Stochastic Delay Difference and Differential Equations: Applications to Financial Markets

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy in Mathematics is entirely my own work, that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge breach any law of copyright, and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

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Contents

Abstract

This thesis deals with the asymptotic behaviour of stochastic difference and functional differential equations of Itô type. Numerical methods which both minimise error and preserve asymptotic features of the underlying continuous equation are studied. The equations have a form which makes them suitable to model financial markets in which agents use past prices. The second chapter deals with the behaviour of moving average models of price formation. We show that the asset returns are positively and exponentially correlated, while the presence of feedback traders causes either excess volatility or a market bubble or crash. These results are robust to the presence of nonlinearities in the traders' demand functions. In Chapters 3 and 4, we show that these phenomena persist if trading takes place continuously by modelling the returns using linear and nonlinear stochastic functional differential equations (SFDEs). In the fifth chapter, we assume that some traders base their demand on the difference between current returns and the maximum return over several trading periods, leading to an analysis of stochastic difference equations with maximum functionals. Once again it is shown that prices either fluctuate or undergo a bubble or crash. In common with the earlier chapters, the size of the largest fluctuations and the growth rate of the bubble or crash is determined. The last three chapters are devoted to the discretisation of the SFDE presented in Chapter 4. Chapter 6 highlights problems that standard numerical methods face in reproducing long–run features of the dynamics of the general continuous–time model, while showing these standard methods work in some cases. Chapter 7 develops an alternative method for discretising the solution of the continuous time equation, and shows that it preserves the desired long–run behaviour. Chapter 8 demonstrates that this alternative method converges to the solution of the continuous equation, given sufficient computational effort.

Chapter 1

Introduction

1.1 Summary and Overview

The thesis examines stochastic systems with memory; initially it involves studying the asymptotic properties of stochastic difference and differential equations. In particular the thesis examines the rate at which the solutions of certain stochastic delay differential equation (SDDE) tend to an infinite limit or the size of fluctuations of the equations. The equations examined all result from models of the inefficient financial markets, in which inefficiency stems from the trading strategy of agents.

The first aim of the thesis is to highlight the fact that different feedback trading strategies led to the same two types of asymptotic behaviour, which depend on parameters in the models. These parameters can be interpreted in terms of the confidence of the traders and the amount of feedback they take from the past. In one regime the models display random walk behaviour which obeys Law of the Iterated Logarithm (LIL)

$$\limsup_{n\to\infty} \frac{X(n)}{\sqrt{2n\log\log n}} = \sigma = -\liminf_{n\to\infty} \frac{X(n)}{\sqrt{2n\log\log n}},$$

where the increments of $X(n)$ (the returns) are not independent. These fluctuations in the returns are quite volatile but controlled. In the other regime however, an event occurs or news enters the market in an initial trading period which triggers a trend to emerge in the returns. This trend encourages feedback traders into further buying (selling) which increases (decreases) the price. This upward (downward) spiral continues and a bubble (crash) ensues.

The second aim is to analyse the discretisation of the continuous-time SDDE which are in keeping with real world problems. We examine the effectiveness of the standard Euler-Maruyama method. This method is inadequate for long-run simulations and we propose an alternative discretisation method which preserves the correct asymptotic behaviour of the continuous equation and also converges like conventional Euler-Maruyama methods.

The type of stability that has been established for this class of equations is important in a variety of real–world problems which involve feedback from the past, and are subject to external random forces. Examples include population biology (Mao [51]),(Mao and Rassias [53, 54]), neural networks (cf. e.g. Blythe et al. [20]), viscoelastic materials subjected to heat or mechanical stress (Drozdov and Kolmanovskii [31]), (Caraballo et al. [27]), (Mizel and Trutzer [61, 62]), or financial mathematics (Anh et al. [1, 2]), (Arrojas et al. [14]), (Hobson and Rogers [38]), and (Cont and Bouchaud [21]).

In particular, stochastic difference and differential equations may be used to model inefficient financial markets. Surveys of financial markets reveal that a persistently high proportion of traders use past prices as a guide to making investment decisions. Such feedback trading strategies are thought to be responsible for speculative asset bubbles and crashes; this feedback behaviour is absent from standard non–delay models. It is therefore plausible to postulate that aggregate demand is functional of past prices. In which case, price dynamics could be modeled by stochastic difference and differential equations.

We first consider a stochastic delay difference equation with a simple trading strategy. We assume that the traders demand for the asset depends on the difference between a weighted average over the last $N_1 - 1$ periods of the cumulative return on the stock in the short run and $N_2 - 1$ in the long-run. We also assume that speculator's react to other random stimuli–"news" which is independent of past returns. This news arrives at time $(n+1)$ adding a further $\xi(n+1)$ to the traders' excess demand, where $\xi \sim N(0,1)$. The stochastic delay difference equation is of the form

$$Y(n+1) = Y(n) + \beta\left(\sum_{j=0}^{N_1-1} w_1(j)Y(n) - \sum_{j=0}^{N_2-1} w_2(j)Y(n-j)\right) + \xi(n+1) \quad n\in\mathbb{N}$$
$$Y(0) = 1, \qquad Y(n) = 0, \quad n < 0.$$

If the trend following speculators do not react very aggressively to the differences between short-run and long-run historical returns then the rate growth of the partial maxima of the solution is the same as that of a random walk. However if the contrary is true, then the returns will tend to plus or minus infinity exponentially fast. The proof of these results involve writing the resolvent $Y(n)$ in terms of the deterministic equation and variation of constants formula. In the latter part of the chapter we study the behaviour of the non–linear stochastic delay difference equation namely

$$X(n+1) = X(n) + \left(\sum_{j=0}^{N_1-1} w_1(j)g(X(n-j)) - \sum_{j=0}^{N_2-1} w_2(j)g(X(n-j)) \right) + \xi(n+1) \quad n \in \mathbb{N}$$
$$X(0) = 1, \qquad X(n) = 0, \quad n < 0,$$

where $g : \mathbb{R} \to \mathbb{R}$ is presumed to have the following property

$$g \in C(\mathbb{R}; \mathbb{R}); \lim_{|x|\to\infty} \frac{g(x)}{x} = \beta \quad \text{for some } \beta \geq 0.$$

i.e. g is asymptotically linear at infinity. We show how the asymptotic behaviour of the non–linear equation mimics that of the linear equation by rewriting the non–linear equation in terms of the linear equation and a controllable equation. By dropping the assumption that the noise is normally distributed and adopting the property that the tails of the noise exhibit polynomial decay then we can show that the system's tails are fatter at the extremes than a normal distribution. If we assume that the cumulative returns follow a correlated random walk then the upper and lower bounds on the rate of growth of the partial maxima, $X(n) - X(n-\Delta)$ are exactly the same as those which apply to the innovation (or "news") process ξ. This indicates that the distribution of the noise term strongly characterizes the distribution of the returns. This proof hinges on writing the equation for the Δ–returns in terms of a difference equation. We also note that in the stable case the Δ-returns are positively correlated at all time horizons. This is the feature of the economic model which is responsible for the excess volatility and bubbles. This proof uses results proved in various lemmas throughout the chapter.

