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Gauge invariant perturbations of self-similar Lemaître-Tolman-Bondi spacetime: Even parity modes with $l \geq 2$

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In this paper we consider gauge invariant linear perturbations of the metric and matter tensors describing the self-similar Lemaître-Tolman-Bondi (timelike dust) spacetime containing a naked singularity. We decompose the angular part of the perturbation in terms of spherical harmonics and perform a Mellin transform to reduce the perturbation equations to a set of ordinary differential equations with singular points. We fix initial data so the perturbation is finite on the axis and the past null cone of the singularity, and follow the perturbation modes up to the Cauchy horizon. There we argue that certain scalars formed from the modes of the perturbation remain finite, indicating linear stability of the Cauchy horizon.

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I. INTRODUCTION

The naked singularities predicted in certain solutions to Einstein's field equations pose a threat to the validity of Penrose's cosmic censorship hypothesis (CCH); indeed, the CCH forbids the existence of naked singularities in generic gravitational collapse. Nonetheless, certain counterexamples do exist in which collapse results in a naked singularity. The best known example is perhaps the Reissner-Nordström solution (in the case $|\text{charge}| < |\text{mass}|$), however other instances would include the Kerr spacetime [1], spacetimes containing colliding plane waves [2], and spacetimes featuring critical collapse [3]. These naked singularities suggest the possibility of information from the singularity escaping to the external universe, resulting in a loss of well-posedness of the field equations. Fortunately, the Reissner-Nordström solution also provides a paradigm for the possible savior of the CCH: perturbations in the metric and matter tensor grow without bound when the Cauchy horizon is approached, with the Cauchy horizon undergoing a "blue-sheet" instability and becoming singular itself (see Chandrasekhar and Hartle [4]; see also Dafermos [5,6] for the Einstein-Maxwell-scalar field case and Poisson and Israel [7]). Thus the naked singularity is unstable under linear perturbations, and these perturbations are essential to give our spacetime the genericity on which the CCH depends; so much so that it could well be that naked singularities are an artifact of the high degree of symmetry of the spacetimes in which they are typically observed.

A class of spacetimes with an additional degree of symmetry in which naked singularities are commonly seen to occur is the class of self-similar spherically symmetric (SSSS) spacetimes, for example, certain classes of self-similar perfect fluid [8] and dust [9] solutions, and the self-similar scalar field [10]. In previous work, the authors have tested the stability of certain members of this class. In [11] a scalar field was used to model a perturbation and was allowed to impinge on the Cauchy horizon of a SSSS spacetime whose matter tensor was unspecified save for satisfying certain energy conditions, and in [12] pointwise bounds are found for a scalar wave impinging on the Cauchy horizon of a SSSS spacetime. In [13] we considered gauge invariant metric and matter perturbations of the self-similar null dust or Vaidya solution. The present paper represents a continuation of this process, in which we consider the perturbations of a more realistic and relevant spacetime, the self-similar timelike dust solution of Lemaître-Tolman-Bondi (LTB). We model metric and matter perturbations to take us away from the high degree of symmetry in the background and, after deriving initial conditions on the axis and past null cone of the origin, allow the perturbations to evolve up to the Cauchy horizon. There we see that certain scalars built from the modes of the perturbations remain finite, indicating that the Cauchy horizon associated with the self-similar LTB spacetime is linearly stable and does not display the blue-sheet instability seen in the Reissner-Nordström solution. The "modes" here refer to the coefficients of the state variables following a Mellin transform. We will use a commoving spatial coordinate r and a similarity variable y that is a time coordinate in the region between the past null cone of the naked singularity and the Cauchy horizon. Then the Mellin transform effects $G(y, r) \rightarrow g(y; s)r^{s-1}$. Then the statement above refers to the behavior of the functions $g(y; s)$: when these satisfy conditions on the axis and along the past null cone that correspond to the presence of an initially finite perturbation, they remain finite at the Cauchy horizon. Thus we demonstrate that a *necessary* condition for the linear stability of the Cauchy horizon is satisfied. The corresponding *sufficient* condition would entail demonstrating that the inverse Mellin transform of an initially finite perturbation remains finite. We discuss this resummation problem below.

The perturbation formalism of Gerlach and Sengupta [14,15] which we use in this work is very robust in that it can be applied to any spherically symmetric background. Moreover, the formalism has been tailored for the longitudinal or Regge-Wheeler gauge which simplifies the matter perturbation terms. Thus this formalism has been used by a number of authors in order to describe perturbations of spherically symmetric spacetimes, among them perturbations of critical behavior in the massless scalar field by Frolov [16,17] and Gundlach and Martín-García [18]; perturbations of timelike dust solutions by Harada *et al.* [19,20]; and perturbations of perfect fluids by Gundlach and Martín-García [21,22]. These analyses (with the exception of Frolov's) primarily rely on numerical evolution of the perturbation equations; there is a gap in the literature with regards to analytic or asymptotic solutions to perturbations of these spacetimes.

In broader terms, perturbations of the metric tensor can be thought of as modeling gravitational waves, an important topic in the current scientific community. This formalism has been used for exactly that purpose by numerous authors such as Harada *et al.* [19,20,23], Sarbach and Tiglio [24], and similar analyses by Nagar and Rezzolla [25]. Gravitational waves manifest themselves at the quadrupole and above, that is multipole mode number $l \geq 2$. Therefore in this work we will consider only those modes $l \geq 2$. In addition, we restrict our analysis to the even parity perturbations as it is in the even sector where the metric and matter perturbations are fully coupled, thus presenting a more substantial and interesting model. In the odd sector, the metric and matter perturbations are coupled but only insofar as the matter perturbation acts as a source term, and obeys a decoupled equation that fully determines the evolution of the matter perturbation. Furthermore, the master equation governing odd parity perturbations takes the form of a wave equation with a source term. This source term is completely and explicitly determined in terms of initial data, and does not give rise to any divergence. Then the perturbation may be dealt with *without* recourse to a Mellin decomposition using the methods of [12] and it seems clear that no instability arises. In order to restrict the length of the present paper, we defer a complete discussion of the odd parity case to a future publication.

The principal finding of this paper is that the Cauchy horizon formed in the collapse of the self-similar Lemaître-Tolman-Bondi spacetime is stable under linear gauge invariant perturbations in the metric and matter tensors, at the level of the Mellin modes as outlined above. In the next section we describe the mathematical background to the stability analysis, namely, we derive the metric and matter tensor for the self-similar timelike dust solution, we outline the perturbation formalism of Gerlach and Sengupta, and we describe two important mathematical tools: the Mellin integral transform and the generalized Frobenius theorem. In Sec. III we test the mode stability of the LTB spacetime by finding asymptotic limits for the perturbation modes on the axis and past null cone of the origin, and under suitable initial conditions allow the perturbation to evolve to the Cauchy horizon and beyond. We use the conventions of Wald [1] and set $G = c = 1$.

II. PRELIMINARIES

A. The self-similar LTB spacetime

The Lemaître-Tolman-Bondi spacetime has been well studied in the literature and we will not derive this solution here (but see, for example, Harada *et al.* [20]); we merely give a summary of the main points:

The LTB solution describes dust particles which move along timelike geodesics in a spherically symmetric spacetime, and thus has a matter tensor of the form

$$t_{\mu\nu} = \rho u_\mu u_\nu,$$

where $u_\mu u^\mu = -1$. We use comoving coordinates t, r with $u^\mu = \delta^\mu_t$ and $u^\mu \nabla_\mu r = 0$, and let $R = R(t, r)$ denote the areal radius. Solving the field equations gives (letting dot and prime denote differentiation with respect to t and r, respectively)

$$\rho = \frac{1}{8\pi} \frac{2m'}{R^2 R'}, \tag{1}$$

where m denotes the Misner-Sharpe mass, and in the marginally bound case $R^3 = \frac{9}{2} m(r)[t_c(r) - t]^2$, with $t_c(r) = \frac{2}{3}\sqrt{r^3/2m}$. Thus once we have specified $m(r)$ (or alternatively $\rho(0, r)$) we have completely determined all the unknowns.

From (1) we see the density diverges when $R = 0$, that is when $t = t_c(r)$. This is the curvature singularity known as the shell-focusing singularity, and we can interpret the function $t_c(r)$ then as the time of arrival of each shell of fluid to the singularity. Note there is an additional singularity known as the shell-crossing singularity when $R' = 0$. We will not consider this singularity as one may extend spacetime nonuniquely through the shell-crossing singularity; see Nolan [26]. To rule out the occurrence of the shell-crossing singularity we take $R' > 0$ for all $r > 0$; see Nolan and Mena [27].

