

# An Efficient Numerical Algorithm for the Transient Analysis of High-Frequency Non-Linear Circuits

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## ABSTRACT

*The paper proposes a new approach for the discrete-time integration of non-linear differential equations that describe the behaviour of high-frequency circuits, in particular those containing complex equivalent-circuit models of microwave transistor devices. The proposed approach reformulates a conventional predictor-corrector method in terms of Padé approximates about each function sample. The method is especially suited to the kind of non-linear stiff differential equations that arise frequently in high-frequency analysis.*

## INTRODUCTION

In the majority of cases, the non-linear simulation of high-frequency systems requires at some point that a numerical solution be found for a system of perhaps highly non-linear differential equations. Usually these equations arise from non-linear equivalent-circuit models for microwave active devices. The central core of any direct time-domain transient circuit simulator, such as SPICE (1), is therefore a numerical integration algorithm. Of course direct time-domain simulators of this kind have difficulties accommodating general linear frequency-domain behaviour at high frequencies, but even in popular alternative simulation strategies such as Harmonic Balance (2-6), a time-domain differential equation solution continues to play a key role.

The nature of the device equivalent-circuit models used is such that 'stiff' differential equations are often found, with very short time-constants arising from internal charge dynamics and efforts to describe non-quasistatic behaviour. However, it may be desired to perform an RF simulation of a circuit using digital modulation formats with long bit sequences, resulting in an extended solution period and excessively slow computation. Most existing circuit simulators use implicit numerical integration techniques with adaptive time stepping. These methods have good stability and accuracy control properties, but require the solution of non-linear algebraic equations at each time-step which can be computationally expensive. Alternatively, predictor-corrector methods can be used. In the

traditional predictor corrector, an explicit method uses polynomial extrapolation to provide an estimate of the solution at the next time step. The estimate is subsequently corrected using one iteration of an implicit formula. In relation to stiff problems, the maximum allowable time-step that can be used with the traditional predictor-corrector techniques such as the Adams-Moulton technique may be unacceptably small.

The current paper presents a different form of predictor-corrector. The predictor stage is similar to that described by Griffith and Nakhla (7) where a Padé approximation is used to predict a solution at the following time-step. The new corrector stage is proposed here, which involves adjusting the output of the predictor stage to increase the accuracy of the local Padé approximation without requiring a higher-order derivative. While the advantages of the proposed method for explicit simulation are illustrated here, the same predictor technique could be used to improve the performance of any one of the standard implicit circuit simulation techniques.

## NUMERICAL INTEGRATION TECHNIQUE

Consider the following initial-value ordinary differential equation system:

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t) \text{ with } \mathbf{y}(t=0) = \mathbf{y}_0 \quad (1)$$

The technique presented in this paper is similar to that of Griffith and Nakhla (7), in building a sequence of local approximations to  $y_i(t)$  in order to provide a solution to (1). These local approximations are Padé approximates and have the following form:

$$y_i(t) = \phi_{m/n} = \frac{\sum_{j=0}^m a_j h^j}{\sum_{j=0}^n b_j h^j} \quad h = t - t_{k-1} \quad (2)$$

The method is advanced in time by using the solution at time  $t$  as the initial condition for the next time-step.

In this paper, the predictor step of the proposed predictor-corrector method employs similar Padé approximates to relate  $y_i(t_{k+1})$  to  $y_i(t_k)$  and a corrector step is developed which further increases the accuracy of the method.

For a fourth-order method, the following predictor step is used:

$$\overline{y}_i(t_{k+1}) = \phi_{2/2} y_i(t_k) = \frac{a_0 + a_1 h + a_2 h^2}{1 + b_1 h + b_2 h^2} y_i(t_k) \quad (3)$$

The proposed corrector for the fourth-order method is as follows:

$$\hat{y}_i(t_{k+1}) = \overline{y}_i(t_{k+1}) - \frac{h^5}{720} f_i'''(t_{k+1}) \quad (4)$$

where  $\overline{y}_i(t_{k+1})$  is obtained from the predictor stage and is an estimate of  $y_i(t_{k+1})$ , the true solution at time  $t_{k+1}$ .

