Bs for the chemistry items did not address the research questions, it was felt that garnering such information could be useful nonetheless.

The Students were administered all of the Tools separately and were allowed approximately thirty minutes to complete each one. Each tool was administered a week apart so as to again avoid the possibility of, for argument sake, Diagnostic Tool 1 helping students to answer the same items in Diagnostic Tool 2 due to a training or recognising-of-patterns effect [10]. By doing this with all of the Diagnostic Tools, it was envisaged that this would provide a more accurate investigation of students' ability to transfer. The percentage of students who answered each item correctly across both studies can be seen in Table 2.9.

Looking at Table 2.9, it can be seen that for Items 1-3, 6-7, 9-11 and 13, students' performance on those items across both studies did not vary much (less than or equal to 10%) in a mathematics context. For Items 4-5, 12-13 and 15, the variation was greater, ranging from 13% to 36%. One of the possible reasons for a large degree of variation in these items could be due to the fact that during Study 2, students were reminded at the start of their chemical kinetics and thermodynamics module of 'how to do' these mathematical items in a mathematics context. This reminder may be the reason why, overall, the correct answering of the mathematical items in a mathematics context increased more so in Study 2 than the correct answering of the mathematical items in a chemistry context.

For the mathematics in the chemistry context, the variation appears greater than the variation in the mathematics context. Only for Items 2 and 6 is it less than 10%, while for the remainder of the items, the variation ranges from 11% to 31%, although it should be noted that for 10 out of 13 of these items, the variation is only between 11% to 16%.

Main Study: Percentage of Students who Answered Each Item Correctly					
		Mathematics Context		**Chemistry Context**	
	Item	**Study 1**	**Study 2**	**Study 1**	**Study 2**
Item 1:	Calculating Slope.	90%	96%	93%	79%
Item 2:	Sketching a Line with Slope.	83%	92%	70%	63%
Item 3:	Sketching a Line with Slope.	80%	87%	70%	58%
Item 4:	Sketching a Line with Slope.	76%	96%	53%	63%
Item 5:	Generating an Expression for Slope.	66%	67%	53%	42%
Item 6:	Generating an Expression for Derivative.	20%	17%	26%	33%
Item 7:	Interpreting Derivative.	73%	67%	83%	96%
Item 8:	Usage of Exponentials.	43%	58%	33%	8%
Item 9:	Usage of Natural Logarithms.	43%	46%	40%	29%
Item10:	Proportionality.	43%	38%	56%	79%
Item 11:	Graphing an Exponential Function	3%	13%	13%	0%
Item 12:	Graphing a Natural Expression.	10%	46%	13%	29%
Item 13:	Graphing a Function.	33%	37%	7%	29%
Item 14:	Evaluation of an Integral.	16%	29%	10%	25%
Item 15:	Graphing an Integral.	13%	29%	10%	25%

Table 2.9 The Percentage of Students who Answered Each Item Correctly in the Main Study.

2.4.3 Data Analysis

2.4.3.1 The Transfer Question

Like the Pilot Study, students were considered to have transferred if they answered the item correctly in the mathematics context and in the corresponding chemistry context. For each Item, testing for statistically significant transfer (that is, an association between answering correctly in a mathematics context and correctly in a chemistry context) was investigated using categorical statistical tests.

For example, for Item 1, students were placed in a contingency table as shown in Table 2.10. The number of students who were correct in both the mathematics context and chemistry context was 25. The p-value of 1 indicated that this number was not significant at a confidence level of 95%. Thus, the conclusion was: if a student answers Item 1 correctly in a mathematics context, they are no more likely to answer it correctly in a chemistry context than a student who answers the item incorrectly in a mathematics context. This type of analysis was carried out for all of the items.

Chemistry Context	**Mathematics Context**		
	Correct	***Incorrect***	***Total***
Correct	25(25.2)	3(2.8)	28
Incorrect	2(1.8)	0(0.2)	2
Total	27	3	30
			p = 1

Table 2.10. Testing for an Association between Being Correct in a Mathematics Context and Being Correct in a Chemistry Context for Item 1 in Study 1.

2.4.3.2 The Explaining and Transfer Question

Ability to Explain in a Mathematics Context and Transfer

The first aspect of the *Explaining and Transfer Question*, namely, do students who can explain their reasoning in a mathematics context associate with transfer, was investigated using the principles of qualitative data analysis as described by Cohen [70]. This type of analysis is used to distil key categories of explanation from students' Part B responses.

The approach comprised of a number of stages: 1) reading students' qualitative responses; 2) identifying the themes/meanings running through these responses; and 3) organising the themes/meanings into categories.

Examples of student responses for Item 1 in Study 1 are shown in Table 2.11; the category allocated to each response is also shown.

Students' Responses for Item 1— Calculating Slope.	**Category**
Change in y increases 4 for every 1 in x.	Refer to how much y increases by for a unit increase in x.
It means the rate at which the line increases.	Refer to the rate at which the line increases.
Is the slope of the line so the value is 4.	Refer to the slope value being 4.
The slope of the line is positive and increasing at a rate.	Refer to the slope as increasing
How much the line increases/decreases.	Refer to the slope as how much the line increases/decreases.

Table 2.11.Examples of Students' Responses and Examples of the Categories Allocated to Students' Responses.

The resultant categories that emerged from Study 1 for Item 1 are shown in the first column of Tables 2.12 and 2.13.

Category	Frequency	Type of Understanding
1. Refer to how much y increases for a unit increase in x.	10	**Ability to Explain**
2. Refer to slope as a measure of steepness.	1	
3. Refer to the rate at which the line increases.	3	
4. Refer to the slope value being 4.	4	
5. Refer to the slope being positive.	1	
6. Provide no reason.	1	**Inability to Explain**
7. Refer to the slope as rising.	1	
8. Refer to the slope as the distance between data points.	1	
9. Refer to the slope as increasing.	3	
10. Refer to the slope as how much the line increases/decreases.	1	
11. Interpret slope as meaning for a y-unit increase, there is a 4 unit x increase.	1	

Table 2.12 The Resultant Categories for the Students in Study 1 who Answered Item 1, Part A Correctly.

Category	Frequency	Type of Understanding
1. Label the data points incorrectly with respect to their insertion into the slope formula and refer to slope as being an angle.	1	**Inability to Explain.**
2. Use the inverse of the slope.	2	

Table 2.13 The Resultant Categories for the Students in Study 1 who Answered Item 1, Part A Incorrectly.

It should be noted that an inter-rater reliability approach [70] was used to decide what the categories were. An inter-rater reliability approach involves a number of researchers to analyse qualitative data for 'categories of meanings' that appear to emerge from the data. The researchers compare what they consider to be categories of meaning with other researchers in order to reach a consensus. A researcher from the School of the Chemical Sciences and a researcher from the School of Mathematical Sciences worked in conjunction with me to reach this consensus. Once the categories were decided upon, an inter-rater reliability approach was again used to decide whether each category was reflective of either an ability to explain or an inability to explain. The categories which were reflective of either an ability to explain or an inability to explain for Item 1 in Study 1 are shown in Tables 2.12 and 2.13. For each Item, testing for the presence of an association between evidencing an ability to explain in a mathematics context and being able to transfer was carried out using categorical statistical tests.

For example, for Item 1, students were placed in a contingency table as shown in Table 2.14. The number of students who evidenced an ability to explain in a mathematics context and who also transferred was 13. Because this number was greater than the expected number of 11.7, an association 'appeared' present. However, the p-value of 0.34 indicated that this 'apparent association' was not significant at a confidence level of 95%. Thus, the conclusion was: if a student evidences an ability to explain their reasoning for the calculation of slope in a mathematics context, they do not associate with the transfer of that item any more so than students who do not evidence an ability to explain their reasoning. This type of analysis was carried out for all of the items which had a Part B.

Ability to Transfer	**Explanation in a Mathematics Context**		
	Ability to Explain	***Inability to Explain***	***Total***
Transferred	13(11.7)	12(13.3)	25
Did not Transfer	1(2.33)	4(2.7)	5
Total	14	16	30
			p = 0.34

Table 2.14.Testing for an Association between Evidencing an Ability to Explain in a Mathematics Context for Item 1 in Study 1 and Being Able to Transfer that Item.

Degree of Explanation in a Mathematics Context and Transfer

To determine the degree to which students explained in a mathematics context, Tall's theory was used. Tall's theory is summarised in Figure 2.3. The cognitive aspect of the theory is highlighted in purple. The theory was used to classify the nature of the mathematical items in Diagnostic Tools 1 and 3 that were used in the Main Study. It was found that all of the mathematical items fell into either one of four categories, namely:

- 1st World;
- 2nd World;
- Movement from the 1st World to 2nd World; or
- Movement from the 2nd World to the 1st World.

The classification of the mathematical items used in Diagnostic Tools 1 and 3 is shown in Table 2.15. Why each item was classified as such is described in Section 3.5 in Chapter 4.

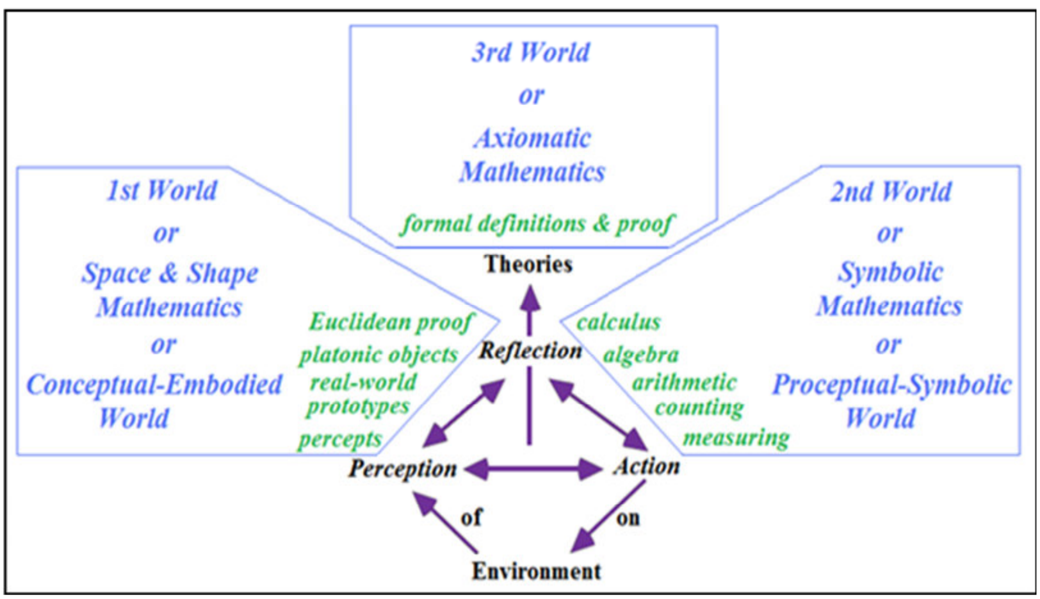


Figure 2.3 Mathematical Concepts Encompassed by Each of the Three Aspects of Mathematics. Adapted from [62]

Item Number	Mathematical Item	Type of Item [62]
1	Calculating Slope.	2^{nd} World
2 and 3	Sketching of Lines with Positive Slope (two items in this regard).	Movement from the 2^{nd} World to the 1^{st} World
4	Sketching of a Line with a Negative Slope.	Movement from the 2^{nd} World to the 1^{st} World
5	Generating an Expression for Slope.	Movement from the 1^{st} World to the 2^{nd} World
6	Generating an Expression for Derivative.	Movement from the 1^{st} World to the 2^{nd} World
7	Interpreting Derivative.	1^{st} World
8	Usage of Exponentials.	2^{nd} World
9	Usage of Natural Logarithms.	2^{nd} World
10	Proportionality.	2^{nd} World
11	Graphing an Exponential Function.	Movement from the 2^{nd} World to the 1^{st} World
12	Graphing a Natural Logarithmic Expression.	Movement from the 2^{nd} World to the 1^{st} World
13	Graphing a Function.	Movement from the 2^{nd} World to the 1^{st} World
14	Evaluation of an Integral.	2^{nd} World
15	Graphing an Integral.	Movement from the 2^{nd} World to the 1^{st} World

Table 2.15 The Mathematical Items Used in the Main Study Classified in Terms of Tall's Theory [62].

The cognitive aspect of Tall's theory (highlighted in purple arrows in Figure 2.3.) was then used to classify the category of explanation which students 'fell into' during the analysis of the first aspect of the *Explaining and Transfer Question*. For the mathematical items that were classified as belonging to the *1st World*, or requiring *Movement from the 1st World to the 2nd World*, it was felt that the students' explanations used during the first aspect of the *Explaining and Transfer Question*, could be re-categorised in terms of Tall's theory as:

> ***A Perception-Action Category of Explanation*** — students evidence an ability to explain how perceptions of embodied mathematical objects can be linked with mathematical objects/symbols that can be acted on. For example, they are able to evidence how a linear graph (embodied mathematical object) can be linked with its symbolic expression: $y = mx + c$ (mathematical object).

For the mathematical items that were classified as belonging to the *2nd World*, or requiring *Movement from the 2nd World to the 1st World*, it was felt that students' explanations used during the first aspect of the *Explaining and Transfer Question* could be re-categorised in terms of Tall's theory as:

> ***An Action-Perception Category of Explanation*** — students evidence an ability to explain how actions on mathematical objects/symbols can be linked with embodied mathematical objects or referents that can be perceived. For example, they are able to link a derivative function (mathematical object) with its derivative graph (embodied mathematical object).

Because the Part B for each mathematical item in Diagnostic Tools 1 and 3 required students to explain their reasoning, the case could be made that students were not explicitly asked to explain their reasoning in terms of *A Perception-Action Explanation* or *An Action-Perception Explanation*. Therefore, the emergence of *Perception-Action Categories of Explanation* or *Action-Perception Categories of Explanation* would be limited.

Thus, for the mathematical items which could be explained with *A Perception-Action Explanation* (Item 5-7 in Table 2.15), it could be argued that the students were asked to explain their reasoning in terms of *An Action-on-Perception Explanation* where the students described how they acted on perceptions. To ask for *A Perception-Action Explanation*, would have required phrasing the Part B as: 'Link the graphical images with precise or general mathematical symbols that explain your answer to Part A'.

Table 2.16 gives an example of what was deemed to be the difference between *An Action-on-Perception Category of Explanation* and a *Perception-Action Category of Explanation* for Item 7—interpreting where the derivative is greater for two points on a graph. However, the difference between *An Action-on-Perception Category of Explanation* and *A Perception-Action Category of Explanation* was for many items not clear-cut and open to interpretation.

Students' Category of Explanation	**Tall Category of Explanation**
Refer to there being a greater change in y for a certain change in x at this point compared to the alternative point.	Perception-Action
Refer to the curve being sharper at this point.	Action-on-Perception

Table 2.16. Examples of Perception-Action and Action-on-Perception Categories of Explanation for Item 7.

For the mathematical items which could be explained with *An Action-Perception Explanation*, (Items 1-4 and Items 8-15 in Table 2.15), it could be argued that students were asked to explain their reasoning in terms of *An Action-on-Action Explanation* where the students described how they acted on symbols/objects. To ask for *An Action-Perception Explanation*, would have required phrasing the Part B as: 'Link the symbols in your answer to Part A with precise or general graphical images that explain your answer to Part A'.

Table 2.17 shows an example of what was deemed to be the difference between *An Action-on-Action Category of Explanation* and *An Action-Perception Category of Explanation* for Item 1— the calculation of slope. However, like the difference between *Action-on-Perception Category of Explanations* and *Perception-Action Category of Explanations*, categorising a student's category of explanation as either an *Action-on-Action Category of Explanation* or an *Action-Perception Category of Explanation* was difficult and open to interpretation.

Student's Category of Explanation	**Tall Category of Explanation**
Refer to how much y increases for a unit increase in x.	Action-Perception
Refer to the slope being positive.	Action-on-Action

Table 2.17. Examples of Action-Perception and Action-on-Action Categories of Explanation for Item 1.

The categorisation of students' explanations in terms of a Tall's category of explanation was independently rated by two other researchers, one from the School of Mathematical Sciences and one from the School of Chemical Sciences. These categories of explanation represented the degree to which students explained in terms of Tall's theory. For each mathematical item, if a high number of students' explanations were in a certain category of explanation, it was investigated to see if these students: 1) associated with the transfer of the item more so than the other students; and 2) (in certain cases) associated with the correct answering of similar items in a mathematics context and with the transfer of these similar items more so than other students.

For example, during Study 1, for Item 1 (the calculation of slope) ten students fell into the category of: *Refer to how much y increases for a unit increase in x*. This was categorised as an *Action-Perception Category of Explanation* in terms of Tall's theory as can be seen in row one of Table 2.17. To see if the students who evidenced this *Action-Perception Category of Explanation* associated with the transfer of the item, more so than students who did not, categorical statistical tests were used. The students were placed in a contingency table as shown in Table 2.18

Ability to Transfer	Explanation in a Mathematics Context		
	Action-Perception	***Other Types of Explanation***	***Total***
Transferred	9(8.33)	16(16.7)	25
Did not Transfer	1(1.67)	4(3.33)	5
Total	10	20	30
			p = 0.64

Table 2.18.Testing for an Association between Evidencing an Action-Perception Category of Explanation in a Mathematics Context for Item 1 in Study 1 and Being Able to Transfer.

The number of students who evidenced an *Action-Perception Category of Explanation* and who also transferred was nine. Because this number was greater than the expected number of 8.33, an association 'appeared' present. However, the p-value of 0.64 indicated that this number was not significant at a confidence level of 95%. Thus, the conclusion was: if a student evidences this *Action-Perception Category of Explanation* in a mathematics context, they do not associate with the transfer of that item any more so than students who do not evidence this category of explanation.

2.5 Chapter Summary

In this chapter, the design and findings from the Pilot Study were discussed. In respect of the *Transfer Question,* using the traditional view of transfer, instances of transfer were observed. However, statistically significant transfer was rare. The lack of any difference in the degree to which presupposed conceptual items were transferred in comparison to presupposed procedural items, suggested that classifying a mathematical item as either procedural or conceptual in nature is not clear-cut. Therefore, the *Conceptual versus Procedural Question* evolved into the *Explaining and Transfer Question* which was investigated in the Main Study.

An additional finding from the Pilot Study was: the problem students have with mathematics in a chemistry context does not appear to be due to an inability to transfer; instead, the problem is most likely due to lack of mathematical knowledge in a mathematics context.

In relation to the Main Study, how the design of these studies was grounded by the results from the Pilot Study was discussed at length, the main changes being the modification of the mathematical items in the diagnostic tools that were used. The *Transfer Question* investigated in the Main Study was of the same form as during the Pilot Study. Moreover, it was investigated in the same manner. The first aspect of the *Explaining and Transfer Question* was investigated using the principles of qualitative data analysis as described by Cohen [70]. The second aspect of the Explaining and Transfer Question was investigated using Tall's theory [62]. In the next Chapter, the findings from the Main Study are described and discussed.

Chapter 3

The Results from the Investigation of the *Transfer Question* and the *Explaining and Transfer Question* during the Main Study.

3.1 Chapter Overview

This chapter discusses the results from the Main Study in terms of the *Transfer Question* and the *Explaining and Transfer Question.* The results from the first aspect of the *Explaining and Transfer Question* (do students who explain their mathematical reasoning in a mathematics context associate with transfer) are discussed in tandem with the results from the *Transfer Question.* The second aspect of the *Explaining and Transfer Question* (do students who evidence a certain degree of explanation in terms of Tall's theory [62] associate with transfer more so than others) is discussed separately.

3.2 The Transfer Question

Table 3.1 shows students' performance during Study 1 for each of the mathematical items. It can be seen that for Items 2-4, 6, 8-10, 13 and 15, (9 out of 15 items), statistically significant transfer was observed (p-value < 0.05). For Item 14, the significance of the transfer observed was borderline (p-value $= 0.06$). For the five remaining items (Items 1, 5, 7, 11 and 12), while non-statistically significant transfer was observed, certain students did indeed transfer the knowledge, as can be seen from column three of Table 3.1.

What emerged from Study 1 was the conclusion that transfer can be observed for all of the items. Looking at column three and four in Table 3.1, the percentage of students answering some of the mathematical items correctly in a mathematics context was low. It was surmised that this might be due to students not remembering the mathematics they had learned from the previous year. Thus, during Study 2, students were reminded of the mathematics they needed to be able to use in a chemistry context. As discussed in the Research Methodology it wanted to be seen if this reminder:1) improved instances of transfer; and 2) had an effect on the statistical significance of the transfer observed. Table 3.2 shows students' performance during Study 2. Transfer was observed for every item (except Item 11). Thus, the instances of transfer observed did not appear to have improved in comparison to the instances observed in Study 1. Also, the instances of statistically significant transfer did not improve (nine instances in Study 1 versus three instances in Study 2. Item 6 was the only item that was transferred, statistically-significant wise, in both Studies).

Mathematical Item	**Observed Frequencies and Expected Frequencies (in Parentheses) in the Contingency Tables.**				**Transfer p-values**
	Correct in MC* and CC**	**Correct in MC and Incorrect in CC**	**Incorrect in MC and Correct in CC**	**Incorrect in MC and CC**	
1. Calculating Slope.	25(25.2)	2(1.8)	3(2.8)	0(0.2)	1.00
2. Sketching a Line with Positive Slope.	20(17.5)	5(7.5)	1(3.5)	4(1.5)	0.02
3. Sketching a Line with Positive Slope.	20(16.8)	4(7.2)	1(4.2)	5(1.8)	0.01
4. Sketching a Line with Negative Slope.	16(12.3)	7(10.7)	0(3.7)	7(3.3)	0.02
5. Generating an Expression for Slope.	13(10.7)	7(9.3)	3(5.3)	7(4.7)	0.12
6. Generating an Expression for Derivative.	4(1.7)	2(4.3)	5(7.3)	20(17.7)	0.04
7. Interpreting Derivative.	19(18.3)	3(3.7)	6(6.7)	2(1.3)	0.60
8. Usage of Exponentials.	10(4.3)	3(8.7)	0(5.7)	17(11.3)	0.00
9. Usage of Natural Logarithms.	9(5.2)	4(7.8)	3(6.8)	14(10.2)	0.01
10. Proportionality.	11(7.4)	2(5.6)	6(9.6)	11(7.4)	0.01
11. Graphing an Exponential Function.	1(0.4)	2(2.6)	3(3.6)	24(23.4)	0.36
12. Graphing a Natural Log Expression.	1(0.4)	2(2.6)	3(3.6)	24(23.4)	0.36
13. Graphing a Function.	4(1.7)	6(8.3)	1(3.3)	19(16.7)	0.03
14. Evaluation of an Integral.	2(0.5)	3(4.5)	1(2.5)	24(22.5)	0.06
15. Graphing an Integral.	2(0.4)	2(3.6)	1(2.6)	25(23.4)	0.04

MC* *- Mathematics Context;* ***CC***** *- Chemistry Context.*

Table 3.1 Results from the Contingency Tables in Study 1 that Were Used to Investigate the Significance of Observed Transfer for Each Item.

Mathematical Item	Observed Frequencies and Expected Frequencies (in Parentheses) in the Contingency Tables.				Transfer p-values
	Correct in MC* and CC**	Correct in MC and Incorrect in CC	Incorrect in MC and Correct in CC	Incorrect in MC and CC	
1. Calculating Slope.	19(18.2)	4(4.8)	0(0.8)	1(0.2)	0.21
2. Sketching a Line with Positive Slope.	13(13.1)	8(7.9)	2(1.9)	1(1.1)	1.00
3. Sketching a Line with Positive Slope.	13(11.7)	7(8.3)	1(2.3)	3(1.7)	0.27
4. Sketching a Line with Negative Slope.	15(13.8)	7(8.3)	0(1.2)	2(0.8)	0.10
5. Generating an Expression for Slope.	10(6.7)	6(9.3)	0(3.3)	8(4.7)	0.01
6. Generating an Expression for Derivative.	4(1.7)	0(2.3)	6(8.3)	14(11.7)	0.02
7. Interpreting Derivative.	15(15.3)	1(0.7)	8(7.7)	0(0.3)	1.00
8. Usage of Exponentials.	2(1.2)	12(12.8)	0(0.8)	10(9.2)	0.49
9. Usage of Natural Logarithms.	10(7.8)	1(3.2)	7(9.2)	6(3.8)	0.08
10. Proportionality.	11(10.3)	2(2.7)	8(8.7)	3(2.3)	0.63
11. Graphing an Exponential Function.	0(0.0)	3(3.0)	0(0.0)	21(21.0)	1.00
12. Graphing a Natural Log Expression.	6(4.6)	6(7.5)	5(6.5)	12(10.6)	0.44
13. Graphing a Function.	4(2.6)	5(6.4)	3(4.4)	12(10.6)	0.36
14. Evaluation of an Integral.	4(1.7)	3(5.3)	2(4.3)	15(12.8)	0.04
15. Graphing an Integral.	3(1.8)	4(5.3)	3(4.3)	14(12.8)	0.31

MC* *- Mathematics Context;* ***CC***** *- Chemistry Context.*

Table 3.2 Results from the Contingency Tables in Study 2 that Were Used to Investigate the Significance of Observed Transfer for Each Item.

3.3 The Explaining and Transfer Question

During Study 1, students were required to explain their reasoning for eleven out of the fifteen mathematical items (Item 1 and Items 6-15). Table 3.3 shows that for eight out of the eleven items requiring an explanation, students who evidenced any type of correct explanation for these items in a mathematics context associated with the transfer of them.

During Study 2, students were required to explain their reasoning for all of the mathematical items. Students who evidenced any form of correct explanation for eleven of the fifteen items in a mathematics context associated with the transfer of them. This is shown in Table 3.4.

Mathematical Item	Observed Frequencies and Expected Frequencies (in Parentheses) in the Contingency Tables.				Transfer p-values
	Ability to Explain and Transfer	Ability to Explain but not Transfer	Inability to Explain and Transfer	Inability to Explain and not Transfer	
1. Calculating Slope.	13(11.7)	1(2.3)	12(13.3)	4(2.7)	0.34
6. Generating an Expression for Derivative.	4(0.8)	2(5.2)	0(3.2)	24(20.8)	0.00
7. Interpreting Derivative.	17(12.0)	1(6.0)	3(8.0)	9(4.0)	0.00
8. Usage of Exponentials.	10(5.7)	7(11.3)	0(4.3)	13(8.7)	0.00
9. Usage of Natural Logarithms.	9(3.9)	4(9.1)	0(5.1)	17(11.9)	0.00
10. Proportionality.	8(3.3)	1(5.7)	3(7.7)	18(13.3)	0.00
11. Graphing an Exponential Function.	0(0.0)	0(0.0)	0(0.0)	30(30.0)	1.00
12. Graphing a Natural Log Expression.	1(0.4)	2(2.6)	3(3.6)	24(23.4)	0.36
13. Graphing a Function.	4(1.7)	6(8.3)	1(3.3)	19(16.7)	0.03
14. Evaluation of an Integral.	2(0.3)	3(4.7)	0(1.7)	25(23.3)	0.02
15. Graphing an Integral.	2(0.3)	2(3.7)	0(1.7)	26(24.3)	0.01

Table 3.3 Results from the Contingency Tables in Study 1 that Were Used to Investigate the Significance of Evidencing an Ability to Explain and Transfer.

Mathematical Item	Observed Frequencies and Expected Frequencies (in Parentheses) in the Contingency Tables.				Transfer p-values
	Ability to Explain and Transfer	**Ability to Explain but not Transfer**	**Inability to Explain and Transfer**	**Inability to Explain and not Transfer**	
1. Calculating Slope.	19(17.4)	3(4.6)	0(1.6)	2(0.4)	0.04
2. Sketching a Line with Positive Slope.	13(11.4)	8(9.6)	0(1.6)	3(1.4)	0.08
3. Sketching a Line with Positive Slope.	12(9.5)	7(9.5)	0(2.5)	5(2.5)	0.04
4. Sketching a Line with Negative Slope.	7(6.2)	3(3.7)	8(8.7)	6(5.3)	0.68
5. Generating an Expression for Slope.	10(6.3)	5(8.8)	0(3.8)	9(5.3)	0.00
6. Generating an Expression for Derivative.	4(0.7)	0(3.3)	0(3.3)	20(16.7)	0.00
7. Interpreting Derivative.	11(6.9)	0(4.1)	4(8.1)	9(4.9)	0.00
8. Usage of Exponentials.	1(0.8)	9(9.2)	1(1.2)	13(12.8)	1.00
9. Usage of Natural Logarithms.	7(3.3)	1(4.7)	3(6.7)	13(9.3)	0.00
10. Proportionality.	11(6.0)	2(7.0)	0(5.0)	11(6.0)	0.00
11. Graphing an Exponential Function.	0(0.0)	1(1.0)	0(0.0)	23(23.0)	1.00
12. Graphing a Natural Log Expression.	3(1.3)	2(3.7)	3(4.7)	16(14.2)	0.08
13. Graphing a Function.	4(1.0)	1(4.0)	1(4.0)	18(15.0)	0.00
14. Evaluation of an Integral.	3(0.8)	2(4.2)	1(3.2)	18(15.8)	0.02
15. Graphing an Integral.	3(0.7)	3(5.3)	0(2.2)	18(15.8)	0.01

Table 3.4 Results from the Contingency Tables in Study 2 that Were Used to Investigate the Significance of Evidencing an Ability to Explain and Transfer.

3.4 Discussion

In terms of the *Transfer Question*, the traditional-view-of-transfer approach used in Study 1 and Study 2 showed that many students can transfer mathematical knowledge.

These findings are at odds with the view of Detterman [71], who according to Lobato [72] describes transfer as rare. Furthermore, the findings do not support the view of Krishner and Whitson [73], who claim that classical/traditional approaches to transfer studies often fail to demonstrate transfer in the laboratory.

In terms of the significance of the transfer observed, for Study 1, statistically significant transfer (p-value < 0.05) was observed for nine out of the fifteen items, borderline transfer ($0.05 \leq$ p-value ≤ 0.1) was observed for two out of the fifteen items while non-statistically significant transfer ($0.1 \leq$ p-value ≤ 1) was found for four out of the fifteen items.

For Study 2, statistically significant transfer (p-value < 0.05) was observed for four out of the fifteen items; borderline transfer ($0.05 \leq$ p-value ≤ 0.1) was observed for two out of the fifteen items, while non-statistically significant transfer ($0.1 \leq$ p-value ≤ 1) was observed for nine out of the fifteen items.

Despite students being reminded of 'how to do' the mathematical items in a mathematics context during Study 2, less statistically significant transfer was observed. There are a number of possible reasons for this:

1) The number of participants in Study 2 was 24 as opposed to 30 in Study 1.
2) Perhaps, in each Study, participants had different mathematical understanding and/or chemistry understanding. Such differences, if they did exist, were not the focus of this research.

It should be noted that the observance of statistically significant transfer for certain items in both studies could be considered to be fallacious. It is well known that a statistically significant association between two variables can be caused by a third variable that influences both of the variables. Even if this is the case (and it was something that the research was not concerned with) the question of what other possible variables could be at play arises. Perhaps the students who transferred had more schooling in mathematics? Perhaps the students who transferred are better than others about quantitative reasoning and representations? Or perhaps the successful students had recently dealt with identical questions where the others had not. Even if this is the case, would such alternative variables be divorced to a large degree from the variable of being correct in a mathematics context? Such a question certainly represents a possible avenue for future research.

In terms of the *Explaining and Transfer Question*, evidencing an ability to explain in a mathematics context may well be a factor in enabling students to transfer this mathematical knowledge to not just a chemistry context but to others. Arguments of a fallacy aside, the findings would very much agree with the view of Bishop [74], who proposes that students will only be able to transfer mathematics if they have developed an appreciation of mathematical meaning, which is dependent on the mathematical environment of which the student is a part. Boaler [33] adds further impetus to the second finding, where she claims that understanding which allows for the development of links between different contexts will most likely develop if students are encouraged to communicate and challenge mathematics.

3.5 Degree to which Students Explained

As stated in Section 2.4.3.2 in Chapter 2, to determine the degree to which students explained, Tall's theory [62] was used. It was found that each of the mathematical items could be classified as: 1st World; 2nd World; Movement from the 1st World to the 2nd World; or Movement from the 2nd World to the 1st World. Why each mathematical item was categorised as such is described in this section.

During the first aspect of the *Explaining and Transfer Question,* students' explanations were categorised. These categories were then categorised in terms of Tall's theory as reflective of one of the following categories of explanation: *an Action-Perception Category of Explanation*; *a Perception-Action Category of Explanation*; *an Action-on-Perception Category of Explanation;* or *an Action-on-Action Category of Explanation*. The categories are abbreviated to AP, PA, P and A respectively in the tables of data categorising students' explanations in terms of Tall's theory for each item. How such categories were distinguished is discussed in Section 2.4.3.2 in Chapter 2.

3.5.1 Item 1— Calculating Slope

Item 1: Calculating Slope

Mathematics Context

(A) Calculate the slope of the straight line from the two points given in Figure 1.

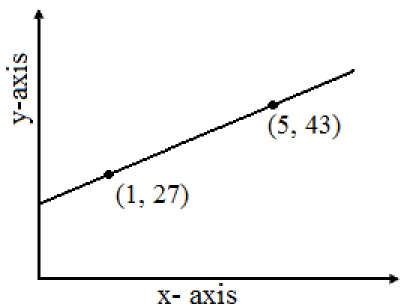


Figure 1

(B) Explain what this number means.

Chemistry Context

(A) Calculate the rate of change of the concentration of the reactant with respect to time over the time interval (Δt) from the two points given in Figure 1.

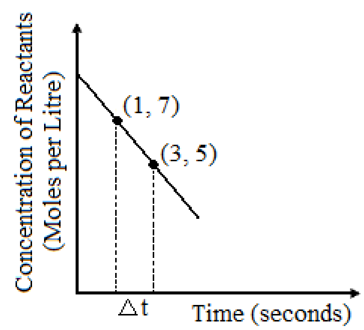


Figure 1

(B) Explain what this value means.

Figure 3.1 Item 1 Used in the Main Study

Mathematical Item	% Correct in a Mathematics Context		% Correct in a Chemistry Context	
	Study 1	Study 2	Study 1	Study 2
1. Calculating Slope	90%	96%	93%	79%

Table 3.5 Students' Performance for Item 1 in the Main Study

Item 1, in both contexts, requires students to recall the formula/technique for calculating slope in a mathematics context (or rate of change in a chemistry context). Upon recollection of the technique, determining the answer requires students to manipulate the symbols/numbers accordingly. Therefore, the item in both contexts could be considered to belong to Tall's 2^{nd} World. The percentage of students who answered this item correctly in a mathematics context can be seen in Table 3.5. Relating Tall's Theory to Item 1, the correct students demonstrated that they could competently *act on their environment in terms of manipulating symbols,* once they recalled the formula for slope. Part B of the item in the mathematics context required students to explain what the number for slope means; thus, in a sense, evidencing whether they were able to reflect upon their actions.

Tables 3.6 and 3.7 show the categories of students' explanations, ascertained during the first aspect of the *Explaining and Transfer Question*. These categories were deemed be reflective of an ability to explain (ATE) or an inability to explain (IATE), as shown in column three and seven in Tables 3.6 and 3.7. The column heading for columns three and seven in Tables 3.6 and 3.7 is 'Exp.', which is an abbreviation for Explanation. The categories were then deemed to be reflective of a category of explanation in terms of Tall's theory as shown in column four and eight in Tables 3.6 and 3.7. The column heading for columns four and eight is 'Tall' which refers to the category of explanation in terms of Tall's theory.

It should be noted that for the correct students who were in the 'provide no reason category', they were deemed to have evidenced an action-on-action category of explanation in terms of Tall's theory because this was the type of understanding they illustrated when they answered Part A correctly. Indeed, this was the type of understanding all the correct students evidenced before answering Part B. However, the correct students who provided a Part B answer had an opportunity to show whether they could evidence a category of explanation that was more than just an action-on-action category of explanation.

Looking at Table 3.7, for the students who answered correctly in Study 1, eleven categories of explanation emerged. Three of these categories were deemed to be reflective of an ability to explain. Fourteen students were in these three categories. In terms of Tall's theory, the three categories were each classified in terms of an action-perception category of explanation. For the other eight categories (of which thirteen students were apart), they were each deemed to be evidence of an inability to explain. Six of these categories were classified in terms of an action-perception category of explanation, while the other two were classified in terms of an action-on-action category of explanation.

In Study 2, for the students who answered correctly, nine categories of explanation emerged. Five of these categories were deemed to be reflective of an ability to explain. Thirteen students were in these categories. In terms of Tall's theory, the five categories were each categorised in terms of an action-perception category of explanation. For the other four categories (of which ten students were apart), they were each deemed to be evidence of an inability to explain. Two of these categories were classified in terms of an action-perception category of explanation, while the other two were classified in terms of an action-on-action category of explanation. Looking at Table 3.6, for the students in Study 1 who answered the item incorrectly (three students), two categories of explanation emerged. Both of these categories were deemed to be reflective of an inability to explain. In terms of Tall's theory, category one was categorised as an action-perception category of explanation because the student in this category referred to slope as being an angle. Category two was categorised as an action-on-action category of explanation because of the reference made by students to use of a formula, similar to the inverse of the slope formula. For the student in Study 2 who answered the item incorrectly their category of explanation was deemed to be reflective of an inability to explain, and, in terms of Tall's theory, was categorised as an action-on-action category of explanation.

Item 1: Calculating Slope
For the Students who Answered Incorrectly in a Mathematics Context

Study 1				**Study 2**			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. Use the inverse of the slope formula.	2	IATE	AA	1. -4; slope decreases.	1	IATE	AA
2. Label the data points incorrectly in terms of the slope formula. Also refer to the slope as an angle.	1	IATE	AP				

Table 3.6. The Incorrect Students' Categories of Explanation for Item 1 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; IATE—inability to explain; Tall.—degree of explanation in terms of Tall's theory; AA—action-on-action; AP—action-perception).

Item 1: Calculating Slope
For the Students who Answered Correctly in a Mathematics Context

Study 1				Study 2			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. Refer to how much y increases for a unit increase in x.	10	ATE	AP	1. Refer to how much y increases for a unit increase in x.	3	ATE	AP
2. Refer to slope as a measure of steepness.	1	ATE	AP	2. The slope represents the rate of increase of the line.	5	ATE	AP
3. Refer to the rate at which the line increases.	3	ATE	AP	3. The bigger the number, the steeper the slope is.	2	ATE	AP
4. Refer to the slope value being 4.	4	IATE	AA	4. The change of x relative to y, how steep a line is.	1	ATE	AP
5. Interpret slope as meaning for a y-unit increase, there is a 4 unit x increase.	1	IATE	AP	5. It is the difference between the heights over the difference between the two length points.	2	ATE	AP
6. Provide no reason.	1	IATE	AA	6. Provide no reason.	1	IATE	AA
7. Refer to the slope as Rising.	1	IATE	AP	7. It means the gradient to the horizontal = 4.	5	IATE	AP
8. Refer to the slope as the distance between data points.	1	IATE	AP	8. Slope of a line is 4; this is the angle at which the line passes through the 2 points.	1	IATE	AP
9. Refer to the slope as increasing.	3	IATE	AP.	9. Means the slope of the line is increasing because it's positive.	3	IATE	AA
10. Refer to the slope as how much the line increases/decreases.	1	IATE	AP				
11. Refer to the slope being positive.	1	IATE	AA				

Table 3.7 The Correct Students' Categories of Explanation for Item 1 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; ATE—ability to Explain; IATE—inability to explain; Tall.—degree of explanation in terms of Tall's theory; AA—action-on-action; AP—action-perception).

Questions Raised

In light of using Tall's theory to categorise a student's category of explanation, a number of questions were raised:

Q.1 *For the students who evidenced a correct action-perception category of explanation for the meaning of slope in the mathematics context for Item 1(Categories 1-3 in Study 1 and Categories 1-5 in Study 2), do they associate with the transfer of Items 1-7 and Item 10 more so than the students who did not evidence a correct action-perception category of explanation?*

The question was investigated for two reasons:

1) All of the students who answered the item correctly in both Study 1 and Study 2, and who were deemed to have evidenced an ability to explain, evidenced a category of explanation which was categorised as an action-perception category of explanation in terms of Tall's theory. This was in contrast to the students who answered the item correctly but who were deemed to have not evidenced an ability to explain their reasoning. The categories of explanation for these students were categorised either in terms of an action-on-action category of explanation or an action-perception category of explanation in terms of Tall's theory, even though these were deemed to be incorrect action-on-action and action-perception categories of explanation.

2) The correct action-perception categories of explanation in both Study1 and Study 2 could be considered (in terms of Tall's theory) to be reflective of an embodied mathematical object-type understanding of slope. If so, it was anticipated that such an explanation may associate with the transfer of slope to other contexts.

The question was investigated for Items 1-7 and Item 10 because these items were considered to have a slope-type element in their makeup. The results from the contingency tables used to investigate this question are shown in Tables 3.8 and 3.9. In these tables, students who evidenced a *Correct Action-Perception Category of Explanation* are referred to as *CAP*, while the *Other Categories of Explanation* are referred to as *OCE*.

It should be noted that for this question and all the questions raised, categorical statistical tests were used when testing *an association* (if it appeared present) for significance. How the tests were used is described in Section 2.4.3.2 in Chapter 2.

Mathematical Item	**Observed Frequencies and Expected Frequencies (in parentheses) in the Contingency Table.**				**Transfer (p-values)**
	CAP in MC* and Transfer	**CAP in MC and No Transfer**	**OCE in MC and Transfer**	**OCE in MC and No Transfer**	
1. Calculating Slope.	13(11.7)	1(2.3)	12(13.3)	4(2.7)	0.34
2. Sketching a Line with Positive Slope.	10(9.3)	4(4.7)	10(10.7)	6(5.3)	0.71
3. Sketching a Line with Positive Slope.	10(9.3)	4(4.7)	10(10.7)	6(5.3)	0.71
4. Sketching a Line with Negative Slope.	10(7.0)	4(7.0)	5(8.0)	11(8.0)	0.03
5. Generating an Expression for Slope.	8(6.1)	6(7.9)	5(6.9)	11(9.1)	0.15
6. Generating an Expression for Derivative.	3(1.9)	11(12.1)	1(2.1)	15(13.9)	0.32
7. Interpreting Derivative.	13(8.9)	1(5.1)	6(10.1)	10(5.9)	0.00
10. Proportionality.	8(5.1)	6(8.9)	3(5.9)	13(10.1)	0.03

MC* *- A Mathematics Context*

Table 3.8 Study 1 Results from the Contingency Tables Used to Investigate Whether Students who Evidenced a Correct Action-Perception Category of Explanation (Referred to as 'CAP') in a Mathematics Context Associated with the Transfer of Items 1-7 and Item 10 more so than the Students who Evidenced Other Categories of Explanation (Referred to as 'OCE').

Mathematical Item	Observed Frequencies and Expected Frequencies (in parentheses) in the Contingency Table.				Transfer (p-values)
	CAP in MC* and Transfer	**CAP in MC and No Transfer**	**OCE in MC and Transfer**	**OCE in MC and No Transfer**	
1. Calculating Slope.	11(10.3)	2(2.7)	8(8.7)	3(2.3)	0.63
2. Sketching a Line with Positive Slope.	8(7.0)	5(6.0)	5(6.0)	6(5.0)	0.43
3. Sketching a Line with Positive Slope.	8(6.5)	5(6.5)	4(5.5)	7(5.5)	0.22
4. Sketching a Line with Negative Slope.	9(8.1)	4(4.9)	6(6.9)	5(4.1)	0.67
5. Generating an Expression for Slope.	7(5.4)	6(7.6)	3(4.6)	8(6.4)	0.24
6. Generating an Expression for Derivative.	4(2.2)	9(10.8)	0(1.8)	11(9.2)	0.10
7. Interpreting Derivative.	11(8.1)	2(4.9)	4(6.9)	7(4.1)	0.03
10. Proportionality.	7(5.2)	5(6.8)	3(4.8)	8(6.2)	0.21

MC* *- A Mathematics Context*

Table 3.9 Study 2 Results from the Contingency Tables Used to Investigate Whether Students who Evidenced a Correct Action-Perception Category of Explanation (Referred to as 'CAP') in a Mathematics Context Associated with the Transfer of Items 1-7 and Item 10 more so than the Students who Evidenced Other Categories of Explanation (Referred to as 'OCE').

For the students in Study 1 (shown in Table 3.8) it can be seen that if students evidenced a *Correct Action-Perception Category of Explanation* for the meaning of slope in a mathematics context, they associated (p-value less than or equal to 0.05) with the transfer of Items 4, 7 and 10, more so than students who evidenced other categories of explanation.

In Study 2 (shown in Table 3.9), students who evidenced a *Correct Action-Perception Category of Explanation* for the meaning of slope in a mathematics context associated with the transfer of Item 7 more so than students who evidenced other categories of explanation.

Q.2 *For the students who evidenced a correct, action-perception category of explanation for the meaning of slope in the mathematics context for Item 1 (Categories 1-3 in Study 1 and Categories 1-5 in Study 2), do they associate with the correct answering of Items 1-7 and Item 10, in a mathematics context more so than the students who did not evidence a correct action-perception category of explanation?*

The question was investigated for two reasons:

1) Items 1-7 and Item 10 were considered to be similar in the sense that they have a slope-type element in their makeup.

2) The correct action-perception categories of explanation evidenced by students in both Study 1 and Study 2 could be considered (in terms of Tall's theory) to be reflective of an embodied mathematical object-type understanding of slope. If so, it was anticipated that students who evidenced such an explanation may associate with the correct answering of Items 1-7 and Item 10 in a mathematics context more so than students who evidenced other categories of explanation.

The results from the contingency tables used to investigate this question are shown in Tables 3.10 and 3.11. In these tables, students who evidenced a *Correct Action-Perception Category of Explanation* are referred to as *CAP*, while the *Other Categories of Explanation* are referred to as *OCE*.

For the students in Study 1(shown in Table 3.10) it can be seen that if these students evidenced a correct action-perception category of explanation for the meaning of slope in a mathematics context, they associated (p-value less than or equal to 0.05) with the correct answering of Items 7 and 10 in a mathematics context, more so than students who evidenced other categories of explanation.

In Study 2 (shown in Table 3.11), students who evidenced a correct action-perception category of explanation for the meaning of slope in a mathematics context appear to associate (if borderline significance [$0.05 \leq$ p-value ≤ 0.1] is accepted), with the answering of Items 6 and 7 in a mathematics context.

Thus, in both Studies, if students evidenced a correct action-perception category of explanation for the meaning of slope in a mathematics context, they appear to associate with the answering of Item 7(interpreting derivative) in a mathematics context, more so than students who evidenced other categories of explanation. There was no evidence (p-values greater than 0.1) of this association for the other items.

Mathematical Item	**Observed Frequencies and Expected Frequencies (in parentheses) in the Contingency Table.**				**Transfer (p-values)**
	CAP in MC* and Correct	**CAP in MC and Incorrect**	**OCE in MC and Correct**	**OCE in MC and Incorrect**	
1. Calculating Slope.	14(12.6)	0(1.4)	13(14.4)	3(1.6)	0.23
2. Sketching a Line with Positive Slope.	11(11.7)	3(2.3)	14(13.3)	2(2.7)	0.64
3. Sketching a Line with Positive Slope.	11(11.2)	3(2.8)	13(12.8)	3(3.2)	1.00
4. Sketching a Line with Negative Slope.	12(10.7)	2(3.2)	11(12.3)	5(3.7)	0.39
5. Generating an Expression for Slope.	10(9.3)	4(4.7)	10(10.7)	6(5.3)	0.71
6. Generating an Expression for Derivative.	4(2.8)	10(11.2)	2(3.2)	14(12.8)	0.38
7. Interpreting Derivative.	13(10.3)	1(3.7)	9(11.7)	7(4.3)	0.04
10. Proportionality.	10(6.1)	4(7.9)	3(6.9)	13(9.1)	0.04

MC* *- A Mathematics Context*

Table 3.10 Study 1 Results from the Contingency Tables Used to Investigate Whether Students who Evidenced a Correct Action-Perception Category of Explanation (Referred to as 'CAP') in a Mathematics Context Associated with the Correct Answering of Items 1-7 and Item 10 in a Mathematics Context more so than the Students who Evidenced Other Categories of Explanation (Referred to as 'OCE').

Mathematical Item	Observed Frequencies and Expected Frequencies (in parentheses) in the Contingency Table.				Transfer (p-values)
	CAP in MC* and Correct	**CAP in MC and Incorrect**	**OCE in MC and Correct**	**OCE in MC and Incorrect**	
1. Calculating Slope.	13(12.5)	0(0.5)	10(10.5)	1(0.5)	0.46
2. Sketching a Line with Positive Slope.	11(11.4)	2(1.6)	10(9.6)	1(1.4)	1.00
3. Sketching a Line with Positive Slope.	11(10.3)	2(2.7)	8(8.7)	3(2.3)	0.63
4. Sketching a Line with Negative Slope.	13(11.9)	0(1.1)	9(10.1)	2(0.9)	0.20
5. Generating an Expression for Slope.	10(8.7)	3(4.3)	6(7.3)	5(3.7)	0.39
6. Generating an Expression for Derivative.	4(2.2)	9(10.8)	0(1.8)	11(9.2)	0.10
7. Interpreting Derivative.	11(8.7)	2(4.3)	5(7.3)	6(3.7)	0.10
10. Proportionality.	9(7.0)	4(6.0)	4(6.0)	7(5.0)	0.12

MC* *- A Mathematics Context*

Table 3.11 Study 2 Results from the Contingency Tables Used to Investigate Whether Students who Evidenced a Correct Action-Perception Category of Explanation (Referred to as 'CAP') in a Mathematics Context Associated with the Correct Answering of Items 1-7 and Item 10 in a Mathematics Context more so than the Students who Evidenced Other Categories of Explanation (Referred to as 'OCE').

Q.3 *For the students who evidenced a Category 1, Action-Perception Category of Explanation for the meaning of slope in a mathematics context for Item 1 (shown in Table 3.7) do they associate with the transfer of Items 1-7 and Item 10 more so than the students who evidenced other categories of explanation?*

The question was investigated for two reasons:

1) The category 1 action-perception category of explanation occurred on ten occasions during Study 1 and on three occasions during Study 2.

2) The inference drawn from such an explanation was that of how these students *might be* visualising or embodying the meaning of slope in terms of Tall's theory. For example, these students might be embodying, for argument sake, a slope value of -3, in terms of 3 units down on the y-axis for every 1 unit across on the x-axis, as shown in Figure 3.2. If so, it wanted to be seen if such an explanation meant that the students associated with transferring Items 1-7 and Item 10 (which all contain a slope-type element in their makeup) more so than students who evidenced other categories of explanation.

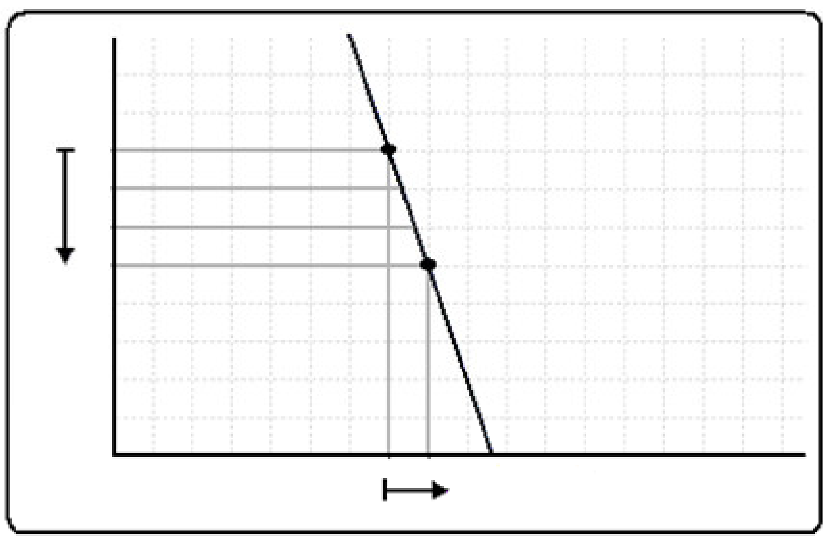


Figure 3.2 The Inferred Embodied Mathematical Object-Type Understanding of Slope Evidenced by Students in Category 1 of Table 3.7.

The results from the contingency tables used to investigate this question are shown in Tables 3.12 and 3.13. The *Category 1 Action-Perception Category of Explanation* is referred to as *C1,* while the *Other Categories of Explanation* are referred to as *Other*.

Mathematical Item	**Observed Frequencies and Expected Frequencies (in parentheses) in the Contingency Table.**				**Transfer (p-values)**
	C1 in MC* and Transfer	**C1 in MC and No Transfer**	**Other in MC and Transfer**	**Other in MC and No Transfer**	
1. Calculating Slope.	9(8.3)	1(1.7)	16(16.7)	4(3.3)	0.64
2. Sketching a Line with Positive Slope.	8(6.7)	2(3.3)	12(13.3)	8(6.7)	0.42
3. Sketching a Line with Positive Slope.	8(6.7)	2(3.3)	12(13.3)	8(6.7)	0.42
4. Sketching a Line with Negative Slope.	8(5.3)	2(4.7)	8(10.7)	12(9.3)	0.06
5. Generating an Expression for Slope.	7(4.3)	3(5.7)	6(8.7)	14(11.3)	0.06
6. Generating an Expression for Derivative.	2(1.2)	8(8.8)	2(2.8)	22(21.2)	0.06
7. Interpreting Derivative.	10(6.3)	0(3.7)	9(12.7)	11(7.3)	0.00
10. Proportionality.	7(3.7)	3(6.3)	4(7.3)	16(12.7)	0.02

MC* *- A Mathematics Context*

Table 3.12 Study 1 Results from the Contingency Tables Used to Investigate Whether Students who Evidenced a Category 1 Action-Perception Category of Explanation (Referred to as 'C1') in a Mathematics Context Associated with the Transfer of Items 1-7 and Item 10 more so than the Students who Evidenced Other Categories of Explanation (Referred to as 'Other').

Mathematical Item	Observed Frequencies and Expected Frequencies (in parentheses) in the Contingency Table.				Transfer (p-values)
	C1 in MC* and Transfer	C1 in MC and No Transfer	Other in MC and Transfer	Other in MC and No Transfer	
1. Calculating Slope.	3(2.4)	0(0.6)	6(16.6)	5(4.4)	1.00
2. Sketching a Line with Positive Slope.	3(1.7)	0(1.3)	11(12.2)	10(8.8)	0.24
3. Sketching a Line with Positive Slope.	3(1.6)	0(1.4)	10(13.4)	11(9.6)	0.22
4. Sketching a Line with Negative Slope.	3(1.8)	0(1.1)	12(13.1)	9(7.9)	0.27
5. Generating an Expression for Slope.	3(1.3)	0(1.7)	7(8.8)	14(12.2)	0.06
6. Generating an Expression for Derivative.	2(0.5)	1(2.5)	2(3.5)	19(17.5)	0.06
7. Interpreting Derivative.	3(1.8)	0(1.1)	12(13.1)	9(7.9)	0.27
10. Proportionality.	3(1.4)	0(1.6)	8(9.6)	13(11.4)	0.08

MC* *- A Mathematics Context*

Table 3.13 Study 2 Results from the Contingency Tables Used to Investigate Whether Students who Evidenced a Category 1 Action-Perception Category of Explanation (Referred to as 'C1') in a Mathematics Context Associated with the Transfer of Items 1-7 and Item 10 more so than the Students who Evidenced Other Categories of Explanation (Referred to as 'Other').

Looking at Table 3.12, for students in Study 1 who gave—what was interpreted as—an embodied mathematical explanation for the meaning of slope, they were more likely to transfer Items 4, 5, 7 and 10, compared with the students who did not evidence an embodied mathematical explanation. It should be noted that the p-values for Items 4 and 5 are borderline ($0.05 \leq$ p-value ≤ 0.1).

In Study 2 (shown in Table 3.13), Items 5, 6 and 10 were deemed to have been transferred by students evidencing an embodied mathematical explanation for the meaning of slope more so than students who did not, even though the p-values were borderline. The p-values for Items 4 and 7 (items which were transferred in Study 1), in addition to the p-values for Items 1-3, suggest that the likelihood of transferring these items, if a student evidenced an embodied mathematical explanation for the meaning of slope was no greater than if a student did not evidence an embodied mathematical explanation. However, in Study 2, the sample was smaller (24 participants as opposed to 30). Moreover, looking at columns three and four of Table 3.13, it can be seen that there were only three students who evidenced the embodied mathematical explanation for the meaning of slope in a mathematics context. Interestingly though, all of these students transferred every item—apart from one of the three students who did not transfer Item 6. Thus, if the sample size was larger, statistically-significant transfer might have been observed.

Q.4 *For the students who evidenced a Category 1, Action-Perception Category of Explanation for the meaning of slope in a mathematics context for Item 1 (shown in Table 3.7) do they associate with the correct answering of Items 1-7 and Item 10 in a mathematics context more so than the students who evidenced other categories of explanation?*

The question was investigated for reasons similar to the investigation of question three. The results from the contingency tables used to investigate this question are shown in Tables 3.14 and 3.15. The *Category 1 Action-Perception Category of Explanation* is referred to as *C1,* while the *Other Categories of Explanation* are referred to as *Other*.

Mathematical Item	Observed Frequencies and Expected Frequencies (in parentheses) in the Contingency Table.				Transfer (p-values)
	C1 in MC* and Correct	**C1 in MC and Incorrect**	**Other in MC and Correct**	**Other in MC and Incorrect**	
1. Calculating Slope.	10(9.0)	0(1.0)	17(8.0)	3(2.0)	0.53
2. Sketching a Line with Positive Slope.	8(8.3)	2(1.7)	17(16.7)	3(3.3)	1.00
3. Sketching a Line with Positive Slope.	8(8.0)	2(2.0)	16(16.0)	4(4.0)	1.00
4. Sketching a Line with Negative Slope.	9(7.7)	4(2.3)	14(15.3)	6(4.7)	0.37
5. Generating an Expression for Slope.	9(6.7)	1(3.3)	11(13.3)	9(6.7)	0.10
6. Generating an Expression for Derivative.	3(2.0)	7(8.0)	3(4.0)	17(16.0)	0.37
7. Interpreting Derivative.	10(7.3)	0(2.7)	12(14.7)	8(5.3)	0.03
10. Proportionality.	7(4.3)	3(5.7)	6(8.7)	14(11.3)	0.06

MC* *- A Mathematics Context*

Table 3.14 Study 1 Results from the Contingency Tables Used to Investigate Whether Students who Evidenced a Category 1 Action-Perception Category of Explanation (Referred to as 'C1') in a Mathematics Context Associated with the Correct Answering of Items 1-7 and Item 10 in a Mathematics Context more so than the Students who Evidenced Other Categories of Explanation (Referred to as 'Other').

Mathematical Item	Observed Frequencies and Expected Frequencies (in parentheses) in the Contingency Table.				Transfer (p-values)
	C1 in MC* and Correct	C1 in MC and Incorrect	Other in MC and Correct	Other in MC and Incorrect	
1. Calculating Slope.	3(2.8)	0(0.1)	20(20.1)	1(0.8)	1.00
2. Sketching a Line with Positive Slope.	3(2.6)	0(0.4)	18(18.4)	3(2.6)	1.00
3. Sketching a Line with Positive Slope.	3(2.4)	0(0.6)	16(16.6)	5(4.4)	1.00
4. Sketching a Line with Negative Slope.	3(2.8)	0(0.3)	19(19.2)	2(1.8)	1.00
5. Generating an Expression for Slope.	3(2.0)	0(1.0)	13(14.0)	8(7.0)	0.53
6. Generating an Expression for Derivative.	2(0.5)	1(2.5)	2(3.5)	19(17.5)	0.06
7. Interpreting Derivative.	3(2.0)	0(1.0)	13(14.0)	8(7.0)	0.53
10. Proportionality.	3(1.62)	0(1.4)	10(11.4)	11(9.6)	0.22

MC* *- A Mathematics Context*

Table 3.15 Study 2 Results from the Contingency Tables Used to Investigate Whether Students who Evidenced a Category 1 Action-Perception Category of Explanation (Referred to as 'C1') in a Mathematics Context Associated with the Correct Answering of Items 1-7 and Item 10 in a Mathematics Context more so than the Students who Evidenced Other Categories of Explanation (Referred to as 'Other').

Table 3.14, for the students in Study 1, suggests that the students who evidenced a category 1, action-perception category of explanation for the meaning of slope, did not associate with the answering of Items 1-6 in a mathematics context any more so than the students who did not evidence this type of explanation. However, for Items 7 and 10, students did associate with answering them correctly if they evidenced the category 1 action-perception category of explanation: was this the case for the students in Study 2?

Looking at Table 3.15, this did not seem to be the case for Study 2. However, it should be noted that for the three students who evidenced a category 1 action-perception category of explanation, all of these students answered Items 7 and 10 correctly in a mathematics context despite the fact that this was not found to be significant; perhaps the smaller sample size in Study 2 was the reason.

Summary:

Q.1 *For the students who evidenced a Correct Action-Perception Category of Explanation for the meaning of slope in the mathematics context for Item 1(Categories 1-3 in Study 1 and Categories 1-5 in Study 2), do they associate with the transfer of Items 1-7 and Item 10 more so than the students who did not evidence a Correct Action-Perception Category of Explanation?*

- In Study 1, students who evidenced this category of explanation were likely to transfer Items 4, 7 and 10 more so than other students. In Study 2, the students who evidenced this category of explanation were likely to transfer Item 7 more so than other students. Thus, across both studies, students who evidenced this category of explanation were likely to transfer Item 7 (interpreting derivative).

Q.2 *For the students who evidenced a Correct, Action-Perception Category of Explanation for the meaning of slope in the mathematics context for Item 1 (Categories 1-3 in Study 1 and Categories 1-5 in Study 2), do they associate with the correct answering of Items 1-7 and Item 10, in a mathematics context more so than the students who did not evidence a Correct Action-Perception Category of Explanation?*

- In Study 1, students who evidenced this category of explanation were likely to answer correctly Items 7 and 10 in a mathematics context more so than other students. In Study 2, the students who evidenced this category of explanation were likely to answer correctly Items 6 and 7 in a mathematics context more so than other students if borderline significance is accepted ($0.05 \leq$ p-value ≤ 0.1). Thus, across both studies, students who evidenced this category of explanation were likely to answer correctly Item 7 (interpreting derivative) in a mathematics context.

Q.3 *For the students who evidenced a Category 1, Action-Perception Category of Explanation for the meaning of slope in a mathematics context for Item 1 (shown in Table 3.7) do they associate with the transfer of Items 1-7 and Item 10 more so than the students who evidenced other categories of explanation?*

- In Study 1, students who evidenced this category of explanation were likely to transfer Items 7 and 10, and Items 4 and 5 more so than other students, if borderline significance is accepted ($0.05 \leq$ p-value ≤ 0.1). In Study 2, the students who evidenced this category of explanation were likely to transfer Items 5, 6 and 10 more so than other students, if borderline significance is accepted ($0.05 \leq$ p-value ≤ 0.1). Thus, across both studies, students who evidenced this category of explanation were likely to transfer Item 5 (generating an expression for slope) and 10 (proportionality) more so than other students, if borderline significance is accepted.

Q.4 *For the students who evidenced a Category 1, Action-Perception Category of Explanation for the meaning of slope in a mathematics context for Item 1 (shown in Table 3.7) do they associate with the correct answering of Items 1-7 and Item 10 in a mathematics context more so than the students who evidenced other categories of explanation?*

- In Study 1, students who evidenced this category of explanation were likely to answer correctly Items 7 and 10 in a mathematics context more so than other students. In Study 2, the students who evidenced this category of explanation were found not likely to answer correctly any of the items in a mathematics context more so than other students.

The findings from these questions inevitably raised the question as to how do the findings fare in the context of existing mathematics-educational literature? Previous literature highlights the importance of a deep understanding of slope. [75, 76, 77, 78].

Aspinwall and Miller [75] argue that a concept image of slope, and indeed derivative, that is limited to a representation involving the manipulation of a formula, could mean that students will find it difficult to understand instantaneous rates of change. As can be seen from the questions raised (Questions 1-4) it was found that the students who evidenced a *correct action-perception category of explanation* for slope tended to associate with the answering of the item on the interpretation of the meaning of derivative (Item 7) in a mathematics context, and also with the transfer of this item.

Gordon [76] found that students who had trouble in interpreting the meaning of slope also had trouble using the general formula for the equation of a line: $y = mx + c$, for predictive questions. For example, when students were faced with the following question:

Brookville College enrolled 2546 student in 1996 and 2702 students in 1998. Assume the enrolment follows a linear growth pattern.

(a) Write a linear equation that gives the enrolment in terms of the year t (let t = 0 represent 1996).

(b) If the trend continues, what will the enrolment be in the year 2016?

(c) What is the slope of the line you found in part (a)?

(d) Explain, using an English sentence, when will the enrolment reach 3500 students?

(e) If the trend continues, when will the enrolment reach 3500 students?

those students which had trouble in answering part (d), also had trouble in answering part (b) and part (e) — the predictive questions. None of the items in this research probed students' understanding of the general formula for the equation of a line or the students' ability to answer predictive questions. The same author also argued, like Aspinwall and Miller [75], that without an understanding of slope, students will have difficulty—if not, be unable—to interpret the significance of the meaning of the derivative of a function.

Again, echoing the findings of Questions 1 and 3, Lobato [34] states that children transfer most successfully when they understand events at a causal level, rather than simply memorising them. Clearly, a deep understanding of slope is important. Is such an understanding taken to be one which is indicative of a correct action-perception category of explanation in terms of Tall's theory?