Thus the line element for marginally bound timelike dust collapse is

$$ds^2 = -dt^2 + R'^2 dr^2 + R^2 d\Omega^2.$$

We, however, are interested in self-similar collapse, and thus we look for a homothetic Killing vector field ξ^a which solves the equation $\nabla_a \xi_b + \nabla_b \xi_a - 2g_{ab} = 0$. If $\xi^a = (\alpha(t, r), \beta(t, r))$, this returns four equations,

$$\dot{\alpha} = 1, \qquad R'^2\dot{\beta} = \alpha',$$
$$\beta R'' + \beta' R' - R' + \alpha \dot{R}' = 0, \qquad \beta R' + \alpha \dot{R} - R = 0.$$

From the first equation we can write $\alpha = t + F_1(r)$, for arbitrary F_1. Since $\beta R'' + \beta' R' = (\beta R')'$, we may combine the third and fourth equations to give $\alpha' = 0$, and thus we may change the origin of t to set $\alpha = t$. The second equation therefore gives $\beta = F_2(r)$, and we can make a coordinate transformation to set $\beta = r$. The remaining equations are

$$t(R')^{\cdot} + r(R')' = 0, \qquad t\dot{R} + rR' - R = 0.$$

The first of these equations is $\xi^a \partial_a R' = 0$ which is solved if and only if R' is a function of a similarity variable, in this case $y = t/r$. Thus if we set $R = rG(y)$, where G is a function of the similarity variable, we have $\partial R/\partial r = G - y(dG/dy)$, which is solely a function of y.

Thus the line element for a self-similar spherically symmetric timelike dust will be

$$ds^2 = -dt^2 + (G - yG')^2 dr^2 + r^2 G^2 dc\Omega^2, \qquad (2)$$

where from now on a prime denotes differentiation with respect to y. We may use this metric now to generate the Einstein tensor and examine the field equations, still using the comoving coordinates. The rr component of the field equations is $G'^2 + 2GG'' = 0$. Integrating yields $GG'^2 = p^2$, where p is some constant. The tt component then gives

$$\rho = \frac{1}{8\pi} \frac{GG'^2}{r^2 G^2 (G - yG')},$$

which is why we chose $GG'^2 = p^2 \geq 0$. Finally integrating this equation and using $R|_{t=0} = r$ we can solve for G as

$$G(y) = (1 - \mu y)^{2/3}, \qquad (3)$$

where $\mu = -\frac{3}{2}p$. We note that flat spacetime is recovered by setting $\mu = 0$.

There is a shell-focusing singularity therefore at $y = \mu^{-1}$. Since

$$\partial R/\partial r = (1 - \mu y)^{2/3}\left(1 + \frac{2}{3}\mu y (1 - \mu y)^{-1}\right),$$

we see that prior to the formation of the shell-focusing singularity, $y < \mu^{-1} \Rightarrow 1 - \mu y > 0$, thus $\partial R/\partial r > 0$. This rules out the formation of shell-crossing singularities prior to the formation of shell-focusing singularities.

The last issue is to examine the causal structure of the spacetime. Radial null geodesics satisfy

$$\frac{dt}{dr} = \pm \frac{\partial R}{\partial r},$$

with the plus and the minus describing ingoing and outgoing null geodesics, respectively. Since $t = yr$ this equation may be rewritten as

$$\frac{dy}{dr} = \frac{1}{r}\left(\pm \frac{\partial R}{\partial r} - y\right).$$

If there is some $y =$ constant which is a root of the right-hand side of this equation, it represents a null geodesic which reaches the singularity in the future/past. Thus the Cauchy horizon, $y = y_c$, is given by the first real positive zero of

$$G - yG' - y = 0, \qquad (4)$$

if one exists, and the past null cone of the origin, $y = y_p$, is given by the root of

$$G - yG' + y = 0. \qquad (5)$$

Since $G = (1 - \mu y)^{2/3}$, we find there is a Cauchy horizon, and therefore a naked singularity, if μ is in the range

$$0 < \mu \leq \mu_*, \qquad \mu_* = \frac{3}{2}(104 - 60\sqrt{3})^{1/3} \approx 0.638014.$$

Moreover, when μ is in this range, we have the following: there is one past null cone of the origin y_p; there is an additional future similarity horizon at $y = y_e > y_c$; as $\mu \to \mu_*$, $y_e \to y_c$; and as $\mu \to 0$, $y_p \to -1$, $y_c \to 1$ and $y_e \to \infty$.

Thus when $0 < \mu < \mu_*$, we have a spacetime with the structure given in Fig. 1. The scaling origin at which the singularity initially forms is the point $(t, r) = (0, 0)$. The apparent horizon forms when $g^{ab}\nabla_a R \nabla_b R = 0$ which is equivalent to $dG/dy = 1$. This occurs at $y = \frac{1}{\mu}(1 - (\frac{2\mu}{3})^3)$, that is, *before* the formation of the shell-focusing singularity at $y = \frac{1}{\mu}$.

B. Gauge invariant perturbations

We will use the formalism of Gerlach and Sengupta [14,15]. This formalism has been well used in the literature and so we will only give an outline here for completeness (but see Gundlach and Martín-García [22] or the authors [13] for a more detailed description).

We perform a $2 + 2$ split of spacetime into a manifold spanned by coordinates $x^A = (t, r)$ denoted $(\mathcal{M}^2, g_{AB})$, crossed with unit two spheres spanned by $x^a = (\theta, \phi)$ coordinates and denoted $(\mathcal{S}^2, \gamma_{ab})$. A spherically symmetric spacetime will therefore have a metric and matter tensor given by

$$g_{\mu\nu}dx^\mu dx^\nu = g_{AB}(x^C)dx^A dx^B + R^2(x^C)\gamma_{ab}dx^a dx^b,$$
$$t_{\mu\nu}dx^\mu dx^\nu = t_{AB}(x^C)dx^A dx^B + \frac{1}{2}t^c{}_c R^2(x^C)\gamma_{ab}dx^a dx^b.$$

Capital Latin indices will denote coordinates on $\mathcal{M}^2$, lower case Latin indices will denote coordinates on $\mathcal{S}^2$, and Greek indices the four-dimensional spacetime [i.e. $x^\mu = (x^A, x^a)$]. R is a function on $\mathcal{M}^2$ and gives the areal radius. Covariant derivatives on $\mathcal{M}$, $\mathcal{M}^2$, and $\mathcal{S}^2$ are, respectively, denoted

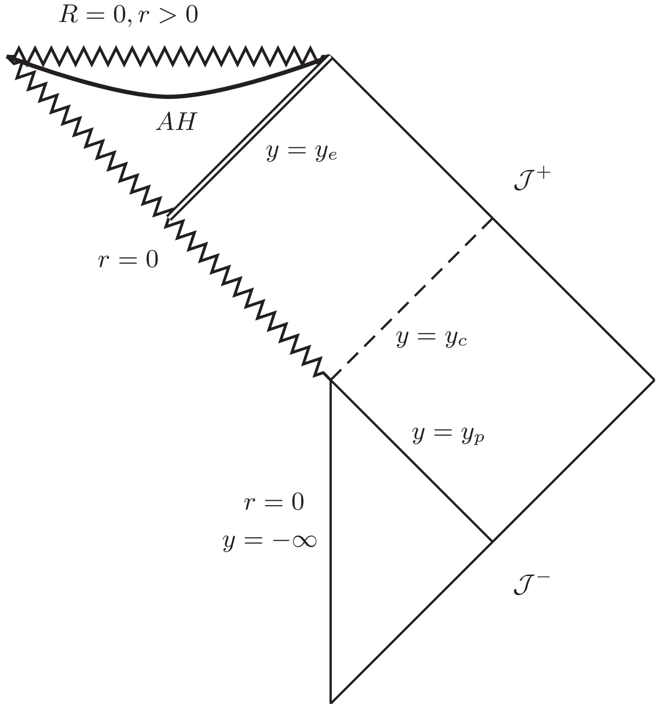

FIG. 1. Conformal diagram for the self-similar LTB admitting a globally naked singularity. There are three similarity horizons at which the similarity coordinate y is null: $y = y_p$ the past null cone, $y = y_c$ shown dashed, and $y = y_e$ shown as a double line. We identify $y = y_c$ as the Cauchy horizon, and will call $y = y_e$ the second future similarity horizon (SFSH). The apparent horizon is shown as a bold curve.

$$g_{\mu\nu;\lambda} = 0, \qquad g_{AB|C} = 0, \qquad g_{ab:c} = 0,$$

and a comma defines a partial derivative.

