Collisionless heating in radio-frequency discharges: A review

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Abstract. Radio-frequency discharges are practically and scientifically interesting. A practical understanding of such discharges requires, among other things, a quantitative appreciation of the mechanisms involved in heating electrons, since this heating is the proximate cause of the ionization that sustains the plasma. When these discharges are operated at sufficiently low pressure, collisionless electron heating can be an important and even the dominant mechanism. Since the low pressure regime is important for many applications, understanding collisionless heating is both theoretically and practically important. This review is concerned with the state of theoretical knowledge of collisionless heating in both inductive and capacitive discharges.

1. Introduction

Radio-frequency discharges [1, 2, 3] are of both practical and scientific interest. Practical interest in such discharges is motivated by economically important applications, of which perhaps the most prominent are concerned with materials processing [4]. The existence and significance of these applications has permitted intense scientific interest in radio-frequency discharges, and this in turn has directed attention to a fascinating array of associated physical phenomena. Understanding these effects has presented both theoretical and experimental challenges, and the still continuing attempts to meet these difficulties has produced a lively, voluminous and occasionally controversial literature. This review is concerned with one aspect of these works, namely, the occurrence of so-called collisionless heating. This is a feature of radio-frequency discharges operated at low-pressure, where collisions involving charged particles are relatively infrequent, and the mean free paths of such particles may be comparable to—or even larger than—the characteristic size of the discharge. In this situation, collisionless heating may be the dominant mechanism sustaining the discharge, and, consequently, understanding and quantifying the collision heating effect may be essential to an approximate understanding of the discharge physics.

Radio-frequency discharges are, of course, excited by an oscillating current, but not all oscillating current discharges are radio-frequency discharges. The term "radio-frequency" discharge is usually understood to apply when the angular excitation frequency, $\omega_{\rm rf}$, is larger than the ion plasma frequency, $\omega_{\rm pi}$, but smaller than the electron plasma frequency, $\omega_{\rm pe}$. From this ordering of frequencies, two convenient theoretical assumptions follow. The first such assumption is that ions do not respond to fields that oscillate at the driving frequency, so ion motions can be calculated using fields that are time-averaged in some sense. The second assumption is that electrons respond instantly to time varying fields, so that the electrons can be taken to be always in equilibrium with the electromagnetic fields. Of course, neither of these assumptions is ever strictly true, and indeed the subject matter of this review can be understood to be a consequence of the failure of the second assumption, as we shall see. We shall, however, suppose that the first assumption holds. This means that we will be mostly concerned with electron physics, taking place against a background of stationary ion density. In experiments and in many computer simulations, there is indeed a flow of ions, for example from the interior towards the boundary of the plasma, but this is not incompatible with the assumption that the ion density is stationary.

The driving frequency $\omega_{\rm rf}$ and the size of the discharge L define natural macroscopic length and time scales. In the low-pressure regime, certain of the natural "microscopic" scales, such as those defined by the mean free paths and the collision frequencies of charged particles, are comparable to the macroscopic variables. This means that the charged particles are typically not in either spatial or temporal equilibrium with the electric and magnetic fields in the plasma, and this is a major cause of theoretical difficulty in understanding radio-frequency discharges, or indeed low-pressure discharges more generally. Under these conditions, the particle kinetics are said to be "non-local" [5, 6]. At the level of electron kinetics, for example, this means that an electron travelling at the typical thermal speed may move a distance comparable with the plasma size during one period of the driving frequency, without experiencing a collision. Clearly, the velocity of such an electron may be influenced by the field at every point along its trajectory. Then consider the population of electrons in some volume element of the plasma. In general, these electrons have all followed different trajectories. The value of some macroscopic quantity in this volume element, such as the current density, is expressed as an average over all the electrons. This average depends on the fields along the trajectory of every electron in the volume element. In general, this means that the current density at a given time in any volume element of the plasma depends on the fields at every position in the plasma and at all earlier times. The implication is that the relationships between many interesting quantities are not simple and algebraic, but are expressed in terms of integrals over the volume of the discharge and over past time. The theoretical difficulties are all too clear.

Collisionless heating can occur in this non-local regime, but only when there are appreciable spatial gradients in the fields. The typical situation in radio-frequency discharges is that the electric field is intensified near the boundaries. Thermal electrons cross this boundary layer in a time that is short compared with the radio-frequency period. Therefore, during a typical transit through the boundary layer, the electric field does not change sign, and the electron experiences an impulse, which is often called a velocity kick. Such an impulse changes the energy of the electron, and this change can be positive or negative, but we will see that, in general, the average effect is to transfer energy from the field to the electrons—there is a heating effect, in other words. The effect is generically known as "transit time heating." Transit time heating can occur whenever spatially inhomogeneous fields are combined with non-local electron kinetics. This situation can arise in both inductive and capacitive radio-frequency discharges, and the electron heating mechanism is essentially the same, although the inhomogeneous fields are produced by different mechanisms. Similarity of basic mechanism does not, however, imply that there are no important differences between these cases. This is because a detailed understanding includes the origin of the field inhomogeneity, and this needs to be considered self-consistently with the non-local electron kinetics. That is, we must solve the coupled system of the electron-kinetic equations and the field equations to obtain a full picture.

The fundamental equations applicable in this situation are, naturally, the Maxwell equations for fields and the Boltzmann equation for electron kinetics. In general, this is a formidable system of equations that can be solved analytically only under special conditions. However, these coupled equations can be solved rather generally, and without further approximations, using computer simulation methods, such as the particle-in-cell algorithm [7, 8, 9, 10]. In principle, such techniques capture all the relevant physics, but in practice, there are a number of disadvantages. Chief among these is the cost—direct solutions of the Boltzmann equation are still challenging when more than one space dimension must be handled. The numerical aspects also require careful handling, if strictly accurate results are desired (and this is often the motivation for using these methods). Moreover, in cases where the computational cost of solving the Boltzmann equation is prohibitive, other methods must be used, and computationally more tractable representations are needed. These can range from fully analytical models, useful for developing scaling relations, to computationally economical approaches, for example using modified moment equations. For these reasons, analytical insight into the nature of collisionless heating mechanisms is desirable, to make clear the basic mechanisms and to guide the development of more elaborate models. The main aim of this review is to discuss the basic mechanisms of collisionless heating in inductive and capacitive discharges, together with the analytical models that embody our present understanding. The most recent review of this area, by Lieberman and Godyak [11], is more than a decade old. There has been considerable progress since then, especially in understanding capacitive discharges. More general information is available in book form [3, 1, 4, 2].

In the next section, we discuss some general issues that apply to both inductive and capacitive discharges. We then turn to specific discussion of the separate cases of inductive and capacitive discharges, and finally offer some concluding remarks.

2. Background

Before proceeding, we should make a terminological point. The effect we propose to discuss is usually called "collisionless heating" or sometime "stochastic heating," which terms are used fairly interchangeably in the literature. These expressions are arguably misleading, in that, in the majority of cases, collisions are important, even if this importance is implicit. For example, in most of the theories that we will discuss, we assume that a well-defined flux of electrons issues from the plasma onto the boundary layer, and that this flux has some definite distribution of velocities, such as a Maxwellian. However, if the electrons are heated in the boundary layer, the flux returning to the plasma cannot have the same velocity distribution. So this scenario implicitly assumes that some mechanism of relaxation occurs in the bulk plasma. Among various possibilities, collisions are probably the most important mechanism in practice. This situation has been called "hybrid" heating [12], because it is possible to envisage a heating mechanism that is strictly collisionless or stochastic, although this probably occurs rarely in practice. The main theme of this review is in fact "hybrid" heating, in this sense.

As we have already said, the basic equations that describe collisionless heating in radio-frequency discharges are the Boltzmann equation and the Maxwell equations. When we consider the separate cases of inductive and capacitive discharges below, we will be using different subsets of the Maxwell equations—the curl equations to describe inductive discharges, and the Poisson equation to describe the capacitive case. What is in common between the two cases is the Boltzmann equation [13, 6], which describes the evolution of the electron distribution function:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{e}{m_e} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) t \nabla_v f = \left(\frac{\delta f}{\delta t} \right)_{\text{coll}} , \tag{1}$$

where the distribution function f is a function of the velocity coordinate $\mathbf{v}$, the space coordinate $\mathbf{r}$ and time. The term on the right hand side of Eq. (1) represents a collision operator, which we have not written out explicitly. In general, the Boltzmann equation coupled to the Maxwell equations can only be solved after substantial analytical simplification, or using numerical methods, or with some combination of these approaches. The classical analytical approach is to deal with perturbations of a uniform plasma, that is to say we write $f(\mathbf{r}, \mathbf{v}, t) = f^{(0)}(\mathbf{v}) + f^{(1)}(\mathbf{r}, \mathbf{v}, t)$, where we assume $f^{(0)} \ll f^{(1)}$ and where $f^{(0)}$ is a given function, usually a Maxwellian. Corresponding perturbations of the field arise, and the basic strategy is to linearize with respect to these perturbations. This leads to a system of linear equations that can be solved systematically. In most cases, this approach leads at least to a good first approximation. As we will see, this is the case for inductive discharges, where the assumption of a spatially uniform plasma does not discard any essential physics. Consequently, classical kinetic theory supplies a good description of kinetic phenomena in inductive discharges. This is not the case for capacitive discharges, where the presence of spatial gradients in the plasma density is essential. For this problem, no truly systematic theoretical approach has yet been found—the attempts we will discuss all involve some *ad hoc* elements, and to this extent are unsatisfactory. Nevertheless, all the models we are going to discuss are in the spirit of this classical approach, in the sense that we assume that there exists a bulk plasma with a well-defined density and temperature, which supplies boundary conditions for the region where heating occurs.

In reality, the electrons in the bulk plasma usually do not have a Maxwellian distribution of velocities, and the operation of collisionless heating mechanisms at the plasma edge is one of the factors that distorts the distribution function [14]. A strictly self-consistent treatment of collisionless heating together with all the factors that affect the electron distribution function is beyond the scope of any known analytical treatment, and is achieved only in self-consistent kinetic models, such as the particle-in-cell simulations mentioned above [7, 8, 9] (and it is not always easy to obtain strictly valid results even then [10]). However, the influence of collisionless heating processes on the shape of the electron energy distribution function can be explored by formulating the influence of the heating mechanism as a diffusion coefficient in velocity or energy space, and inserting that coefficient into a suitable formulation of the Boltzmann equation [5, 6, 12]. Evidently, some understanding of the nature of the heating mechanism is a prerequisite for this approach. The scope of this review is limited to exploration of the mechanisms themselves, in the context of rather simple models of the bulk plasma, and we will not, therefore, discuss the question of how collisionless heating influences the electron energy distribution function beyond the remarks already made, but we note that there an extensive literature on this important and interesting topic [15]. Valuable but not specially recent reviews of these and related issues are by Kolobov and Godyak [5] and Kortshagen *et al* [6].

