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Physical interpretation of gauge invariant perturbations of spherically symmetric space-times

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By calculating the Newman-Penrose Weyl tensor components of a perturbed spherically symmetric space-time with respect to invariantly defined classes of null tetrads, we give a physical interpretation, in terms of gravitational radiation, of odd parity gauge invariant metric perturbations. We point out how these gauge invariants may be used in setting boundary and/or initial conditions in perturbation theory.

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I. INTRODUCTION

Perturbation theory in general relativity is complicated by the issue of coordinate freedom in the unperturbed background space-time $(M,\bar{g}_{\mu\nu})$. If one formally adds a perturbation to the metric $\bar{g}_{\mu\nu}\rightarrow g_{\mu\nu}=\bar{g}_{\mu\nu}+h_{\mu\nu}$, it is not necessarily true that one has moved to a different space-time: $g_{\mu\nu}$ may be the metric $\bar{g}_{\mu\nu}$ written in a different coordinate system, which is related to the original coordinates by an infinitesimal coordinate transformation. This is known as the identification gauge problem. The gauge freedom represented by such infinitesimal coordinate transformations must be dealt with carefully. One way to do this is to treat the perturbation problem in hand using identification gauge invariant (IGI) quantities [1]. For perturbations of spherically symmetric space-times, a complete set of such quantities representing metric and matter perturbations, and the corresponding IGI perturbation equations, have been given by Gerlach and Sengupta (GS) [2]. We review their formalism in Sec. II below.

This formalism has been applied in many different areas, for example in studies of nonspherical stellar collapse [3–5], critical collapse [6–9], phenomenology of naked singularities [10], black holes [11–13], cosmology [14–16], nonlinear perturbation theory [17] and perturbations of gauge fields [18,19]. These studies have generally extracted the physical significance of the metric perturbations, e.g. by calculating the radiated power of gravitational waves [10] or by making the connection with the more familiar Regge-Wheeler-Zerilli and Teukolsky perturbation formalisms [11]. Nevertheless, a general and direct interpretation of the full set of IGI metric perturbations has not been given. The aim in the present paper is to attempt to do so by calculating the Newman-Penrose (NP) Weyl tensor components of the perturbed space-time. The type-N component, which represents transverse gravitational waves has previously been calculated in Refs. [10] and [11]. In carrying out this calculation, one encounters another type of gauge problem, namely the freedom of choice in the null tetrad of the perturbed space-time.

Stewart and Walker [1] discussed this additional gauge invariance, and concluded that the only Weyl scalars that are both IGI and tetrad gauge invariant (TGI) are the type-N terms, and furthermore, that these terms can only be gauge invariant if the background is of Petrov type D or conformally flat. These include all spherically symmetric space-times. (We use the phrase "gauge invariant" to refer to a quantity which is both tetrad and identification gauge invariant.) Consequently, any attempt to attach physical significance to the full set of perturbed Weyl scalars seems doomed. However, as we will see below, this is not the case for odd perturbations (see Ref. [2] and Sec. II below). In this case, there is sufficient geometric information in the background that is invariant with respect to the generators of odd perturbations to enable the construction of gauge invariant perturbed Weyl scalars. This will allow the interpretation of the metric perturbations in terms of longitudinal and transverse waves propagating in the inward and outward radial null directions of the spherically symmetric background and in terms of a perturbation of the Coulombic interaction. As in the analysis of Ref. [1], this will involve the choice of a special class of tetrads, but one which admits an IGI description. We follow the curvature, tetrad and NP conventions of Ref. [20].