Previous work [76] has found, as supported by the findings in Table 3.5 of this study, that most students can calculate slope with little difficulty. However, in terms of interpreting the significance of the slope value, research indicates that students interpret the meaning of slope as a difference (the difference in y-values divided by the difference in x-values) rather than a ratio of differences — more specifically, the ratio of the change in value of the dependent variable, for each one unit change in the corresponding independent variable [42, 68, 77]. If the latter type of explanation indicates a deep understanding of slope, then it may not be surprising why so few students evidenced it during both Study 1 and Study 2, as can be seen from Tables 3.6 and 3.7.

Stump [78] found that only one student gave a specific numerical response for the meaning of slope, a response—from Stump's perspective—indicative of realising that 'm' is the ratio of the change in values of the dependent variable for each one unit change in the corresponding independent variable. The response was as follows: "say up 1, over . . ." Interestingly, Stump also found that students could mention ratios and rates in their responses for what a slope value means without using any specific numbers—perhaps the usage of the words 'ratios' and 'rates', by these students, were used without any deep understanding of what these words stand for in the context of explaining the meaning of slope. Such uncertainty was the reason why it was decided to classify students in Categories 2-3 for Study 1, and students in Categories 2-5 for Study 2 (as seen in Table 3.7), as evidencing a correct action-perception category of explanation for the meaning of slope, and thus ask Questions 1 and 2.

The findings of this study, and the literature on slope, suggested that students' understanding of slope must be improved. In the Intervention Chapter, Section 5.2.2.1, it can be seen that students' attention was directed, in visual terms, to a unit per unit comparison as to what it is a slope value means. Simply focusing on the calculation of the quotient in the slope formula does not suffice in ensuring that students interpret slope as a ratio [34]. Also, Lobato [68] puts forward the case for directing students' attention to the co-ordination of co-varying quantities, for to do so, will mean that students are more likely to generalise slope as a ratio.

3.5.2 Item 2—Sketching a Line with Positive Slope

Item 2: Sketching a Line with Positive Slope

Mathematics Context

L_1 as shown in Figure 1, passes through the Point ‘P’ and has a slope = 2.

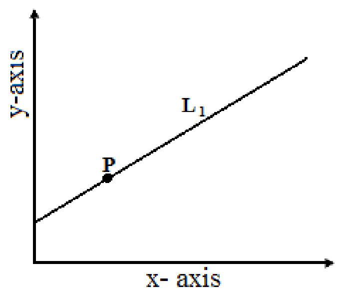


Figure 1

(A) Sketch in Figure 1: a line (L_2) that passes through the point P and has slope = 3.

(B) Explain your reasoning

Chemistry Context

The Line L_1 in Figure 1 shows the graph of the concentration of product with respect to time over a certain time interval (Δt). It has a value for the rate of change = 3.

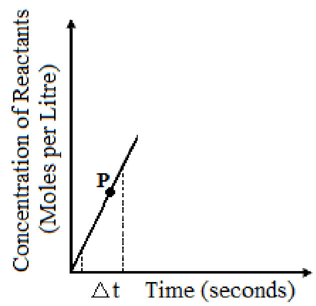


Figure 1

(A) Sketch in Figure 1: a line (L_2) that passes through the point P and has a value for the rate of increase of the product with respect to time = 4.

(B) Explain your reasoning.

Figure 3.3 Item 2 Used in the Main Study.

Mathematical Item	% Correct in a Mathematics Context		% Correct in a Chemistry Context	
	Study 1	Study 2	Study 1	Study 2
2. Sketching a Line with Positive Slope.	83%	92%	70%	63%

Table 3.16 Students’ Performance for Item 2 in the Main Study.

Item 2 requires students to interpret a value for slope (or rate of change in a chemistry context) and sketch what this value represents graphically. Therefore, the item in both contexts could be considered to be similar to movement from Tall's *2nd World* to *1st World.* The percentage of students in both studies who answered the item correctly in a mathematics context can be seen in Table 3.16. Thus, the students who answered correctly demonstrated that they could competently translate a *value for slope/rate of change*, into an 'embodied mathematical object'. The categories of explanation for the students who answered the item correctly in Study 2 are shown in Table 3.17. The Part B responses for the students who answered the item incorrectly in Study 2 are shown in Table 3.18 in tandem with the graphs these students drew for Part A. The graphs that the incorrect students drew in Study 1 are also shown in Table 3.18.

Item 2: Sketching a Line with Positive Slope
For the Students who Sketched the Line Correctly in a Mathematics Context

Study 2

Category	No.	Exp.	Tall.
1. Because the slope is 3, the line will increase at a steeper angle.	1	ATE	AP
2. L_2 has a greater slope, so its rate of increase will be higher, therefore the line will be steeper.	10	ATE	AP
3. In L_1, the rate of increase of y with respect to x is 2. In L_2, it is higher (3), which implies a steeper slope.	2	ATE	AP
4. Higher gradient than L_1	3	ATE	AP
5. More increasing than L_1 as L_2 has a higher slope.	2	ATE	AP
6. $\mathrm{Tan}^{-1}\,3 = 71.56^{\circ}$	1	ATE	AP
7. Slope is sharper; quicker change; $\frac{\Delta y}{x} = \text{larger}$.	2	ATE	AP

Table 3.17 The Correct Students' Categories of Explanation for Item 2 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; ATE—ability to Explain; Tall.—degree of explanation in terms of Tall's theory; AP—action-perception).

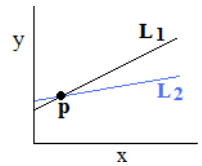
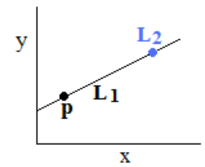
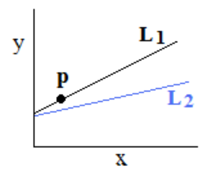
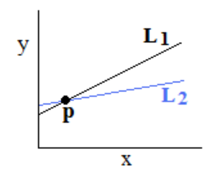
Item 2: Sketching a Line with Positive Slope For the Students who Sketched the Line Incorrectly in a Mathematics Context							
Study 1				**Study 2**			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. 	3	IATE	AP	1. Refer to L_2 passing through P and as increasing from L_1. 	1	IATE	AP
2. 	1	IATE	AP	2. Refer to the line being steeper when the slope value is smaller. 	2	IATE	AP
3. No graph drawn.	1	IATE	N/A				

Table 3.18. The Incorrect Students' Categories of Explanation for Item 2 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; IATE—inability to explain; Tall.—degree of explanation in terms of Tall's theory; AP—action-perception).

Looking at Table 3.17, for the students who answered the item correctly in Study 2, all of these students evidenced an ability to explain their answer, and were deemed to have evidenced an action-perception category of explanation in terms of Tall's theory. The most frequent of these categories of explanation was that of referring to the rate of increase for L_2 as greater than for L_1 (Category 2). In Table 3.18, the incorrect students demonstrated that they clearly were unable to interpret the value for the slope and sketch what this value meant. However, because the incorrect students in Categories 1 and 2 in both Study 1 and Study 2 drew a graph, the categories were deemed to be evidence of an action-perception category of explanation in terms of Tall's theory, albeit an incorrect action-perception category of explanation. Interestingly, two of the students in Category 2 of Study 2, articulated that a line is steeper when its slope value is 'lower'.

3.5.3 Item 3—Sketching a Line with Positive Slope

Item 3: Sketching a Line with Positive Slope

Mathematics Context

L_1 as shown in Figure 1, passes through the Point ‘P’ and has a slope = 2.

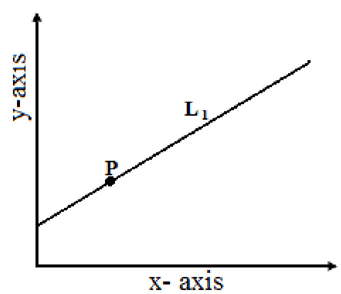


Figure 1

(A) Sketch in Figure 1: a line (L_3) that passes through the point P and has slope = 1.

(B) Explain your reasoning

Chemistry Context

The Line L_1 in Figure 1 shows the graph of the concentration of product with respect to time over a certain time interval (Δt). It has a value for the rate of change = 3.

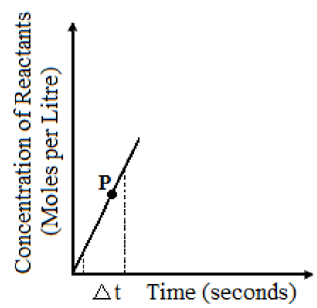


Figure 1

(A) Sketch in Figure 1: a line (L_3) that passes through the point P and has a value for the rate of increase of the product with respect to time = 1.

(B) Explain your reasoning.

Figure 3.4 Item 3 Used in the Main Study.

Mathematical Item	% Correct in a Mathematics Context		% Correct in a Chemistry Context	
	Study 1	Study 2	Study 1	Study 2
3. Sketching a Line with Positive Slope.	80%	87%	70%	58%

Table 3.19 Students’ Performance for Item 3 in the Main Study.

Similar to Item 2, Item 3 requires students to interpret a value for slope (or rate of change in a chemistry context) and sketch what it represents graphically. Therefore, like Item 2, the item in both contexts could be considered to be similar to movement from Tall's *2nd World* to *1st World.* The percentage of students in both studies who answered the item correctly in a mathematics context can be seen in Table 3.19. Thus, the students who answered correctly demonstrated that they could competently translate a *value for slope/rate of change*, into an 'embodied mathematical object'. The categories of explanation for the students who answered the item correctly in Study 2 are shown in Table 3.20. The categories of explanation for the students who answered the item incorrectly in Study 2 are shown in Table 3.21 in tandem with the graphs these students drew. The graphs that the incorrect students drew for Part A in Study 1 are also shown in Table 3.21.

Looking at Table 3.20, for the students who answered the item correctly in Study 2, all of these students — bar one — evidenced an ability to explain their answer, and were deemed to have evidenced an action-perception category of explanation in terms of Tall's theory. The most frequent of these categories of explanation was that of referring to the slope of L_3 as less steep (Category 6). The second most frequent categories of response were: Category 3; referring to the rate of increase of L_3 as less when compared with the rate of increase for L_1 and Category 5; referring to the gradient of L_3 as less when compared with the gradient of the line L_2 which they drew in Item 2. Interestingly, the explanation: *less change in y per x-value* (Category 1) was similar to the *Category 1 Action-Perception Category of Explanation*, observed for Item 1 (as can be seen in Table 3.7).

In Table 3.21, the incorrect students demonstrated that they clearly were unable to interpret the value for the slope and sketch what this value meant for the line in question. However, because the students in Categories 1-3 in Study 1 and in Categories 1-4 in Study 2 drew a graph, the categories were deemed to be evidence of an action-perception category of explanation in terms of Tall's theory, albeit an incorrect action-perception category of explanation. Two of the students in Category 3 of Study 2, articulated that a line is steeper when its slope value is 'lower'. These students were the same students who provided the same category of explanation for Item 2 in Study 2.

Item 3: Sketching a Line with Positive Slope
For the Students who Sketched the Line Correctly in a Mathematics Context

Study 2

Category	No.	Exp.	Tall.
1. Less change in y per x-value	1	ATE	AP
2. The line will decrease at a less steeper angle than the +2 line.	1	ATE	AP
3. It has a lower slope so its rate of increase is lower, therefore the line will be less steep.	3	ATE	AP
4. +1 so it again goes up from left to right and is more gradual than L_1 and L_2.	2	ATE	AP
5. Lesser gradient than that for L_2.	3	ATE	AP
6. L_3; slope is less; should be less steep.	4	ATE	AP
7. L_3 has lower slope so increases slower than L_1 and L_2.	1	ATE	AP
8. The line (L_3) passes through P, but has a lower slope of +1.	1	ATE	AP
9. Because slope is +1, the degree of incline of L_3 should be smaller than L_1; and because L_3 must pass 'p', the y-intercept must change again to accommodate for this.	1	ATE	AP
10. Its slope is lower so the line has a shallower rise as the values for y are lower and/or the x-values are higher.	1	ATE	AP
11. No reason	1	IATE	AP

Table 3.20 The Correct Students' Categories of Explanation for Item 3 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; ATE—ability to Explain; IATE—Inability to Explain; Tall. — degree of explanation in terms of Tall's theory; AP—action-perception).

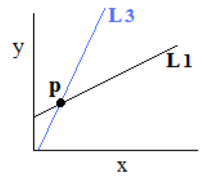
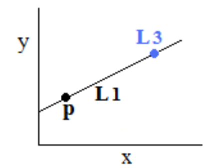
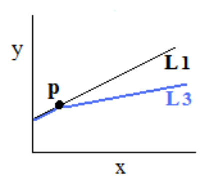
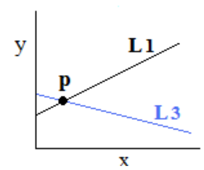
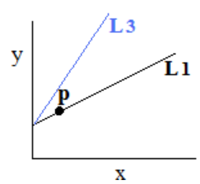
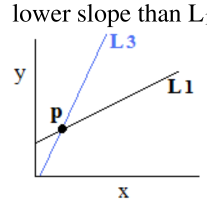
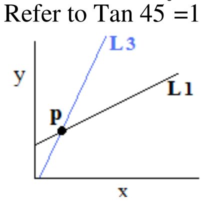
Item 3: Sketching a Line with Positive Slope For the Students who Sketched the Line Incorrectly in a Mathematics Context							
Study 1				**Study 2**			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. 	2	IATE	AP	1. Refer to L_3 passing through P and increasing but not as far as L_1. 	1	IATE	AP
2. 	1	IATE	AP	2. Refer to L_3 as lying beneath +2 as its slope is not as steep. 	1	IATE	AP
3. 	1	IATE	AP	3. Refer to L_3 as steeper which means it has a lower slope than L_1. 	2	IATE	AP
4. No graph drawn.	2	IATE	N/A	4. Refer to Tan $45^{\circ}=1$ 	1	IATE	AP

Table 3.21.The Incorrect Students' Categories of Explanation for Item 3 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; IATE—inability to explain; Tall.—degree of explanation in terms of Tall's theory; AP—action-perception).

3.5.4 Item 4 — Sketching a Line with Negative Slope

Item 4: Sketching a Line with Negative Slope

Mathematics Context

L_1 as shown in Figure 1 passes through the Point ‘P’ and has a slope = 2.

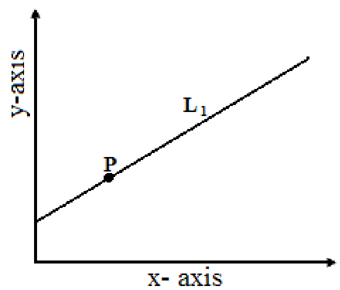


Figure 1

(A) Sketch in Figure 1: a line (L_4) that passes through the point P and has slope = -1.

(B) Explain your reasoning

Chemistry Context

The line in Figure 1 shows the graph of concentration of reactant with respect to time over a certain interval (Δt) . Its rate of decrease over this interval is equal to 2.

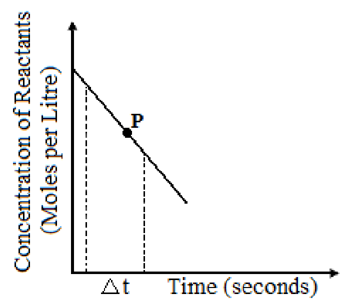


Figure 1

(A) Sketch in Figure 1: a line (L_4) that passes through the point P and has a value for the rate of decrease of the reactant with respect to time = 1.

(B) Explain your reasoning.

Figure 3.5 Item 4 Used in the Main Study.

Mathematical Item	% Correct in a Mathematics Context		% Correct in a Chemistry Context	
	Study 1	Study 2	Study 1	Study 2
4. Sketching a Line with Negative Slope.	77%	96%	53%	63%

Table 3.22 Students’ Performance for Item 4 in the Main Study.

Similar to Items 2 and 3, Item 4, requires students to interpret a value for slope (or rate of change in a chemistry context) and sketch what it represents graphically. Therefore, like Items 2 and 3, the item in both contexts could be considered to be similar to movement from Tall's *2nd World* to *1st World.* The percentage of students in both studies who answered the item correctly in a mathematics context is shown in Table 3.22. Thus, like for Items 2-3, certain students evidenced that they could not competently translate a *value for slope/rate of change*, into an 'embodied mathematical object'. The categories of explanation for the students who answered the item correctly in Study 2 are shown in Table 3.20. The categories of explanation for the students who answered the item incorrectly in Study 2 are shown in Table 3.23 in tandem with the graphs these students drew. The graphs that the incorrect students drew for Part A in Study 1 are also shown in Table 3.24.

Looking at Table 3.23, for the students who answered the item correctly in Study 2, all of these students in Categories 1-6 were deemed to have evidenced an ability to explain, and to have evidenced a correct action-perception category of explanation in terms of Tall's theory. The most frequent of these categories (Category 5) was that of referring to the slope of L_4 as a reversed gradient because it [the slope value] is negative. Interestingly, the explanation: *decrease in y per x-value* (Category 1) was provided by the same student who provided the explanation: *less change in y per x-value* when they answered Item 3 correctly, as can be seen in Category 1 of Table 3.20.

The students in Categories 7-11 were deemed to have evidenced an inability to explain, and to have evidenced an incorrect action-perception category of explanation in terms of Tall's theory. The most frequent of these categories of explanation (Category 7) was that of referring to the slope of L_4 as decreasing, suggesting that the students do not have a deep understanding of the meaning of slope, or perhaps they do, and this explanation was just a figure of speech.

In Table 3.24, the incorrect students demonstrated that they clearly were unable to interpret the value for the slope and sketch what this value meant for the line in question. However, because the students in Categories 1-3 in Study 1 and the student in Category 1 in Study 2 drew a graph, the categories were deemed to be evidence of an action-perception category of explanation in terms of Tall's theory, albeit incorrect action-perception categories of explanation.

Item 4: Sketching a Line with Negative Slope For the Students who Sketched the Line Correctly in a Mathematics Context Study 2			
Category	**No.**	**Exp.**	**Tall.**
1. Decrease in y for increase in x.	1	ATE	AP
2. When slope is minus, the rate of change is a decrease.	1	ATE	AP
3. Would have the same angle as L_3, only inverted as slope is negative.	2	ATE	AP
4. Negative slope, so is decreasing at the same rate as L_3 is increasing.	1	ATE	AP
5. Reversed gradient as its negative.	4	ATE	AP
6. Negative slope implies: NE→ SW	1	ATE	AP
7. The slope of the line will be decreasing.	8	IATE	AP
8. The slope is negative and must have a negative value on either axis at some point.	1	IATE	AP
9. Point 2 is before point p. The x_2-x_1 value is negative, giving the slope a negative answer.	1	IATE	AP
10. Slope is decreasing as it's a negative value i.e., the degree of incline must be smaller than zero. Also, the y-intercept must change to accommodate for this.	1	IATE	AP
11. No reason	1	IATE	AP

Table 3.23 The Correct Students' Categories of Explanation for Item 4 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; ATE—ability to Explain; IATE—Inability to Explain; Tall. — degree of explanation in terms of Tall's theory; AP—action-perception).

Item 4: Sketching a Line with Negative Slope
For the Students who Sketched the Line Incorrectly in a Mathematics Context

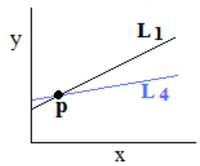
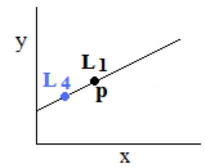
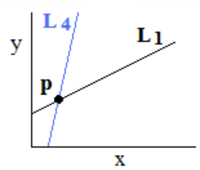
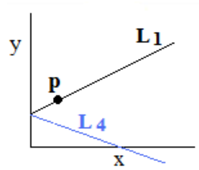
Study 1				**Study 2**			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. 	1	IATE	AP	1. Refer to L_4 as decreasing but still passing through P. 	1	IATE	AP
2. 	2	IATE	AP	2. Provide no drawing and no reasoning.	1	IATE	N/A
3. 	1	IATE	AP				
4. No graph or reasoning provided	3	IATE	N/A				

Table 3.24.The Incorrect Students' Categories of Explanation for Item 4 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; IATE—inability to explain; Tall.—degree of explanation in terms of Tall's theory; AP—action-perception; N/A—not applicable).

3.5.5 Item 5—Generating an Expression for Slope

Item 5: Generating an Expression for Slope

Mathematics Context

(A) Using the notation in the diagram in Figure 1, write down an expression for the slope of a line connecting B-C.

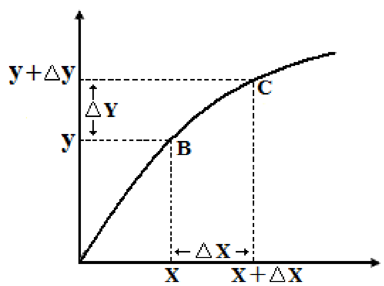


Figure 1

(B) Explain your reasoning

Chemistry Context

(A) Figure 1 shows the change of concentration of product (P) over time (t). Using the notation in the diagram, write down an expression for the average rate of change of product (P) between B and C.

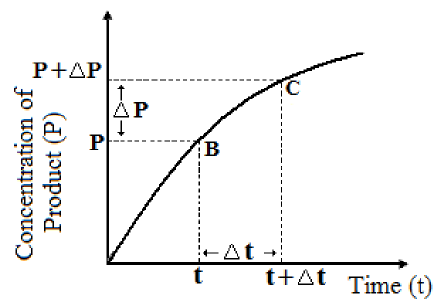


Figure 1

(B) Explain your reasoning.

Figure 3.6 Item 5 Used in the Main Study.

Mathematical Item	% Correct in a Mathematics Context		% Correct in a Chemistry Context	
	Study 1	Study 2	Study 1	Study 2
5. Generating an Expression for Slope.	66%	67%	53%	42%

Table 3.25 Students' Performance for Item 5 in the Main Study.

Item 5 requires students to recall the embodied mathematical object-type image for slope (or average rate of change in a chemistry context), and then symbolise this image appropriately. Therefore, the item in both contexts could be considered to resonate with movement from Tall's *1st World* to *2nd World.* The percentage of students in both studies who answered the item correctly in a mathematics context can be seen in Table 3.25. Thus, many students could not competently recall the embodied mathematical object-type image for slope/rate of change, and then translate this image into the symbols which represent it. The categories of explanation provided by the students who answered the item correctly in Study 2 are shown in Table 3.26. The categories of explanation provided by the students who answered the item incorrectly in Study 2 are shown in Table 3.27; the incorrect students' responses for Part A in Study 1 are also shown in this table.

Item 5: Generating an Expression for Slope
For the Students who Answered Correctly in a Mathematics Context

Study 2

Category	No.	Exp.	Tall.
1. Slope is the difference of the y-values divided by the difference of the x-values.	4	ATE	PA
2. The slope equation: $\frac{y_2 - y_1}{x_2 - x_1}$ simplifies to $\frac{\Delta y}{\Delta x}$.	2	ATE	PA
3. Slope formula is: $\frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y + y - y}{\Delta x + x - x} = \frac{\Delta y}{\Delta x}$	7	ATE	PA
4. Slope = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Changein y}}{\text{Changein x}} = \frac{\Delta y}{\Delta x}$	2	ATE	PA
5. No reason.	1	IATE	PA

Table 3.26 The Correct Students' Categories of Explanation for Item 5 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; ATE—ability to Explain; IATE—Inability to Explain; Tall. — degree of explanation in terms of Tall's theory; PA—perception-action).

Looking at Table 3.26, for the students who answered the item correctly in Study 2, all of these students in Categories 1-4 were deemed to have evidenced an ability to explain, and to have evidenced a correct perception-action category of explanation in terms of Tall's theory. The most frequent of these categories (Category 3) was that of referring to the slope formula in order to explain how to generate an expression for slope. The second most frequent category (Category 1) was that of referring to slope as the difference of the y-values divided by the difference of the x-values. The student in Category 5 provided no reason despite answering Part A correctly. They were deemed to have evidenced an inability to explain, but as having evidenced a perception-action category of explanation in terms of Tall's theory because they demonstrated this when they answered Part A correctly.

Item 5: Generating an Expression for Slope
For the Students who Answered Incorrectly in a Mathematics Context

Study 1				**Study 2**			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. m =	1	IATE	PA	1. The equation: $\frac{y_1 - b}{x_1 - c}$ would give values that include B & C, that would give the slope of a line.	1	IATE	PA
2. $\Delta x + \Delta y$	1	IATE	PA	2. Δx is the relative change in respect to Δy (the change of y); x, y factor of change.	1	IATE	PA
3. $\frac{y + \Delta y}{x + \Delta x}$	1	IATE	PA	3. Differentiation implies finding slope	1	IATE	PA
4. $\frac{\Delta x}{\Delta y}$	1	IATE	PA	4. m= $\frac{x + \Delta x - y + \Delta y}{x - y}$	1	IATE	PA
5. $\frac{(y + \Delta y) - y}{(x + \Delta x) - x}$	1	IATE	PA	5. Refer to the two points corresponding to two points on the graph, so the slope can be worked out to be those two points.	1	IATE	PA
6. $\frac{dy}{dx}$; derivative	1	IATE	PA	6. Did not answer Part A or provide a reason in Part B.	3	IATE	N/A
7. $\frac{(x + \Delta x) - x}{(y + \Delta y) - y}$	1	IATE	PA				
8. Did not answer Part A or provide a reason in Part B.	3	IATE	N/A				

Table 3.27 The Incorrect Students' Categories of Explanation for Item 5 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; IATE—inability to explain; Tall.—degree of explanation in terms of Tall's theory; PA—perception-action; N/A—not applicable).

In Table 3.27, for Study 2, the incorrect students' categories of explanation evidenced that these students could not explain how the embodied mathematical object-type image for slope, in a mathematics context, can be symbolised appropriately. Despite this inability to explain, the students' categories of explanation were each classified as perception-action in terms of Tall's theory, albeit incorrect perception-action categories of explanation.

The incorrect students' responses to Part A in Item 1 for Study 1, also demonstrated that they were unable to recall the embodied mathematical object-type image for slope in a mathematics context and symbolise it appropriately. However, because these students attempted to symbolise an expression for slope, they were considered to have evidenced a perception-action category of explanation in terms of Tall's theory, even though these categories were incorrect perception-action categories of explanation.

The percentage of students who answered Item 5 correctly in a mathematics context, in both studies, in comparison to the higher percentage of the same students who could answer Items 1-4, raises the question of why this is so. Perhaps, with respect to Item 1, correctly performing 'actions on the symbols relevant to calculating slope' is easier for the students in comparison to moving from an embodied mathematical object-type understanding of slope to a symbolic one (Item 5). However, such a claim is at odds with the findings for Items 2-4, which all required actions to be linked with embodied mathematical objects in the form of a graph. Even though the correct response rates for each of Items 2-4 were less compared to Item 1, they were still greater than the correct response rate for Item 5. Perhaps, through familiarity, students can remember the answer for Items 2-4, in terms of the shape of the appropriate graph—or hazard a guess? Or perhaps the students considered Items 1-5 to be similar, and by the time they answered Item 5, they had lost interest.

3.5.6 Item 6—Generating an Expression for Derivative

Item 6: Generating an Expression for Derivative

Mathematics Context

Using the notation in the diagram in Figure 1, write down an expression for the slope of a line connecting B-C.

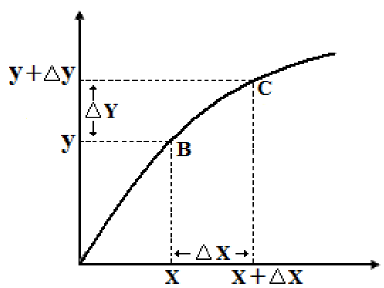


Figure 1

(A) Using your answer, explain how you could generate the derivative $\left(\frac{dy}{dx}\right)$ at B.

Chemistry Context

Figure 1 shows the change of concentration of product (P) over time (t). Using the notation in the diagram, write down an expression for the average rate of change of product (P) between B and C.

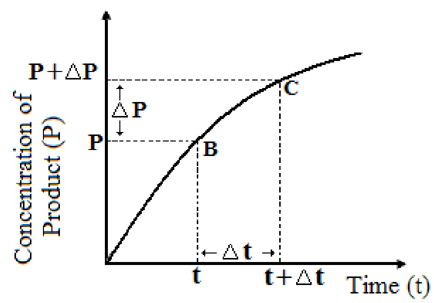


Figure 1

(A) Using your answer, explain how you could generate the instantaneous rate of change $\left(\frac{dP}{dt}\right)$ at B.

Figure 3.7 Item 6 Used in the Main Study.

Mathematical Item	% Correct in a Mathematics Context		% Correct in a Chemistry Context	
	Study 1	**Study 2**	**Study 1**	**Study 2**
6. Generating an Expression for Derivative.	20%	17%	26%	33%

Table 3.28 Students' Performance for Item 6 in the Main Study.

Item 6 in the mathematics context requires students to explain the generation of the derivative at the point in question, in one of either two ways: 1) state that finding the limit of the quotient $\frac{\Delta y}{\Delta x}$ as Δx approaches zero and the related Δy approaches zero, produces a value for the *slope of the tangent/derivative* at the point B; or 2) state that the derivative can be found by finding the slope of a tangent at the point B. In a similar vein, in a chemistry context, the same type of explanations could be evidenced in respect of how the instantaneous rate of change $\frac{dP}{dt}$ is generated.

Therefore, the item in both contexts could be considered to be similar to movement from Tall's *1st World* to *2nd World.* The percentage of students in both studies who answered the item correctly in a mathematics context can be seen in Table 3.28. Thus, many students demonstrated that they could not competently recall the embodied mathematical object for *derivative/instantaneous rate of change*, and describe, in symbolic terms, how such an object is generated. The categories of explanation for the students who answered the item correctly in Study 1 and Study 2 are shown in Table 3.29. The categories of explanation for the students who answered the item incorrectly in Study 1 and Study 2 are shown in Table 3.30.

Item 6: Generating an Expression for Derivative
For the Students who Answered Correctly in a Mathematics Context

Study 1				**Study 2**			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. Refer to differentiating.	2	ATE	PA	1. The instantaneous rate has to be obtained. This is the rate of change of y with respect to x at a point P.	1	ATE	PA
2. Refer to finding the slope of a tangent at the point.	4	ATE	PA	2. Find the slope of the tangent to the curve at B.	3	ATE	PA

Table 3.29 The Correct Students' Categories of Explanation for Item 6 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; ATE—ability to Explain; Tall. — degree of explanation in terms of Tall's theory; PA—perception-action).

Looking at Table 3.29, for the students who answered the item correctly in both studies, the categories of explanation which reflected these students' responses, were each deemed to be reflective of a perception-action category of explanation in terms of Tall's theory. The most frequent of these categories (Category 2) which emerged in both studies was that of referring to finding the slope of the tangent to the curve at B.

Item 6: Generating an Expression for Derivative
For the Students who Answered Incorrectly in a Mathematics Context

Study 1				Study 2			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. Allude to what a derivative is but not how to generate it.	3	IATE	PA	1. $\frac{y+\Delta y-y}{x+\Delta x-x}=\frac{dy}{dx}$ at B	5	IATE	PA
2. Refer to Integration	1	IATE	PA	2. Refer to filling the point B into the equation for the slope.	1	IATE	PA
3. Do not know.	1	IATE	N/A	3. $\frac{y_2-y_1}{x_2-x_1}$	1	IATE	PA
4. Do not remember.	2	IATE	N/A	4. At B, go across to the y-axis to find dy, then down to the x-axis to find dx.	1	IATE	PA
5. Refer to substituting Δy with the y-value at B and the Δx with x-value at B.	3	IATE	N/A	5. Refer to using the equation of the line.	2	IATE	PA
6. Inability to distinguish between $\frac{dy}{dx}$ and $\frac{\Delta y}{\Delta x}$.	1	IATE	PA	6. $\frac{d^2y}{dx^2}=$ Quotient rule	1	IATE	PA
7. Did not provide an answer to Part A or provide a reason.	13	IATE	N/A	7. Did not provide an answer to Part A or provide a reason.	8	IATE	N/A
				8. If the origin is treated as x,y, B becomes x + Δx, y +Δy; slope becomes derivative.	1	IATE	PA

Table 3.30 The Incorrect Students' Categories of Explanation for Item 6 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; IATE—inability to explain; Tall.—degree of explanation in terms of Tall's theory; PA—perception-action; N/A—not applicable).

In both studies, some of the most frequent categories which emerged (Categories 3-4 and Category 7 in Study 1, and Category 7 in Study 2) as a result of the students answering the item incorrectly indicated that students were unable to provide any kind of answer for Part A of the item. These categories can be seen in Table 3.30. The remainder of the categories in Table 3.30 indicated that students were unable to explain how to generate the derivative at the point in question. Despite these categories not being evidence of an ability to explain, they were classified as perception-action categories of explanation in terms of Tall's theory, albeit incorrect perception-action categories of explanation.

3.5.7 Item 7

Item 7 (shown in Figure 3.8) requires students to recall the embodied mathematical object, in graphical terms, for the *derivative /instantaneous rate of change* at a particular point, and subsequently interpret the meaning behind this object (namely, the slope of the tangent) in order to answer the item correctly. Therefore, the item in both contexts could be considered to reside within Tall's *1st World,* and may or may not require students to move into the *2nd World* when comparing the slopes of the tangents in either a mathematics context or chemistry context. The percentage of students in both studies who answered the item correctly in a mathematics context can be seen in Table 3.31. The categories of explanation for the students who answered the item correctly in Study 1 and Study 2 are shown in Table 3.32. The categories of explanation for the students who answered the item incorrectly in Study 1 and Study 2 are shown in Table 3.33.

Looking at Table 3.32, for the students who answered the item correctly, the most frequent category of response (Category 2 in both studies) was that of referring to the point having a bigger slope and/or greatest rate of change. The second most frequent category of response (Category 1 in both studies) referred to the slope of the tangent at the point as greater. All of the categories which were deemed as evidence of an ability to explain were classified as perception-action categories of explanation in terms of Tall's theory. In Study 1, Categories 4-5 were deemed as evidence of an inability to explain. One of these categories (Category 4) was classified as an action-on-perception category of explanation. In Study 2, Category 4 was deemed as evidence of an inability to explain, but was classified as an incorrect perception-action category of explanation in terms of Tall's theory.

Item 7: Interpreting Derivative

Mathematics Context

(A) Figure 1 shows the graph of y against x. At which point, A or B, does the greatest value of $\frac{dy}{dx}$ occur?

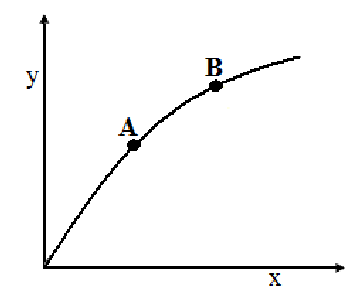


Figure 1

(B) Explain your reasoning.

Chemistry Context

(A) For a particular reaction:

$$A + B \rightarrow P$$

where A and B are reactants and P is product, Figure 1 shows the graph of concentration of product (P) against time (t). At which point, E or F, does the greatest increase in concentration of product with respect to time occur?

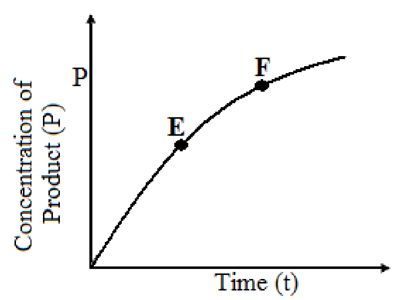


Figure 1

(B) Explain your reasoning.

Figure 3.8 Item 7 Used in the Main Study.

Mathematical Item	% Correct in a Mathematics Context		% Correct in a Chemistry Context	
	Study 1	Study 2	Study 1	Study 2
7. Interpreting Derivative.	73%	67%	83%	96%

Table 3.31 Students' Performance for Item 7 in the Main Study.

Item 7: Interpreting Derivative For the Students who Answered Correctly in a Mathematics Context							
Study 1				**Study 2**			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. Refer to the slope of the tangent at this point being greater.	4	ATE	PA	1. Refer to the derivative meaning the tangent to the line at a particular point, and state the tangent at A has a greater slope than the tangent at B.	4	ATE	PA
2. Refer to the point having a bigger slope and/or the greatest rate of change.	11	ATE	PA	2. Refer to the slope being steeper, and that this is why the value [slope value] will be higher.	6	ATE	PA
3. Refer to there being a greater change in y for a certain change in x at this point, compared to the alternative point.	3	ATE	PA	3. Refer to the derivative representing the rate of change. The A value has a higher rate of change compared to B.	1	ATE	PA
4. Refer to the curve being sharper at this point.	1	IATE	P	4. Refer to the line increasing more at the Point A, from 0 to A than from A to B.	5	IATE	PA
5. Refer to guessing and/or provide no reason.	3	IATE	N/A				

Table 3.32 The Correct Students' Categories of Explanation for Item 7 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; ATE—ability to Explain; IATE—Inability to Explain; Tall. — degree of explanation in terms of Tall's theory; PA—perception-action; P—action-on-perception; N/A—not applicable).

In Table 3.33, for the students who answered the item incorrectly in both studies, the categories of explanation which these students evidenced were all deemed to be evidence of an inability to explain. In Study 1, Categories 1-2 and Category 5 were deemed to be evidence of action-on-perception categories of explanation, while Category 4 was considered to be evidence of a perception-action category of explanation. In Study 2, all of the categories, apart from Category 3, were considered to be evidence of perception-action categories of explanation in terms of Tall's theory.

Item 7: Interpreting Derivative For the Students who Answered Incorrectly in a Mathematics Context							
Study 1				**Study 2**			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. Refer to the point being a greater distance from the x-axis and/or the origin.	4	IATE	P	1. Refer to B having a higher slope than A, and refer to B's gradient being much larger.	4	IATE	PA
2. Refer to the graph slowly levelling off.	1	IATE	P	2. Refer to the derivative as the slope m.	1	IATE	PA
3. Provide no answer and no reason	1	IATE	N/A	3. Provide no answer and no reason.	2	IATE	N/A
4. State that the slope is increasing at the same rate at both points.	1	IATE	PA	4. Refer to the derivative being normally associated with a small change, and state that there is less of a change with B.	1	IATE	PA
4. State that the graph is more curved at this point.	1	IATE	P				

Table 3.33 The Incorrect Students' Categories of Explanation for Item 7 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; IATE—inability to explain; Tall.—degree of explanation in terms of Tall's theory; PA—perception-action; P—action-on-perception; N/A—not applicable).

In both studies, the percentage of students who answered Item 7 correctly in a mathematics context was greater than the percentage of students who answered Item 6 correctly in a mathematics context. This raises the question of why, especially since Items 6 and 7 both test students' understanding of derivative.

Aspinwall and Miller [75] argue that for the majority of students, their understanding of derivative is limited to a representation involving the manipulation of a formula; the findings in respect of Item 6 (which does not test a students' ability to differentiate a function) would uphold this point of view. Cetin [79] also agrees with this, describing how students, who can only differentiate an algebraic function, and not link this differentiation to anything, will not be aware that the derivative of the function represents the instantaneous rate of change of the function at any particular point. Thompson [80] found that many students interpret the derivative as how fast the function is changing, without, more importantly, interpreting the derivative as representative of the amount of change in one quantity in relation to a change in another. Perhaps the reason for this is that many students are only able to think in terms of amounts, rather than in terms of rates of change of amounts [81].

Another reason for the majority of students not being able to answer Item 6 may be because the item can be interpreted as implicitly requiring the explanation of the limiting process behind the generation of the derivative at a particular point. Students' difficulties with understanding limits, in the context of calculus, are well documented [57,82,83]. The term *dy/dx* also causes serious problems for students, with a large percentage of students failing to realise that *dy/dx* is the ratio of the quotient of two infinitesimally small increments [81]. If such difficulties may explain students' performance in respect of Item 6, then why the comparatively better performance for Item 7?

Two possible reasons for this may be: 1) the absence of a need to understand the limiting process involved in generating a derivative in order to explain and 2) students possess the ability to recall that the derivative at any particular point is equivalent to finding the slope of a tangent at that point without necessarily being able to explain how to generate the slope of the tangent at that point. This type of explanation is implicitly required (although not necessarily) for the answering of Item 6.

Looking at Tables 3.3 and 3.4, it can be seen that for Items 6 and 7, students who evidenced an ability to explain correctly in the mathematics context in both studies, associated with the transfer of these items. In Table 3.29, it can be seen that for Item 6, the most frequent perception-action category of explanation, in both studies, was that of referring to finding the slope of the tangent at the point. It could be argued that this is an embodied mathematical object-type image of derivative, which allows the students to transfer. For Item 7, in both studies, the two most frequent perception-action categories of explanation are those in Categories 1-2, (as can be seen in Table 3.32). These categories highlight the importance of understanding the significance of the derivative at a particular point in terms of the slope of the tangent at that point.

3.5.8 Item 8—Usage of Exponentials

Item 8: Usage of Exponentials

Mathematics Context	**Chemistry Context**
Given: $\text{Ln}\,y = c - mx$	A student is studying the chemical reaction: $A + B \rightarrow P$, where A and B are reactants and P is the product. After graphing the Ln of the concentration of A, obtained at different times (i.e. the graph of $\text{Ln}[A]_t$ against time (t)), the student finds that the graph corresponds to the relation given below, showing that the rate of the reaction is 1st order with respect to A. $\text{Ln}[A]_t = \text{Ln}[A]_0 - kt$
(A) Derive an expression for y.	**(A)** Derive an expression for $[A]_t$
(B) Explain your reasoning.	**(B)** Explain your reasoning.

Figure 3.9 Item 8 Used in the Main Study.

Mathematical Item	**% Correct in a Mathematics Context**		**% Correct in a Chemistry Context**	
	Study 1	**Study 2**	**Study 1**	**Study 2**
8. Usage of Exponentials	43%	58%	33%	8%

Table 3.34 Students' Performance for Item 8 in the Main Study.

Item 8 requires students to recall the inverse relationship that exists between an exponential function to the base e and its corresponding, inverse natural logarithmic function. The item resides within Tall's *2nd World*. Such a classification does not necessarily imply that the students do not need an embodied mathematical object-type image of the inverse relationship; instead, such an understanding is not required to be evidenced by the students in order to answer the item. The percentage of students in both studies who answered the item correctly in a mathematics context can be seen in Table 3.34. The categories of explanation for the students who answered the item correctly in Study 1 and Study 2 are shown in Table 3.35. The categories of explanation for the students who answered the item incorrectly in Study 1 and Study 2 are shown in Table 3.36.

Item 8: Usage of Exponentials
For the Students who Answered Correctly in a Mathematics Context

Study 1				**Study 2**			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. Refer to 'exponential' as the opposite of Ln/Log	2	ATE	A	1. Refer to getting y on its own by multiplying both sides by e.	8	ATE	A
2. State that the Ln can be cancelled by getting the exponent of both sides.	5	ATE	A	2. State that Ln and e^x are inverses of each other, and that this makes it possible to get y on its own.	3	ATE	A
3. Refer to using the log and/or indice rules.	5	ATE	A	3. Provide no reason.	1	IATE	A
4. Word the exponential-natural logarithmic relationship incorrectly.	1	IATE	A				
5. Provide no reason	1	IATE	A				

Table 3.35 The Correct Students' Categories of Explanation for Item 8 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; ATE—ability to explain; IATE—inability to explain; Tall. — degree of explanation in terms of Tall's theory; A—action-on-action).

Looking at Table 3.35, in both studies the majority of students were deemed to have evidenced an ability to explain their reasoning, apart from the student in Category 5 in Study 1 and the student in Category 3 in Study 2. All of the correct categories of explanation were deemed to be evidence of an action-on-action type of explanation in terms of Tall's theory. The two most frequent categories of explanation in Study 1 were

Categories 2-3, and in Study 2, Categories 1-2. All of these categories either referred to the fact that e^x and Ln x are inverses or that y can be obtained by use of ‘e’.

Item 8: Usage of Exponentials For the Students who Answered Incorrectly in a Mathematics Context							
Study 1				**Study 2**			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. Realise that it is necessary to ‘exponentiate’; however, they make errors in the symbolic manipulation.	4	ATE	A	1. Realise that applying ‘exponentiation’ removes Ln but produce the following: $y = e^c - e^{mx}$	4	IATE	A
2. Divide the right-hand-side of the expression by Ln.	4	IATE	A	2. Refer to Ln’s and e’s cancelling, and produce the following: $y = e\ (c\text{-}mx)$	4	IATE	A
3. Refer to getting the natural log of both sides in order to cancel each log out.	1	IATE	A	3. Refer to rearranging into the following: $y = \frac{c - mx}{\text{Ln}}$	1	IATE	A
4. Refer to dividing the right-hand-side by Ln, believing ‘Ln’ to become Log when brought across the equal sign.	2	IATE	A	4. Refer to Ln & e as the inverses of each other, and produce the following: $y = e^c + e^{-mx}$	1	IATE	A
5. Refer to multiplying the right-hand-side by Ln.	2	IATE	A	5. Refer to using the exponential in order to get rid of the y, and produce the following: $e^y = \text{-}mx + c$	1	IATE	A
6. Provide no workings and no reason.	3	IATE	A	6. State the inverse of Ln is e, and produce the following: $y = e^c - e^{mx}$	1	IATE	A

Table 3.36 The Incorrect Students’ Categories of Explanation for Item 8 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; IATE—inability to explain; ATE—Ability to Explain; Tall.—degree of explanation in terms of Tall’s theory; A—action-on-action).

Looking at Table 3.36, for the students who answered incorrectly in both studies, in only one of these categories (Category 1) during Study 1, did students evidence an ability to explain. All of the other categories suggested an inability to explain and were categorised in terms of Tall’s theory as action-on-action categories of explanation.

3.5.9 Item 9 — Usage of Natural Logarithms

Item 9: Usage of Natural Logarithms

Mathematics Context

Given:

$$y = y_0 e^{-2x}$$

(A) Derive an expression for x in terms of y and y_0.

(B) Explain your reasoning.

Chemistry Context

For a reaction:

$$A + B \rightarrow P,$$

where A and B are reactants and P is product, the concentration of reactant B after a certain time ($[B]_t$) is given as a function of time in the following expression:

$$[B]_t = [B]_0 e^{-kt}$$

where $[B]_0$ and k are the initial concentration of reactant B and rate constant respectively.

(A) Derive an expression for k.

(B) Explain your reasoning.

Figure 3.10 Item 9 Used in the Main Study.

Mathematical Item	% Correct in a Mathematics Context		% Correct in a Chemistry Context	
	Study 1	Study 2	Study 1	Study 2
9. Usage of Natural Logarithms.	43%	46%	40%	71%

Table 3.37 Students' Performance for Item 9 in the Main Study.

Like Item 8, Item 9 requires students to recall the inverse relationship that exists between an exponential function to the base e and its corresponding, inverse natural logarithmic function. Therefore, the item resides within Tall's *2nd World*. Again, such a classification does not necessarily imply that the students do not need an embodied mathematical object-type image of the inverse relationship; instead such an understanding is not required to be evidenced by the students in order to answer the item.

The percentage of students in both studies who answered the item correctly in a mathematics context can be seen in Table 3.37. Thus, the majority of students demonstrated that they do not know the inverse relationship which exists between an exponential function to the base e and its corresponding, natural logarithmic function. The categories of explanation for the students who answered the item correctly in Study 1 and Study 2 are shown in Table 3.38. The categories of explanation for the students who answered the item incorrectly in Study 1 and Study 2 are shown in Table 3.39.

Looking at Table 3.38, for the students who answered the item correctly in Study 1, the only category of explanation to emerge was that of referring to the insertion of 'Ln' or using log rules (Category 1). This was deemed to be evidence of an ability to explain, and considered to be evidence of an action-on-action category of explanation in terms of Tall's theory. In Study 2, of the five categories of explanation to emerge, three of these (Categories 1-3), were considered evidence of an ability to explain; all of these categories referred to the use of natural logs to reach the answer, and were considered to be evidence of an action-on-action category of explanation in terms of Tall's theory. Categories 4-5, while considered not to be evidence of an ability to explain, were classified as action-on-action categories of explanation in terms of Tall's theory because the students in these categories clearly evidenced this type of explanation when they answered Part A correctly.

In Table 3.39, for the students who answered the item incorrectly in both studies, the categories of explanation which these students evidenced were all deemed to be evidence of an inability to explain. In Study 1, the most frequent category of explanation (Category 1) suggested that students realised that 'Ln' had to be used, but were unable to do so. The explanation was categorised as an action-on-action category of explanation in terms of Tall's theory; Categories 3-4 were also classified as such. In Study 2, Categories 1-3 suggested that students in these categories realised that natural logs had to be used but were unable to do so, while Categories 4-6 suggested that students did not realise that natural logs had to be used. All of these categories of explanation (except Category 5) were classified as an action-on-action category of explanation in terms of Tall's theory.

Item 9: Usage of Natural Logarithms
For the Students who Answered Correctly in a Mathematics Context

Study 1				**Study 2**			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. Refer to inserting 'Ln' or using log rules to isolate the power, thus allowing one to derive an expression for x using algebraic methods.	13	ATE	A	1. Refer to how the exponential function can be removed by applying the natural log function.	5	ATE	A
				2. Refer to: $e^{\ln(a)} = a$; $Ln^{ea} = a$ Ln xy = Ln x + Ln y Ln x/y = Ln x – Ln y	2	ATE	A
				3. Refer to applying simple division of y_o to both sides to make the equation as simple as possible; apply the inverse of natural log to isolate -2x; and to get x, on its own, divide by -2 .	1	ATE	A
				4. Refer to not understanding the formula, and just putting x on the left on its own.	1	IATE	A
				5. No reason.	2	IATE	A

Table 3.38 The Correct Students' Categories of Explanation for Item 9 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; ATE—ability to explain; IATE—inability to explain; Tall. — degree of explanation in terms of Tall's theory; A—action-on-action).

Item 9: Usage of Natural Logarithms For the Students who Answered Incorrectly in a Mathematics Context							
Study 1				Study 2			
Category	No.	Exp.	Tall.	Category	No.	Exp.	Tall.
1. Realise that ‘Ln’ must be inserted to isolate the x; however, they make manipulation errors, most notably in relation to using the laws of logarithms.	10	IATE	A	1. Refer to: $x = \frac{Lny_0 - Lny}{2} = \frac{1}{2}\frac{y_0}{y}$	1	IATE	A
2. Provide no workings and no reason.	2	IATE	N/A	2. Refer to using log rules.	1	IATE	A
3. Try to solve the expression algebraically, without the insertion of natural logarithms.	3	IATE	A	3. Refer to: Ln y = Ln y_0. Ln e^{-2x} ; Ln y = Ln y_0 -2x.	1	IATE	A
4. Refer to having learned how to solve the expression, but having never really understood why it worked.	2	IATE	A	4. Refer to: $\frac{1}{2y_0} = x$, and stated that x is rearranged in order to be left on its own.	1	IATE	A
				5. Provide no answer and no reason.	6	IATE	N/A
				6. Refer to the square root letting the power ‘go’.	3	IATE	A

Table 3.39 The Incorrect Students’ Categories of Explanation for Item 9 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; IATE—inability to explain; Tall.—degree of explanation in terms of Tall’s theory; A—action-on-action; N/A—not applicable).

Item 8 and Item 9 can be considered to be related to each other. It was observed that the students in Study 1 who evidenced an ability to explain their reasoning for Item 8 in a mathematics context associated with the transfer of that item; the students who evidenced an ability to explain their reasoning in a mathematics context for the same item in Study 2 did not. For Item 9, students in both Study 1 and Study 2 who evidenced an ability to explain in a mathematics context associated with the transfer of that item. The results for both Items 8 and 9 can be seen in Tables 3.3 and 3.4.

Ignoring the lack of a statistically-significant association between evidencing an ability to explain in a mathematics context and transfer for Item 8 in Study 2, for the students who evidenced an ability to explain Items 8 and 9 in a mathematics context, the categories of explanation which these students evidenced in terms of Tall's theory suggested that evidencing an action-on-action category of explanation appears to be a sufficient explanation in order to transfer.

Despite this, the majority of students in both studies were unable to answer the items correctly in a mathematics context. Weber [84] articulates that not much is known about the way in which students come to a meaningful understanding of the usage of exponential and logarithmic functions. Despite this, Weber articulates that exponential and logarithmic expressions play a critical role in mathematics courses necessary for college. If this is the case, then why did the students in Study 1 and Study 2 appear unable to describe the inverse relationship which exists between an exponential function and its natural logarithmic function in terms of an embodied mathematical object-type understanding? (where the embodied mathematical object is the graph of the two functions showing the mirror image of one in respect of the other—or in other words, the inverse relationship between the two).

Perhaps the reason was because of the fact that this type of explanation was not explicitly asked for in the Part B of Items 8 and 9. Or, perhaps the focus of students' learning in respect of the two functions, resides too much in Tall's 2nd World/symbolic branch of mathematics? The authors De Pierro and Garafala [7] would support this view. They articulate that the majority of students are unable to describe the symbols: $Log_b N = L$ in terms of the question: what power (L) the base (b), has to be raised to in order to produce the number (N)— in other words, the logarithm of L to the base b. Furthermore, students are unable to describe the symbols: $b^L = N$ in terms of the question: what number (L) the number (b) has to be raised to in order to produce N—in other words, the antilog of N to the base b. Re-iterating what was stated in the Introduction, the authors state that the usage of the irrational number e as the base for the natural logs is a mystery to students.

3.5.10 Item 10—Proportionality

Item 10: Proportionality

Mathematics Context	**Chemistry Context**
The derivative of a particular function: $y = f(x)$ with respect to x is denoted: $\frac{dy}{dx}$	The rate law for a particular reaction: $A + B \rightarrow P$, where A and B are reactants and P is product is given as:
It is found that $\frac{dy}{dx}$ is proportional to y^2	$\text{Rate} = k[A]^2$
Thus $\frac{dy}{dx} = ky^2$	when the concentration of B is held constant. 'k' is the rate constant.
where k is the constant of proportionality.	
(A) What happens to the value of the derivative if y is doubled?	**(A)** What happens to the value of the rate if [A] is doubled?
(B) Explain your reasoning.	**(B)** Explain your reasoning.

Figure 3.11 Item 10 Used in the Main Study.

Mathematical Item	**% Correct in a Mathematics Context**		**% Correct in a Chemistry Context**	
	Study 1	**Study 2**	**Study 1**	**Study 2**
10. Proportionality	43%	38%	56%	79%

Table 3.40 Students' Performance for Item 10 in the Main Study.

Item 10 requires students to move from Tall's *2nd World* to *1st World*, embodying the symbols as numbers in order to *operate/act* on them. The students have to interpret *dy/dx,* and *rate,* as a number which is related to *y* and *[A]* respectively—the terms *y* and *[A]* also having to be interpreted as numbers. The students must then realise: 1) the *dy/dx number* and *rate number* are affected by changes in *y* and *[A]* respectively; 2) due to the nature of the functional relationship between *dy/dx* and *y* or the *rate number* and *[A],* the doubling of *y* or *[A]* will respectively result in *dy/dx* or the *rate* quadrupling.

The percentage of students in both studies who answered the item correctly in a mathematics context can be seen in Table 3.40. The categories of explanation for the students who answered the item correctly in Study 1 and Study 2 are shown in Table 3.41. The categories of explanation for the students who answered the item incorrectly in Study 1 and Study 2 are shown in Table 3.42.

Looking at Table 3.41, for the students who answered the item correctly in Study 1, seven categories of explanation emerged, four of which (Categories 1-4) were considered evidence of an ability to explain. These four categories were deemed to be evidence of action-perception categories of explanation in terms of Tall's theory. The most frequent category (Category 2) referred to the fact the y is squared. Categories 5-7 were deemed to be evidence of an inability to explain, and were each classified as action-on-action categories of explanation in terms of Tall's theory. In Study 2, four categories of explanation emerged, the first three of which were considered to be evidence of an ability to explain, and were in turn classified as action-perception categories of explanation in terms of Tall's theory. The most frequent of these categories (Category 1) referred to the quadrupling satisfying the conditions of the equation.

In Table 3.42, for the students who answered incorrectly in Study 1, the seven categories that emerged were each considered as evidence of an inability to explain. All of these categories (except Category 6) were classified as action-on-action categories of explanation in terms of Tall's theory. The most frequent category of explanation (Category 5), which students evidenced, referred to the derivative doubling because it is directly proportional to y. This indicated that the students do not understand the term 'directly proportional'. The second most frequent category of explanation (Category 1), which students evidenced, referred to the derivative as doubling because the other side of the equation doubled.

For the incorrect students in Study 2, the four categories of explanation which emerged were all considered as evidence of an inability to explain, and classified as action-on-action categories of explanation in terms of Tall's theory. The most frequent of these categories (Category 2) referred to the derivative as doubling because it is proportional to y squared. Thus, somewhat like the incorrect students in Category 5 of Study 1, these students indicated that they do not understand proportionality in the context of this question.

Item 10: Proportionality
For the Students who Answered Correctly in a Mathematics Context

Study 1				Study 2			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. Refer to the quadrupling as satisfying the conditions of the equation.	1	ATE	AP	1. Refer to the quadrupling as satisfying the conditions of the equation.	8	ATE	AP
2. Refer either directly or indirectly to the fact that y is squared.	6	ATE	AP	2. Refer to y being squared as the reason for the derivative increasing.	3	ATE	AP
3. State the derivative is proportional to y^2.	1	ATE	AP	3. Produce the following: y^2, $2^2 = 4$; $4\ \alpha \frac{dy}{dx}$; k constant.	1	ATE	AP
4. Refer to thinking of the law of indices.	1	ATE	AP	4. State that the reaction is second order.	1	IATE	A
5. Provide no reason.	2	IATE	A				
6. Refer to 'it' being a second-order reactant.	1	IATE	A				
7. Refer to the derivative of $2y^2$ as 4y.	1	IATE	A				

Table 3.41 The Correct Students' Categories of Explanation for Item 10 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; ATE—ability to explain; IATE—inability to explain; Tall. — degree of explanation in terms of Tall's theory; A—action-on-action; AP—action-perception).

Item 10: Proportionality For the Students who Answered Incorrectly in a Mathematics Context							
Study 1				**Study 2**			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. State that if one side of the equation is doubled, the other side must be doubled.	4	IATE	A	1. State that if the derivative is doubled, the value of k is halved.	1	IATE	A
2. State that the derivative is proportional to y^2 but are unable to translate this statement into the correct answer.	2	IATE	A	2. State that if the derivative is proportional to y^2, it will double, as k remains constant.	8	IATE	A
3. State that it will double and provide no reason.	1	IATE	A	3. State that the derivative depends on k.	1	IATE	A
4. State that it will half and provide no reason.	1	IATE	A	4. State the bigger y gets, the smaller the derivative gets.	1	IATE	A
5. State that it will double because the derivative is directly proportional to y.	6	IATE	A				
6. Provide no workings and no reason.	3	IATE	N/A				

Table 3.42 The Incorrect Students' Categories of Explanation for Item 10 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; IATE—inability to explain; Tall.—degree of explanation in terms of Tall's theory; A—action-on-action; N/A—not applicable).

The categories of explanation for the correct students (shown in Table 3.41) suggest that evidencing an action-perception category of explanation in terms of Tall's theory is necessary in order to transfer. The analysis of Item 1 (Section 3.5.1) revealed that students who evidenced a *Category 1, Action-Perception Category of Explanation* for the meaning of slope in a mathematics context (as shown in Table 3.7) associated with the transfer of Item 10 more so than students who did not evidence this category of explanation. Perhaps this suggests that these students have thought of the slope in terms of two numbers (for example, a slope value of two meaning two units up for every one unit across), and somehow used this object-type understanding of slope to answer Item 10.

3.5.11 Item 11—Graphing an Exponential Function

Item 11: Graphing an Exponential Function

Mathematics Context

Given:

$$y = y_0 e^{-2x}$$

(A) Draw a graph that represents the relationship in Figure 1. Label the axis accordingly.

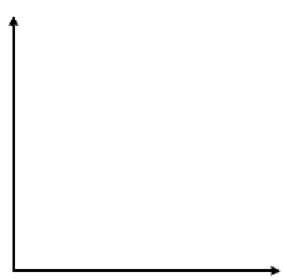


Figure 1

(B) Explain your reasoning

Chemistry Context

For a reaction:

$$A + B \rightarrow P,$$

where A and B are reactants and P is product, the concentration of reactant B after a certain time ($[B]_t$) is given as a function of time in the following expression:

$$[B]_t = [B]_0 e^{-kt}$$

where $[B]_0$ and k are the initial concentration of reactant B and rate constant respectively.

(A) Draw a graph that represents this expression in Figure 1. Label the axis accordingly.

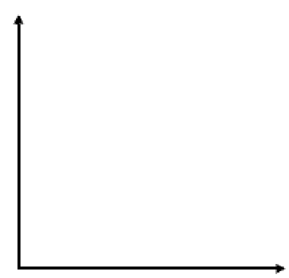


Figure 1

(B) Explain your reasoning.

Figure 3.12 Item 11 Used in the Main Study.

Mathematical Item	% Correct in a Mathematics Context		% Correct in a Chemistry Context	
	Study 1	Study 2	Study 1	Study 2
11. Graphing an Exponential Function	3%	13%	13%	0%

Table 3.43 Students' Performance for Item 11 in the Main Study.

Item 11 requires students to move from Tall's *2nd World* to *1st World.* The percentage of students in both studies who answered the item correctly in a mathematics context can be seen in Table 3.43. The categories of explanation for the students who answered the item correctly in Study 1 and Study 2 are shown in Table 3.44. The categories of explanation for the students who answered the item incorrectly in Study 1 and Study 2 are shown in Tables 3.45 and 3.46.

Looking at Table 3.44, for the one student in Study 1 and the two students in Study 2 who were in the 'no reasoning' category of explanation, they were categorised as having evidenced an inability to explain but as having evidenced an action-perception category of explanation in terms of Tall's theory. Such a category was classified as 'action-perception' because when the students answered Part A correctly, they evidenced the possession of an action-perception category of explanation in terms of Tall's theory irrespective of whether they explained their reasoning in Part B.

Item 11: Graphing an Exponential Function
For the Students who Answered Correctly in a Mathematics Context

Study 1				Study 2			
Category	No.	Exp.	Tall.	Category	No.	Exp.	Tall.
1. No reasoning.	1	IATE	AP	1. Any positive number (excluding zero) introduced into the formula will give a rapidly decreasing y-value.	1	ATE	AP
				2. No reasoning.	2	IATE	AP

Table 3.44 The Correct Students' Categories of Explanation for Item 11 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; ATE—ability to explain; IATE—inability to explain; Tall. — degree of explanation in terms of Tall's theory; AP—action-perception).

Looking at Tables 3.45 and 3.46, it is clear from the categories of explanation that emerged, that these students were unable to graph the exponential function or explain how to do so. The most frequent category in both studies was Category 1 (as seen in Table 3.45); students in this category did not draw a graph or explain how to do so.

Item 11: Graphing an Exponential Function
For the Students who Answered Incorrectly in a Mathematics Context

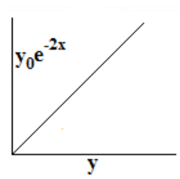
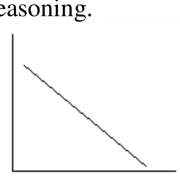
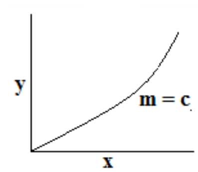
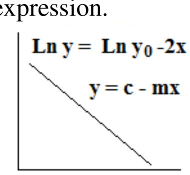
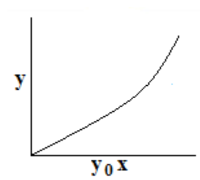
Study 1				**Study 2**			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. Do not draw a graph and do not provide a reason.	12	IATE	N/A	1. Do not draw a graph and do not provide a reason.	8	IATE	N/A
2. Draw the following graph, and provided indecipherable reasoning. 	5	IATE	AP	2. State that if $y = y_0e^{-2x}$ it means they're proportional. 	2	IATE	AP
3. Draw the following graph, and provided indecipherable reasoning. 	1	IATE	AP	3. Draw the following graph and state they have no idea. 	2	IATE	AP
4. Draw the following graph, and provide reasoning in the form a natural logarithmic expression. 	1	IATE	AP	4. Draw the following graph and state that slope is exponential. 	2	IATE	AP

Table 3.45 The Incorrect Students' Categories of Explanation for Item 11 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; IATE—inability to explain; Tall.—degree of explanation in terms of Tall's theory; AP—action-perception; N/A—not applicable).

Item 11: Graphing an Exponential Function
For the Students who Answered Incorrectly in a Mathematics Context

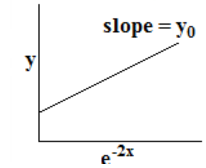
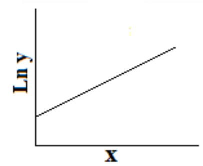
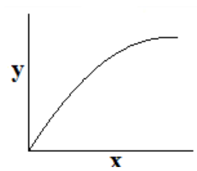
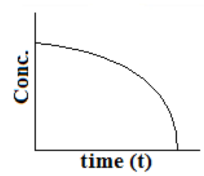
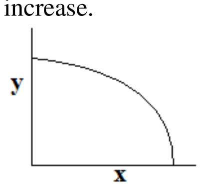
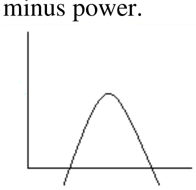
Study 1				Study 2			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
5. Labelled the x-y axes inappropriately, did not draw a graph and provided an indecipherable reason.	2	IATE	N/A	5. State that the formula is a line. 	2	IATE	AP
6. Did not draw a graph and stated that they forget 'how to do' graphs.	3	IATE	N/A	6. Draw the following graph and provide no reasoning. 	2	IATE	AP
7. Stated that the graph they drew was a guess. 	3	IATE	AP	7. Draw the following graph and state that the other graphs were like that. 	1	IATE	AP
8. Stated that y-values drop as x-values increase. 	2	IATE	AP	8. Draw the following graph and say it's to a minus power. 	2	IATE	AP

Table 3.46 The Incorrect Students' Categories of Explanation for Item 11 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; IATE—inability to explain; Tall.—degree of explanation in terms of Tall's theory; AP—action-perception; N/A—not applicable).