In chapter 3 the continuous linear analogue of the model in the previous chapter is considered. However we have chosen to model the speculators' behaviour using finite measures rather than through fixed delays of continuous averages of past returns. This allows us to capture a very wide variety of moving average–type strategies within the same model including continuous time moving averages. It also highlights the fact that the manner in which traders compute their moving averages is unimportant in determining the ultimate dynamics. This is important in any mathematical model in economics, as model assumptions are unlikely to be satisfied in reality, rendering general models which are robust to changes in the assumptions particularly desirable. Chapter 4 considers the continuous nonlinear analogue of chapter 2. The proofs of the main theorems in these two chapters mimic the proofs from chapter 2.

In chapter 5 we assume that the traders adopt a new trading strategy where trades are based on a comparison of current prices with a reference level. As with the previous equations we assume that traders can respond to "news" where ξ is assumed to be either heavy or thin tailed. This type of speculative behaviour makes it reasonable to incorporate a maximum functional of the process on the right–hand side of the stochastic difference equation. For these reasons we are led to analyse a stochastic functional difference equation of the form

$$X(n+1) = X(n) + \alpha X(n) + \beta \max_{n-N \leq j \leq n} X(j) + \xi(n+1) \quad n \geq 0,$$

where ξ is the news and α and $\beta > 0$ are constants which model the trading behaviour of the various classes of speculators. If the speculators do not react very aggressively to the difference between the maximum and the current returns the rate of growth of the partial maxima of the solution is the same as that of the noise term where the ξ are assumed to be either thin–tailed or heavy–tailed. However if the contrary is true, the volatility in the market increases and subsequently the upper and lower bounds of the noise are pushed higher and lower respectively. This causes a bubble or crash to occur. All the results are proved by contradiction.

In chapter 6 we consider the discretisation of two continuous stochastic delay equations

$$dX(t) = \left(\sum_{j=1}^{m} \alpha_j g(X(t-\theta_j)) - \sum_{j=1}^{p} \beta_j g(X(t-\tau_j)) \right) dt + \sigma\, dB(t), \quad t \geq 0$$

and

$$dX(t) = \left(\int_0^{\tau_1} w_1(s) g(X(t-s))\, ds - \int_0^{\tau_2} w_2(s) g(X(t-s))\, ds \right) dt + \sigma\, dB(t), \quad t \geq 0. \tag{1.1.1}$$

in which the weight functions w_1 and w_2 are *continuous*. First we show that the simple Euler method discretises the first equation successfully and that the asymptotic behaviour of the discretisation preserves that of the continuous equation. However this is not the case for the second equation. The problem arises because zero is a solution of the characteristic equation of the underlying linear differential resolvent. Moreover zero can be the solution with largest real part. The Euler method does not ensure that the underlying discretised characteristic equation has a unit solution. Although it is possible to modify the Euler method so this will be the case, errors arising from truncation and rounding–off make it unsuccessful in practice. Only with the removal of this unit solution is the correct asymptotic behaviour displayed. This is shown in the penultimate chapter. Theorems in this chapter are derived in a manner similar to that of chapter 2, 3 and 4.

The final chapter shows that it is possible to perform a uniform discretisation of (1.1.1) which preserves the positivity and exponentially fading memory present in the autocovariance function of increments of the process X. This ensures that the discretisation captures the short–run and long–run asymptotic behaviour of the continuous equation. In the final chapter we even show that this discretisation method obeys

$$\lim_{h \to 0^+} \mathbb{E} \left[\max_{0 \leq t \leq T} |\bar{X}_h(t) - X(t)|^2 \right] = 0, \quad \text{for any } T > 0, \tag{1.1.2}$$

where $\bar{X}_h$ is a piecewise constant process defined on $[0, T]$ for which $\bar{X}_h(t) = \widehat{X}_h([t/h])$ for $t \geq 0$. The condition (1.1.2) is enjoyed by conventional Euler–Maruyama methods for stochastic functional differential equations. This result is proved by adopting similar techniques to those employed by Mao.

1.2 Preliminaries

In this section, we give some results, notation and terminology from real and stochastic analysis that will be used throughout the thesis.

Let $\mathbb{N}$ denotes the integers $0, 1, 2, \ldots$, and $\mathbb{R}$ the real line. A real sequence $a = \{a(n) : n \in \mathbb{N}\}$ obeys $a \in \ell^1(\mathbb{N}; \mathbb{R})$ if $\sum_{n \in \mathbb{N}} |a(n)| < \infty$.

Landau notation. In subsequent work it is necessary to characterise the asymptotic behaviour of functions. The Landau notation often helps in this regard by means of the symbols o and O.

The symbol O is used to describe an asymptotic upper bound for the magnitude of a function in terms of another, usually simpler function. So for example suppose $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to (0, \infty)$ are measurable functions and that $g(t) = O(f(t))$ as $t \to \infty$. This notation signifies that $g(t)$ exhibits a growth that is limited to that of the function f according to

$$\limsup_{t \to \infty} \frac{|g(t)|}{f(t)} < +\infty.$$

On the other hand, if $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to (0, \infty)$ are measurable functions are such that $g(t) = o(f(t))$ as $t \to \infty$, then g is related to f according to

$$\lim_{t \to \infty} \frac{g(t)}{f(t)} = 0.$$

Gronwall's Inequality. If $Q : [0, \infty) \to [0, \infty)$ is continuous and is such that

$$Q(t) \leq A + \int_0^t B(t') Q(t')\, dt' \quad \text{for all } t$$

where B is a non–negative locally integrable function and $A > 0$ is some constant, then

$$Q(t) \leq A \exp\left(\int_0^t B(t')\,dt'\right), \quad t \geq 0.$$

Z–Transform. The *z–transform* of a sequence $x(n)$ which is identically zero for negative integers n is defined by

$$\widetilde{x}(z) = \sum_{j=0}^{\infty} x(j) z^{-j} \quad \text{for all } z \in \mathbb{C}.$$