We write a nonspherical metric and matter perturbation as

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}(t, r, \theta, \phi),$$
$$t_{\mu\nu} = \tilde{t}_{\mu\nu} + \Delta t_{\mu\nu}(t, r, \theta, \phi),$$

where from now on an over-tilde denotes background quantities. The spherical harmonics form a basis for functions, and from the spherical harmonics we can construct bases for vectors,

$$\{Y_{,a}; S_a \equiv \epsilon_a{}^b Y_{,b}\} \tag{6}$$

and tensors,

$$\left\{Y\gamma_{ab}; Z_{ab} \equiv Y_{,a:b} + \frac{1}{2}l(l+1)Y\gamma_{ab}; S_{(a:b)}\right\}, \tag{7}$$

where we have suppressed the mode numbers l, m, $X_{(ab)} = \frac{1}{2}(X_{ab} + X_{ba})$ is the symmetric part of a tensor, and ϵ_{ab} is the antisymmetric pseudotensor with respect to $\mathcal{S}^2$ such that $\epsilon_{ab:c} = 0$. Using these, we decompose the perturbation in terms of scalar, vector, and tensor objects defined on $\mathcal{M}^2$, times scalar, vector, and tensor bases defined on $\mathcal{S}^2$.

We write the metric and matter perturbation as

$$h_{\mu\nu} = \begin{pmatrix} h_{AB}Y & h_A Y_{,a} \\ Symm & R^2(KY\gamma_{ab} + GZ_{ab}) \end{pmatrix},$$
$$\Delta t_{\mu\nu} = \begin{pmatrix} \Delta t_{AB}Y & \Delta t_A Y_{,a} \\ Symm & R^2\Delta t^1 Y\gamma_{ab} + \Delta t^2 Z_{ab} \end{pmatrix},$$

where, as mentioned previously, we confine our interest to even parity perturbations; that is, those defined using bases Y, $Y_{,a}$, $Y\gamma_{ab}$, and Z_{ab}. From these, we construct a set of gauge invariant scalars, vectors, and tensors given by

$$\left.\begin{aligned} k_{AB} &= h_{AB} - (p_{A|B} + p_{B|A}) \\ k &= K - 2v^A p_A \end{aligned}\right\}\text{(metric)} \tag{8a}$$

$$\left.\begin{aligned} T_{AB} &= \Delta t_{AB} - \tilde{t}_{AB|C}p^C - \tilde{t}_A{}^C p_{C|B} - \tilde{t}_B{}^C p_{C|A} \\ T_A &= \Delta t_A - \tilde{t}_A{}^C p_C - R^2(\tilde{t}^a{}_a/4)G_{,A} \\ T^1 &= \Delta t^1 - (p^C/r^2)(R^2\tilde{t}^a{}_a/2)_{,C} + l(l+1)(\tilde{t}^a{}_a/4)G \\ T^2 &= \Delta t^2 - (R^2\tilde{t}^a{}_a/2)G \end{aligned}\right\}\text{(matter)}, \tag{8b}$$

where $p_A = h_A - \frac{1}{2}R^2 G_{,A}$, and $v_A = R_{,A}/R$.

We may then recast the perturbation equations entirely in terms of these gauge invariant quantities (see Appendix A). Finally we must consider what to measure on the relevant surfaces to test for stability. As explained in [13], our "master" function will be

$$\delta P_{-1} = |\delta\Psi_0 \delta\Psi_4|^{1/2}, \tag{9}$$

where

$$\delta\Psi_0 = \frac{Q_0}{2r^2}\tilde{\ell}^A\tilde{\ell}^B k_{AB}, \qquad \delta\Psi_4 = \frac{Q_0^*}{2r^2}\tilde{n}^A\tilde{n}^B k_{AB}, \tag{10}$$

with

$$Q_0 = \tilde{w}^a\tilde{w}^b Y_{:ab},$$

where $\tilde{\ell}^\mu$, $\tilde{n}^\mu$, $\tilde{m}^\mu = r^{-1}\tilde{w}^\mu(\theta, \phi)$, and $\tilde{m}^{*\mu}$ are a null tetrad of the background and the $*$ represents complex conjugation. We note that δP_{-1} is a fully gauge invariant scalar, being both identification and tetrad gauge invariant (see [13]).

C. The Mellin transform

The Mellin transform is an integral transform related to the Laplace transform and is particularly useful for equations deriving from self-similar spacetimes. It is defined by

$$G(y; s) = \mathbb{M}[g(x, r)](r \to s) := \int_0^\infty g(y, r) r^{s-1} dr \quad (11)$$

with $s \in \mathbb{C}$. For this transform to exist, there will be a restriction on the allowed values of s, typically to lie in a strip in the complex plane with $\sigma_1 < \mathrm{Re}(s) < \sigma_2$. The inverse Mellin transform is given by

$$g(y, r) = \mathbb{M}^{-1}[G(y; s)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^s G(y; s) ds, \quad (12)$$

where $c \in \mathbb{R}$ is such that $\sigma_1 < c < \sigma_2$. To recover the original function from the Mellin transform, we integrate over the vertical contour in the complex plane of s given by $\mathrm{Re}(s) = c$. We emphasize, as this will be crucial later, that we do not integrate over all values of s in the interval $\sigma_1 < \mathrm{Re}(s) < \sigma_2$, only over the vertical contour defined by a specific value of $\mathrm{Re}(s)$ in this interval, which we are free to choose.

The perturbation equations will reduce to systems of ordinary differential equations (ODE's) in the individual modes of the Mellin transformed variables. Resumming the modes to recover the original function is an extremely complicated task and is beyond the scope of this paper (although see [13] for a discussion), however we will point out that although the finiteness of each mode is not a sufficient condition for the finiteness of the resummed original function, it is a necessary condition. Thus we will adopt the following minimum stability requirement: for the inverse Mellin transform to exist on a surface we must have each mode of the Mellin transformed quantity finite on that surface.

D. Extension to the Frobenius theorem

The theorem of Frobenius is particularly useful in finding power series solutions to ordinary differential equations near regular singular points. Consider the following nth order ODE in $f(x)$; there is a regular singular point at $x = 0$ if the ODE is of the form

$$x^n f^{(n)}(x) + x^{n-1} b_1(x) f^{(n-1)}(x) + \cdots + b_n(x) f(x) = 0, \quad (13)$$

with each b_i analytic at $x = 0$. We can Taylor expand each b_i about $x = 0$, and we denote such as expansion as

$$b_i(x) = \sum_{m=0}^{\infty} b_{i,m} x^m.$$

The so-called indicial exponents (see below) determine the leading behavior of the series solutions. It is well-known that when the indicial exponents repeat the solution must contain a logarithmic term, and when they differ by integers the solution may or may not contain a logarithmic term (see, for example, [28]). To clarify the structure of the solution when the roots differ by integers we give the following theorem due to Littlefield and Desai [29].