The student in Category 4 in Table 3.45 evidenced an ability to represent the relationship, but was deemed to have answered incorrectly because they did not graph an exponential function — the answer that was looked for.

3.5.12 Item 12 – Graphing a Natural Logarithmic Expression

Item 12: Graphing a Natural Logarithmic Expression

Mathematics Context

Given the relationship:

$$\mathrm{Ln}\,y = c - mx$$

(A) Draw a graph that represents the relationship in Figure 1. Label the axis accordingly.

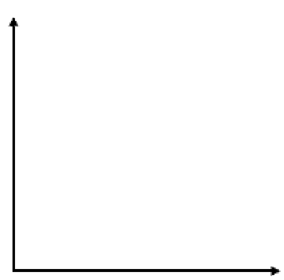


Figure 1

(B) Explain your reasoning

Chemistry Context

A student is studying the chemical reaction:

$$A + B \rightarrow P,$$

where A and B are reactants and P is the product. After graphing the Ln of the concentration of A, obtained at different times: (i.e. the graph of $\mathrm{Ln}[A]_t$ against time (t)), the student finds that the graph corresponds to the relationship:

$$\mathrm{Ln}[A]_t = \mathrm{Ln}[A]_0 - kt$$

showing that the rate of the reaction is 1^{st} order with respect to A.

(A) Sketch the relationship in Figure 1. Label the axis appropriately.

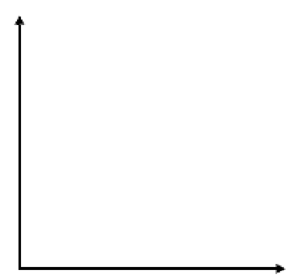


Figure 1

(B) Explain your reasoning.

Figure 3.12 Item 12 Used in the Main Study.

Mathematical Item	% Correct in a Mathematics Context		% Correct in a Chemistry Context	
	Study 1	**Study 2**	**Study 1**	**Study 2**
12. Graphing a Natural Logarithmic Expression	10%	46%	13%	29%

Table 3.47 Students' Performance for Item 12 in the Main Study.

Item 12 requires students to move from Tall's *2nd World* to *1st World.* The students have to be competent in interpreting a number of things: 1) interpret: $Ln\ y\ /\ Ln\ [A]_t$ as similar to y (in terms of a *Generalised Dependent Variable*); 2) interpret $c\ /\ Ln\ [A]_0$ as similar to c (in terms of a *Generalised Constant*); 3) Interpret $-m\ /\ -k$ as similar to m (in terms of a *Generalised Slope*); and 4), interpret $x\ /\ t$ as similar to x (in terms of a *Generalised Independent Variable*).

Once students interpret the above, they then have to interpret the linear relationship that exists between $Ln\ y$ and x, or $Ln\ [A]_t$ and t, where $Ln\ y$ or $Ln\ [A]_t$ is similar to a dependent variable, while x or t is similar to an independent variable. Upon interpreting all of this, the students should be able to graphically represent the item in both contexts.

The percentage of students in both studies who answered the item correctly in a mathematics context can be seen in Table 3.47. The categories of explanation for the students who answered the item correctly in Study 1 and Study 2 are shown in Table 3.48. The categories of explanation for the students who answered the item incorrectly in Study 1 and Study 2 are shown in Tables 3.49 and 3.50.

Looking at Table 3.48, for the students who answered the item correctly in Study 1, all of these students evidenced the same category of explanation (Category 1). It is interesting to note that the students in this category drew the correct graph because they interpreted the value for the slope as negative. The category was classified as evidence of an ability to explain, and classified as an action-perception category of explanation in terms of Tall's theory. For the students in Study 2, three categories of explanation emerged. Two of these categories (Categories 1-2) were deemed to be evidence of an ability to explain, and classified as action-perception categories of explanation in terms of Tall's theory. Interestingly, in these categories, like in Category 1 of Study 1, the students referred to the slope of the line as negative.

Item 12: Graphing a Natural Logarithmic Expression
For the Students who Answered Correctly in a Mathematics Context

Study 1				**Study 2**			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. State that the slope of the expression is negative.	3	ATE	AP	1. State the slope is a minus.	1	ATE	AP
				2. State that Ln y is an integer; c is a point at which the line crosses the x-axis; and -m is the coefficient of x, so the slope is negative.	4	ATE	AP
				3. Provide no reason.	1	IATE	AP

Table 3.48 The Correct Students' Categories of Explanation for Item 12 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; ATE—ability to explain; IATE—inability to explain; Tall. — degree of explanation in terms of Tall's theory; AP—action-perception).

In Tables 3.49 and 3.50, the categories of explanation evidenced by the incorrect students in Study 1 and Study 2 were all considered as an inability to explain. All of these categories, (apart from Category 3 in both studies) were classified as evidence of an action-perception category of explanation in terms of Tall's theory, albeit the categories were incorrect action-perception categories of explanation. Interestingly, in both studies, certain students drew the correct shape of the graph, but labelled the axes incorrectly. This can be seen in Category 1 for Study 1 and in Categories 4-8 and Category 12 for Study 2. Other students in both studies drew a linear graph with positive slope, as can be seen in Category 2 in Study 1 and in Categories 1 and 11 for Study 2. Lastly, in both studies, certain students drew a decreasing exponential function, as can be seen in Category 4 for Study 1, and in Categories 9 and 10 for Study 2.

Item 12: Graphing a Natural Logarithmic Expression
For the Students who Answered Incorrectly in a Mathematics Context

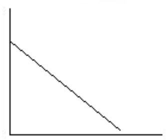
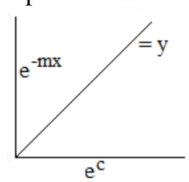
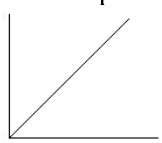
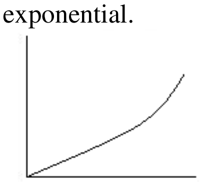
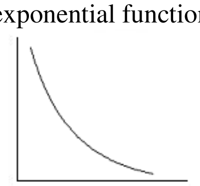
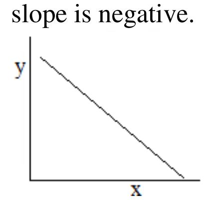
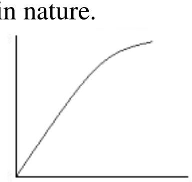
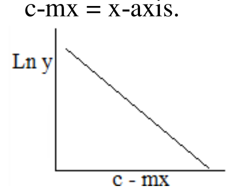
Study 1				**Study 2**			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. Graph the expression; however, the axes are not labelled appropriately. 	4	ATE	AP	1. Draw a linear-type graph with a positive slope. 	1	IATE	AP
2. Equate the expression with a linear-type expression that has a positive slope. 	6	IATE	AP	2. State that the equation in terms of y means the graph is exponential. 	1	IATE	AP
3. Provide no graph and no reason.	13	IATE	N/A	3. Provide no graph and no reason.	3	IATE	N/A
4. Graph a decreasing exponential function. 	3	IATE	AP	4. State that they only understand that the slope is negative. 	1	IATE	AP
5. Describe the expression as curved in nature. 	1	IATE	AP	5. State that: Lny = y-axis; and, c-mx = x-axis. 	2	IATE	AP

Table 3.49 The Incorrect Students' Categories of Explanation for Item 12 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; ATE—ability to explain; IATE—inability to explain; Tall.—degree of explanation in terms of Tall's theory; AP—action-perception; N/A—not applicable).

Item 12: Graphing a Natural Logarithmic Expression
For the Students who Answered Incorrectly in a Mathematics Context

Study 2

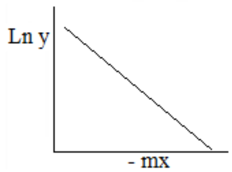
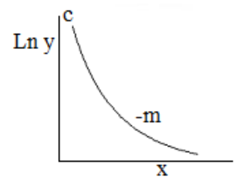
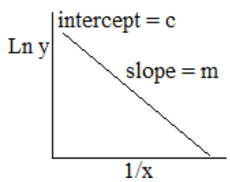
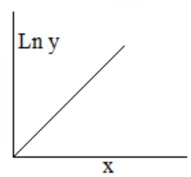
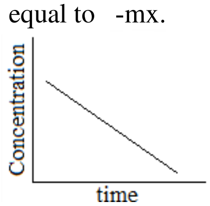
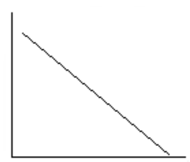
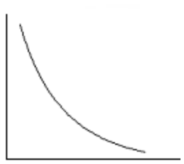
Category	No.	Exp.	Tall.	Category	No.	Exp.	Tall.
6. State that y = c - mx is a straight line and Ln y would be on the y-axis while -mx would be on the x-axis. 	2	IATE	AP	10. Refer to the expression as similar to the expression for the equation of a line. 	3	IATE	AP
7. State: the inverse of x for Ln. 	1	IATE	AP	11. State that 'y = mx + c' implies a straight line. 	1	IATE	AP
8. Refer to the negative slope being equal to -mx. 	1	IATE	AP	12. State that if the graph is exponential it becomes a straight line. 	1	IATE	AP
9. Draw a decreasing exponential function and provide no reason. 	2	IATE	AP				

Table 3.50 The Incorrect Students' Categories of Explanation for Item 12 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; ATE—ability to explain; IATE—inability to explain; Tall.—degree of explanation in terms of Tall's theory; AP—action-perception; N/A—not applicable).

4.5.13 Item 13 — Graphing a Function

Item 13: Graphing a Function

Mathematics Context

(A) Sketch in Figure 1, the graph of the function:

$$y = \frac{1}{x} \text{ for } x > 0$$

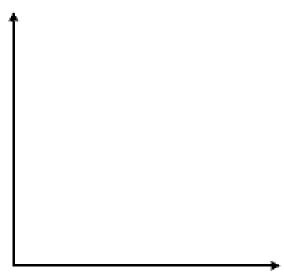


Figure 1

(B) Explain your reasoning

Chemistry Context

(A) Sketch in Figure 1, the graph of P versus V, for:

$$0\text{m}^3 < V < 5\text{m}^3$$

given the relationship:

$$P = \frac{1}{V}$$

This relationship comes from the ideal gas law applied to an isothermal system. For this example, *nRT* has the constant value of 1kJ.

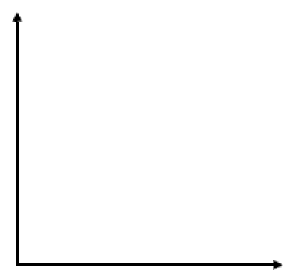


Figure 1

(B) Explain your reasoning.

Figure 3.14 Item 13 Used in the Main Study.

Mathematical Item	% Correct in a Mathematics Context		% Correct in a Chemistry Context	
	Study 1	Study 2	Study 1	Study 2
13. Graphing a Function.	33%	37%	17%	29%

Table 3.51 Students' Performance for Item 13 in the Main Study.

Like Items 11-12, Item 13 requires students to move from Tall's 2^{nd} *World* to 1^{st} *World*. The students must interpret the functional relationship between y and x, and transform it into its graphical representation (embodied mathematical object). The percentage of students in both studies who answered the item correctly in a mathematics context can be seen in Table 3.51. The categories of explanation for the students who answered the item correctly in Study 1 and Study 2 are shown in Table 3.52. The categories of explanation for the students who answered the item incorrectly in Study 1 and Study 2 are shown in Tables 3.53 and 3.54.

Looking at Table 3.52, the students in Category 2 for Study 1, and the students in Categories 3-4 for Study 2, evidenced an inability to explain. However, they were deemed to have evidenced an action-perception category of explanation in terms of Tall's theory because they answered Part A correctly.

Item 13: Graphing a Function
For the Students who Answered Correctly in a Mathematics Context

Study 1				**Study 2**			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. State that as x increases, y decreases and/or insert x-values into the function in order to calculate the corresponding y-values.	9	ATE	AP	1. State that the function passes through the point (1, 1) and that the x and y axes are asymptotes.	2	ATE	AP
2. Draw the correct graph but provide no reason.	1	IATE	AP	2. Refer to x increasing while y decreases. Also refer to x and y as inverses; as one increases, the other decreases.	3	ATE	AP
				3. State that y is the opposite of x.	1	IATE	AP
				4. Provide No reason.	2	IATE	AP

Table 3.52 The Correct Students' Categories of Explanation for Item 13 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; ATE—ability to explain; IATE—inability to explain; Tall. — degree of explanation in terms of Tall's theory; AP—action-perception).

Item 13: Graphing a Function
For the Students who Answered Incorrectly in a Mathematics Context
Study 1

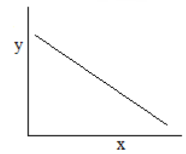
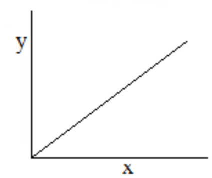
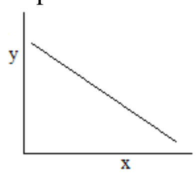
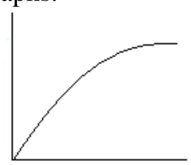
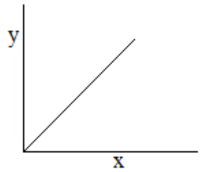
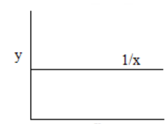
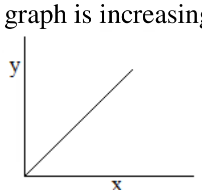
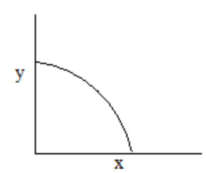
Category	No.	Exp.	Tall.	Category	No.	Exp.	Tall.
1. State that y and x are inversely proportional and/or as x increases, y decreases. 	5	IATE	AP	6. State that as x increases, y decreases. 	3	IATE	AP
2. State that the graph has a negative slope. 	2	IATE	AP	7. State that they do not really understand graphs. 	2	IATE	AP
3. Provide no reason. 	3	IATE	AP	8. State that x is greater than zero, unknown and somewhere along the y-axis. 	1	IATE	AP
4. State that the slope of the function given is positive and therefore the graph is increasing. 	2	IATE	AP	9. State that the relationship is logarithmically proportional. 	1	IATE	AP
5. No graph or reason.	1	IATE	N/A				

Table 3.53 The Incorrect Students' Categories of Explanation for Item 13 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; IATE—inability to explain; Tall.—degree of explanation in terms of Tall's theory; AP—action-perception; N/A—not applicable).

Item 13: Graphing a Function
For the Students who Answered Incorrectly in a Mathematics Context

Study 2

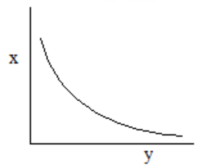
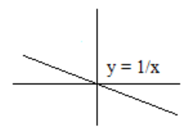
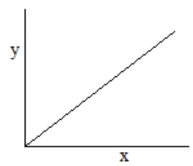
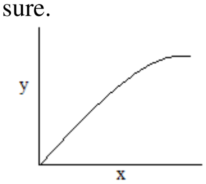
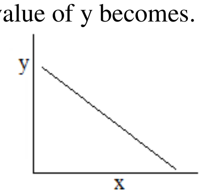
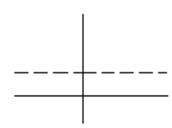
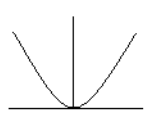
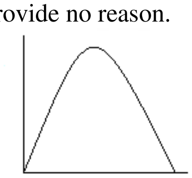
Category	No.	Exp.	Tall.	Category	No.	Exp.	Tall.
1. State that 1/x will never reach zero. 	2	IATE	AP	6. Provide no reason. 	2	IATE	AP
2. Provide no reason. 	2	IATE	AP	7. State that they are not sure. 	1	IATE	AP
3. State that the greater x becomes, the lower the value of y becomes. 	2	IATE	AP	8. Provide no reason. 	2	IATE	AP
4. Provide no reason. 	1	IATE	AP	9. Provide no reason. 	1	IATE	AP
5. Provide no graph or reason.	3	IATE	N/A				

Table 3.54 The Incorrect Students' Categories of Explanation for Item 13 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; IATE—inability to explain; Tall.—degree of explanation in terms of Tall's theory; AP—action-perception; N/A—not applicable).

Tables 3.53 and 3.54 show the categories of explanation provided by the students who answered the item incorrectly. All of these categories were deemed to be evidence of an inability to explain, and all of the categories (apart from Category 5 in both Study 1 and Study 2) were categorised as evidence of an action-perception category of explanation in terms of Tall's theory, even though, these were incorrect action-perception categories of explanation. In both studies:

- Certain students drew a linear graph with a negative slope. This can be seen in Categories 1-2 of Table 3.53 for Study 1 and in Category 3 of Table 3.54 for Study 2;
- Certain students drew a linear graph with positive slope. This can be seen in Categories 3-4 and in Category 6 of Table 3.53 for Study 1 and in Category 2 of Table 3.54 for Study 2; and
- Certain students provided no graph or reason, as can be seen in Category 5 of Table 3.53 for Study 1 and in Category 5 of Table 3.54 for Study 2.

Items 11-13 all require an ability to move from Tall's *2nd World* to *1st World*. The majority of students in both studies were unable to answer these items correctly in a mathematics context which raises the question: why?

Perhaps the teaching focus for Items 11-13 is too algebraic in nature. Research by Leinhardt et. al [85]supports this view. They articulate that the formal definition of a function (or algebraic expression) is algebraic in spirit, with its graphical depiction taking a minor role. Potgieter et. al [37] also found that the majority of students' work with functions is restricted to the algebraic domain. If this is so, then it might be expected that students would perform better on items that are algebraic in nature. Clearly, this was found not always to be the case for the algebraic (2nd World) items in this study; for example, students' performance for Items 8 and 9 in comparison to students performance for Items 2 and 3.

3.5.14 Item 14—Evaluation of an Integral

Item 14: Evaluation of an Integral

Mathematics Context	Chemistry Context
(A) Evaluate the integral: $\int_1^3 \frac{1}{x}dx$	**(A)** Calculate the work done when the volume of a gas in a reversible isothermal gas expansion increases from:1m^3(V_1) to 3m^3(V_2), given that the work will be equal to the expression: $$w = -\int_{V_1}^{V_2} \frac{1}{V}dv$$ where V_1 (1m^3) is the initial volume of the gas and V_2 (3m^3) is the final volume of the gas. The minus sign is used to denote the fact that the work leaves the system.
(B) Explain your reasoning	**(B)** Explain your reasoning.

Figure 3.15 Item 14 Used in the Main Study.

Mathematical Item	% Correct in a Mathematics Context		% Correct in a Chemistry Context	
	Study 1	**Study 2**	**Study 1**	**Study 2**
14. Evaluation of an Integral.	16%	29%	10%	25%

Table 3.55 Students' Performance for Item 14 in the Main Study.

Item 14 requires students to work within Tall's 2^{nd} *World* because the students must be able to ac*t on symbols*; they must realise that the integral of *1/x* is *Ln x* and that the integral of *1/V* is *Ln V*. The percentage of students in both studies who answered the item correctly in a mathematics context can be seen in Table 3.55. The categories of explanation for the students who answered the item correctly in Study 1 and Study 2 are shown in Table 3.56. The categories of explanation for the students who answered the item incorrectly in Study 1 and Study 2 are shown in Table 3.57

Looking at Table 3.56, the students who were deemed to have evidenced an ability to explain in both studies evidenced the same category of explanation (Category 1). This category was deemed to be evidence of an action-on-action category of explanation in terms of Tall's theory. The two students who did not provide a reason in Study 2 (those in Category 2) were still considered to have evidenced an action-on-action category of explanation in terms of Tall's theory because they answered Part A correctly.

Item 14: Evaluation of an Integral
For the Students who Answered Correctly in a Mathematics Context

Study 1				**Study 2**			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. State that the integral of '1/x' is 'Ln x'.	4	ATE	A	1. State that the integral of '1/x' is 'Ln x'.	5	ATE	A
				2. Provide no reason.	2	IATE	A

Table 3.56 The Correct Students' Categories of Explanation for Item 14 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; ATE—ability to explain; IATE—inability to explain; Tall. — degree of explanation in terms of Tall's theory; A—action-on-action).

For the students who answered the item incorrectly (shown in Table 3.57) the categories of explanation provided by these students were considered to be evidence of an inability to explain. In both studies:

- Certain students substituted the values for the limits into the integrand and subtracted the lower-limit integrand value from the upper-limit integrand value. This can be seen in Category 1 for both Study 1 and Study 2;
- Certain students evidenced that they realised they had to integrate but were unable to do so. This can be seen in Category 2 for Study 1 and in Category 5 for Study 2; and

- Certain students provided no working or explanation, as can be seen in Category 4 of Table 3.57 for both Study 1 and Study 2.

Item 14: Evaluation of an Integral
For the Students who Answered Incorrectly in a Mathematics Context

Study 1				Study 2			
Category	**No.**	**Exp.**	**Tall.**	**Category**	**No.**	**Exp.**	**Tall.**
1. Substitute the values for the limits into the integrand and subtract the lower-limit integrand value from the upper-limit integrand value.	4	IATE	A	1. Substitute the values for the limits into the integrand and subtract the lower-limit integrand value from the upper-limit integrand value.	4	IATE	A
2. Use the power rule for differentiation in various, incorrect forms to integrate the integrand.	8	IATE	A	2. State that the integral is Log x + c. When the values one and three are added in, this is the result obtained.	2	IATE	A
3. Try to differentiate the integrand.	2	IATE	A	3. Refer to finding the curve of the graph.	1	IATE	AP
4. No workings and no reason.	9	IATE	N/A	4. Provide no answer and no reason.	6	IATE	N/A
5. Subtract 1 from 3—perhaps indicating that they perceive the anti-derivative for the integrand in question to be x.	2	IATE	A	5. Integrate incorrectly; sub-in the limits, and take the lower away from the upper to give a value of 0.5.	3	IATE	A
6. Realise the integral is Ln x but do not substitute for the limits; instead, they state that they forget how to do integration.	1	IATE	A	6. Refer to the Integral changing the equation given.	1	IATE	A

Table 3.57 The Incorrect Students' Categories of Explanation for Item 14 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; IATE—inability to explain; Tall.—degree of explanation in terms of Tall's theory; A—action-on-action; AP—action-perception; N/A—not applicable).

3.5.15 Item 15 – Graphing an Integral

Item 15: Graphing an Integral

Mathematics Context

(A) Indicate in Figure 1, the area corresponding to the integral:

$$\int_{1}^{3} \frac{1}{x} dx$$

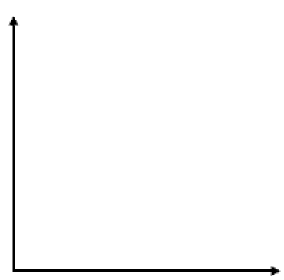


Figure 1

(B) Explain your reasoning

Chemistry Context

(A) The relationship:

$$P = \frac{1}{V},$$

where P is the pressure of a gas, and V is its volume, represents the ideal gas law applied to an isothermal system. For this example, *nRT* has the constant value of 1kJ. Indicate in Figure 1, the area corresponding to the integral:

$$w = -\int_{V_1}^{V_2} \frac{1}{V} dV$$

which represents the work done by the system (the gas) in expanding from an initial volume:

($V_1 = 1m^3$) to a final volume ($V_2 = 3m^3$),

for a reversible isothermal gas expansion. The minus sign is used to denote the fact that the work leaves the system.

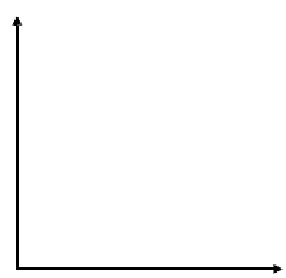


Figure 1

(B) Explain your reasoning.

Figure 3.16 Item 15 Used in the Main Study.

Mathematical Item	% Correct in a Mathematics Context		% Correct in a Chemistry Context	
	Study 1	Study 2	Study 1	Study 2
15. Graphing an Integral.	13%	29%	10%	25%

Table 3.58 Students' Performance for Item 15 in the Main Study.

Item 15 requires students to move from Tall's *2nd World* to *1st World*. The students must interpret the symbolic expression in terms of how it relates to finding an area between the function and the x-axis, bounded by the limits in question. The percentage of students who answered the item correctly in a mathematics context can be seen in Table 3.58. The categories of explanation for the students who answered the item correctly in Study 1 and Study 2 are shown in Table 3.59. The categories of explanation for the students who answered the item incorrectly in Study 1 and Study 2 are shown in Table 3.60 and 3.61

Item 15: Graphing an Integral
For the Students who Answered Correctly in a Mathematics Context

Study 1				Study 2			
Category	No.	Exp.	Tall.	Category	No.	Exp.	Tall.
1. Draw the correct diagram, and are able to state the integral is representative of the area within the limits in question.	4	ATE	AP	1. Refer to the integral meaning the area under the curve between the limits 1 and 3.	7	ATE	AP
				2. Refer to the slope of the graph as already being determined; and that all one needs to do is put in the limits, which only apply to the x-axis.	1	IATE	AP

Table 3.59 The Correct Students' Categories of Explanation for Item 15 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; ATE—ability to explain; IATE—inability to explain; Tall. — degree of explanation in terms of Tall's theory; AP—action-perception).

Looking at Table 3.59, the category of explanation provided by the students in Study 1 who answered the item correctly in a mathematics context was considered to be evidence of an ability to explain. A similar category (Category 1) was provided by the correct students in Study 2. The student in Category 2 of Study 2 was deemed not to have evidenced an inability to explain. All of the categories of explanation were classified as action-perception categories of explanation in terms of Tall's theory because this was the type of explanation students evidenced when they answered Part A correctly.

Item 15: Graphing an Integral
For the Students who Answered Incorrectly in a Mathematics Context

Study 1

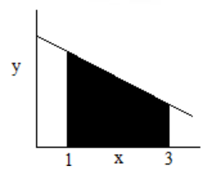
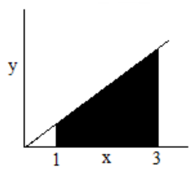
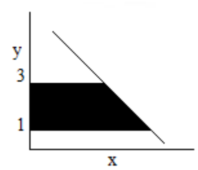
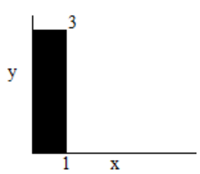
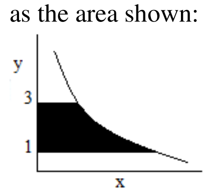
Category	No.	Exp.	Tall.	Category	No.	Exp.	Tall.
1. Refer to the integral as the area shown: 	2	IATE	AP	5. Refer to the integral as the area shown: 	2	IATE	AP
2. Refer to the integral as the area shown: 	1	IATE	AP	6. Refer to the integral as the area shown: 	1	IATE	AP
3. Appear to think the integral is the area of a rectangle with dimensions similar to the limits of the integral in question. 	2	IATE	AP	7. Refer to the integral as the area shown: 	1	IATE	AP
4. Provide no graph and no reason.	17	IATE	N/A				

Table 3.60 The Incorrect Students' Categories of Explanation for Item 15 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; IATE—inability to explain; Tall.—degree of explanation in terms of Tall's theory; AP—action-perception; N/A—not applicable).

Item 15: Graphing an Integral
For the Students who Answered Incorrectly in a Mathematics Context

Study 2

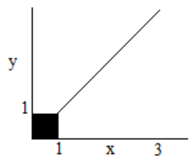
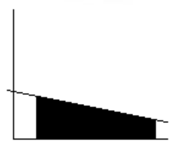
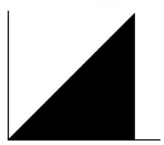
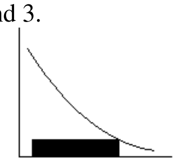
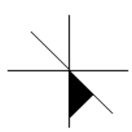
Category	No.	Exp.	Tall.	Category	No.	Exp.	Tall.
1. State that the answer is equal to 0.5. 	1	IATE	AP	6. State that the integral is equal to the area under the curve. 	1	IATE	AP
2. Refer to the integral giving the area under the curve y = 1/x. 	1	IATE	AP	7. State that the integral is from 1-3. 	2	IATE	AP
3. State that the integral is the area between 3 and 1. 	1	IATE	AP	8. Refer to the integral as the area under the curve, just between 1 and 3. 	1	IATE	AP
4. State that the area corresponding to the integral is equal to the area below the curve. 	1	IATE	AP	9. Refer to Integration as finding the area under the curve. 	1	IATE	AP
5. Provide no graph and no reason.	7	IATE	N/A				

Table 3.61 The Incorrect Students' Categories of Explanation for Item 15 (Abbreviations: No.—number of students in each category; Exp.—form of explanation; IATE—inability to explain; Tall.—degree of explanation in terms of Tall's theory; AP—action-perception; N/A—not applicable).

Looking at the categories of explanation which the incorrect students provided, (as shown in Tables 3.60 and 3.61), it is clear that most of the students realised that an area had to be highlighted for Item 15. However, in Categories 6-7 for Study 1, students highlighted an area between a graph and the y-axis. Also, most of the categories evidenced that students were unable to depict the graph relevant to the depiction of the integral for the function 1/x. This was not a surprise given that the majority of the students were unable to answer Item 13 which required the depiction of the function 1/x.

In respect of Items 14 and 15 (items concerned with integration) previous research [53, 55,58] has found that most students can perform integration routinely, and yet not realise what they are doing. Looking at Table 3.56, it can be seen that the majority of the students who correctly evaluated the integral in Item 14 gave an explanation for *what* they did but not *why* they did it. Students' performance in evaluating the integral appears to be at odds with the view that students can perform the technique of integration routinely, irrespective of whether they can explain why they use integration [86]. Perhaps the reason for this was simply due to students being unable to remember the integral of $\frac{1}{x}$. The findings of De Pierro and Garafala would uphold this view: they state that students rarely understand why the integral of $\int_1^x \frac{1}{t}\,dt = \text{Ln } x$.

Students' performance for Item 15 suggests that the majority of students are unable to graphically represent a definite integral. Previous research has also found this to be true [53, 86,87]. However, looking at students' categories of response for the Item (Tables 3.60 and 3.61), it can be seen that many students realised that the integral is concerned with evaluating an 'area'; however, the students were unable to draw the correct graph. Perhaps this is again due to poor awareness of functions in terms of graphs, as is indicated by students' results for Items 11-13. Even if this is the case, Bressoud [8] claims that it has not been traditional to test students' graphical meaning of integrals so this could be an equally valid reason as to why the majority of the students could not answer Item 15 correctly.

3.6 Chapter Summary

In terms of the *Transfer Question*, transfer was observed for the majority of items in both studies. For certain items, more so in Study 1 than in Study 2, the instances of transfer observed was found to be significant. For Study 2, the reminder of the mathematics that students need to use in a chemistry context did not improve the instances of transfer observed when compared to Study 1. Neither did the reminder improve the instances of statistically significant transfer.

In terms of the 1st aspect of the *Explaining and Transfer Question*, for many of the items, in both studies, those students who evidenced an ability to explain in a mathematics context, associated with the transfer of the item more so than the students who did not.

In terms of the 2nd aspect of the *Explaining and Transfer Question,* the main findings that emerged were:

1) The students who evidenced an *action-perception category of explanation* for the meaning of slope (Item 1) in a mathematics context were likely to answer Item 7 (Interpreting Derivative) in a mathematics context, more so than students who did not evidence the same category of explanation. These same students were likely to be able to transfer Item 7 to a chemistry context, more so than students who did not evidence the same action-perception category of explanation.

2) The students who evidenced a category 1 action-perception category of explanation for the meaning of slope (Item 1) in a mathematics context, tended to associate with the transfer of Items 5 and 10, in both studies. These same students appear to be able to answer Items 7 and 10 in a mathematics context more so than other students.

3) For Item 6, the most frequent perception-action category of explanation, in both studies, was: *find the slope of the tangent at the point*. It could be argued that perhaps this is an embodied mathematical object-type image of derivative, which allows the students to transfer.

4) Evidencing an action-on-action category of explanation for Items 8 and 9 appears to be a sufficient explanation in order to transfer these items.

5) Many students were unable to answer Items 11-13 in a mathematics context correctly. One of the reasons proposed for this, was: the focus of functions/algebraic expressions is often divorced from the graphical nature of such functions/algebraic expressions. However, it was found that for many of the algebraic items, students' performance for these items was not much different from their performance in 'graphicacy-type' items.

6) Many students were unable to answer Items 14-15 (Integration Items) correctly in a mathematics context, irrespective of whether the item was algebraic or graphic in nature.

Overall, for the majority of the Items, certain students appear to be able to transfer them, suggesting that the problems students have with mathematics in a chemistry context, may not be a transfer one, but instead be because of insufficient mathematical knowledge in a mathematics context.

In the next chapter, how these results informed the design of an *Intervention* aimed at: 1) improving students' understanding of *Slope*, *Derivative* and *Integral*, in a mathematics context; and 2) improving students ability to transfer *Slope*, *Derivative* and *Integral*, is discussed. Also, the evaluation of the impact of the Intervention programme is discussed.

Chapter 4

The Development of an Intervention Designed to Improve Students' Mathematical Understanding, and Ability to Transfer.

4.1 Chapter Overview

The results from Study 1 and Study 2, discussed in Chapter 3, suggested that the problem students have with mathematics in a chemistry context is not necessarily a transfer problem, but rather due to insufficient mathematical knowledge. Consequently, mathematical interventions aimed at improving students understanding of slope, derivative and integration were designed. The interventions were trialled over two years. Trial 1 occurred during the academic year 09/10 and Trial 2 occurred during the academic year 10/11.

This chapter discusses Trial 1 in terms of: 1) the sample of students involved; 2) the design of the four mathematical interventions which were implemented with students (one intervention on slope, two interventions on derivative and one intervention in respect of integration); 3) the evaluation of the trial; and 4) the conclusions which arose from the trial.

Trial 2 is also discussed in terms of: 1) the sample of students involved; 2) the evaluation of the trial; and 3) the conclusions which arose from the trial. Lastly, the conclusions which arose from both Trial 1 and Trial 2 are discussed in a chapter summary.

4.2 The Intervention — Trial 1

In the Main Study (discussed in Chapter 3) it was observed that if students were able to explain their reasoning for an item in a mathematics context, they tended to associate with transferring it. Thus, it was decided to design mathematical interventions which aimed to improve students' understanding in a mathematics context. It was envisaged that so doing would: 1) improve students' ability to answer a particular item correctly; 2) improve students' ability to explain their reasoning; and 3) improve students' ability to transfer.

4.2.1 The Sample

First-year science students were asked to participate voluntarily in four workshops (which contained the mathematical interventions) aimed at improving students' understanding of certain mathematical concepts. All of the students were undertaking one of the following science programmes: Chemical and Pharmaceutical Science, Analytical Science, Environmental Science and Health, Common Entry into Science, Biotechnology or Genetics and Cell Biology. Eighteen students agreed to participate. Each of the workshops contained an intervention in the form of a series of 'guided worksheets' for the students to complete by themselves. Each workshop was of an hour duration, and the students received a nominal payment for participation. My role in the workshops consisted of dealing with questions in respect of difficulties which the students may have had in the interpretation of the worksheets.

4.2.2 Methodology

The mathematical interventions were designed in such a way that required students to explain their reasoning in a mathematics context. But what constitutes evidencing an ability to explain? Tall's theory [62] was used in this regard. The mathematical intervention focused on improving students' understanding of Items 1-7 and Items 14-15 (shown in Appendix D); the reasons for the focus on these items are discussed in Section 3.5 in Chapter 3. The design of the interventions was focused on developing the cognitive process associated with Tall's theory. This is highlighted in Figure 1.5 in Chapter 1.

Insofar as was possible, all of the interventions aimed to embed mathematical concepts such as slope, derivative and integration in an *environment/context* that was *visual/graphical* in nature. This embedding aimed to embody the mathematical concept in the form of an embodied mathematical object, which in turn could be perceived by the students and then acted upon in the form of symbols – in effect, moving from Tall's 1st World to 2nd World. It was thought that utilising such an approach would improve students' ability to explain in the form of Action-Perception explanations or Perception-Action explanations. While this approach constituted the essence of the design of the mathematical interventions, other views on how to teach calculus, as deduced from mathematics-educational literature in general, were analysed for relevance.

A view shared by some researchers [26, 27] is that the teaching of calculus concepts should be one that is driven by the *re-invention / historical evolution* of the concepts. Such a re-invention does not necessarily have to mirror the exact historical evolution of the concept [27]. Instead, when deciding on how to introduce a concept, it is more important to be aware that while such an introduction may be experientially real from the researchers' perspective, it may in fact, be not so from the students' perspective [27]. For example, when introducing the definition of electric current in terms of it meaning the quantity of charge (Q) passing a point in a circuit per second, a teacher (or researcher) might consider using the flow of water in the form of a diagram as an experientially real analogy. However, the students looking at such a diagram might not experience it as similar to the definition of current. They might just see water flowing, as opposed to what the teacher wants them to see: namely, a number of units of water (akin to units of charge (Q)) flowing past a point within a one second interval (akin to the quantity of charge (Q) flowing past a point in a circuit per second). Also, a certain number of assumptions have to be made as to what the students know — or should know — when designing any mathematical intervention. Assumptions in this regard were made for the design of all the interventions.

Embedding the mathematical concepts in a *context/environment* that allowed students to develop a qualitative idea of the concept before it was introduced more formally (in the form of symbols) echoes with the views of Gravemeijer and Doorman [27]. They highlight the importance of encouraging students to develop qualitative notions about mathematical concepts in order to promote understanding of them.

By using an *environment/context* that was not strictly mathematical in nature to introduce each of the concepts, it was envisaged that this would allow the concept to be understood better in an abstract mathematical context, thus making it more transferable. Lobato and Siebert [34] argue that being able to do this is important, but at the same time challenging and oftentimes difficult.

In all the interventions, students were asked 'guiding questions' so as to allow them to construct the knowledge of the mathematical concept for themselves. Such 'guiding-without-telling' questions [88] were balanced by summary sections of each intervention, summary sections which summarised what the students had learned—or should have learned. This balancing act, which Wagner et al. [88] describe as not typically a part of an instructor's repertoire, was challenging to strike.

Lastly, the author Orton [89] underscores the importance of a lengthy focus on graphs and rates of change, before attempting to introduce calculus. This focus was accomplished with the design of the *Slope Intervention*. Its design was informed not just by the views of Orton, but also by the results from the Main Study and previous literature. The four mathematical interventions are each discussed separately. The intervention on slope is discussed in Section 4.2.2.1, the two interventions on the meaning of derivative are discussed in Sections 4.2.2.2 and 4.2.2.3, while the intervention on integration is discussed in Section 4.2.2.4.

4.2.2.1 The Slope Intervention

The slope-type items used in the Main Study were Items 1-7 and Item 10 (these items can be seen in Appendix D). During Study 1 of the Main Study, it was observed that for the items related to slope, those students who evidenced an ability to explain Items 6-7 and Item 10 in a mathematics context associated with the transfer of these items more so than students who did not. Items 2-5 did not require students to explain their answer. During Study 2 of the Main Study, those who evidenced an ability to explain Items 1-3, 5-7 and 10, associated with the transfer of them. The overall conclusion from the Main Study was that if a student explains their reasoning for these items in a mathematics context, they will tend to transfer them. Tall's theory was used in an attempt to understand the degree to which students explained the slope-type items.

In respect of students' categories of explanation for Item 1 (shown in Table 4.1), it was found (as discussed in Section 3.5.1 of Chapter 3) that for the students who evidenced an action-perception category of explanation in terms of Tall's theory for the meaning of slope, they were likely to answer Item 7 (Interpreting Derivative) in a mathematics context more so than other students. They were also more likely to transfer Item 7.

Table 4.1 shows the correct action-perception categories of explanation which emerged for Item 1. Looking at the *Category 1, Action-Perception Category of Explanation* for the meaning of slope, the students who evidenced such a category of explanation in the Main Study associated with the transfer of Items 5 and 10 more so than other students.

These students also associated with the answering of Items 7 and 10 in a mathematics context and (if statistical significance is ignored for Study 2) all of the Study 2 students in this category answered Item 7 and 10 correctly in a mathematics context. Thus, these findings encouraged an intervention on improving students' understanding of slope in an action-perception way, with a particular emphasis on the *Category 1, Action-Perception Category of Explanation*. It was felt that such an emphasis may improve students' understanding of Items 5, 7 and 10. During the Main Study, these items were answered not so well in a mathematics context.

Item 1: Calculating Slope
For the Students who Answered Correctly in a Mathematics Context

Study 1				**Study 2**			
Category	**No.**	**Und.**	**Tall.**	**Category**	**No.**	**Und.**	**Tall.**
1. Refer to how much y increases for a unit increase in x.	10	ATE	AP	1. Refer to how much y increases for a unit increase in x.	3	ATE	AP
2. Refer to slope as a measure of steepness.	1	ATE	AP	2. The slope represents the rate of increase of the line.	5	ATE	AP
3. Refer to the rate at which the line increases.	3	ATE	AP	3. The bigger the number, the steeper the slope is.	2	ATE	AP
				4. The change of x relative to y, how steep a line is.	1	ATE	AP
				5. It is the difference between the heights over the difference between the two length points.	2	ATE	AP

Table 4.1 The Correct Action-Perception Categories of Explanation for Item 1.

The interesting findings in relation to the students who evidenced a *Category 1, Action-Perception Category of Explanation* for the meaning of slope were the main reasons for a mathematical intervention on slope. Other reasons included the views of Rasmussen and King [26] who, as stated in the literature review, found that if students have to conceptualise a situation in a way that involves a rate, it is non-trivial for the students. Thompson and Silvermann [90] articulate that students' success in terms of integration can only begin in secondary school if rate of change is taught substantively, while Lobato and Siebert [34] argue that the mathematical concept of slope has not been effectively taught to students in a manner which allows them to generalise it. The mathematical intervention in respect of slope is shown in Appendix E. The intervention consisted of three sections.

Section 1

The first section aimed to encourage students to think about the meaning of slope in the context of a measure of steepness. Questions 1-9 were designed to foster this and thus echo with the sentiments of Stump [78, p. 87] who articulates that an essential question to ask students as they observe the graphs of linear functions is “what does the slope represent in the context of the situation?” Even if the students were unable to fully answer Section 1, they were instructed to complete Section 2, and then return to Section 1.

Section 2

Section 2 emphasised the definition of slope and the meaning of steepness. In order to do this, a real-world context was chosen, namely a function representing the volume of water in a tank at different times. It was stressed that the slope gives a measure of how much a line increases or decreases in the vertical direction (Δy) for an increase of Δx in the horizontal direction. This focus would be supported by the work of Lobato [72, p. 297], who states that in respect of slope: “papers indicate that revised curricula materials tend to focus attention on the co-varying quantities rather than on the location of something vertical and something horizontal in each new problem”. This focusing was undertaken in tandem with the embodied object aspect of Tall’s theory in mind.

The *Category 1 Action-Perception Category of Explanation* (shown in Table 4.1) unearthed during the Main Study was built upon in Section 2. As can be seen from Figure 4.1 (the figure used in Section 2) the slope calculated between the Points A and B was visually represented in an embodied mathematical object-type manner; in effect linking the symbolic actions needed to calculate the slope value with visual referents in the form of squares. The meaning of the slope value was further emphasised in terms of ratio, as shown in Figure 4.2. This emphasis on slope in terms of a ‘ratio as measure’ has been encouraged by previous researchers [72, 91]. The meaning of slope was then divorced from the real-world context and abstracted into a mathematics context, as can be seen from Figures 5 and 6 in Appendix E.

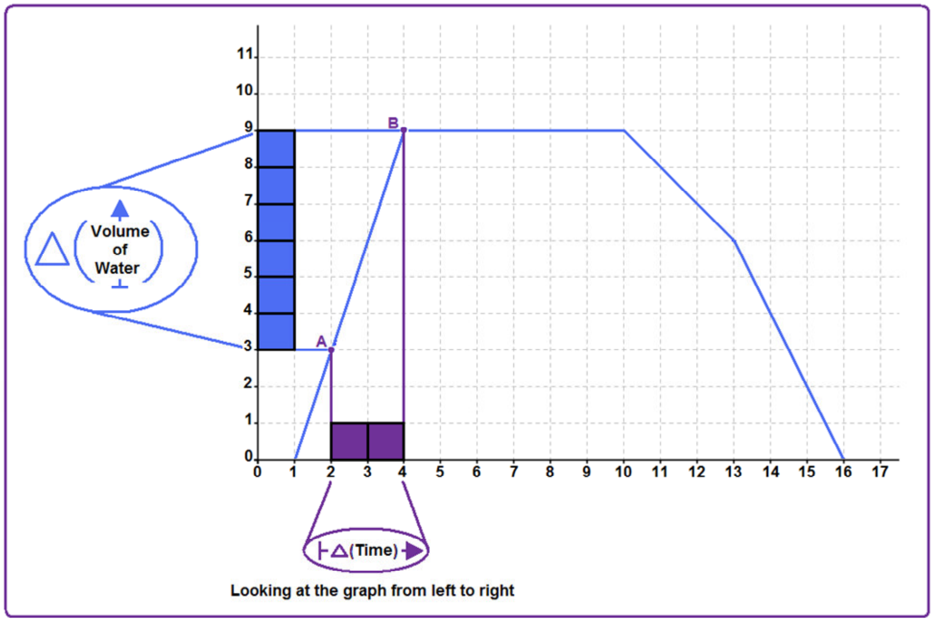


Figure 4.1 The Embodied Mathematical Object-Type Visualisation of Slope Used in the Trial 1 Intervention.

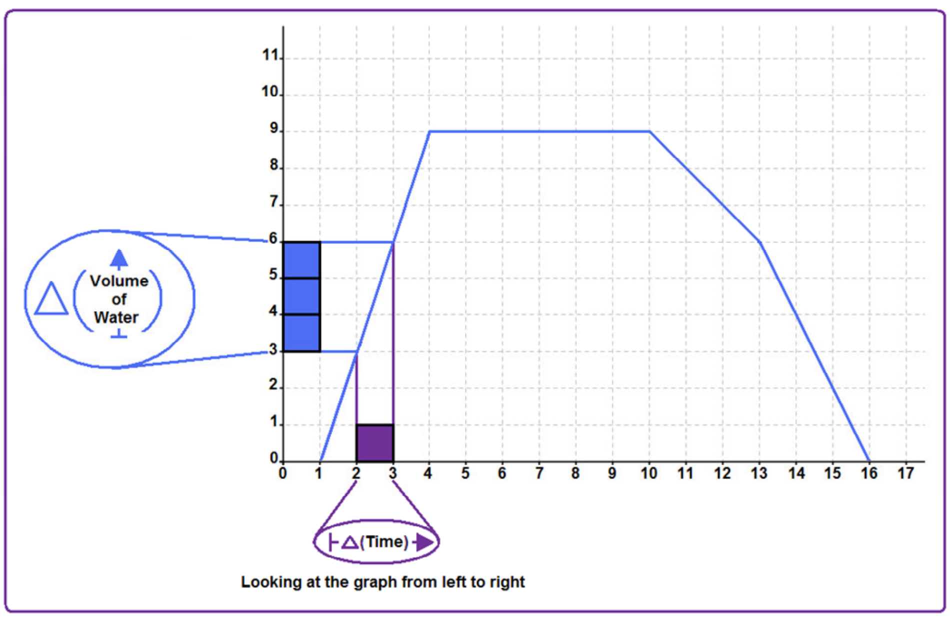


Figure 4.2 The Embodied Mathematical Object-Type Visualisation of Slope as Ratio Used in the Trial 1 Intervention.

The idea of slope being equal to steepness with a sign was also covered in Section 2. Question 10 was included as an exercise to enforce the embodied mathematical object-type understanding of slope developed in Section 2. The students were presented with a graphical description of a walker descending in height 0.25 metres for every one metre they moved forward—the graphical depiction of which is shown in Figure 4.3. The students were asked to highlight the change in the walker's horizontal direction, (the walker being initially at Point A), when the walker descended one metre in the vertical direction. The highlighting that students were expected to produce is shown in Figure 4.4.

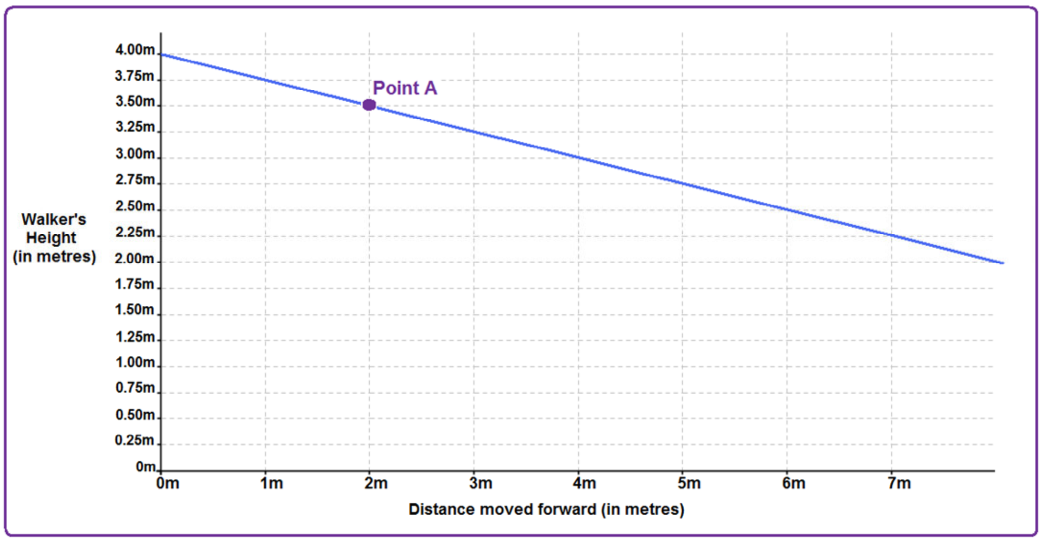


Figure 4.3 The Graphical Depiction of a Walker Descending in Height 0.25 Metres for Every Metre they Move Forward.

Section 3

Lastly, Section 3, similar to Section 1, albeit in a different context, was designed for students to re-enforce their understanding of slope in light of Section 2. It was hoped that the section would enable the students to explain the meaning of slope in an embodied mathematical object-type way for Questions 3-8.

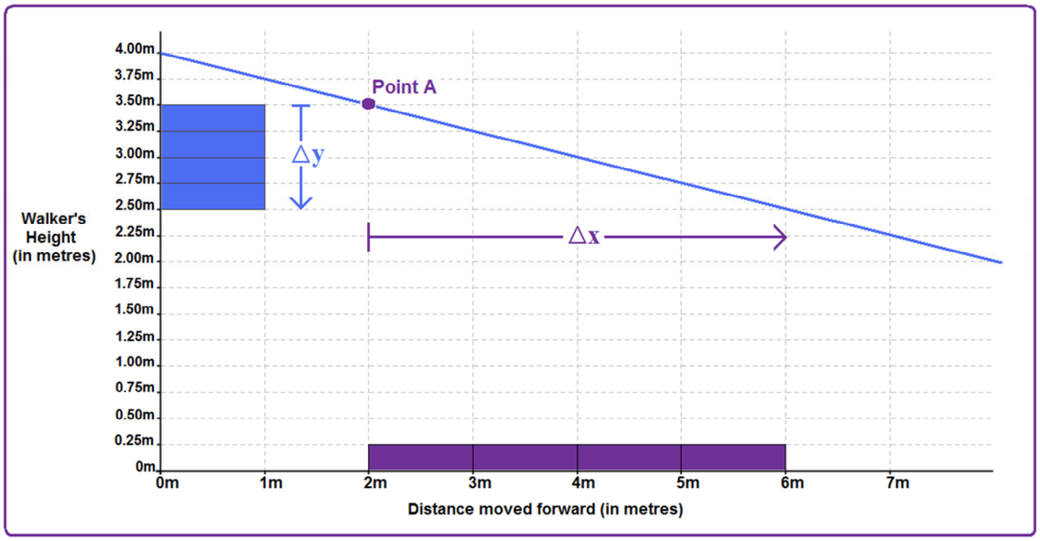


Figure 4.4 The Embodied Mathematical Object-Type Depiction of a Walker Descending in Height One Metre for Every Four Metres they Move Forward.

4.2.2.2 The Meaning-of-Derivative Intervention

Students' performance for Items 6 and 7 (which can be seen in Tables 3.1 and 3.2. in Chapter 3) was the reason for the design of an intervention aimed at improving students' understanding of derivative. As is discussed in Section 3.5.7 in Chapter 3, for Items 6 and 7 (items on generating an expression for derivative and interpreting derivative respectively), students who evidenced an ability to explain the items in a mathematics context, associated with the transfer of those items. This was observed during both Study 1 and Study 2 in the Main Study.

It was found that one of the most frequent perception-action categories of explanation for Item 6, in both studies, was that of referring to finding the slope of the tangent at a point. It was speculated that such a perception-action category of explanation is an embodied mathematical object-type image of derivative which allows students to transfer Item 6. A similar perception-action category of explanation was found to be given by students for Item 7, thus allowing them to transfer that item. In light of these findings, it was decided to improve students' understanding of derivative in terms of:

1) the meaning of derivative in respect of an embodied mathematical object-type image of slope; and

2) the meaning of derivative as a function in respect of the derivative function allowing the determination of the slope of a tangent at any particular point on the function that was differentiated.

This two-pronged approach to the intervention was situated in a real-life context, focusing on the underlying quantities being modelled: Rasmussen [92] claims that this type of approach generates more flexible, notational [algebraic] schemes for important mathematical concepts amongst students. The intervention on the *Meaning of Derivative* can be seen in its entirety in Appendix F; the three sections in this intervention are described.

Sections 1 and 3

Like the intervention on slope, Section 1 aimed to encourage the students to think about the meaning of derivative; Questions 1 and 2 were used in this regard. Section 2 was designed to help the students answer the questions in Section 1 if they were unable to do so. Section 3 re-enforced the ideas in Section 2, by way of requiring students to apply the ideas to a chemistry context.

Section 2

The relationship between the surface area of a balloon and its radius was used as a real-world context to promote students' understanding of derivative. The graph of the relationship produces a curve. Care was taken that a real-world functional relationship be chosen which was not linear in nature. A linear relationship has a constant rate of change, and, as Orton [89] articulates, the distinction between an average rate of change and instantaneous rate of change at a point (or derivative at a point), if a linear relationship is used, has little meaning to some students.

In order to answer Question 1 in the intervention, it was stressed that students need to estimate the slope of a tangent at the point in question. Such estimation was (without being made explicit), an exercise in finding the limiting value for the slope of a set of secants as Δx approached zero). Rasmussen [92] articulates that students need considerable help in this area. This was accomplished through making such a process visual in nature, as can be seen in Figure 4.5 which was one of the figures used. The visual nature is in effect an embodied mathematical object-type image of slope, building on the slope intervention.

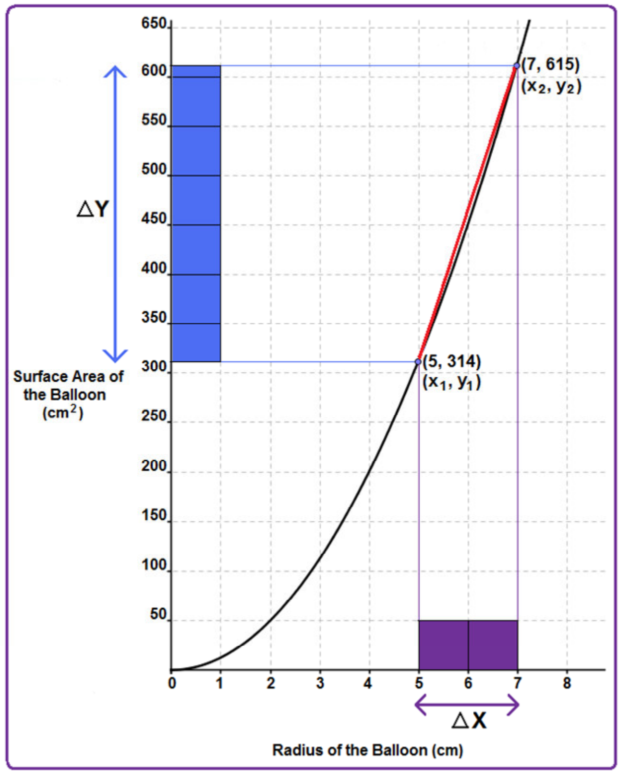


Figure 4.5 Emphasising the Visual Nature for the Estimate of the Slope of the Tangent at the Point (5,314).

The requirement of a function as only being differentiable at a point if its right-hand derivative at the point is equal to its left-hand derivative at the point was incorporated in Section 2, as can be seen from Figure 5 in Appendix F. The students had to estimate the slope of the tangent line at the point in question, as positive values of Δx approach zero and negative values of Δy approach zero, as shown in Tables 4.2 and 4.3.

Figure	Δx	Δy	Value for Slope
4	2cm	$301cm^2$	151
6	1cm	$138cm^2$	138
-	0.0001cm	$0.0125cm^2$	125.6649
-	0.00001cm	$0.00125cm^2$	125.6637
-	0.000001cm	$0.0001256cm^2$	125.6632
-	-	-	-
-	-	-	-
8	Approaches zero, yet does not reach zero.	Approaches zero, yet does not reach zero.	

*Table 4.2 The Table Used to Show that as a **Positive Value of Δx** Becomes Extremely Small (Infinitesimal), Δy Becomes Extremely Small (Infinitesimal), and thus, the More Accurate Becomes the **Estimate** for the Value of the Slope of the Tangent Line at the Point (5, 314)*

Figure	Δx	Δy	Value for Slope
5	-2cm	$-201cm^2$	101
-	-1cm	$-113cm^2$	113
-	-0.0001cm	$-0.012566cm^2$	125.6624
-	-0.00001cm	$-0.0012566cm^2$	125.6637
-	-0.000001cm	$-0.00012566cm^2$	125.6635
-	-	-	-
8	Approaches zero, yet does not reach zero.	Approaches zero, yet does not reach zero.	

*Table 4.3 The Table Used to Show that as a **Negative Value of Δx** Becomes Extremely Small (Infinitesimal), Δy Becomes Extremely Small (Infinitesimal), and thus, the More Accurate Becomes the **Estimate** for the Value of the Slope of the Tangent Line at the Point (5, 314)*

The summary section dealt with the estimation process the students were using in terms of Tall's 2^{nd} World. The technique of estimation was now referred to as finding the limiting value of the expression: $\left.\frac{dy}{dx}\right|_{x=5cm} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$,

at the point: x = 5cm. The shape of the tangent at the point: x = 5cm was now referred to as the derivative at this point.

4.2.2.3 The Derivative-as-a-Function Intervention

Items 6 and 7, as shown in Appendix D, did not explicitly test students' understanding of the derivative of a function in terms of the derivative being a function in its own right. Nevertheless, it was decided to make students aware of this. Previous research encouraged this stance. Asiala et al. [93] found that some students equate the derivative of a function with the equation for the line tangent to the graph of the function at a given point. Furthermore, Rasmussen [92] claims that for students to conceptualise the derivative of a function as a function in its own right, is by no means trivial. Bearing these findings in mind, an intervention was designed. The *Derivative-as-a-Function Intervention* can be seen in its entirety in Appendix G; the three sections in this intervention are described.

Sections 1 and 3

Question 1 in Section 1 asked students to differentiate a function and explain its meaning. Question 2 asked the students to choose which graph represented the derivative which they found in Question 1. These two questions tested whether the students realised that the derivative they calculated was: 1) a function that allows the calculation of a derivative at any particular point and 2) a function which can be represented graphically.

Section 2 was designed to help the students answer the questions in Section 1 if they had been unable to do so. Section 3, similar to Section 1, reinforced the ideas in Section 2 by way of requiring students to apply these ideas to a chemistry context.

Section 2

Like in the intervention on the meaning of derivative, the relationship between the surface area of a balloon and its radius was used as the context to embed the intervention within. The intervention revolved around Question 1: *by how much is the surface area of the balloon increasing when the radius of the balloon is instantaneously passing through any particular value*? It was stressed that a function would be generated by taking a number of points on the function: $y = 4\pi x^2$ in order to find out how much the surface area of the balloon increased (Δy), when the radius of the balloon instantaneously passed through a particular radius value; Sections 2.1-2.3

required students to do this. The same sections also reinforced what students should have learnt in the intervention on the meaning of derivative, by way of re-iterating the estimation/limiting process associated with finding the slope of a tangent at any particular point on a function. Section 2.4 tabulated the results of the work that the students should have garnered from Section 2.1-2.3 – as shown in Table 4.4. The results were plotted in the form of *dy/dx* against *r*, and it was shown that what appeared to be a linear relationship, was indeed so, as shown in Figure 4.6.

Point	***Radius of the Balloon (cm)***	$\frac{dy}{dx}$
(2,50)	2	50
(4,201)	4	100
(6,452)	6	150

Table 4.4 The Tabulation of the Results the Students Should have Garnered from the Completion of Sections 2.1-2.3.

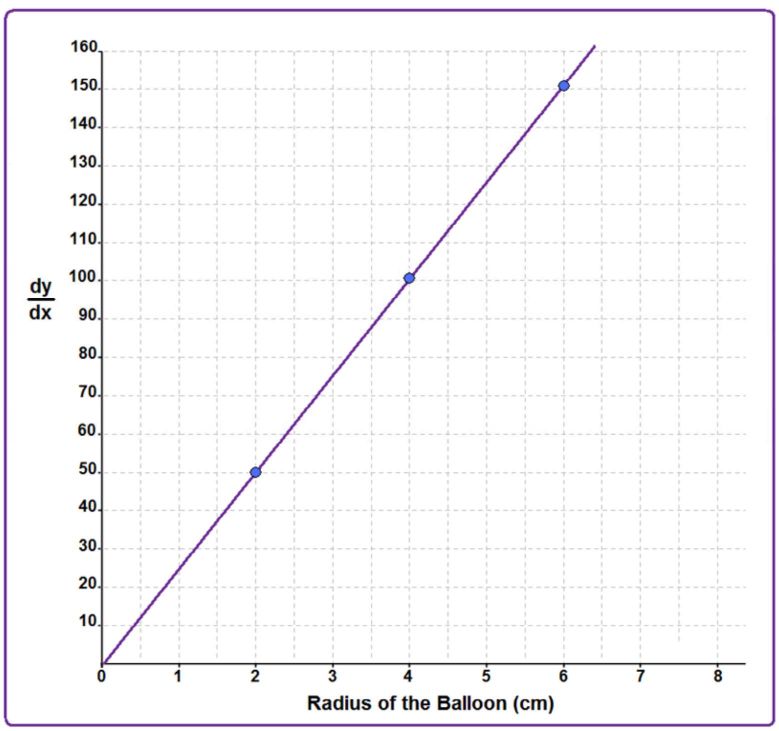


Figure 4.6 The Plotting of the Results the Students should have Garnered from Sections 2.1-2.3

Questions 5-6 were designed to reinforce the idea that the derivative of a function is a function in its own right. The emphasis of the questions was one in which the derivative function allows the calculation of the rate of change of one quantity with respect to another for any particular point of the function that was differentiated. The graph of the function and the graph of its derivative function were shown side-by-side, as can be seen in Figure 4.7, in order to stress the visual nature of the derivative function. Lastly, what a derivative function is used for was re-iterated.

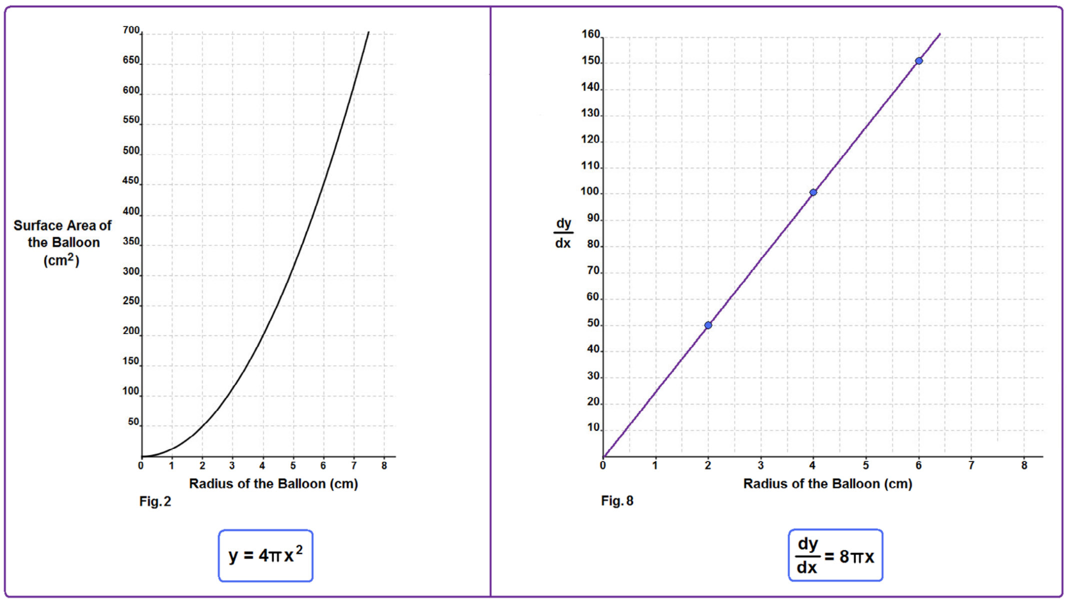


Figure 4.7 The Graph of the Function: $y = 4\pi x^2$ *, and the Graph of its Derivative Function.*

4.2.2.4 The Integration Intervention

In the Main Study, Items 14 and 15 tested students' understanding of the evaluation of an integral, and the graphical representation of an integral respectively. Students' results for these items can be seen in Tables 3.1 and 3.2 in Chapter 3; such results were the reason for the design of an intervention aimed at improving students' understanding of integration. As discussed in Section 3.5.15 in Chapter 3, students who evidenced an ability to explain these items in a mathematics context tended to associate with transferring them.