II. GERLACH-SENGUPTA FORMALISM

For convenience, we give a brief review of the formalism introduced by Gerlach and Sengupta [2], following the presentation of Martin-Garcia and Gundlach [8]. The metric of a spherically symmetric space-time M^4 can be written as

$$ds^2=g_{AB}(x^C)dx^A dx^B+r^2(x^C)\gamma_{ab}dx^a dx^b, \qquad (1)$$

where g_{AB} is a Lorentzian metric on a 2-dimensional manifold with boundary M^2 and γ_{ab} is the standard metric on the unit 2-sphere S^2. Capital Latin indices represent tensor indices on M^2, and lower case Latin indices are tensor indices on S^2. $r(x^C)$ is a scalar field on M^2. 4-dimensional space-time indices will be given in Greek. The covariant derivatives on M^4, M^2 and S^2 will be denoted by a semicolon, a vertical and a colon respectively. ϵ_{AB} and ϵ_{ab} are covariantly constant antisymmetric unit tensors with respect to g_{AB} and γ_{ab}. We define

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$$v_A = \frac{r_{|A}}{r}, \tag{2}$$

$$V_0 = -\frac{1}{r^2} + 2v^A{}_{|A} + 3v^A v_A. \tag{3}$$

Writing the stress-energy tensor as

$$t_{\mu\nu} dx^\mu dx^\nu = t_{AB}(x^C) dx^A dx^B + Q(x^C) r^2 \gamma_{ab} dx^a dx^b, \tag{4}$$

the Einstein equations of the spherically symmetric background read

$$G_{AB} = -2(v_{A|B} + v_A v_B) + V_0 g_{AB} = 8\pi t_{AB}, \tag{5}$$

$$\frac{1}{2} G^a_a = -\mathcal{R} + v^A{}_{|A} + v^A v_A = 8\pi Q, \tag{6}$$

where $G^a_a = \gamma^{ab} G_{ab}$ and $\mathcal{R}$ is the Gaussian curvature of M^2.

Spherical symmetry of the background allows us to expand the perturbed metric tensor in terms of spherical harmonics. Writing $Y = Y_l^m$ and suppressing the indices l,m throughout, we have the following bases for scalar, vector and tensor harmonics respectively: $\{Y\}$, $\{Y_a := Y_{:a}, S_a := \epsilon_a^b Y_b\}$ and $\{Y\gamma_{ab}, Z_{ab} := Y_{a:b} + [l(l+1)/2] Y\gamma_{ab}, S_{a:b} + S_{b:a}\}$. These are further classified depending on the transformation properties under spatial inversion $\vec{x} \to -\vec{x}$: a spherical harmonic with index l is called even if it transforms as $(-1)^l$ and is called odd if it transforms as $(-1)^{l+1}$. In the bases above, Y, Y_a and Z_{ab} are even and $S_a, S_{(a:b)}$ are odd.

The perturbation $\delta g_{\mu\nu}$ of the metric tensor can then be decomposed as

$$\delta g_{AB} = h_{AB} Y, \tag{7}$$

$$\delta g_{Ab} = h_A^E Y_{:b} + h_A^O S_b, \tag{8}$$

$$\delta g_{ab} = r^2 K \gamma_{ab} Y + r^2 G Z_{ab} + 2h S_{(a:b)}. \tag{9}$$

The superscripts E,O stand for even and odd respectively. Note that h_{AB}, $\{h_A^E, h_A^O\}$ and $\{K,G,h\}$ are respectively a 2-tensor, vectors and scalars on M^2. A similar decomposition of the perturbation of the stress-energy tensor is made:

$$\delta t_{AB} = \Delta t_{AB} Y, \tag{10}$$

$$\delta t_{Ab} = \Delta t_A^E Y_{:b} + \Delta t_A^O S_b, \tag{11}$$

$$\delta t_{ab} = r^2 \Delta t^3 \gamma_{ab} Y + r^2 \Delta t^2 Z_{ab} + 2\Delta t S_{(a:b)}. \tag{12}$$

In this case, Δt_{AB}, $\{\Delta t_A^E, \Delta t_A^O\}$ and $\{\Delta t^3, \Delta t^2, \Delta t\}$ are respectively a 2-tensor, vectors and scalars on M^2.