Theorem 1 (*n*th order Frobenius theorem)—Let $f(x)$ solve an ODE of the form (13). Then the indicial equation is

$$I_n(\lambda) \equiv \lambda(\lambda - 1) \ldots (\lambda - n + 1) + b_{1,0}\lambda(\lambda - 1) \ldots (\lambda - n + 2) + \cdots + b_{n-1,0}\lambda + b_{n,0},$$

whose roots are the indicial exponents. Collect together the indicial exponents which differ by integers into groups, and order the elements of each group as

$$\{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$$

such that $\lambda_i > \lambda_{i+1}$. Then the solution corresponding to λ_1 is $f_1(x) = \sum_{m=0}^{\infty} A_m x^{m+\lambda_1}$, and a linearly independent solution corresponding to λ_j is

$$f_j(x) = K_1 \log^{j-1} x \sum_{m=0}^{\infty} A_m x^{m+\lambda_1} + \sum_{i=2}^{j} \Big(\beta_i K_i \log^{(j-i)} x \sum_{m=0}^{\infty} \frac{\partial(i-1)}{\partial\lambda(i-1)} \times [(\lambda - \lambda_i) A_m]_{\lambda i} x^{m+\lambda_i} \Big)$$

where

$$K_i = \lim_{\lambda \to \lambda_j} \Big(\frac{\partial^{(i-1)}}{\partial \lambda^{(i-1)}} \Big[(\lambda - \lambda_j)^{(j-1)} \frac{A_{\delta_i}}{A_0} \Big] \Big), \qquad K_j = 1$$

and

$$\delta_i = \lambda_i - \lambda_j \epsilon \mathbb{N},$$

$$\beta_i = \frac{(j-1)(j-2)\ldots(j-i+1)}{i-1}, \qquad \beta_1 = \beta_j = 1.$$

III. PERTURBATIONS OF SELF-SIMILAR LTB SPACETIME

We will consider only modes $l \geq 2$. We assume the perturbed matter tensor remains that of dust,

$$\tilde{t}_{\mu\nu} + \Delta t_{\mu\nu} = (\tilde{\rho} + \delta\rho)(\tilde{u}_\mu + \delta u_\mu)(\tilde{u}_\nu + \delta u_\nu).$$

If we write the angular part in terms of the spherical harmonics, $\delta\rho = \varrho Y$ and $\delta u_\mu = \zeta_A Y + \zeta Y_{,a}$, then for $\tilde{u}_\mu + \delta u_\mu$ to be a unit, future pointing, timelike geodesic of the perturbed spacetime, as the conservation equation $\nabla^\mu t_{\mu\nu} = 0$ implies must be the case, we must have $\delta u_\mu = \gamma_{,A} Y + \gamma Y_{,a}$ for some scalar γ. Additionally, we must have

$$h_{tt} = -2\gamma_{,t}. \quad (14)$$

Using the Regge-Wheeler gauge (in which $h_A = G = 0 = p_A$), we may write the gauge invariant matter objects as

$$T_{AB} = \begin{pmatrix} \varrho + 2\tilde{\rho}\gamma_{,t} & \tilde{\rho}\gamma_{,r} \\ \rho\gamma_{,r} & 0 \end{pmatrix}, \qquad T_A = \begin{pmatrix} \tilde{\rho}\gamma \\ 0 \end{pmatrix}, \qquad T^1 = T^2 = 0.$$

Next we calculate the full set of perturbed field equations as given in (A1). We use the equation $k_A{}^A = 0$ to remove $k_{rr} = R'^2 k_{tt}$, and we use the tt component of (A1a) to define ϱ in terms of the other perturbation variables.

Thus we have a set of five second order, coupled, linear, partial differential equations in the four unknowns $\{k_{tt}, k_{tr}, k, \gamma\}$, and the two dependent variables t, r. (Had we removed k_{tt} with $k_{tt} = -2\gamma_{,t}$ (14), these would be third order in γ.) In this set of equations we make the change of coordinates

$$(t, r) \rightarrow \left(y = \frac{t}{r}, r\right), \tag{15}$$

and then perform a Mellin transform over r, reducing the problem to five second order ordinary differential equations in y, the similarity variable, and parametrized by s, the transform parameter. The Mellin transforms of our unknowns can be written

$$k_{tt} = r^s A(y; s), \qquad k_{tr} = r^s B(y; s),$$
$$k = r^s K(y; s), \qquad \gamma = r^{s+1} H(y; s),$$

thus the four unknowns of our set of ODE's are $\{A, B, K, H\}$.

The future pointing ingoing and outgoing radial null geodesic tangents of the background spacetime in t, r coordinates are

$$\tilde{\ell}^A = \frac{1}{\sqrt{2}R'}(R', 1), \qquad \tilde{n}^A = \frac{-1}{\sqrt{2}R'}(-R', 1)$$

respectively, since we restrict $R' > 0$ to avoid the shell-crossing singularity occurring before the shell-focusing singularity. The $\mathcal{M}^2$ portion (i.e. neglecting the angular part) of the perturbed Weyl scalars then becomes

$$\delta\Psi_0 = \frac{1}{r^2}\left(k_{tt} + \frac{k_{tr}}{R'}\right), \qquad \delta\Psi_4 = \frac{1}{r^2}\left(k_{tt} + \frac{k_{tr}}{R'}\right).$$

After a change of coordinates, and Mellin transform, we may write each mode of these scalars in terms of A and B. We define a new variable $D = A + B/(G - yG')$, and the scalars' modes simplify to

$$\delta\Psi_0 = r^{s-2}D, \qquad \delta\Psi_4 = r^{s-2}(2A - D),$$

and $\delta P_{-1} = |\delta\Psi_0 \delta\Psi_4|^{1/2}$. We must find solutions to the set of ODE's and use them to evaluate these modes on the relevant surfaces.

We can write this set of second order ODE's as a first order linear system

$$Y' = M(y)Y, \tag{16}$$

where a prime denotes differentiation with respect to y, and $Y = (A, D, K, H)^T$. We note that one of the equations in the system is $H' = -A/2$, and thus we have recovered (14), since $\partial/\partial t = \frac{1}{r}\partial/\partial y$. Because of its length, we give the components of the matrix M in Appendix B.

Examining the leading order coefficient matrix near the axis reveals that the axis corresponds to an irregular singular point of the system (16), with multiple zero eigenvalues, and a number of off-diagonal entries in its Jordan normal form; all of which make the system methods used in [13] very unattractive. In any case, we anticipate that the system methods would break down when eigenvalues of leading matrices differ by integers, suggesting we would at some stage need to decouple an equation in one variable, and use its solution as an inhomogeneous term to integrate the other equations. We will sketch the decoupling of an equation in H.

This system can be written as four first order equations,

$$h_1(A, D, K, H, A') = 0, \qquad h_2(A, D, K, H, D') = 0,$$
$$h_3(A, D, K, H, K') = 0, \qquad h_4(A, H') = 0.$$

We solve the first equation for $D = f_1(A, K, H, A')$ and substitute this into the other three equations, giving

$$h_5(A, K, H, A', K', H', A'') = 0,$$
$$h_6(A, K, H, A', K') = 0, \qquad h_4(A, H') = 0.$$

Combining h_5 and h_6 to remove K' means we can solve for $K = f_2(A, H, A', H', A'')$, and we are left with two equations,

$$h_7(A, H, A', H', A'', H'', A''') = 0, \qquad h_4(A, H') = 0.$$

Finally we remove A for a fourth order ODE in H,

$$h_0(H, H', H'', H''', H'''') = 0.$$

The other variables can be calculated when the solutions for H are found, as

$$A = g_1(H'), \qquad D = g_2(H, H', H'', H'''),$$
$$K = g_3(H, H', H'', H'''),$$

and thus we can write the scalars $\delta\Psi_{0,4}$ in terms of H and its derivatives.

Having derived the necessary equations, we pause to outline our general strategy for studying linear stability. The perturbation equations comprise a first order system of equations in certain perturbation variables $X(y, r)$ from which we can extract physically significant quantities. Of principal importance here is δP_{-1}. We seek to impose initial and boundary conditions on the perturbations that correspond to the most general, initially finite perturbation that satisfies appropriate conditions at the axis. This perturbation is allowed to evolve up to the Cauchy horizon, and we then try to determine if the perturbation has remained finite. The question of what is meant by "finite" is important. A minimal condition is that δP_{-1} be bounded on the past null cone. This however would allow for an infinite energy content over the past null cone—or over a spacelike surface an arbitrarily short time to the future of the past null cone. This leads naturally to the consideration of the L^2 norm of the perturbation. This is finite if and only if (by Plancherel's theorem) the L^2 norm of the Fourier transform of the perturbation is finite. But the Fourier transform is related to the Mellin transform by the complex rotation $s \to iz$ and $r = \ln\rho$. So we are led naturally to consider finiteness of the Mellin transform (a weaker condition than finiteness of the L^2 norm) as a minimal condition for finiteness of the perturbation. This is the condition on which we will focus: the perturbation $X(y, r)$ will be referred to as finite at time y_0 if the Mellin transform $x(y; s)$ is finite at time $y = y_0$.