3. Inductive discharges

An inductive discharge operates on simple principles: an oscillating current passed through an antenna produces time varying electric and magnetic fields outside the antenna, and a plasma can sustain itself by absorbing energy from these fields [1, 4]. In the radio-frequency regime that we are discussing, electromagnetic waves cannot propagate through unmagnetized plasma: The electromagnetic disturbance is consequently confined to a boundary layer. If the plasma is cold and collisionless, we can quickly see that the length scale of this boundary layer is the classical skin depth. Within the plasma, the displacement current is negligible, and Maxwell's equations are approximately

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \tag{2}$$

$$\nabla \times \mathbf{H} = \mathbf{J}, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4}$$

$$\nabla \cdot \mathbf{D} = \frac{\rho}{\epsilon_0} \tag{5}$$

where $\mathbf{J}$ is the conduction current density in the plasma, ρ is the charge density and $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{H}$ are the usual electromagnetic fields.

Inductive discharge experiments usually, and apparently necessarily, have a complex geometry, and probably most, if not all, such experiments have three dimensional aspects. Moreover, the process of driving an oscillating current through the coil produces voltages that can be large, so that the plasma interacts with the coil electrostatically as well as electromagnetically. These are major issues in developing a detailed understanding of particular inductive discharge experiments, but peripheral complications when thinking about collisionless heating. We therefore want to a consider a model from which these effects are absent, and the simplest such model is planar and one dimensional. For analytical purposes we can go slightly further and consider a slab model, in which the plasma density is uniform. This model—or minor variations thereof—captures the main physics effects we wish to discuss, but at the price of introducing a degree of abstraction that may obscure the relationship between the model and experiments. We will return later to this issue.

Consider then a plasma slab, extended along the x axis [16]. We assume that the plasma is quasineutral, and that the inductive excitation is by a plane wave with an electric field on the y axis and a magnetic field on the z axis. In this model, there are no electric fields along the x axis, and we need consider only those components of the curl equations that involve E_y and B_z. If we further assume that time dependent quantities oscillate at the driving frequency, $\omega_{\rm rf}$, then

$$\frac{d^2\tilde{E}_y}{dx^2} - i\mu_0\omega_{\rm rf}\tilde{J}_y = 0 \tag{6}$$

where $\tilde{E}_y$ and $\tilde{J}_y$ are the amplitudes of the indicated electric field and current density components. If the plasma is cold and collisionless, then the electron momentum balance equation is

$$\frac{\partial u}{\partial t} = -\frac{e}{m_e}E_y, \tag{7}$$

where u is the electron drift velocity, so with harmonic time variation

$$\tilde{J}_y = \sigma\tilde{E}_y = \frac{ne^2}{im\omega_{\rm rf}}\tilde{E}_y, \tag{8}$$

so that

$$\frac{d^2\tilde{E}_y}{dx^2} - \frac{\omega_{\rm pe}^2}{c^2}\tilde{E}_y = 0. \tag{9}$$

If the plasma slab is contained in the region $0 \leq x \leq L$, with boundary conditions $\tilde{E}_y(x=0) = \tilde{E}_0$, and $\tilde{E}_y(x=L) = 0$, then

$$\tilde{E}_y(x) = \tilde{E}_0\frac{\sinh([L-x]/\delta_c)}{\sinh(L/\delta_c)}, \tag{10}$$

where $\delta_c = c/\omega_{\rm pe}$ is the collisionless skin depth. The typical spatio-temporal variation of the field in this case is shown in Fig. 1. We can define the electrical response of this system in terms of its surface impedance

$$\zeta = \frac{E_y(x=0)}{\mathcal{J}_z(x=0)} = i\omega\mu_0\delta_c \tanh\left(L/\delta_c\right) \tag{11}$$

where $\mathcal{J}_z$ is a surface current, such that

$$\mathcal{J}_z = H_z(x=0) = -\int_0^L J_y(x')dx'. \tag{12}$$

In this simple model, the surface impedance, which is expressed in ohms, is pure imaginary, because there is no dissipation.

We are going to open the discussion of collisionless heating with a rather elementary treatment [17], which illustrates the main ideas without too much mathematical complexity. The electric field in the slab model is always inhomogeneous in space, and in the limit that $\delta_c \ll L$, there is a distinct boundary layer of the kind assumed in our opening discussion. In this case, we can write the electric field as

$$E_y(x,t) = \tilde{E}_0 \exp(-x/\delta_c)\sin(\omega_{\rm rf}t). \tag{13}$$

Let us now consider an electron with a thermal speed v_x interacting with this boundary layer. Suppose the electron approaches from $x=\infty$, is specularly reflected at the origin, and returns to $x=\infty$. This electron undergoes a velocity change given by

$$\Delta v = -\frac{e\tilde{E}_0}{m_e}\int_{-\infty}^{\infty} dt' \exp(-s(t')/\delta_c)\sin(\omega_{\rm rf}t'+\phi_0), \tag{14}$$

where the position of the electron is $s(t)$ and ϕ_0 controls the relative phase of the interaction between the electrons and the field. If we adopt a convention that the electron is reflected from $x=0$ at $t=0$, then

$$s(t) = \begin{cases} -v_x t & t<0 \\ v_x t & t>0 \end{cases} \tag{15}$$

so that

$$\begin{aligned} \Delta v_y = & -\frac{e\tilde{E}_0}{m_e}\left[\int_0^{\infty} dt' \exp(-t/\tau)\sin(\omega_{\rm rf}t'+\phi_0) \right. \\ & \left. + \int_{-\infty}^{0} dt' \exp(t/\tau)\sin(\omega_{\rm rf}t'+\phi_0)\right] \qquad (16) \\ = & \frac{e\tilde{E}_0}{m_e}\frac{2\tau\sin\phi_0}{1+(\tau\omega_{\rm rf})^2} \qquad (17) \end{aligned}$$

where $\tau = \delta_c/v_x$ is the electron transit time through the boundary layer. If the electron under consideration has a thermal velocity v_y in the field direction, then the velocity after an interaction with the boundary layer is $v_y + \Delta v$, and the corresponding energy change is

$$\Delta\epsilon = (v_y+\Delta v_y)^2 - v_y^2 = \Delta v^2 - 2v_y\Delta v_y. \tag{18}$$

The phase average of this quantity is of primary interest, and this is

$$\langle \Delta\epsilon \rangle = \int_0^{2\pi} d\phi_0' \left(\Delta v^2 - 2 v_y \Delta v \right) \tag{19}$$

$$= \frac{e^2 \tilde{E}_0^2}{m_e} \frac{\tau^2}{\left[1 + (\tau\omega_{\mathrm{rf}})^2\right]^2}, \tag{20}$$

and this clearly is a positive quantity. To see why this is so, consider a pair of electrons entering the boundary layer at the same phase, and with y velocity components of equal magnitude but opposite sign. Both electrons will experience the same impulse, Δv_y, so the change of energy in this case is

$$\Delta\epsilon = \frac{1}{2} m_e \left[(v_y + \Delta v_y)^2 + (-v_y + \Delta v_y)^2 - 2 v_y^2 \right] = m_e \Delta v_y^2, \tag{21}$$

so a sufficient condition for positive energy transfer is that the y velocity distribution is symmetrical about 0. A final step in this calculation is to average Eq. (20) over the distribution of x velocities, $f(v_x)$ to obtain the heating power per unit area:

$$P = \int_0^{\infty} dv_x' f(v_x') v_x \langle \Delta\epsilon \rangle \tag{22}$$

$$= \frac{e^2 \tilde{E}_0^2 \delta_c}{4\pi m_e} \frac{4\delta_c}{\bar{v}} \left[(1 + \bar{\tau}\omega_{\mathrm{rf}}) \exp(\bar{\tau}\omega_{\mathrm{rf}}) \mathrm{Ei}(\bar{\tau}\omega_{\mathrm{rf}}) - 1 \right] \tag{23}$$

where we have assumed that $f(v_x)$ is Maxwellian, $\bar{v} = \sqrt{8T/\pi m_e}$, and $\bar{\tau} = 2\delta_c/\sqrt{\pi}\bar{v}$. $\mathrm{Ei}(x)$ is the exponential integral defined as

$$\mathrm{Ei}(x) = \int_x^{\infty} \frac{\exp(-x')}{x'} dx'. \tag{24}$$

This calculation shows that the interaction of thermal electrons with an electric field localised in a boundary layer typically leads to power absorption by the electrons from the field. We have so far assumed that the structure of the boundary layer is not affected by this warm plasma effect, but in general this is not true. To achieve a quantitatively accurate understanding of this collisionless heating effect, we must attempt a self-consistent calculation, including the influence of warm plasma effects on the fields.

3.1. Kinetic theory

The field structure is changed because thermal electrons transport not just energy but also momentum along the x axis. This means that the current density at each location is not only a function of the electric field at the same place, as in Eqs. (7) and (8), but is some function of the electric field globally. At the microscopic level, this is because the population of electrons in any given region of the plasma are following trajectories given by Eq. (15), and their y velocity components in effect are given by an integral along these trajectories, where the limits of the integration extend from the present to $t = -\infty$. In a macroscopic formulation, and assuming again harmonic time variation, we can express this formally by replacing Eq. (8) with

$$\tilde{J}_y = \int_{-\infty}^{\infty} \Sigma(x - x') \tilde{E}_y(x') dx', \tag{25}$$

where Σ is a non-local conductivity [18]. For the purpose of evaluating this integral, we must extend the definition of $E_y(x)$ over the whole x axis using periodic images. This requirement is a consequence of assuming specularly reflecting boundaries. We shall see that the spatial distribution of the current density can be distorted quite grossly by these thermal effects, as Fig. 2 shows (although this is an extreme example).