The set of numbers z in the complex plane where $\widetilde{x}(z)$ converges is called the *region of convergence* where

$$\lim_{j\to\infty} \left| \frac{x(j+1)}{x(j)} \right| = R.$$

Then by the *Ratio Test*, the infinite series $\widetilde{x}(z)$ *converges* if

$$\lim_{j\to\infty} \left| \frac{x(j+1)z^{-j-1}}{x(j)z^{-j}} \right| < 1 \qquad |z| > R$$

and *diverges* if

$$\lim_{j\to\infty} \left| \frac{x(j+1)z^{-j-1}}{x(j)z^{-j}} \right| > 1 \qquad |z| < R.$$

Properties of the z-transform. If α and $\beta \in \mathbb{R}$

$$(\widetilde{\alpha x + \beta y})(z) = \alpha \widetilde{x}(z) + \beta \widetilde{y}(z).$$

If $x(-i) = 0$ for $i = 1, 2, \cdots, k$ then

$$z[x(n-k)] = z^{-k} \widetilde{z}(x) \qquad \text{for } |z| > R$$

and

$$z[x(n+k)] = z^k \widetilde{z}(x) - \sum_{r=0}^{k-1} x(r) z^{k-r} \qquad |z| > R.$$

The Final Value Theorem. Suppose $L = \lim_{z\to 1}(z-1)\widetilde{x}(z)$ is finite. Then $\lim_{n\to\infty} x(n) =: x(\infty)$ is finite and $x(\infty) = L$.
The *convolution* of $f = \{f(n) : n \in \mathbb{N}\}$ and $g = \{g(n) : n \in \mathbb{N}\}$, $f * g$, is a sequence defined by $n \in \mathbb{N}$.

$$(f * g)(n) = \sum_{k=0}^{n} f(n-k) g(k) = \sum_{k=0}^{n} f(k) g(n-k).$$

The z-transform of the convolution is then given by

$$(\widetilde{x * y})(z) = \widetilde{x}(z) \widetilde{y}(z).$$

Integrable Functions in the Deterministic Sense. If J is an interval in $\mathbb{R}$ and V a finite–dimensional normed space, with norm $||\cdot||$, then $C(J,V)$ denotes the family of continuous functions $\phi : J \to V$. The space of Lebesgue integrable functions $\phi : (0,\infty) \to V$ will be denoted by $L^1((0,\infty),V)$, where

$$\int_0^\infty ||\phi(t)||\, dt < \infty$$

The space of Lebesgue square–integrable functions $\phi : (0,\infty) \to V$ will be denoted by $L^2((0,\infty),V)$ where

$$\int_0^\infty ||\phi(t)||^2\, dt < \infty.$$

Where V is clear from the context it is omitted from the notation. Note that a function of domain J that belongs to $L^1(K,V)$ for every compact subset K of J is known as a locally integrable function.

Convolutions. The *convolution* of $A : [0,\infty] \to \mathbb{R}$ and $B : [0,\infty] \to \mathbb{R}$ is denoted by $A * B$ and defined by the function given by

$$(A * B)(t) = \int_0^t A(t-s)B(s)\, ds, \qquad t \geq 0$$

Laplace Transform. The Laplace transform of the function $x : [0,\infty) \to \mathbb{R}$ is defined as

$$\widehat{x}(z) = \int_0^\infty x(t)e^{-zt}\, dt$$

If $\epsilon \in \mathbb{R}$ and $\int_0^\infty ||x(t)||e^{\epsilon t}\, dt < \infty$ then $\widehat{x}(z)$ exists for Re$z \geq \epsilon$ and is analytic for Re$z > \epsilon$. The following property of the Laplace transform is useful: if x and y $\in L^1(0,\infty)$ then

$$\widehat{(x * y)}(z) = \widehat{x}(z)\widehat{y}(z) \quad \text{Re} z \geq 0.$$

1.2.1 Stochastic Preliminaries

Random Variables. A random variable is an $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}$. Every random variable X induces a probability measure μ_x on the Borel sets $\mathbb{B}$ of $\mathbb{R}$ where $\mu_x(\mathbb{B}) = \mathbb{P}[\omega : X(\omega) \in \mathbb{B}]$, If X is *integrable* with respect to the probability measure; that is if

$$\int_\Omega ||X(\omega)||\, d\mathbb{P}(w) < \infty$$

then the expectation of X can be expressed as

$$\mathbb{E}[X] = \int_\Omega ||X(\omega)||\, d\mathbb{P}(w) = \int_0^\infty x\, d\mu_x(x).$$

Distributions. The *distribution function* of a random variable X is the function $F : \mathbb{R} \to [0,1]$ given by $F(x) = \mathbb{P}(X \leq x)$. The sequence of random variables $X_1, X_2, \cdots$ (with corresponding distributions functions $F_1, F_2 \cdots$) has a *limiting distribution* denoted F if $\lim_{n\to\infty} F_n = F$.

Stochastic Processes. A *stochastic process* is a family $\{X(t)\}_{t\geq 0}$ of $\mathbb{R}^n - valued$ random variables. It is *continuous* for all $\omega \in \Omega$ if the function $t \to X(t,\omega)$ is continuous. It is $\mathcal{F}(t)$ - adapted if $X(t)$ is $\mathcal{F}(t)$- measurable for every t. It is said to be *increasing* if $X(t,\omega)$ is nonnegative, nondecreasing and right continuous on $t \geq 0$ for almost all $\omega \in \Omega$. It is a process of *finite variation* if $X(t) = \overline{A}(t) - \widehat{A}(t)$ where both $\{\overline{A}(t)\}$ and $\{\widehat{A}(t)\}$ are increasing processes.

Standard Brownian Motion. If $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$ is a filtered probability space then a one–dimensional standard Brownian motion $\{B(t)\}_{t\geq 0}$ is a process which has the following properties: $B(0) = 0$; the increments $B(t) - B(s)$ is normally distributed with mean 0 and variance $t - s$ where $0 \leq s < t < \infty$; the increment $B(t) - B(s)$ is independent of $\mathcal{F}_s$ where $0 \leq s < t < \infty$. We often take as filtration $(\mathcal{F}(t))_{t\geq 0}$ the natural filtration of the Brownian motion. In this thesis, this is the filtration with respect to which processes are adapted. It will be denoted by $\{\mathcal{F}^B(t)\}_{t\geq 0}$ throughout.