A. Axis

We consider first the axis, $y = -\infty$. We make the transformation $y = -1/w$ to put the axis at $w = 0$, and then the transformation $w = z^3$ to ensure integer exponents in the series expansions about the axis of the coefficients of the differential equation. We find $z = 0$ is a regular singular point of the fourth order ODE in H, which we will write as

$$\sum_{j=0}^{4} z^j[h_j + O(z)]\frac{d^j H}{dz^j} = 0, \qquad h_j \neq 0. \tag{17}$$

The indicial exponents near $z = 0$ are

$$\{-3, 2, -4 - l - 3s, -3 + l - 3s\}. \tag{18}$$

The ambiguity of the value of s complicates matters regarding the position of logarithmic terms in the full solution, so we will begin by reminding ourselves of the two conditions the solutions must solve:

(1) The solution must exist; that is we must be able to recover the original function from its Mellin transform. Our minimum stability requirement for this to hold is that an acceptable solution is one which does not diverge on the relevant surface.

(2) The solution must be such that each mode of $\delta\Psi_{0,4}$ is finite on the axis.

Consider the indicial exponent -3. Regardless of the values of the other exponents, the corresponding solution will contain at least the series $\sum_{m=0}^{\infty} A_m z^{m-3}$. This is certainly not convergent; it diverges at $z = 0$ ($A_0 \neq 0$). Thus we must not consider this solution.

In examining requirement 2, we expand the coefficients of H and its derivatives in $\delta\Psi_{0,4}$ around $z = 0$, and we find the dominant term is

$$\delta\Psi_{0,4} \sim r^{s-2} z^4 H'.$$

Consider the indicial exponent $l - 3s - 3$. The contribution to the general solution corresponding to this eigenvalue is

$$(\text{Logarithmic terms}) \times (\text{Series}) + \sum_{m=0}^{\infty} A_m z^{m+l-3s-3},$$

where the first portion of this solution depends on the other eigenvalues, and may not even be present. Near $z = 0$, we find $\delta\Psi_{0,4} \sim z^{l-6}$ due to the second term. We would certainly expect these scalars to be finite on the axis for the quadrupole and other modes with $l < 6$, thus we must rule out this solution.

Similarly for the indicial exponent $-4 - l - 3s$, we find $\delta\Psi_{0,4} \sim z^{-l-7}$ near $z = 0$. Thus we must also rule out this solution.

Finally for indicial exponent 2, we see the solution

$$H = \sum_{m=0}^{\infty} A_m z^{m+2} \tag{19}$$

is convergent (near $z = 0$) $\forall\ s$, and thus satisfies our minimum stability requirement. Further, the scalars $\delta\Psi_{0,4}$ will be finite on the axis for $\mathrm{Re}(s) \geq 1/3$. Thus we have found a one-parameter family of solutions near the axis.

B. Past null cone

The past null cone, $y = y_p$, is the real, negative root of $G - yG' + y = 0$, where $G(y) = (1 - \mu y)^{2/3}$. There is only one real root (when $0 < \mu < \mu_*$), and it is parametrized by μ. Thus we may write

$$G - yG' + y = (y - yp)F(y), \qquad F(y_p) \neq 0.$$

y_p is a very cumbersome surd, and is quite difficult to work with. Instead, we draw out the nature of the coefficients of the H-equation by using $G'(y_p) = (G(y_p) + y_p)/y_p$. We find that setting $G' = (G + y)/y$ makes each coefficient vanish, except for the coefficient of the highest derivative. Thus we may write the H-equation as

$$\begin{aligned}(G - yG' + y)&[m_0 + O(y - y_p)]H^{(4)} \\ &+ [n_0 + O(y - y_p)]H^{(3)} + [p_0 + O(y - y_p)]H^{(2)} \\ &+ [q_0 + O(y - y_p)]H' + [r_0 + O(y - y_p)]H = 0,\end{aligned}$$

where m_0, n_0, etc. are the first nonzero terms in the series expansions about the past null cone.

We may write this in canonical form as

$$\begin{aligned}(y - y_p)^4 H^{(4)} &+ (y - y_p)^3 b_1(y) H^{(3)} \\ &+ (y - y_p)^2 b_2(y) H'' + \cdots = 0.\end{aligned}$$

If the series expansions of the b_i about the past null cone are denoted $b_i = \sum_{j=0}^{\infty} b_{i,j}(y - y_p)^j$, then the first few terms in the expansions of the b_i about $y = y_p$ are

$$
\begin{aligned}
&b_{1,0}=n_0/(m_0F_p) \qquad b_{1,1}=\cdots\\
&b_{2,0}=0 \qquad b_{2,1}=p_0/(m_0F_p) \qquad b_{2,2}=\cdots\\
&b_{3,0}=0 \qquad b_{3,1}=0 \qquad b_{3,2}=q_0/(m_0F_p) \qquad b_{3,3}=\cdots\\
&b_{4,0}=0 \qquad b_{4,1}=0 \qquad b_{4,2}=0 \qquad b_{4,3}=r_0/(m_0F_p)\\
&b_{4,4}=\cdots, \qquad (20)
\end{aligned}
$$

where $F_p = F(y_p)$. Therefore $y = y_p$ is a regular singular point of this ordinary differential equation, and the indicial equation for a fourth order ODE is

$$
\lambda(\lambda-1)(\lambda-2)(\lambda-3)+b_{1,0}\lambda(\lambda-1)(\lambda-2)+b_{2,0}\lambda(\lambda-1)+b_{3,0}\lambda+b_{4,0}=0.
$$

Thus the indicial exponents are

$$
\{0, 1, 2, 3-b_{1,0}\equiv\sigma\}.
$$

To determine what exactly σ is, we note

$$
F_p = F(y_p) = \lim_{y\to y_p}\frac{G-yG'+y}{y-y_p} = 1-y_pG''(y_p),
$$

using l'Hôpital's rule, and thus

$$
\sigma = 3-\left[\frac{7-s+2(\frac{y}{G}+\frac{G}{y})}{1-yG''}\right]_{y=y_p}. \qquad (21)
$$

We note that $\sigma = s$ in the limit $\mu\to 0$.

We may find the solutions due to these indicial exponents from the analysis in Sec. II D. Let us consider first the case $\sigma\notin\mathbb{Z}$. We group together the indicial exponents as

$$
\{2, 1, 0\}, \qquad \{\sigma\}
$$

since $\sigma\notin\mathbb{Z}$. Then, according to Theorem 1, the general solution has the form

$$
\begin{aligned}
H|_{y=y_p} = {}& h_1\left[\sum_{m=0}^{\infty}A_m(y-y_p)^{m+2}\right]\\
&+h_2\left[K_1\log x\sum_{m=0}^{\infty}A_m(y-y_p)^{m+2}+\sum_{m=0}^{\infty}B_m\right.\\
&\left.\times(y-y_p)^{m+1}\right]+h_3\left[\bar{K}_1\log^2 x\sum_{m=0}^{\infty}A_m(y-y_p)^{m+2}\right.\\
&+\bar{K}_2\beta_2\log x\sum_{m=0}^{\infty}B_m(y-y_p)^{m+1}+\sum_{m=0}^{\infty}C_m\\
&\left.\times(y-y_p)^m\right]+h_4\left[\sum_{m=0}^{\infty}D_m(y-y_p)^{m+\sigma}\right], \qquad (22)
\end{aligned}
$$

where the h_i are constants of integration and we have used an overbar to distinguish the K coefficients in the second and third solution.

We see the general solution contains three logarithmic terms, each multiplied by a constant. For the fourth order ODE in H we are considering here, these constants are (again from Theorem 1)

$$
\begin{aligned}
K_1 &= \lim_{\lambda\to 1}[(\lambda-1)A_1(\lambda)] = \frac{(b_{3,1}+b_{4,1})}{(2b_{1,0}-2)},\\
\bar{K}_1 &= \lim_{\lambda\to 0}[\lambda^2A_2] = \frac{b_{4,1}(b_{3,1}+b_{4,1})}{(2-b_{1,0})(2b_{1,0}-2)},\\
\bar{K}_2 &= \lim_{\lambda\to 0}\left[\frac{d}{d\lambda}(\lambda^2A_1)\right] = \frac{b_{4,1}}{(2-b_{1,0})},
\end{aligned}
$$

where we have set $A_0 = 1$. Crucially, since (20) $b_{3,1} = b_{4,1} = 0$, each of these terms vanish, and thus when $\sigma\notin\mathbb{Z}$, we have a general solution

$$
\begin{aligned}
H|_{y=y_p} = {}& h_1\sum_{m=0}^{\infty}A_m(y-y_p)^{m+2}+h_2\sum_{m=0}^{\infty}B_m(y-y_p)^{m+1}\\
&+h_3\sum_{m=0}^{\infty}C_m(y-y_p)^m+h_4\sum_{m=0}^{\infty}D_m(y-y_p)^{m+\sigma},
\end{aligned}
\qquad (23)
$$

with each series linearly independent. Our minimum stability requirement for these solutions will be satisfied for $\mathrm{Re}(\sigma) > 0$.