We can approach this problem by considering how Eq. (7) should be amended to account for longitudinal momentum transport. The relevant equation, which can be obtained directly as a moment of the Boltzmann equation [19], is

$$\frac{\partial u_y}{\partial t} + \frac{1}{m_e n}\frac{\partial S}{\partial x} = -\frac{e}{m_e}E_y, \tag{26}$$

where S is the so-called shear stress given by

$$S = m_e \int d^2 v v_x v_y f. \tag{27}$$

In the language of conventional fluid mechanics, the new term in Eq. (26), the gradient of the shear stress, represents a viscosity. The shear stress clearly vanishes if f is uniform in real space and isotropic in velocity space, and therefore any calculation of S must allow, implicitly or explicitly, that f is not an isotropic Maxwellian. Under these circumstances it is natural to proceed from the Boltzmann equation, which under the conditions of interest here is

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} - \frac{e}{m_e}E_y \frac{\partial f}{\partial v_y} = \nu\left(f_0 - f\right). \tag{28}$$

We will assume that the distribution function f can be written as

$$f = f_0 + f_1, \tag{29}$$

where f_0 is an isotropic Maxwellian and f_1 is a small perturbation due to the radio-frequency field. We will also treat E_y as a small perturbation. If we assume that these are plane wave perturbations with angular frequency ω and wave number k, then we can find at once a relationship between the amplitudes of these perturbations

$$\tilde{f}_1 = \frac{\tilde{E}_y}{i(kv_x - \omega) + \nu}\frac{\partial f_0}{\partial v_y}. \tag{30}$$

From this equation we can find an expression for the amplitude of perturbations in the drift velocity

$$\tilde{u}_y = \frac{1}{n}\int v_y \tilde{f}_1 d^3\mathbf{v} \tag{31}$$

$$= \frac{\sqrt{\pi}}{m_e v_T}\frac{1}{k}\exp\left(-Z^2\right)\left[\frac{k}{|k|} + i\,\mathrm{erfi}(Z)\right]\tilde{E}_y \tag{32}$$

$$= \sigma_T \tilde{E}_y \tag{33}$$

where σ_T is the transverse conductivity and $Z = (\omega + i\nu)/kv_T$. The non-local conductivity in Eq. (25) can be expressed in terms of this transverse conductivity as

$$\Sigma(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sigma_T(|k|, \omega)\exp(ikx)dk. \tag{34}$$

With the help of a linearized version of Eq. (26), we can also obtain

$$\tilde{S} = \left[\frac{1}{\sqrt{\pi}}\frac{\exp(Z^2)}{i|k|/k - \mathrm{erfi}(Z)} + Z\right] v_T m_e n \tilde{u}_y. \tag{35}$$

The dimensionless parameter Z that appears in these equations can be understood as measuring the degree of non-locality. When collisional effects dominate, $|Z| \gg 1$, and

$$\tilde{S} \approx -ikm_e n v_T^2 \tilde{u}_y / 2\nu \tag{36}$$

which is equivalent to the real space expression

$$S = \mu \frac{\partial u_y}{\partial x} \tag{37}$$

where $\mu = m_e n v_T^2/2\nu$ is the Navier-Stokes viscosity. In an inductive discharge where collisionless heating dominates, the opposite limit applies, and we obtain

$$\tilde{S} = -\frac{m_e n v_T}{\sqrt{\pi}}\frac{ik\tilde{u}_y}{|k|} \tag{38}$$

with real space equivalent

$$S = -\frac{m_e n v_T}{\pi^{\frac{3}{2}}}\int_{-\infty}^{\infty}\frac{u_y(x+x') - u_y(x-x')}{x'}dx'. \tag{39}$$

This expression, like Eq. (25), explicitly displays the non-local character of the underlying physics, in the form of the integration over all space. We note that expressions such as Eq. (39 can be the basis for fluid-like treatments of these kinetic phenomena [20, 21, 19]. Such approaches can be valuable in large-scale simulation, where strictly kinetic approaches are difficult to apply.

We are now in a position to solve Eq. (6) using the non-local expression for the current density, Eq. (25). This solution can be carried out using Fourier transforms [22], and gives the formal result

$$\tilde{E}_y = -i\frac{\omega}{\pi c}B(+0)\int_{-\infty}^{\infty} dk \frac{\exp(ikx)}{k_x^2 - i(4\pi\omega/c^2)\sigma_T(|k_x|,\omega)}., \tag{40}$$

where $B(+0)$ denotes the magnetic field at the plasma surface. This result does not lead to an elementary expression for the field, but it can be used to obtain a result for surface impedance, which is

$$\zeta = \frac{2\mu_0\omega\delta_a}{3}\left(\frac{1}{\sqrt{3}} + i\right) \tag{41}$$

where

$$\delta_a = \left[\left(\frac{2T}{\pi m_e}\right)^{\frac{1}{2}}\frac{c^2}{\omega_{\mathrm{pe}}^2\omega}\right]^{\frac{1}{3}} \tag{42}$$

is a length, usually called the anomalous skin depth. The effect we are discussing is known as the anomalous skin effect [23]. The anomalous skin depth, however, does not have the same geometrical significance as the classical skin depth, because Eq. (40) does not describe an exponential decay in space. On the contrary, the spatial structure of the fields is complex, to the extent that there exist zones of negative time averaged electron heating deep within the plasma. The existence of these features has been confirmed experimentally, by direct measurement of the fields with a B-dot probe [24]. Some of this data is reproduced in Fig. 4. There is also reasonable agreement between the calculated and experimental inferred ratio of total power to collisional power [25]. The calculation we have just outlined applies to a half-infinite plasma, and will apply to finite sized plasmas if the skin depth is small compared to the size of the plasma. Not all experiments and simulations satisfy this condition, but there exist generalisations of this theoretical argument that extend to finite sized plasma, as well as to plasmas with finite collision frequency [26, 27]. These arguments are complicated, and they lead to results too complex to be presented here.

The main limitation of linearized theories such as those we have been describing is that they do not fully account for the influence of the induced magnetic field. When this is relatively strong, the electrons in the skin layer are magnetized, which inhibits the motion of the electrons through the skin layer, and reduces the collisionless heating effect [28, 29, 30].

4. Capacitive discharges

A capacitive discharge [3, 1, 2] is excited by applying a voltage to an electrode directly in contact with the plasma. In the usual way, a sheath forms where the plasma is in contact with the electrode. Since the electron density in the sheath is small, the electrical current passing between the electrode and the plasma through the sheath is substantially displacement. Consequently, large voltages are produced across the sheath (relative to the electron temperature) and there is a large, time varying, disturbance of the plasma density, as shown in Fig. 6. This means that there is a complicated, time varying, electric field in the plasma adjacent to the electrode. Hot electrons can gain energy by traversing this field, in much the same way as they do in the skin layer in inductive discharges. Collisionless heating can then occur by the same basic mechanism (transit time) that we saw in inductive discharges. There is experimental evidence for this effect [3, 31, 32], but not a detailed experimental characterisation, because no one has yet succeeded in making direct measurements of fields and currents in capacitive discharges, comparable with those that exist for inductive discharges [24, 25]. Other kinds of experiments, such as phase resolved optical emission spectroscopy [33], also provide evidence that we have basically correct ideas about these heating effects, but again, such measurements do not directly corroborate the theories that we are about to discuss. For these reasons, we are often reliant on simulation evidence to validate theories. A typical simulation result is in Fig. 8, which shows the characteristic outcome at low pressure, with heating localised in the sheath region, and a zone in the bulk plasma where the net heating effect is either small or negative.

Arriving at a quantitative understanding of collisionless effects in capacitive discharges is challenging, because of the complicated nature of the disturbance produced by the sheath voltage. This disturbance is too large to be treated as a small perturbation of a uniform plasma, so the assumptions employed in the conventional methods of plasma kinetic theory are not satisfied, and these methods cannot be applied in any straightforward way. As we shall see, there are several approaches to these quite severe theoretical difficulties, none entirely satisfactory. Clearly, one would like to see a theory developed from the basic equations, in this case the Vlasov-Poisson system, either exactly or using an approximation scheme that is both internally consistent and demonstrably valid. At present, all theories involve incompatible assumptions, or *ad hoc* elements, or both. It is, therefore, not easy to show that any of these theories has an *a priori* superiority, since choosing among inconsistencies and *ad hoc* elements can easily become a matter of taste. One can, of course, also investigate the degree of agreement between theories and experiments or simulations, and this may be a basis for distinguishing between theories. We will have more to say about this below.

In principle, a comprehensive theory of a radio frequency sheath could encompass both the electron and ion dynamics, and collisionless heating would emerge naturally from a consideration of the sheath structure. In practice, no one has advanced such a comprehensive theory. Instead, the usual approach is to develop a theory of the sheath structure that considers primarily the ion dynamics, and to use such a theory as a foundation for a theory of collisionless heating. In the regime where collisionless heating is likely to be important, a reasonable approximation is to treat the ions as responding only to time-averaged fields. This approach is valid when the ion transit time across the sheath is many times larger than the period of the radio-frequency voltage. Under these conditions, the ion density and the ion flux are stationary in time, although of course there is a constant flow of individual ions from the bulk plasma towards the electrode. Although the ion density is nearly constant in this case, the electron density is violently modulated by the applied voltage, and the region where the electron density is disturbed is typically much larger than the Debye length, the characteristic size of a quiescent sheath. In spite of the large voltages that are involved, the electron fluid is almost in instantaneous equilibrium with the potential, and can be represented as in Boltzmann equilibrium. These approximations—Boltzmann electrons and stationary ions—frame a convenient model of the sheath, which can be solved analytically [34]. This model features a single dimensionless control parameter:

$$H == \frac{\tilde{J}^2}{\pi\epsilon_0 T\omega^2 n_0} = \frac{1}{\pi}\frac{\tilde{s}_0^2}{\lambda_D^2}, \tag{43}$$

As this parameter controls the spatial structure of the sheath, it has a major influence on collisionless heating of electrons. A related parameter is the ratio of the electron drift velocity to the thermal velocity, which is effectively a normalized current density:

$$\delta = \frac{\tilde{u}}{\bar{v}}. \tag{44}$$

where $\tilde{u}$ is the electron drift velocity amplitude and $\bar{v} = \sqrt{8T/\pi m_e}$ is the electron thermal velocity. The case $\delta \to 0$ essentially corresponds to the limit of Boltzmann equilibrium for electrons. For any value of H likely to be experimentally accessible,

$\delta \ll 1$. Most experiments are conducted in a regime where $H \lesssim 10$. Note that the quantities in Eq. (44) are specified in the bulk plasma, and the inequality does not necessarily hold everywhere in the sheath region, where the electron density is smaller and the drift velocity amplitude correspondingly larger. Observations of relatively large electron drift velocities in the sheath region are not, therefore, necessarily incompatible with the foregoing remarks.

We have said that a radio-frequency sheath has a complicated spatio-temporal structure. Some appreciation of this structure is necessary background to an understanding of the associated collisionless heating effect. The main feature is of course the oscillation of the potential across the sheath. For reasons already explained, this oscillation does not directly affect the ion density, which is approximately stationary. However, there is a large modulation of the electron density, because the electron fluid is approximately in Boltzmann equilibrium with the potential. An example is shown in Fig. (6). At all times, there is somewhere a transition between a region of positive space charge, and a region of quasi-neutral plasma. The characteristic size of this transition region is the Debye length. Naturally, the location of the transition region varies with time. However, one can identify a fixed point dividing the sheath region from the bulk plasma—at this fixed point the Bohm criterion is satisfied. One can, therefore, usefully refer to the entire space between the Bohm point and the electrode as the "sheath region," keeping in mind that for most of the time this region is partly filled with plasma. Only when the sheath potential is a maximum do the Bohm point and the transition between space-charge and plasma approximately coincide. We generally assume that the plasma density and electron temperature at the Bohm point supply appropriate "bulk plasma" boundary conditions for the sheath region. For most purposes, we further assume that the plasma density beyond the Bohm point is effectively constant, so we will not usually distinguish carefully between quantities defined at the sheath edge and further into the bulk plasma. This model of the sheath structure is sketched in Fig. 7.