Martingales. The stochastic process $M = \{M(t)\}_{t\geq 0}$, defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$ is said to be a martingale with respect to the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ if $M(t)$ is $\mathcal{F}(t)$–measurable for all $t \geq 0$, $\mathbb{E}[|M(t)|] < \infty$ for all $t \geq 0$ and for all $0 \leq s \leq t$

$$\mathbb{E}\left[M(t)|\mathcal{F}(s)\right] = M(s) \qquad a.s.$$

Furthermore if the process M is a real–valued square integrable martingale then there exists a unique, adapted, increasing., integrable process such that the process $\{M(t)^2 - \langle M(t)\rangle\}_{t\geq 0}$ is a martingale which vanishes at $t = 0$. The process $\langle M\rangle$ is known as the *quadratic variation* of M. The asymptotic behavior of the quadratic variation characterises the asymptotic behavior of the martingale, this is seen in the Martingale convergence Theorem which is stated precisely in Lemma 2.3.1. A random variable $\tau : \Omega \to [0,\infty]$ is called a stopping time if $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}(t)$ for an $t \geq 0$. A right continuous adapted process $M = \{M(t)\}_{t\geq 0}$ is a local martingale if there exists a nondecreasing sequence of stopping times $\{\tau_k\}_{k\geq 0}$ with $\tau_k \to \infty$ as $k \to \infty$ almost surely such that every $\{M(\tau_k) \wedge t\}_{t\geq 0}$ is a martingale.

Stochastic Integrability and Convergence. Due to the random nature of stochastic processes various definition of stochastic integrability exist. A stochastic process $X(t)$ is integrable with respect to the probability measure if $\mathbb{E}[X(t)] < \infty$ for each $t > 0$; it is square integrable if $\mathbb{E}[X(t)]^2 < \infty$ for each $t \geq 0$.

One–dimensional Itô calculus and integration. In this thesis we study *scalar* stochastic functional differential equations, and consider the initial data for our price processes to be known. Therefore, we need only consider the scalar theory of Itô integration and Itô processes with deterministic initial values.

We now define our terms more precisely. Let $T > 0$. Suppose B is a one–dimensional standard Brownian motion and $g = \{g(t) : t \in [0,T]\}$ is a scalar process adapted to the natural filtration $\{\mathcal{F}^B(t)\}_{t\geq 0}$ generated by B. Suppose also that

$$\int_0^T g(s)^2\,ds < +\infty, \quad \text{a.s.}$$

Then the Itô integral of g is denoted by

$$\int_0^t g(s)\,dB(s), \quad t \in [0,T].$$

Furthermore if

$$\mathbb{E}\int_0^T g(s)^2\,ds < +\infty,$$

then *Itô's Isometry* holds: this is the identity

$$\mathbb{E}\left[\left(\int_0^t g(s)\,dB(s)\right)^2\right] = \mathbb{E}\left[\int_0^t g(s)^2\,ds\right], \quad 0 \leq t \leq T.$$

Suppose that X is a scalar $\mathcal{F}^B$–adapted process with deterministic initial value $X(0)$. Then $X = \{X(t) : 0 \le t \le T\}$ is an *Itô process* if there exist adapted scalar processes f and g obeying

$$\int_0^T |f(s)|\,ds < +\infty, \quad \int_0^T g(s)^2\,ds < +\infty, \quad \text{a.s.}$$

such that

$$X(t) = X(0) + \int_0^t f(s)\,ds + \int_0^t g(s)\,dB(s), \qquad t \in [0,T]. \tag{1.2.1}$$

The equivalent stochastic differential shorthand notation used to express this is given by

$$dX(t) = f(t)\,dt + g(s)\,dB(t) \qquad t \in [0,T].$$

If we have an Itô process X, we can transform it using a stochastic version of the chain rule called *Itô's formula*. Let $F \in C^2(\mathbb{R};\mathbb{R})$ and X be the Itô process defined by (1.2.1). Then $F(X)$ is an Itô process and for each $T > 0$ we have

$$F(X(t)) = F(X(0)) + \int_0^t \left(F^{'}(X(s))f(s) + \frac{1}{2}F^{''}(X(s))g^2(s)\right)\,ds + \int_0^t F^{'}(X(s))g(s)d\,B(s) \quad 0 \le t \le T.$$

Laws of the Iterated Logarithm. The law of the iterated logarithm is the name given to several theorems which describe the magnitude of the fluctuations of a random walk. Let S_n be the sum of n independent and identically distributed random variables with mean zero and finite variance σ^2. Then

$$\limsup_{n\to\infty} \frac{S_n}{\sigma\sqrt{2n\log\log n}} = 1 = -\liminf_{n\to\infty} \frac{S_n}{\sigma\sqrt{2n\log\log n}} \qquad a.s.$$

It should be noted that this Law can also be applied in continuous time to standard Brownian motion. If B is a standard Brwonian motion, then

$$\limsup_{t\to\infty} \frac{B(t)}{\sigma\sqrt{2t\log\log t}} = 1 = -\liminf_{t\to\infty} \frac{B(t)}{\sigma\sqrt{2t\log\log t}} \qquad a.s.$$

1.2.2 Useful Results

This section contains results that are used throughout the chapters.

Burkholder Davis Gundy Inequality. Let B be a standard one–dimensional Brownian motion and H be an $\mathcal{F}^B$ adapted process such that $\mathbb{E}[\int_0^T H(s)^2\,ds] < +\infty$. Then for any $p \ge 0$, there exists $C_p > 0$ independent of H, T and B such that

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t H(s)\,d\,B(s)\right|^p\right] \le C_p\mathbb{E}\left[\left(\int_0^t H(s)^2\,ds\right)^{\frac{p}{2}}\right].$$

A special case is when $p = 2$ and $C_p = 4$ and is known as *Doob's Inequality*.
Chebyshev Inequality. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability triple. If an $\mathcal{F}$-measurable random variable X is such that $\mathbb{E}[|X|^p] < \infty$, then for all $k > 0$ we have

$$\mathbb{P}(|X| \ge k) \le \frac{\mathbb{E}[|X|^p]}{k^p}.$$

Dirac Measure. A Dirac measure σ_x for any measurable set A is defined as

$$\sigma_x(A) = \begin{cases} 1, & \text{for all } t \in A \\ 0, & \text{otherwise} \end{cases}$$