A complete set of identification gauge invariant variables is produced as follows. An infinitesmal coordinate transformation on the background is generated by a vector field $\vec{\xi}$. Again, we can decompose into even and odd harmonics and consider separately the transformations generated by the 1-form fields

$$\boldsymbol{\xi}^E = \xi_A(x^C) Y dx^A + \xi^E(x^C) Y_{:a} dx^a, \tag{13}$$

$$\boldsymbol{\xi}^O = \xi^O S_a dx^a. \tag{14}$$

From the transformed versions of the metric perturbations, one can construct combinations which are independent of the coefficients of $\vec{\xi}$. These combinations are then identification gauge invariant. Writing

$$p_A = h_A^E - \frac{r^2}{2} G_{|A}, \tag{15}$$

a complete set of IGI metric perturbations is given by

$$k_{AB} = h_{AB} - 2p_{(A|B)}, \tag{16}$$

$$k_A = h_A^O - h_{|A} + 2h v_A, \tag{17}$$

$$k = K + \frac{l(l+1)}{2} G - 2v^A p_A. \tag{18}$$

Similarly, a complete set of IGI stress-energy tensor perturbations may be constructed. We will not give these here, but refer the reader to Refs. [2] or [8]. The full set of IGI perturbation equations may also be found in these references; we will not use these equations in the present paper.

An important point to note is that this formalism is incomplete for $l=0$ and for $l=1$. For $l=0,1$, G and h are not defined, being coefficients of zero, and so should be considered to be zero. The same holds for h_A^E, h_A^O when $l=0$. Thus the gauge invariants cannot be constructed. However it is convenient to use the same variables (16)–(18) for all values of l. For $l=0,1$, these variables are only partially IGI and so gauge fixing is required. This does not affect the calculation below.

To conclude this section, we point out the existence of a preferred gauge in which $h = G = h_A^E = 0$. This is the Regge-Wheeler (RW) gauge. This has the advantage that the bare perturbations of Eqs. (7)–(9) match identically the IGI perturbations.

III. NULL TETRADS AND WEYL SCALARS

It is convenient to introduce coordinates $x^\mu = (\theta, \phi, u, v)$ on the spherically symmetric background, with $\mu = 1$–4 in the order shown. u,v are null coordinates on M^2 which we take to increase into the future. Furthermore, we specify that u,v are respectively retarded and advanced time coordinates, so that u (respectively v) labels the future (respectively past) null cones of the axis $r=0$. Then the background line element can be written as

$$ds^2 = -r^2(u,v) d\Omega^2 + 2e^{-2f(u,v)} du dv,$$

where the only coordinate freedom corresponds to the relabeling $u \to U(u), v \to V(v)$ of the spherical null cones. We introduce the null tetrad

$$\bar{m}_\mu = \frac{r}{\sqrt{2}}(\delta^1_\mu + i \sin\theta \delta^2_\mu), \tag{19}$$

$$\bar{m}^*_\mu = \frac{r}{\sqrt{2}}(\delta^1_\mu - i \sin\theta \delta^2_\mu), \tag{20}$$

$$\bar{n}_\mu = e^{-f}\delta^4_\mu, \tag{21}$$

$$\bar{l}_\mu = e^{-f}\delta^3_\mu, \tag{22}$$

so that

$$\bar{g}_{\mu\nu} = 2\bar{l}_{(\mu}\bar{n}_{\nu)} - 2\bar{m}_{(\mu}\bar{m}^*_{\nu)}. \tag{23}$$

Here and throughout, the overline indicates a background quantity and the asterisk represents complex conjugation. With respect to this tetrad, there is only one nonvanishing Weyl tensor component;

$$\bar{\Psi}_2 = \frac{1}{6r^2}[2re^{2f}(r_{,uv} + rf_{,uv}) - 1 - 2e^{2f}r_{,u}r_{,v}]$$

$$= \frac{1}{6}\left(\mathcal{R} + \frac{1}{r}\Box_2 r - \frac{1}{r^2}(1+\chi)\right), \tag{24}$$

where $\Box_2$ is the d'Alembertian of M_2 and $\chi = g^{AB}r_{,A}r_{,B}$. Under general Lorentz transformations of the null tetrad, this term is not invariant. However, due to spherical symmetry, there is an invariant class of null tetrads, namely that which takes the two real members of the tetrad to be the repeated principal null directions of the Weyl tensor (the ingoing and outgoing radial null directions). Specifying that we always do this, the only allowed Lorentz transformations are spin boosts which involve

$$\bar{l}^\mu \to a^2 \bar{l}^\mu, \quad \bar{n}^\mu \to a^2 \bar{n}^\mu, \quad \bar{m}^\mu \to e^{2i\omega}\bar{m}^\mu, \tag{25}$$

where a, ω are arbitrary. $\bar{\Psi}_2$ is invariant under these transformations. Henceforth, a null tetrad $\{\bar{m}^\mu, \bar{m}^{*\mu}, \bar{n}^\mu, \bar{l}^\mu\}$ for the background will always be taken to lie in this class. Without loss of generality, we can always take $\bar{n}^\mu$ to point in the radial ingoing null direction and $\bar{l}^\nu$ to point in the radial outgoing null direction.