Now we examine the scalars $\delta\Psi_{0,4}$ near the past null cone, and we find we can write

$$
\delta\Psi_0 \sim c_1H + c_2H' + c_3H'' + c_4(y-y_p)H^{(3)}, \qquad (24)
$$

with a similar expression for $\delta\Psi_4$. The scalars are automatically finite on $y = y_p$ for the first three series in (23). For the fourth series, we find surprisingly that $c_3\sigma(\sigma-1)+c_4\sigma(\sigma-1)(\sigma-2) = 0$ for both $\delta\Psi_0$ and $\delta\Psi_4$; that is the coefficient of the leading term, which goes like $(y-y_p)^{\sigma-2}$, vanishes exactly. Thus for finite scalars on the past null cone due to the fourth solution, we require only $\mathrm{Re}(\sigma) > 1$.

Now let us consider $\sigma\in\mathbb{Z}$. Firstly if $\sigma < 0$, the minimum stability requirement is not met and we certainly cannot recover γ from H via the inverse Mellin transform; thus we consider $\sigma\geq 0$. Now we note an important point regarding the Frobenius method: if two indicial exponents differ by an integer, the solution corresponding to the lowest index may contain a logarithmic term; however if two indicial exponents are equal, the second solution *must* contain a logarithmic term.

If $\sigma = 0$, then there will be a solution which has leading term $\ln(y-y_p)$, which diverges at the past null cone, and thus the minimum stability requirement is not satisfied. If $\sigma = 1$, the corresponding leading term is $(y-y_p)\ln(y-y_p)$, which is finite in the limit $y\to y_p$. Thus we only consider $\sigma > 0$, when $\sigma\in\mathbb{Z}$.

When calculating the scalars $\delta\Psi_{0,4}$, we see from (24) that if $\sigma = 1$, then $H'\sim\ln(y-y_p)$, and thus we must discount $\sigma = 1$. Again, when $\sigma = 2$, we find $H''\sim\ln(y-y_p)$, however for $\sigma\geq 3$ we have $\delta\Psi_{0,4}\sim O(1)$.

Thus for the scalars to be finite on the past null cone, we require s to be such that $\mathrm{Re}(\sigma) > 1$, with the exception of $\sigma = 2$.

C. Cauchy horizon

The Cauchy horizon, denoted by $y = y_c$, is the first real root of $G - yG' - y = 0$ where $G = (1 - \mu y)^{2/3}$ and, as described in Sec. II A, exists and is unique for $0 < \mu < \mu_*$. The situation on the Cauchy horizon is very similar to the past null cone: we obtain a fourth order ODE in H with $y = y_c$ as a regular singular point, use series expansions about $y = y_c$ of the coefficients of the differential equation in the form (20), and find indicial exponents $\{0, 1, 2, \bar{\sigma}\}$ where

$$\bar{\sigma} = 3 - \left[\frac{7 - s - 2(\frac{y}{G} + \frac{G}{y})}{1 + yG''}\right]_{y=yc}, \qquad \lim_{\mu\to 0} \bar{\sigma} = s. \tag{25}$$

When $\bar{\sigma} \notin \mathbb{Z}$, all the logarithmic terms in the general solution vanish as at the past null cone. The scalar $\delta\Psi_0$ can be written near $y = y_c$ as

$$\delta\Psi_0 \sim \bar{c}_1 H + \bar{c}_2 H' + \bar{c}_3 H'' + \bar{c}_4 (y - y_c) H^{(3)},$$

with a similar expansion for $\delta\Psi_4$. Again, the coefficient of the leading term due to the solution due to the indicial exponent $\bar{\sigma}$ vanishes, and we find the scalars will be finite on the Cauchy horizon if and only if $\mathrm{Re}(\bar{\sigma}) > 1$, when $\bar{\sigma} \notin \mathbb{Z}$.

When $\bar{\sigma} \in \mathbb{Z}$, we find, for the same reasons as at the past null cone, we must rule out $\bar{\sigma} \leq 1$; when $\bar{\sigma} = 2$ the scalars diverge like $\ln(y - y_c)$; and when $\bar{\sigma} \geq 3$ the scalars are finite on the Cauchy horizon.

Let us consider first the clearer picture, when neither σ nor $\bar{\sigma}$ are integers. Both σ and $\bar{\sigma}$ are parametrized by s and μ, and thus we can plot the line in the $\mathrm{Re}(s)$, μ parameter space where $\sigma = 1$ and $\bar{\sigma} = 1$. We give this schematically in Fig. 2 for $0 < \mu < \mu_*$.

We interpret this plot so: for every μ, if $\mathrm{Re}(s)$ is such that the point $(\mu, \mathrm{Re}(s))$ is above the line $\sigma = 1$, the scalars will be finite on the past null cone. Similarly, if $\mathrm{Re}(s)$ is such that the point $(\mathrm{Re}(s), \mu)$ is above the line $\bar{\sigma} = 1$, the scalars will be finite on the Cauchy horizon. As the $\bar{\sigma} = 1$ line is always below the $\sigma = 1$ line for $0 < \mu < \mu_*$, this means that *all* perturbations which are finite on the past null cone at the level of the modes of the Mellin transform will be finite on the Cauchy horizon at the same level, when $\sigma, \bar{\sigma} \notin \mathbb{Z}$. It remains to consider the problem of resummation; this is discussed below.

When $\sigma, \bar{\sigma} \notin \mathbb{Z}$, the picture is a touch more intricate, due to the fact that $\sigma = 2$ or $\bar{\sigma} = 2$ will give a divergence in the scalars. Consider Fig. 3, and let us choose a particular value for μ, μ_0 where $0 < \mu_0 < \mu_*$. The solid portion of the line $\mu = \mu_0$ represents all the allowable values (from the point of view of initial data) of $\mathrm{Re}(s)$ for this

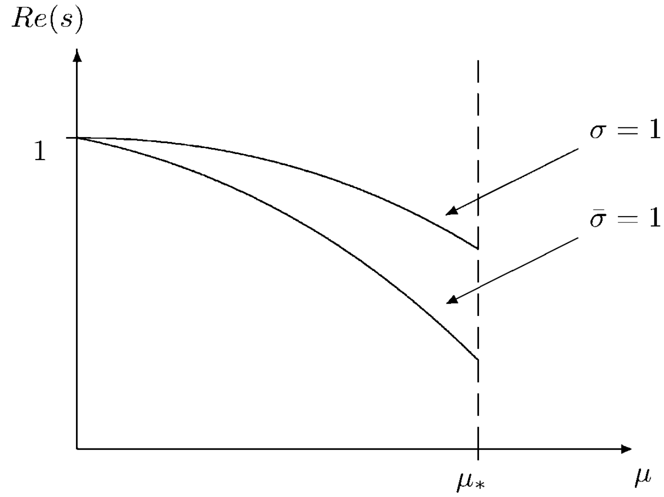

FIG. 2. The lines $\sigma = 1$ and $\bar{\sigma} = 1$ plotted in the $\mathrm{Re}(s)$, μ parameter space for $0 < \mu < \mu_*$.

μ_0, with the exception of where the line intersects $\sigma = 2$. We see that this line must intersect $\bar{\sigma} = 2$ at some point (μ_0, s^*), represented by the black dot in Fig. 3.