Fatou's Lemma. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability triple. Suppose that $(Y(n))_{n\geq 0}$ is a sequence of non-negative random variables with $Y(n)$ being $\mathcal{F}$–measurable for each $n \geq 0$. Then

$$\mathbb{E}[\liminf_{n\to\infty} Y(n)] \leq \liminf_{n\to\infty} \mathbb{E}[Y(n)].$$

First Borel-Cantelli Lemma. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability triple. If $(E_n : n \geq 1)$ be a sequence of events such that each $E_n \in \mathcal{F}$ and $\sum_{n=1}^{\infty} \mathbb{P}(E_n) < \infty$, then

$$\mathbb{P}(E_n, i.o.) = 0,$$

where $\{E_n \, i.o.\}$ is the event that the events E_n are realised infinitely often.
Second Borel-Cantelli Lemma. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability triple. If $(E_n : n \geq 1)$ is a sequence of independent events such that each $E_n \in \mathcal{F}$ and

$$\sum_{n=1}^{\infty} \mathbb{P}(E_n) = \infty$$

then

$$\mathbb{P}(E_n, i.o.) = 1.$$

Fubini's Theorem. If f is a bounded measurable function on $[0, T] \times [0, T]$ and

$$Z_u(t) = \int_0^t f(s, u)\, dB(s), \quad 0 \leq t \leq T$$

is continuous from the right and has left limits for each $u \in [0, T]$. Then

$$\int_0^t \left(\int_0^t f(s, u)\, du \right) dB(s) = \int_0^t \left(\int_0^t f(s, u)\, dB(s) \right) du \quad \text{for all } t \in [0, T].$$

Mill's Estimate. Let Z be a standard normal variable. Then

$$\lim_{x\to\infty} \frac{\mathbb{P}[|Z| > x]}{\frac{2}{\sqrt{2\pi}} \cdot \frac{1}{x} \cdot e^{-\frac{1}{2}x^2}} = 1.$$

The following result, which appears as e.g. Corollary 4.1.3 in Chow and Teicher [28] is used throughout the thesis. To use it we need a preliminary definition.

Definition 1.2.1. Given any non–negative constants $(b_n)_{n\geq 0}$ a continuous function b on $[0, \infty)$ is called an **extension** of $(b_n)_{n\geq 0}$ to $[0, \infty)$ if $b(n)_{n\geq 0} = b_n$ for $n \in \mathbb{N}$. Moreover, when $(b_n)_{n\geq 0}$ is strictly monotone, b is called a **strictly monotone extension** of $(b_n)_{n\geq 0}$ if it is both strictly monotone and an extension on $(b_n)_{n\geq 0}$.

Lemma 1.2.1. *Let $(b_n)_{n\geq 0}$ be a strictly monotone increasing sequence with $b_n \geq 0$ and $b_n \to \infty$ as $n \to \infty$. and let b be a strictly monotone extension of (b_n) to $[0, \infty)$. Then for any a.s. non–negative random variable ζ*

$$\sum_{n=1}^{\infty} \mathbb{P}[\zeta \geq b_n] \leq \mathbb{E}[b^{-1}(\zeta)] \leq \sum_{n=0}^{\infty} \mathbb{P}[\zeta > b_n].$$

We use this result in various sections of the thesis to show that the expected value of a function of each member of a sequence of identically and independently distributed noise terms is either finite or infinite. The existence of this generalised moment can then be related to the size of the large fluctuations of the process by means of the Borel–Cantelli lemmata.

1.3 Definition of inefficiency in financial markets

According to Fama [32, 33], when efficiency refers only to historical information which is contained in every private trading agent's information set, the market is said to be *weakly efficient* (cf.[42, Definition 10.17]. Weak efficiency implies that successive price changes (or returns) are independently distributed. Formally, let the market model be described by a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that trading takes place in continuous time, and that there is one risky security. Let $h > 0$, $t \geq 0$ and let $r_h(t+h)$ denote the return of the security from t to $t+h$, and let $S(t)$ be the price of the risky security at time t. Also let $\mathcal{F}(t)$ be the collection of historical information available to every market participant at time t. Then the market is weakly efficient if

$$\mathbb{P}[r_h(t+h) \leq x | \mathcal{F}(t)] = \mathbb{P}[r_h(t+h) \leq x], \quad \forall\, x \in \mathbb{R},\ \ h > 0,\ \ t \geq 0.$$

Here the information $\mathcal{F}(t)$ which is publicly available at time t is nothing other than the generated σ-algebra of the price $\mathcal{F}^S(t) = \sigma\{S(u) : 0 \leq u \leq t\}$. An equivalent definition of weak efficiency in this setting is that

$$r_h(t+h) \text{ is } \mathcal{F}^S(t)\text{-independent, for all } h > 0 \text{ and } t \geq 0. \tag{1.3.1}$$

Geometric Brownian Motion is the classical stochastic process that is used to describe stock price dynamics in a weakly efficient market. More concretely, it obeys the linear SDE

$$dS(t) = \mu S(t)\, dt + \sigma S(t)\, dB(t), \quad t \geq 0 \tag{1.3.2}$$

with $S(0) > 0$. Here $S(t)$ is the price of the risky security at time t, μ is the appreciation rate of the price, and σ is the volatility. It is well-known that the logarithm of S grows linearly in the long-run. The increments of $\log S$ are stationary and Gaussian, which is a consequence of the driving Brownian motion. That is, for a fixed time lag h,

$$r_h(t+h) := \log \frac{S(t+h)}{S(t)} = (\mu - \frac{1}{2}\sigma^2)h + \sigma(B(t+h) - B(t))$$

is Gaussian distributed. Clearly $r_h(t+h)$ is $\mathcal{F}^B(t)$-independent, because B has independent increments. Therefore if $\mathcal{F}^B(t) = \mathcal{F}^S(t)$, it follows that the market is weakly efficient. To see this, note that S being a strong solution of (1.3.2) implies that $\mathcal{F}^S(t) \subseteq \mathcal{F}^B(t)$. On the other hand, since

$$\log S(t) = \log S(0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma B(t), \quad t \geq 0,$$

we can rearrange for B in terms of S to get that $\mathcal{F}^B(t) \subseteq \mathcal{F}^S(t)$, and hence $\mathcal{F}^B(t) = \mathcal{F}^S(t)$. Due to this reason, equation (1.3.2) has been used to model stock price evolution under the classic Efficient Market Hypothesis.