This complicated sheath structure is frequently simplified by assuming that the transition zone between the space-charge sheath and the quasi-neutral plasma can be replaced by a point, which we will call the electron sheath edge [3, 34]. With this approximation, there is a step in the electron density at the electron sheath edge. An influential idea in the theory of collisionless heating in capacitive discharges is associated with this representation of the electron density. The idea is that this step presents a moving potential barrier to electrons, so that electrons repulsed from the barrier can gain or lose energy during the interaction [35, 34]. When the sheath is expanding, the electrons collectively gain energy from the sheath field, and when the sheath is contracting, the electrons collectively lose energy. This is easy to see from the expression for the reflected velocity v_r of a particle with velocity v reflected from a barrier moving with velocity u_w:

$$v_r = -v + 2u_w. \tag{45}$$

This is a model with intuitive appeal, but there are difficulties in elaborating a detailed theory along these lines, as we shall see. In any case, Eq. (45) is an ansatz in need of justification, rather than a self-evident truth. Before turning to detailed discussion, we will note an important qualitative feature of the heating effect, namely that under typical conditions, the heating and cooling effects are almost equal. This observation highlights a basic theoretical difficulty, that one is attempting to construct an approximate theory that will extract a small average effect from a large oscillating background. The difficulty is obvious—the average effect might be expected to be highly sensitive to the choice of the approximations.

The case of greatest theoretical interest is the unmagnetized discharge excited by a single frequency. We shall therefore devote the greatest amount of space to this case, in the next section. In the following section we will comment relatively briefly on several more complicated cases.

4.1. Single Frequency Discharges

4.1.1. Classical kinetic theory Landau [36] was the first to suggest that non-Ohmic electron heating could occur in the presence of an oscillatory current crossing a plasma boundary. The physical situation considered by Landau was that of a quasi-neutral plasma slab of uniform density n_b, with electron temperature T_b and fixed ions. This plasma slab fills the half-plane where $x \geq 0$, and the boundary at $x = 0$ is assumed to specularly reflect electrons, such that the electron drift velocity at $x = 0$ vanishes identically. A sinusoidal current density with amplitude $\tilde{J}$ and angular frequency ω flows into the plasma across this boundary. Clearly, this current must be wholly displacement at $x = 0$ and, if $\omega < \omega_{\rm pe}$, the current will become wholly conduction as $x \to \infty$. Although Landau's calculation was kinetic, a simpler model illustrates the structure of his solution. If we linearize the first two moments of the Vlasov equation with respect to an electron number density perturbation and a drift velocity with respective amplitudes $\tilde{\delta n}$ and $\tilde{u}$, and assume that these perturbations oscillate in time at angular frequency ω, we can readily show that

$$\lambda_D^2 \frac{d^2\tilde{u}}{dx^2} - \left(1 - \frac{\omega^2}{\omega_{\rm pe}^2}\right)\tilde{u} = \tilde{u}_\infty \qquad (46)$$

where $\tilde{J} = -en\tilde{u}_b$. If $\omega \ll \omega_{\rm pe}$, then

$$\tilde{u} \approx \tilde{u}_\infty \left[1 - \exp(-x/\lambda_D)\right], \qquad (47)$$

i.e., displacement current is confined to a region at the edge of the plasma with a width $\sim \lambda_D$. (If $\omega > \omega_{\rm pe}$, Langmuir waves will be launched into the plasma.) In Landau's kinetic analysis, a collisionless heating effect appears that can be understood as a transit time phenomenon. Rukhadze [37] later obtained a formula for the power dissipated per unit area by this mechanism that can be written

$$\langle P \rangle = \frac{4}{\pi} n_b \bar{v}_b T_b \delta^2 = \frac{8}{\pi}\delta^2 Q_b, \qquad (48)$$

where Q_b is the heat flux incident from the plasma onto the boundary, $\bar{v}_b = \sqrt{8T_b/\pi m_e}$ is a thermal velocity, and $\delta = \tilde{u}_b/\bar{v}_b$ is the dimensionless parameter introduced above.

While this argument surely demonstrates the possibility of collisionless heating at the boundary of a current-carrying plasma, it is not acceptable as a model for collisionless heating in a capacitive discharge. Indeed, it is difficult to envisage any experimental situation for which its assumptions would be valid. A real plasma in contact with a boundary forms a sheath, as discussed above, and consequently there are large gradients of density as the boundary is approached. These steep gradients exist in a region of width $\sim \lambda_D$, *i.e.*, of the same characteristic size as the disturbance described by the Landau model. So it is doubtful if any gaseous plasma is accurately described by this theory, although there might be solid state plasmas compatible with these assumptions.

However, this line of theoretical investigation has recently been revisited by Kaganovich *et al* [38, 15]. Their approach aims to treat the sheath structure by the assuming the existence of a step function in the plasma density at the ion sheath edge, so that there is a uniform density n_b in the bulk plasma and a uniform density n_s in the sheath, with a step at the interface between the plasma and the sheath. In effect, this approach reformulates the boundary conditions in the Landau model to permit some fraction of the electrons reaching the boundary of the bulk plasma to penetrate into a sheath region, where they experience a velocity change given by Eq. (45). This sheath region is used to develop boundary conditions for a plasma bulk that can be treated as a slab. Therefore, this argument leads to a uniform plasma model that can be treated using classical kinetic theory. A parameter of this theory is the ratio of the sheath plasma density to the bulk plasma density n_s/n_b. Clearly, as a theory of the heating effect alone, this is rather rough, at least because the parameter n_s/n_b is a crude characterisation of the sheath structure. What this theory does offer is a careful treatment of kinetic effects in the bulk plasma [39]. Kaganovich *et al* [38, 15] point out that the kinetic solution can be written, *c.f.*, Eq. (47):

$$\tilde{u} \approx \tilde{u}_\infty \left[1 - \exp(-x/\lambda_D)\right] + u_t(x), \tag{49}$$

where u_t represents an intermediate scale solution with a characteristic length scale $\bar{v}/\omega$, which is a kinetic effect and as such absent from the moment analysis. This intermediate scale solution represents a disturbance of the bulk plasma produced by pulses of fast electrons emitted by the sheath as a side effect of collisionless heating. These disturbances decay by phase mixing (*i.e.*, they fade, because they consist of a bunch of electrons travelling at different speeds) on the indicated length scale. In most capacitive discharges, this length scale is large compared with the electrode separation, so the effect is "observed" (so far only in simulations) as regions of negative time averaged power deposition in the bulk plasma [39]. A useful conclusion from this work is that the intermediate scale solution has little effect on the net energy transfer to electrons—which is helpful to other theoretical approaches that consider only the sheath region.

4.1.2. Hard wall models Godyak introduced the "hard wall" model [35, 3], in which the electron sheath edge is regarded as a rigid barrier—the "hard wall"—that specularly reflects incident electrons, with a change of velocity assumed to be given by Eq. (45).

Since the barrier is in motion, there will generally be an exchange of energy between the electron and the sheath. In the model developed by Godyak, the electron flux incident on the sheath is assumed to have a non-drifting velocity distribution function $g_s(v)$ such that the instantaneous power per unit area transferred between the electrons and the sheath edge moving at velocity, u_w, is

$$P = \int_{u_w}^{\infty} \frac{1}{2} m_e \left(v_r^2 - v^2\right)(v - u_w) g_s(v) dv \tag{50}$$

$$= -2m_e \int_{u_w}^{\infty} u_w (v - u_w)^2 g_s(v) dv. \tag{51}$$

With the further assumptions that $g_s(v)$ can be approximated by a Maxwellian with temperature T_b and a density appropriate to the sheath edge, and that $u_w \ll \bar{v}_b$, so that the lower limit of integration can be set to zero, an integration over velocity and an average over time can be carried out to obtain the power per unit area:

$$\langle P \rangle = \frac{1}{2} m_e u_b^2 \bar{v}_b = \frac{8}{\pi} Q_b \delta^2, \tag{52}$$

which result is identical with Eq. (48). (This is presumably fortuitous.) An important variation of this model was given by Lieberman [34], who used a different form for the velocity distribution in Eq. (50), designed to take into account two effects neglected by Godyak: that the electron densities at the ion sheath edge and the electron sheath edge are different, and that the electron drift velocity is not zero. Lieberman uses

$$f_s(v) = \frac{n_s}{n_b} f_b(v - u_b), \tag{53}$$

where quantities with subscript b are defined at the ion sheath edge, and quantities with subscript s are defined at the electron sheath edge. When this expression is inserted into Eq. (50), again with the approximation that $\bar{v}_b \gg u_s$, one obtains the time averaged power:

$$\langle P \rangle = \frac{3\pi}{32} H m n_b \bar{v}_b \tilde{u}_b^2 = \frac{3}{2} Q_b \delta^2 H, \tag{54}$$

where H is given by Eq. (43). Eq. (54) has been widely adopted, and is the canonical expression found in the modern literature. And this is not unreasonable, since Eq. (54) is easy to evaluate and is fairly consistent with simulation and experimental data, insofar as systematic comparisons exist. The most recent variation of on this approach is due to Kaganovich *et al* [15]. Their model differs from Eq. 52 in that the energy transfer between the electrons and the sheath is expressed in the frame of reference in which the bulk electrons are non-drifting. The rationale for this change is unclear, but the effect is to modify the result obtained by Lieberman at small H:

$$\langle P \rangle = \frac{3}{2} Q_b \delta^2 H G(H), \tag{55}$$

where $G(H)$ denotes an integral that cannot be expressed analytically, but is well approximated by $H/(H + 1.1)$.