We write a null tetrad of the perturbed space-time as $\{\vec{m}, \vec{m}^*, \vec{n}, \vec{l}\}$, with

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} = -2m_{(\mu}m^*_{\nu)} + 2l_{(\mu}n_{\nu)}, \tag{26}$$

where $l_\mu = \bar{l}_\mu + \delta l_\mu$ and similar for other tetrad members. The condition (26) is an underdetermined linear system for the perturbations δl_μ (etc.) in terms of the metric perturbations, corresponding to the gauge freedom of Lorentz transformations. In order that the Weyl scalars calculated below have an invariant meaning, we must choose the tetrad (or more correctly, class of tetrads) in an invariant way, as was done above for the background.

The Weyl scalars are given by

$$\Psi_0 = C_{\mu\nu\lambda\sigma}l^\mu m^\nu l^\lambda m^\sigma, \tag{27}$$

$$\Psi_1 = C_{\mu\nu\lambda\sigma}l^\mu m^\nu l^\lambda n^\sigma, \tag{28}$$

$$\Psi_2 = C_{\mu\nu\lambda\sigma}l^\mu m^\nu n^\lambda m^{*\sigma}, \tag{29}$$

$$\Psi_3 = C_{\mu\nu\lambda\sigma}l^\mu n^\nu m^{*\lambda} n^\sigma, \tag{30}$$

$$\Psi_4 = C_{\mu\nu\lambda\sigma}n^\mu m^{*\nu} n^\lambda m^{*\sigma}. \tag{31}$$

With our choice of background tetrad, we find that these yield

$$\delta\Psi_0 = \delta C_{\mu\nu\lambda\sigma}\bar{l}^\mu \bar{m}^\nu \bar{l}^\lambda \bar{m}^\sigma, \tag{32}$$

$$\delta\Psi_1 = -a\bar{\Psi}_2 + \delta C_{\mu\nu\lambda\sigma}\bar{l}^\mu \bar{m}^\nu \bar{l}^\lambda \bar{n}^\sigma, \tag{33}$$

$$\delta\Psi_2 = b\bar{\Psi}_2 + \delta C_{\mu\nu\lambda\sigma}\bar{l}^\mu \bar{m}^\nu \bar{n}^\lambda \bar{m}^{*\sigma}, \tag{34}$$

$$\delta\Psi_3 = -c\bar{\Psi}_2 + \delta C_{\mu\nu\lambda\sigma}\bar{l}^\mu \bar{n}^\nu \bar{m}^{*\lambda} \bar{n}^\sigma, \tag{35}$$

$$\delta\Psi_4 = \delta C_{\mu\nu\lambda\sigma}\bar{n}^\mu \bar{m}^{*\nu} \bar{n}^\lambda \bar{m}^{*\sigma}, \tag{36}$$

where

$$a = \bar{m}_\mu \delta l^\mu, \tag{37}$$

$$b = \bar{n}_\mu \delta l^\mu + \bar{l}_\mu \delta n^\mu - \bar{m}_\mu \delta m^{*\mu} - \bar{m}^*_\mu \delta m^\mu, \tag{38}$$

$$c = \bar{m}^*_\mu \delta n^\mu. \tag{39}$$

The gauge invariance of $\delta\Psi_0$ is demonstrated as follows. (An identical argument applies for $\delta\Psi_4$.) We see from above that this term depends only on the perturbed Weyl tensor and on the background tetrad. Both these terms are fixed once the background and tetrad have been specified and the perturbation has been added in any particular gauge. Thus Ψ_0 is a TGI scalar. Then IGI follows from the Stewart-Walker lemma [1] (see also Sec. 1.6 of Ref. [20]) which we state in this form:

Lemma 1. *The linearized perturbation of a geometric quantity Q with background value $\bar{Q}$ is IGI if it satisfies*

$$\mathcal{L}_{\vec{\xi}}\bar{Q} = 0$$

for all generators $\vec{\xi}$ of infinitesmal coordinate transformations of the background space-time.