Chapter 2

Fat Tails and Bubbles in a Discrete Time Model of an Inefficient Financial Market

2.1 Introduction

In recent years much attention in financial economics has focussed on the trading strategies of investors. Classical models of financial markets assume that agents are rational, have homogeneous preferences, and do not use historical market data in making their investment decisions. An important and seminal collection of papers summarising this position is [29].
Econometric evidence of market returns (cf., e.g. [48]) and analysis of the behaviour of real traders reveal a more complex picture. Traders often employ rules of thumb which do not conform to notions of rational behaviour based on knowledge of the empirical distribution of returns (cf., e.g. [43]). Moreover, many traders use past prices as a guide to the evolution of the price in the future, with strategies using the crossing of short–run and long–run price averages being very popular (cf., e.g. [64]). Linear continuous–time stochastic models of markets in which agents use past prices to determine their demand, but in which traders discount past returns using an exponentially fading memory, include [21, 38].
In this chapter, we present a stochastic difference equation model of an inefficient financial market. The model is informationally inefficient in the sense that past movements of the stock price have an influence on future movements. We assume that there is trading at intervals of one time unit with prices fixed in the intervening period. The inefficiency stems from the presence of trend–following speculators whose demand for the asset depends on the difference between a short–run and long–run weighted average of the cumulative returns on the stock over the last N_1 and N_2 periods, where $N_2 > N_1$. More precisely, if $X(n)$ is the cumulative return up to time n, the planned excess demand just before trading at time $n+1$ is $\sum_{j=0}^{N_1-1} w_1(j)g(X(n-j)) - \sum_{j=0}^{N_2-1} w_2(j)g(X(n-j))$ where $\sum_{j=0}^{N_m-1} w_m(j) = 1$ for $m = 1, 2$ and g is an increasing function. In other words, speculators buy when the short–run average is above the long–run average, and sell when the short–run average is below the long–run average. Speculators react to other random stimuli— "news"— which is independent of past returns. This news arrives at time $n+1$, adding a further $\xi(n+1)$ to the traders' excess demand. Prices increase when there is excess demand (resp. fall when there is excess supply), with the rise (resp. fall) being larger the greater the excess demand (resp. supply). Hence, the price adjustment at time $n+1$ is given by

$$X(n+1) = X(n) + \sum_{j=0}^{N_1-1} w_1(j)g(X(n-j)) - \sum_{j=0}^{N_2-1} w_2(j)g(X(n-j)) + \xi(n+1). \tag{2.1.1}$$

We need one final assumption on the weights w_1 and w_2 so that the weighted average with weight w_1 represents a short–run average, while the weighted average with weight w_2 represents a long–run average. If the terminology "short–run" and "long–run" is to be meaningful, the short–run average should always give a greater cumulative weight to the last n units of information than the long–run average for any n; mathematically, this means that

$$\sum_{j=0}^{n} w_1(j) \geq \sum_{j=0}^{n} w_2(j), \quad n = 0, 1, \ldots, N_1 - 1. \tag{2.1.2}$$

We study the almost sure asymptotic behaviour as $n \to \infty$ of solutions of (2.1.1) under the assumption (2.1.2). Roughly speaking, we show that the market either follows a correlated random walk, or experiences a crash or bubble.
This chapter shows three things: firstly, if the trend–following speculators do not react very aggressively to differences between the short–run and long–run returns, then the rate of growth of the partial maxima of the solution is the same as that of a random walk. Therefore, to a first approximation, the market appears efficient. However, the size of these largest fluctuations is greater in the presence of trend–following speculators than in their absence, where the market only reacts to "news". Hence, the presence of these

speculators tends to increase market volatility, as well as causing correlation in the returns. These results hold if g is linear (Theorem 2.3.1), or $g(x) = \beta x + o(x)$ as $x \to \infty$ for some $\beta \geq 0$ sufficiently small (Theorem 2.5.1). Moreover, when g is linear the returns follow a random walk plus a stationary process. Secondly, when g is linear and the trend–following speculators behave aggressively, the returns will tend to plus or minus infinity exponentially fast (Theorem 2.3.2). This is a mathematical realisation of a stock market bubble or crash. The result also holds for the nonlinear equation when $g(x) = \beta x + O(|x|^\nu)$ as $x \to \infty$ for some $\nu \in (0, 1)$ (Theorem 2.5.2).

Thirdly, in the case when the cumulative returns follow a correlated random walk, we show for both linear (Theorem 2.6.1) and nonlinear (Theorem 2.6.2) models that the Δ–returns $X(n) - X(n - \Delta)$ are subject to the same size of large excursions as the "news" process ξ, under the assumption that the distribution of each variate in ξ has polynomially decaying tails. In other words, we show that the Δ–returns behave in a way which is consistent with a stationary process in which each variate is distributed with polynomially decaying tails. In the context of financial markets, such returns' distributions are often called "fat–tailed" or "heavy–tailed". In this situation, where "news" is an additive perturbation, and the function g is not too far from linear, the heavy–tailed returns arise from the "news" process being heavy–tailed rather than from the nature of the interaction of the speculators.

This chapter is a generalisation of work of Appleby & Swords [12] which proves analogues of Theorems 2.3.1, 2.3.2 and 2.5.1 in the simpler case when the speculators' trading strategy is based on comparing the current returns with a weighted average of past returns. Moreover, there are no analogues of Theorems 2.5.2, 2.6.1 or 2.6.2 in [12].

If we define $T_2 = \sum_{j=1}^{N_2-1} jw_2(j)$, $T_1 = \sum_{j=1}^{N_1-1} jw_1(j)$, the market experiences a bubble or crash, or a correlated random walk, depending on whether $\beta(T_2 - T_1)$ is greater than or less than unity. $T_2 - T_1$ is positive on account of (2.1.2). Large values of β correspond to aggressive or confident speculative behaviour; if $g(x) = \beta x$ for example, the planned excess demand of traders is β times the difference between the short–run and long–run weighted averages of returns. Therefore, for larger β a smaller signal from the market is required to produce a given response from the traders.