The design of the integration intervention can be seen in Appendix H. It should be noted that Sections 1 and 3 of the intervention were not in the same form as Sections 1 and 3 of both the slope and derivative interventions. The Section 1 and Section 3 for the slope and derivative interventions encouraged students to think about the concept the interventions sought to improve students' understanding of. Such an approach was not used for the integration intervention as it was felt that doing so would make the intervention too long.

Section 1

Dijksterhuis [94] cited in Gravemeijer and Doorman [27], is quoted as saying that fundamental theorems are understood intuitively before they are proven. Fostering this intuitive approach in terms of the fundamental theorem of calculus was undertaken using the physical quantities of velocity, distance and time. These physical quantities were expressed graphically because the authors Gravemeijer and Doorman argue that the use of visual referents helps students to focus on the mathematics in question [27].

The connection between velocity, distance and time was thought to offer the students a meaningful context to interpret the significance of integration. In many respects, it sought to re-invent the way in which mathematicians first unearthed an intuitive idea of the fundamental theorem of calculus by dint of *obtaining/recovering* information about a quantity of interest through its rate-of-change equation [26]. Because the authors Rasmussen and King [26] articulated that it takes today's students some thought and reflection to deal sensibly with using rate-of-change equations to obtain a quantity of interest, guiding questions (or what were deemed to be guiding questions) were incorporated in both Sections 1 and 2.

The velocity function in Section 1 was linear so as to avoid the need to calculate lower and upper bounds in order to estimate the 'perceptual area' between the graph, the x-axis, and the limits in question. The avoidance of lower and upper bounds during the introductory stage (Section 1) of the intervention was considered to be of importance in order for students to be 'intuitively struck' in terms of what integration is.

The term 'perceptual area' refers to the everyday interpretation of area, be that in terms of square centimetres, square metres or square kilometres for example. The term does not refer to the physical quantities produced by integrating this ‘perceptual area’ which could be work if the integral of pressure as a function of volume is integrated or (as in the case of the context used in this intervention) displacement — produced as a consequence of integrating velocity as a function of time.

Question 2: *In terms of a physical interpretation, what does the value for the area you have calculated mean? Does it give you the displacement of the body (the distance the body travels in a certain direction) between t = 0 seconds and t = 6 seconds?,* was included to make the students realise that the value they calculated was a physical quantity, namely displacement as opposed to a *perceptual area*. This approach was further justified by the authors Thompson and Silvermann [90], as they stress that students must see the perceptual area under a curve as representing a quantity other than an area.

The students’ tabulation of the perceptual area between the graph and the x-axis for the velocity function: $v = 2t$, as time varies (shown in Table 4.5) and its graphical depiction (shown in Figure 4.8) aimed to:

1) make the students aware that a new functional relationship had been generated —displacement versus time; and

2) this functional relationship, when differentiated, produced the original function which the students had integrated (albeit the students had not been explicitly told that they were integrating it) over specific intervals.

Figure	***Length of the base of the triangle.***	***Height of the triangle***	***Area/ displacement***
2	6 seconds	12 metres per second	36 metres
3	5 seconds		
4	4 seconds		
5	3 seconds		
6	2 seconds		
7	1 seconds		

Table 4.5 The Table Students Tabulated for the Perceptual Area between the Graph and the X-Axis for the Velocity Function: v = 2t, as Time Varied.

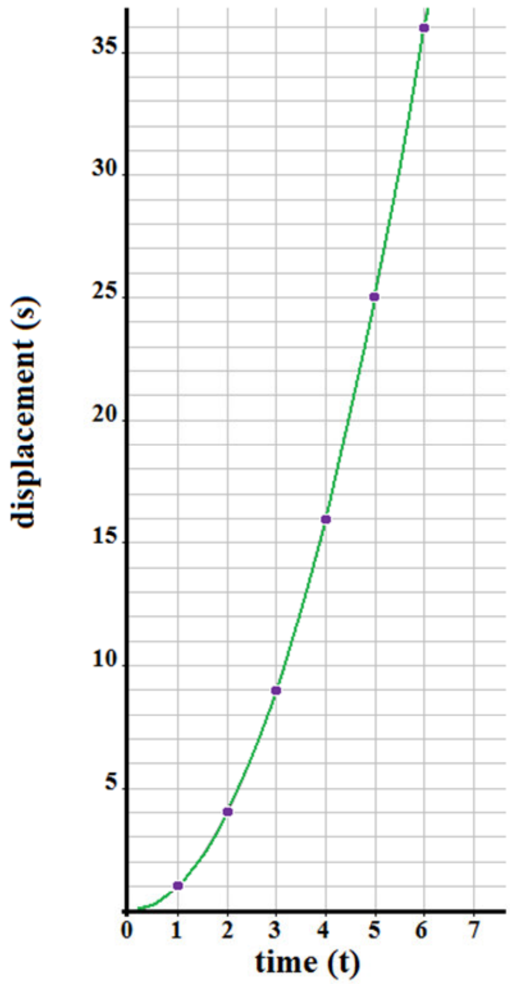


Figure 4.8 The Graphical Depiction of the Perceptual Area between the Graph and the X-Axis for the Velocity Function: v = 2t, as Time Varies.

The summary of Section 1 re-iterated what the students had been doing and observing. More importantly, it put forward an inference:

If given any function f(x), such as the one shown in Figure 4.9, then perhaps the shaded area is found by using a function F(x) that satisfies : $f(x) = \dfrac{dF(x)}{dx}$.

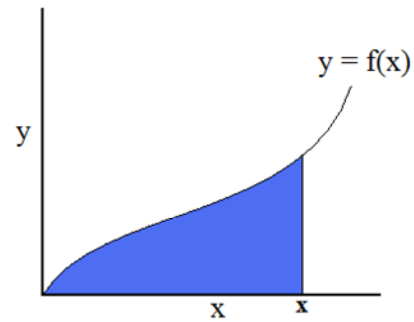


Figure 4.9 The Function Students were Shown in Light of the Suggested Inference Concerning the Integration of any Function.

Section 2

Section 2 considered the aforementioned *inference/proposition* put forward in Section 1 in more depth, this time through the form of a function that produced the shape of a curve when graphed. Question 5: *Can you calculate the area underneath the graph between time t = 0 seconds and time t = 4 seconds, as highlighted in Figure 12 (the figure can be seen in Appendix H)?,* was inquiry in nature. How such an area could be estimated by way of rectangles was described.

Questions 7-11 were included in order for students to realise that when the base width of the rectangles involved in both the lower-sum and upper-sum estimates approach zero, the numbers of rectangles increases (approach an infinite number), and the estimate for the area between the curve and the x-axis becomes more accurate.

Such an emphasis was deemed important because Orton [86] observed that in respect of an integral evaluating the area between a curve and the x-axis, many students do not understand the limiting process involved. Aspinwall and Miller [75] also found that many students are able to respond with correct answers to problems involving integrals [supposedly evaluating definite integrals], yet they do not understand how upper and lower sums create bounds for the value of an integral.

Students used the expression: $Area = b^3 + b$ to evaluate the area between the curve and the horizontal axis for the function: $v = 3t^2 + 1$, as b varied. The curve is shown in Figure 4.10 .

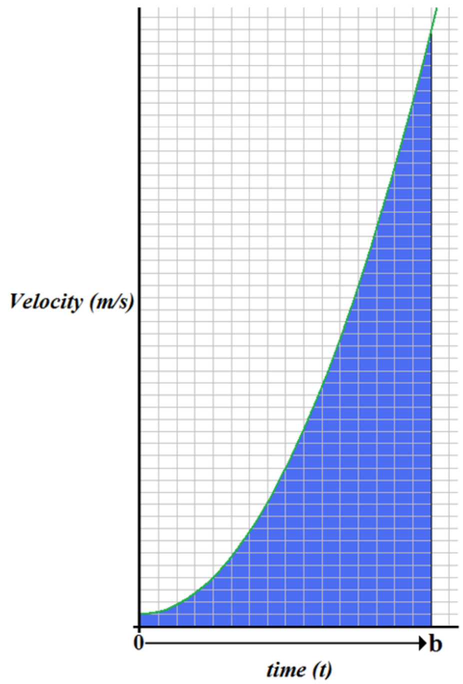


Figure 4.10 The Area between the Graph: $v=3t^2+1$, and the x-axis over *any Interval from t = 0 Seconds to t = b Seconds.*

The exact generation of the expression: $Area = b^3 + b$, was not completely shown, for to do so would have involved a lengthy explanation, involving the use of series which was beyond the scope of the intervention. Instead, it was stated that the expression for the upper-sum estimation of the area could be shown to be:

$$b^3 + b + \frac{3b^3}{2n} + \frac{b^3}{2n^2},$$

which in turn produced the expression: $b^3 + b$, as n approached infinity. Students were then instructed to calculate the area between the function shown in Figure 4.10 and the horizontal axis, between: zero seconds and three seconds; zero seconds and two seconds; and zero seconds and one second. Subsequent to this, the students had to tabulate the results and were instructed to plot the results in the form of a graph as shown in Figure 4.11.

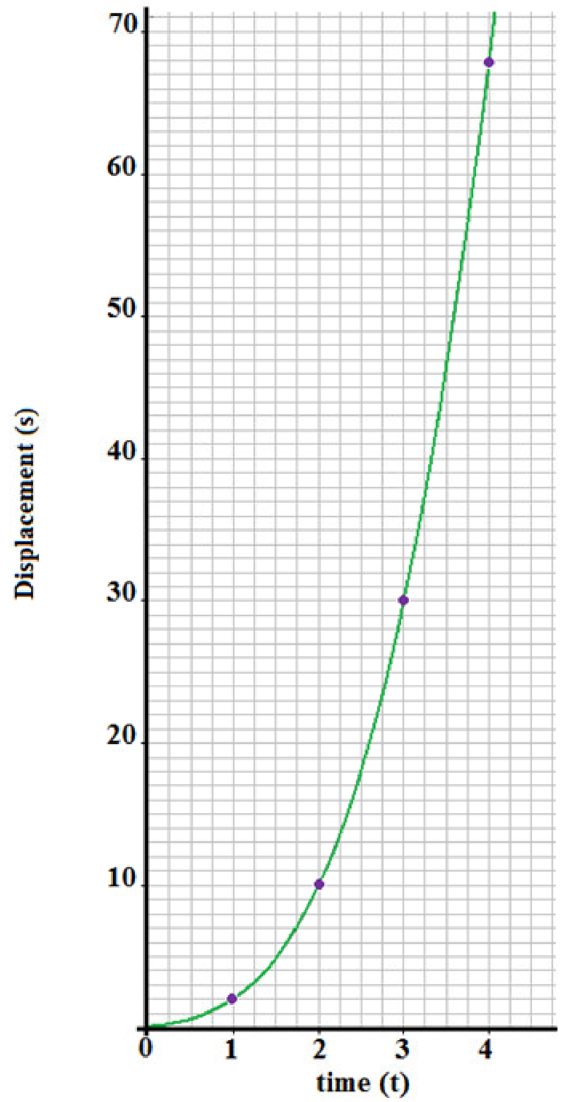


Figure 4.11 The Graphical Depiction of the Perceptual Area the Students Calculated between the Function: $v = 3t^2 + 1$ and the Horizontal Axis.

Like in Section 1, the tabulation and graphing of results was designed to make the students realise:

1) they had generated a new functional relationship — displacement versus time; and

2) the new functional relationship, when differentiated, produced the original function which the students had integrated.

Section 3

Section 3 (or the summary section) generalised what the students should have observed in Sections 1 and 2. They should have realised that given any function (f(x)), it is possible to generate a function (F(x)) which can be used to evaluate the perceptual area between the function (f(x)) and the x-axis, bounded by limit values. Furthermore, they should have realised that the technique used to generate this function (F(x)) is referred to as integration, and that when this function (F(x)) is differentiated, it produces the function (f(x)) that was integrated.

The notation used to symbolise integration was introduced and explained in Section 3. Taking an example of a function in a mathematics context, its anti-derivative function was shown. It was then emphasised that the anti-derivative function could be used to calculate the area highlighted between the function and the x-axis within certain limits. Lastly, the equation:

$$\int_a^b f(x)dx = \mathrm{F(b)} - \mathrm{F(a)},$$

was stated to emphasise how to calculate the area of a function f(x) between any limit values, a and b. Such an algebraic emphasis of integration was only made explicit in Section 3, because previous research articulated that for some students, the introduction of the meaning of integration can be obscured by algebraic manipulation [86].

4.2.3 Evaluation of Trial 1

The following questions were used to evaluate Trial 1:

1) What were the students' opinions about the usefulness of the mathematical interventions?

2) What was the students' understanding of slope, derivative and integration in a mathematics context?

3) Were students able to transfer to a chemistry context?

Ascertaining the students' opinions and the students' mathematical understanding in respect of slope, derivative and integration in a mathematics context took place through the form of student interviews during the academic year 09/10. Determining students' ability to transfer to a chemistry context was investigated through the form of a questionnaire, conducted during the academic year 10/11.

4.2.3.1 Opinions and Mathematical Understanding

Method of Investigation

Of the 18 students who completed the mathematical interventions, three students agreed to be interviewed. Students' opinions as to whether the interventions improved the students' understanding of slope, derivative and integration more so than when the students encountered the concepts in school, university lectures and university tutorials, were garnered. These thoughts were garnered by asking questions one to six and questions ten to twelve, as can be seen in Table 4.6. To investigate the students' mathematical understanding, students were asked to complete a number of questions (shown later in this section). Upon completing these questions, students were asked questions seven to nine and questions thirteen to fifteen as shown in Table 4.6. These questions were asked to ascertain the students' opinions in terms of whether the interventions helped them to answer questions related to slope, derivative and integration more so than what the students learnt in school, university lectures and university tutorials.

Interview Questions to Determine Students' Opinions
Q.1 Did your understanding of slope change from what you learnt in school to what you learnt in lectures during your first year at University? If so, how? If not, why not?
Q.2 Did your understanding of slope change from what you learnt in lectures at University compared to what you learnt in tutorials at University? If so, how? If not, why not?
Q.3 Did your understanding of slope change from what you learnt in lectures & tutorials compared to what you learnt in the mathematics-intervention workshops? If so, how? If not, why not?
Q.4 Did your understanding of derivative change from what you learnt in secondary school to what you learnt in lectures during your first year at University? If so, how? If not, why not?
Q.5 Did your understanding of derivative change from what you learned in lectures at University compared to what you learned in tutorials at University? If so, how? If not, why not?
Q.6 Did your understanding of derivative change from what you learned in lectures & tutorials compared to what you learned in the mathematics-intervention workshops? If so, how? If not, why not?
Q.7 Did what you learnt at school help you answer questions 1-3?
Q.8 Did what you learnt in lectures and tutorials at university help you to answer questions 1-3?
Q.9 Did what you learnt in the interventions help you to answer questions 1-3?
Q.10 Did your understanding of integration change from what you learnt in school to what you learnt in lectures during your first year at University? If so, how? If not, why not?
Q.11 Did your understanding of integration change from what you learnt in lectures at University compared to what you learnt in tutorials at University? If so, how? If not, why not?
Q.12 Did your understanding of integration change from what you learnt in lectures & tutorials compared to what you learnt in the mathematics-intervention workshops? If so, how? If not, why not?
Q.13 Did what you learnt at school help you answer questions 1-5?
Q.14 Did what you learnt in lectures and tutorials at university help you to answer questions 1-5?
Q.15 Did what you learnt in the integration intervention help you to answer questions 1-5?

Table 4.6 The Questions asked to Garner Students' Opinion.

The interviews were undertaken individually by a postgraduate student not directly involved in the intervention, and unfamiliar to the students, so as to make the interviews as objective as possible. All the interviews were recorded and independently rated for reliability of interpretation by two independent researchers—one from the School of Mathematical Sciences and one from the School of Chemical Sciences. An interview-evaluation rubric was designed to ensure that: 1) the interviews were close to ideal; and 2) the interview analysis determined students' opinions. The interview-evaluation rubric was applied to each question when analysing the recorded interviews; it is shown in Figure 4.12.

The criteria that is italicised in Figure 4.12 were used to evaluate if the interview was close to ideal. Cohen et al. [70] specify some of the criteria that can be used to gauge whether an interview is ideal, i.e.; 1) the extent of spontaneous, rich, specific and relevant answers from the interviewee; 2) the shortness of the interviewer's questions; 3) the length of the interviewee's answers; and 4) the degree to which the interviewer follows up, and clarifies the meaning of relevant aspects of interviewee's answers.

The interview-evaluation rubric was used to analyse the interviews in parallel with a number of other stages described by Cohen et al. [70], namely:

I. Bracketing—what it is that the interviewee is saying.

II. Listening to the interviewee for a sense of whole.

III. Delineating units of meaning relevant to the research question [students' opinions in this case].

IV. Verifying the units of relevant meaning—getting other researchers to carry out the above procedures.

V. Clustering units of relevant meaning.

VI. Determining themes from clusters of meaning.

VII. Writing a summary of each individual interview.

VIII. Writing up a *composite summary* of all the interviews which accurately captures the essence of the phenomenon being investigated.

The criteria highlighted in bold in Figure 4.12, enabled the carrying out of Stages I - VIII. The last stage of this analysis (Stage VIII) is discussed in respect of each question asked.

The Interview-Evaluation Rubric	
Criteria	**Observations**
Facial & Bodily Expression of Interviewee	
Language of Interviewer Understandable	
Interviewee at Ease throughout	
Specific and relevant answers given by Interviewee	
Interviewee Answers—Short or Long	
Did the Interviewer Look for Clarification of Certain Answers	
Did the Interviewer Interpret the Student's Answer Throughout	
Was the Answer 'Self-Communicating' (contain a story within itself)	
Units of Meaning to Emerge from this Question	
What was Said on the Whole	
Illuminating Quotations	
The Interpersonal, Interactional, Communicative and Emotional Aspect	

Figure 4.12 The Interview-Evaluation Rubric that was Applied to the Interview Process.

Results

Students' Opinions: Slope

Tables 4.7 summarises what were deemed to be the main points which emerged from the three students' answering of Questions 1-3 (shown in Table 4.6).

Q.	Student 1	Student 2	Student 3
1	• "Kind of developed a little bit". • Understood it more. • "School—here's a slope; calculate it. Actually understood what slope meant here".	• Understanding of slope did not really change. • "It was all the same"; "We used a lot of the same formulas"; " I already know this."	• Understanding of slope changed. • "In secondary school, it was just: y = mx+ c or something like that. I've a different perspective now"
2	• "Methods in lectures sink in more after you go to tutorials". • "Learnt way more in tutorials".	• Understanding of slope did not really change. • "Don't remember doing anything that I don't already know".	• Understood more. • Doing questions meant that they got a better understanding of it.
3	• They moved from understanding of slope, calculation wise, to what it actually means. • "More so in the interventions that I learnt what the slope was".	• "Looking at the graph and actually seeing if it's steep or not — never really thought of slope that way". • "All I saw was a formula: never related it to pictures".	• "Why you do it this way [calculation]—found this helpful." • "In secondary school, you're just told how to do it, but you don't understand why you do it."

Table 4.7 Summary of Students' Opinions in Respect of the Slope Intervention. The points which are surrounded by quotation marks are what were deemed to be illuminating quotations. The points which do not contain quotation marks represent the essence of what it was that the students were deemed to be saying.

Looking at Table 4.7, for Question 1, two out of the three students (Students 1 and 2) thought that their understanding of slope changed in lectures compared to what they learned at school; the other student, Student 2, said that their understanding remained the same.

For Question 2, Students 1 and 3 considered their understanding to have changed in tutorials when compared with lectures. However, for these students, from the summary of their responses, it can be argued that they equate a change in understanding with an ability ‘to do questions’ and ‘use methods’. Student 2 felt that their understanding did not really change.

In respect of Question 3, all of the students agreed that their understanding of slope changed from what they learned in lectures and tutorials, compared to what they learned in the mathematics-intervention workshop. What the students consistently reported when asked this question was that they understood more of the ‘why’—in terms of why slope is used, and what slope means.

Students’ Opinions: Derivative

Tables 4.8 summarises what were deemed to be the main points which emerged from the three students’ answering of Questions 4-6 (shown in Table 4.6).

Looking at the students’ answers for Question 4 in Table 4.8, Students 2 and 3 thought that their understanding of derivative changed from what they learnt in secondary school to what they learnt in lectures. Student 3 said that they understood more of the ‘why’ as opposed to the ‘how’; however, while Student 2 claimed that their understanding improved, they did not qualify what they meant by improved. Student 1 felt that their understanding in lectures did not improve.

For Question 5, Students 2 and 3 articulated that they felt their understanding changed in tutorials from what they learnt in lectures. It is interesting to note that what the students consider to be a change in understanding is equated with ‘how to do’ questions or ‘put things into practice’. Student 1 felt their understanding did not improve; they “already knew how to do it”.

In respect of Question 6, for Students 2 and 3, their understanding improved in the mathematics-intervention more so than in lectures and tutorials. They articulated that they understood more of the 'why'. Student 1 felt that their understanding did not improve.

Q.	Student 1	Student 2	Student 3
4	• "In Leaving Cert, I understood how to do it." (derivative). • "Nobody done anything new here in comparison to what we would have done for Leaving Cert."	• Did not really understand differentiation or derivatives in secondary school. • "I thought they did everything kind of differently in lectures."	• Understood 'why' as opposed to 'how'. • "In secondary school, it was just finding: $\frac{dy}{dx}$." • "I understand what I'm doing and why I'm doing it."
5	• Their understanding of derivative did not improve in tutorials compared with lectures. • "Already knew how to do it"; "Didn't really learn anything".	• Thought the tutorials were really helpful; people were helpful. • "I suppose I learned how to do the questions in the tutorials".	• In tutorials, you put things into practice; in lectures it was just about the notes. • Learnt more in tutorials.
6	• Understanding of derivative did not improve in the intervention compared with lectures and tutorials.	• "I think the maths intervention was about teaching you how it all began". • "I really only knew the formulas and how to do it before the maths intervention".	• Understanding slightly improved. • "Never knew that: $\frac{dy}{dx} = m$ in secondary school.

Table 4.8 Summary of Students' Opinions in Respect of the Derivative Interventions. The points which are surrounded by quotation marks are what were deemed to be illuminating quotations. The points which do not contain quotation marks represent the essence of what it was that the students were deemed to be saying.

Summary: Students' Opinions; Slope and Derivative

In respect of the slope intervention, all of the students found it beneficial in terms of *why* slope is used as opposed to *how* it is used. Students' 'how-type of understanding' was the type of understanding that they felt changed throughout lectures and tutorials.

In respect of the derivative interventions, two out of the three students felt that it was beneficial—beneficial in terms of understanding *why* derivatives are used, as opposed to *how*. Thus, in terms of students' opinions, it would appear that the interventions on slope and derivative were, on the whole, found to be of use by the students. Whether the interventions improved students' ability to answer questions relevant to slope and derivative in a mathematics context is discussed.

Mathematical Understanding: Slope and Derivative

The three questions used to probe students' understanding of slope and derivatives are shown in Appendix I. Table 4.9 shows the students' performance in respect of these questions.

Question	Student 1	Student 2	Student 3
1	×; Graph A. f(x) is an x^2 graph; when differentiated, it becomes 2x.	√; Because of the slope of the curve, I would assume the slope is positive and the line has a positive slope.	×; Graph A, because from what I remember from lectures, the U-curve is positive so the slope would be positive.
2	×; B < C < A	×; B < C < A	×; B < C < A
3	√ It means it is a straight line (not a curve).	× $\frac{dy}{dx}=0$. It is at the origin (point 0, 0).	×; It means that on a graph, the point is equal to the x-axis.

Table 4.9 Students' Results in Respect of the Trial 1 Slope and Derivative Mathematical Understanding Questions. Answering Correctly is Denoted as √, *while answering incorrectly is denoted as* ×.

For question one, two of the students did not provide a correct answer. Student 2 may have understood why Graph A in Figure 2 (in Appendix I) is the correct answer. Their statement/reason: “because of the slope of the curve, I would assume the slope is positive and the line has a positive slope”, could be interpreted as meaning the slope of the derivative function for the graph in Figure 1 (shown in Appendix I) is positive and therefore Graph B is the correct answer. Alternatively, it could be interpreted to mean the slope of the curve in Figure 1 is positive [which it clearly isn’t], and therefore, because Graph B has a positive slope, it is the correct answer. Such reasoning is clearly flawed.

For question two, all of the students provided the same, incorrect answer. However, they may have confused the definition of slope with the definition of steepness, dealt with in Section 2 of the slope intervention. If the question was worded: *rank the steepness of the tangents to the graph of f(x) at Points A-C in Figure 1, in increasing order*, then all the students would have answered the question correctly. Contradictory as it may seem, all of the students answering question two incorrectly was nonetheless encouraging because the primary focus of the slope intervention was not on the distinction between slope and steepness, but on the visual representation of the Δy increase or decrease for a line, for an increase of Δx, in the horizontal direction.

For question three, Student 1 answered it correctly. Students 2 and 3 provided incorrect answers, surprising, given that these students were able to answer question two in terms of ranking the tangents in increasing order of steepness. The argument, as to why this was so, may be that the slope of a line with a value equal to zero was not explained in the slope intervention. However, the purpose of the slope intervention was for students to be able to deduce what any particular slope value means, even a slope value of zero.

It would appear that the slope and derivative interventions did not help students to answer questions one to three very well. In spite of this, all the students agreed (when they answered opinion Questions 1-6) that the slope intervention could improve a students’ understanding, while two out of the three students agreed that the derivative intervention could do likewise. Students were asked an additional three questions (Questions 7-9) in light of completing the slope-and-derivative questions in Appendix I.

These questions were asked in order to ascertain the students’ opinions in terms of whether the interventions improved students’ mathematical understanding in respect of questions related to slope and derivative more so than what the students learnt in school, university lectures and university tutorials. The results from Questions 7-9 are summarised in Table 4.10.

Q.	Student 1	Student 2	Student 3
7	• Understanding did not improve more at university compared to what it was in school.	• What they learnt at school did not help them.	• What they learnt at school did not really help to answer the question. • "Can't really remember secondary school."
8	• No improvement.	• Did not help.	• Lecture and tutorials helped.
9	• In lectures, nothing stuck. • Meaning of slopes and derivatives was not covered. • "We never did graphs like this in terms of explaining slopes and derivatives."	• "Looked at graphs more in the maths intervention". • "I still really wouldn't know the answers to those questions".	• "I can't really recall."; "I don't know any specifics."; • "Think it was kind of going over what we did in lectures and tutorials."

Table 4.10 Summary of Students' Opinions in Respect of the Intervention's Effect on Their Mathematical Understanding in Terms of Slope and Derivative.

Looking at Table 4.10, in respect of Question 7, all of the students felt that what they learnt at school did not help them to answer the slope-and-derivative questions. For Question 8, two out of the three students felt that what they learnt in lectures and tutorials was of no help. Student 1, when answering Question 6 (shown in Table 4.8), articulated that the intervention on derivative did not improve their understanding. However, interestingly, they articulated otherwise when answering Question 9 (as can be seen in Table 4.10). Student 2, who felt that the slope intervention and derivative interventions improved their understanding, stated "I still really wouldn't know the answers to those questions [the questions used to probe students' understanding of slope and derivative, located in Appendix I]". Student 3, while agreeing that the slope and derivative interventions improved their understanding before they answered the slope-and-derivative questions, now stated that they “can’t really recall, I don’t know any specifics”.

Students’ Opinions: Integration

Table 4.11 summarises what were deemed to be the main points which emerged from the three students’ answering of Questions 10-12 (shown in Table 4.6).

Looking at the students’ answers for Question 10 in Table 4.11, Students 2 and 3 had not covered integration in school; Student 1 stated that their understanding improved; however, they appeared to equate a change in understanding with learning different methods.

For Question 11, Students 1 and 3 conveyed that their understanding changed, but they equated this change in understanding with ‘doing questions’. Student 2’s understanding did not change much in the tutorials; however, they too equated understanding with practising questions.

In respect of Question 12, Students 2 and 3 felt that their understanding of integration changed in the intervention compared to what they learnt in lectures and tutorials. Student 2 stated that they “should have learnt the maths intervention before going into the formulas”, implying that lectures and tutorials were perhaps too algebraic in focus. However, the same student found it hard to relate the integration intervention to the “paper” and “stuff” [their tutorial questions and examination]. Student 3 found the intervention to be like a “kind of review”, stating that it was similar to what was already covered. Student 1’s understanding of integration did not change throughout the intervention. They stated that their understanding of the integration intervention was not checked—however, the intervention was designed to be self-directed in nature.

To summarise: It would appear that during lectures and tutorials, the students’ understanding of integration improved in terms of ‘how to do it’. Only one student (Student 2) thought that the integration intervention was beneficial. Student 3 saw it as a review, and Student 1 did not consider it to be of any benefit. Whether the interventions improved students’ ability to answer questions relevant to integration in a mathematics context is discussed.

Q.	Student 1	Student 2	Student 3
10	• Understanding of integration changed. • "Learnt different methods of integration."	• Understanding changed, because she did not do integration in school.	• They had no idea of what integration was in school—did not cover it.
11	• Learnt more in tutorials. • "Learnt more in tutorials because I'm actually doing the question myself."	• Understanding did not change much in tutorials. • Practising questions made them understand them more.	• Understanding improved. • "Could not understand why u-substitution went the way it did until tutorials."
12	• No difference in their understanding of integration during the intervention. • "I just had my understanding of it." • Their understanding of the integration intervention wasn't checked. • "I don't remember the maths intervention because we weren't being graded." • "I don't know if what I was thinking was right or wrong."	• Understanding changed. • "Basics are probably important [referring to intervention] because you'll understand the formulas better." • "Should have learnt the maths intervention before going into the formulas". • "Suppose I should know the basics but I didn't really". • "Found it difficult to relate the maths-intervention questions to the paper and stuff".	• For the first couple of weeks they just could not understand integration. • Intervention helped. • Intervention was similar to what was already covered. • Tend to forget certain parts from lectures and tutorials. • "Intervention was kind of a review."

Table 4.11 Summary of Students' Opinions in Respect of the Integration Intervention. The points which are surrounded by quotation marks are what were deemed to be illuminating quotations. The points which do not contain quotation marks represent the essence of what it was that the students were deemed to be saying.

Mathematical Understanding: Integration

The five questions used to probe students' understanding of integration are shown in Appendix J. Table 4.12 shows the students' performance in respect of these questions.

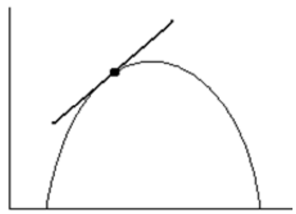
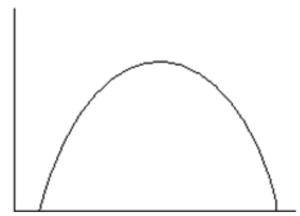
Question	Student 1	Student 2	Student 3
1	×; 	×; no diagram	×; 
2	×: B; Graph B, as the y-value must be twice the x-value.	×: B; F(x) is positive, so it cannot be Graph D. f(x) = y, which is not = 0, so it cannot be Graph C. The formula is similar to a graph, so it cannot be Graph A, so it must be Graph B.	× C; $\frac{d2x}{dx}=2$ $\frac{dy}{dx}=f(x)$ would mean that it would be the same, so I think it is C, a straight line.
3	√ ;The anti-derivative of f(x) between x-values b and a.	×; It represents $\frac{dy}{dx}$, which is the slope.	×; $\int f(b)dx - \int f(a)dx$.
4	×; The function to be integrated.	×; The x-value signifies where the graph meets the x-axis.	×; x is a constant seen as b-a from the above equation.
5	×; derivative.	×; a constant.	×; the derivative of x.

Table 4.12 Students' Results in Respect of the Trial 1 Integration Mathematical Understanding Questions. Answering Correctly is Denoted as √, while answering incorrectly is denoted as ×.

It would appear that the integration intervention did not improve students' mathematical understanding in this regard. The question that arises is: why? A possible reason may be the fact that during the integration intervention, students were not asked questions similar to the five questions in Appendix J. The counter-argument to this is: the integration intervention should have prepared the students to answer such questions. Further, suggested reasons as to the students' poor performance are articulated in the Conclusions and Implications.

Students were asked an additional three questions (Questions 13-15) in light of completing the integration questions in Appendix J. These questions were asked to ascertain the students' opinions in terms of whether the interventions improved students' mathematical understanding in respect of questions related to integration more so than what the students learnt in school, university lectures and university tutorials. The results from the asking of Questions 13-15 are summarised in Table 4.13.

Q.	**Student 1**	**Student 2**	**Student 3**
13	• What they learnt at school did not help. • "At school, we never did area with anti-derivatives."	• What they learnt at school did not help.	• They did not cover integration at secondary school.
14	• Tutorials helped them to answer the question.	• "Lectures did not help in any of the graph questions." • "The only question the lectures helped in was question three."	• "Thought it did, until I did that question."; "better understanding in tutorials."
15	• The intervention did not really help. • "I don't remember doing anything in the maths intervention that I could use."	• "The intervention should have helped me with the graphs." • " Don't remember how to graph the integration thing."	• "It helps but I haven't answered the question."

Table 4.13 Summary of Students' Opinions in Respect of the Intervention's Effect on Their Mathematical Understanding in Terms of Integration.

Looking at Table 4.13 in respect of Question 13, for Student 1, they felt that what they learnt in school did not help them to answer the integration questions in Appendix J. For Students 2 and 3, their response was similar; however, they were not exposed to integration in school. For Question 14, Student 1 felt that what they learnt in tutorials was of help in answering the integration questions. However, Student 2 did not think this. Student 3 thought they had a better understanding in tutorials, but as can be seen in Table 4.12, they answered none of the integration questions correctly. Lastly, for Question 15, Student 1 felt that the integration intervention did not help them to answer the questions. Somewhat on a positive note, Students 2 and 3 articulated that the intervention should have helped them, but it did not. Possible reasons as to why this was so are articulated in the Conclusions and Implications.

Summary: Opinions and Mathematical Understanding

In terms of the students' opinions about the usefulness of the mathematical interventions:

- all of the students found the slope intervention to be beneficial;
- two out of the three students considered the derivative intervention to be beneficial; and
- in respect of the integration intervention, one student considered it to be useful, one student seen it as a review of what they had already done, while the other did not find it to be of benefit.

In terms of what was the students' understanding of slope, derivative and integration in a mathematics context:

- the results in Tables 4.9 and 4.12 suggest that the interventions did not have an impact in this regard.

However, all of the students articulated that the interventions on slope and derivative improved their understanding despite not being able to answer correctly, the slope and derivative questions in Appendix I. As shown in Table 4.13, two out of the three students articulated that the intervention on integration should have helped with the answering of the integration questions in Appendix J. Students' ability to transfer to a chemistry context in light of the intervention is now discussed.

4.2.3.2 Transfer Ability

Method of Investigation

Of the eighteen students who completed the mathematical interventions during the first year of their studies, fifteen of these students were successful in progressing to second year. Of these fifteen students, eight had to complete a chemical kinetics and thermodynamics module. These eight students were approached in order to complete a questionnaire. Four of the students agreed to do so. None of the four students had participated in the interviews that were used to ascertain the students' opinions about the usefulness of the interventions.

Determining students' transfer ability took place over two assessments. The questionnaire used in the assessments contained some of the items used in the Main Study (as can be seen in Appendix D). Items 1-7 were used to investigate the students' ability to transfer items relevant to slope and derivative. Items 14-15 were used to test students' transfer ability in respect of integration. The items in the mathematics context were administered separately from the items in the chemistry context, over the space of a week, in order to avoid a recognising-of-patterns effect [10]. Because only four students completed the questionnaire, transfer (if there so happened to be any) was not tested for significance. Instead, the study was more qualitative in nature. The four students' ability to transfer each item was compared with the transfer that was observed for these items in Study 1 and Study 2 in the Main Study.

Results

The results for the four students' ability to transfer Items 1-7 and Items 14-15 are shown in Table 4.14.

Mathematical Item	Correct in MC* and CC**	Correct in MC and Incorrect in CC	Incorrect in MC and Correct in CC	Incorrect in MC and CC
1. Calculating Slope.	4	0	0	0
2. Sketching a Line with Positive Slope.	4	0	0	0
3. Sketching a Line with Positive Slope.	3	0	1	0
4. Sketching a Line with Negative Slope.	3	1	0	0
5. Generating an Expression for Slope.	4	0	0	0
6. Generating an Expression for Derivative.	1	1	0	2
7. Interpreting Derivative.	2	1	1	0
14. Evaluation of an Integral.	1	0	0	3
15. Graphing an Integral.	1	0	1	2

MC* *- Mathematics Context;* ***CC***** *- Chemistry Context.*

Table 4.14 Results for the Trial 1 Students' Ability to Transfer Each Item.

Certain students were able to transfer certain mathematical items. However, because only four students completed the questionnaire, it was difficult to conclude whether the interventions improved students' ability to transfer in comparison to the transfer observed in Study 1 and Study 2. It was found:

- In terms of the Items related to slope — Items 1-5 — three out of the four students transferred Items 3-4, while all of the students transferred Items 1, 2 and 5. It could be surmised that the intervention on slope was successful in promoting transfer. However, these items were transferred better than any other items in both Study 1 and Study 2, so it is difficult to gauge whether the slope intervention made any difference.

- For Item 6— the item on the generation of an expression for derivative — only one student transferred. Low numbers of students also transferred this item in both Study 1 and Study 2. Thus, it would appear that the intervention on derivative had limited success in improving students' ability to transfer.

- Item 7 was transferred by an extra student when compared with Item 6. Similarly, during Study 1 and Study 2, more students transferred Item 7 than Item 6. On the whole, for Items 6-7: the interventions on derivative were of limited success in improving students' ability to transfer.

- For Items 14 and 15, the intervention on integration was also of limited success in improving students' ability to transfer when compared with the transfer observed for these items during Study 1 and Study 2.

4.2.4 Conclusion: Trial 1

The conclusion that arose from Trial 1 is discussed in the context of the questions that were used to evaluate it:

- In terms of: *what were the students' opinions about the usefulness of the mathematical interventions?,* the majority of the students articulated that the interventions were of benefit in improving their understanding.

- In terms of: *what was the students' understanding in respect of slope, derivative and integration in a mathematics context*?, the students appeared unable to answer the questions in respect of slope, derivative and integration despite claiming that the interventions should have helped them to do so.

- In terms of: *were the students able to transfer to a chemistry context?*, one year on, certain students did transfer certain items. However, it was difficult to determine whether the intervention improved transfer in comparison to the transfer observed for these items during Study 1 and Study 2.

Despite the findings in respect of students' mathematical understanding, it was decided to trial the intervention again—Trial 2. The reasons for this were:

- The sample of students used in the evaluation of Trial 1 was small. Therefore, the results from the evaluation may not have reflected the general impact of the intervention.

- All of the students who participated in Trial 1 agreed that the interventions were beneficial.

4.3 The Intervention — Trial 2

For Trial 2, it was decided to trial the mathematical interventions amongst first-year students with average mathematical knowledge. The term 'average mathematical knowledge' was defined as those students who obtained Grades A-C_3 in Ordinary-Level Mathematics in the Irish Leaving Certificate examination. It was felt that if the interventions produce an impact in terms of: 1) students' mathematical understanding and; 2) students' ability to transfer, then, they would have the most effect on these students.

4.3.1 The Sample

60 students were classified as possessing average mathematical knowledge. All of the students were undertaking one of the following science programmes: Chemical and Pharmaceutical Science, Analytical Science, Environmental Science and Health, Common Entry into Science, Biotechnology or Genetics and Cell Biology. All of the students were asked to participate in the intervention. Six students agreed to do so. Of these six students, all of them completed the interventions in respect of slope and derivative, while five of them completed the integration intervention. The students were paid a nominal amount for their participation.

4.3.2 Methodology

The design of the slope, derivative and integration interventions remained unaltered from the design used in Trial 1; the rationale behind the design of the interventions can be seen by referring to Section 4.2.2. All of the students undertook the four interventions a week apart, over the space of a month. Students were given an hour to complete each intervention. The evaluation of the interventions took place separately from the administration of the interventions. Each student who completed the interventions was evaluated separately. The evaluation of each student took no longer than an hour.

4.3.3 Evaluation of Trial 2

The following questions were used to evaluate Trial 2:

1) What were the students' opinions about the usefulness of the mathematical interventions?

2) What was students' understanding in respect of slope, derivative and integration in a mathematics context? Additionally: Were the students able to link their mathematical actions in a mathematics context with referents—be that embodied mathematical objects or mathematical objects?

3) Were students able to transfer to a chemistry context? Additionally: For the students who were able to link their mathematical actions with objects, were they more likely to transfer to a chemistry context?

Garnering students' opinions in Trial 2 was probed using a questionnaire in respect of each mathematical intervention. The questionnaire focused on establishing the students' opinions about each of the interventions in their own right. This approach was in contrast to the approach taken in Trial 1. In Trial 1, students' opinions as to whether each intervention improved the students' understanding of slope, derivative and integration more so than when the students encountered the concepts in school, university lectures and university tutorials was probed.

The students' mathematical understanding and transfer ability in light of the interventions was investigated using a combination of mathematical questions and a Think-Aloud Protocol [95]. This investigation was slightly different from how students' mathematical understanding and transfer ability was investigated in Trial 1. An insight of students' understanding of slope, derivative and integration in terms of Tall's theory wanted to be gained. It wanted to be seen if students were able to link mathematical actions in respect of slope, derivative and integration to referents—be that in terms of embodied mathematical objects or mathematical objects; this was of course the main aim of the mathematical interventions (to improve students' ability to explain the concepts). Furthermore, it wanted to be seen if students who could link mathematical actions to referents/objects had greater transfer ability in comparison to students who could not.

4.3.3.1 Opinions

Method of Investigation

In order to gauge the students' opinions in respect of the interventions, a series of questions in the form of a questionnaire were formulated for each of the interventions. The questions asked for each intervention were similar, and can be seen in Table 4.15. The questions in Table 4.15 pertain to the evaluation of the slope intervention. Question 1: *The workshop increased my understanding of slope*, would have been worded similarly when gauging the students' opinions in respect of the derivative interventions or the integration intervention—the word 'slope' being replaced with the word 'derivative' or the word 'integration'. These slight adjustments were made to all of the questions, depending on the nature of the mathematical intervention that was evaluated.

The Questions Used in the Slope Questionnaire
Q.1 *The workshop increased my understanding of slope.*
Q.2 *I had a good understanding of Slope before the workshop and therefore I learnt very little from it.*
Q.3 *Before the workshop, I could only apply the slope formula in problems, but now, I understand why I apply the slope formula.*
Q.4 *The workshop was clear and it could be a useful resource for students to access themselves.*
Q.5 *The time allocated to the workshop on slope was: too short, about right or too long.*
Q.6 *Any other comments.*

Table 4.15 The Questions Asked in the Slope Questionnaire.

The questions incorporated a Likert scale, where the scale included the categories of: *Strongly Disagree*, *Disagree*, *Undecided*, *Agree* and *Strongly Agree*. Questions 1 and 2 were linked. The reason for this is discussed in the context of the questions used for the evaluation of the slope intervention.

For Question 1: *The workshop increased my understanding of slope,* some of the reasons why students might *Disagree* or *Strongly Disagree* when answering this question were anticipated; Question 2: *I had a good understanding of slope before the workshop and therefore I learnt very little from it,* was one of these anticipated reasons.

For the students who answered *Strongly Disagree*, *Disagree* or *Undecided* for Question 1, it was anticipated that they would answer *Agree* or *Strongly Agree* for Question 2. For the students who *Agreed* or *Strongly Agreed* for Question 1, it was anticipated that these students would answer *Strongly Disagree* or *Disagree* for Question 2.

Question 3 focused on probing whether the intervention cultivated students' understanding in terms of why they use a slope formula. Question 4 was designed to gauge the students' opinions as to whether the workshop could be a useful resource, while Question 5 gauged the students' views on whether enough time was allocated to the intervention. Lastly, Question 6 was an any-other-comments type question.

Results

Students' Opinions: Slope

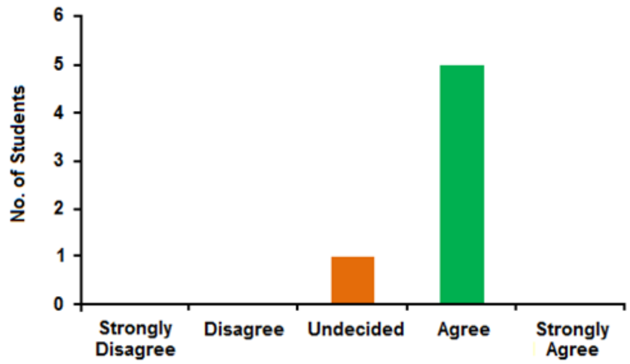
The results from Questions 1 and the results form Question 2 are shown in Figure 4.13. Of the six students who completed the intervention, five agreed that the workshop increased their understanding while one student was undecided. Of the five students who agreed, three of them disagreed with Question 2, as was anticipated, while the other two were undecided; despite these two students being undecided, at least they were not in the *Agree* or *Strongly Agree* category, which would have meant that they contradicted their answer to Question 1. For the student who answered undecided when answering Question 1, they answered *Agree* for Question 2, thus explaining their answer of *Undecided* for Question 1.

Looking at the results for Questions 3 to 5 in Figure 4.14, the majority of students agreed that: 1) the slope intervention improved their understanding of why they apply the slope formula; 2) the intervention could be a useful resource for students to access by themselves; and 3) the time allocated to the intervention was about right.

For the any-other-comments type question, shown in Figure 4.15, one student found the intervention extremely beneficial, while another deemed it to be revision—even though this was not the aim of the intervention.

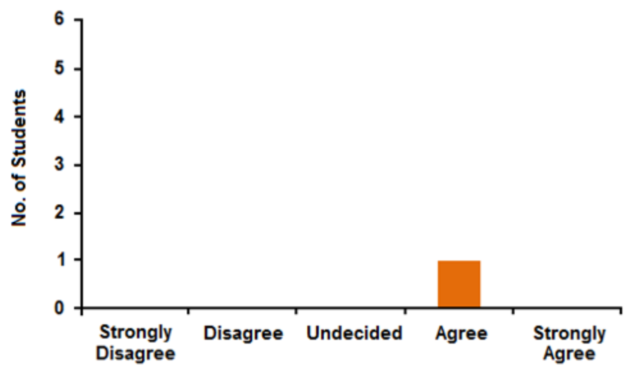
Slope Questionnaire: Results from Questions 1 and 2

Q. 1 *The workshop increased my understanding of slope.*



Q.2 *I had a good understanding of Slope before the workshop and therefore I learnt very little from it.*

For the Students who Answered Undecided in Question 1



For the Students who Answered Agreed in Question 1

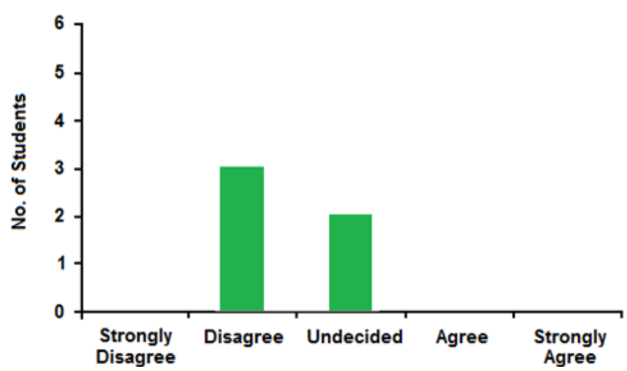
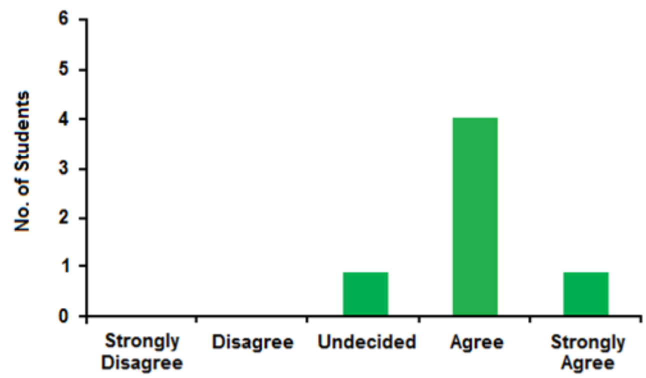


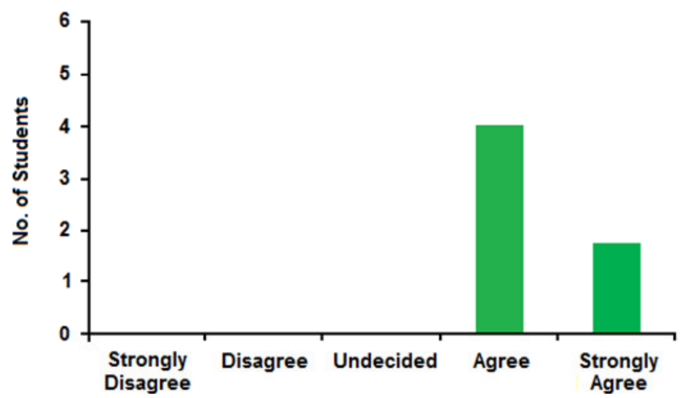
Figure 4.13 The Results from the Slope Questionnaire for Questions 1 and 2.

Slope Questionnaire: Results from Questions 3-5

Q. 3 *Before the workshop, I could only apply the slope formula in problems, but now, I understand why I apply the slope formula.*



Q.4 *The workshop was clear and it could be a useful resource for students to access themselves.*



Q.5 *The time allocated to the workshop on slope was:*

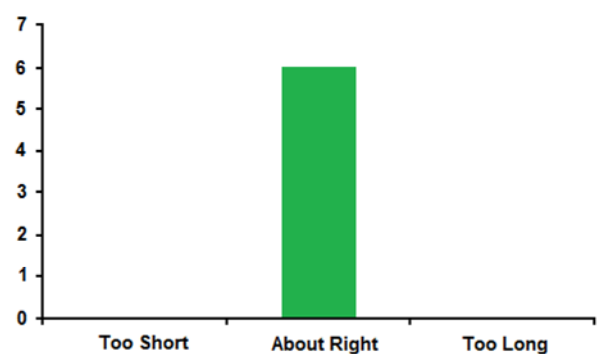


Figure 4.14 The Results from the Slope Questionnaire for Questions 3-5.

Slope Questionnaire: Results from Question 6

Q. 6 *Any other comments:*

- *The workshop helped me in my maths and physics labs. It was extremely beneficial.*
- *If it was on twice a week, I would have gone. The slope was revision but still helpful.*

Figure 4.15 The Results from the Slope Questionnaire for Question 6.

Students' Opinions: Derivative

The results from Question 1 and the results from Question 2 are shown in Figure 4.16. Of the six students who completed the intervention, when answering Question 1, four agreed that the workshop increased their understanding while two were undecided. Of the four students who agreed, they all answered either *Strongly Disagree*, *Disagree* or *Undecided* for Question 2, as was anticipated. For the two students who were undecided for Question 1, these students were in the *Disagree* category for Question 2, suggesting that they may really be in the *Agree* category for Question 1.

The results for Questions 3-5 are shown in Figures 4.17. For Question 3, only three out of the six students either *Agreed* or *Strongly Agreed* that they understood why they find a derivative; the other three students were in the *Undecided* category. For Question 4, the majority of the students agreed that the workshops on the derivative could be a useful resource for students to access, while four out of six students found the time allocated to the workshops on derivative to be about right, with—interestingly—the other two students stating that they found it too short.

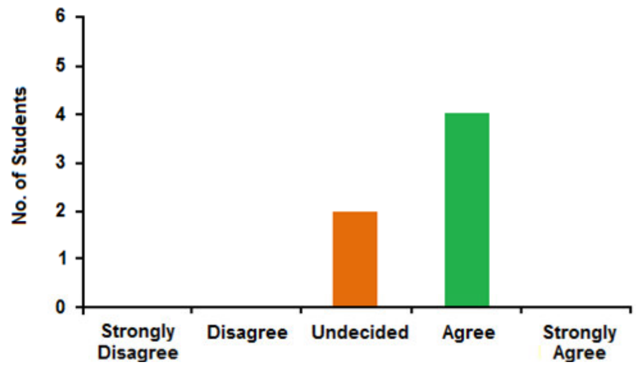
For the any-other-comments type question (shown in Figure 4.18), a response that was not to be expected in light of the intervention was:

It was a help but I'm still slightly confused with the topic.

Such a statement is perhaps not a surprise given the results that emerged from the evaluation of Trial 1 in respect of students' ability to answer questions related to derivatives. These results can be seen by referring to Section 4.2.3.1.

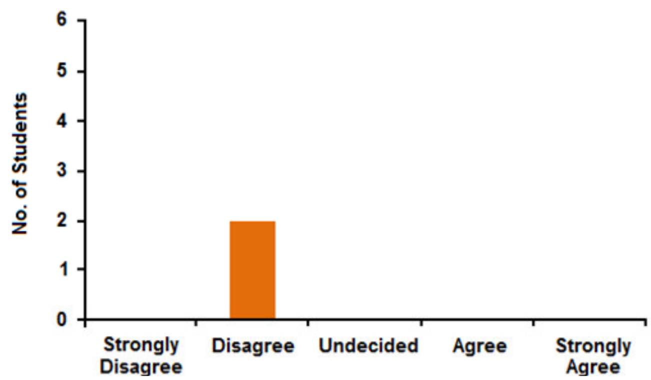
Derivative Questionnaire: Results from Questions 1 and 2

Q. 1 *The workshop increased my understanding of Derivative.*



Q.2 *I had a good understanding of derivative before the workshops and therefore I learnt very little from them.*

For the Students who were Undecided in Q. 1



For the Students who Agreed in Q.1

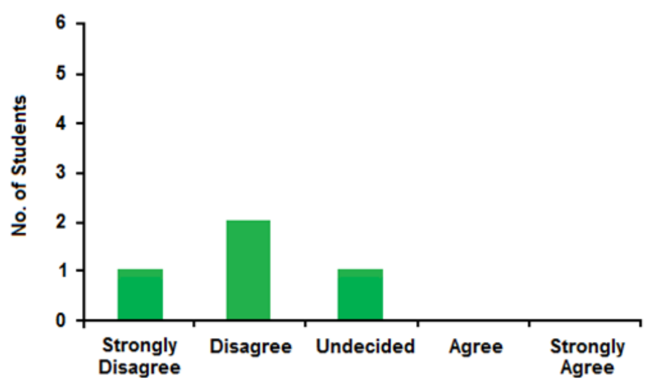
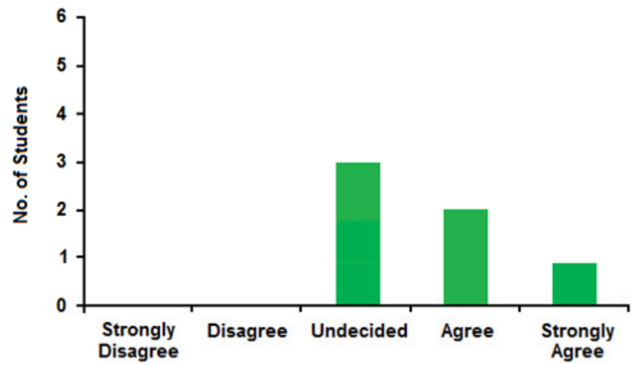


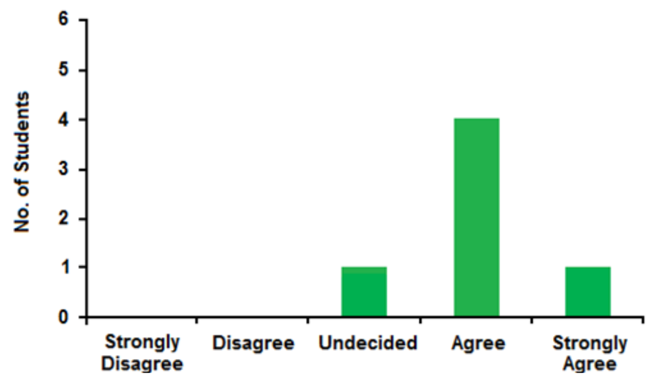
Figure 4.16 The Results from the Derivative Questionnaire for Questions 1 and 2.

Derivative Questionnaire: Results from Questions 3-5

Q. 3 *Before the workshops, I could only apply the technique of finding a derivative in problems, but now, I understand why I find a derivative.*



Q.4 *The workshops were clear and it could be a useful resource for students to access themselves.*



Q.5 *The time allocated to the workshop on slope was:*

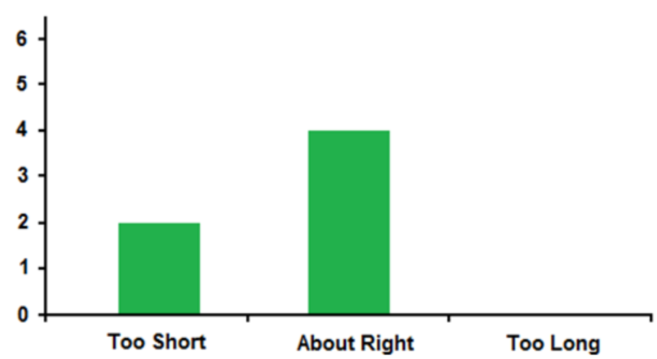


Figure 4.17 The Results from the Derivative Questionnaire for Questions 3-5.

Derivative Questionnaire: Results for Question 6
Q. 6 *Any other comments:* • *I found this harder than the slope problems but once it was explained to me, I understood.* • *It was a help but I'm still slightly confused with the topic.*

Figure 4.18 The Results from the Derivative Questionnaire for Question 6.

Students' Opinions: Integration

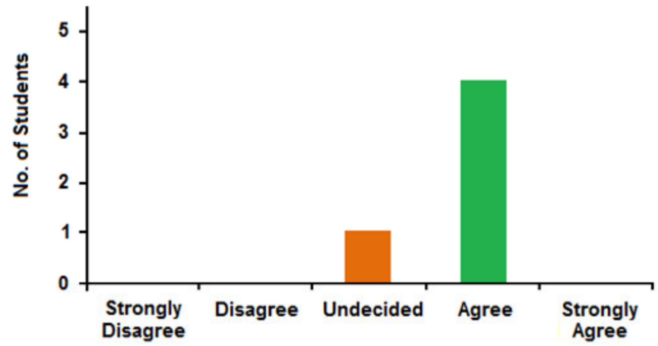
The results from Questions 1 and the results from Question 2 are shown in Figure 4.19. Of the five students who undertook the intervention, four agreed that the workshop increased their understanding while one student was undecided. Of the four students who agreed, three of these either disagreed or were undecided for Question 2. For the student who was undecided for Question 1, they were in the *Disagree* category for Question 2, suggesting that they probably should have been in the *Agree* category for Question 1.

Looking at the results for Questions 3-5 in Figure 4.20, for Question 3, only two out of five students agreed that they understood why they apply the technique of integration. Nonetheless, for Question 4, all the students agreed that the workshop on integration was a useful resource for students to access by themselves. For Question 5, all agreed that the time allotted to the workshop was about right.

No striking comments emerged from the asking of the any-other-comments type question (Question 6), the results of which are shown in Figure 4.21.

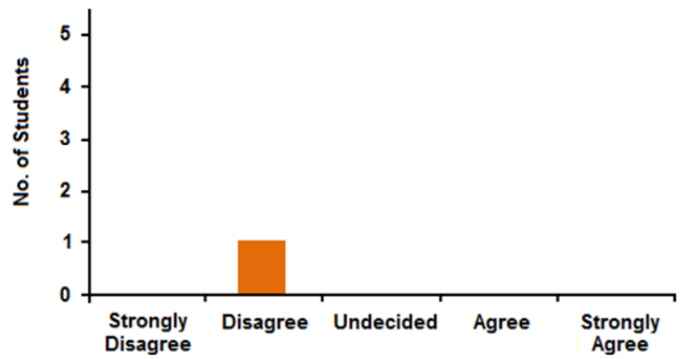
Integration Questionnaire: Results from Questions 1 and 2

Q. 1 *The workshop increased my understanding of Integration.*



Q.2 *I had a good understanding of integration before the workshop and therefore I learnt very little from it.*

For the Students who were Undecided in Q. 1



For the Students who Agreed in Q.1

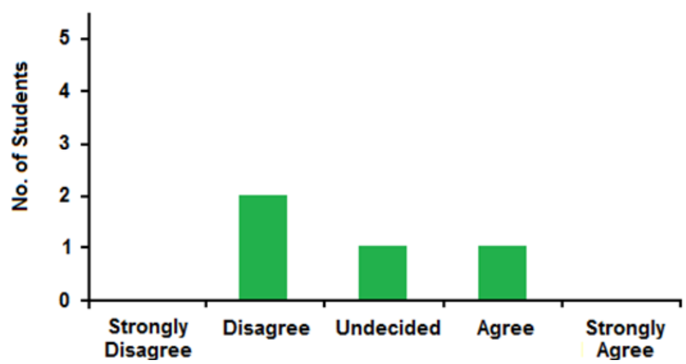
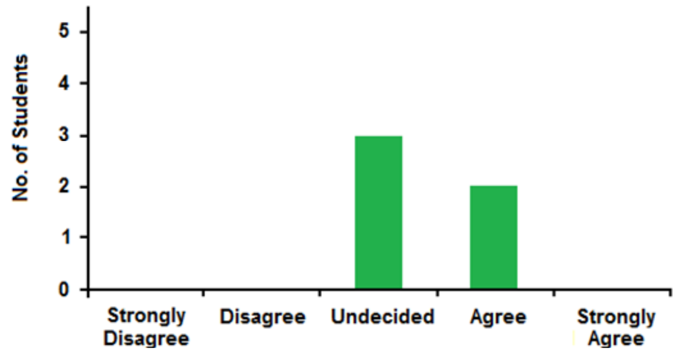


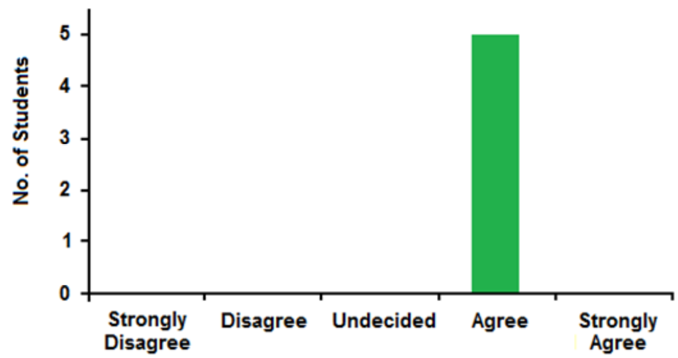
Figure 4.19 The Results from the Integration Questionnaire for Questions 1-2.

Integration Questionnaire: Results from Questions 3-5

Q. 3 *Before the workshops, I could only apply the technique of integration in problems, but now I understand why I use this technique.*



Q.4 *The workshops were clear and it could be a useful resource for students to access themselves.*



Q.5 *The time allocated to the workshop on slope was:*

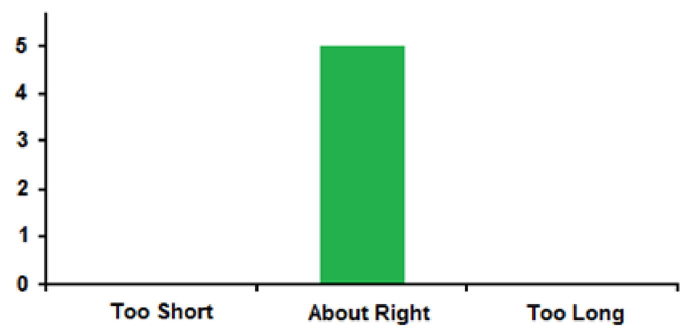


Figure 4.20 The Results from the Integration Questionnaire for Questions 3-5.

Integration Questionnaire: Results from Question 6

Q. 6 *Any other comments:*

- *I understand integration much better after doing these workshops.*
- *I found the gap was too big between each session; I would have benefited more from it, if it was all covered in the same week.*

Figure 4.21 The Results from the Integration Questionnaire for Question 6.

Summary: Students' Opinions

- In terms of Questions 1-2, which were basically asking the students the same question, namely: *did the intervention improve their understanding*, this was found, in general terms, to be the case for all the interventions.

- In terms of Question 3, the majority of the students agreed that their understanding in relation to why they use slope improved, but in respect of the interventions on derivative and integration, this was not the case. The latter result was somewhat surprising as the majority of students conveyed in Questions 1-2 that their understanding improved in respect of these concepts. Perhaps what they equated with understanding for Questions 1-2 was 'a how- to-do type of understanding' as opposed to a 'why-type of understanding'.

- For Question 4, the majority of the students either agreed or strongly agreed that all the interventions could be a useful resource for students to access by themselves.

- For Question 5, in terms of the interventions on slope and integration, the majority of the students considered the time allocated to each of the interventions to be 'about right'. For the interventions on derivative, two of the students felt that the time allocated was too short.

- In relation to the any-other-comments type question (Question 6), one notable comment emerged from the intervention on derivative: a student found the intervention to be a help but was still slightly confused with the topic.

4.3.3.2 Mathematical Understanding

As stated in Section 4.3.3., students' understanding of questions relevant to slope, derivative and integration in a mathematics context in light of the interventions was determined. Also, students' ability to link mathematical actions in a mathematics context with referents—be that embodied mathematical objects or mathematical objects—was determined. To investigate these questions, a series of mathematical questions were designed, and administered to the students.

Method of Investigation

Students' understanding of slope, derivative (in terms of the meaning of a derivative value and the meaning of a derivative function) and integration was investigated in a mathematics context. Four questions were designed in order to do so. The questions are located in Appendix K. Each of the questions contained a Part A and a Part B. The Part A allowed the investigation of whether the mathematical interventions improved students' understanding in respect of questions relevant to slope, derivative and integration in a mathematics context, while the Part B allowed the investigation of whether the students were able to link their mathematical actions in a mathematics context with referents—be that embodied mathematical objects or mathematical objects.