This allows one to characterize all IGI quantities [1].

Lemma 2. *The linearized perturbation of a geometric quantity Q with background value $\bar{Q}$ is IGI if one of the following holds:*

(1) $\overline{Q}=0$,
(2) $\overline{Q}$ *is a constant scalar*,
(3) $\overline{Q}$ *is a constant linear combination of products of Kronecker deltas.*

Lemma 1 is trivially satisfied by Ψ_0 as it vanishes in the background; hence full gauge invariance follows. As noted in the Introduction, it is only Ψ_0 and Ψ_4 which satisfy the requirements of being both tetrad and identification gauge invariant. Gauge invariance of these terms has long been recognized and used; see e.g. Ref. [21]. The form of these terms in GS variables has been given in Refs. [10] and [11].

Equations (32)–(36) and (37)–(39) clearly rule out the possibility of all the Weyl scalars being TGI in general. However if we consider odd and even perturbations separately, some progress can be made.

A. Odd perturbations

In an arbitrary gauge, we have $h_{AB}=h_A^E=G=K=0$ for odd perturbations. Infinitesimal co-ordinate transformations of odd parity are generated by 1-form fields of the form (14). We can write down an "odd perturbations only" version of the Stewart-Walker lemma:

Lemma 3. *The linearized perturbation of a geometric quantity Q with background value $\overline{Q}$ is IGI with respect to odd perturbations if it satisfies*

$$\mathcal{L}_{\vec{\xi}_O}\overline{Q}=0$$

for all generators $\vec{\xi}_O$ of infinitesmal coordinate transformations of odd parity of the background space-time.

The form (14) of these generators yields the following useful result:

Corollary 1. *Let $\overline{S}(x^D)$ and $\overline{T}_{AB\cdots C}(x^D)$ be respectively a scalar and a covariant tensor field on M^2 and define a tensor field $\overline{T}_{\alpha\beta\cdots\gamma}$ on M^4 by padding out with zeros. Then both $\overline{S}$ and $\overline{T}_{\alpha\beta\cdots\gamma}$ are IGI with respect to odd perturbations*.

Proof: Vanishing of the Lie derivative of $\overline{S}$ along $\vec{\xi}_O$ is immediate. Also,

$$\begin{aligned}\mathcal{L}_{\vec{\xi}_O}\overline{T}_{\alpha\beta\cdots\gamma}&=\overline{T}_{\alpha\beta\cdots\gamma,\nu}\xi_O^\nu+\overline{T}_{\nu\beta\cdots\gamma}\xi^\nu_{O,\alpha}+\cdots+\overline{T}_{\alpha\beta\cdots\nu}\xi^\nu_{O,\gamma}\\&=\overline{T}_{\alpha\beta\cdots\gamma,A}\xi_O^A+\overline{T}_{a\beta\cdots\gamma}\xi^a_{O,\alpha}+\cdots+\overline{T}_{\alpha\beta\cdots a}\xi^a_{O,\gamma}\\&=0.\end{aligned}$$

Quantities of particular relevance to us that satisfy this corollary are the background Weyl scalar $\overline{\Psi}_2$ and the tetrad members $\overline{\mathbf{l},\mathbf{n}}$. Note that it is crucial that we consider the tetrad members as 1-forms. Corollary 1 does not apply to contravariant tensor fields. Hence the perturbed quantities δl_μ, δn_μ are IGI with respect to odd perturbations. (Note however that $\delta l^\mu,\delta n^\mu$ are not IGI.) This allows us to make a gauge invariant choice of the tetrad members l_μ,n_μ in the perturbed space-time. This choice will strongly constrain, in a gauge invariant manner, the perturbations δm_μ via Eq. (26). Furthermore, the parts of δm_μ not fixed by the choice of δl_μ do not make any contribution to the perturbed Weyl scalars (32)–(36). Thus subject to a choice of the IGI terms $\delta l_\mu,\delta n_\mu$ (which is analogous to the choice of tetrad in the background), the perturbed Weyl scalars are TGI. When we add in the fact that $\overline{\Psi}_2$ satisfies Corollary 1, we have our main result.