The "hard wall" heating model is often likened to the astrophysical phenomenon known as Fermi acceleration [11], in which charged particles are accelerated by reflection from moving magnetic structures. However, there is an important difference: There is a self-consistency condition applicable to capacitive discharges for which no analogy exists in Fermi acceleration. This condition is continuity of current in the sheath. The models we have been discussing essentially ignore this consideration, and thereby set aside a worrying difficulty, namely that it is difficult to avoid the conclusion that the time-averaged heating effect ought to be zero. To see this, we begin by noting that it is always possible to write the incident particle velocity in Eq. (45) as the sum of the drift velocity of the incoming particle distribution, u_s, and a random thermal component v', so that $v = u_s + v'$. Similarly, we can always write the incoming particle velocity distribution function $f(v)$ in terms of a non-drifting distribution function g such that $g(v' + u_s) = f(v)$ where $\int_{-\infty}^{\infty} v' g(v') dv' = 0$. Then

$$P = \int_{u_w}^{\infty} \frac{1}{2} m_e \left(v_r^2 - v^2\right)(v - u_w) f(v) dv \tag{56}$$

$$= -2m_e \left\{ \int_{u_w - u_s}^{\infty} (u_w - u_s)(v' + u_s - u_w)^2 g(v') dv' + \int_{u_w - u_s}^{\infty} u_s (v' + u_s - u_w)^2 g(v') dv' \right\}. \tag{57}$$

In the Godyak model $u_s = 0$, and in the Lieberman model $u_s = u_b$, where u_b is the drift velocity at the ion sheath edge. Either of these choices leads to a discontinuity of current at the electron sheath edge. In fact, the only choice consistent with current conservation at the electron sheath edge is $u_s = u_w$, and in this case the first integral vanishes and the second becomes

$$P = -2m_e u_s \int_0^{\infty} v'^2 g(v') dv' \tag{58}$$

$$= -2n_s u_s T_b \tag{59}$$

$$= -2n_b u_b T_b, \tag{60}$$

where T_b is the effective electron temperature in the bulk plasma (which is assumed constant). Since n_b is independent of time and u_b is a periodic function averaging to zero, this result does not vanish instantaneously, but it is zero when averaged over time [40]. The physical significance of this outcome can be appreciated by considering the hard wall velocity exchange formula, Eq. (45), when $v = u_s + v'$ and $u_w = u_s$. In this case $v_r = u_s - v'$, whereupon it becomes clear that when current is conserved only the sign of the thermal part of the particle velocities is exchanged upon reflection from the moving sheath, so that the distribution of thermal velocities g is the same for particles leaving the sheath as for those entering. This is why the first integral in Eq. (57) is identically zero. Although the energy of electrons does change on reflection at the electron sheath edge, this is necessary to preserve the drift energy, and is not to be considered a heating effect, any more than the oscillation of electrons in the bulk plasma is considered a heating effect in the absence of collisions. Thus, the non-vanishing term in Eq. (57) is a contribution to the oscillation of the electron drift energy, its presence required by the assumption of a point-like electron sheath edge. To show a heating effect, a mechanism must exist for modifying the distribution of thermal velocities. This argument does not, of course, call into question the existence of a collisionless heating effect. What it does show is that non-zero time averaged heating only occurs in the presence of some asymmetry with respect to the expanding and contracting phases of the sheath. The existing "hard wall" models achieve this asymmetry by means of an arbitrary inconsistency: a discontinuity in current across the sheath edge. One apparently needs a significantly more powerful underlying model to show a non-zero heating effect without this inconsistency.

Apart from the analytic studies we have been discussing, the literature contains many investigations of electron heating by oscillating sheaths using different kinds of computer simulation that do not include self-consistent electric fields [41, 42, 43, 44, 45, 46]. Generally, in the absence of self-consistent fields, current will not be conserved, and in view of the preceding discussion one cannot expect quantitatively correct results in such cases. Thus, while we do not propose to examine these works in detail, one should be cautious in accepting that they give valid accounts of the the phenomena they discuss.

4.1.3. Kinetic fluid models The difficulties so far discussed have motivated efforts to develop other types of models of collisionless heating. One such approach is based on so-called kinetic fluid equations, which might more properly be called kinetic moment equations, since the basic equations are developed from velocity moments of the Boltzmann equation. At least the first four velocity moments of this equation have a familiar physical interpretation. They represent the particle number density, the momentum density, the energy density and the heat flux. However, each moment equation contains a term involving the next higher moment. For example, the second moment equation, which is in essence an energy conservation equation, contains a heat flux term. Such terms have to be eliminated if the moment equations are to be of practical use, since otherwise an infinite hierarchy of equations arises. This consideration leads to the closure problem, which in this case involves expressing the heat flux in terms of the zeroth, first and second moments. The conventional closure approximations are constructed with the assumption that the mean free path is short compared to other interesting lengths, and this argument leads to a closed set of moment equations that are valid essentially at high pressure. However, this is not, as is often assumed, an intrinsic limitation of moment equations, but a particular property of this closure. Other closures may lead to systems of moment equations that are valid at low pressure, and even in the collisionless case [20].

With these ideas in mind, the kinetic-fluid approach to collisionless heating in capacitive sheaths embarks from the second moment equation [47, 48, 40]:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}nT\right)+\frac{\partial}{\partial x}\left(\frac{3}{2}nuT+Q\right)-u\frac{\partial}{\partial x}\left(nT\right)=0, \tag{61}$$

where the zeroth and first moment equations have been used to eliminate the electric field and terms involving the drift energy, *i.e.*, quadratic terms in u. The closure problem is represented here by the term involving the heat flux, Q. The kinetic fluid treatment of Gozadinos *et al* [40] proceeds by assuming that the effective temperature T can be modelled as constant in space. This assumption permits Eq. (61) to be integrated analytically over the sheath region, to give an ordinary differential equation for T with coefficients that depend on the plasma density. These coefficients are assumed to be supplied by some sheath structure model, such as that of Lieberman. The closure problem then entails expressing Q_b, the heat flux at the ion sheath edge, in terms of other known quantities. This can be done uniquely by imposing various self-consistency conditions to find

$$Q = Q_b \left(\frac{T}{T_b}\right)\left(1 - \frac{T}{T_b}\right), \tag{62}$$

which leads to an ordinary differential equation for the normalized temperature $\tau = T/T_b$:

$$\delta\left\{(1+\cos\theta)\frac{d\tau}{d\theta} + 2\tau\sin\theta\ln\eta\right\} + \tau(\tau - 1) = 0, \tag{63}$$

where as before $\delta = \tilde{u}_b/\bar{v}_b$ and $\eta = n_s/n_b$. Since $\delta \ll 1$, the solution can be conveniently approximated as

$$\tau = \tau^{(0)} + \delta\tau^{(1)} + \delta^2\tau^{(2)} + \ldots. \tag{64}$$

By requiring that Eq. (63) be satisfied separately for each power of δ, we obtain a hierarchy of equations that determine the functions $\tau^{(n)}$:

$$\tau^{(0)} = 1 \tag{65}$$

$$\tau^{(1)} = -2\sin\theta\ln\eta \tag{66}$$

$$\tau^{(2)} = -(1+\cos\theta)\frac{d\tau^{(1)}}{d\theta}. \tag{67}$$

Then we can express the power transfer to electrons as

$$\langle P\rangle = -\langle Q\rangle \tag{68}$$

$$= Q_b\left\langle \delta\tau^{(1)} + \delta^2\left(\tau^{(2)} + {\tau^{(1)}}^2\right)\right\rangle + O(\delta^3) \tag{69}$$

$$\approx \frac{3}{2}Q_b\delta^2\frac{24H}{55+H}, \tag{70}$$

where again the final expression is a heuristic fit to an integral that is not analytic.

This approach does not involve any explicit assumption about the form of the electron energy distribution function, although it is important to note that it cannot consistently be supposed to be a drifting Maxwellian, because the heat flux must vanish in that case. This treatment also avoids flagrant inconsistencies of the kind that appear in "hard wall" treatments of the problem. However, there are difficulties. Most notably, the assumption of a spatially uniform effective temperature has not been rigorously justified, and the closure given by Eq. (62) has heuristic elements. A further weakness—inherited from the Lieberman sheath model—is the concept of a point-like electron sheath edge, which is not an especially good description, especially when H is not much larger than unity, which is the case of primary practical interest.

4.1.4. Discussion In the three preceding sections, we have outlined the main line of the arguments used in three different theoretical approaches to describing collisionless heating in capacitive discharges. Despite the different physical concepts and mathematical ideas that appear in these treatments, there is no essential disagreement about the nature of the effect—only about the most appropriate mathematical description. As we have seen, it is not easy to argue for the superiority of any of these models on *a prior* grounds, because all involve either inconsistencies or contain heuristic elements—which are effectively unquantified approximations. Moreover, significant improvements in any of these theories do not seem imminent. The hard-wall approach appears not powerful enough to be the basis of a consistent theory. In kinetic theory and the kinetic-fluid approach, it is not easy to see how to avoid the heuristic elements in a mathematically tractable fashion, and indeed it is also possible that a better sheath model than that of Lieberman is needed, in particular, because the representation of the electron sheath edge as a step function may be one of the less satisfactory approximations that feature in all models. Further work exposing more clearly the limitations of these models would be desirable, and might lead to improvements. Meanwhile, one can ask the practical question: Which of the expressions canvassed above is the most useful? In answering this question it seems legitimate to look to comparisons with simulations or experiments. As there are no experiments that serve the purpose, simulations are the only choice. In this connection, we note that all theoretical approaches lead to a result that can be written

$$\langle P\rangle = \frac{3}{2}Q_b\delta^2 HG(H). \tag{71}$$

Physically, this is the product of a heat flux, the (normalized) square of the current density and a function of H that can be reasonably interpreted as a sheath structure factor, taking into account the effect of the sheath voltage on the spatio-temporal structure of the sheath. That all theories agree on the form given by Eq. (71) facilitates both comparisons of theories and theories with simulations. Furthermore, if the theories are right in agreeing on this form, then we can set aide the differences between the theories and simply choose $G(H)$ to agree with simulation data. In effect, this approach has been pursued in recent work by Kawamura *et al* [49]. Their conclusion is that their simulation data indeed take the form of Eq. (71), and that Eq. (55) best agrees with their numerical values. These data are reproduced in Fig. 9. On these grounds, Eq. (55) is the most appropriate practical formula, and in this context, formal criticism is not relevant. Clearly, an improved formula might be found by generating a larger simulation database, verifying (or perhaps disproving) that the form of Eq. (71) fits the data, and then proposing a purely empirical expression in place of Eq. (55).

4.2. More complex cases

The preceding sections have addressed the case of an unmagnetized capacitive discharge excited by a single frequency. This section will comment briefly on four more complicated cases. Namely, when two sheaths interact (so that bounce resonances may occur), when a magnetic field is present, when multiple excitation frequencies are deliberately employed, and when a circuit resonance introduces a high-frequency component into the current wave form. All these cases have received some attention in the literature, but, for reasons of space, we do not propose to discuss them in great detail here.

4.2.1. Interacting sheaths The discussion above implicitly assumes that we are interested in only one sheath, which is at the boundary of a practically infinite plasma. This is theoretically convenient, not least because it permits the simple assumption that electrons impinging on the sheath have a well-defined velocity distribution, such as a Maxellian. However, in practice, capacitive discharges are of decidedly finite extent, and interaction between sheaths may occur. Roughly speaking, if the distance between the sheaths is smaller than the electron mean free path, the sheaths may not be independent [12]. In particular, the possibility of a "bounce" resonance arises, in which a group of electrons is resonantly heated by repeated interaction with opposing sheaths. The relevant parameter here is the so-called "bounce frequency," usually denoted by Ω. Such a resonance can occur only in a region of parameter space that is not common in experiments, so that this effect seems to be primarily of theoretical interest, at present.