The term $T_2 = \sum_{j=1}^{N_2-1} jw_2(j)$ is in $[1, N_2 - 1]$, and the greater weight that traders give to returns further back in time, the larger T_2 becomes. Therefore, T_2 is a measure of the effective length of the "long–run" memory of the traders; in a similar manner, T_1 is a measure of the effective length of the "short–run" memory of the traders. The larger the difference between $T_2 - T_1$ the more readily the market leaves the correlated random walk regime and enters the bubble or crash regime. Moreover, even within the random walk regime, the large fluctuations become more extreme the larger that $T_2 - T_1$ becomes. It may be seen that a large value of $T_2 - T_1$ arises, for example, when traders base their short–run average on returns over a very short time–horizon, but whose long–run average gives significant weight to returns from the relative distant past. This strategy can obviously introduce significant feedback from the distant past, so causing trends from the returns in the past to persist for long periods of time, which tends to cause the formation of bubbles or crashes. To take a simple example, if traders make their decisions based only on a comparison of returns $N_1 - 1$ periods ago with returns $N_2 - 1$ periods ago, where $N_1 < N_2$, then we have $T_1 = N_1 - 1$ and $T_2 = N_2 - 1$, and so bubbles form if $\beta(N_2 - N_1) > 1$ while we have a correlated random walk if $\beta(N_2 - N_1) < 1$.

This chapter has the following structure; Section 2 gives notation and supporting results, the asymptotic behaviour of the cumulative returns in the linear equation and the probability of a bubble or a crash is studied in Section 3; Section 4 studies the asymptotic behaviour of the autocovariance function of the linear equation, the corresponding results for the nonlinear equation are in Section 5; Section 6 is concerned with the large deviations of the Δ–returns and Section 7 contains the proofs of supporting lemmas.

2.2 Background Material

$\mathbb{N}$ denotes the integers $0, 1, 2, \ldots$, and $\mathbb{R}$ the real line. A real sequence $a = \{a(n) : n \in \mathbb{N}\}$ obeys $a \in \ell^1(\mathbb{N}; \mathbb{R})$ if $\sum_{n\in\mathbb{N}} |a(n)| < \infty$. The convolution of $f = \{f(n) : n \in \mathbb{N}\}$ and $g = \{g(n) : n \in \mathbb{N}\}$, $f * g$, is a sequence defined by $(f * g)(n) = \sum_{k=0}^{n} f(n-k)g(k)$, $n \in \mathbb{N}$. Let $\tilde{x}(z)$ denote the z-transform of x. Let $\beta > 0$, $N_1, N_2 \in \mathbb{N}$ with $N_2 > N_1$ and $w_m = \{w_m(n) : n = 0, \ldots, N_m - 1\}$, $m = 1, 2$ be sequences obeying

$$w_m(n) \geq 0, \quad n = 0, \ldots, N_m - 1; \quad \sum_{n=0}^{N_m-1} w_m(n) = 1, \quad m = 1, 2; \tag{2.2.1a}$$

$$\sum_{j=0}^{n} w_1(j) \geq \sum_{j=0}^{n} w_2(j), \quad j = 0, \ldots, N_1 - 1. \tag{2.2.1b}$$

The resolvent $r = \{r(n) : n \geq -N_2\}$ is a scalar sequence defined by

$$r(n+1) = r(n) + \beta \left(\sum_{j=0}^{N_1-1} w_1(j) r(n-j) - \sum_{j=0}^{N_2-1} w_2(j) r(n-j) \right), \quad n \in \mathbb{N} \tag{2.2.2a}$$

$$r(0) = 1, \quad r(n) = 0, \quad n < 0. \tag{2.2.2b}$$

Lemma 2.2.1. *Let $\beta > 0$, N_1 and N_2 be positive integers with $N_1 > N_2$, w_1 and w_2 obey* (2.2.1)*, and r be defined by* (2.2.2)*.*

(a) r is a non–decreasing sequence with $r(n) > 0$ for $n \in \mathbb{N}$.

(b) If

$$\beta \left(\sum_{j=0}^{N_2-1} j w_2(j) - \sum_{j=0}^{N_1-1} j w_1(j) \right) < 1, \tag{2.2.3}$$

then

$$\lim_{n \to \infty} r(n) = \frac{1}{1 - \beta \left(\sum_{j=0}^{N_2-1} j w_2(j) - \sum_{j=0}^{N_1-1} j w_1(j) \right)} =: r^*, \tag{2.2.4}$$

and $\delta = \{\delta(n) : n \geq -N_2\} \in \ell^1(\mathbb{N}; \mathbb{R}^+)$, where $\delta(-N_2) = 0$, $\delta(n+1) = r(n+1) - r(n)$ for $n \geq -N_2$.

(c) If

$$\beta \left(\sum_{j=0}^{N_2-1} j w_2(j) - \sum_{j=0}^{N_1-1} j w_1(j) \right) > 1, \tag{2.2.5}$$

there exists $\alpha \in (0, 1)$ defined by

$$\alpha^{-1} = 1 + \beta \left(\sum_{k=0}^{N_1-1} \alpha^k w_1(k) - \sum_{k=0}^{N_2-1} \alpha^k w_2(k) \right) \tag{2.2.6}$$

such that $\lim_{n \to \infty} \alpha^n r(n) = R^*$, where $R^* > 0$ is given by

$$R^* = \frac{1}{(1 - \alpha)(1 + \beta \alpha \sum_{j=1}^{\infty} j \alpha^j w(j))}, \tag{2.2.7}$$

where $w(n) := \sum_{j=0}^{n \wedge (N_2-1)} w_1(j) - w_2(j)$, and $w_1(n) := 0$ for $n \geq N_1$.

Remark 2.2.1. For the proof of this lemma we write the resolvent $r(n)$ in terms of a new function $\delta(n)$. We then use this information to prove part (a). For part (b) we can compute the limit of $\delta(n)$ and use this information to derive an explicit formula for the limit of $r(n)$. By applying the final value theorem to the z-transform of $\delta(n)$ we compute an explicit formula for its rate of exponential growth. Rewriting the exponential rate of growth of $r(n)$ in terms of $\delta(n)$ we can derive the exponential rate of growth of $r(n)$ thus proving part (c).