Proposition 1. *The perturbed Weyl scalars* (32)–(36) *are identification and tetrad gauge invariant with respect to odd perturbations*.

We can now calculate these gauge invariant terms. We repeat that two tetrad choices must be made: (i) we specify that the background tetrad uses the principal null directions as its real members and (ii) we must specify the gauge invariant terms $\delta l_\mu,\delta n_\mu$. We note however that $\delta\Psi_0$ and $\delta\Psi_4$ depend only on the first choice. In fact the same is true for $\delta\Psi_2$: using Eq. (26), we can show that

$$b=-\overline{g}^{\mu\nu}\delta g_{\mu\nu}.$$

Thus there is no contribution to $\delta\Psi_2$ from the perturbed tetrad.

The most obvious gauge invariant choice for the perturbation of the real members of the null tetrad is $\delta l_\mu=\delta n_\mu=0$. Working in the RW gauge, we can then solve Eq. (26) for δm_μ; as noted above, *any* particular solution of this system yields the same Weyl scalars. Then we calculate the Weyl scalars, and to conclude, write these in terms of the IGI quantities of Sec. II. The result is

$$\delta\Psi_0=\frac{Q_0}{2r^2}\overline{l^A}\,\overline{l^B}k_{A|B},\tag{40}$$

$$\delta\Psi_1=\frac{Q_1}{r}\left[(r^2\Pi)_{|A}\overline{l^A}-\frac{4}{r^2}k_A\overline{l^A}\right],\tag{41}$$

$$\delta\Psi_2=Q_2\Pi,\tag{42}$$

$$\delta\Psi_3=\frac{Q_1^*}{r}\left[(r^2\Pi)_{|A}\overline{n^A}-\frac{4}{r^2}k_A\overline{n^A}\right],\tag{43}$$

$$\delta\Psi_4=\frac{Q_0^*}{2r^2}\overline{n^A}\,\overline{n^B}k_{A|B},\tag{44}$$

where

$$\Pi=\epsilon^A_B(r^{-2}k^B)_{|A}$$

is the scalar introduced in Ref. [2] which appears in the master equations for odd perturbations. The angular coefficients here are given by

$$Q_0=-2w^aw^bS_{a:b},\tag{45}$$

$$Q_1=-\frac{1}{4}w^aS_a,\tag{46}$$

$$Q_2 = -\frac{i}{4} l(l+1) Y, \tag{47}$$

where $w^a = r^{-1}\overline{m}^a$. We can now give an interpretation of the gauge invariant metric perturbation k_A based on these scalars using the work of Szekeres [22]. The scalars Ψ_0, Ψ_4 are independent of the choice of perturbation in the tetrad and so depend only on our choice of background tetrad which, as argued above, may be considered to be invariant. Thus these two terms represent pure transverse gravitational waves propagating in the radial inward (respectively outward) null directions. We note that the formulas (40) and (44) have been given previously in Ref. [11].

Similarly, Ψ_2 is independent of the choice of tetrad perturbation. Thus this term invariantly describes a perturbation of the Coulomb component of the gravitational field.

The scalars Ψ_1, Ψ_3 depend on the choice of tetrad perturbation. However with our gauge invariant choice described above, we can state that the relevant coefficients represent pure longitudinal gravitational waves propagating in the radial inward (respectively outward) null directions.

We note that these statements are valid for $l \geqslant 2$. The angular coefficient Q_0 vanishes identically for $l=1$. Thus the vanishing of the terms $\delta\Psi_0$ and $\delta\Psi_4$ for $l=1$ is gauge invariant (and of course entirely expected: we only expect these gravitational radiation terms to switch on for the quadrupole and higher moments, $l \geqslant 2$). For $l=1$, Π is IGI but k_A is not so. Hence $\delta\Psi_2$ is gauge invariant, but $\delta\Psi_1$ and $\delta\Psi_3$ are not.