4.2.2. Magnetisation An obvious complication is the presence of a magnetic field perpendicular to the electric field—such a field magnetizes the charged particles, which inhibits their motion along the electric field. In particular, the motion of electrons is appreciably changed for quite weak magnetic fields. Experimental and simulation evidence [50, 51] seems to show that imposing such a magnetic field causes a rather rapid transition into a discharge mode that is dominated by Ohmic heating. For typical experimental parameters, a magnetic field such that $\omega_c \sim \omega_m athrmrf$ is enough to induce such a transition [50, 51]. Indeed, for large enough ω_c, $\langle P_{\mathrm{stoch}}\rangle/\langle P_{\mathrm{ohmic}}\rangle \sim 1/\omega_c^6$. The physical reason for this behaviour is that magnetized electrons cannot freely pass into and out of the sheath region, so the transit time effect is suppressed. A different view of this problem is taken in Lieberman *et al* [52]. That work deals with the case of a relatively strong magnetic field, and emphasises the possibility of magnetized electrons making multiple interactions with the sheath. This approach *et al* uses the ideas of the "hard wall" model, and may be exposed to the criticisms discussed above. The results obtained appear incompatible with the simulation results of Turner *et al* [50], but further work might resolve this discrepancy. Understanding of the unmagnetized case has advanced since these works appeared, and in view of this and the present topical interest in magnetized capacitive discharges, this issue probably warrants re-investigation.

4.2.3. Multiple frequencies In recent years, capacitive discharges excited by two or three frequencies have attracted interest as plasma sources for microelectronic processing applications. These discharges can be treated in much the same way as single frequency discharges, but new effects arise because of the nonlinear interaction of the frequencies through the sheath. The Lieberman sheath model has been extended to the dual frequency case, and this supplies a useful starting point for investigations of heating effects. A typical dual-frequency discharge is excited by frequencies ω_l and ω_h, with $\omega_l \ll \omega_h$. This normally implies $V_h \ll V_l$ but $J_h \gg J_l$. These orderings show that the current is predominantly at the higher frequency, but the voltage is dominated by the lower frequency. It is therefore not specially surprising to find that

$$\langle P \rangle \approx \frac{3}{2} Q_b \delta_h^2 H_l G_{dual}(H_l), \tag{72}$$

where H_l is given by Eq. (43) using the lower frequency current density, and δ_h by Eq. (44) using the higher frequency current density. In effect, the sheath structure factor is a function of the lower frequency current density [10, 49]. The form of $G_{dual}(H_l)$ is such that $G_{dual}(H_l) \gg G(H_h)$, so that there can be a considerable enhancement of the collisionless heating effect when two frequencies are combined, relative to either frequency acting alone [53]. (A similar phenomenon occurs in connection with Ohmic heating under the same conditions [49, 54, 55])

Although capacitive discharges are usually assumed to be electrostatic phenomena, multi-frequency discharges often exhibit electromagnetic effects. Usually, these are expected when the electromagnetic wavelength becomes comparable with the plasma size, however the relevant wavelength for this comparison can be considerably smaller that in vacuum, so that the frequency where such effects appear can smaller than might be naively expected. There is a large literature on this topic, which has recently been reviewed by Chabert [56]. No one has yet shown that there are any heating mechanisms unique to this case (although there might be).

4.2.4. Circuit resonances Recent years have seen renewed interest in the effect commonly called the "series resonance" [3] A capacitive discharge, viewed as a combination of lumped circuit elements, may be represented as series combination of an inductance (physically associated with electron mass) and a capacitive (physically associated with the space charge in the sheaths). These elements produce a resonance, with a frequency approximately given by

$$\omega \approx \omega_{\mathrm{pe}} \sqrt{\frac{2s}{L}} \tag{73}$$

where s is the width of a sheath and L is the length of the bulk plasma. The idea of exciting a discharge at this resonant frequency has attracted intermittent interest [57, 58], and the effect may be exploited as a diagnostic [59], but most recently the suggestion has been made that this resonance is rather ubiquitously excited in capacitive discharges [60], and a certain amount of experimental and theoretical evidence supports this proposition. The suggested excitation mechanism is by harmonics of the driving frequency produced by circuit nonlinearities, such as the sheath capacitance. However, it is not clear that any new heating mechanism is associated with the series resonance—power absorption is apparently due to the usual Ohmic and collisionless processes [61, 62]. Consequently, this is in essence a circuit effect that can produce a large current at a particular frequency, and this large current may indeed cause an important amount of electron heating. However, the mechanisms by which this heating occurs seem to be the same as those already discussed.

5. Concluding Remarks

The situation for inductive and capacitive discharges is different. Collisionless heating in inductive discharges is susceptible to the tools of classical kinetic theory, because spatial gradients of the plasma density–which of course exist to some degree in most circumstances of practical interest—are not intrinsic to the effect. Consequently, there exist useful formulae describing the power absorption by inductive discharges, applicable in most cases. These are cumbersome in their most general form, but evaluating them is fairly minor difficulty. The main results of these theories have been validated by experiments. There is, therefore, not much doubt that collisionless heating in inductive discharges is well understood, and indeed it is difficult to see that more work is urgently needed. Conclusions of this type have a habit of being wrong, and if this is so, probably new work will be motivated by some innovative application directing attention to a neglected part of parameter space. Capacitive discharges are a different case, because spatial gradients are essential to the effect, and the gradients involved are too large to be treated as any kind of perturbation. This has so far thwarted all efforts to develop a mathematically systematic theory. However, despite conceptual and mathematical differences between the theories, all lead to a result with essentially the same form, and this form suggests a relatively simple parameterization. Comparison with simulations allows us to propose with some confidence a practical formula, even though this formula is without a rigorous mathematical basis. Consequently, it is tempting to announce that the research programme directed at understanding collisionless heating in capacitive discharhes has also been brought to a satisfactory conclusion, but this is probably at least slightly premature. Further investigations into the validity of Eq. (55) should lead to at least minor improvements, and might indentify regimes where that formula is inappropriate. Moreover, we should not give up the aspiration to develop a mathematically satisfactory theory—one day this might be achieved. There is also the consideration that work on applications of capacitive discharges is particularly vigorous at present, and such application studies tend to uncover weaknesses in theoretical knowledge. There could, therefore, be surprises in store.

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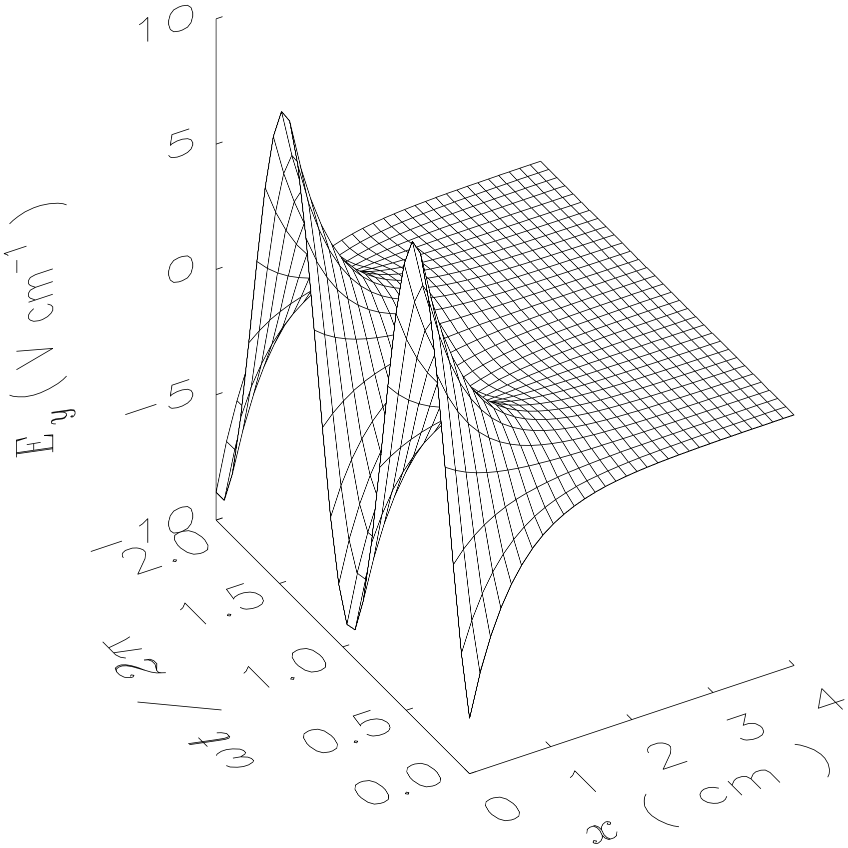

Figure 1. Typical electric field in a slab model of an inductive discharge, from a particle-in-cell simulation. The localisation of the field in the skin depth layer at the plasma boundary is evident.

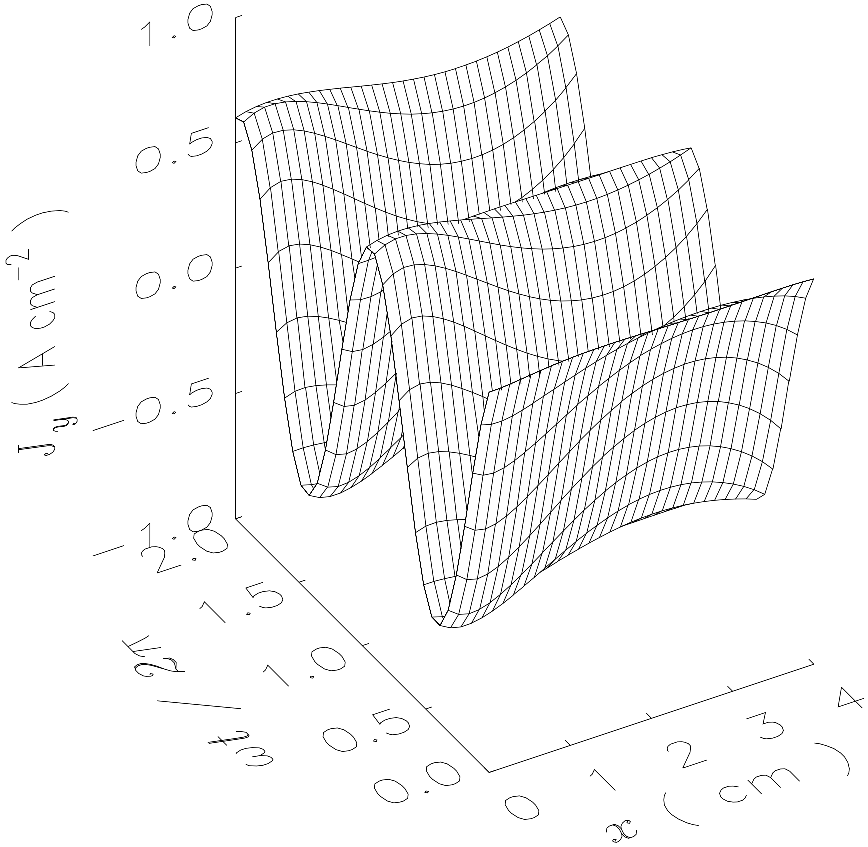

Figure 2. . The current density in a slab model of an inductive discharges in the anomalous regime. Clearly, in comparison with Fig. 1, which shows the same conditions, the spatial distribution of the current density is drastically changed.