Using Tall's theory, the Part A of each question was classified as belonging to Tall's 2nd World. Therefore, the ability of students to answer each Part A was reflective of performing actions in a mathematical environment (as highlighted by the blue arrow in Figure 4.22). Students' ability to reflect on these mathematical actions (as highlighted by the green arrow in Figure 4.22) in terms of objects was probed in the Part B of each question.

For the Part B of each question, a *Think-Aloud Protocol* [95] was used to analyse the students' answers. When using the Think-Aloud Protocol, students are asked to explain what it is that they are doing and/or thinking, to themselves. An example of how to 'think-aloud' must be demonstrated to the students. Using a Dictaphone, the students' explanations were recorded. By not interrupting the students as they 'thought-aloud', the interviewer avoided interfering with the students' thought process.

For each Part B, there were a series of images which students had to explain. These images were considered to be embodied mathematical objects which the students could relate their mathematical actions in Part A to. It could be argued that these images guided the students towards linking their actions with objects, and this is true. However, the logic behind such a step was: the avoidance of interrupting the students' 'think-aloud' process by way of the students not having to ask the researcher questions such as how many diagrams they should draw. The analysis of the recorded interviews was undertaken using an inter-rater reliability approach [70].

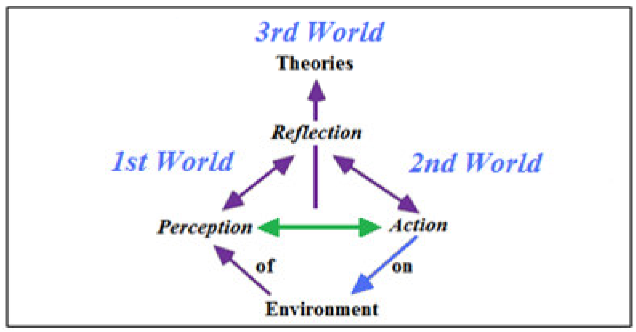


Fig. 4.22 The Cognitive Stages of Tall's Theory that were Probed (Highlighted in Blue and Green Arrows).

The Slope Question

The Part A aspect of the slope question is shown in Figure 1 in Appendix K. For the Part B aspect of the question, it wanted to be seen if students could link their actions (the calculation of slope) with mathematical objects/images as shown in Figures 2 and 3 in Appendix K. An example of the distinction between a student who was deemed to have evidenced an ability to link their actions with objects, from a student who did not, is shown in Table 4.16

Students were expected to link the slope value of three with three units up on the y-axis for every one unit across on the x-axis; and realise that this ratio of 3:1 was preserved in both Figures 2 and 3. Determining if students could do so involved an inter-rater reliability approach [70].

Part B Response	Linked Actions with Objects?
Am, both figures have the same value for the slope because no matter what two points you choose on the same line, they'll always give you the same value for the slope.	×
Although Figure 2 and Figure 3 . . . the points between . . . the distance between the two points are different, the ***ratio*** *in the change of x and y is the same, so therefore in Figure 2, the change for y = 9 and the change for y is 3 and the change for x equals 1, giving the same slope and . . . the same answer.*	√

Table 4.16 The Distinction between a Student who Evidenced an Ability to Link their Actions with Objects, from a Student who Did Not.

There were two questions in respect of the derivative. The first question probed students' understanding of the meaning of derivative, while the second question probed the students' understanding of the meaning of a derivative function.

The Meaning-of-Derivative Question

The Part A aspect of the question was:

$$\textit{Given the function: } y = 0.5x^2 \textit{, find the derivative } \frac{dy}{dx} \textit{ when } x = 0.3.$$

The Part B aspect required students to link the mathematical actions involved in finding the derivative value for the function: $y = 0.5x^2$ *when* $x = 0.3$, to the limiting process involved in such an action, namely: finding the limit of the quotient $\frac{\Delta y}{\Delta x}$ as Δx approaches zero and the related Δy approaches an infinitesimally small value, thus producing a value for the *slope of the tangent/derivative* at the point which has an x-coordinate equal to 0.3. The mathematical objects/images deemed suitable for explaining the process behind generating the derivative in Part A are shown in Figures 4-7 in Appendix K. Determining if students could do so involved an inter-rater reliability approach [70]. Examples of some of the students' responses for Part B are shown in Table 4.17. None of the students were judged to have evidenced an ability to link mathematical actions in Part A with objects in Part B.

Part B Response	Linked Actions with Objects?
To get the slope of a curve, you have to draw a line because you can't . . . am . . . get the slope . . . of a curve, so that's why you have to draw a tangent to the line and by doing that you find . . . am . . . the slope and the derivative.	×
I'm looking at four figures of curves on line graphs. Each of these curves includes a red line, cutting the curves at either one or more places. The derivative of the red . . . the derivative for each of the curves in these figures can be found by locating the slope for the red lines that intersect them.	×

Table 4.17 Two Student Responses for Part B of the Meaning-of-Derivative Question.

The Derivative-as-a-Function Question

The Part A aspect of the question was:

$$\textit{Find the derivative function} \frac{dy}{dx} \textit{ for the function: } y = 0.5x^2.$$

The Part B aspect required students to: describe ***how and why*** the graph of the derivative function which they found in Part A (shown in Figure 8 in Appendix K) could be used to find the slope of the tangents: L_1, L_2 and L_3 (shown in Figures 9-11 respectively in Appendix K) on the function: $y = 0.5x^2$. The Figures 8-11 were considered to be mathematical objects/images which explain the significance of the actions required to find the derivative function for the function: $y = 0.5x^2$. Determining if students could do so involved an inter-rater reliability approach [70].Examples of some of the students' responses for Part B are shown in Table 4.18. None of the students were judged to have evidenced ability to link mathematical actions in Part A, with objects in Part B.

Part B Response	Linked Actions with Objects?
By drawing the tangent on the curve and picking two points on the tangent . . . ah, you can get the slope of . . . the . . . am . . . derivative function.	×
The graph of the derivative function can be used to find the slope of the tangents as the slopes touch the lines in each figure L_1, L_2, L_3 in one position. This is due to the constant change in $\frac{dy}{dx}$ *. . . when graphed gives a constant curve and no straight line.*	×

Table 4.18 A Sample of Student Responses for Part B of the Derivative-as-Function Question.

The Integration Question

The Part A aspect of the question was:

$$\textit{Evaluate the integral: } \int_{1}^{6} (-x^2 - 10x + 150)\,dx$$

The Part B aspect of the question required the students to: 1) link the evaluation of the integral by drawing a sketch of its graphical representation in Figure 12, as shown in Appendix K; and 2) describe how the area is evaluated using the graph of the anti-derivative function (shown in Figure 13 in Appendix K).Because the Part B did not require the students to think-aloud, there was no need to use an inter-rater reliability approach. None of the students were able to answer Part B, so there are no examples of their responses.

Results

The students' results for the slope, derivative and integration questions are shown in Tables 4.19-4.22.

The Slope Question		
Student	*Part A:* Ability to Perform the Correct Mathematical Actions	*Part B:* Ability to Link the Part A Mathematical Actions with Referents
1	√	×
2	√	×
3	√	√
4	√	√
5	√	√
6	√	√

Table 4.19 Students' Performance in Respect of the Slope Question.

The Meaning-of-Derivative Question		
Student	*Part A:* Ability to Perform the Correct Mathematical Actions	*Part B:* Ability to Link the Part A Mathematical Actions with Referents
1	√	×
2	√	×
3	√	×
4	√	×
5	√	×
6	√	×

Table 4.20 Students' Performance in Respect of the Meaning of Derivative Question.

The Derivative-as-Function Question		
Student	*Part A:* Ability to Perform the Correct Mathematical Actions	*Part B:* Ability to Link the Part A Mathematical Actions with Referents
1	√	×
2	√	×
3	√	×
4	√	×
5	√	×
6	√	×

Table 4.21 Students' Performance in Respect of the Derivative-as-Function Question.

The Integration Question		
Student	*Part A:* Ability to Perform the Correct Mathematical Actions	*Part B:* Ability to Link the Part A Mathematical Actions with Referents
1	×	×
2	×	×
3	×	×
4	×	×
5	×	×
6	×	×

Table 4.22 Students' Performance in Respect of the Integration Question.

In terms of the slope question, meaning-of-derivative question and derivative-as-a-function question, all of the students were able to answer the Part A of the questions—the mathematical action-type questions. An ability to answer Part B for these questions—link mathematical actions with referents/objects—was only evidenced by four students for the slope question. None of the students evidenced understanding in terms of being able to link mathematical actions with referents/objects for the meaning-of-derivative question and the derivative-as-a-function question. In terms of the integration question, none of the students were able to complete the Part A or the Part B.

4.3.3.3 Transfer Ability

Students' ability to transfer, and whether students who linked mathematical actions with objects were more likely to transfer than other students, was determined. To investigate these questions, a series of transfer questions were used. These transfer questions were similar to the questions used to investigate students' mathematical understanding in a mathematics context; however, the questions were now in the form of a chemistry context. The transfer questions were administered to the students a week after they completed the questions used to investigate their mathematical ability. Thus, in contrast to Trial 1, where students' ability to transfer was evaluated one year later, during Trial 2, students' transfer ability was evaluated within the same year as when they participated in the intervention.

Method of Investigation

The transfer questions on slope, derivative (in terms of the meaning of a derivative value and derivative function) and integration mirrored the Part A aspect of these questions in a mathematics context (the questions used to investigate the effect of the interventions on students' mathematical understanding). Students who answered the Part A aspect of the questions in both the mathematics context and chemistry context correctly were deemed to have transferred. Because the number of students who completed the intervention was small, transfer—if it so happened to be observed—was not tested for significance.

The Slope Transfer Question

Students were presented with a question in the following form: Figure 4.23 shows the graph of the 'Volume of a Gas in Litres' against 'Temperature in Degrees Celsius'. The graph stems from the Ideal Gas Law which states that for an ideal gas, when the number of moles (n) of the gas, and the atmospheric pressure (P) remain constant, the volume (V) of the gas, in litres, is directly proportional to the temperature (T) of the gas, in Degrees Celsius or Degrees Kelvin. The relationship can be expressed with the following functional relationship:

$$V = \frac{nRT}{P}$$

where R is the universal gas constant. Students were then presented with the graph in Figure 4.23 and were asked to calculate the slope of the line using the two points in question.

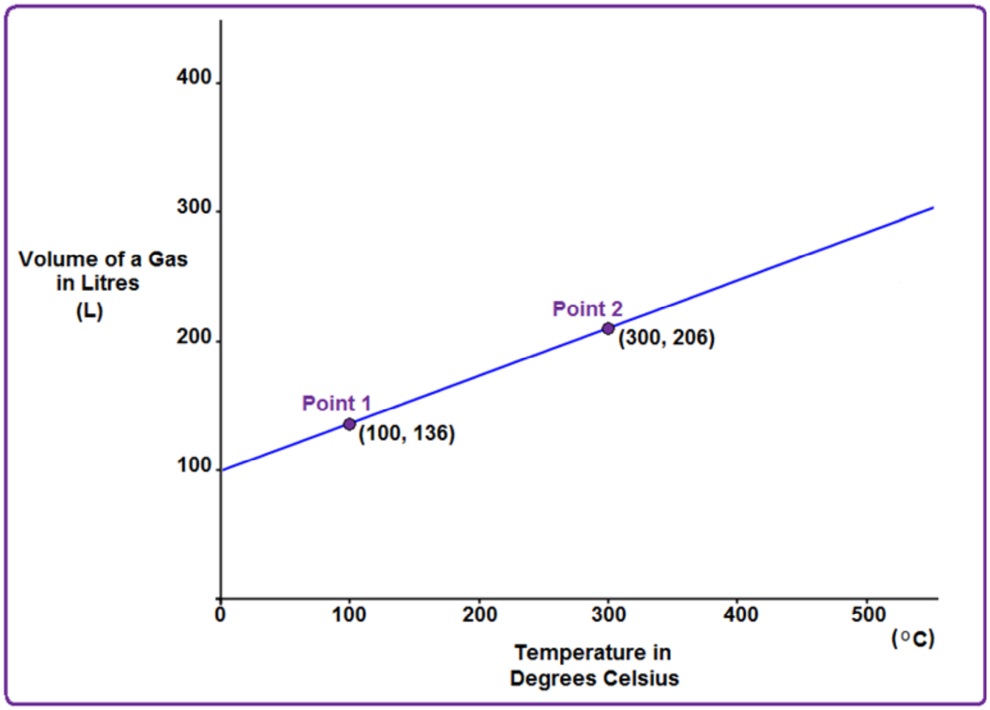


Figure 4.23 The Graph Students were Shown for the Transfer Question on Slope.

The Meaning-of-Derivative Question

Students were presented with a question in the following form: Figure 4.24 shows the graph of the 'Volume of a Gas in Litres' against its 'Pressure in Atmospheres'. The graph stems from the ideal gas law, which states that for an ideal gas, when the number of moles (n) of the gas and the temperature (T) remain constant, the volume (V) of the gas is inversely proportional to the pressure (P) of the gas. The relationship can be expressed with the following functional relationship:

$$V = \frac{nRT}{P}$$

Students were then presented with the graph of the function in Figure 4.24. They were told 'nRT' is equal to 1 and that the functional relationship could be written as follows:

$V = \frac{1}{P} = P^{-1}$. Subsequent to this, the students were then asked to find the derivative $\frac{dV}{dP}$, when $P = 0.5$.

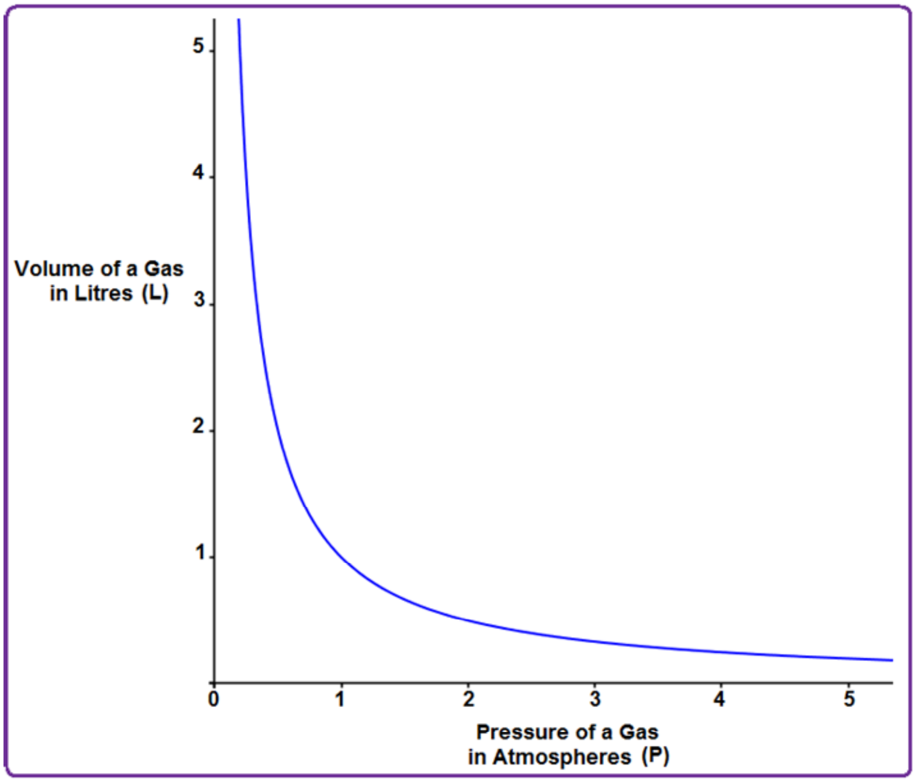


Figure 4.24 The Graph Students were Shown for the Meaning-of-Derivative Question.

The Derivative-as-a-Function Question

Students were presented with the same functional relationship and graph, shown in Figure 4.24. However, this time, the students were asked to calculate the derivative function $\frac{dV}{dP}$ for the function: $V = \frac{1}{P} = P^{-1}$

The Integration Question

Students were presented with Expression 1. They were told: 1) the pressure (P) of a gas inside an ideal cylinder (which allows for expansion due to it being frictionless) is inversely proportional to the volume (V) of the gas; and 2) it can be shown that: $P = k\frac{1}{V}$, where 'P' is the pressure of the gas (in Newtons per square metre);'V' is the volume (in metres cubed) of the cylinder which the gas occupies, and k is a constant equal to ‘nRT’ with, in this case, a value equal to 1 . The students were asked to evaluate the integral in Expression 1. Furthermore, they were told that Ln V is the integral of $\frac{1}{V}$.

$$\int_{0.002}^{0.006} \frac{1}{V} dV$$

Expression 1

Results

The results in respect of students’ ability to transfer are shown in Tables 4.23-4.26.

	The Calculation of Slope	
Student	*Correct in a Mathematics Context*	*Correct in a Chemistry Context*
1	√	×
2	√	×
3	√	√
4	√	√
5	√	√
6	√	√

Table 4.23 Students’ Ability to Transfer the Calculation of Slope.

	The Calculation of a Derivative Value	
Student	*Correct in a Mathematics Context*	*Correct in a Chemistry Context*
1	√	×
2	√	√
3	√	√
4	√	×
5	√	×
6	√	√

Table 4.24 Students’ Ability to Transfer the Calculation of a Derivative Value.

The Calculation of a Derivative Function		
Student	*Correct in a Mathematics Context*	*Correct in a Chemistry Context*
1	√	×
2	√	√
3	√	√
4	√	×
5	√	×
6	√	√

Table 4.25 Students' Ability to Transfer the Calculation of a Derivative Function.

The Evaluation of an Integral		
Student	*Correct in a Mathematics Context*	*Correct in a Chemistry Context*
1	×	×
2	×	×
3	×	√
4	×	×
5	×	×

Table 4.26 Students' Ability to Transfer the Evaluation of an Integral.

For the calculation-of-slope question, four out of the six students transferred it. All of the students answered the question on the calculation of derivative correctly in a mathematics context; with three of these students being able to transfer. Similarly, all of the students could calculate the derivative function in a mathematics context correctly; with three of these students being able to transfer. None of the students could answer the question on the evaluation of an integral in a mathematics context correctly; however, one of the students answered it correctly in a chemistry context.

The results in respect of whether students who were able to link their mathematical actions with objects were more likely to transfer than students who could not are shown in Tables 4.27-4.30.

The Calculation-of-Slope Question		
Student	*Part B:* Ability to Link Mathematical Actions with Referents	Transfer
1	×	×
2	×	×
3	√	√
4	√	√
5	√	√
6	√	√

Table 4.27 Students' Ability to Transfer the Calculation of Slope in Light of Linking Mathematical Actions with Referents.

The Calculation of Derivative Value		
Student	*Part B:* Ability to Link Mathematical Actions with Referents	Transfer
1	×	×
2	×	√
3	×	√
4	×	×
5	×	×
6	×	√

Table 4.28 Students' Ability to Transfer the Calculation of Derivative Value in Light of Linking Mathematical Actions with Referents.

The Calculation of Derivative Function		
Student	*Part B:* Ability to Link Mathematical Actions with Referents	Transfer
1	×	×
2	×	√
3	×	√
4	×	×
5	×	×
6	×	√

Table 4.29 Students' Ability to Transfer the Calculation of a Derivative Function in Light of Linking Mathematical Actions with Referents.

The Evaluation of an Integral		
Student	*Part B:* Ability to Link Mathematical Actions with Referents	Transfer
1	×	×
2	×	×
3	×	×
4	×	×
5	×	×

Table 4.30 Students' Ability to Transfer the Evaluation of an Integral in Light of Linking Mathematical Actions with Referents.

Interestingly, looking at Table 4.27, for the four out of six students who were able to link mathematical actions with referents for the calculation of slope in a mathematics context, all of these students transferred. The two students who were not able to link mathematical actions with referents were correct in the mathematics context, but unable to transfer.

For the calculation-of-derivative-value question and the calculation-of-derivative-function question (shown in Tables 4.28 and 4.29 respectively), all of the students were not able to link mathematical actions with referents. However, for both questions, the students answered the question correctly in a mathematics context, with three students being able to transfer.

For the evaluation-of-integral question (shown in Table 4.30), all of the students were not able to link mathematical actions with referents, and none of the students were able to transfer. Furthermore, none of these students were able to calculate the integral correctly in a mathematics context or the chemistry context.

4.3.4 Conclusion: Trial 2

The conclusion that arose from Trial 2 is discussed in the context of the questions used to evaluate it:

- In terms of: *what were the students' opinions about the usefulness of the mathematical interventions,* the majority of the students articulated that the interventions were of benefit in improving their understanding. However, what they appear to equate understanding with is 'how to do' as opposed to 'why'.

- In terms of: *what was students' understanding in respect of slope, derivative and integration in a mathematics context*, the results were mixed. All of the students were able to answer the Part A of the slope question, the meaning-of-derivative question and the derivative-as-a-function question. None of the students were able to answer the Part A of the integration question. It should be noted that the Part A of all these questions were classified as belonging to Tall's 2nd World, thus requiring students to perform mathematical actions.

- In terms of: *were the students able to link mathematical actions in a mathematics context with referents*, only four students were able to do so for one of the questions, namely the slope question.

- In terms of: *were the students able to transfer to a chemistry context*, the interventions on slope, meaning of derivative and derivative as a function may or may not have been of help. For the integration intervention, the students were unable to answer in a mathematics context and therefore could not be expected to transfer.

- Lastly, in terms of: *did students who were able to link their mathematical actions with objects, transfer more so than students who did not link their mathematical actions with objects,* this appeared to be the case for only one of the items—the item on slope.

4.4 Chapter Summary

It can be seen that the design of the mathematical interventions was informed by Tall's theory of mathematics education. Particular attention was paid to the cognitive aspect of Tall's theory. Insofar as was possible, the starting points for the introduction of each intervention was an *image/embodied mathematical object*. The *images/embodied mathematical objects* were embedded in what were deemed to be real-world contexts.

The results from the evaluation of Trial 1 of the Intervention were mixed. Students' did not appear to understand questions in respect of slope, derivative and integration, in a mathematics context. Certain students were able to transfer items related to slope, derivative and integration. However, it was not possible to determine whether the interventions played a part in students' transfer ability, as transfer for items related to slope, derivative and integration was observed in Study 1 and Study 2.

The results from Trial 2 of the Intervention were somewhat similar to those from Trial 1. The majority of the students agreed that the interventions were beneficial. The majority of the students were able to answer questions related to slope and derivative, but were unable to answer the integration question. Whether the interventions had an effect in this regard is not possible to determine.

In Trial 2, the interventions did not appear to have an effect on students' ability to link mathematical actions with referents for derivative and integration questions. However, for the slope question, certain students were able to link their actions with referents; furthermore, all of these students were able to transfer this question, so the intervention on slope may have had a positive effect on students' understanding of slope.

Lastly, in Trial 2, certain students were able to transfer questions related to slope and derivative, but not the integration question. Whether students' ability to transfer slope and derivative was a consequence of the interventions, was not possible to determine.

Conclusions and Implementations

The research was undertaken in two phases. In Phase 1, the *Transfer Question* and the *Explaining and Transfer Question* were investigated. In Phase 2, the effect of an Intervention programme designed to improve students' mathematical ability, with a view that this would improve students' ability to transfer, was investigated. Overall, the research undertaken contributes to knowledge in the field of mathematical transfer in a number of ways. Firstly, the research design behind the investigation of the research questions can be used to investigate undergraduate students' ability to transfer other mathematical knowledge in chemistry. Such a design could also be used to investigate undergraduate students' ability to transfer mathematical knowledge to contexts other than a chemistry context. The results from such investigations can be used to inform the design of interventions aimed at improving students' mathematical understanding, as was the case in Phase 2 of this project.

In terms of the *Transfer Question*, it was found that transfer can occur, and for certain mathematical items this transfer is significant. During the Pilot Study, students' ability to transfer nine mathematical items was probed. Transfer was observed for Items 1-7 but not for Items 8-9. Statistically significant transfer was only observed for one of these items (Item 7). During Study 1 in the Main Study, students' ability to transfer fifteen mathematical items was investigated. Transfer was observed for all of these items. For nine of the items, the transfer observed was significant, while the significance of the transfer observed for one of the items was borderline ($0.05 < \text{p-value} < 0.1$).

The observation of transfer in this study is at odds with the views of Detterman [71], and Krishner and Whitson [73] who claim that traditional approaches to transfer studies often fail to demonstrate transfer. However, transfer was consistently observed across both the Pilot Study and Main Study. From a Barnett and Ceci perspective [40], perhaps the reason why transfer was observed is because the transfer was near as opposed to far? Or, perhaps transfer is context and concept dependant? The investigation of the *Transfer Question* also raised a number of other questions:

- For the students who were able to answer correctly in a mathematics context but not in a chemistry context, why was this so? Perhaps the students made minor errors in a chemistry context?; or perhaps the students need an understanding of the chemistry concepts in the chemistry context?

- For the students who were able to answer correctly in a chemistry context but not in a mathematics context, why was this so? The Part B results for the students who answered incorrectly in a mathematics context would suggest that this was not because these students made minor errors in a mathematics context. Perhaps the students simply remembered the mathematical knowledge which they were exposed to in a chemistry context without any real understanding of it?

Future research could use the actor-oriented view of transfer to investigate these questions. The overall conclusion reached from the investigation of the *Transfer Question* was: the problem which students have with mathematics in a chemistry context may not always be due to an inability to transfer. Instead, the problem is due, in significant part, to a lack of mathematical knowledge in a mathematics context, thus agreeing with the main conclusion of Potgieter et al. [37] in their investigation of undergraduate students' ability to transfer mathematical knowledge, relevant to the Nernst equation, from a mathematics context to a chemistry context.

During Study 2, when students were reminded of 'how to do' the mathematical items in a mathematics context before being presented with the mathematical items in a chemistry context, transfer was observed for fourteen out of fifteen of the items. Statistically significant transfer was observed for two of the items. Less instances of statistically significant transfer was observed in Study 2, when compared with Study 1. Moreover, reminding students of 'how to do' mathematics in a mathematics context, before investigating if the students can transfer the knowledge, does not appear to be a factor which improves students' ability to transfer. Such a finding contradicts the sentiments of Hann and Polik [6] who argue that including a mathematics review session near the beginning of a chemistry course improves students' ability to transfer.

In terms of the *Explaining and Transfer Question*, the precursor to this question (the *Conceptual versus Procedural Question*) was investigated during the Pilot Study. It was found that based on the presupposition that procedural knowledge is symbolic in nature and conceptual knowledge is graphical in nature, conceptual knowledge is not transferred by students any more so than procedural knowledge. This suggests that the views that conceptual knowledge in a mathematics context is graphical in nature while procedural knowledge is symbolic in nature [17,37,57] may not be correct. It may be not possible to objectively classify knowledge in a mathematics context as procedural or conceptual. Also, the view by Hiebert and Lefevre [51] that conceptual knowledge transfers more easily than procedural knowledge is open to question. If conceptual mathematical knowledge and procedural mathematical knowledge were defined differently during the Pilot Study, then conceptual mathematical knowledge might have been found to transfer more easily than procedural. Nonetheless, the *Conceptual versus Procedural Question* evolved into the *Explaining and Transfer Question*.

In terms of the 1st aspect of the *Explaining and Transfer Question*, in Study 1, it was found that students who explained their reasoning in a mathematics context for eight out of the eleven mathematical items requiring an explanation associated with the transfer of these items. In Study 2, students were required to explain their reasoning for all of the mathematical items in a mathematics context. Students who evidenced any form of explanation for eleven of these fifteen items in a mathematics context associated with the transfer of them. Thus, it was concluded that a possible reason as to why students can transfer is because they can explain their reasoning in a mathematics context. This finding adds to knowledge in the field of transfer, in terms of determining, what Barnett and Ceci [40] would term the factors which influence students' ability to transfer.

To determine the degree to which students explained (the 2nd aspect of the *Explaining and Transfer Question*) Tall's theory [62] was used. A number of interesting findings in respect of Item 1 (Calculating Slope) emerged. It was found that during Study 1 and Study 2, students who evidenced a correct action-perception category of explanation for the meaning of slope in a mathematics context (Categories 1-3 for Study 1 and Categories 1-5 for Study 2, as shown in Table 1) were more likely to transfer Item 7 (Interpreting Derivative) than other students; furthermore, they were more likely to answer Item 7 correctly in a mathematics context. Could it be that explaining the calculation of slope in such a manner is necessary to transfer Item 7 (which is related to slope) to not just a chemistry context but to other contexts such as physics and business for example? Also, could it be that such an understanding is a key ingredient in developing what Gill [17] terms 'Graphicacy', which he found to be an apparent factor in explaining students' ability to transfer in mathematics? Future research could investigate this.

Study 1		Study 2	
Category	**Frequency**	**Category**	**Frequency**
1. Refer to how much y increases for a unit increase in x.	10	1. Refer to how much y increases for a unit increase in x.	3
2. Refer to slope as a measure of steepness.	1	2. The slope represents the rate of increase of the line.	5
3. Refer to the rate at which the line increases.	3	3. The bigger the number, the steeper the slope is.	2
		4. The change of x relative to y, how steep a line is.	1
		5. It is the difference between the heights over the difference between the two length points.	2

Table 1 The Correct Action-Perception Categories of Explanation Furnished by Students for Item 1.

In both Study 1 and Study 2, the Category 1 students (as can be seen in Table 1) were likely to transfer Item 5 (Generating an Expression for Slope) and Item 10 (Proportionality) more so than other students (if borderline significance is accepted [$0.05 < \text{p-value} < 0.1$]). For the students who evidenced this category of explanation in a mathematics context, in Study 1, these students were likely to answer Item 7 (Interpreting Derivative) and Item 10 (Proportionality) in a mathematics context more so than other students. In Study 2, students were not likely to transfer any of the items related to slope more so than other students. Thus, it would appear that these correct categories of explanation (action-perception categories of explanation in terms of Tall's theory) may indicate what previous literature would term a deep understanding of slope [75,76,77,78]. However, it is important to note that when applying Tall's theory, such an application was subjective. External validation of such an application would be needed.

In Phase 2, mathematical interventions in respect of slope, derivative and integration were designed to improve students' mathematical understanding. The findings from Phase 1 of the research project informed the design of these interventions. It was anticipated that the interventions would improve students' ability to transfer. The mathematical interventions, in Trial 1 and Trial 2, were evaluated in terms of: 1) did the students find the interventions useful; 2) what was the students' understanding of slope, derivative and integration in a mathematics context and 3) were the students able to transfer to a chemistry context.

During both Trial 1 and Trial 2, the students articulated that they found the Intervention to be of benefit. In terms of the Intervention having an effect on students' understanding of slope, derivative and integration in a mathematics context, this appeared not to be the case in Trial 1. For Trial 2, the intervention may have had an effect on students' understanding of slope and derivative (many students answered questions in respect of these concepts correctly in a mathematics context). In Trial 2, the interventions did not have an effect on students' understanding of integration. How such interventions could be redesigned in light of these results is discussed. In many ways, modifying the interventions as a result of their implementation is reflective of the Realistic Mathematics Education (RME) movement in mathematics education [27]. Indeed, the following suggested modifications of the mathematical interventions add to existing knowledge in the field of RME.

In respect of the *Slope Intervention* which appeared to have a limited impact on the understanding of some students, perhaps redesigning the intervention, with an emphasis on developing the idea that rate (or slope) can vary, would be beneficial. This would have involved using curves as well as lines. Oehrtman et al. [65] state that students are slow to develop an ability to interpret varying rates of change over intervals of a function's domain. If a focus was put on this, perhaps students would have been able to understand both the slope question and derivative questions better? Confrey and Smith [91] also stress the importance of developing the idea that rate can vary, believing that ability to recognise variation in a rate of change is essential for the transition to calculus. The slope intervention only focused on what Confrey and Smith [91] would term the ratio concept of slope (the slope of a line as opposed to the slopes at various points on a curve).

For the students who did not appear to benefit from the derivative and integration interventions, perhaps designing an intervention aimed at improving these students' understanding of function before the students are exposed to the derivative and integration interventions would be beneficial. Oehrtman et al.[65, p.151] state that "a strong understanding of the function concept is essential for any student hoping to understand calculus". The idea of variable and co-variation could also be targeted if such an intervention were to be designed. Confrey and Smith [91] believe that a co-variation approach is central to the rate concept. Furthermore, Oehrtman et al. [65] articulate that a co-variation view of function has been found to be essential for understanding critical concepts of calculus, for example; average and instantaneous rates of change. Carlson [96, p.141] also states that "function constructs among students develop slowly and their development appears to be facilitated by reflection and constructive activities".

In respect of the students who did not benefit from the derivative interventions, perhaps emphasising the idea that rate can vary, in a redesigned slope intervention, and exposing these students to an intervention on function, would be beneficial for these students. Also, there may be a need to design a separate intervention on the meaning of limit. Elai et al. [97] argue that the limit concept is a fundamental concept, and a failure to grasp it will mean that students will not have an understanding of continuity and derivative. Moreover, Orton [86] states that the topic of limits seems to be neglected in spite of the fact that they are "important to a real understanding of integration and differentiation" [86, p.5].

In relation to the integration intervention, the fact that there was a lack of emphasis on the variation of rate, function and limits may have had a knock-on effect in terms of students not being able to engage with the integration intervention, and apply it to questions. Also, the use of the terms velocity, speed and displacement may have confused students. Rowland and Jovanoski [81], in their research on student understanding of kinematics graphs, velocity and acceleration, found that many students cannot distinguish between distance, velocity and acceleration. Thus, in a future integration intervention, it would be wise to ascertain whether students are able to distinguish the terms velocity, displacement and time.

All of the interventions were insofar as possible visual in nature. Such an emphasis was due to the views of various theories on how students learn mathematics, claiming that visualisation is important in improving students' understanding. Despite this emphasis, the impact of the interventions was limited; Piaget's theory of Cognitive Development may explain why.

From a Piagetian perspective, the interventions would require students to be at a formal-operational stage of cognitive development in order to complete them. Work by McCormack [98] in respect of measuring the cognitive development levels/stages of a sample of 1st year university science students, showed that almost 70% of students were at a level capable of formal operational thought. However, only a very small minority of the sample of students (7%) showed capability of *late formal operational thought* necessary for meaningful engagement and understanding of many scientific and mathematical concepts such as proportionality and modelling.

Perhaps the students who participated in the Intervention were at an early level of formal operational thought or a concrete stage of cognitive development? If so, it would suggest that concrete referents (referents which students can physically manipulate) in respect of each of the concepts may have needed to be designed. Studies have shown [99] that a pictorial and visual focus does not improve students' understanding of abstract mathematical ideas any more than a symbolic approach; instead, it is concrete referents over an extended period of time which improves students' understanding of mathematical concepts. Designing such 'manipulatives' could be an avenue for future research. It is important to note that such a development does not preclude the symbolic aspect of mathematics. Rather, it is important for students to see the two-way relationship between concrete materials/manipulatives and the symbolic systems which they represent [100]. How exactly, such referents could be designed is an area that could be researched. Also, whether such manipulatives improve students' ability to transfer could be investigated.

In respect of the interventions effecting students' ability to transfer, during both Trial 1 and Trial 2, certain students were able to transfer mathematical items related to slope and derivative. However, it is not possible to say whether the interventions had a direct impact on students' ability to transfer these items because during the Main Study, many students were able to transfer items related to slope and derivative.

For the questions related to integration, only one student transferred during Trial 1 and no student transferred during Trial 2. It would appear that the intervention on integration had a limited effect in this regard. However, during the Main Study, few students transferred the items related to integration, so it is difficult to gauge the effectiveness of the integration intervention. Future research could take an actor-oriented approach in terms of ascertaining what it is that the students see as similar or different between the mathematics context and chemistry context for each transfer item. Such an approach could also allow the investigation of what the students' views are in terms of the interventions helping the students to see two contexts as similar in order to transfer.

During Trial 2, there was a particularly interesting finding that emerged during the investigation of students' ability to transfer slope. All of the students who evidenced an ability to link their mathematical actions with referents associated with transfer. From a Lobato perspective [34], the intervention on slope may be an effective instructional treatment that enables students to transfer to different contexts, thus adding to literature in the field of mathematical instructional materials that aim to promote such an occurrence. Perhaps if students could have linked their mathematical actions for items on derivative and integration, they may have transferred? How exactly this can be accomplished, remains the preserve of future research.

To summarise, the key findings from this research are:

- The problems which undergraduate students have with mathematics in a chemistry context appear not to be due solely to students' inability to transfer. Instead, the problems are due, to a significant degree, to a lack of mathematical knowledge.
- Students can successfully transfer some mathematical knowledge and skills.
- Evidencing an ability to explain in a mathematics context appears to be a factor which underpins successful transfer by students.

- Designing mathematical interventions that are strongly visual in nature may have an effect on students' understanding of slope and derivative questions in a mathematics context. They may also have an effect on students' ability to transfer slope and derivative questions. In terms of the intervention having an effect on students' understanding of integration in a mathematics context, and students' ability to transfer integration, the interventions did not appear to have an effect during this research.

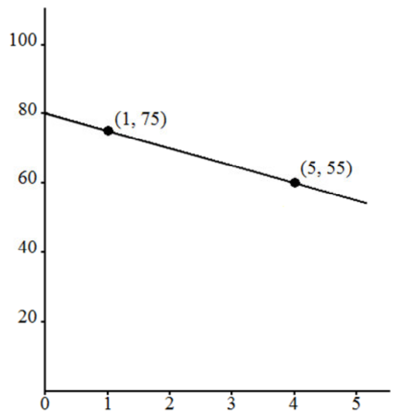
Appendices

Appendix A – Pilot Study Mathematical Items

Mathematical Items Used in Diagnostic Tools 1 and 2

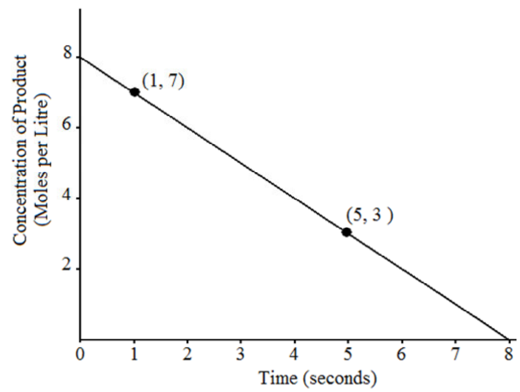
Item 1: Calculating Slope

Mathematics Context



(A) Calculate the slope for the line between the point (x, y) = (1,75) and the point (x, y) = (5,55).

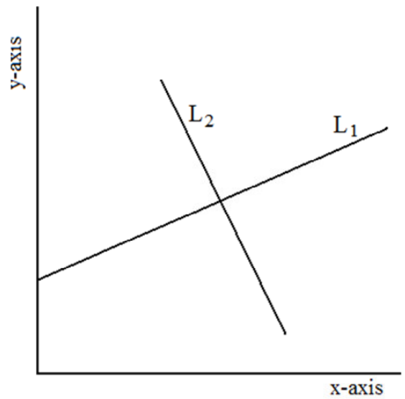
Chemistry Context



(A) Determine the rate of change of reactant between one second and five seconds. Express your answer using appropriate units.

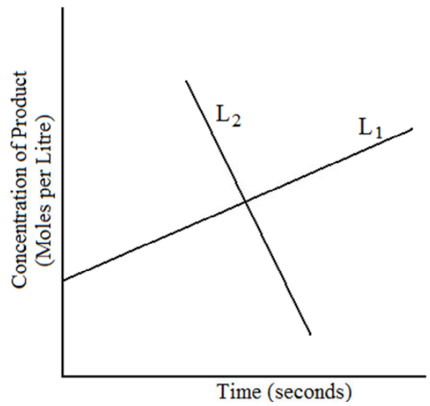
Item 2: Determining which Line has the Greatest Rate of Change

Mathematics Context



(A) Which of the two lines has greater slope, L_1 or L_2?

Chemistry Context



(A) The line L_1 represents the rate of change of product with respect to time, while the line L_2 represents the rate of change of reactant with respect to time. Which line L_1 or L_2 has the greatest rate of change with respect to time?

Item 3: Differentiation

Mathematics Context

(A) By differentiating the expression:

$$f(x) = 3x^2 + 1$$

find the derivative of f(x) at x = 3.

Chemistry Context

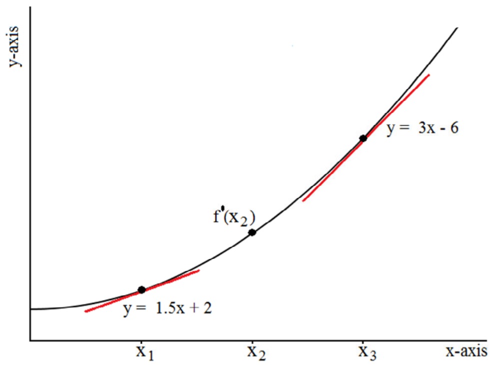
(A) The change in the concentration of a reactant with respect to time is given by the following expression:

$$R(t) = -3t^2 + 1$$

Using differentiation, find the instantaneous rate of change of reactant after four seconds.

Item 4: Graphical Interpretation of the Meaning of Derivative

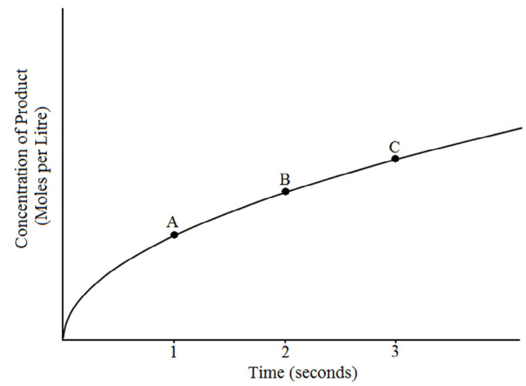
Mathematics Context



(A) Chose a value from the list of values given (A, B or C) for the derivative of the graph at the point X_2.

(A) = 1 (B) = 3.5 (C) 1.75

Chemistry Context



(A) Using the graph shown, rank the instantaneous rates of change of product after one second, two seconds and three seconds in order of increasing magnitude.

Use the notation: $P'(1)$, $P'(2)$ and $P'(3)$ to represent the instantaneous rate after 1, 2 and 3 seconds respectively.

Item 5: Multiplication of Fractions

Mathematics Context

(A) Express the following in its simplest form:

$$\frac{2}{3} \times \frac{1}{4} = ?$$

Chemistry Context

(A) Express the following in its simplest form:

$$\frac{2k_a k_b [O_2]}{k_b} \times \frac{[NO]^2}{[O_2]} = ?$$

Item 6: Use of Exponent Laws

Mathematics Context

(A) Express the following in its simplest form:

$$\frac{a^{5/2}}{b} \times \frac{c}{a^2} = ?$$

Chemistry Context

Rate of formation of HBr:

$$\frac{2k_b \left(\frac{k_a}{k_b}\right)^{1/2} [H_2][Br_2]^{3/2}}{[Br_2] + \frac{k_c}{k'_b}[HBr]}$$

(A) Derive an expression for the initial rate of formation of HBr if the concentration of [HBr] becomes much smaller than [Br_2], so much so that the value for the term:

$$\left(\frac{k_c}{k'_b}[HBr]\right)$$

can be taken to be zero in comparison with [Br_2].

Mathematical Items Used in Diagnostic Tools 3 and 4

Item 7: Graphing a Function

Mathematics Context

(A) Sketch (in Figure 1), the graph of the function:

$$y = \frac{1}{x} \quad \text{for } x > 0$$

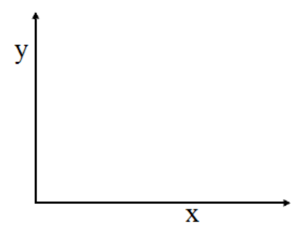


Figure 1

Chemistry Context

(A) Sketch (in Figure 1), the graph of P versus V, for $0 \text{ m}^3 < V < 5 \text{ m}^3$ given the relationship: $P = \frac{nRT}{V}$, where nRT has a constant value = 1kJ.

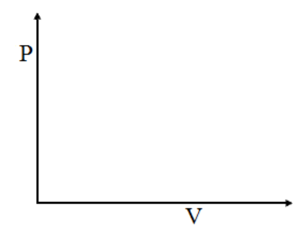


Figure 1

Item 8: Evaluation of an Integral

Mathematics Context

(A) Evaluate the integral:

$$\int_{1}^{3} \frac{1}{x} dx$$

Chemistry Context

(A) According to the ideal gas equation, the pressure of a gas is given by:

$$P = \frac{nRT}{V} \quad \textit{Eqn. 1}$$

For a reversible isothermal gas expansion, the variable in *Eqn.1* is volume (V), while *nRT* remains constant. When *nRT* is equal to 1kJ, the relationship in *Eqn. 1* can be written as follows:

$$P = \frac{1}{V}.$$

Calculate the work done when the volume of a reversible isothermal gas increases from:

$$1m^3 \text{ to } 3m^3,$$

given that the work will be equal to the integral of the expression:

$$w = \int_{V_1}^{V_2} \frac{1}{V} dV$$

where:

$$V_1 = 1m^3 \text{ and } V_2 = 3m^3,$$

and the minus sign is used to denote the fact that the work leaves the system.

Item 9: Graphing an Integral

Mathematics Context

(A) Draw a diagram (in Figure 1) that represents the area corresponding to the integral:

$$\int_1^3 \frac{1}{x}dx$$

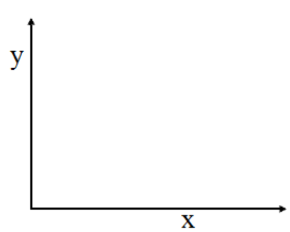


Figure 1

Chemistry Context

(A) According to the ideal gas equation, the pressure of a gas is given by:

$$P = \frac{nRT}{V} \quad \textit{Eqn. 1}$$

For a reversible isothermal gas expansion, the variable in *Eqn.1* is volume (V), while *nRT* remains constant. When *nRT* is equal to 1kJ, the relationship in *Eqn. 1* can be written as follows:

$$P = \frac{1}{V}.$$

Indicate in Figure 1, the area corresponding to the integral:

$$w = -\int_{V_1}^{V_2} \frac{1}{V}dV$$

which represents the work done by the system (the gas) in expanding from an initial volume:

($V_1 = 1m^3$) to a final volume ($V_2 = 3m^3$),

for a reversible isothermal gas expansion. The minus sign is used to denote the fact that the work leaves the system.

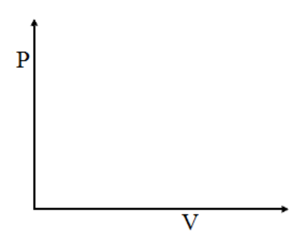


Figure 1

Appendix B – Piaget's Theory of Cognitive Development

The Sensori-Motor Period — The First Stage of Cognitive Development:

The Sensori-Motor Period underlies all subsequent stages of cognitive development. It occurs over the space of the first two years of a child's life. Piaget divides it into six sub-stages. Within these sub-stages, a child's 'object concept' and 'concept of causality' undergo development. For Piaget, the evolution of these concepts are "two of the most important indicators of intellectual development during this period" [63, p.35].

Stage 1

Object Concept

During this stage, a child has no awareness of objects, and is unable to differentiate between him/her and their environment. For example, any object presented to the child encourages a similar reflexive response such as sucking or grasping—the responses are undifferentiated.

Concept of Causality

Interestingly, the child is "totally egocentric" [63, p.35] during this stage, being not aware of causality.

Sage 2

Object Concept

A child begins to look at objects which they hear. For Piaget, this indicates that the child is beginning to co-ordinate their vision and hearing schema. Furthermore, the child may continue to follow the path of an object with their eyes after it has disappeared from view.

Concept of Causality

Wadsworth does not describe explicitly what happens a child's concept of causality during this stage. However, he draws attention to the 'concept of intentionality' which may implicitly be describing an aspect of the concept of causality at this stage. For Piaget, at this stage, a child's behaviour still lacks 'intention' whereby they imitate behaviour of their own accord in order to attain ‘certain ends’. Perhaps what could be inferred from this statement is that the child is still egocentric and may not be able to see how they can cause something to happen.

Stage 3

Object Concept

The child is capable of anticipating or predicting the “positions [that] objects will pass through while they are moving” [63, p.46]. In addition, their awareness of objects as possessing a degree of permanence is developing.

Concept of Causality

The child still remains egocentric, but to a lesser degree than in previous stages. They see themselves as the primary cause of all activity.

Stage 4

Object Concept

At this stage, the shape and size of objects develop a sense of stabilisation for the child. This new awareness of object permanence is different from that of early stages. Wadsworth [63, p.48] describes how Piaget uses the example of a rattle to explicate the point. If a rattle is placed under a carpet or rug in front of a child before Stage 4, they will not look for it; however, at Stage 4, they will. In spite of this will to search, the child may only search for objects where they are habitually observed to disappear, as distinct from where they have been observed by the child at a specific moment in time to disappear.

Concept of Causality

A child’s concept of causality is becoming less egocentric. They begin to “discover that a spatial contact exists between cause and effect” [63, p.52]. Any object can be “a source of activity” [63, p.52], as distinct from a child’s body, which was deemed by the child to be the cause of all activity in their previous stages of development.

Stage 5

Object Concept

A child learns to account for sequential displacements of an object. A child can search for objects in the position resulting from their last visible displacement, as distinct from an habitual displacement. For example, when a rattle is hidden in A, it is searched for in A, as distinct from place B, where it might habitually be viewed to be hidden.

While this sequential displacement comes to the fore at this stage, it is not yet fully developed. It is limited in the sense that the child is only competent in following displacements that appear visible, as distinct from displacements which appear invisible. Put simply, if a child views an object to be hidden in A, they will search for it in A. If they do not find it there, they may not realise that it could have moved to a new location called B, [displaced to B] — a displacement which they did not observe.

Concept of Causality

A child's concept of causality becomes aware that people apart from themselves can affect activity. They also retain the view that other objects can cause activity.

Stage 6

Object Concept

The child maintains 'images' of objects when they are absent [63, p.59]. In addition to following sequential displacements, they are now in a position to follow invisible displacements. In summary, the child knows that objects are permanent.

Concept of Causality

The child possesses the means to reconstruct causes "in the presence of their effects alone without having perceived the action of those causes" [63, p.59]. Just as they are able to infer the causation of effects from observing effects alone, they become capable of predicting the effects of objects acting in a certain way. In summary, they are capable of causal deduction [supposedly in the physical-object sense] and are as Wadsworth describes, [63, p60] "no longer restricted to perception or sensori-motor utilisation of the relations of cause to effect".

The Preoperational Thought Period — The Second Stage of Cognitive Development

This period is characterised by the development of symbolic representation, which, as Wadsworth describes, [63, p.65] facilitates "the very rapid conceptual development that takes place during this period". The period typically lasts from age two to age seven. Piaget views the development of language during the preoperational period "as a gradual transition from egocentric speech to intercommunicative speech" (the distinction between the two terms being described later on).

Furthermore, as language develops and its use becomes intercommunicative by the child, its use helps to add impetus to the development of conceptual activity more rapidly than sensori-motor operations allow. "Language permits the child to simultaneously handle many elements in an organised manner" [63, p.68], as compared with sensori-motor intelligence, which "proceeds in a one-step-at-a-time fashion" [63, p.68].

The reason Piaget *creates/has found* this 'preoperational stage' is because, based on his observations, it accounts for the transition children undergo in moving from the sensori-motor stage of cognitive development to the concrete operational stage of cognitive development (the third stage of cognitive development), which encompasses logical development.

For Piaget, there are obstacles, which have to be overcome before students can reach the concrete-operational stage. These obstacles and how they are overcome are described in the preoperational stage of cognitive development.

Egocentrism

The egocentricity of the preoperational child means that they do not reflect on their thoughts. Their style of communication is egocentric. It is only with the development of intercommunicative behaviour amongst their peers that their cognitive egocentrism dissolves.

It should be noted that Wadsworth states that Piaget views egocentrism as a "characteristic that pervades thought, in some way, in all periods of development" [63, p.71] [presumably all periods of cognitive development].

The knock-on effect of their egocentrism is that it encourages students/children to assimilate more so than accommodate and thus prevents cognitive development. Thus, it is an obstacle that must be overcome by the child if he/she is to transition him or herself from the sensori-motor period to the period of concrete operations.

Transformation

This is characterised by an inability amongst the child to "move from a particular perceptual event to a particular perceptual event" [63, p.73] via integrating the "series of events in terms of any beginning-end relationship" [63, p.73]. This raises the question of what is a perceptual event? Perhaps a definition of the verb 'to perceive' may go some way towards answering this question. To perceive is defined as: to become aware of, know, or identify by means of the senses; to recognise, discern, envision or understand. This deficiency in pursuing transformations, in terms of linking them, "inhibits the development of logic in thought" [63, p. 73], within children.

Centration

This obstacle manifests itself in the form of when a "child is presented with a visual stimulus and they tend to centre or fix their attention on a limited perceptual aspect of the stimulus" [63, p.74]. It is best explained with an example that Piaget uses.

For example, if a child is asked to compare two rows of like objects in which one row contains nine objects and the other (a longer row) contains only seven objects, albeit spread further apart, the child of four to five years of age, typically selects the perceptually longer row as having more objects. Interestingly, Wadsworth points out that "this will occur even when the child knows cognitively that nine is more than seven" [63, p.74]. Perceptual evaluation [supposedly in this sense, whereby the child focuses on length as opposed to objects in each row] dominates cognitive evaluation.

According to Piaget, it is only when the child reaches the age of six or seven, that they are able to solve such a problem and thus “reach the point where cognitions assume their proper position with respect to perceptions in thought” [63, p.74].

Reversibility

When thought is reversible, it means that the child can follow a line of reasoning back to its origin. This type of thought has been observed by Piaget as lacking “in all cognitive activity of the preoperational child” [63, p.76].

An example which explains a child’s inability to reverse is as a follows: a child without reversible thought, when shown two equal-length rows of eight coins each, will agree that each row has the same number of coins. When one of the rows becomes lengthened, they will no longer agree that each of the rows contains the same number of coins in each row. Lack of reversibility is part of the problem— “they cannot maintain the equivalence of number in the face of perceptual change” [63, p.76].

The reason why Wadsworth states that ‘lack of reversibility is part of the problem’, as opposed to ‘lack of reversibility is the problem’ in solving the above scenario is because “Piaget’s concepts of egocentrism, centration, transformation and reversibility are closely related” [63, p.76].

A lessening in egocentrism requires the child to decentre more and attend to transformations. “All this, in turn, makes thought more reversible” [63, p.76]. Therefore, the term ‘lack of reversibility is part of the problem’, as opposed to ‘lack of reversibility is the problem’, is warranted.

Conservation

Wadsworth [63, p.76] defines conservation as the “conceptualisation (schematisation) that the amount or quantity of a matter stays the same regardless of any changes in shape or position”. For the preoperational child, they typically cannot conserve, that is to say, they cannot hold one dimension invariant (be that in terms of number, mass, area, and volume) in the face of changes in other dimensions. However, by the end of the preoperational period, some conservation structures are usually developed.

Interestingly, Wadsworth states that Piaget's theory in tandem with research in connection to 'conservation learning' makes evident that "the application of conservation principles to different types of problems usually follows a sequence" [63, p.84] of the form, whereby students conserve number first (5-6 years), mass second (7-8 years), area third (7-8 years), weight fourth (9-10 years) and volume fifth (11-12 years). A brief description of these conservational abilities follows.

Conservation of Number

A four to five-year-old who is presented with a row of checkers, and who is asked to construct a row that is the same, will typically construct a row of the same length. However, the row may not correspond to the number of checkers in the previous model. On the other hand, a five to six-year-old will use one-to-one correspondence to make each row equal in number and length. Interestingly, if they see one row lengthened or transformed, without any change in the number of elements, the child declares they are no longer equivalent. In a nutshell, the preoperational child holds the view that "the rows are equivalent only as long as there is visual correspondence in the length of arrays/columns" [63, p.79].

At the end of the preoperational period, the child will have learned to conserve number while simultaneously being able to decentre their perceptions [supposedly in terms of not solely focusing on what has changed], attending to transformations and reversing operations.

Conservation of Area

As with conservation-of-number problems, the preoperational child fails to conserve area in the face of perceptual change, due to not being able to decentre. However, around the age of seven or eight, conservation of area is usually attained.

Conservation of Volume

As with conservation of number and conservation of area, the preoperational child is not able to focus on, for example, the constant volume of a liquid while the container that it is placed in varies in terms of shape. It is not until the concrete-operational period is reached (ages seven to eleven) that volume conservation is acquired by the child.

Appendix C – Statistical Tests

In order to test whether a student who was correct in a mathematics context for a particular item tended to correctly answer (or was associated with correctly answering) the corresponding item in a chemistry context, categorical statistical tests were used.

The Chi-Squared Test

The workings of the test are explained in the context of Item 10 (Proportionality) which was used in the research.

Chemistry Context	**Mathematics Context**	
	Correct	**Incorrect**
Correct	C C	I C
Incorrect	C I	I I

Table 1. The Contingency Table for Item 10 (Proportionality).

Table 1 is representative of the number of possible outcomes that can occur when we investigate if students can answer a mathematical item correctly in a mathematics context and in its corresponding chemistry context. A student can either:

- answer correctly in a mathematics context and a chemistry context (denoted 'C C')
- answer correctly in a mathematics context and incorrectly in a chemistry context (denoted 'C I')
- answer incorrectly in a mathematics context and incorrectly in a chemistry context (denoted 'I I')
- answer incorrectly in a mathematics context and correctly in a chemistry context (denoted 'I C')

Figure 1 represents these four possible outcomes in a sample space.

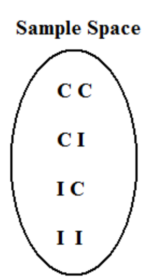


Figure 1. Possible Outcomes in Table 1 in the Form of a Sample Space.

It can be seen that with random answering, the following probabilities would be expected:

Event (A): Probability that a student answers correctly in a mathematics context: $\frac{2}{4} = \frac{1}{2}$

Event (B): Probability that a student answers correctly in a chemistry context: $\frac{2}{4} = \frac{1}{2}$

Event (A & B): Probability that a student answers both correctly in a mathematics context and in the corresponding chemistry context: $\frac{1}{4}$

The probability of Event A & B is also equal to the probability of Event A multiplied by Event B, which is equal to: $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

There are probabilities for other events, which can be deduced from the sample space in Figure 1. However, these other events were not of concern.

The probabilities for Event A, Event B and Event A & B are what would be expected if the outcomes shown in Figure 1, for Table 1, occur randomly. However, the outcomes will probably not occur randomly, primarily because students will have some mathematical knowledge and chemistry knowledge, which, for argument sake, should affect the probability of Event A & B occurring. Because of this in-built non-randomness, we took a sample of the population (in our case 30 students) and looked for a frequency distribution of possible outcomes amongst that sample. Such an approach yielded Table 2.

Chemistry Context	**Mathematics Context**		**Total**
	Correct	**Incorrect**	
Correct	11	6	17
Incorrect	2	11	13
	13	17	30

Table 2. Frequency Distribution of Possible Outcomes for a Sample of 30 Students.

The visualisation of the frequency of the possible outcomes, which we observed amongst our sample is shown in Figure 2.

C C	C C	C C	I C	I I	I I
C C	C C	C I	I C	I I	I I
C C	C C	C I	I C	I I	I I
C C	C C	I C	I C	I I	I I
C C	C C	I C	I I	I I	I I

Figure 2. Visualisation of the Frequency of Possible Outcomes Observed.

From Figure 2, it can be seen that for the 30 outcomes, in 13 of these outcomes, students answered the mathematical item correctly; this is highlighted in blue in Figure 3.

C C	C C	C C	I C	I I	I I
C C	C C	C I	I C	I I	I I
C C	C C	C I	I C	I I	I I
C C	C C	I C	I C	I I	I I
C C	C C	I C	I I	I I	I I

Figure 3. The 13 Students who Answered Item 10 Correctly in a Mathematics Context (Highlighted in Blue).

Looking at Figure 3, we can see that the probability of answering correctly in a mathematics context (Event A) is now: $\frac{13}{30}$

Likewise, the probability of answering correctly in a chemistry context (Event B, as shown in red in Figure 4) is now: $\frac{17}{30}$

C C	C C	C C	I C	I I	I I
C C	C C	C I	I C	I I	I I
C C	C C	C I	I C	I I	I I
C C	C C	I C	I C	I I	I I
C C	C C	I C	I I	I I	I I

Figure 4. The 17 Students who Answered Item 10 Correctly in a Chemistry Context (Highlighted in Red).

To summarise thus far:

From our sample of students, we expected the probability of the following events to be representative of the entire population of students:

Event (A): Probability that a student answers correctly in a mathematics context: $\frac{13}{30}$

Event (B): Probability that a student answers correctly in a chemistry context: $\frac{17}{30}$

Event (A & B): Probability that a student answers both correctly in a mathematics context and in the corresponding chemistry context is equal to the probability of Event A multiplied by Event B, which is equal to: $\frac{13}{30} \times \frac{17}{30} = 0.24$

If the probability of Event A & B are independent (or can both occur by chance alone), then we expected 0.24 times the sample of 30 students to answer correctly in the mathematics context and chemistry context, due to chance alone.

Therefore in Figure 5, we would have expected (0.24 x 30) students to answer correctly in both the mathematics context and chemistry context, due to chance alone; this number of students is approximately 7. If we look at Figure 5, we see that 11 students (as highlighted in green) actually answered correctly in both a mathematics context and chemistry context.

C C	C C	C C	I C	I I	I I
C C	C C	C I	I C	I I	I I
C C	C C	C I	I C	I I	I I
C C	C C	I C	I C	I I	I I
C C	C C	I C	I I	I I	I I

Figure 5 The 11 Students (Highlighted in Green) who Answered Item 10 Correctly in both a Mathematics Context and Chemistry Context.

The following question arose: *Do the 11 students who answered correctly in both contexts, show that there is an association between students answering correctly in a mathematics context and in the corresponding chemistry context?*; in other words, if you can answer the item in a mathematics context, are you also likely to answer the item in its corresponding chemistry context more so than by chance alone?

Perhaps our observed frequency of 11 (instead of the expected frequency of 7) was due to sampling variation? If so, then there must be some sampling distribution, which takes account of the variability of expected frequencies; furthermore, this sampling distribution must have a mean. This sampling distribution is called a chi-squared distribution. It depends, not just on the sample size in question but also on the degrees of freedom. The degrees of freedom depend on the number of cells within a table. The formula for calculating such degrees of freedom is:

$$df = (r-1)\,(c-1)$$

where df is the degrees of freedom;

r is the number of rows in the table;

and c is the number of columns in the table.

In a nutshell, the degrees of freedom indicate how many parameters are needed to determine all the comparisons for describing the table. For example, in our table, if we know the value in Cell 1 (Highlighted in Table 3) in tandem with the row totals and column totals, then we can deduce the value for Cell 2 (Highlighted in Table 3)—the cell value we compare the Cell-1 value against, when looking for significance. Likewise, for a three by two table, if we know the value in Cell 1 and Cell 2, in tandem with the row totals and column totals, then we can deduce the value for Cell 3; thus a three-by-two table has two degrees of freedom and two-by-two table has one degree of freedom. Chi-squared distributions for various degrees of freedom are shown in Figure 6.

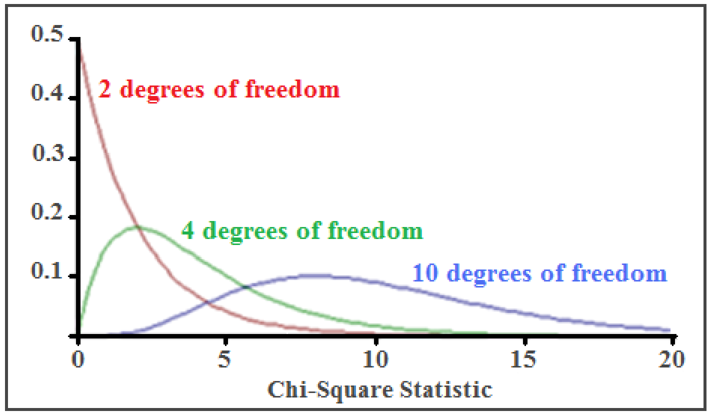


Figure 6. Chi-Squared Distributions for various degrees of freedom. Adapted from [101].

The approximate shape of the chi-squared distribution that was applicable to our study is shown in Figure 7.

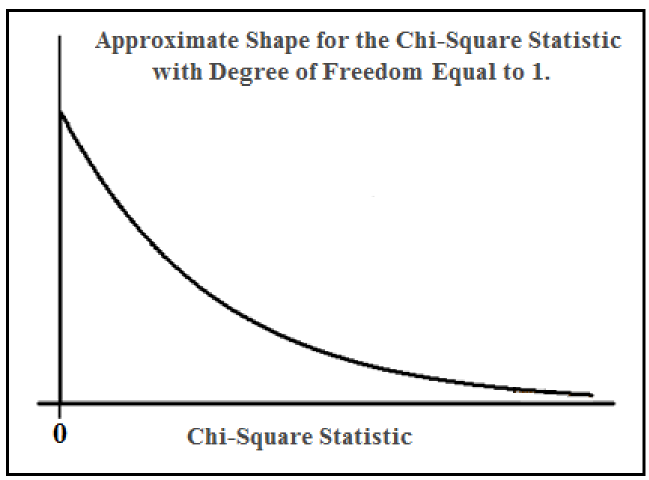


Figure 7. The Approximate Shape of the Chi-Squared Distribution Applicable to our Research.

Looking at Figure 7, we can see that the value for the chi-squared distribution varies—the larger the value, the more likely that the observed frequency is not due to chance alone; or in other words, the likelihood of an association is greater. In our investigation of a possible association, we looked for a chi-squared value with a probability of less than or equal to 0.05. This is shown in Figure 8.