We note also that Eqs. (40)–(44) completely specify the gauge invariant metric perturbation; that is, these equations may be solved for k_A in terms of $\delta\Psi_{1-4}$. In particular, vanishing of the perturbed Weyl scalars at a point of space-time implies vanishing of k_A at that point.

B. Even perturbations

For even perturbations, we set $h_A^O = h = 0$. Infinitesmal coordinate transformations of even parity are generated by 1-forms of the form (13). The ‘‘even perturbations only’’ version of Lemma 3 is immediate. The following result describes the terms additional to those described by Lemma 2 which become IGI when we restrict to even perturbations.

Lemma 4. *Let $\overline{Q}(x^\mu)$ and $\overline{v}_\mu(x^\nu)$ be respectively a scalar and a 1-form defined on M^4. Then the linear perturbations of $\overline{Q}$ and $\overline{v}_\mu$ are IGI with respect to even perturbations if $\overline{Q} = \overline{Q}(x^a)$ with*

$$\gamma^{ab}\overline{Q}_{,a} Y_b = 0,$$

$\overline{v}_A = 0$ *and*

$$\overline{v}_a = \lambda S_a,$$

where $\lambda(x^b)$ satisfies

$$Y_a Y^a \lambda_{,b} Y^b + Y_{a:b}(Y^a Y^b - S^a S^b)\lambda = 0.$$

There are no vector fields $\overline{v}^\mu$ which are IGI with respect to even perturbations.

Note that it possible to construct covariant tensor fields of higher rank which are IGI by taking tensor products of the 1-forms described by the lemma.

Proof: The proof for the scalar case is immediate. In the 1-form case, the result follows by writing down the equations $\mathcal{L}_{\vec{\xi}_E}\overline{v}_\mu = 0$. This equation must hold for all $\vec{\xi}_E$ with 1-form equivalents given by Eq. (13). We obtain $\overline{v}_A = 0$ by considering particular forms of ξ^μ. We also obtain $\overline{v}_a = \overline{v}_a(x^b)$ and $Y^a\overline{v}_a = 0$. Since we are in 2 dimensions and $Y^a S_a = 0$, this implies that we can write $\overline{v}_a = \lambda(x^b) S_a$. The remaining conditions reduce to the linear partial differential equation for λ given in the statement.

Unlike the corresponding situation for odd perturbations, there is no hope of constructing useful gauge invariant background terms from the quantities described in this lemma. In particular, it is not possible to use the 1-forms described in the lemma to construct some of the null tetrad members. This is essentially because one cannot have any x^A dependence in the gauge invariant terms. Thus we can summarize as follows.

Proposition 2. *$\delta\Psi_0$ and $\delta\Psi_4$ are the only perturbed Weyl scalars that are identification and tetrad gauge invariant with respect to even perturbations.*

For completeness, we give these terms which have been given previously in Ref. [11]:

$$\delta\Psi_0 = \frac{1}{2r^2}\overline{l}^A\overline{l}^B k_{AB}(w^a w^b Y_{:ab}), \tag{48}$$

$$\delta\Psi_4 = \frac{1}{2r^2}\overline{n}^A\overline{n}^B k_{AB}(w^{*a} w^{*b} Y_{:ab}). \tag{49}$$

For the lowest multipole moments $l = 0,1$, the angular coefficients here vanish identically, and so the vanishing of $\delta\Psi_0$ and $\delta\Psi_4$ is gauge invariant.