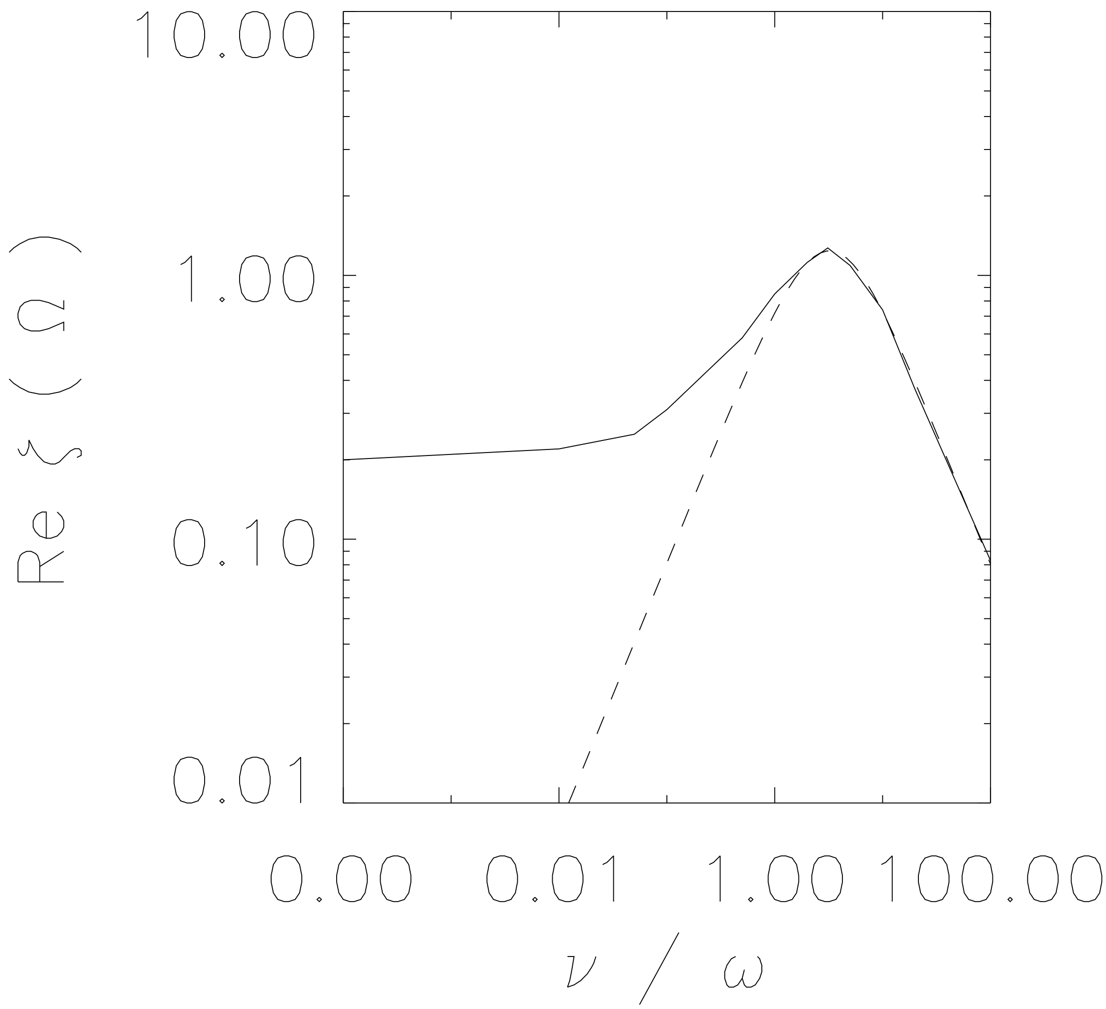

Figure 3. Comparison of particle-in-cell simulation results (solid line) with analytical theory for a cold plasma (dashed line), for the real part of the surface impedance (from Turner [16]). This data clearly shows the onset of the anomalous regime when the electron collision frequency ν is less than the driving frequency ω.

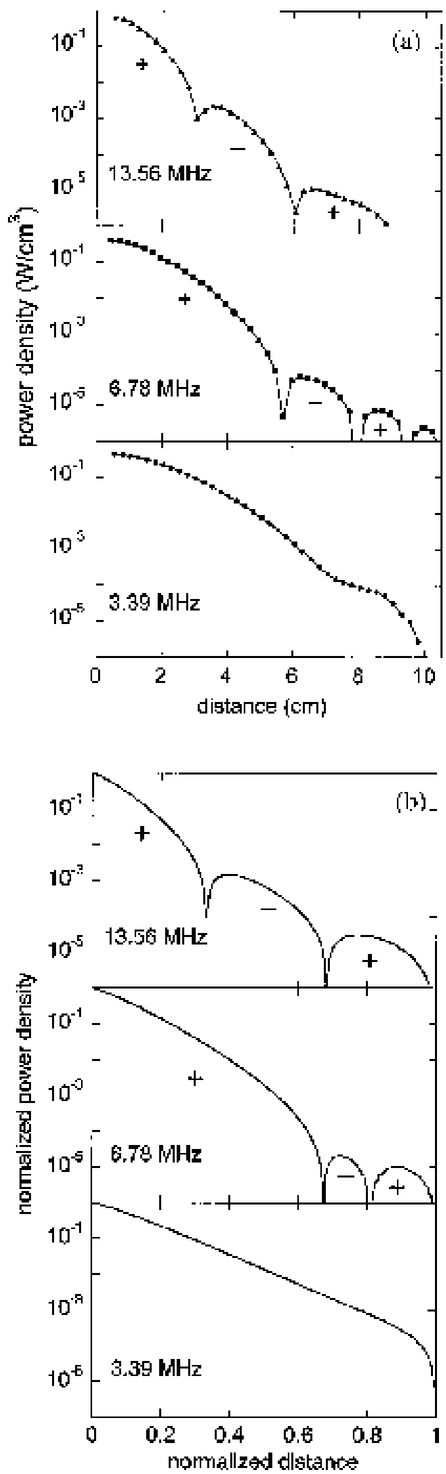

Figure 4. Comparison of experimental (above) and theoretical (below) results for the time averaged power deposition in an inductively coupled discharge [24]. The alternating layers of positive and negative power deposition suggest the complicated spatial and temporal structure of the underlying fields and currents

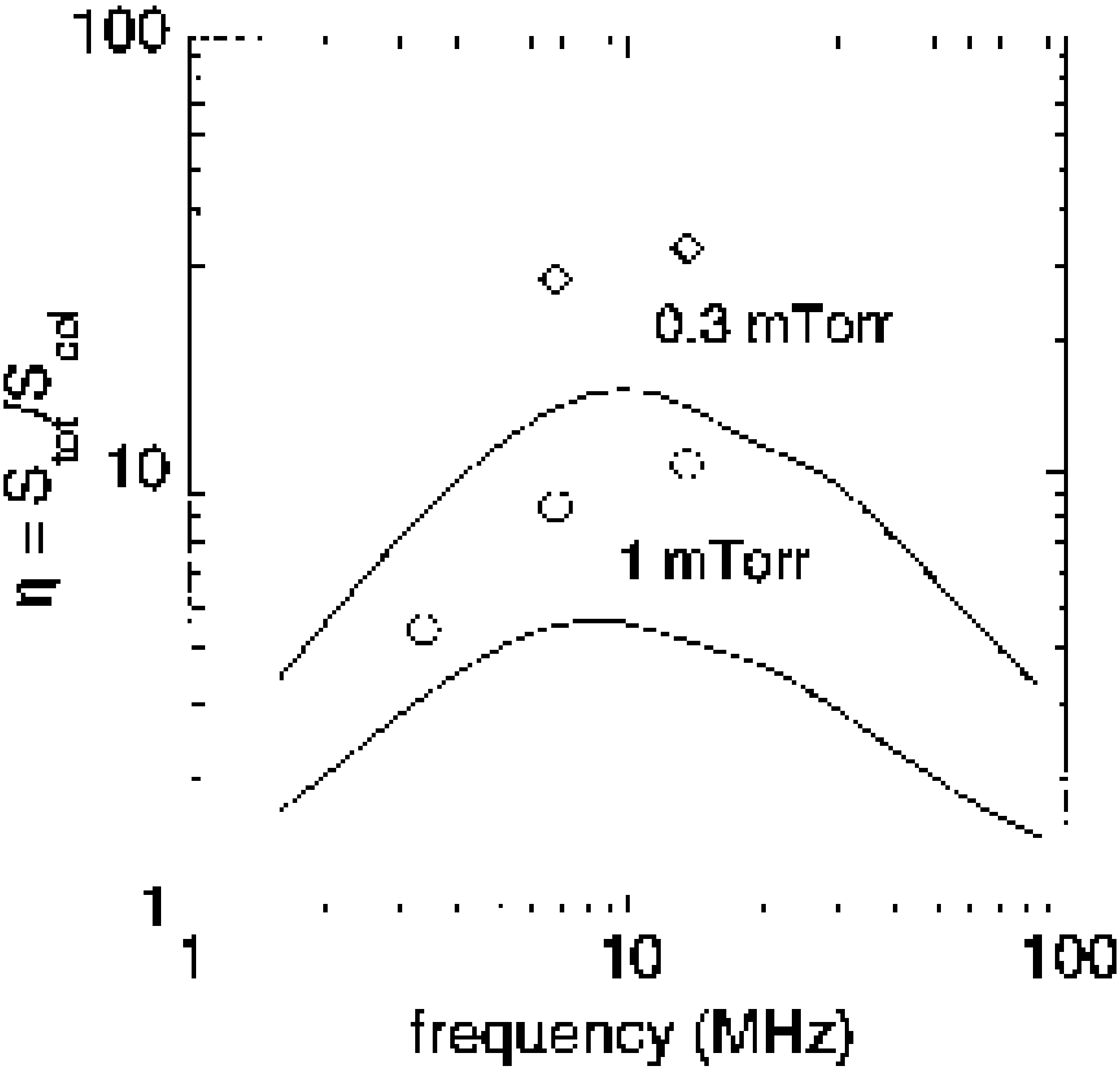

Figure 5. The ratio of total power to collisionless power for an inductively coupled discharge, at two pressures, as a function of excitation frequency. The points are experimental data, and the lines are theoretical. These data were obtained in a cylindrical chamber of diameter 19.8 cm and axial length 10.5 cm, with excitation by a planar coil attached to one face of the chamber. Further details are given in the original work by Godyak *et al* [25] and in references therein