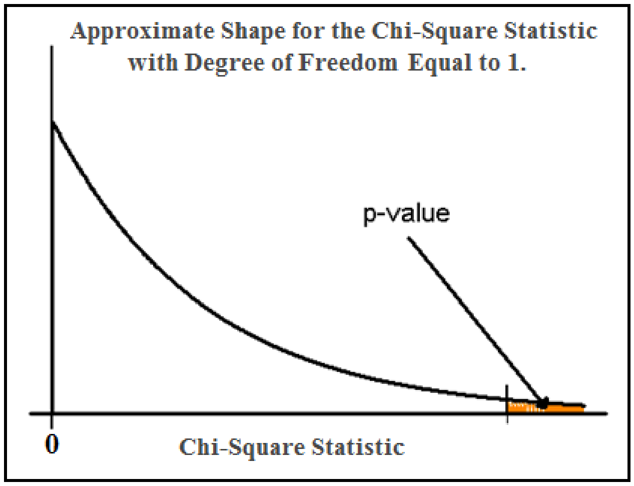


Figure 8. Visual Depiction of a Chi-Squared Value with a Probability of less than or equal to 0.05, for a Table with Degrees of Freedom Equal to One.

The next question that arises is how do we calculate the chi-squared statistic for Table 2?

Firstly, we must calculate the expected frequencies for each cell (labelled Cell 1, Cell 2, Cell 3 and Cell 4) as shown below in Table 3.

Chemistry Context	**Mathematics Context**		**Total**
	Correct	**Incorrect**	
Correct	Cell 1	Cell 2	17
Incorrect	Cell 3	Cell 4	13
	13	17	30

Table 3. The Cell 1, Cell 2, Cell 3 and Cell 4 in Table 2.

The methodology for finding the expected frequency in each cell is the same. For Cell 1, this is equivalent to finding the probability of being correct in the maths context and chemistry context, and multiplying this probability by our sample number (namely 30). The probability for Cell 1 is:

$$\frac{13}{30} \times \frac{17}{30} = 0.24$$

We then multiply this probability by the population total to give us the expected frequency of students in that cell:

$$30 \times \frac{13}{30} \times \frac{17}{30} = 7.36$$

Our calculations can be shortened into the following formula for expected cell count for any cell:

$$\text{Expected cell count for any cell} = \frac{\textit{(Row total)} \times \textit{(Column total)}}{\textit{Total Sample Size}}$$

Thus, Table 4, shows the observed counts and expected counts (in parentheses) for Table 2.

Chemistry Context	**Mathematics Context**		**Total**
	Correct	Incorrect	
Correct	11(7.37)	6(9.63)	17
Incorrect	2(5.63)	11(7.37)	13
	13	17	30

Table 4. Observed Counts and Expected Counts (in Parentheses) for Table 2.

The next step in determining whether there is an association between being correct in the mathematics context and the corresponding chemistry context is to sum the square of the difference between the observed frequency and expected frequency in each cell of the table. This sum is the chi-squared statistic for the table. Its formula is as follows:

$$\chi^2 = \sum \frac{(\text{observed} - \text{expected count})^2}{\text{expected count}}$$

Thus, for our table:

$$\chi^2 = \sum \frac{(11 - 7.37)^2}{7.37} + \frac{(6 - 9.63)^2}{9.63} + \frac{(2 - 5.63)^2}{5.63} + \frac{(11 - 7.37)^2}{7.37}$$

$$= 1.78 + 1.36 + 2.34 + 1.78 = 7.26$$

The final step is to determine where this chi-squared value lies on the chi-squared distribution and the probability of being at or beyond this value. As it transpired, this chi-squared value happened to have a p-value of: *8.1E-03,* thus indicating significance. The conclusion that was reached for this Item was: there is strong evidence to suggest that if a student answers Item 10 correctly in a mathematics context, they will associate with answering the equivalent mathematical item in a chemistry context correctly.

This type of analysis was performed for each of our items. If any of the expected frequencies in our two-by-two table happened to be less than five, we used a more precise categorical test, called Fisher's Exact Test. The principles of both the Chi-Squared Test and Fisher's Exact Test are summarised.

Chi-Squared Test

Assumptions:

- We have two categorical variables; in our case, being correct or incorrect in the mathematics context is the first variable, while being correct or incorrect in the chemistry context is the second.
- We have a random sample.
- The expected frequencies are greater than or equal to five in all cells (if not, we use Fisher's Exact Test).

Hypotheses:

H_0: The two variables are independent; in other words, one will answer correctly in both contexts (or whatever possible outcome you want to chose) due to chance alone.

H_a: The two variables are dependent (associated); for example, the likelihood that a student will answer correctly in both contexts is greater than chance alone.

Test Statistic:

$$\chi^2 = \sum \frac{(observed\ count - expected\ count)^2}{expected\ count}$$

where expected count = (row total x column total)/total sample size.

P-value: **A** Right-tail probability above the observed χ^2 value for the chi-squared distribution with df = (r-1) (c-1).

Conclusion: Reject H_0 when the p-value $\leq$ the significance level (such as 0.05).

Fisher's Exact Test

Assumptions:

- We have two categorical variables; in our case, being correct or incorrect in the mathematics context is the first variable, while being correct or incorrect in the chemistry context is the second.
- We have a random sample.
- The expected frequencies are less than or equal to five in one cell or more [102].

Hypotheses:

H_0: The two variables are independent.

H_a: The two variables are dependent (associated).

Test Statistic [103]**:**

$$\sum p = \frac{(\text{row 1 total})!(\text{row 2 total})!(\text{column 1 total})!(\text{column 2 total})!}{(\text{sample size})!} \times \sum \frac{1}{a!b!c!d!}$$

where *p* is probability, *a!* is the Cell 1 total, *b!* is the Cell 2 total, *c!* is the Cell 3 total and *d!* is the Cell 4 total.

The test statistic is the summation of the probabilities of all possible two-by-two contingency tables with a cell frequency equal to or smaller than the smallest expected frequency observed (keeping the row and column totals fixed, as above).

P-value: The test statistic is the p-value.

Conclusion: If the $\sum p$ is less than the significance level chosen, we may reject the null hypothesis—independence between the two categorical variables.

Appendix D – Main Study Mathematical Items

Mathematical Items Used in Diagnostic Tools 1 and 2

Item 1: Calculating Slope

Mathematics Context

(A) Calculate the slope of the straight line from the two points given in Figure 1.

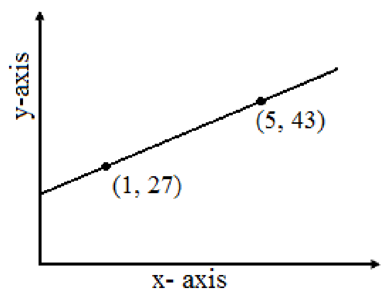


Figure 1

(B) Explain what this number means.

Chemistry Context

(A) Calculate the rate of change of the concentration of the reactant with respect to time over the time interval (Δt) from the two points given in Figure 1.

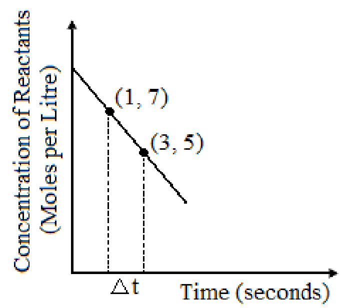


Figure 1

(B) Explain what this value means.

Item 2: Sketching a Line with Positive Slope

Mathematics Context

L_1 as shown in Figure 1, passes through the Point 'P' and has a slope = 2.

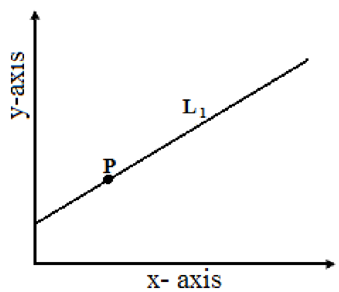


Figure 1

(A) Sketch in Figure 1: a line (L_2) that passes through the point P and has slope = 3.

(B) Explain your reasoning

Chemistry Context

The Line L_1 in Figure 1 shows the graph of the concentration of product with respect to time over a certain time interval (Δt). It has a value for the rate of change = 3.

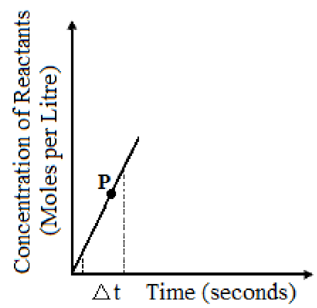


Figure 1

(A) Sketch in Figure 1: a line (L_2) that passes through the point P and has a value for the rate of increase of the product with respect to time = 4.

(B) Explain your reasoning.

Item 3: Sketching a Line with Positive Slope

Mathematics Context

L_1, in Figure 1, passes through the point 'P', and has a slope = 2.

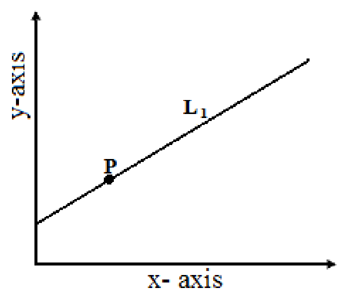


Figure 1

(A) Sketch in Figure 1: a line (L_3) that passes through the point P, and has slope = 1.

(B) Explain your reasoning

Chemistry Context

The Line L_1 in Figure 1 shows the graph of the concentration of product with respect to time over a certain time interval (Δt). It has a value for the rate of change = 3.

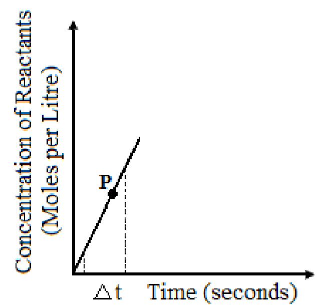


Figure 1

(A) Sketch in Figure 1: a line (L_3) that passes through the point P, and has a value for the rate of increase of the product with respect to time = 1.

(B) Explain your reasoning.

Item 4: Sketching a Line with Negative Slope

Mathematics Context

L_1, as shown in Figure 1, passes through the Point 'P' and has a slope = 2.

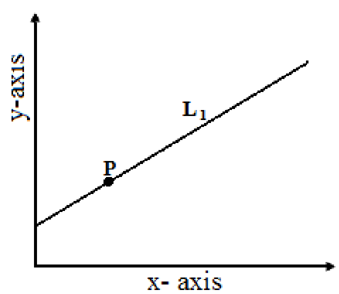


Figure 1

(A) Sketch in Figure 1: a line (L_4) that passes through the point P, and has slope = -1.

(B) Explain your reasoning

Chemistry Context

The line in Figure 1 shows the graph of concentration of reactant with respect to time over a certain interval (Δt). Its rate of decrease over this interval is equal to 2.

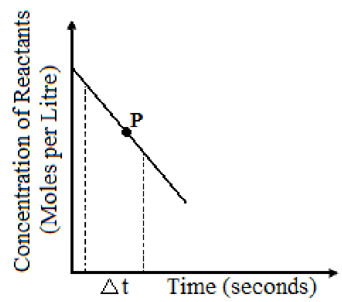


Figure 1

(A) Sketch in Figure 1: a line (L_4) that passes through the point P, and has a value for the rate of decrease of the reactant with respect to time = 1.

(B) Explain your reasoning.

Item 5: Generating an Expression for Slope

Mathematics Context

(A) Using the notation in the diagram in Figure 1, write down an expression for the slope of a line connecting B-C.

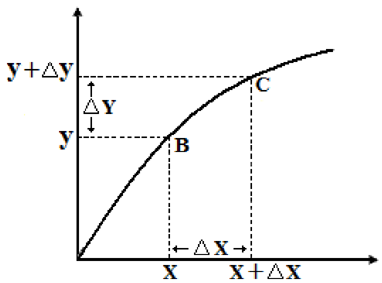


Figure 1

(B) Explain your reasoning

Chemistry Context

(A) Figure 1 shows the change of concentration of product (P) over time (t). Using the notation in the diagram, write down an expression for the average rate of change of product (P) between B and C.

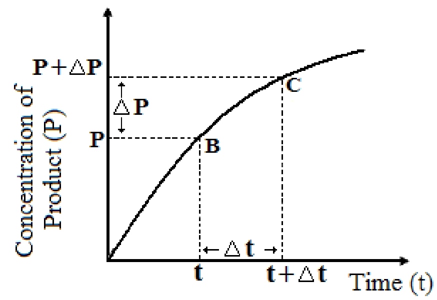


Figure 1

(B) Explain your reasoning.

Item 6: Generating an Expression for Derivative

Mathematics Context

Using the notation in the diagram in Figure 1, write down an expression for the slope of a line connecting B-C.

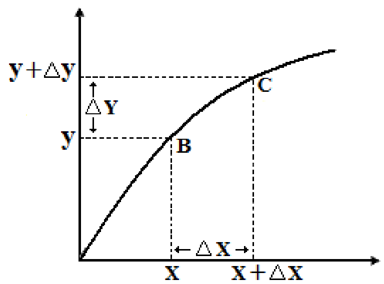


Figure 1

(A) Using your answer, explain how you could generate the derivative $\left(\frac{dy}{dx}\right)$ at B.

Chemistry Context

Figure 1 shows the change of concentration of product (P) over time (t). Using the notation in the diagram, write down an expression for the average rate of change of product (P) between B and C.

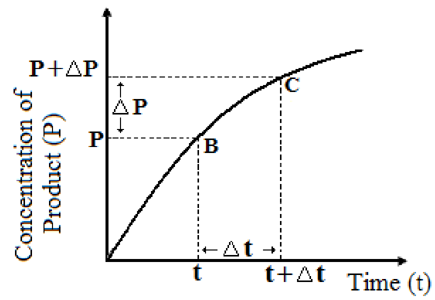


Figure 1

(A) Using your answer, explain how you could generate the instantaneous rate of change $\left(\frac{dP}{dt}\right)$ at B.

Item 7: Interpreting Derivative

Mathematics Context

(A) Figure 1 shows the graph of y against x. At which point, A or B, does the greatest value of $\frac{dy}{dx}$ occur?

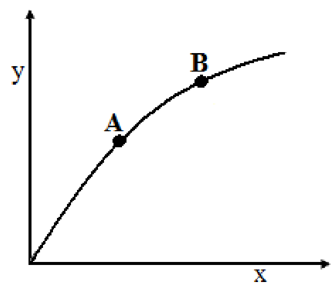


Figure 1

(B) Explain your reasoning.

Chemistry Context

(A) For a particular reaction:

$$A + B \rightarrow P$$

where A and B are reactants and P is product, Figure 1 shows the graph of concentration of product (P) against time (t). At which point, E or F, does the greatest increase in concentration of product with respect to time occur?

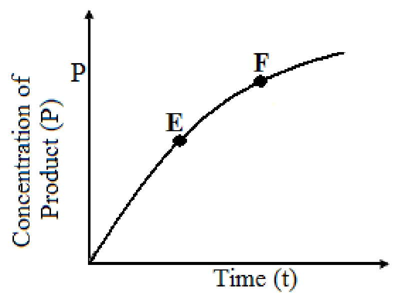


Figure 1

(B) Explain your reasoning.

Item 8: Usage of Exponentials

Mathematics Context	Chemistry Context
Given: $$\text{Ln}\,y = c - mx$$	A student is studying the chemical reaction: $$A + B \rightarrow P,$$
(A) Derive an expression for y.	where A and B are reactants, and P is the product. After graphing the Ln of the concentration of A, obtained at different times (i.e. the graph of $\text{Ln}[A]_t$ against time (t)), the student finds that the graph corresponds to the relation given below, showing that the rate of the reaction is 1st order with respect to A. $$\text{Ln}[A]_t = \text{Ln}[A]_0 - kt$$
	(A) Derive an expression for $[A]_t$
(B) Explain your reasoning.	**(B)** Explain your reasoning.

Item 9: Usage of Natural Logarithms

Mathematics Context

Given:

$$y = y_0 e^{-2x}$$

(A) Derive an expression for x in terms of y and y_0.

(B) Explain your reasoning.

Chemistry Context

For a reaction:

$$A + B \rightarrow P,$$

where A and B are reactants and P is product, the concentration of reactant B after a certain time ($[B]_t$) is given as a function of time in the following expression:

$$[B]_t = [B]_0 e^{-kt}$$

where $[B]_0$ and k are the initial concentration of reactant B and rate constant respectively.

(A) Derive an expression for k.

(B) Explain your reasoning.

Item 10: Proportionality

Mathematics Context

The derivative of a particular function:

$$y = f(x)$$

with respect to x is denoted:

$$\frac{dy}{dx}$$

It is found that $\frac{dy}{dx}$ is proportional to y^2

$$\text{Thus } \frac{dy}{dx} = ky^2$$

where k is the constant of proportionality.

(A) What happens to the value of the derivative if y is doubled?

(B) Explain your reasoning.

Chemistry Context

The rate law for a particular reaction:

$$A + B \rightarrow P,$$

where A and B are reactants and P is product is given as:

$$\text{Rate} = k[A]^2$$

when the concentration of B is held constant. 'k' is the rate constant.

(A) What happens to the value of the rate if [A] is doubled?

(B) Explain your reasoning.

Item 11: Graphing an Exponential Function

Mathematics Context

Given:

$$y = y_0 e^{-2x}$$

(A) Draw a graph that represents the relationship in Figure 1. Label the axis accordingly.

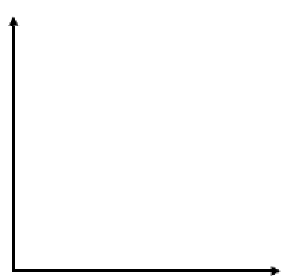


Figure 1

(B) Explain your reasoning

Chemistry Context

For a reaction:

$$A + B \rightarrow P,$$

where A and B are reactants and P is product, the concentration of reactant B after a certain time ($[B]_t$) is given as a function of time in the following expression:

$$[B]_t = [B]_0 e^{-kt}$$

where $[B]_0$ and k are the initial concentration of reactant B and rate constant respectively.

(A) Draw a graph that represents this expression in Figure 1. Label the axis accordingly.

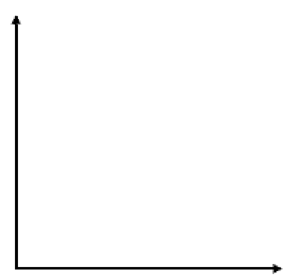


Figure 1

(B) Explain your reasoning.

Item 12: Graphing a Natural Logarithmic Expression

Mathematics Context

Given the relationship:

$$\mathrm{Ln}\,y = c - mx$$

(A) Draw a graph that represents the relationship in Figure 1. Label the axis accordingly.

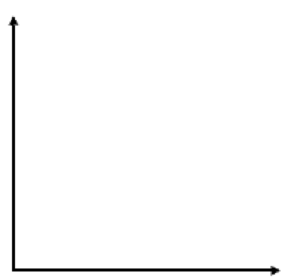


Figure 1

(B) Explain your reasoning

Chemistry Context

A student is studying the chemical reaction:

$$A + B \rightarrow P,$$

where A and B are reactants, and P is the product. After graphing the Ln of the concentration of A, obtained at different times: (i.e. the graph of $\mathrm{Ln}[A]_t$ against time (t)), the student finds that the graph corresponds to the relationship:

$$\mathrm{Ln}[A]_t = \mathrm{Ln}[A]_0 - kt$$

showing that the rate of the reaction is 1^{st} order with respect to A.

(A) Sketch the relationship in Figure 1. Label the axis appropriately.

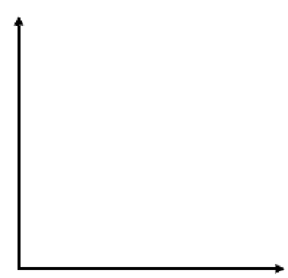


Figure 1

(B) Explain your reasoning.

Item 13: Graphing a Function

Mathematics Context

(A) Sketch in Figure 1, the graph of the function:

$$y = \frac{1}{x} \text{ for } x > 0$$

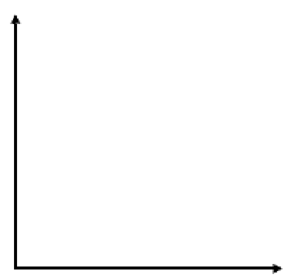


Figure 1

(B) Explain your reasoning

Chemistry Context

(A) Sketch in Figure 1, the graph of P versus V, for:

$$0\text{m}^3 < V < 5\text{m}^3$$

given the relationship:

$$P = \frac{1}{V}$$

This relationship comes from the ideal gas law applied to an isothermal system. For this example, *nRT* has the constant value of 1kJ.

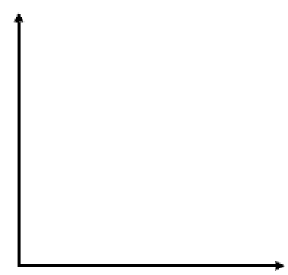


Figure 1

(B) Explain your reasoning.

Item 14: Evaluation of an Integral

Mathematics Context	Chemistry Context
(A) Evaluate the integral: $$\int_1^3 \frac{1}{x}dx$$	**(A)** Calculate the work done when the volume of a gas, in a reversible isothermal gas expansion, increases from:$1m^3(V_1)$ to $3m^3(V_2)$, given that the work will be equal to the expression: $$w = -\int_{V_1}^{V_2} \frac{1}{V}dv$$ where V_1 ($1m^3$) is the initial volume of the gas, and V_2 ($3m^3$) is the final volume of the gas. The minus sign is used to denote the fact that the work leaves the system.
(B) Explain your reasoning	**(B)** Explain your reasoning.

Item 15: Graphing an Integral

Mathematics Context

(A) Indicate in Figure 1, the area corresponding to the integral:

$$\int_1^3 \frac{1}{x}dx$$

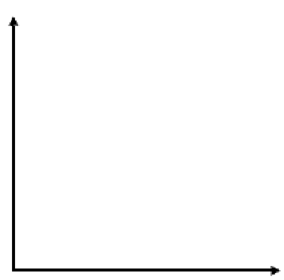


Figure 1

(B) Explain your reasoning

Chemistry Context

(A) The relationship:

$$P = \frac{1}{V},$$

where P is the pressure of a gas, and V is its volume, represents the ideal gas law applied to an isothermal system. For this example, *nRT* has the constant value of 1kJ. Indicate in Figure 1, the area corresponding to the integral:

$$w = -\int_{V_1}^{V_2} \frac{1}{V}dV$$

which represents the work done by the system (the gas) in expanding from an initial volume:

($V_1 = 1m^3$) to a final volume ($V_2 = 3m^3$),

for a reversible isothermal gas expansion. The minus sign is used to denote the fact that the work leaves the system.

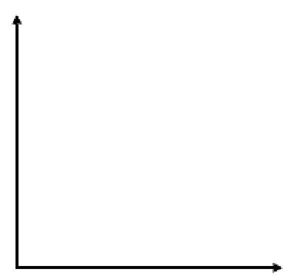


Figure 1

(B) Explain your reasoning.

Appendix E – Slope Intervention

Section 1:

A walker crosses a hill, as shown in the graph in Figure 1. They move from Point A to Point B and so forth, all the way to Point G.

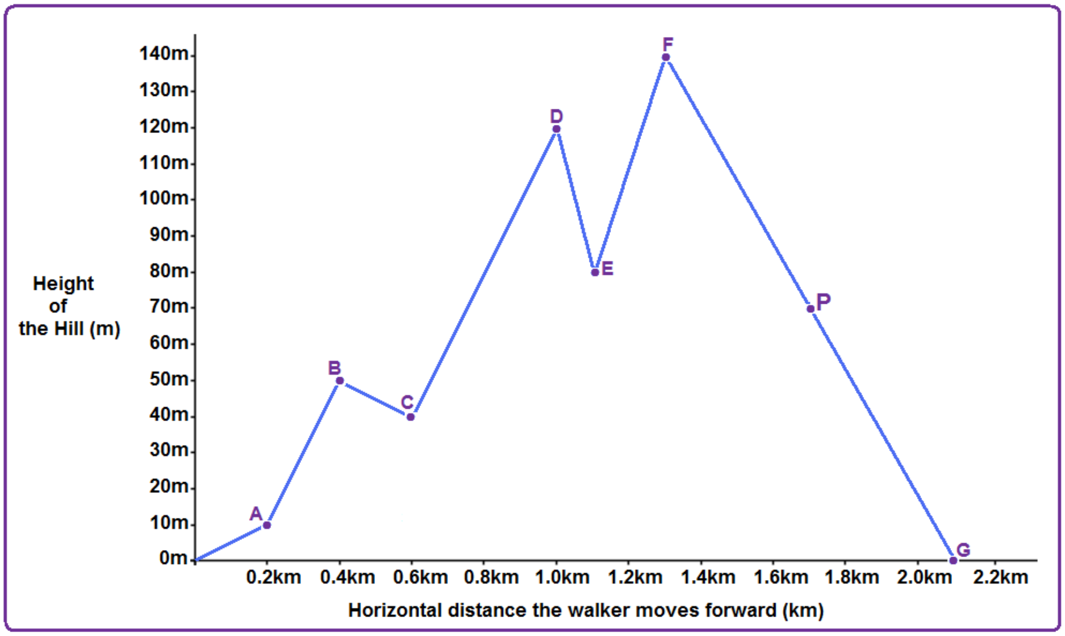


Figure 1

The x-y co-ordinates for the Points A, B, C, D, E, F and G, in Figure 1, are as follows:

A: (0.2km, 10m) B: (0.4km, 50m) C: (0.6km, 40m) D: (1km, 120m)

E: (1.1km, 80m) F: (1.3km, 140m) G: (2.1km, 0m)

Q.1 Between which points on the graph does the walker move uphill?

____

Q.2 Between which points on the graph does the walker move downhill?

____

____

Q.3 Which is steeper, the line CD or the line DE?

Please explain your reasoning.

Q.4 Is the slope of the line AB the same as the slope of the line CD?

Please explain your reasoning.

Q.5 Calculate the slope of the line FG.

Please explain what your answer means.

Q.6 Calculate the slope of the line FG at the Point P, as shown in Figure 1.

Please explain your reasoning.

Q.7 Read the following conversation between two students, Tom and Kate.

Kate: Tom, what value did you get for the slope of the line DE?

Tom: I got *minus 400 metres per kilometre* — and you?

Kate: *Plus 400 metres per kilometre*. Your answer must be wrong. This is how I calculated the value for slope:

Change in the walker's height:

120m – 80m = 40m

Change in the horizontal distance that the walker moves forward:

1.1km – 1.0km = 0.1km

Therefore, the value for slope is:

$$\frac{\text{Change in the walker's height}}{\text{Change in the horizontal distance the walker moves forward}} = \frac{40\text{m}}{0.1\text{km}} = 400\text{m/km}$$

Do you agree with my reasoning, Tom?

Tom: Not quite. This is how I did it:

Change in the walker's height:

80m – 120m = -40m

Change in the horizontal distance that the walker moves forward:

1.1km – 1.0km = 0.1km

Therefore, the value for slope is:

$$\frac{\text{Change in the walker's height}}{\text{Change in the horizontal distance the walker moves forward}} = \frac{-40\text{m}}{0.1\text{km}} = -400\text{m/km}$$

Agree?

Kate: I'm not sure if I do.

After reading the above conversation, what is the value for the slope of the line DE?

Q.8 In terms of the slope of the line DE, how can you ensure that your value will be interpreted as meaning that the walker moves downhill in Figure 1?

Please explain your reasoning.

Q.9 Rank the lines AB, BC, CD, DE, EF, and FG in ***increasing order*** of steepness.

Please explain your reasoning.

Section 2:

Review of Part I in terms of the Definition of Slope and the Meaning of 'Steepness'

Definition and Meaning of Slope:

We use the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

to calculate the slope of a line, where (x_1,y_1) and (x_2,y_2) are any two points on the line.

The slope gives us a measure of how much the line increases ***or*** decreases in the vertical direction (Δy) for an increase of Δx in the horizontal direction (**NB** an increase of Δx always means an increase moving from left to right.).

Any points on a line can be used to calculate the slope of that line.

Figure 2 shows the graph of a function that represents the volume of water in a tank at different times. The slope of the line AB in Figure 2 is calculated as follows:

Taking the Points A and B to be the points (x_1, y_1) and (x_2, y_2) respectively.

$$m = \frac{9m^3 - 3m^3}{4\text{hours} - 2\text{hours}} = \frac{6m^3}{2\text{hours}} = \frac{\Delta \text{Volume of Water}}{\Delta \text{Time}}$$

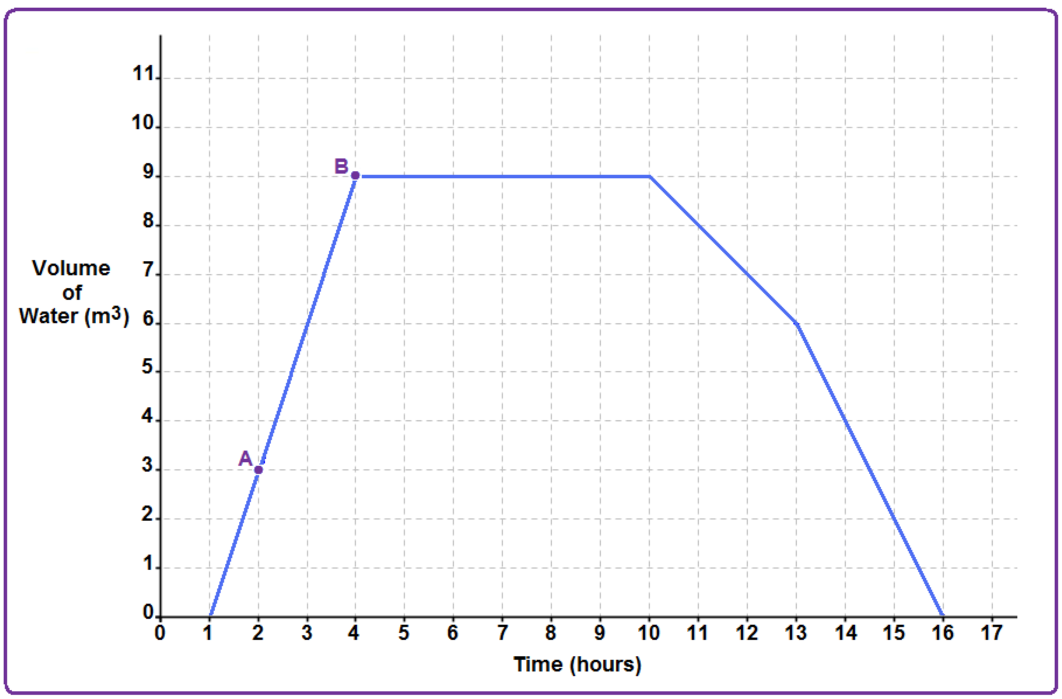


Figure 2

The visual meaning of $\frac{6m^3}{2\text{hours}} = \frac{\Delta \text{Volume of Water}}{\Delta \text{Time}}$ is shown in Figure 3.

We can see in Figure 3 that for:

$$\Delta \text{Time} = 2 \text{ hours},$$

the corresponding change in the volume of water = $6\,m^3$.

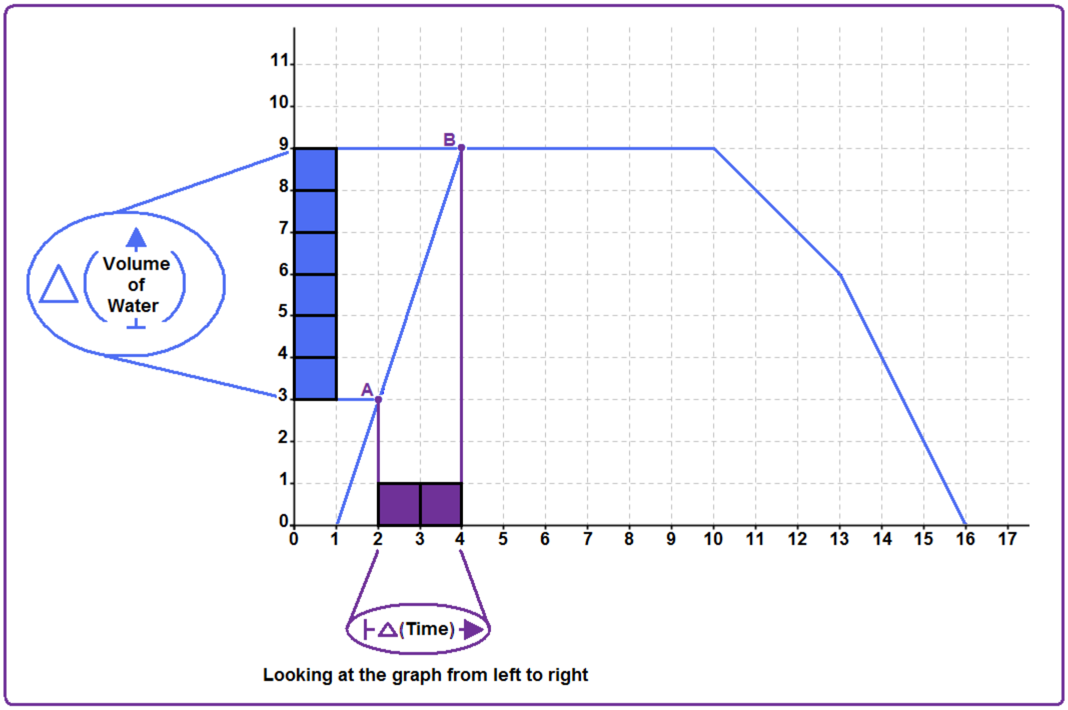


Figure 3

We can see in Figure 4 that for:

$$\Delta \text{Time} = 1 \text{ hour},$$

the corresponding change in the volume of water = $3\,m^3$.

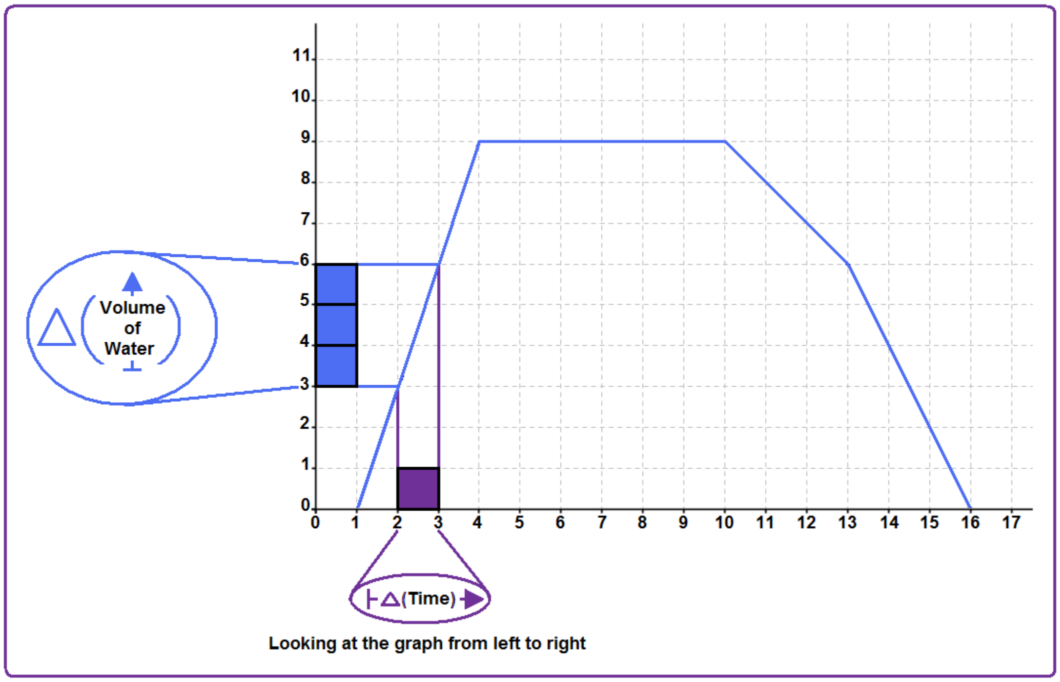


Figure 4

In the formula for slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

By taking $\Delta x = 1$, we see that:

$$m = \frac{\Delta y}{1} = \text{Change in y corresponding to a 1 unit increase in x.}$$

The line L1, in Figure 5, which has a slope of *m = 0.5*, represents the above description of slope in a visual sense. We can see in Figure 5 that for a ***unit increase*** on the x-axis, there is a corresponding increase of 0.5 units in the value of y.

In contrast, looking at the line L2 in Figure 6 which has a slope of *m =* ***-3***, we see that for a ***unit increase*** on the x-axis, there is a corresponding decrease of 3 units in the value of y.

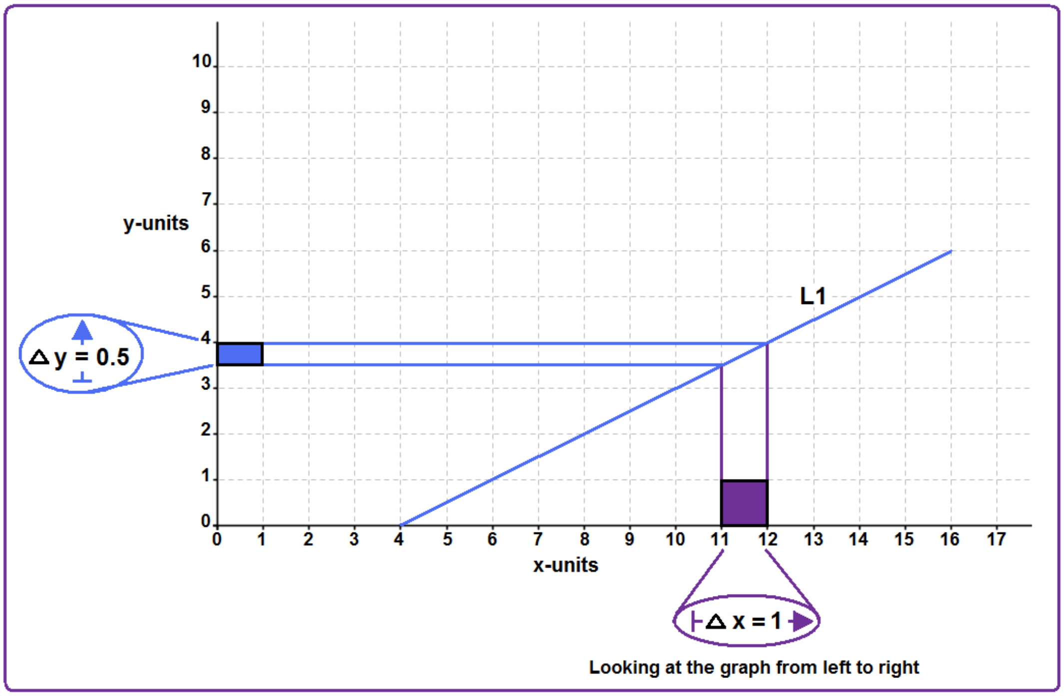


Figure 5

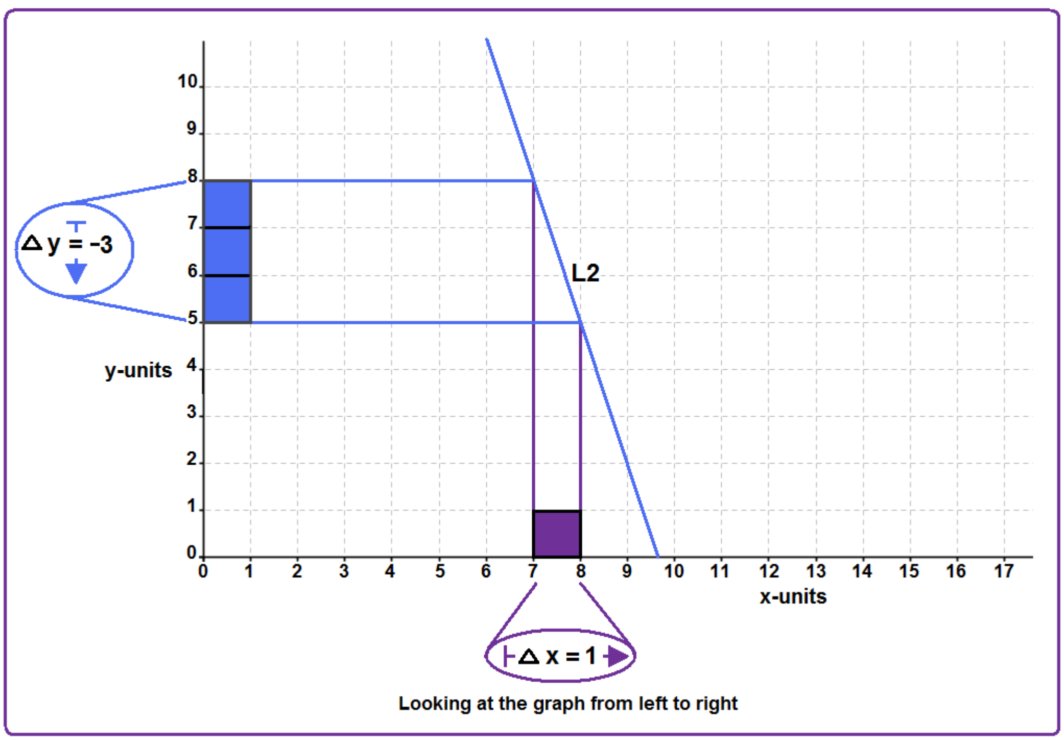


Figure 6

Slope is Equal to Steepness with a Sign:

By looking at Figure 5 and Figure 6, we can clearly see that L2 is steeper than L1. However,

$$\text{Slope of L2} < \text{Slope of L1}$$

because: $$-3 < 0.5$$

To measure *steepness* rather than slope, we look at the *magnitude* or *absolute value* of the slope. For example, we have already seen in Figure 5, that the line L1 has a value for slope equal to ***+0.5***.

In contrast, the slope of the line L2 in Figure 6 has a value for slope equal to ***-3***.

Then, we see:

The magnitude of the slope of L1

$= \left|\text{slope of L1}\right|$

$= \left|+0.5\right|$

$= +0.5$

The magnitude of the slope of L2

$= \left|\text{slope of L2}\right|$

$= \left|-3\right|$

$= +3$

Thus, taking the absolute values for the slopes of line L1 and line L2 reflects the fact that L2 is steeper than L1.

For the line L1, in Figure 5, we can see that for a ***unit increase*** on the x-axis, a corresponding 0.5 unit change occurs in the value of y. However, for the line L2 in Figure 6, for a unit increase on the x-axis, a corresponding 3 unit change occurs in the value of y.

In other words, the change in y for a ***unit increase*** on the x-axis is greater for the line L2 than for the line L1; this makes L2 steeper than L1.

Q.10

A walker's change in height (in metres) per unit distance that they move forward is found to be ***minus*** *0.25 metres per metre*. This can be interpreted from Figure 7.

The walker is at ***Point A*** in Figure 7. They continue moving along the hill, moving *-1metre* in the vertical direction. What is their corresponding change in the horizontal direction?

Highlight this change in metres that the walker moves forward along with the corresponding change of minus 1 metre in their height, on the graph in Figure 7.

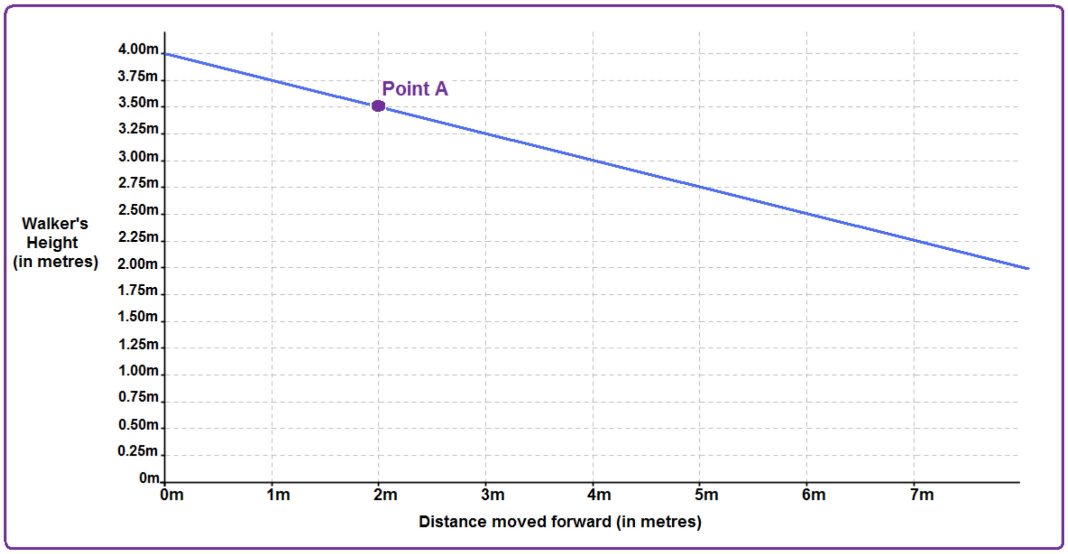


Figure 7

Section 3:

The temperature (°C) inside a room was recorded during a certain period of time (in hours). The data was plotted as shown in Figure 8.

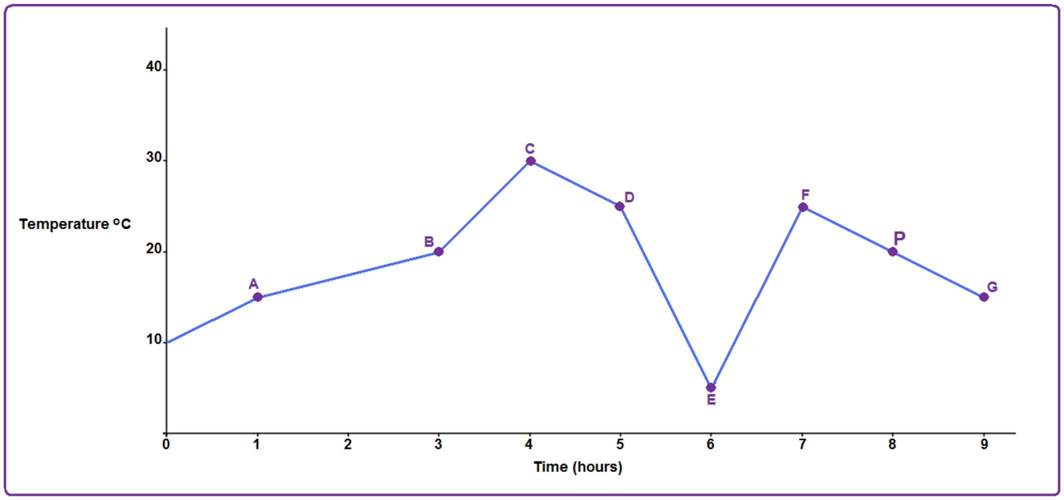


Figure 8

The x-y co-ordinates for the Points A, B, C, D, E, F and G in Figure 8 are as follows:

A: (1hr, 15°C) B: (3hrs, 20°C) C: (4hrs, 30°C) D: (5hrs, 25°C)

E: (6hrs, 5°C) F: (7hrs, 25°C) G: (9hrs, 15°C)

Q.1 Between which points on the graph does the temperature increase?

____

Q.2 Between which points on the graph does the temperature decrease?

____

Q.3 Which is steeper, the line BC or the line CD?

____

Please explain your reasoning

Q.4 Is the slope of the line AB the same as the slope of the line BC?

Please explain your reasoning.

Q.5 Calculate the slope of the line FG.

Please explain what your answer means.

Q.6 Calculate the slope of the line FG at the Point P, as shown in Figure 8.

Please explain your reasoning.

Q.7 What is the slope of the line DE?

In terms of this value, how can you ensure that it will be interpreted as meaning that the temperature decreases between the Points D and E?

Please explain your reasoning.

Q.8 Rank the lines AB, BC, CD, DE, EF, and FG in ***increasing order*** of steepness.

Please explain your reasoning.

Appendix F – Meaning-of-Derivative Intervention

Section 1:

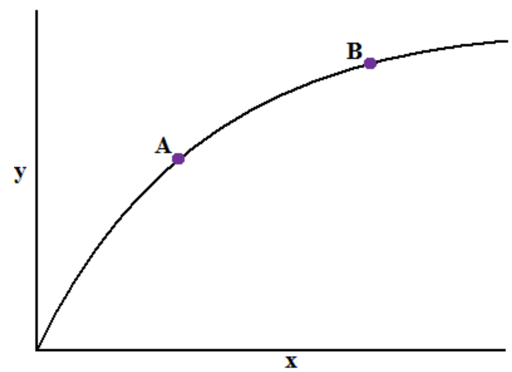


Figure 1

Figure 1 shows the graph of y against x.

Q. 1 At which Point, A or B, is the value for y the greatest?

____

Q. 2 At which Point, A or B, does the greatest value of $\frac{dy}{dx}$ occur?

____

Give a reason for your answer.

____

____

____

____

____

____

Section 2:

As a balloon is inflated or deflated, its surface area (in centimetres squared (cm^2)) is proportional to the square of the radius of the balloon (in centimetres (cm)). This relationship is plotted in Figure 2. Its algebraic representation is of the form:

$$y = 4\pi x^2$$

Expression 1

Where 'y' is equal to the surface area of the balloon and 'x' is equal to the radius of the balloon.

Let us consider when the balloon is inflating:

We can see, for example, from Figure 2, that when the radius of the balloon is 5cm, the corresponding surface area of the balloon is approximately (all such figures in this exercise are rounded-off to the nearest whole number) _____cm^2. We can also see that the surface area of the balloon changes as its radius changes.

Consider Question 1:

Q.1 *By how much is the surface area of the balloon increasing (Δy) at the instant that the radius of the balloon passes through 5cm?*

Answering such a question is equivalent to finding the slope of a straight line 'touching' the graph at the point (5cm, 314cm^2), as shown in Figure 3. Such a straight line is called the tangent line at the point in question.

Finding the slope of such a tangent line involves a process of estimation.

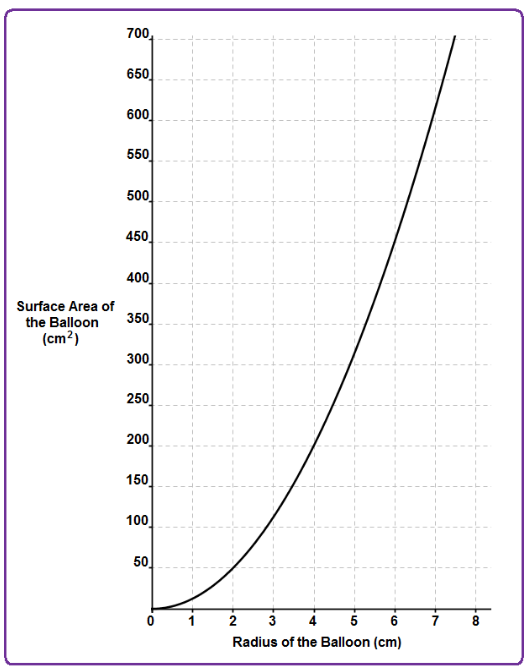


Figure 2

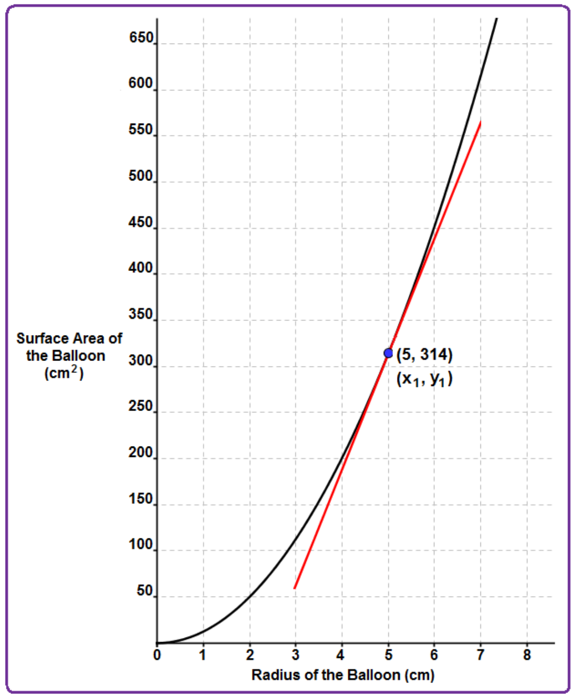


Figure 3

For our first estimate, let's find the slope of the line extending from the point (5, 314) to the point (7, 615), as shown in Figure 4. The slope of this line **estimates** the slope of the tangent line at the point (5, 314).

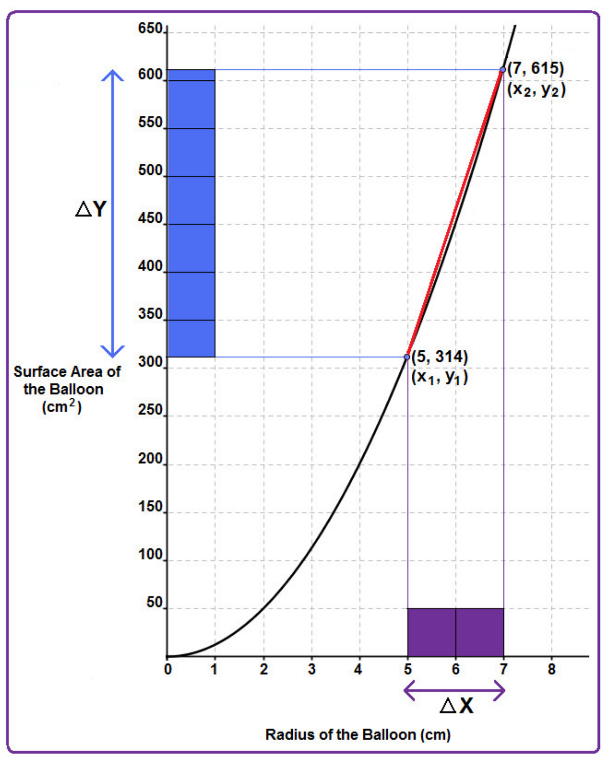


Figure 4

Finish off the calculation for the slope of this line in Expression 2 and the conclusion, which follows from it:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{615\text{cm}^2 - 314\text{ cm}^2}{7\text{cm} - 5\text{cm}} = \frac{\_\_\_\_\_\ \text{cm}^2}{2\text{cm}} = \frac{\Delta y}{\Delta x} = \_\_\_\_ \qquad \textit{Expression 2}$$

Conclusion:

This means that at the point (5, 314), we **estimate** the surface area of the balloon to increase by/change by ______cm^2 for a 1cm increase in the radius.

Notice how the value for Δx in Expression 2 is positive (namely 2cm). The value for Δx can also be negative; Figure 5 shows an example of when this can be the case.

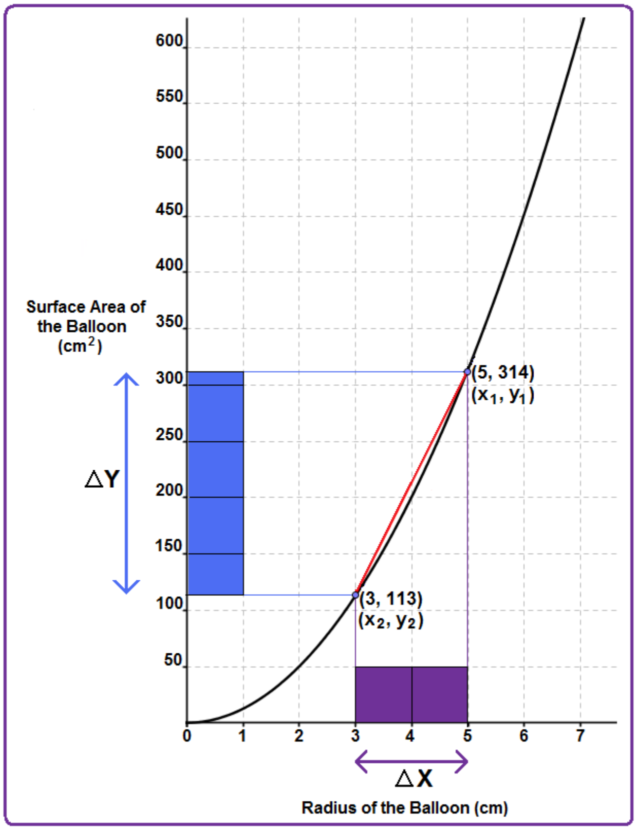


Figure 5

Again, let's estimate the slope of the tangent line at the point (5, 314) by finding the slope of the line extending from the point (5, 314) to the point (3, 113), as shown in Figure 5. The slope of this line **estimates** the slope of the tangent line at the point (5, 314).
Finish off the calculation for the slope of this line, in Expression 3, and the conclusion which follows from it:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{113\text{cm}^2 - 314\text{cm}^2}{3\text{cm} - 5\text{cm}} = \frac{\_\_\_\_\_\ \text{cm}^2}{-2\text{cm}} = \frac{\Delta y}{\Delta x} = \_\_\_\_ \qquad \textit{Expression 3}$$

Conclusion:

This means that at the point (5, 314), we **estimate** the surface area of the balloon to increase by/change by ______cm ^{2} for a 1cm increase in the radius.

Let's return to estimating the slope of the tangent line at the point (5, 314), when Δx remains a positive value, but decreases in size.

In Figure 6, we find the slope of the line extending from the point (5, 314) to the point (6, 452) so that Δx is now 1cm instead of 2cm. Calculating the slope of this line **estimates** the slope of the tangent line at the point (5, 314) more accurately than calculating the slope of the previous line, as shown in Figure 4.

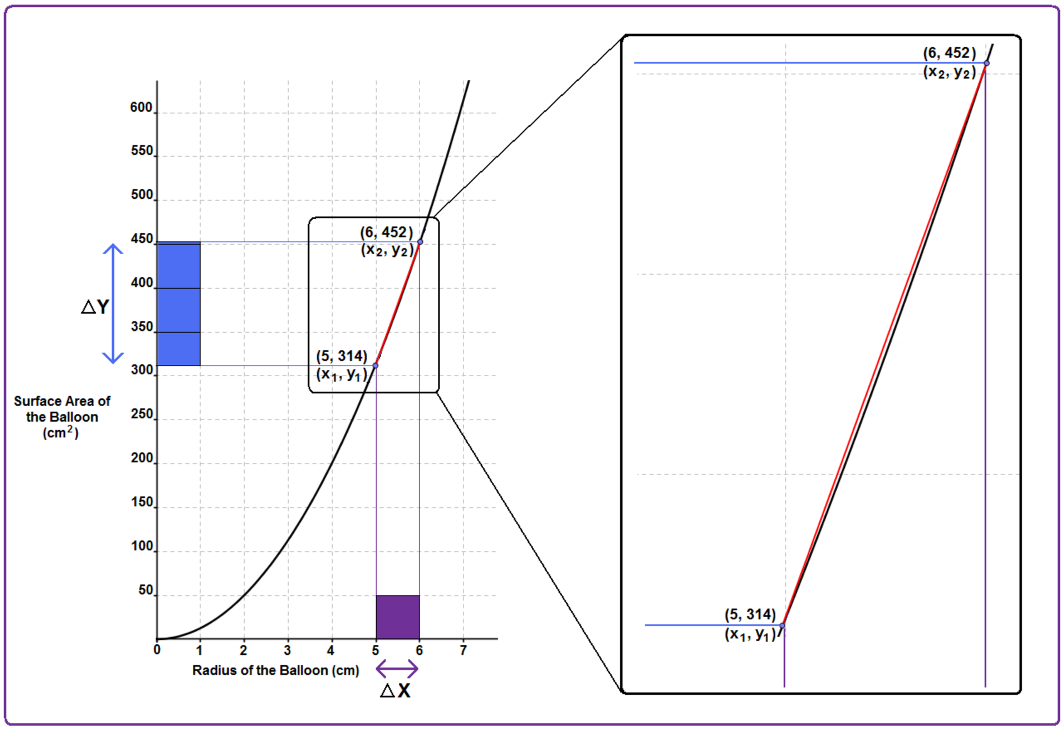


Figure 6

Finish off the calculation for the slope of this line, in Expression 4, and the conclusion which follows from it:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\_\_\_\_\_\_\text{cm}^2 - \_\_\_\_\_\_\text{cm}^2}{6\text{cm} - 5\text{cm}} = \frac{\_\_\_\_\text{cm}^2}{1\text{cm}} = \frac{\Delta y}{\Delta x} = \_\_\_\_ \quad \textbf{\textit{Expression 4}}$$

Conclusion:

This means that at the point (5, 314), we **estimate** the surface area of the balloon to increase by/change by ______cm ^{2} for a 1cm increase in the radius.

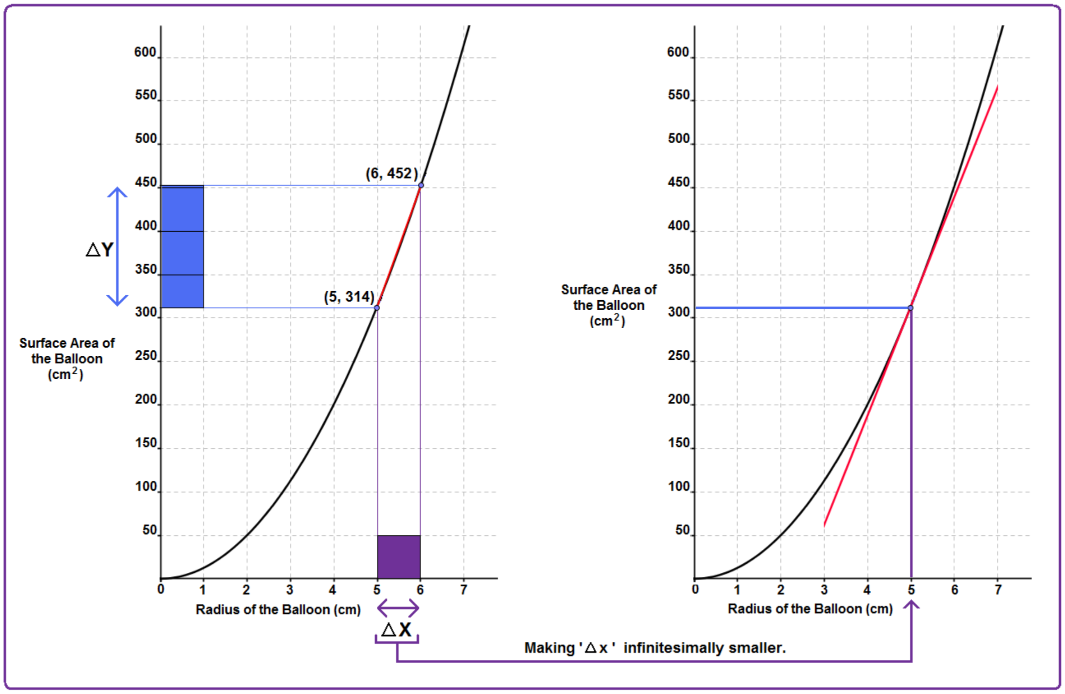


Figure 7

Let's try to find the slope of the line extending from the point (5, 314) to a point where the difference between the starting x-point and the 'new x-point' is infinitesimally small, as shown in Figure 7.
Likewise, the corresponding difference between the starting y-point and the 'new y-point' is infinitesimally small, as shown in Figure 7. The slope of this line **best estimates** the slope of the tangent line at the point (5, 314).

Instead of attempting to calculate the slope of the tangent line at the point (5, 314), it is easier to use our previous estimations, as summarised in Table 1* and Table 2* to predict what the value of this slope is.

*It should be noted that there are extra values which have been calculated for the **estimate** of the slope of the tangent line at the point (5, 341), as Δx 'approaches zero'/ 'an infinitesimally small value'. For the sake of brevity, these estimates have not been shown graphically.

Table 1 shows us that as a ***positive value of Δx*** becomes extremely small (infinitesimal), Δy becomes extremely small (infinitesimal). The smaller Δx and Δy become, the more accurate becomes the **estimate** for the value of the slope of the tangent line at the point (5, 314).

Fill in the missing value for the estimate of the slope (correct to the nearest whole number) in the last row of column four of Table 1.

Figure	Δx	Δy	Value for Slope
4	2cm	301cm^2	151
6	1cm	138cm^2	138
-	0.0001cm	0.0125cm^2	125.6649
-	0.00001cm	0.00125cm^2	125.6637
-	0.000001cm	0.0001256cm^2	125.6632
-	-	-	-
-	-	-	-
8	Approaches zero, yet does not reach zero.	Approaches zero, yet does not reach zero.	

Table 1

Table 2 shows us that as a ***negative value of Δx*** becomes extremely small (infinitesimal), Δy becomes extremely small (infinitesimal). Again, the smaller Δx and Δy become, the more accurate becomes the **estimate** for the value of the slope of the tangent line at the point (5, 314).

Fill in the missing value for the estimate of the slope (correct to the nearest whole number) in the last row of column four of Table 2.

Figure	Δx	Δy	Value for Slope
5	-2cm	-201cm^2	101
-	-1cm	-113cm^2	113
-	-0.0001cm	-0.012566cm^2	125.6624
-	-0.00001cm	-0.0012566cm^2	125.6637
-	-0.000001cm	-0.00012566cm^2	125.6635
-	-	-	-
8	Approaches zero, yet does not reach zero.	Approaches zero, yet does not reach zero.	

Table 2

Q.2 *How does the value for the estimate of the slope of the tangent line at the point (5, 314), as positive values of Δx approach zero compare with the value for the estimate of the slope of the tangent line at the same point, as negative values of Δx approach zero?*

____

____

As can be seen in Table 1 and Table 2, we predict the slope of the tangent line at the point (5, 314), to be approximately (correct to the nearest whole number) ________, as positive or negative values of Δ x 'approach zero'/ 'become infinitely small'.

The geometrical representation of the tangent line at the point (5, 314), is shown in Figure 8. The slope of the tangent line in Figure 8 is also known as the derivative $\frac{dy}{dx}$ at this point.