The rationale for the choice of corrector is as follows: Consider the Euler method:

$$\overline{y}_i(t_{k+1}) = y_i(t_k) + hf(t_k, y_i(t_k)) \quad (5)$$

The Trapezoidal Rule is:

$$\overline{y}_i(t_{k+1}) = y_i(t_k) + \frac{h}{2} [f(t_k, y_i(t_k)) + f(t_{k+1}, y_i(t_{k+1}))] \quad (6)$$

Now consider a predictor-corrector method that uses eqn. 5 as a predictor and uses eqn. 6 as a corrector. If this predictor-corrector is applied to the test function  $y=e^{-t}$ , it is observed that the result is equivalent to initially employing an explicit second-order Taylor series expansion, i.e.

$$\overline{y}_i(t_{k+1}) = y_i(t_k) + hf(t_k) + \frac{h^2}{2} f'(t_k) \quad (7)$$

Returning to the method specified in eqns. 3 and 4, a Padé approximate of order  $n$  matches the first  $n+1$  coefficients (time-domain moments) of a Taylor series expansion. It also provides additional terms. Considering the test function  $e^{-t}$ , a fourth-order Padé approximate is given by:

$$\overline{y}_i(t_{k+1}) = \phi_{2/2} y_i(t_k) = \frac{1 - 0.5h + 0.0833h^2}{1 + 0.5h + 0.0833h^2} y_i(t_k) \quad (8)$$

This function matches the first five coefficients of a Taylor series expansion for  $e^{-t}$ . It also produces additional terms, the first of which is:

$$T_6 = \frac{-1}{144} h^5 y_i(t_k) \quad (9)$$

However, the correct sixth coefficient in a Taylor series expansion for  $e^{-t}$  is:

$$T_6 = \frac{-1}{120} h^5 y_i(t_k) \quad (10)$$

Noting the observation regarding the Euler predictor-corrector, a corrector is chosen so as to match  $T_6$  for the particular test function,  $y=e^{-t}$ , without requiring a higher-order derivative. Hence, the choice of corrector specified in eqn. 4.

## ILLUSTRATIVE NUMERICAL EXAMPLE

As an initial example, the following well-known classic equation system is used (as given in Stabrowski (8)). Equations (11) and (12) constitute a stiff system of differential equations.

$$\frac{du}{dt} = 998u + 1998v \quad u(0)=1.0 \quad (11)$$

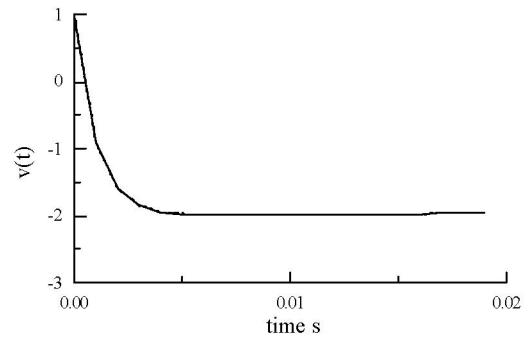
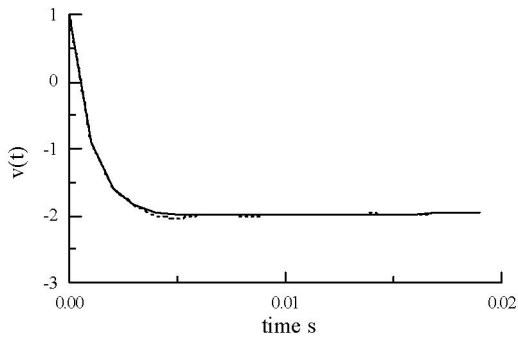
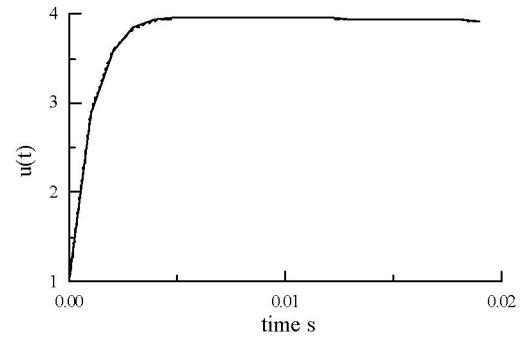
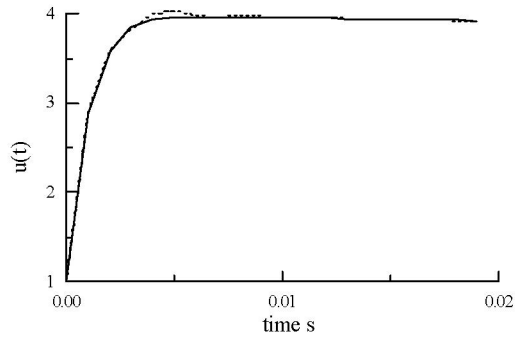
$$\frac{dv}{dt} = -999u - 1999v \quad v(0)=1.0 \quad (12)$$