This point represents a precise value of s for which, if we were to perform the inverse Mellin transform over the vertical contour in the complex plane of s given by $\mathrm{Re}(s) = s^*$, the perturbation variables thus returned would generate finite scalars $\delta\Psi_{0,4}$ on the past null cone of the origin, but *diverging* scalars on the Cauchy horizon. However, we maintain this is not enough to conclude the Cauchy horizon is unstable, for the following reasons:

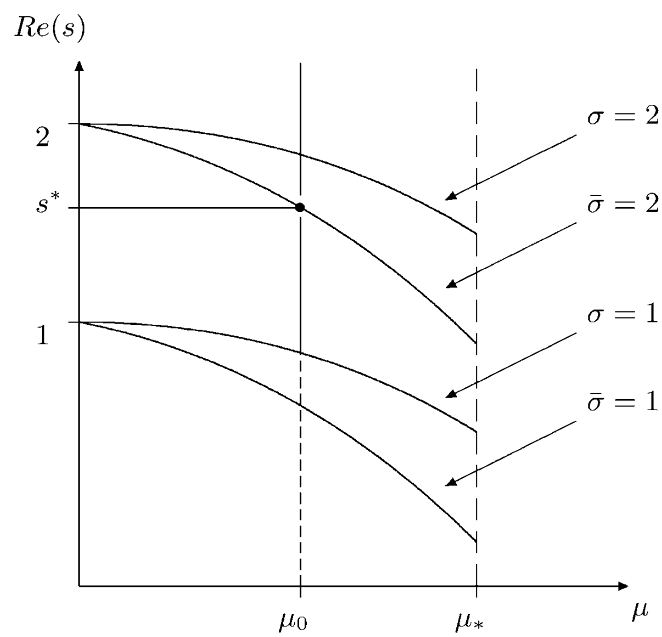

FIG. 3. The lines $\sigma = 1, 2$ and $\bar{\sigma} = 1, 2$ plotted in the $\mathrm{Re}(s)$, μ parameter space for $0 < \mu < \mu_*$.

(1) From our definition of $\bar{\sigma}$ (25), for $\bar{\sigma} = 2$, a real integer, we require $s = s^* \in \mathbb{R}$. Thus there is only a single, isolated point in the s complex plane at which s is such that $\bar{\sigma} = 2$, and it lies on the real axis. When performing the inverse Mellin transform, we must integrate over the contour $\mathrm{Re}(s) = s^*$ in the complex plane, where $\varsigma_1 < s^* < \varsigma_2$, as in Fig. 4. Thus the function γ is recovered as

$$\gamma(y, r) = \frac{1}{2\pi i} \int_{s^*-i\infty}^{s^*+i\infty} r^s H(y; s) ds.$$

A well-known theorem in complex analysis (Cauchy's integral theorem) states that we may continuously deform the contour of integration if the region thus swept out does not contain any poles. From our solution for H when $s = s^*$ (and thus $\bar{\sigma} = 2$), we see that the integrand has no poles due to the value of y. That the integrand has no poles due to the value of s is a technically very difficult question to address fully, and is beyond the scope of this paper; however, some analysis in this direction was carried out in Sec. 6 of [13], and there is evidence that no poles would be encountered in the general solution for H.

Thus when performing the inverse Mellin transform we may avoid the single, isolated point which makes the scalars diverge.

(2) The diverging mode corresponds to a single isolated point, that is a set of zero measure, in the s plane. This is not generic in any sense; to conclude an unstable Cauchy horizon we would be looking for an extended region in the s plane in which the modes diverge.

(3) Note that $\bar{\sigma} = 2$ lies between $\sigma = 1$ and $\sigma = 2$. Thus the value $\mathrm{Re}(s) = s^*$ means noninteger exponents in the solution for H near the past null cone; that is, the solution is nonanalytic. From the point of view of critical collapse, we would restrict our initial data to only consider analytic perturbations, and thus would avoid the diverging mode altogether. However assuming analytic initial data is a very strong restriction that we do not feel is warranted in the present case.

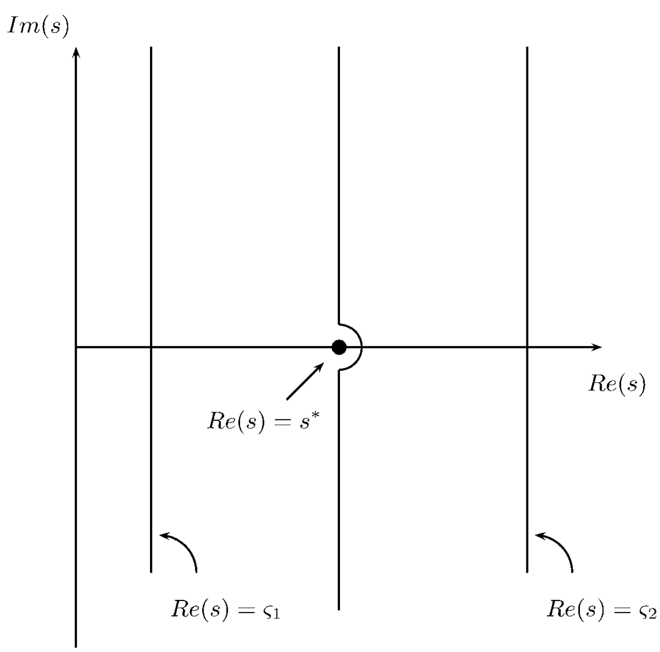

FIG. 4. Integrating over a contour in the complex plane of s.

For these three reasons we conclude that the Cauchy horizon formed in the collapse of self-similar timelike dust is stable at the level of the modes of the Mellin transform under even parity perturbations with $l \geq 2$.

D. SFSH

On the second future similarity horizon (SFSH), denoted $y = y_e$, we find indicial exponents for the fourth order ODE in H as $\{0, 1, 2, \bar{\bar{\sigma}}\}$, where

$$\bar{\sigma} = 3 - \left[\frac{7 - s - 2(\frac{y}{G} + \frac{G}{y})}{1 + yG''}\right]_{y=ye}, \qquad \lim_{\mu\to 0} \bar{\sigma} = -1. \tag{26}$$

Again we find the scalars go like $(y - y_e)^{\bar{\sigma}-1}$. We may write

$$\bar{\sigma} - 1 = \alpha(\mu)s + \beta(\mu).$$

Our initial data confined $\mathrm{Re}(s) > 0$, and it is easily found that for $0 < \mu < \mu_*$, both $\alpha(\mu)$ and $\beta(\mu)$ are always negative. Thus the scalars $\delta\Psi_{0,4}$ will diverge on the SFSH for *all* values of $\mathrm{Re}(s)$ allowed by initial data (in contrast to the Cauchy horizon).

We conclude that the second future similarity horizon formed in the collapse of self-similar timelike dust is unstable at the level of the modes of the Mellin transform under even parity perturbations with $l \geq 2$.

IV. CONCLUSIONS

We have examined the linear stability of the Cauchy horizon which may form in the collapse of the self-similar Lemaître-Tolman-Bondi spacetime, due to even parity perturbations of the metric and matter tensors of multipole mode $l \geq 2$. We have found that the Cauchy horizon is linearly stable at the level of the modes of the Mellin transform of the perturbation variables. However, interestingly, the second future similarity horizon which follows the Cauchy horizon is unstable.

A crucial question then is whether the same result applies to the full perturbation—that is, to the resummed Mellin modes. This is a highly nontrivial question. We note two possible approaches. One would be to try to determine the asymptotic behavior of the solutions of the s-parametrized system of ODE's for large values of $|s|$, with a view to showing that the solutions fall off at a rate that would guarantee existence of the contour integral giving the inverse Mellin transform. Another would be to employ the energy methods of [12] to directly study the evolution of the perturbation without recourse to the Mellin transform. This approach is currently begin used by one of us (BCN) to study rigorously the even parity perturbations of self-similar Vaidya spacetime. In both cases, there are significant technical obstacles. For the asymptotic analysis, one would need global information about how the different independent solutions of the s-parametrized ODE's at different singular points are related to one another. For the energy methods, the transition from a scalar wave equation to a first order hyperbolic system gives rise to significant additional difficulties, principally in determining an appropriate energy functional. It is hoped that developing the appropriate ''technology'' for Vaidya spacetime (a simpler case) will yield results applicable to the present case.

However, we maintain that the results derived here are of physical relevance. We have found that a nontrivial necessary condition for linear stability is satisfied. Furthermore and for example, an initially finite perturbation constructed from a finite number of Mellin modes will remain finite when it impinges on the Cauchy horizon.

Finally, we note that our results here mirror exactly those relating to the stability of the self-similar Vaidya spacetime previously studied by the authors, namely, that the naked singularity survives the perturbation but only does so for a finite time. This adds further weight to the observation of the authors in [13] that perhaps a generic feature of naked singularities in self-similar spacetimes is the linear stability of ''fan''-type similarity horizons (the Cauchy horizon) and instability of ''splash''-type similarity horizons (the SFSH), to use the terminology of Carr and Gundlach [30].