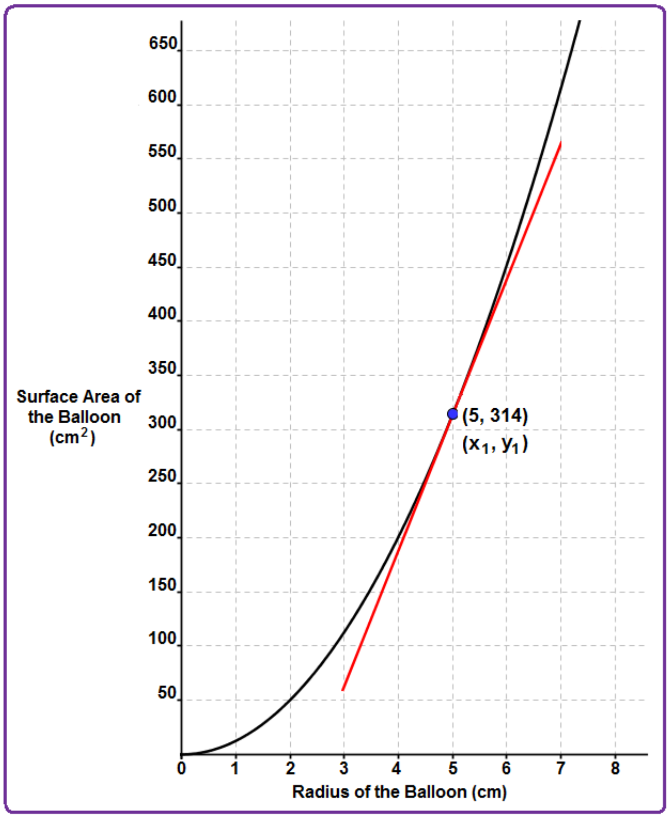


Figure 8

Thus, in Figure 8, *'the slope of the tangent line at the point (5, 314)'*/ '***derivative at the point (5, 314)'*** has a value of ______. We ask what does this mean in the context of our original question, namely:

Q.1 *By how much is the surface area of the balloon increasing (Δy), at the instant that the radius of the balloon passes through 5cm?*

Answer:

We see from Table 1 and Table 2 that as the radius of the balloon '*passes through*' /'*increases through'* the value r = 5cm, we predict the surface area of the balloon to increase by 126cm^2 per cm. In other words, we predict the slope of the tangent line at the point (5, 314) to be 126.

Summary:

The technique, which we used in this section in order to find the slope of the tangent line *touching* the graph at the point (5cm, 314cm^2) and which thus allowed us to find the derivative at this point can be written in mathematical symbols as follows:

$$\left.\frac{dy}{dx}\right|_{x=5cm} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

where:

- $\frac{dy}{dx}$ at the point x = 5cm refers to the value for the slope of the tangent line touching the graph at the point (5cm, 314cm^2).

- The expression $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ refers to the process of finding the ***limiting value*** for our estimate of the slope of the tangent line (equals the derivative) at the point (5cm, 314cm^2), as either positive or negative values of Δx approach zero. The ***limiting value*** in our case was determined by predicting the value for slope in the last row of the fourth column in both Table 1 and Table 2.

Section 3:

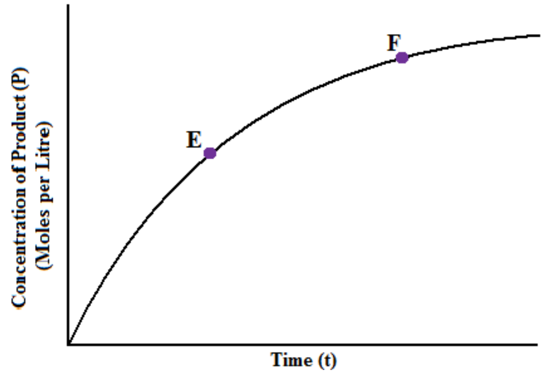


Figure 9

For a particular reaction: A+B $\rightarrow$P, where A & B are reactants, and P is product, Figure 9 shows the graph of concentration of product (P) against time (t).

Q.1 At which point, E or F, does the concentration of product (P) have a greater value?

____

Q. 2 At which point, E or F, does the greatest value of $\frac{dP}{dt}$ occur?

____

Give a reason for your answer.

____

____

____

____

____

____

If you struggled to complete Section 1 before completing Section 2 and Section 3, perhaps attempt Section 1 again.

Appendix G – Derivative-as-a-Function Intervention

Section 1:

Q. 1 Find the derivative with respect to x for the following function:

$$y = 2x$$

$$\frac{dy}{dx} = \_\_\_\_\_\_\_\_$$

Explain your reasoning:

____

____

Q.2 Which of the following graphs (Graph A, B or C) in Figure 1 represents $\frac{dy}{dx}$?

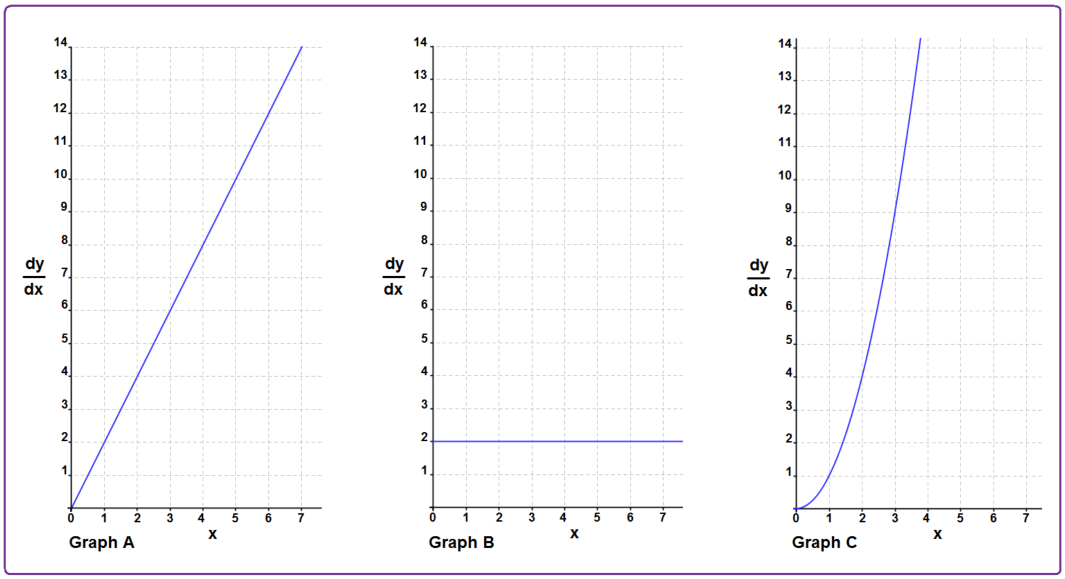


Figure 1

Explain your reasoning:

____

____

____

Section 2:

As a balloon is inflated or deflated, its surface area (in centimetres squared (cm^2)) is proportional to the square of the radius of the balloon (in centimetres (cm)). This relationship is plotted in Figure 2. Its algebraic representation is of the form:

$$y = 4\pi x^2$$

where 'y' is equal to the surface area of the balloon, and 'x' is equal to the radius of the balloon.

We can see, for example, from Figure 2, that when the radius of the balloon is 2cm, the corresponding surface area of the balloon is approximately (all such figures in this exercise are rounded-off to the nearest whole number) ________cm^2. We can also see that the surface area of the balloon changes as its radius changes.

Consider Question 1:

> ***Q.1*** *By how much is the surface area of the balloon increasing, when the radius of the balloon is instantaneously passing through any particular value?*
>
> ***or, in other words,***
>
> *we want a function, where the x-inputs are the 'radius of the balloon' and the y-outputs are 'how much the surface area of the balloon is increasing (Δy) when the radius of the balloon is instantaneously passing through any particular radius value'.*

We can attempt to generate such a function by taking a number of points on our original function—the first of which is graphically shown in Figure 3—and finding out: *how much the surface area of the balloon is increasing (Δy) when the radius of the balloon is instantaneously passing through a particular radius value.*

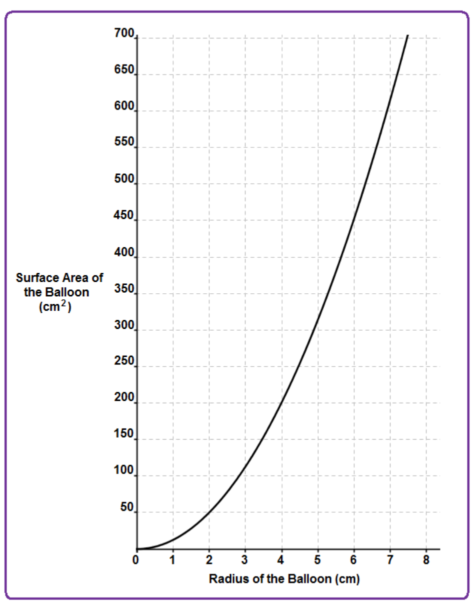


Figure 2

Section 2.1 — 1st Point:

Q.2 *By how much is the surface area of the balloon increasing (Δy) at the instant the radius of the balloon passes through 2cm?*

Answering such a question is equivalent to finding the slope of a straight line 'touching' the graph at the point (2cm, 50cm^2), as shown in Figure 3. The line at the point in question is called the tangent line.

Finding the slope of such a tangent involves a process of estimation.

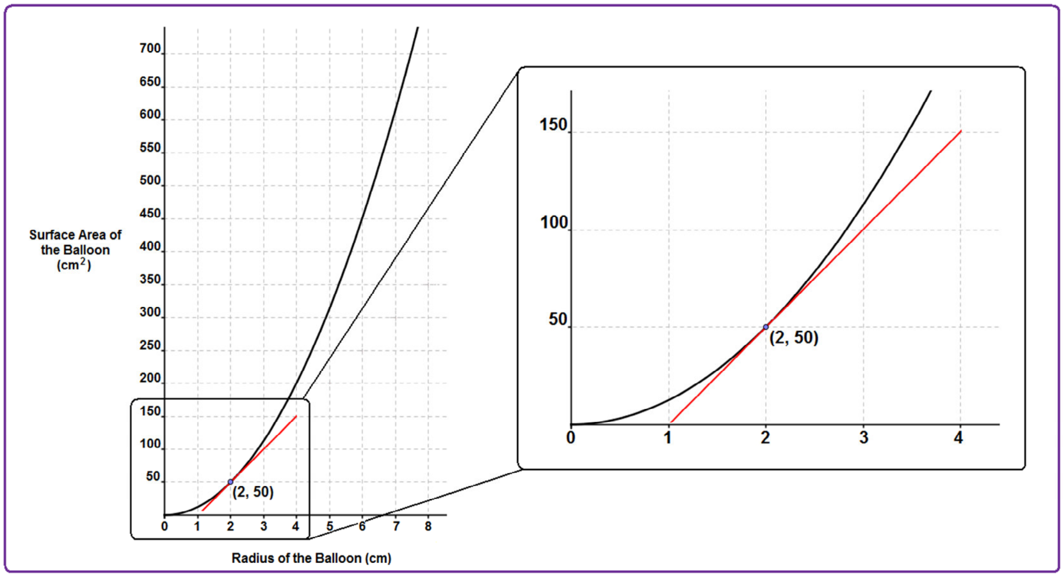


Figure 3

The process of estimating the slope of the tangent line at the point (2, 50), is summarised in both Table 1 and Table 2. Table 1 shows us that as a ***positive value of Δx*** becomes extremely small (infinitesimal), Δy becomes extremely small (infinitesimal). The smaller Δx and Δy become, the more accurate becomes the **estimate** for the value of the slope of the tangent line at the point (2, 50). Fill in the missing value for slope (correct to the nearest whole number) in the last row of column three of the table.

Δx	**Δy**	**Value for Slope**
1cm	62.8318	62.8318
0.0001cm	5.0266×10^{-3}	50.2667
0.00001cm	5.0265×10^{-4}	50.2656
-	-	-
-	-	-
Approaches zero, yet does not reach zero.	Approaches zero, yet does not reach zero.	

Table 1

Table 2 shows us that as a ***negative value of Δx*** becomes extremely small (infinitesimal), Δy becomes extremely small (infinitesimal). Again, the smaller Δx and Δy become, the more accurate becomes the **estimate** for the value of the slope of the tangent line at the point (2, 50). Fill in the missing value for slope (correct to the nearest whole number) in the last row of column three of the table.

Δx	Δy	Value for Slope
-1cm	-37.6991	37.6991
-0.0001cm	-50.2642	50.2642
-0.00001cm	-50.2653	50.2553
-	-	-
-	-	-
Approaches zero, yet does not reach zero.	Approaches zero, yet does not reach zero.	

Table 2

From the value for slope predicted in the last row of column three in both Table 1 and Table 2, we predict the value of the slope of the tangent line at the point (2, 50) to be, approximately, (correct to the nearest whole number) ________ as positive or negative values of Δx 'approach zero'/ 'become infinitely small'.

The geometrical representation of the tangent line is shown in Figure 4. The slope of the tangent line in Figure 4 is also known as the derivative $\frac{dy}{dx}$ at this point.

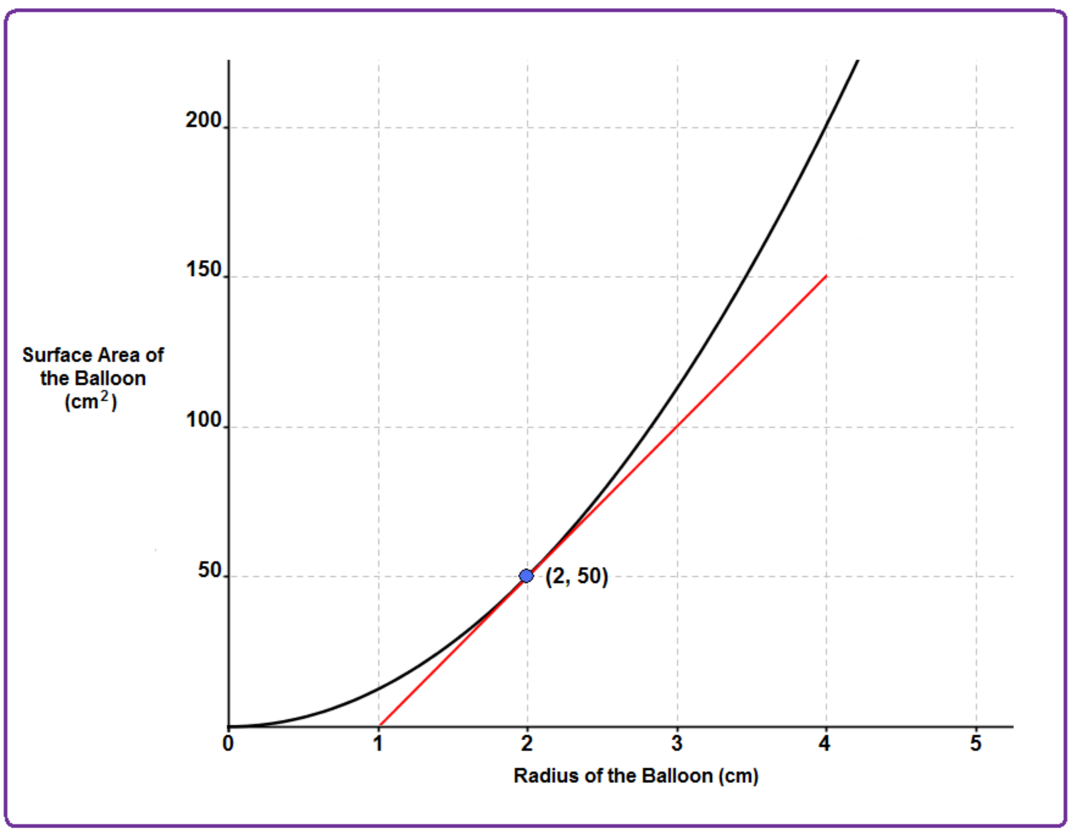


Figure 4

In Figure 4, the slope of the *tangent line*/***derivative at the point (2, 50)*** has a value of _____. We ask what this means in the context of question two, namely:

Q.2 *By how much is the surface area of the balloon increasing (Δy) at the instant the radius of the balloon passes through 2cm?*

Answer:

We see from both Table 1 and Table 2 that as the radius of the balloon *passes through/increases through* the value r = 2cm, we predict the surface area of the balloon to increase by $50cm^2$ per cm. In other words, we predict the slope of the tangent line at the point (2, 50) to be 50.

Summary:

The technique which we used in this section in order to find the slope of the tangent line *touching* the graph at the point (2cm, $50cm^2$), and which thus allowed us to find the derivative at this point, can be written in mathematical symbols, as follows:

$$\left.\frac{dy}{dx}\right|_{x=2cm} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Where:

- $\frac{dy}{dx}$ at the point x = 2cm refers to the value for the slope of the tangent line touching the graph at the point (2cm, $50cm^2$).
- The expression $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ refers to the process of finding the ***limiting value*** for our estimate of the slope of the tangent line (equals the derivative) at the point (2cm, $50cm^2$), as either positive or negative values of Δx approach zero. The ***limiting value*** in our case was determined by predicting the value of the last row in the third column of both Table 1 and Table 2.

Section 2.2 — 2nd Point:

Q.3 *By how much is the surface area of the balloon increasing (Δy) at the instant the radius of the balloon passes through 4cm?*

Answering such a question is equivalent to finding the slope of a straight line 'touching' the graph at the point (4cm, 201cm^2), as shown in Figure 5. Again, such a straight line is called the tangent line at the point in question.

Again, finding the slope of such a tangent line involves a process of estimation.

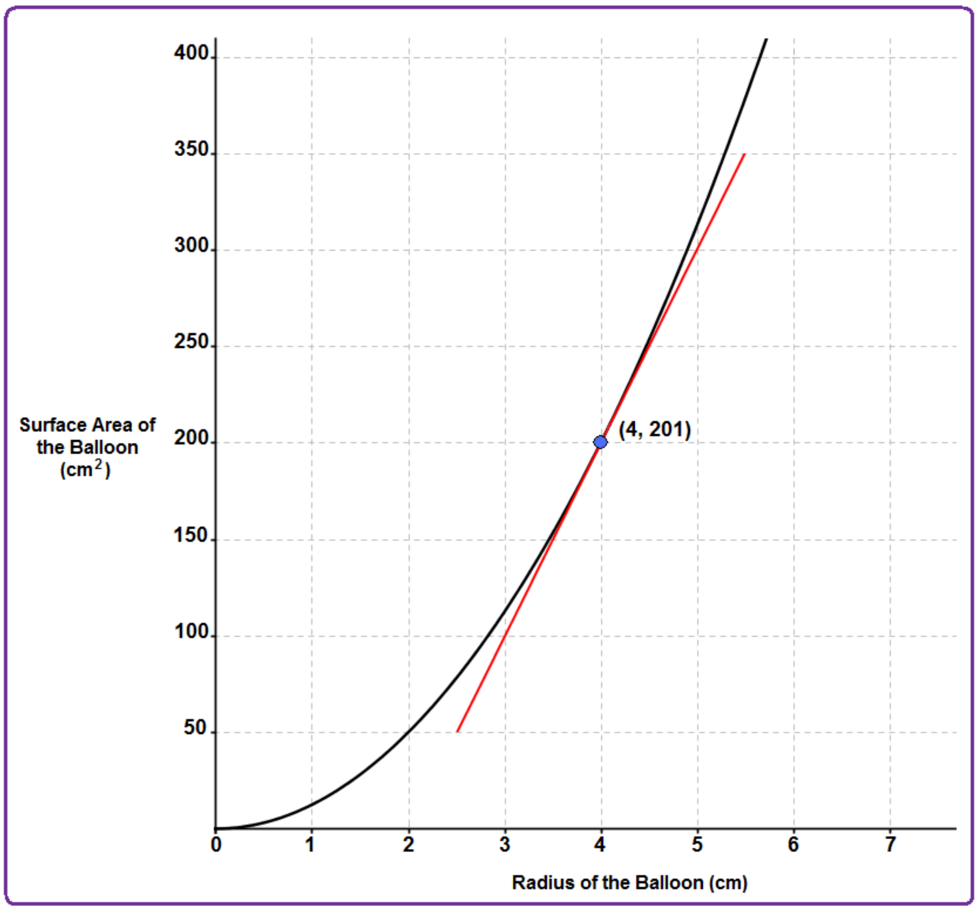


Figure 5

Such a process of estimation is, for the sake of brevity, summarised in Table 3. Fill in the missing value for slope (correct to the nearest whole number) in the last row of column two and column four of the table.

Δx	**Value for Slope**	**Δx**	**Value for Slope**
1cm	113.0973	-1cm	87.9645
0.0001cm	100.5322	-0.0001cm	100.5297
0.00001cm	100.5310	-0.00001cm	100.5309
-	-	-	-
-	-	-	-
-	-	-	-
Approaches zero, yet does not reach zero.		Approaches zero, yet does not reach zero.	

Table 3

From the value for slope predicted in the last row of column two and four in Table 3, we predict the value of the slope of the tangent line at the point (4, 201) to be, approximately (correct to the nearest whole number) ________, as positive or negative values of Δx 'approach zero'/ 'become infinitely small'.

The geometrical representation of the tangent line is shown in Figure 5. Again, the slope of the tangent line in Figure 5 is also known as the derivative $\frac{dy}{dx}$ at this point.

In Figure 5, the slope of the *tangent line*/***derivative at the point (4,201)*** has a value of ________. Again, we ask, what does this mean in the context of question three, namely:

Q.3 *By how much is the surface area of the balloon increasing (Δy) at the instant the radius of the balloon passes through 4cm?*

Answer:

We see from Table 3 that as the radius of the balloon *passes through*/*increases through* the value r = 4cm, we predict the surface area of the balloon to increase by 100cm^2 per cm. In other words, we predict the slope of the tangent line at the point (4, 201) to be 100.

Section 2.3 — 3rd Point:

Q.4 *By how much is the surface area of the balloon increasing (Δy) at the instant the radius of the balloon passes through 6cm?*

Again, answering such a question is equivalent to finding the slope of a straight line 'touching' the graph at the point (6cm, 452cm^2), as shown in Figure 6. Again, such a straight line is called the tangent line at the point in question.

Again, finding the slope of such a tangent involves a process of estimation.

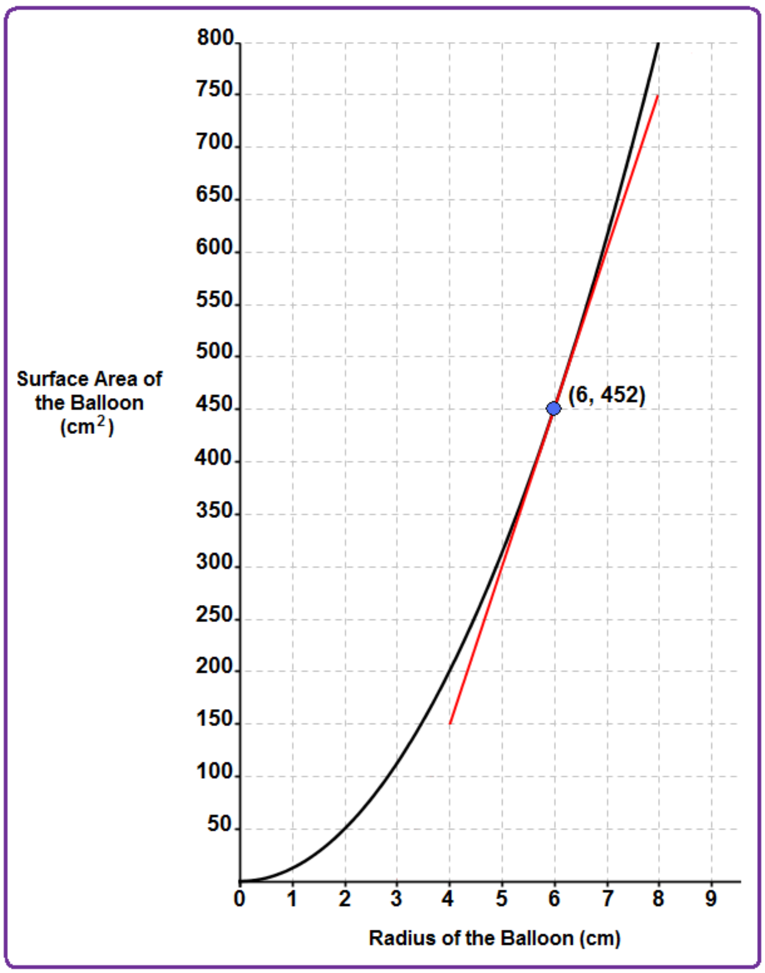


Figure 6

Such a process of estimation is, for the sake of brevity, again summarised, as shown in Table 4. Fill in the missing value for slope (correct to the nearest whole number) in the last row of column two and column four of the table.

Δx	Value for Slope	Δx	Value for Slope
1cm	163.3628	-1cm	138.2300
0.0001cm	150.7977	-0.0001cm	150.7951
0.00001cm	150.7964	-0.00001cm	150.7964
-	-	-	-
-	-	-	-
-	-	-	-
Approaches zero, yet does not reach zero.		Approaches zero, yet does not reach zero.	

Table 4

From the value for slope predicted in the last row of column two and four in Table 4, we predict the value of the slope of the tangent line at the point (6, 452) to be, approximately (correct to the nearest whole number) ________, as positive or negative values of Δx 'approach zero'/ 'become infinitely small'.

The geometrical representation of the tangent line is shown in Figure 6. Again, the slope of the tangent line in Figure 6 is also known as the derivative $\frac{dy}{dx}$ at this point.

In Figure 6, we can see that the slope of the *tangent line/**derivative at the point 6, 452)*** has a value of _______. Again, we can ask what does this mean in the context of question four, namely:

Q.4 *By how much is the surface area of the balloon increasing (Δy) at the instant the radius of the balloon passes through 6cm?*

Answer:

We see from Table 4 that as the radius of the balloon *passes through/increases through* the value r = 6cm, we predict the surface area of the balloon to increase by 151cm^2 per cm. In other words, we predict the slope of the tangent line at the point (6, 452) to be 151.

Section 2.4—Returning to Our Original Question:

Q.1 *By how much is the surface area of the balloon increasing when the radius of the balloon is instantaneously passing through any particular value?*

or, in other words,

How do we generate a function, where the x-inputs are the 'radius of the balloon' and the y-outputs are 'how much the surface area of the balloon is increasing (Δy) when the radius of the balloon is instantaneously passing through any particular value'. Let's tabulate our work from Section 2.1-2.3 in Table 5.

Point	***Radius of the Balloon (cm)***	$\frac{dy}{dx}$
(2, 50)	2	50
(4, 201)	4	100
(6, 452)	6	150

Table 5

Plotting the results of the 2nd and 3rd columns of Table 5 produces the following as shown in Figure 7.

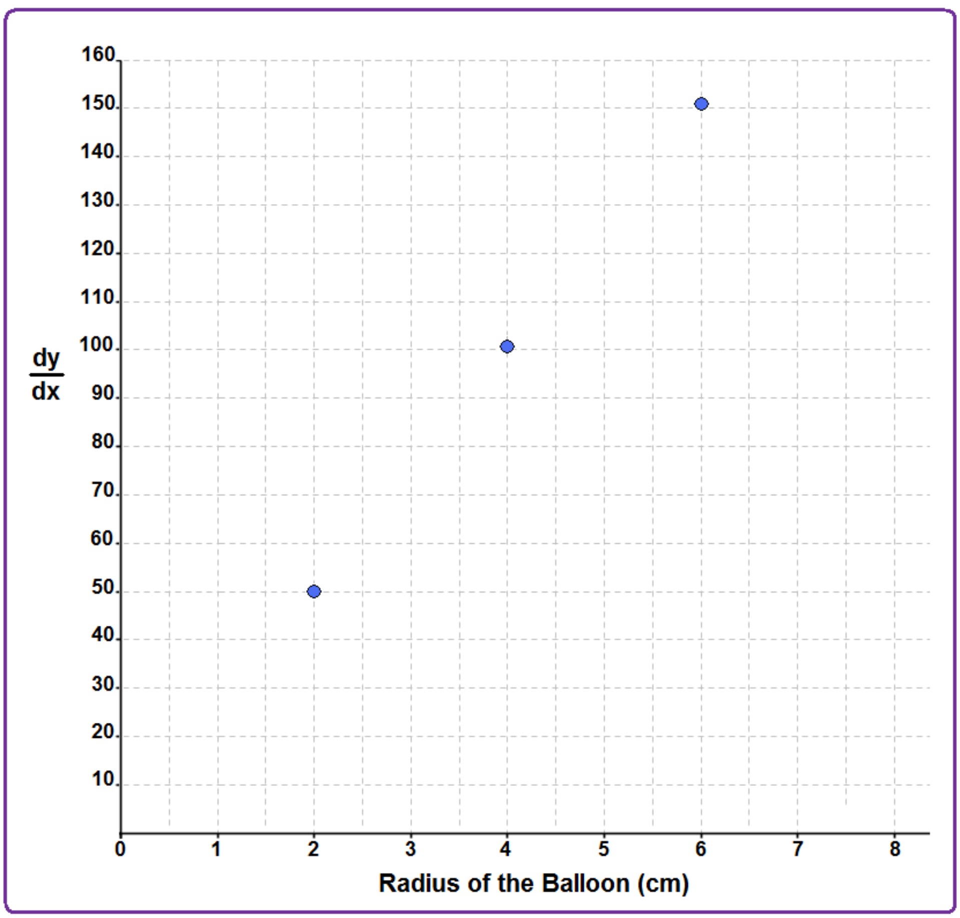


Figure 7

We can see that the points appear to be in a line; this raises the question of whether or not the derivative of the surface area of the balloon with respect to the radius is linear. Let's return to the function for the surface area of the balloon, as shown in Expression 1.

$$y = 4\pi x^2 \qquad \textit{Expression 1}$$

Finding the derivative of y with respect to x produces:

$$\frac{dy}{dx} = 8\pi x \qquad \textit{Expression 2}$$

From the points plotted in Figure 7, it is clear that we are plotting the derivative of the surface area of the balloon against its radius. What appears to be a linear relationship is confirmed after we differentiate Expression 1 to produce Expression 2. Expression 2 is analogous to y = mx: where m is equivalent to 8π; x is equivalent to x and $\frac{dy}{dx}$ is equivalent to y.

Thus, the 'apparent' linear relationship in Figure 7 is, in fact, so, and can be graphed as shown in Figure 8.

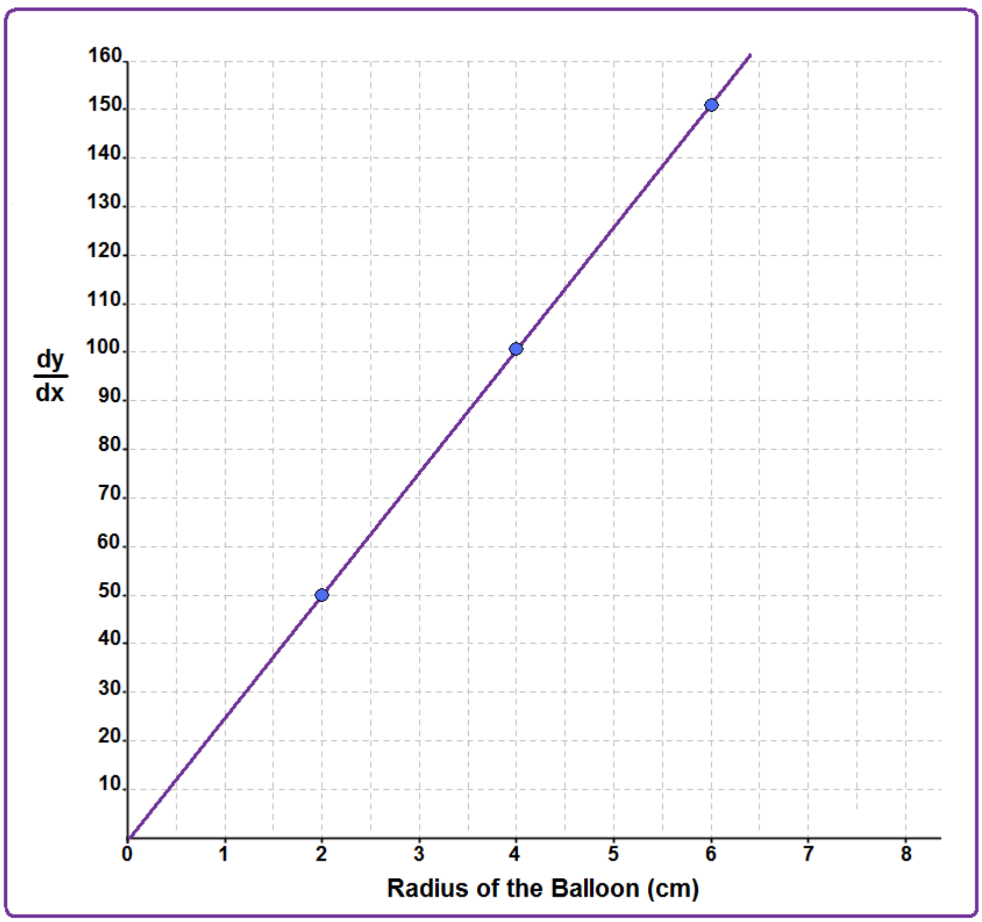


Figure 8

Q.5 By how much is the surface area of the balloon increasing per cm when the radius of the balloon is instantaneously passing through 5cm?

Q.6 Using Expression 2, can you calculate how much the surface area of the balloon is increasing when the radius of the balloon is instantaneously passing through 8cm?

Let's compare Figure 8 with the graph of the surface area of the balloon as a function of its radius. (Figure 2). Such a comparison yields Figure 9.

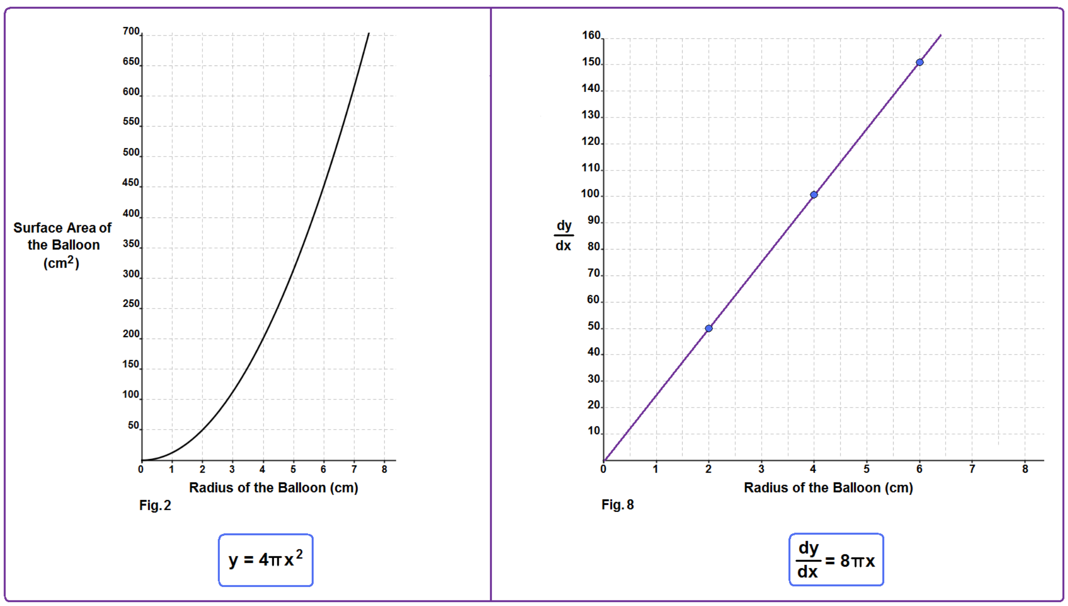


Figure 9

We can see more clearly that Figure 8 in Figure 9 is the graph of the derivative of the expression: $\mathbf{y}=4\pi x^2$. The algebraic expression for the derivative of the expression in Figure 2 is $\frac{dy}{dx} = 8\pi x$.

The derivative graph in Figure 8 can be read in order to tell us: *How much the surface area of the balloon is increasing when the radius of the balloon is instantaneously passing through any particular radius value; the algebraic part of the derivative (Expression 2), namely* $\frac{dy}{dx} = 8\pi x$ *also tells us this when we substitute the radius value in question into the expression.*

Section 3:

Q. 1 Find the derivative of the following function with respect to T:

$$p = \frac{nrT}{v}$$

where $\frac{nr}{v}$ is a constant, with a value equivalent to 2.

$$\frac{dp}{dT} = \_\_\_\_\_\_\_\_$$

Explain your reasoning:

____

____

Q. 2 Which of the following graphs (Graph A, B or C), in Figure 10, represents $\frac{dp}{dT}$?

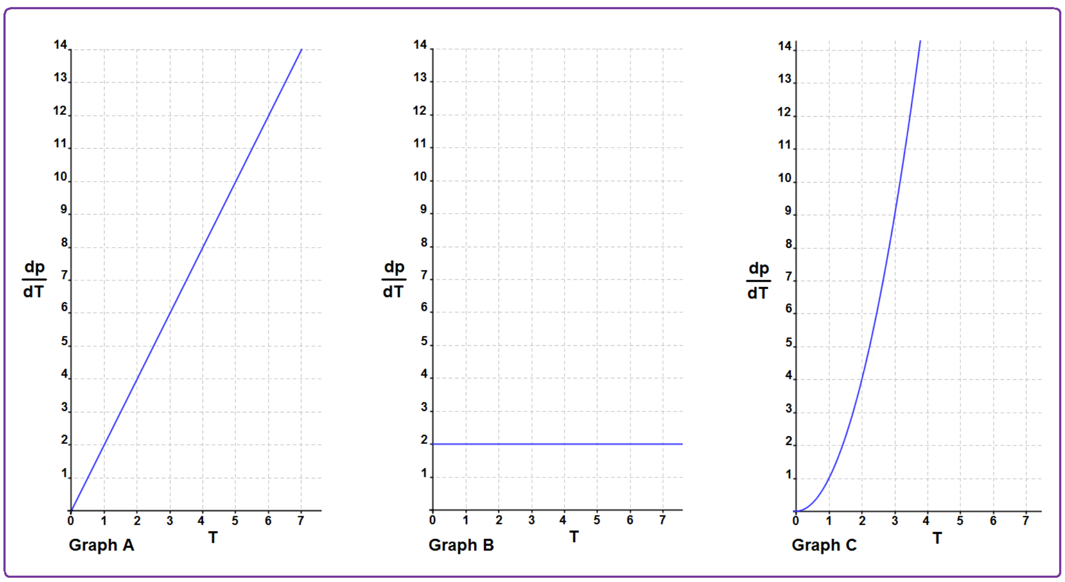


Figure 10

Explain your reasoning:

____

____

____

Appendix H – Integration Intervention

Section 1:

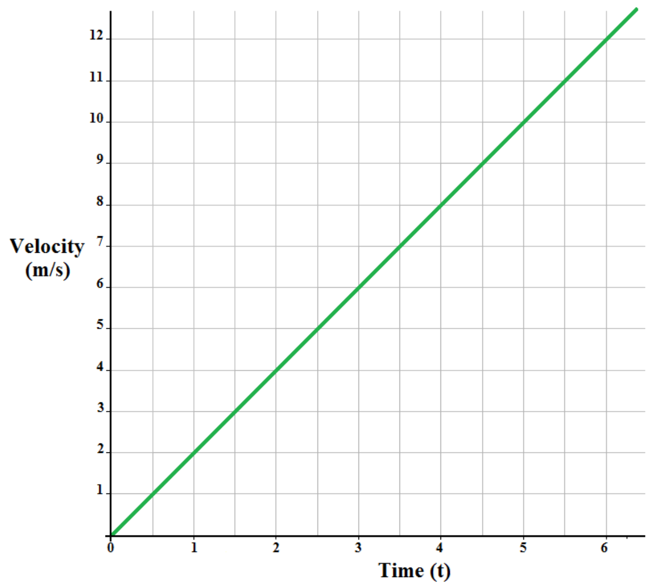


Figure 1

Figure 1 shows the velocity of a moving body as a function of time: $v=2t$, where velocity v is in units of metres per second and time t is in seconds.

***Q.1** Calculate the area underneath the graph between time t = 0 seconds and time t = 6 seconds, as highlighted in Figure 2. As the area to be calculated has the shape of a triangle, you can use this formula:*

$$\text{Area of a triangle} = \frac{1}{2}(\text{base} \times \text{perpendicular height})$$

where:

- The base of the triangle is the length of time from t = 0 seconds to t = 6 seconds; namely it is 6 seconds (6s).
- The perpendicular height of the triangle is the velocity at time t = 6 seconds; namely, it is 12 metres per second (12m/s).

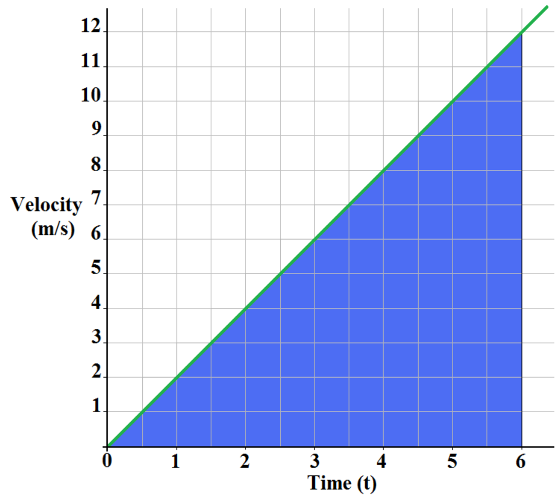


Figure 2

Thus:

$$\text{The area} = \frac{1}{2}(\_\_\_\_ \times \_\_\_\_) = \_\_\_\_\_$$

Q.2 *In terms of a physical interpretation, what does the value for the area you have calculated mean? Does it give you the displacement of the body (the distance the body travels in a certain direction) between t = 0 seconds and t = 6 seconds?*

____

____

Find the area underneath the graph and between the horizontal axis as time varies, as shown in Figures 3, 4, 5, 6 and 7. Use the units ‘m/s’ for the height of the triangle and ‘s’ for the base.

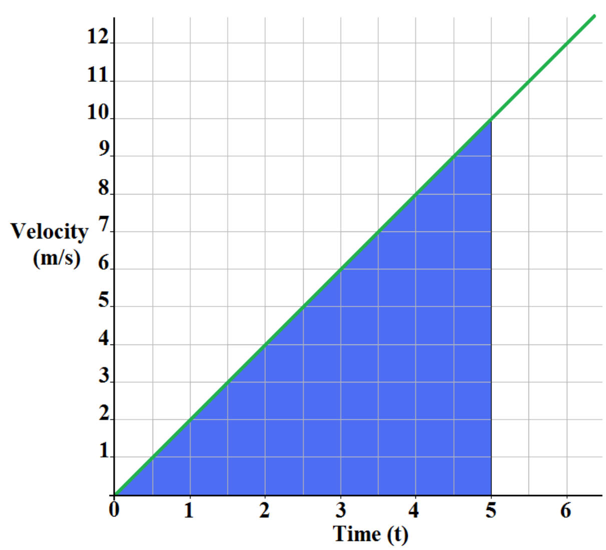


Figure 3

$$\text{The area in Figure 3} = \frac{1}{2}(\_\_\_\_ \times \_\_\_\_) = \_\_\_\_\_$$

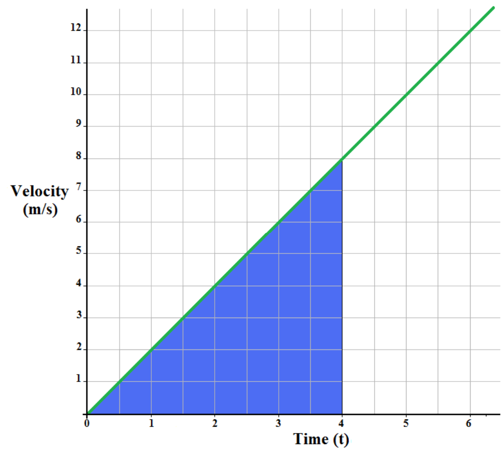


Fig. 4

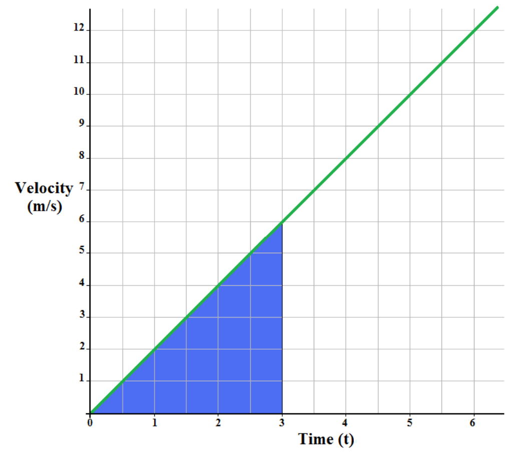


Fig. 5

The area in Figure 4 = $\frac{1}{2}$(_____ × _____) = ______

The area in Figure 5 = $\frac{1}{2}$(_____ × _____) = ______

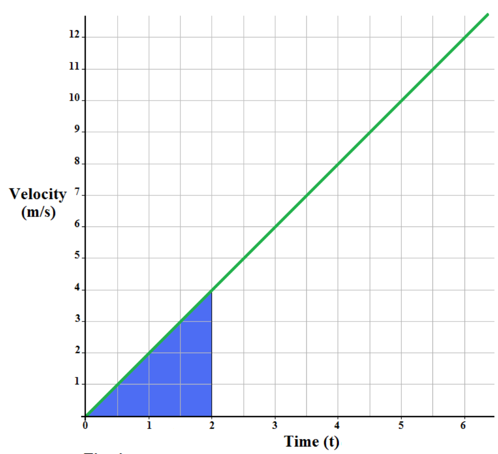


Fig. 6

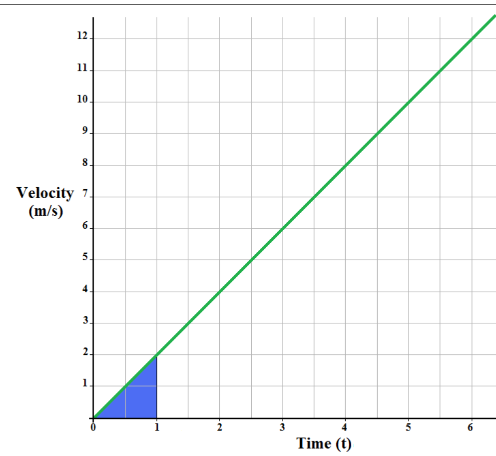


Fig.7

The area in Figure 6 = $\frac{1}{2}$(_____ × _____) = ______

The area in Figure 7 = $\frac{1}{2}$(_____ × _____) = ______

Tabulate your results in Table 1 for the 'area underneath the graph and between the horizontal axis' as time varies, which you calculated for the Figures 3, 4, 5, 6 and 7.

Figure	*Length of the base of the triangle.*	*Height of the triangle*	*Area/ displacement*
2	6 seconds	12 metres per second	36 metres
3	5 seconds		
4	4 seconds		
5	3 seconds		
6	2 seconds		
7	1 seconds		

Table 1

Plotting the results of the 'area' of the triangle in Figure 1, as the base of the triangle (time) varies, will produce the following set of points highlighted in Figure 8.

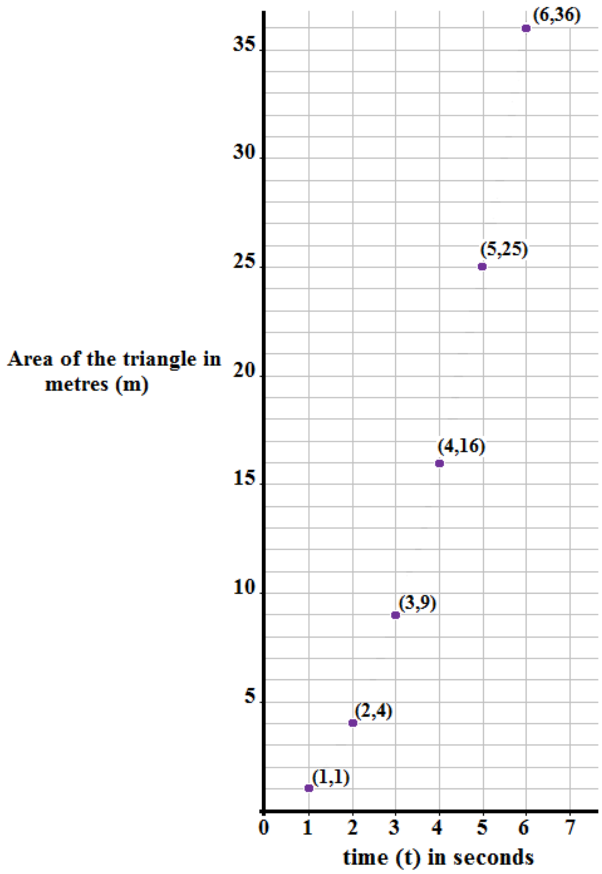


Figure 8

Observing Figure 8, we can join the points and thus produce the shape of the graph which represents the areas of the triangles in Figures 1-7, as a function of the base of the triangle (time) varying. This graph is shown in Figure 9.

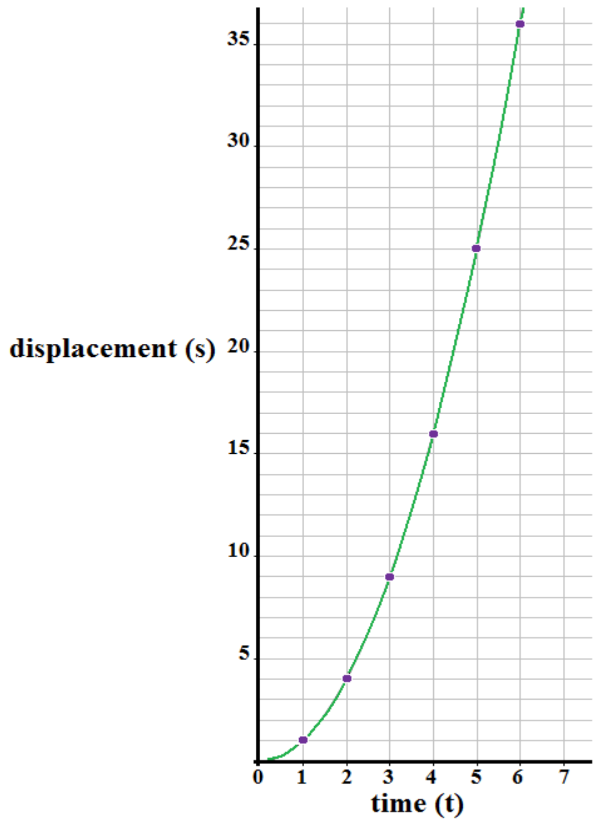


Figure 9

Because the area in Figure 9 is representative of the displacement of the body as a function of time, the graph is labelled accordingly. The displacement is denoted s and time t, where t is in seconds. The shape of the graph in Figure 9 appears to be reflective of a squared-type relationship between the displacement of the body and time. Thus, the functional relationship in Figure 9 is of the form: $s = t^2$.

To summarise:

Finding the area underneath our original function: $v=2t$ in Figure 1, as a function of the base/time varying, generates the following function:

$$s = t^2,$$

where s = displacement and t = time.

Q.3 Find the derivative of s with respect to t for the function:

$$s = t^2,$$

$$\frac{ds}{dt} = \_\_\_\_$$

$\frac{ds}{dt}$ can also be written as v.

You should see that after differentiating we get the function graphed in Figure 1, namely the function which we aimed to find the area underneath as the value for the base/time varied.

Summary:

- We started with the function $v=2t$
- We found that the area under the graph for the above function as t varies is given by the function $s = t^2$.
- $v=2t$ and $s = t^2$ are related by the fact that $v = \frac{ds}{dt}$
- **Inference Thus Far:**

  If we are given any function f(x), such as the one shown in Figure 10, then maybe the shaded area is found by using a function F(x) that satisfies: $f(x) = \frac{dF(x)}{dx}$

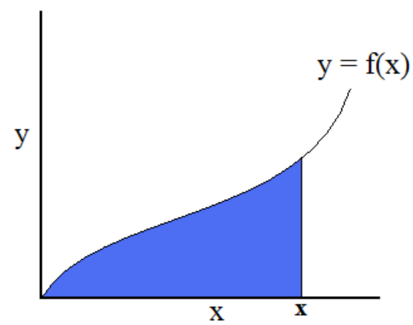


Figure 10

Section 2:

Next, we consider a different case where the function in Figure 11 is of the form:

$$v = 3t^2 + 1. \quad \textit{\textbf{Expression 1}}$$

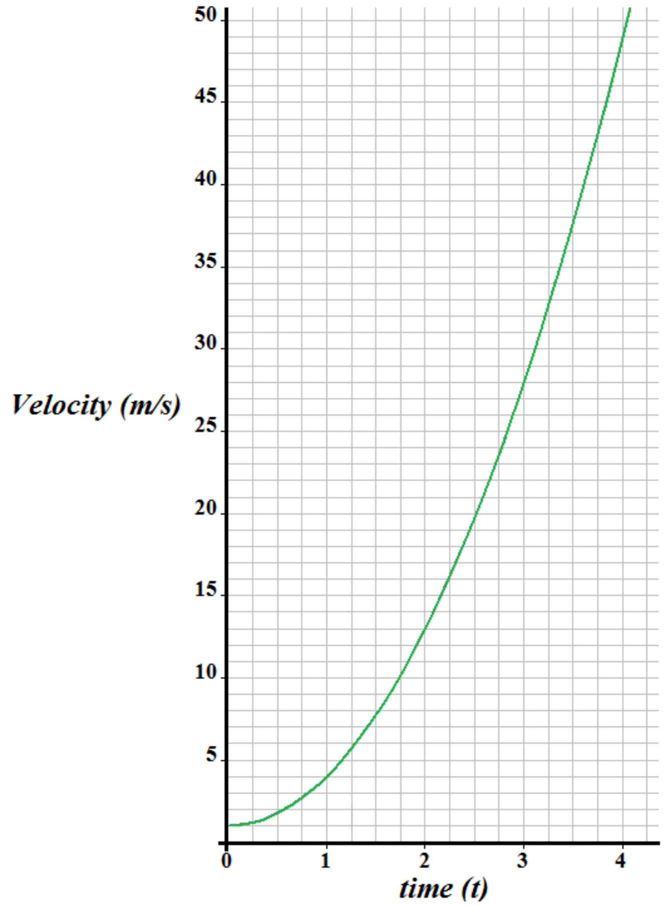


Figure 11

Q.5 *Can you calculate the area underneath the graph and between time t = 0 seconds and time t = 4 seconds, as highlighted in Figure 12?*

____

____

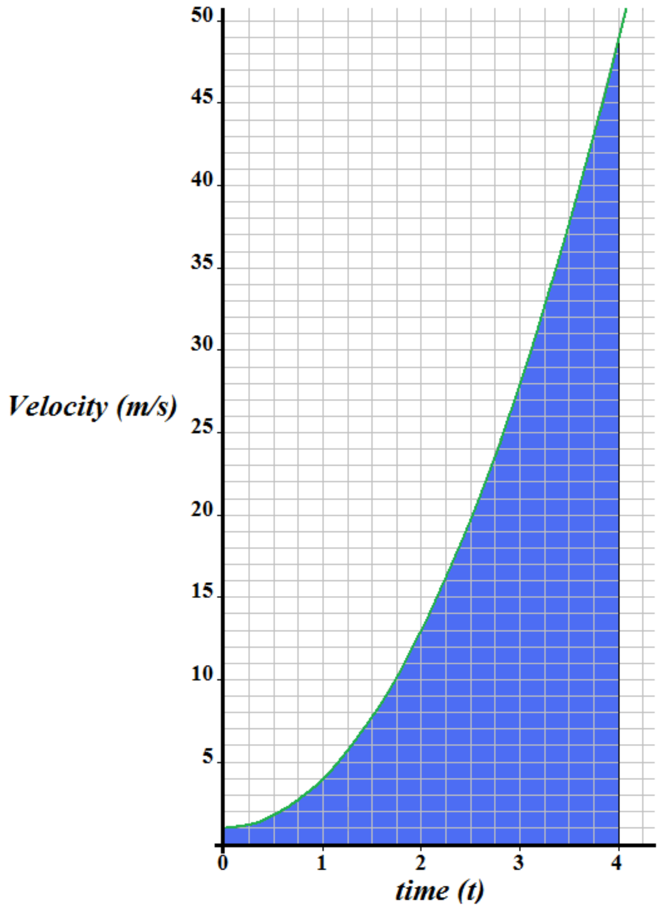


Figure 12

As the area to be calculated does not have the shape of a triangle, we must use a different approach to calculate the area.

Q.6 How do you think we might find the area?

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We can do this by finding the area of a number of rectangles which touch the graph from either above or below. This statement is now made more explicit.

Section 2.1:

The Upper-Sum Estimation:

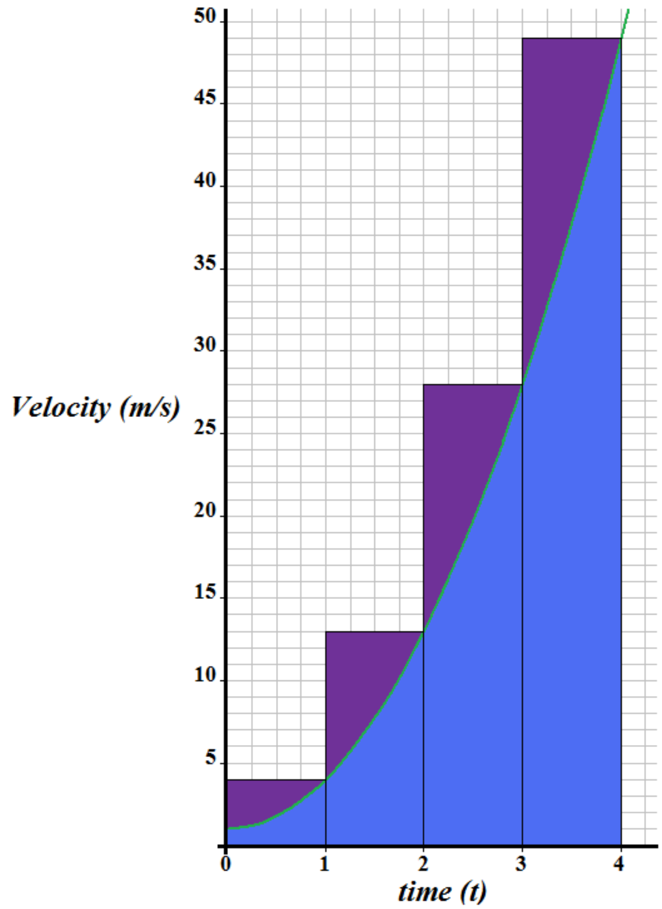

Figure 13

Figure 13 shows four rectangles that together contain the region whose area we want to estimate.The purple part of each rectangle lies outside the area. Each rectangle has width of 1 second. The height of each rectangle is obtained by evaluating the function at the right endpoint of the base of each rectangle*. The total area of the sum of the 4 rectangles over-estimates the area of the region we want to find.
The area is approximately equal to:

Σ Area of the rectangles

= [(4m/s).(1s) + (13m/s).(1s) + (28m/s).(1s) + (49m/s).(1s)]

= ?

where the symbol 'Σ' (pronounced 'sigma') signifies the fact that we are summing the area of a finite number of rectangles; in this case four.

* We note that in this case, this is the maximum value of v over the base of the rectangle.

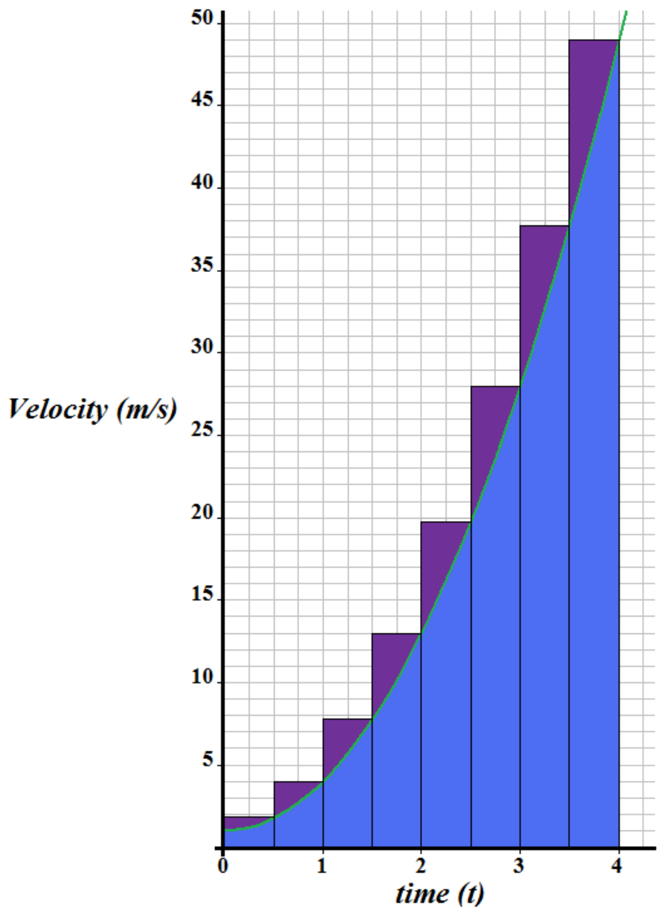

Figure 14

Figure 14 shows eight rectangles that together contain the region whose area we want to estimate.The purple part of each rectangle lies outside the area. Each rectangle has width of 0.5 seconds. The height of each rectangle is obtained by evaluating the function at the right endpoint of the base of each rectangle. The total area of the sum of the eight rectangles over-estimates the area of the region we want to find.

The area is approximately equal to:

Σ Area of the rectangles

= [(1.75m/s).(0.5s) + (4m/s).(0.5s) + (7.75m/s).(0.5s) + (13m/s).(0.5s) + (19.75m/s).(0.5s) + (28m/s).(0.5s) + (37.75m/s).(0.5s) + (49m/s).(0.5s)]

= ?

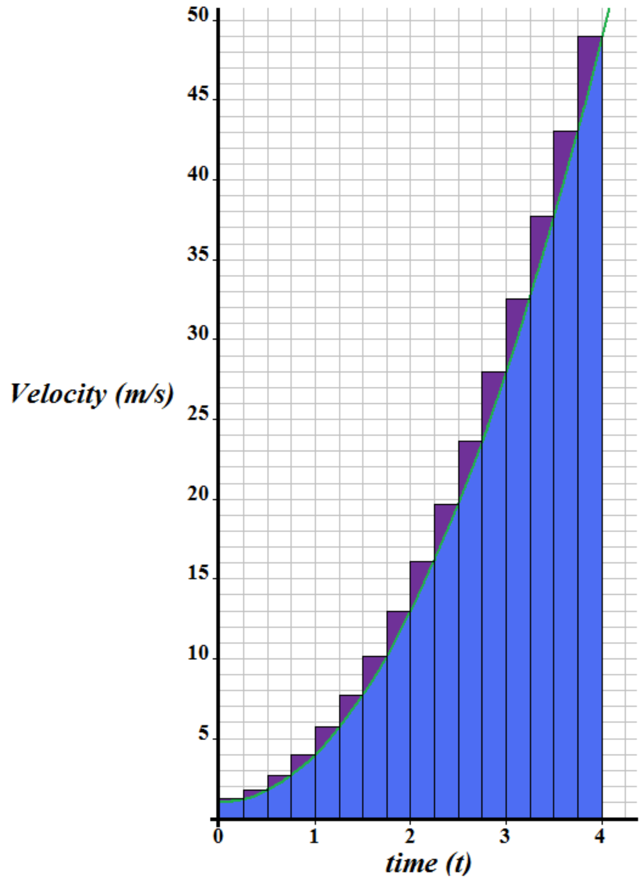

Figure 15

Figure 15 shows 16 rectangles that together contain the region whose area we want to estimate.The purple part of each rectangle lies outside the area. Each rectangle has width of 0.25 seconds. The height of each rectangle is again obtained by evaluating the function at the right endpoint of the base of each rectangle.

Q.7 Which of the estimates will be the best: Figure 13, Figure 14 or Figure 15?

____

Explain

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____

Q.8 Will the best estimate, identified in question 7, be an overestimate or an underestimate?

____

Explain

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____

Section 2.2:

The Lower-Sum Estimation:

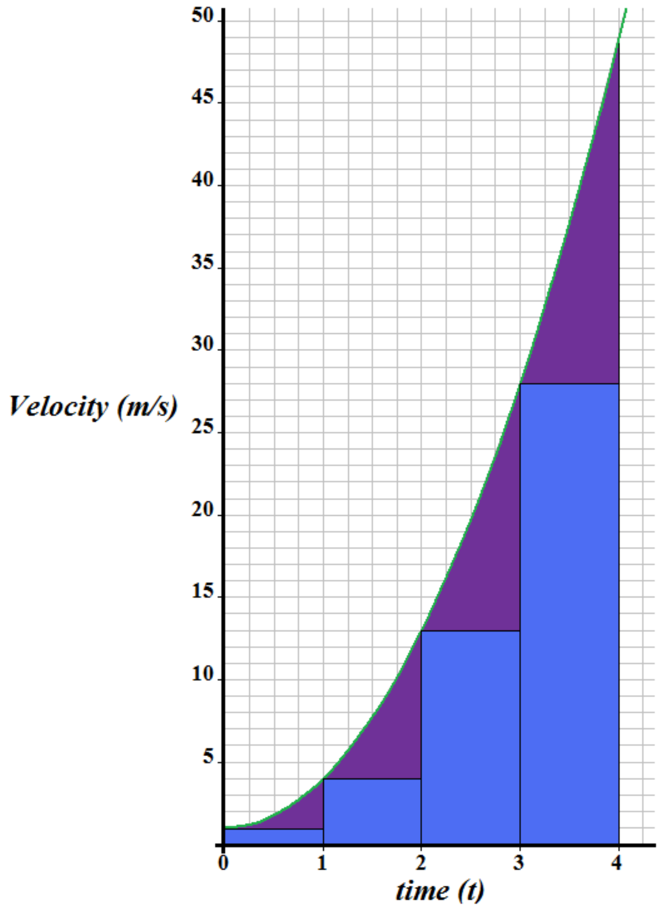

Figure 16

Figure 16 shows four rectangles ***inside*** the region whose area we want to estimate.The purple area in the figure is the area which our rectangles fail to enclose. When the rectangles are summed together, they underestimate the area of the region we want to find. Each rectangle has a width of one second. The height of each rectangle is obtained by evaluating the function at the left endpoint of the base of each rectangle*. The total area of the sum of the four rectangles under-estimates the area of the region we want to find.