The analytical solution is given by:

$$u(t) = 4e^{-t} - 3e^{-1000t} \quad (13)$$

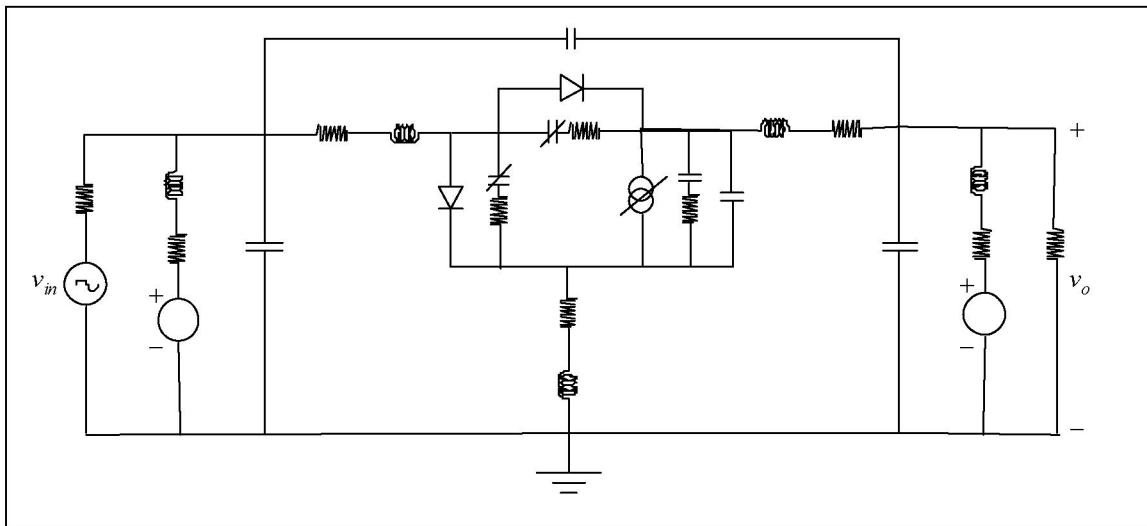
$$v(t) = -2e^{-t} + 3e^{-1000t} \quad (14)$$

Fig. 1 shows, superimposed on the analytical solution, the result computed with the Adams-Moulton predictor corrector for a step-size of 0.001s. Fig. 2 shows the corresponding result computed with the new predictor corrector. As evidenced by these results, the new technique is superior for the given step-size and therefore permits a significantly larger step-size for a comparable level of accuracy.



**Fig.1** Results computed with Adams-Moulton predictor-corrector

**Fig.2** Results computed with new predictor-corrector

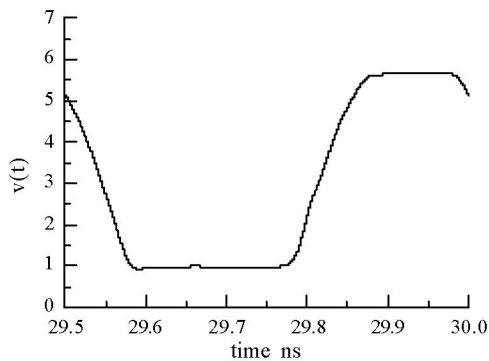


**Fig. 3** Simple MESFET Amplifier

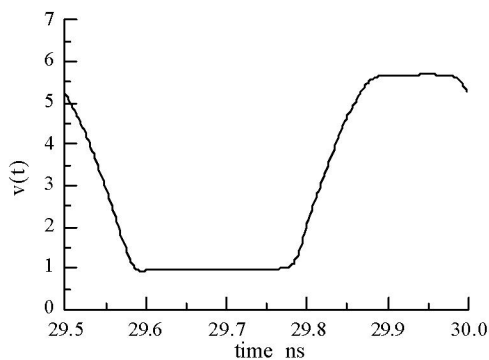
### A STIFF TRANSISTOR CIRCUIT

The simple single-ended MESFET amplifier shown in Fig. 3 is taken as a suitable test circuit for the proposed technique. The amplifier is described by ten non-linear differential equations which are stiff in nature. Fig. 4 shows a short segment of the output voltage  $v_o(t)$  obtained with the Adams-Moulton predictor-corrector

method. A time-step of  $\sim 0.12\text{ps}$  was required to solve the given system of equations with this technique. Fig. 5 shows the corresponding result obtained with the new predictor-corrector method. The new technique permits the use of a step-size **seven** times larger, ie  $\sim 0.85\text{ps}$ , for a comparable level of accuracy. This constitutes a remarkable saving in computing time.



**Fig. 4** Result with Adams-Moulton technique



**Fig. 5** Result with new technique

## CONCLUSION

The paper has proposed a new predictor-corrector method for the numerical integration of stiff differential equations. Illustrative examples have shown that the technique allows a significant increase in step-size when compared to a traditional predictor-corrector such as the Adams-Moulton technique. The technique is particularly suitable for CAD packages as it obviates the need for the solution of a non-linear set of algebraic equations at each time step.

## ACKNOWLEDGMENT

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