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APPENDIX A: PERTURBATION EQUATIONS

We give here the full set of perturbation equations for the gauge invariant quantities defined in Sec. II B. Note we only consider multipole modes $l \geq 2$ and thus all equations are valid.

$$
\begin{aligned}
&2v^C(k_{AB|C} - k_{CA|B} - k_{CB|A} + 2\tilde{g}_{AB}k_{CD}{}^{|D}) - 2\tilde{g}_{AB}v^C k_D{}^D{}_{|C} \\
&+\tilde{g}_{AB}\left(\frac{l(l+1)}{r^2} + \frac{1}{2}(\tilde{G}_C{}^C + \tilde{G}_a{}^a) + \tilde{\mathcal{R}}\right)k_D{}^D + 2(v_A k_{,B} + v_B k_{,A} + k_{,A|B}) \\
&+\tilde{g}_{AB}(2v^{C|D} + 4v^C v^D - \tilde{G}^{CD})k_{CD} - \tilde{g}_{AB}\left(2k_{,C}{}^{|C} + 6v^C k_{,C} - \frac{(l-1)(l+2)}{r^2}k\right) \\
&-\left(\frac{l(l+1)}{r^2} + \tilde{G}_C{}^C + \tilde{G}_a{}^a + 2\tilde{\mathcal{R}}\right)k_{AB} = -16\pi T_{AB},
\end{aligned} \tag{A1a}
$$

$$
\begin{aligned}
&-\left(k_C{}^C{}_{|D}{}^{|D} + \tilde{\mathcal{R}}k_C{}^C - \frac{l(l+1)}{2r^2}k_C{}^C\right) - (k_{,C}{}^{|C} + 2v^C k_{,C} + \tilde{G}_a{}^a k) \\
&+(k_{CD}{}^{|C|D} + 2v^C k_{CD}{}^{|D} + 2(v^{C|D} + v^C v^D)k_{CD}) = -16\pi T^1,
\end{aligned} \tag{A1b}
$$

$$
k_{,A} - k_{AC}{}^{|C} + k_C{}^C{}_{|A} - v_A k_C{}^C = -16\pi T_A, \tag{A1c}
$$

$$
k_A{}^A = -16\pi T^2. \tag{A1d}
$$

Here $\mathcal{R}$ is the Gaussian curvature of $\mathcal{M}^2$, the manifold spanned by the time and radial coordinates, and thus equals half the Ricci scalar of $\mathcal{M}^2$; also $\tilde{G}_{\mu\nu}$ is the Einstein tensor of the background spacetime.

APPENDIX B: COEFFICIENTS OF THE FIRST ORDER LINEAR MATRIX EQUATION

We give here the coefficients of the first order linear system $Y' = MY$ of Sec. III:

$$M_{11} = G^4(l + l^2 - 2s^2 + 2sG') - G^3 y\left(l + l^2 + 2(s^2 + s - 1) - 4\left(1 - l - l^2 - \frac{s}{2} + s^2\right)G' + (-1 + 8s)G'^2\right)$$
$$+ G^2y^2(l + l^2 + 2s + (-6 + 3l + 3l^2 + 4s + 2s^2)G' + (-14 + 6l + 6l^2 + s - 2s^2)G'^2 + 2(-2 + 5s)G'^3)$$
$$+ y^4G'(-2 + l + l^2 + (-4 + l + l^2)G' + (-4 + l + l^2)G'^2 + (-5 + l + l^2)G'^3 - G'^4) - Gy^3(-2 + l + l^2$$
$$+ 2(-2 + l + l^2 + s)G' + 3(-3 + l + l^2 + s)G'^2 + (-14 + 4l + 4l^2 - s)G'^3 + 4(-1 + s)G'^4)/G(2G^2(1 + s)$$
$$- 2G(1 + s)yG' + y^2G'^2)(G - yG' - y)(G - yG' + y)$$

$$M_{12} = G^4(l + l^2 - 2s^2) - 2G^3(-2 + 2l + 2l^2 + s - 2s^2)yG' + y^4G'^2(-4 + l + l^2 + (-5 + l + l^2)G'^2)$$
$$+ Gy^3G'(-2(-2 + l + l^2 + s) + (14 - 4l - 4l^2 + s)G'^2)$$
$$+ G^2y^2(l + l^2 + 2s + (-14 + 6l + 6l^2 + s - 2s^2)G'^2)/G(2G^2(1 + s) - 2G(1 + s)yG' + y^2G'^2)(G - yG' - y)$$
$$\times (G - yG' + y)(yG' - G)$$

$$M_{13} = (G - yG')(2G^3sG' + (-2 + l + l^2)y^3(1 + G'^2) + G^2y(l + l^2 + 2(-1 + s + s^2) - 5sG'^2)$$
$$+ Gy^2G'(-2(-2 + l + l^2 - s) + 3sG'^2))/G(2G^2(1 + s) - 2G(1 + s)yG' + y^2G'^2)(G - yG' - y)(G - yG' + y)$$

$$M_{14} = -2G'^2(G^3(-1 + s) - 2G^2(-2 + s)yG' + y^3G'(2 + G'^2) + Gy^2(-3 - s + (-4 + s)G'^2))/G(2G^2(1 + s)$$
$$- 2G(1 + s)yG' + y^2G'^2)(G - yG' - y)(G - yG' + y)$$

$$M_{21} = (G - yG')(G - y(1 + G'))(y^2(-2 + l + l^2 - 2G')G'(1 + G') + G^2(l + l^2 + 2s + 2sG') - Gy(-2 + l + l^2$$
$$+ 2(-2 + l + l^2 + s)G' + (-3 + 2s)G'^2))/G(G - yG' + y)(2G^2(1 + s) - 2G(1 + s)yG' + y^2G'^2)$$

$$M_{22} = -2G^4(l + l^2 - 2s^2) - y^4(2(-4 + l + l^2) - G')G'^3(1 + G') + 2G^3y(l + l^2 + 2s + 2(-2 + 2l + 2l^2 + s - 2s^2)G'$$
$$- (1 + s)G'^2) + 2G^2y^2G'(4 - 3l - 3l^2 - 4s + (13 - 6l - 6l^2 - 2s + 2s^2)G' + 2(1 + s)G'^2)$$
$$+ Gy^3G'^2(6l + 6l^2 + 4(-4 + s) + (-26 + 8l + 8l^2)G' - (3 + 2s)G'^2)/2G(2G^2(1 + s)$$
$$- 2G(1 + s)yG' + y^2G'^2)(G - yG')(G - yG' + y)$$

$$M_{23} = (G - yG')^2(-2G^2s(1 + s - G') - (-2 + l + l^2)y^2(1 + G')$$
$$+ Gy(-2 + l + l^2 - 2sG' - 3sG'^2))/G(G - yG' + y)(2G^2(1 + s) - 2G(1 + s)yG' + y^2G'^2)$$

$$M_{24} = -2G'^2(-G + yG')(-(G^2(-1 + s)) + y^2G'(2 + G') + Gy(-3 - s + (-3 + s)G'))/G(G - yG' + y)(2G^2(1 + s)$$
$$- 2G(1 + s)yG' + y^2G'^2)$$

$$M_{31} = (-(y^2(-2 + l + l^2 - 2G')G'(1 + G')) - G^2(l + l^2 + 2s + 2sG') + Gy(-2 + l + l^2 + 2(-2 + l + l^2 + s)G'$$
$$+ (-3 + 2s)G'^2))/G(2G^2(1 + s) - 2G(1 + s)yG' + y^2G'^2)$$

$$M_{32} = (G(l + l^2 + 2s) - (l^2 + l - 4)yG')/G(2G^2(1 + s) - 2G(1 + s)yG' + y^2G'^2)$$

$$M_{33} = (-2G^2sG' + (-2 + l + l^2)y^2G' - Gy(-2 + l + l^2 - 3sG'^2))/G(2G^2(1 + s) - 2G(1 + s)yG' + y^2G'^2)$$

$$M_{34} = (2G'^2(-(G(3 + s)) + 2yG'))/G(2G^2(1 + s) - 2G(1 + s)yG' + y^2G'^2)$$

$$M_{41} = -1/2 \qquad M_{42} = 0 \qquad M_{43} = 0 \qquad M_{44} = 0.$$

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