The area is approximately equal to:

Σ Area of the rectangles

= [(1m/s).(1s) + (4m/s).(1s) + (13m/s).(1s) + (28m/s).(1s)]

= ?

* We note that in this case, this is the minimum value of v over the base of the rectangle.

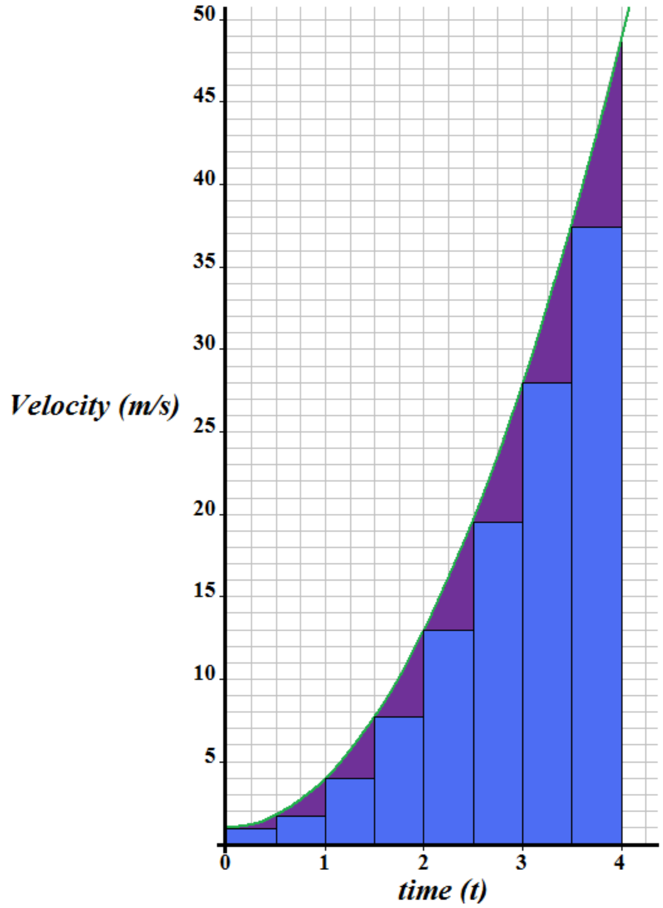

Figure 17

Figure 17 shows 8 rectangles ***inside*** the region whose area we want to estimate. When the rectangles are summed together, they again underestimate the area of the region we want to find. Each rectangle has width of 0.5 seconds. Again, the height of each rectangle is obtained by evaluating the function at the left endpoint of the base of each rectangle.

The area is approximately equal to:

Σ Area of the rectangles

= [(1m/s).(0.5s) + (1.75m/s).(0.5s) + (4m/s).(0.5s) + (7.75m/s).(0.5s) + (13m/s).(0.5s) + (19.75m/s).(0.5s) + (28m/s).(0.5s) + (37.75m/s).(0.5s)]

= ?

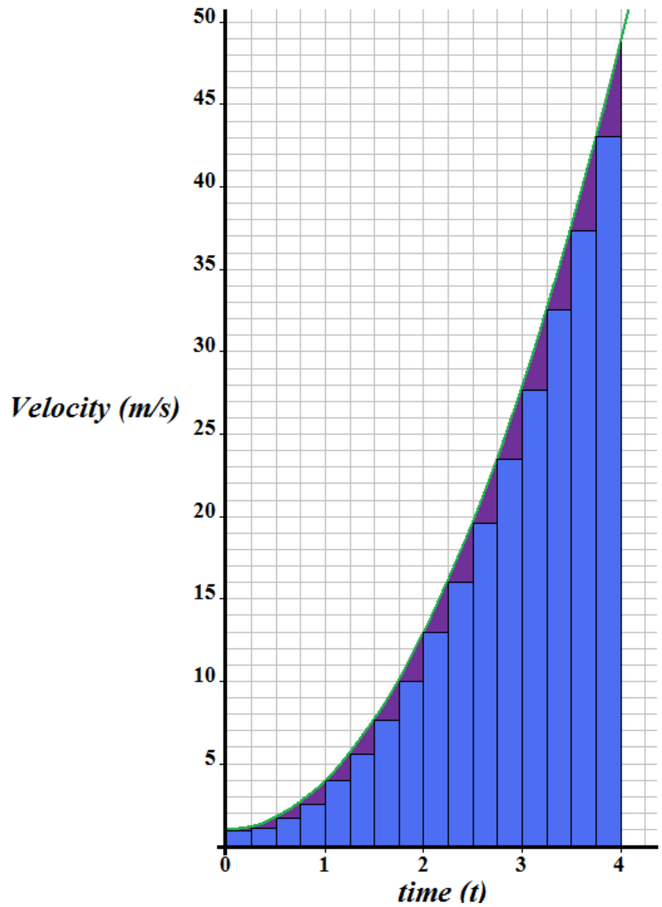

Figure 18

Figure 18 shows 16 rectangles that when summed together again underestimate the area of the region whose area we want to estimate. Each rectangle has width of 0.25 seconds. The height of each rectangle is obtained by evaluating the function at the left endpoint of the base of each rectangle.

Q.9 Which of the estimates will be the best: Figure 16, Figure 17 or Figure 18?

Explain

Q.10 Will the estimate identified in Question 9 be an overestimate or an underestimate?

Explain

Conclusion:

The estimate for the area of the region in Figure 12 is somewhere between the lower and upper-sum estimations for the area:

Lower-sum estimation **< area of the region <** Upper-sum estimation

Table 2 shows the values of the lower and upper-sum estimations for the area of the region in Figure 12 using up to 1000 rectangles.

Number of subintervals/rectangles	***Base Width of Each Rectangle***	***Lower Sum / Under-Estimate***	***Upper Sum/Over-Estimate***	***Area of the region***
4	1	46.00m	94.00m	*Between 46m and 94m*
8	0.5	56.50m	80.50m	*Between 56.50m and 80.50m*
16	0.25	62.12m	74.12m	*Between 62.12m and 74.12m*
50	0.08	66.09m	69.93m	*Between 66.09m and 69.93m*
100	0.04	67.04m	68.96m	*Between 67.04m and 68.96m*
1,000	0.004	67.90m	68.09m	*Between 67.90m and 68.09m*

Table 2

Q.11 How do you think we could get a precise value for the area of the region in Figure 12?

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Q.12 Would taking a value for the ***limit,*** as the ***base width of each rectangle*** goes to zero and the ***number of rectangles*** goes to infinity, give us the precise value for the area of the region in Figure 12?

____

____

Explain

____

____

Let's investigate ***Question 12*** further; more specifically let's investigate the value for ***the limit of the upper-sum estimation*** as the base width of each rectangle goes to zero and the number of rectangles goes to infinity.

With n rectangles, the upper-sum estimation can be shown to be equal to:

$$b^3 + b + \frac{3b^3}{2n} + \frac{b^3}{2n^2} \qquad \textbf{\textit{Expression 2}}$$

where b is the value in seconds for which the area between the velocity function and the horizontal axis from time t = 0 seconds to time t = b seconds is to be calculated. In our case b has a value of four seconds; this is shown in Figure 19.

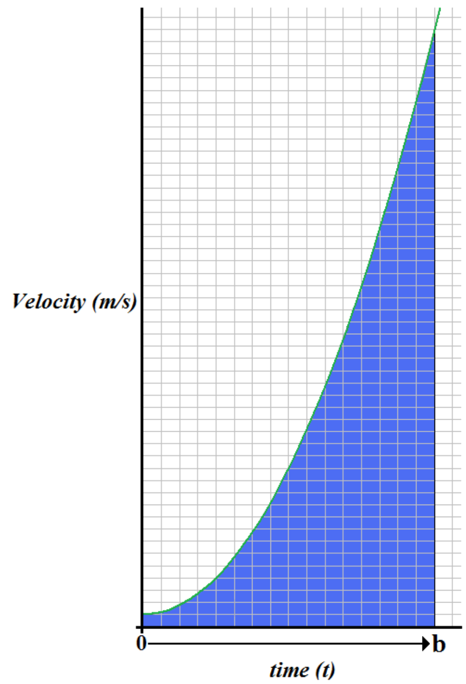

Figure 19

Q.13 As n increases without bound, what do you think happens to the value of the terms $\dfrac{3b^2}{2n}$ and $\dfrac{b^3}{2n^2}$?

____

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Thinking about this question should make you realise that the value for each term becomes very small and approaches zero. Thus, the value for expression 2 leads to:

$$\text{Area} = b^3 + b \qquad \textbf{\textit{Expression 3}}$$

Expression 3 allows us to calculate the area under the graph of $v = t^2$ and the horizontal axis over any interval from t = 0 seconds to t = b seconds, where b can be any positive number (see Figure 19). In our case, we wanted to calculate the value for the area from 0 seconds to 4 seconds.

The area (in units of metres) is equal to $4^3 + 4 =$ ___*m*

Q.14 How do you think you might calculate the area between the function and the horizontal axis between 0 seconds and 3 seconds, 2 seconds or 1 second, as shown in Figure 20, 21 and 22 respectively?

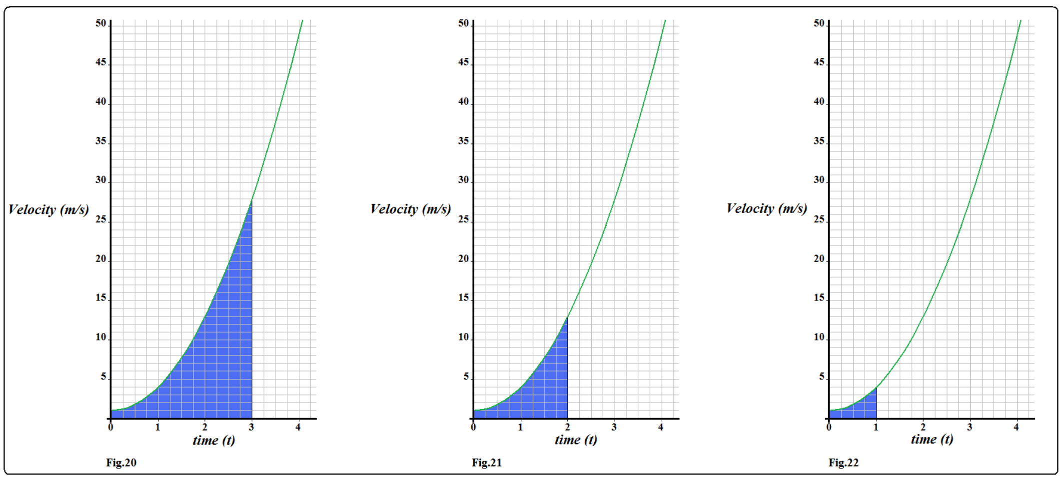
____

____

Tabulate your results in Table 3.

Figure	***Length of interval***	***Area/ displacement***
11	4 seconds	68 metres
20	3 seconds	
21	2 seconds	
22	1 seconds	

Table 3



Plotting the 'area values' between the function: $v = 3t^2 + 1$ and its horizontal axis as time varies, produces the following set of points in Figure 23.

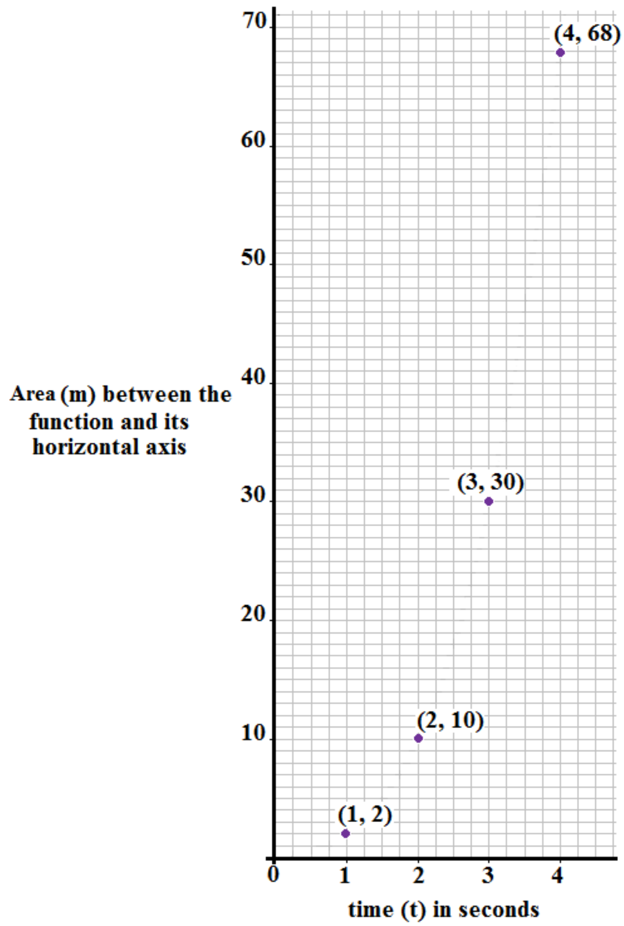


Figure 23

We can now, with more certainty, feel that the shape of the graph produced by joining the points will be of the form governed by the function which we generated in Expression 3—the result of finding the limit of the area of an infinite number of rectangles as their base width became infinitely smaller—namely:

$$\text{Area} = b^3 + b$$

Such a function is graphed in Figure 24, where b is replaced with t. We are only concerned with the shape of the graph in the 1st quadrant of the Cartesian plane.

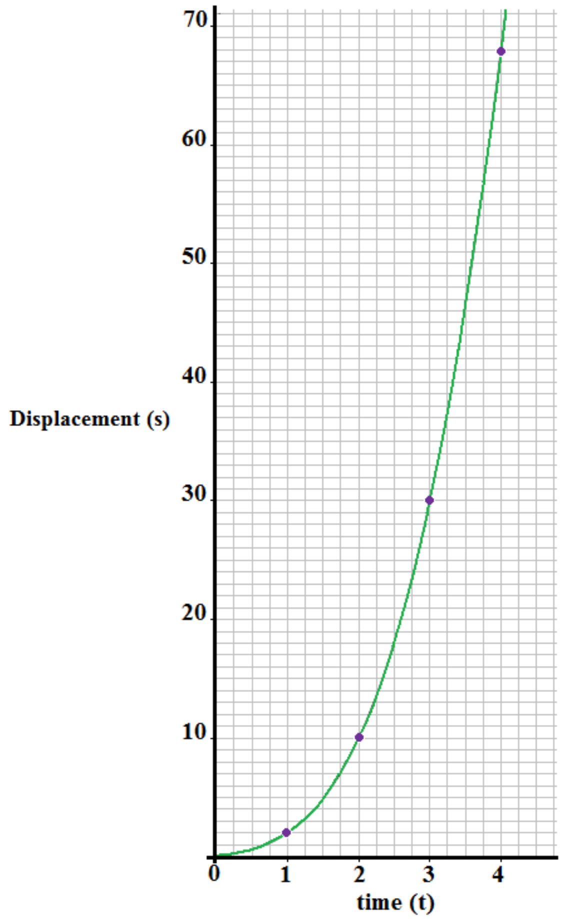


Figure 24

Because the area in Figure 24 is representative of the displacement of the body as a function of time, the graph is again labelled accordingly. The displacement is denoted *s* and time *t*.
Thus, the functional relationship in Figure 24 is of the form: $s = t^3 + t$

Q.15 Differentiate:

$$s = t^3 + t$$

$$v = \frac{ds}{dt} = \_\_\_\_\_$$

You should find that after differentiating the above function with respect to time, we get the function in Expression 1, namely the function we wished to find the area underneath as the value for the base (b)/time (t) varied.

Thus, it would appear that our inference in Section 1 is justified, namely:

> If we are given any function such as the one shown in Figure 25, then the shaded area is found by using a function F(x) that satisfies: $f(x) = \frac{dF(x)}{dx}$.

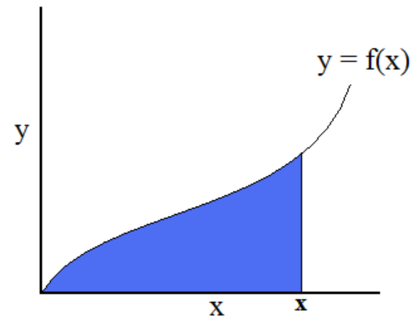


Figure 25

Section 3 — Summary:

In general terms, if we are given a function: y = f(x) — for argument sake, y = 2x — as shown in Figure 26, to calculate the area between the graph of the function and the horizontal axis over any particular interval, (as in Figure 27 for example), we must integrate the function.

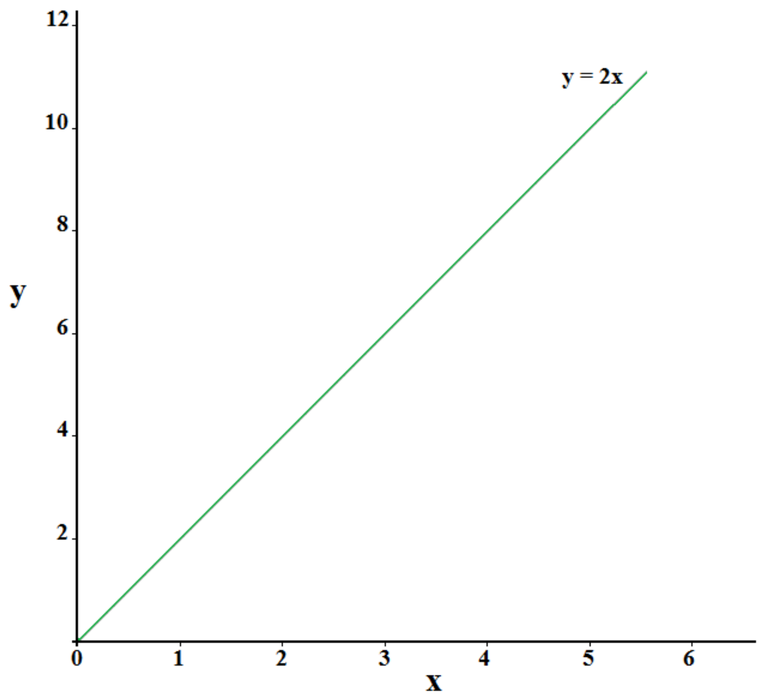


Figure 26

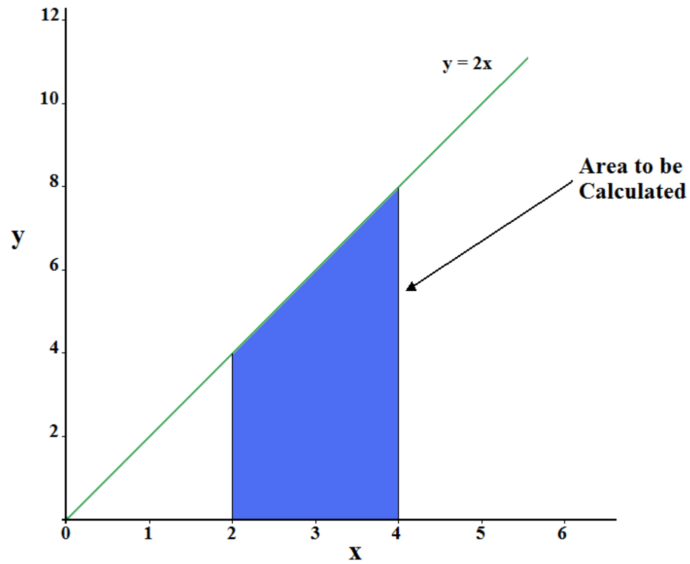


Figure 27

We use the symbols:

$$\int_{2}^{4} 2x \, dx$$

to represent calculating the area in Figure 27.

- The word integrate refers, as we have seen in Section 2, to finding the sum of the area of an infinite number of rectangles between the function and the horizontal axis of the interval in question. The elongated ∫ signifies the sum of this infinite number of rectangles.
- The numbers ‘2’ and ‘4’ are the limits of integration or the interval over which we want to find the area of between the function and its horizontal axis.
- The ‘2x’, or in more general terms, f(x), gives us the height of each individual rectangle.
- ‘dx’ signifies the infinitesimal width of each of the infinite number of rectangles.

The integral of $\int_{2}^{4} 2xdx$ is equivalent to subtracting the red region in Figure 28 from the entire region (both red and blue in Figure 28). This is also shown in Figure 29 and Figure 30.

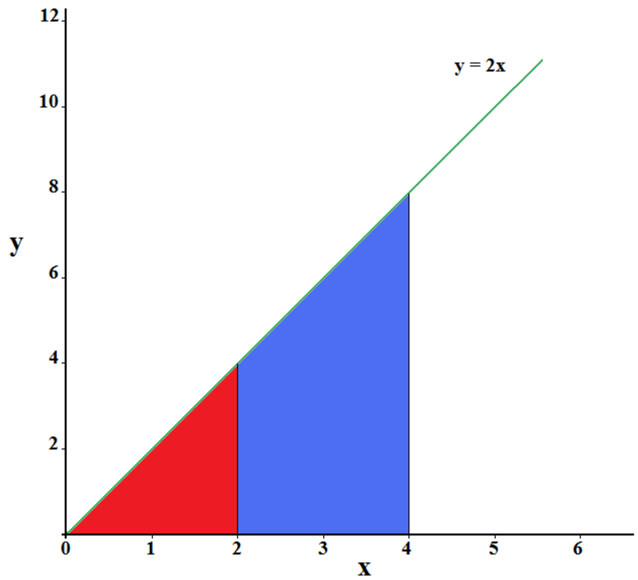


Figure 28

Area in figure 29 subtracted from area in figure 30 equals the area in figure 31

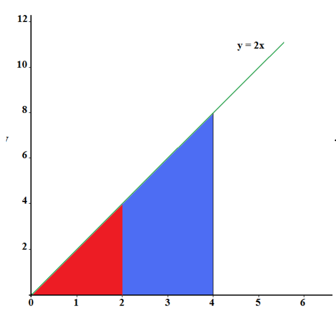


Fig. 30

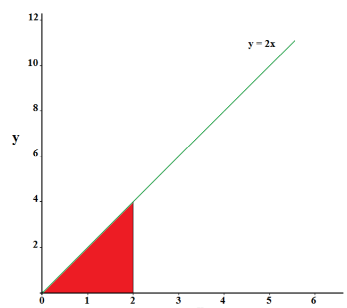


Fig. 29

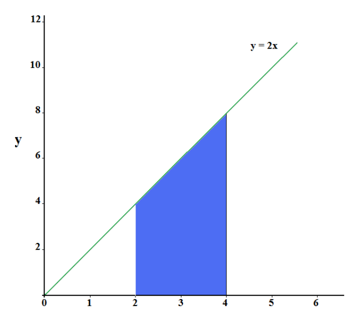


Fig.31

Using our work from Section 1 and Section 2 we can find, algebraically, the area for the integral in Figure 27.

We want to calculate:

$$\int_{2}^{4} 2x \, dx$$

- In this case, the function to be integrated is f(x) = 2x, also known as the integrand.
- The limits are x = 2 and x = 4. This means that the area to be calculated lies between x=2 and x = 4.
- We want to find a function F(x) such that $\frac{dF(x)}{dx} = 2x$.

  We can use $F(x) = x^2$; such a function is also known as the anti-derivative or the integral.
- When we put in the limits x = 2 and x = 4 into $F(x) = x^2$, and subtract the lower-limit value from the upper-limit value, we get the following result:

$$\left[x^2\right]_2^4 = 4^2 - 2^2 = 16 - 4 = 12 \qquad \textbf{\textit{Expression 4}}$$

where 12 is the value for the area between the function 2x and the horizontal axis within the limits of x=2 and x=4. Figure 32a and Figure 32b make this more explicit.

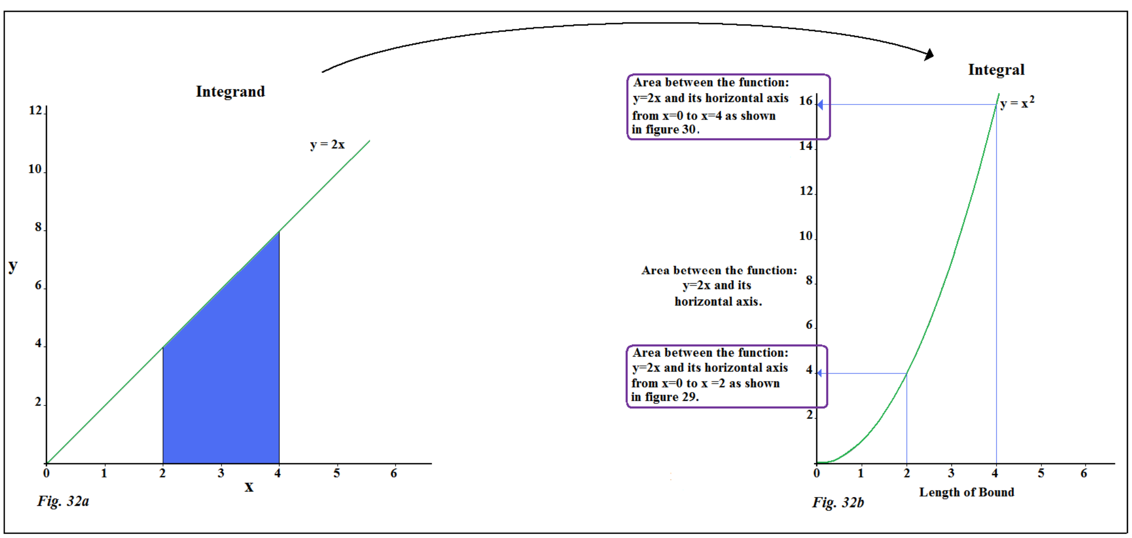


Fig. 32a *Fig. 32b*

In more general terms, for any function to be integrated, as shown:

$$\int_a^b f(x)dx$$

- The function/integrand to be integrated is f(x).
- The limits are x = a and x = b, where *a* and *b* represent any number.
- The area lies between the function f(x) and the x-axis, bounded by the limits x = a and x = b.
- We want to find a function F(x) such that: $\frac{dF(x)}{dx} = f(x)$. Using such a function allows us to calculate the area in question whereby we input *a* and *b* into the function and subtract the lower-limit value (a) from the upper-limit value (b).

Thus:

$$\int_a^b f(x)dx = F(b) - F(a),$$

Where F(x) is a function such that: $\frac{dF(x)}{dx} = f(x)$.

Appendix I – Questions Used to Probe Students' Understanding of Slope and Derivative in Trial 1

Q.1 Let f(x) be a function that is graphed in Figure 1. Which of the graphs in Figure 2 represents the graph of the derivative of f(x)?

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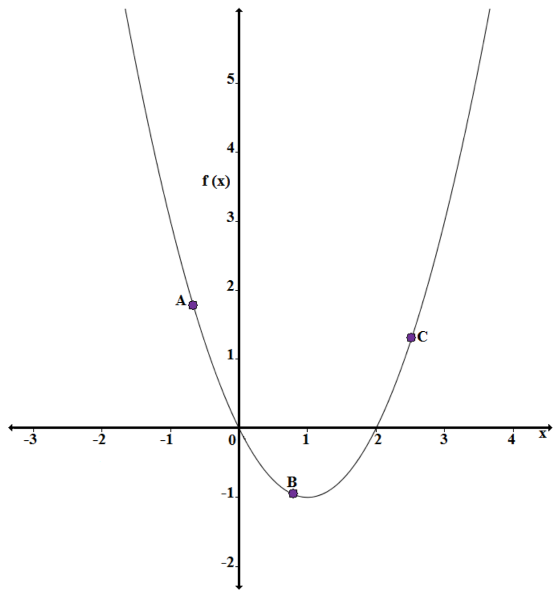


Figure 1

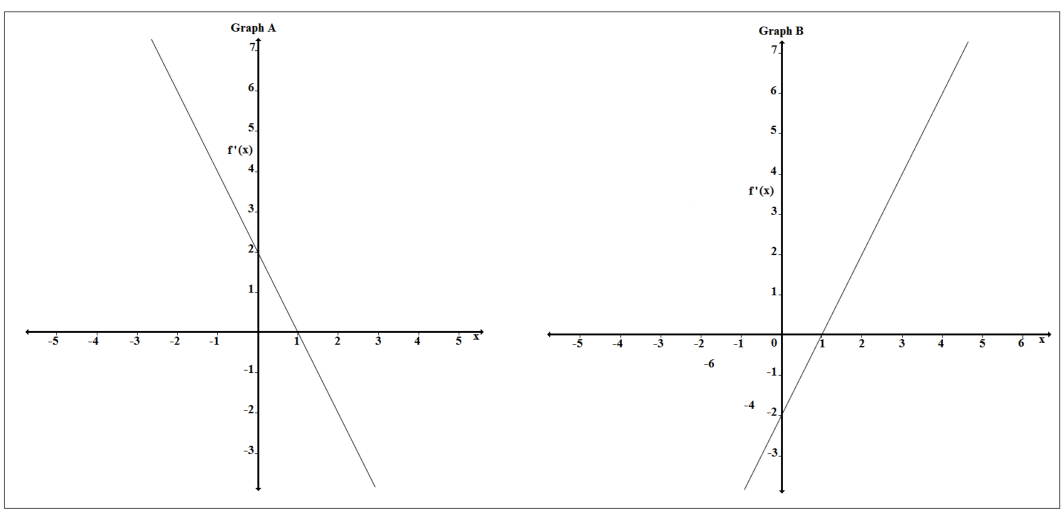


Figure 2

Explain your answer to Question 1.

Q.2 Rank the slope of the tangents to the graph of f(x) at the Points A-C in Figure 1 in increasing order.

Q.3 What does it mean for a line to have a slope equal to zero?

Appendix J – Questions Used to Probe Students' Understanding of Integration in Trial 1

$$A = \int_{1}^{4} x\,dx$$

Expression 1

Q.1 Sketch on the coordinate diagram in Figure 1, the area represented by Expression 1.

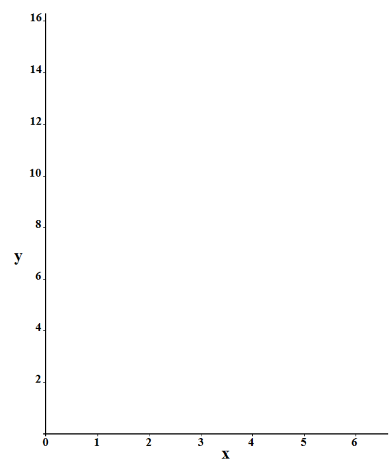


Figure 1

Q.2 Let f(x) = 2x.

Which graph in Figure 2 is the graph of a function F(x) that satisfies $\frac{dF(x)}{dx} = f(x)$?

____

Explain your reasoning.

____

____

____

____

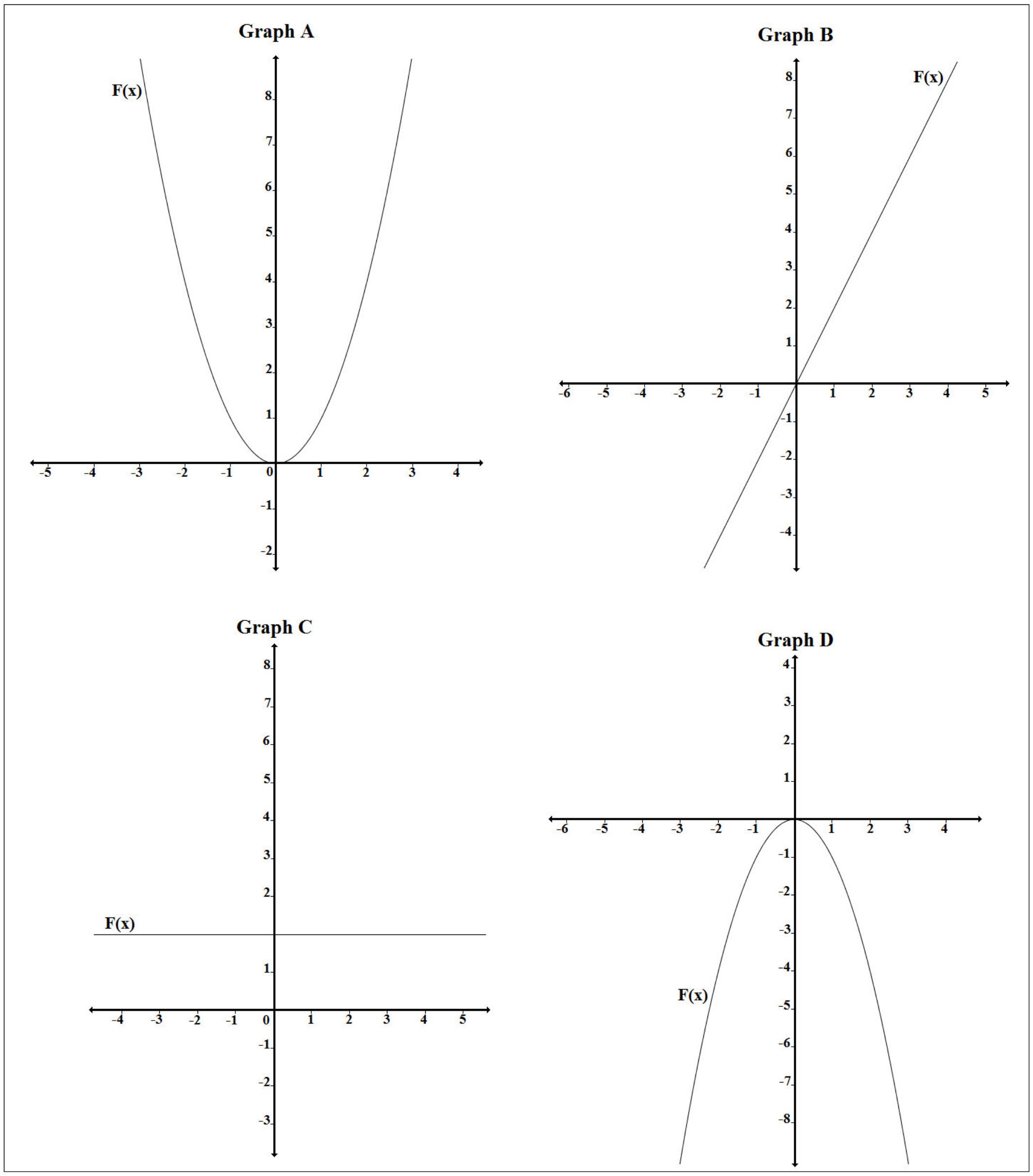


Figure 2

Q.3 What does the definite integral: $\int_{a}^{b} f(x)dx$ represent?

Q.4 What does the ‘*x*’ signify in Expression1?

Q.5 What does the ‘*dx*’ signify in Expression 1?

Appendix K – Questions Used to Probe Students' Understanding of Slope, Derivative and Integration in Trial 2

Q.1 (A) Using the values for the two points on the line in Figure 1, calculate the slope of the line.

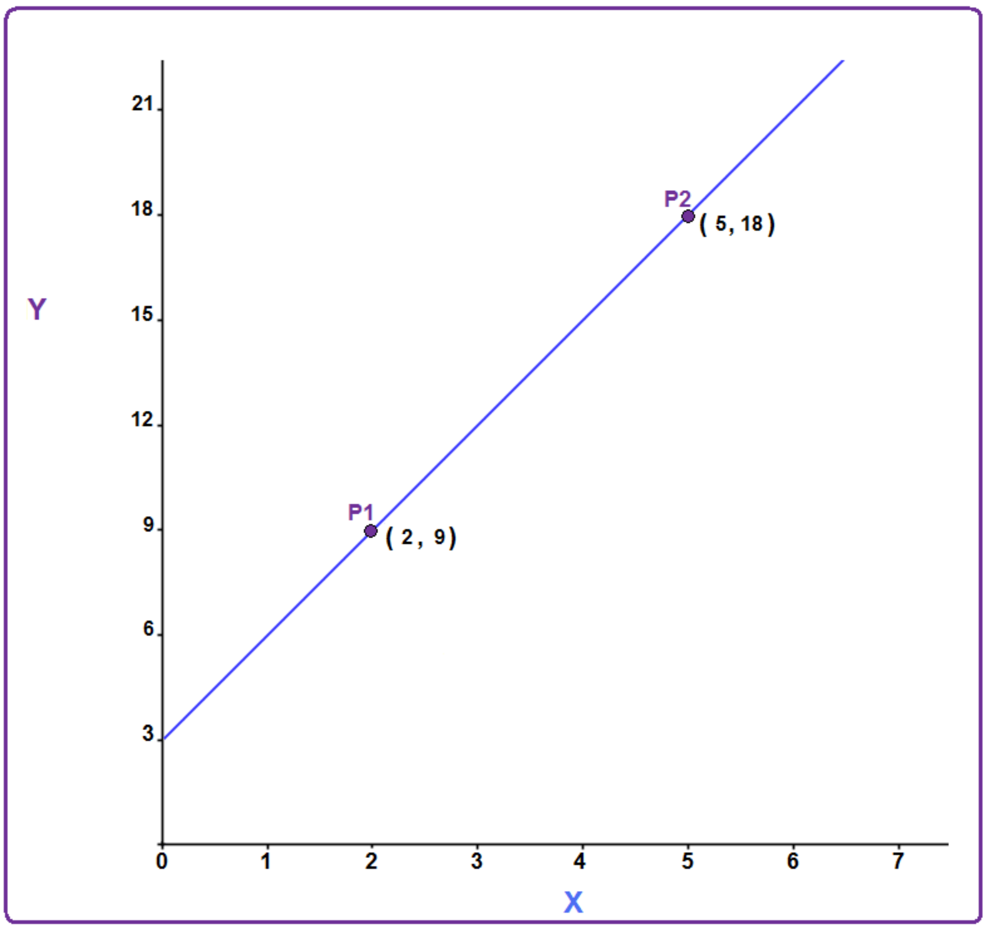


Figure 1

(B) Looking at Figure 2 and Figure 3, describe why both figures have the same value for slope.

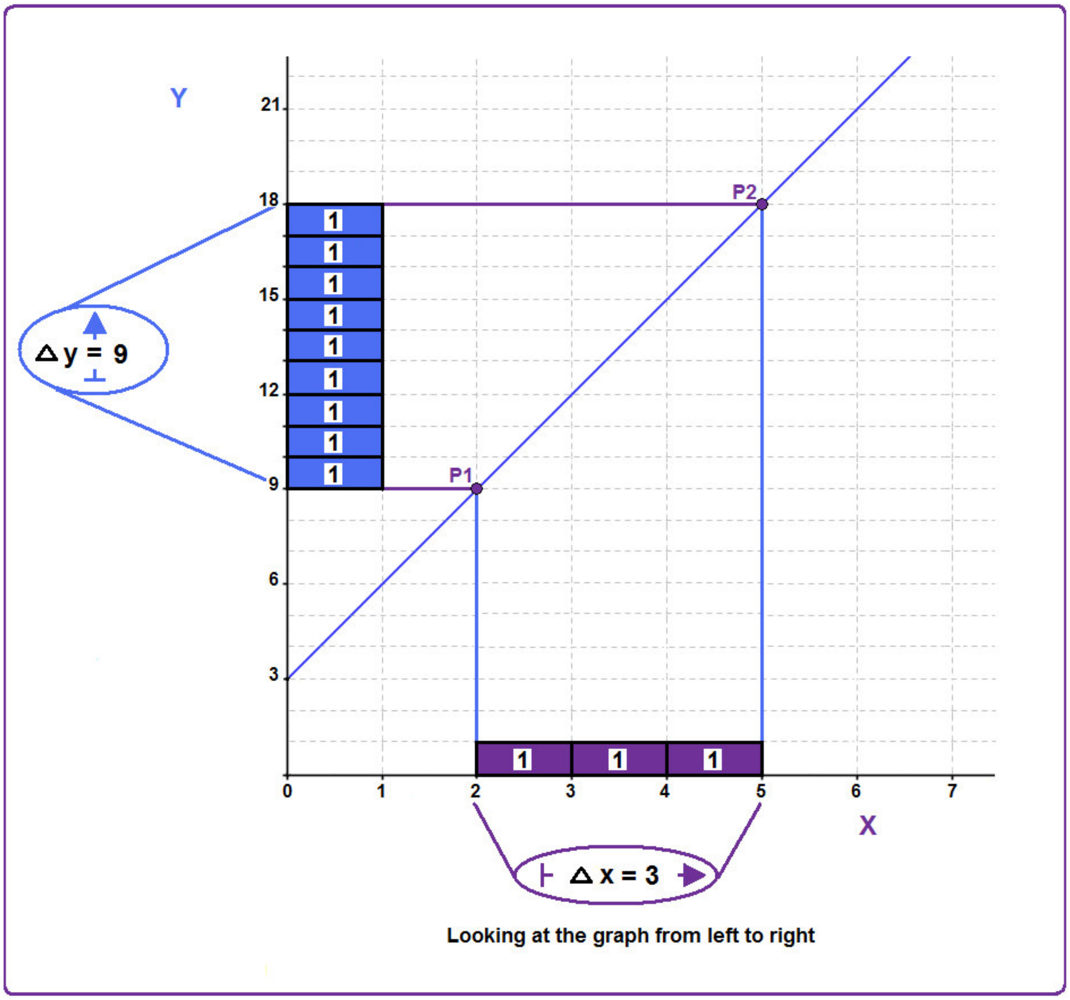

Y
21
18
15
12
9
6
3
P2
P1
Δ y = 9
1
0
1
2
3
4
5
6
7
X
Δ x = 3
Looking at the graph from left to right

Figure 2

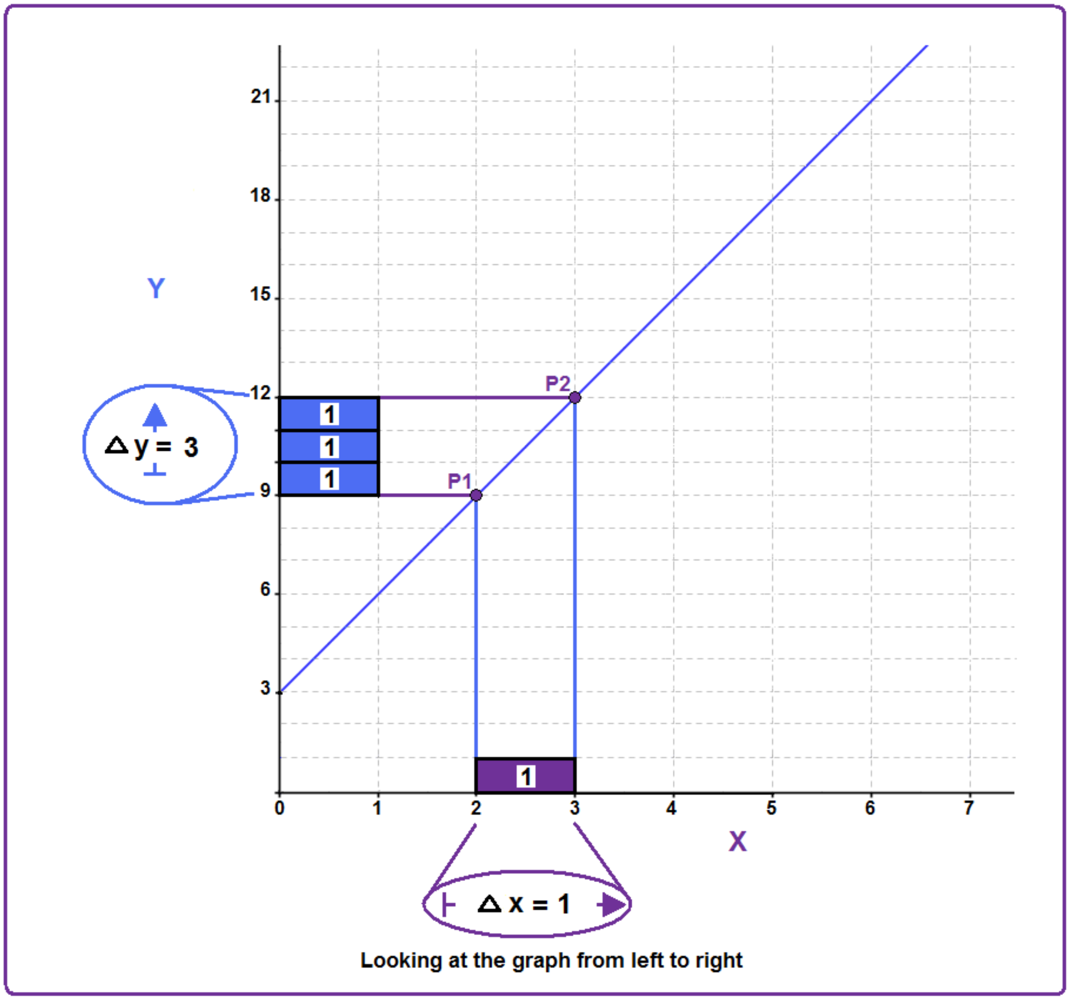

21
18
Y
15
12
P2
1
1
1
Δy = 3
9
P1
6
3
1
0
1
2
3
4
5
6
7
X
Δx = 1
Looking at the graph from left to right

Figure 3

Q.2 (A) Given the function: $y = 0.5x^2$, find the derivative $\frac{dy}{dx}$ when $x = 0.3$.

(B) Looking at Figures 4, 5, 6 and 7, describe how these Figures can be used to explain the meaning of the derivative found in Part A.

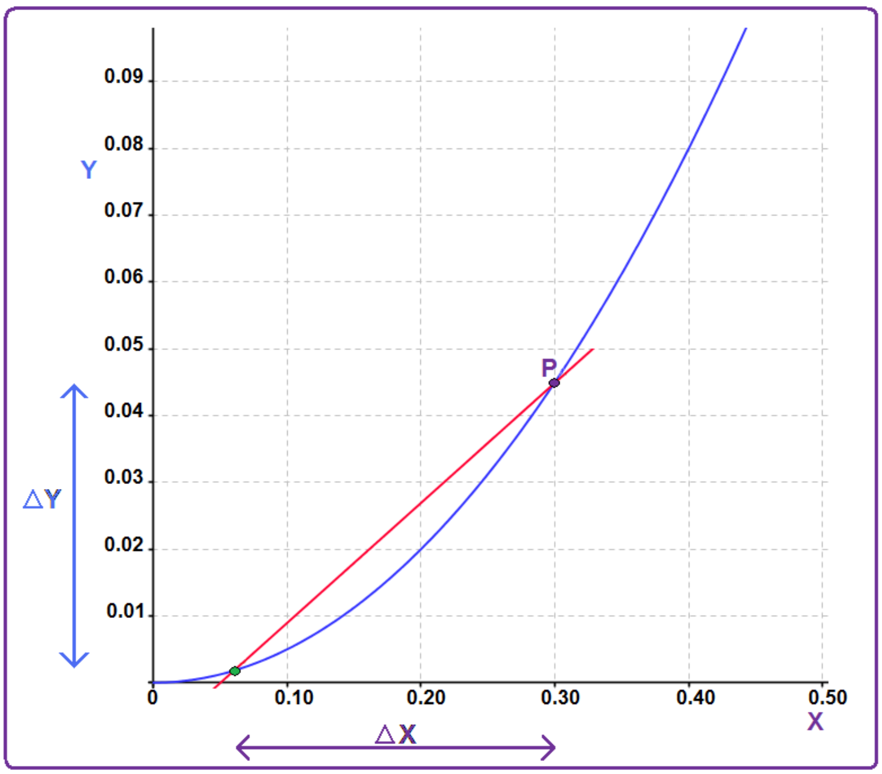


Figure 4

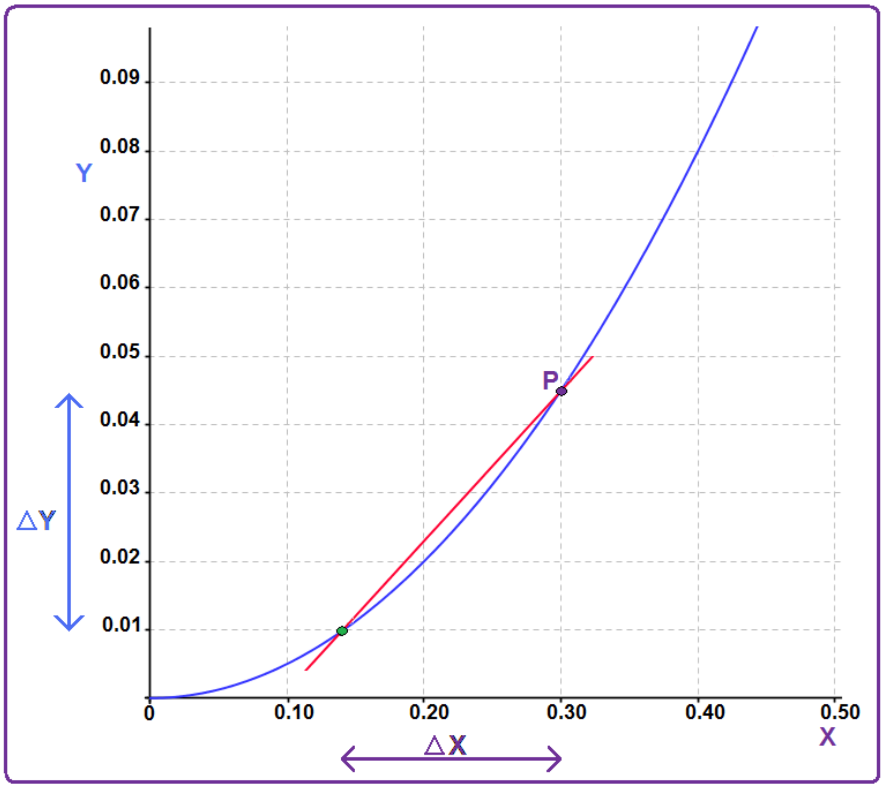


Figure 5

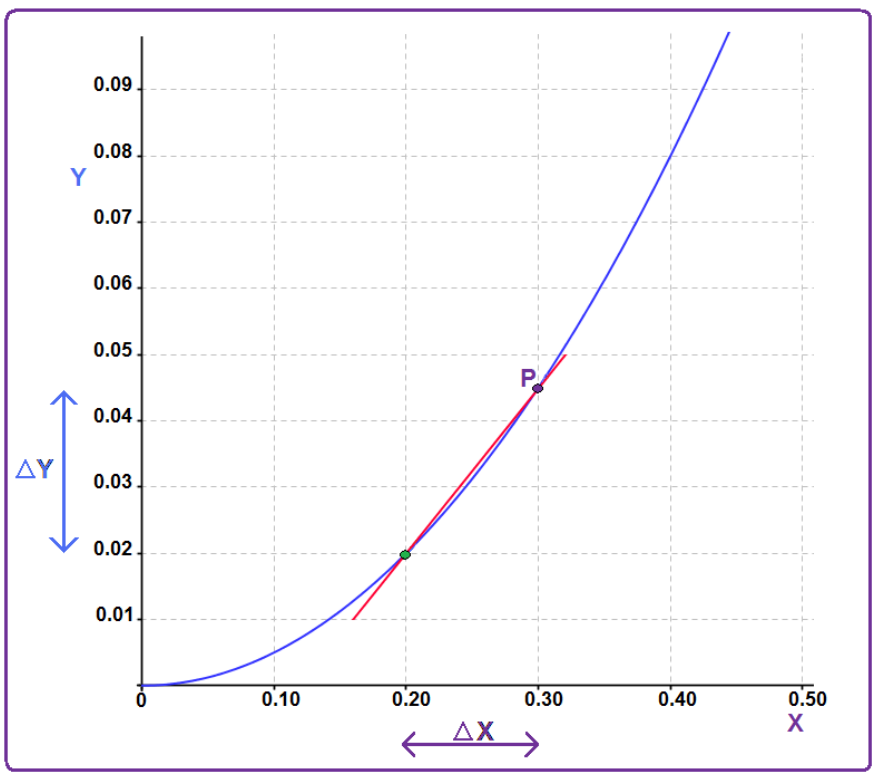
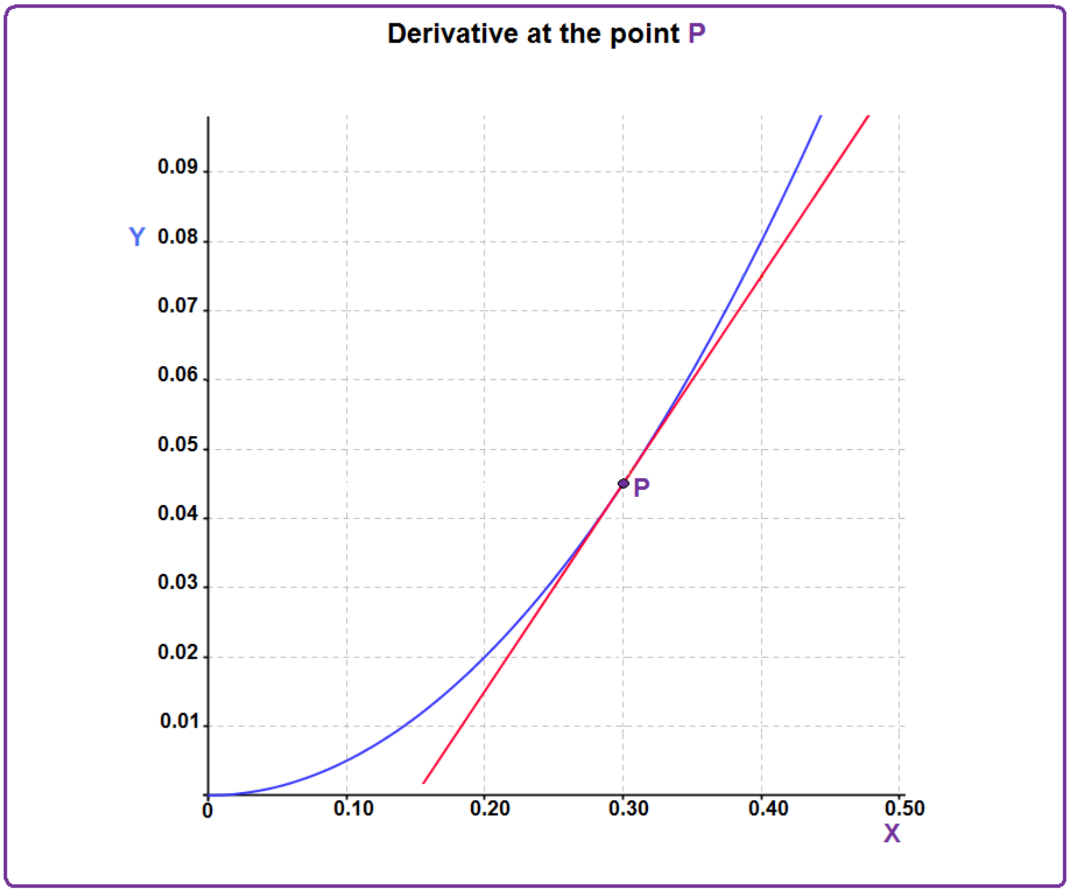


Figure 6



Derivative at the point P
Y
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.10
0.20
0.30
0.40
0.50
X
P

Figure 7

Q.3 (A) Find the derivative function $\frac{dy}{dx}$ for the function: $y = 0.5x^2$.

(B) Figure 8 shows the graph of the derivative function for the function: $y = 0.5x^2$. Describe ***how and why*** the graph of the derivative function can be used to find the slope of the tangents: L_1, L_2 and L_3 on the function $y = 0.5x^2$, as shown in Figure 9, 10 and 11 respectively.

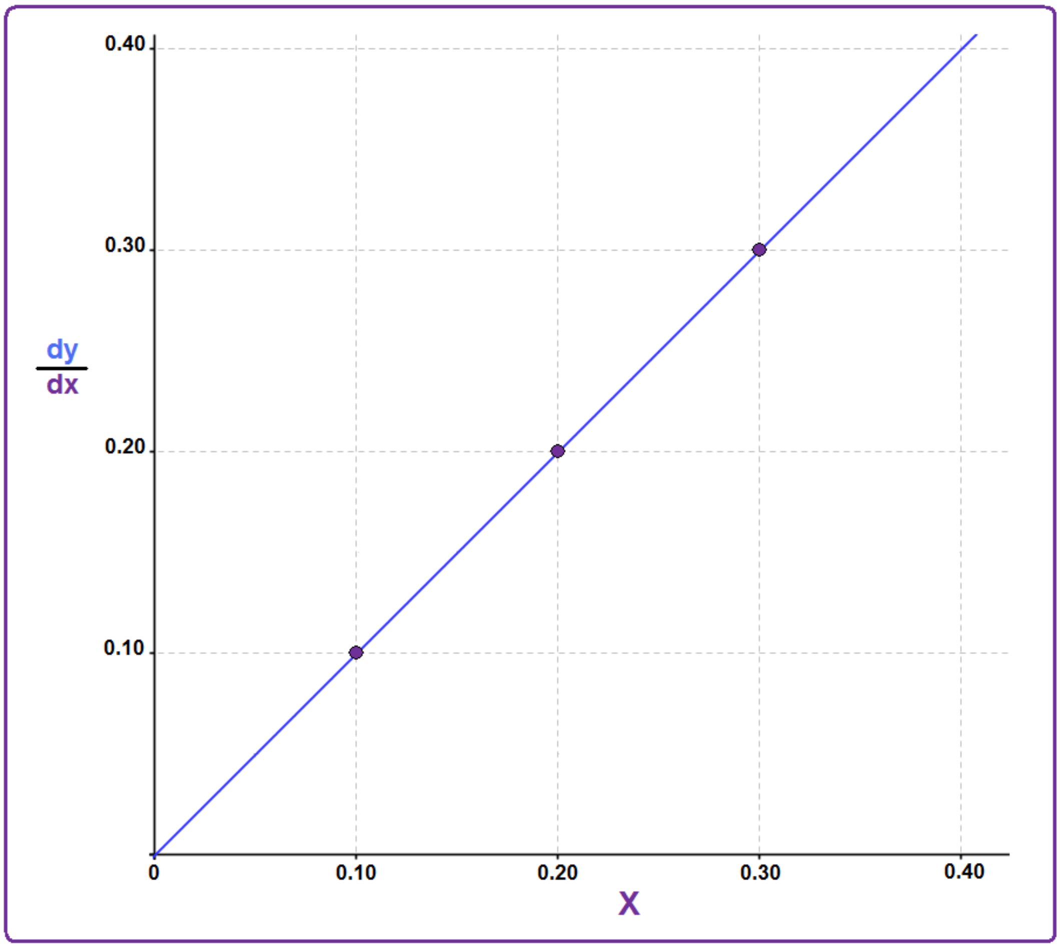


Figure 8

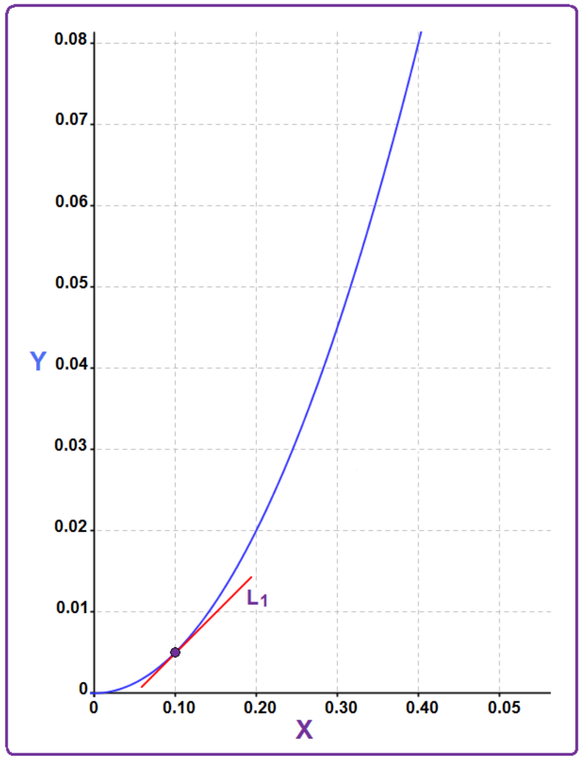


Figure 9

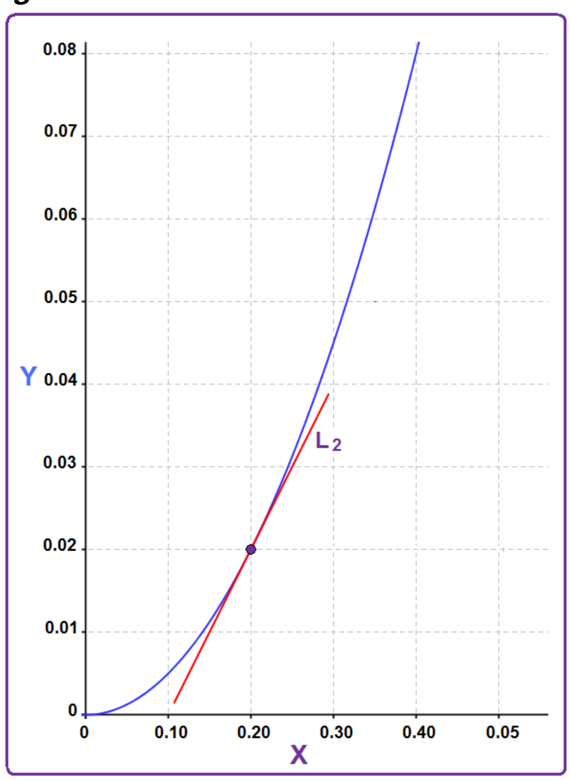


Figure 10

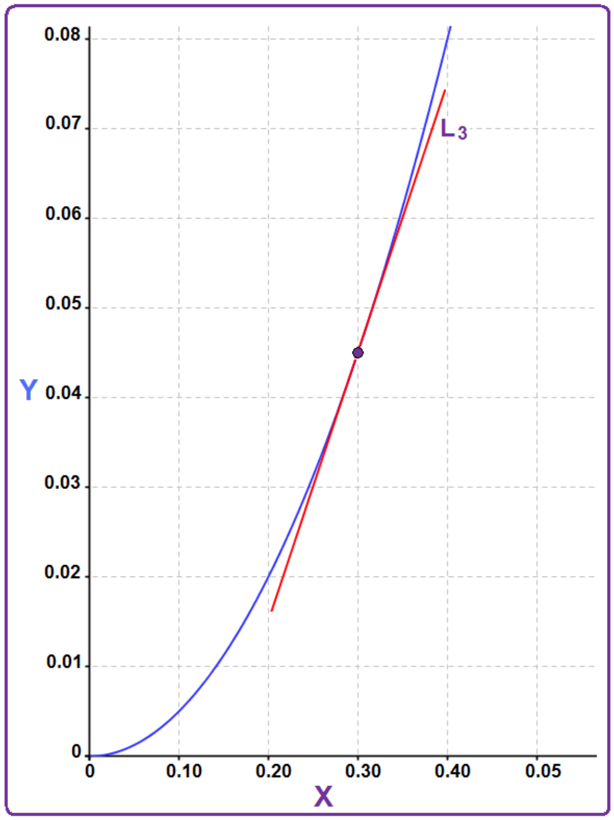


Figure 11

Q.4 (A) Evaluate the integral:

$$\int_{1}^{6} (-x^2 - 10x + 150)dx$$

(B) Use Figure 12 to graphically depict the integral:

$$\int_{1}^{6} (-x^2 - 10x + 150)dx$$

Use Figure 13 to illustrate how the function: $F(x) = \dfrac{-x^3}{3} - 10x + 150x + c$,

where we assume 'c' to be equal to zero, allows us to evaluate the integral:

$$\int_{1}^{6} (-x^2 - 10x + 150)dx \,.$$

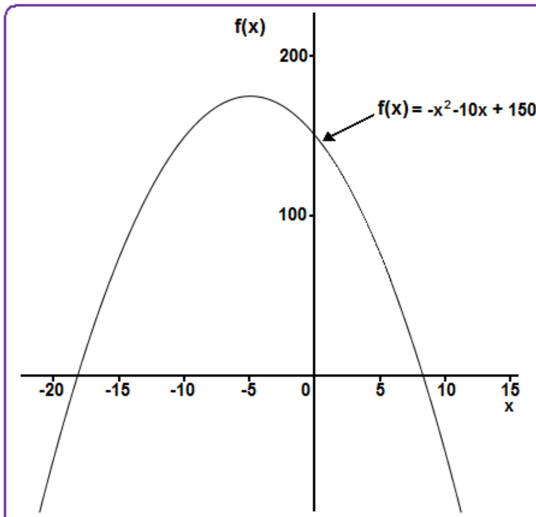


Figure 12

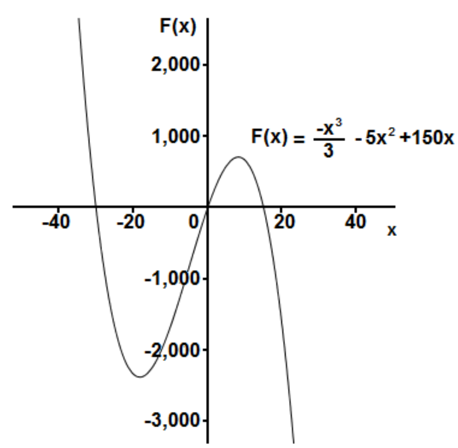


Figure 13

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