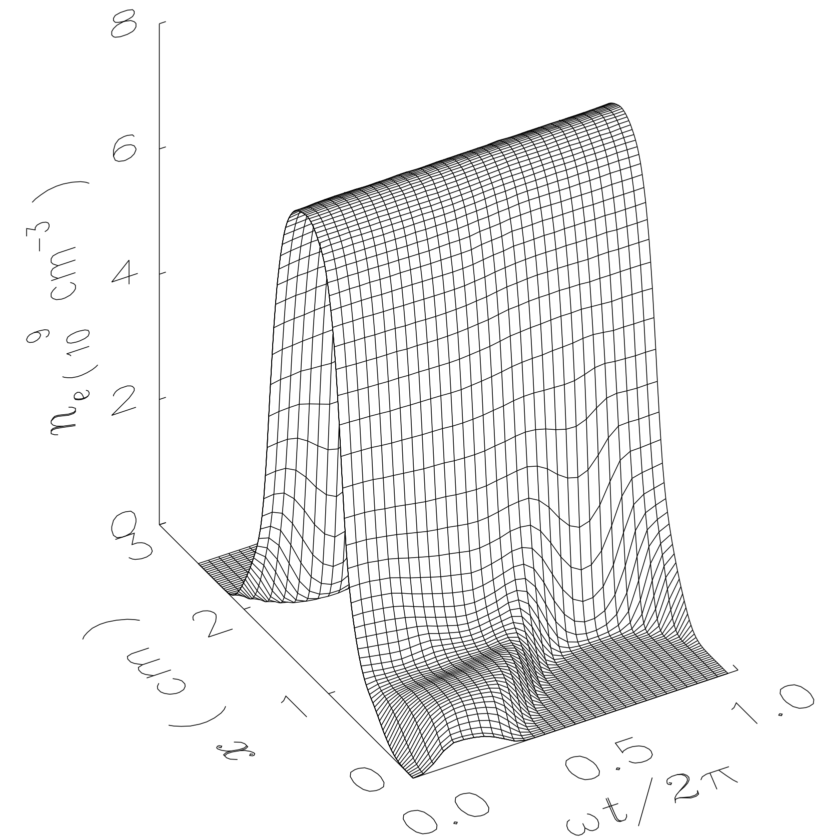

Figure 6. Particle-in-cell simulation showing the typical behaviour of the electron density as a function of space and time through a single cycle of the driving frequency. The large modulation in sheath regions is evident.

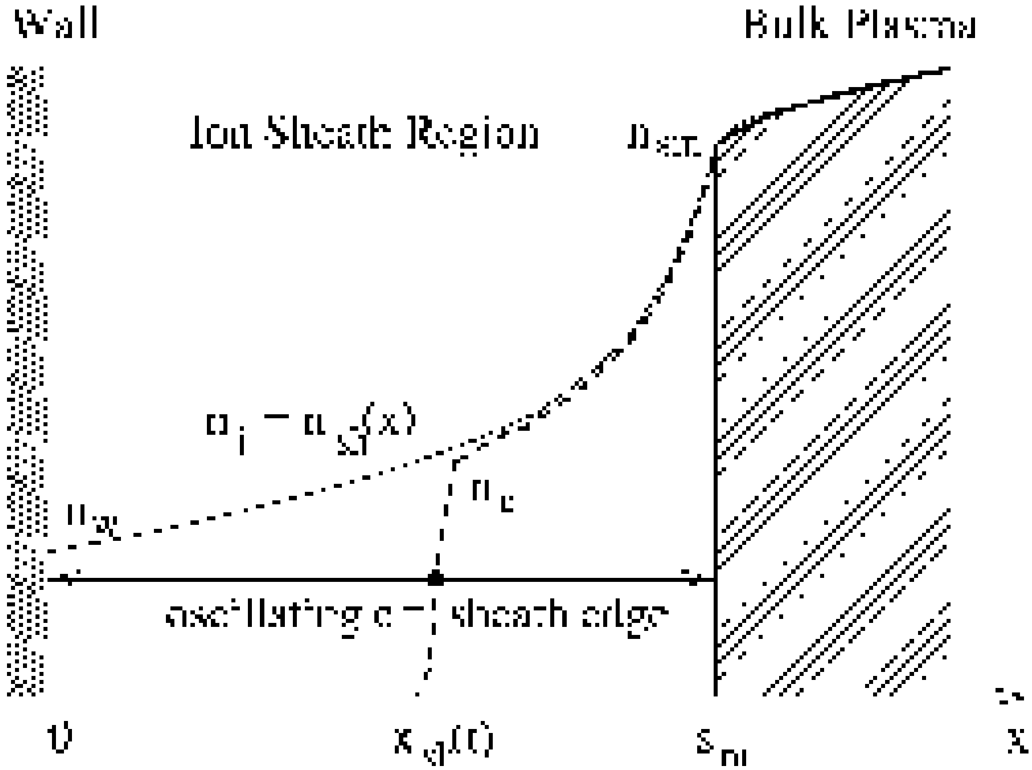

Figure 7. Indicative diagram of a capacitive sheath at some particular phase of the voltage, showing the approximately stationary ion density and the instantaneous electron density. The (stationary) ion sheath edge, or Bohm point, is at s_m and the (moving) electron sheath edge is at x_{sh}. Note this sketch greatly exaggerates the sharpness of the transition between plasma and sheath, for typical values of the H parameter, *i.e.* $H \lesssim 10$. (Figure from Kawamura *et al* [49])

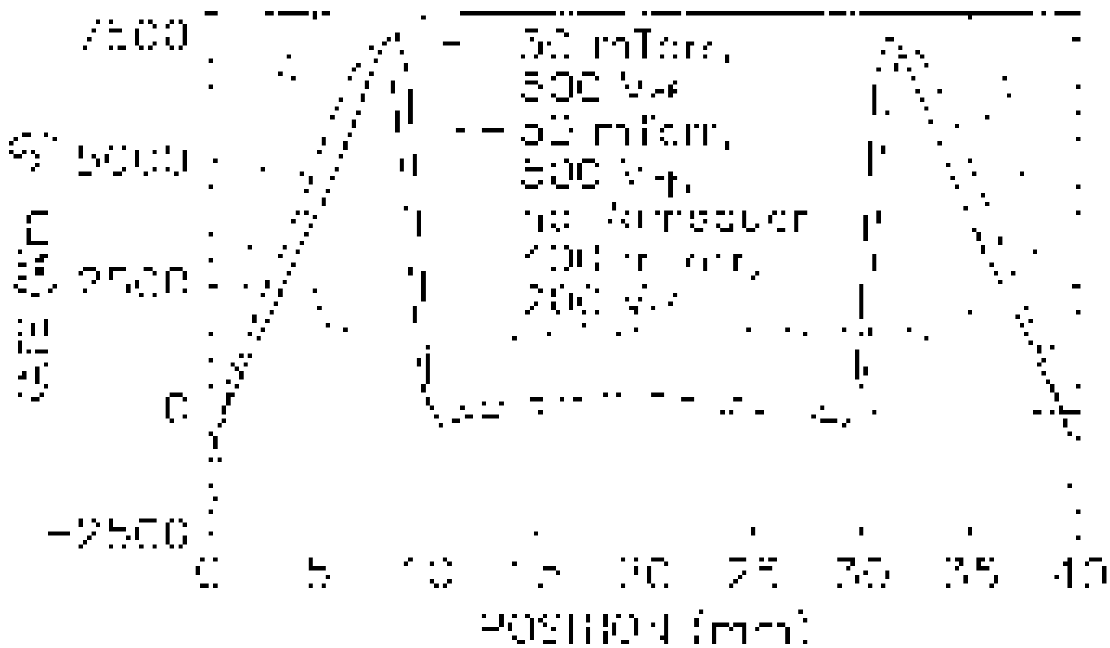

Figure 8. Time averaged electron heating in a capacitive radio-frequency discharge in argon, from a particle-in-cell simulation by Surendra and Graves [39]. This data shows the typical spatial pattern observed in such simulations at low pressure—positive heating localized in the sheath regions, and small or negative heating in the bulk plasma.

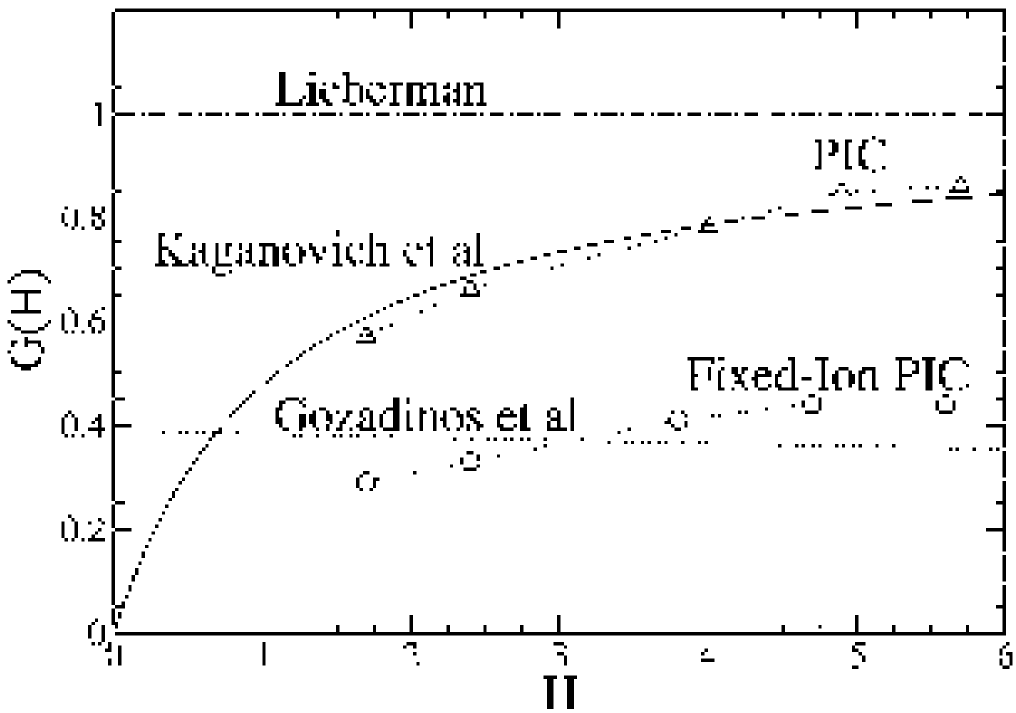

Figure 9. Comparison of various theoretical expressions with particle-in-cell simulations of collisionless heating in a capacitive discharge, from Kawamure *et al* [49]. The curve denoted "Lieberman" corresponds to Eq. (54), that denoted "Kaganovich *et al*" to Eq. (55) and that denoted "Gozadinos *et al*" to Eq. (70). The two sets of simulation points correspond to normal simulations with moving ions, and restricted simulations with fixed ions. Clearly, the first case corresponds most closely to the situation found in experiments, and Eq. (55) is evidently the most useful practical formula.