IV. CONCLUSIONS

We have investigated the possibility of giving a gauge invariant physical interpretation of gauge invariant metric perturbations of spherically symmetric space-times by considering the perturbed Weyl scalars. This turns out to be possible only for the case of odd perturbations; however in this case, it transpires that all the perturbed Weyl scalars are identification and tetrad gauge invariant, and so the physical interpretation of the metric terms can be made. One can therefore immediately see the contribution of a particular metric perturbation to ingoing and outgoing longitudinal and transverse gravitational waves, and to the Coulombic interaction term. We anticipate that this will be of use in various different studies, for example in our ongoing work on the stability of Cauchy horizons in self-similar collapse [23]. The expressions (40)–(44) can be used to set coordinate independent and gauge invariant boundary conditions for perturbations, and can also be used as indicators of instability in different regimes (for example if such terms diverge in the approach to a singularity or to a Cauchy horizon). Care is needed here however. While the terms (40)–(44) indicate the presence or otherwise of various gravitational waves and Coulomb-type perturbations, they should not be used to determine magnitudes. This is crucial in setting boundary conditions, where one typically imposes a condition on the limiting behavior of a physically significant quantity. This is because of the scale covariance in the scalars resulting from the spin boosts (25): under these Lorentz transformations, we have

$$\delta\Psi_n \to a^{2-n}\delta\Psi_n, \quad n=0,\dots,4.$$

[For convenience, we have set $\omega=0$ in Eq. (25) as this will not affect magnitudes.] However this shows that the following GI first-order quantities have physically significant magnitudes, and so can be used for setting boundary conditions:

$$\delta P_{-1} = |\delta\Psi_0 \delta\Psi_4|^{1/2},$$

$$\delta P_0 = \delta\Psi_2,$$

$$\delta P_1 = |\delta\Psi_1 \delta\Psi_3|^{1/2}.$$

All three provide terms useful for the analysis of odd perturbations, while the first can also be used for even perturbations (and indeed in more general contexts [24]).

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[1] J.M. Stewart and M. Walker, Proc. R. Soc. London **A341**, 49 (1974).
[2] U.H. Gerlach and U.K. Sengupta, Phys. Rev. D **19**, 2268 (1979).
[3] T. Harada, H. Iguchi, and M. Shibata, Phys. Rev. D **68**, 024002 (2003).
[4] K.H. Lockitch, J.L. Friedman, and N. Andersson, Phys. Rev. D **68**, 124010 (2003).
[5] J.M. Martin-Garcia and C. Gundlach, Phys. Rev. D **64**, 024012 (2001); C. Gundlach and J.M. Martin-Garcia, *ibid.* **61**, 084024 (2000).
[6] D. Garfinkle, C. Gundlach, and J.M. Martin-Garcia, Phys. Rev. D **59**, 104012 (1999).
[7] A.V. Frolov, Phys. Rev. D **59**, 104011 (1999).
[8] J.M. Martin-Garcia and C. Gundlach, Phys. Rev. D **59**, 064031 (1999).
[9] C. Gundlach, Phys. Rev. D **57**, 7075 (1998).
[10] H. Iguchi, T. Harada, and K.-I. Nakao, Prog. Theor. Phys. **101**, 1235 (1999); **103**, 53 (2000).
[11] O. Sarbach and M. Tiglio, Phys. Rev. D **64**, 084016 (2001).
[12] G. Allen, K. Camarda, and E. Seidel, gr-qc/9806014.
[13] M. Siino, Prog. Theor. Phys. **99**, 1 (1998); Phys. Rev. D **59**, 064006 (1999).
[14] H. Kodama, A. Ishibashi, and O. Seto, Phys. Rev. D **62**, 064022 (2000).
[15] K. Tomita, Phys. Rev. D **56**, 3341 (1997).
[16] A. Ishibashi and H. Ishihara, Phys. Rev. D **56**, 3446 (1997).
[17] K. Nakamura, Prog. Theor. Phys. **110**, 723 (2003).
[18] C. Moreno and O. Sarbach, Phys. Rev. D **67**, 024028 (2003).
[19] O. Sarbach, M. Heusler, and O. Brodbeck, Phys. Rev. D **62**, 084001 (2000); O. Brodbeck, M. Heusler, and O. Sarbach, Phys. Rev. Lett. **84**, 3033 (2000).
[20] J. Stewart, *Advanced General Relativity* (Cambridge University Press, Cambridge, England, 1991).
[21] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1983).
[22] P. Szekeres, J. Math. Phys. **6**, 1387 (1965).
[23] B.C. Nolan and T.J. Waters, Phys. Rev. D **66**, 104012 (2002).
[24] C. Beetle and L.M. Burko, Phys. Rev. Lett. **89**, 271101 (2002).