Proof. We assume $N_1 > 1$. r is non–decreasing if δ is non–negative. Extend w_1 to $\{N_1, \dots, N_2 - 1\}$ by $w_1(n) = 0$ for $n = N_1, \dots, N_2 - 1$. To prove part (a), we put $r(n) = \sum_{j=-N_2+1}^{n} \delta(j)$ into (2.2.2) for $n \geq 0$, we have

$$\sum_{j=-N_2+1}^{n+1} \delta(j) - \sum_{j=-N_2+1}^{n} \delta(j) = \\ + \beta \left(\sum_{k=0}^{N_1-1} w_1(k) \sum_{j=-N_2+1}^{n-k} \delta(j) - \sum_{k=0}^{N_2-1} w_2(k) \sum_{j=-N_2+1}^{n-k} \delta(j) \right),$$

and hence

$$\delta(n+1) = \beta \left(\sum_{j=0}^{N_1-1} \sum_{k=-N_2+1}^{n-j} w_1(j)\delta(k) - \sum_{j=0}^{N_2-1} \sum_{k=-N_2+1}^{n-j} w_2(j)\delta(k) \right).$$

Using the fact that $w_1(n) = 0$ for $n \geq N_1$, and by reversing the order of summation, we have

$$\begin{aligned} \sum_{j=0}^{N_1-1} \sum_{k=-N_2+1}^{n-j} w_1(j)\delta(k) &= \sum_{j=0}^{N_2-1} \sum_{k=-N_2+1}^{n-j} w_1(j)\delta(k) \\ &= \sum_{k=-N_2+1}^{n} \left\{ \sum_{j=0}^{(n-k)\wedge(N_2-1)} w_1(j) \right\} \delta(k), \end{aligned}$$

and similarly we have

$$\sum_{j=0}^{N_2-1} \sum_{k=-N_2+1}^{n-j} w_2(j)\delta(k) = \sum_{k=-N_2+1}^{n} \left\{ \sum_{j=0}^{(n-k)\wedge(N_2-1)} w_2(j) \right\} \delta(k).$$

By defining

$$w(n) = \sum_{j=0}^{n\wedge(N_2-1)} w_1(j) - w_2(j), \tag{2.2.8}$$

we have $\delta(n+1) = \beta \sum_{k=-N_2+1}^{n} w(n-k)\delta(k)$ for $n = 0, 1, 2, \dots$. Since $\delta(n) = 0$ for $n < 0$ and $\delta(0) = 1$,

$$\delta(n+1) = \beta \sum_{k=0}^{n} w(n-k)\delta(k), \quad n = 0, 1, 2, \dots; \quad \delta(0) = 1. \tag{2.2.9}$$

By (2.2.1a), for $n \geq N_2 - 1$ we have $w(n) = 0$. For $0 \leq n \leq N_1 - 1$, as $N_1 < N_2$, we have $w(n) = \sum_{j=0}^{n} w_1(j) - w_2(j)$, and so by (2.2.1b), $w(n) \geq 0$ for $n = 0, 1, \dots, N_1 - 1$. Finally for $N_1 \leq n \leq N_2 - 2$, we have

$$w(n) = \sum_{j=0}^{n} (w_1(j) - w_2(j)) = \sum_{j=0}^{N_1-1} w_1(j) + \sum_{j=N_1}^{n} w_1(j) - \sum_{j=0}^{n} w_2(j) = 1 - \sum_{j=0}^{n} w_2(j),$$

and so $w(n) \geq 0$ for $n = N_1, \dots, N_2 - 2$. Hence $w(n) \geq 0$ for all $n \geq 0$, and so $\delta(n) \geq 0$ for all $n \geq 0$, proving (a).

To prove part (b), we note first that $w \in \ell^1(\mathbb{N}; [0, \infty))$. If $\beta \sum_{n=0}^{\infty} w(n) < 1$, then

$$\begin{aligned} \sum_{n=0}^{\infty} \delta(n) - 1 = \sum_{n=0}^{\infty} \delta(n+1) &= \beta \sum_{n=0}^{\infty} \sum_{k=0}^{n} w(n-k)\delta(k) \\ &= \beta \sum_{n=0}^{\infty} w(n) \sum_{j=0}^{\infty} \delta(j), \end{aligned}$$

and $\sum_{n=0}^{\infty} \delta(n) =: r^*$ is finite, with $r^* = 1 + \beta r^* \sum_{n=0}^{\infty} w(n)$. Since $w(n) = 0$ for $n \geq N_2 - 1$ and w obeys (2.2.8),

$$\begin{aligned}
\sum_{n=0}^{\infty} w(n) &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n \wedge (N_2-1)} (w_1(j) - w_2(j)) \right) \\
&= \sum_{j=0}^{N_2-2} \sum_{n=j}^{N_2-2} w_1(n) - \sum_{j=0}^{N_2-2} \sum_{n=j}^{N_2-2} w_2(n) \\
&= \sum_{j=0}^{N_2-2} (N_2 - j - 1) w_1(j) - \sum_{j=0}^{N_2-2} (N_2 - 1 - j) w_2(j).
\end{aligned}$$

Now

$$\sum_{j=0}^{N_2-2} (N_2 - 1 - j) w_2(j) = \sum_{j=0}^{N_2-1} (N_2 - 1 - j) w_2(j) = N_2 - 1 - \sum_{j=0}^{N_2-1} j w_2(j)$$

and similarly for $\sum_{j=0}^{N_2-2} (N_2 - 1 - j) w_1(j)$. As (2.2.1a) holds, we get

$$\sum_{n=0}^{\infty} w(n) = \sum_{j=0}^{N_2-1} j w_2(j) - \sum_{j=0}^{N_1-1} j w_1(j),$$

proving (2.2.4). For part (c), the condition that $\beta \sum_{n=0}^{\infty} w(n) > 1$ implies there is a unique $\alpha \in (0,1)$ such that $\beta\alpha \sum_{j=0}^{\infty} w_\alpha(j) = 1$, where $w_\alpha(n) = \alpha^n w(n)$. Indeed, such an α must satisfy (2.2.6). Now, by multiplying across (2.2.9) by α^{n+1}, we get $\delta_\alpha(n+1) = \beta\alpha \sum_{j=0}^{n} w_\alpha(j)\delta_\alpha(n-j)$ where $\delta_\alpha(n) = \alpha^n \delta(n)$. Now taking the z-transforms yields

$$\begin{aligned}
z(\widetilde{\delta_\alpha}(z)) - 1 &= \beta\alpha \sum_{j=0}^{n} w_\alpha(j) z^{-j} \widetilde{\delta_\alpha}(z) \\
1 &= \widetilde{\delta_\alpha}(z) \left(z - \beta\alpha \sum_{j=0}^{n} w_\alpha(j) z^{-j} \right) \\
1 &= \lim_{z \to 1} \widetilde{\delta_\alpha}(z) \left(z - \beta\alpha \sum_{j=0}^{n} w_\alpha(j) z^{-j} \right).
\end{aligned}$$

Now by the final value theorem $\lim_{z\to 1} \widetilde{x}(z)(1 - z^{-1}) = \lim_{n\to\infty} x(n)$ where $\lim_{z\to 1}(1 - z^{-1}) = 0$. Then

$$\begin{aligned}
\lim_{z \to 1} \left(z - \beta\alpha \sum_{j=0}^{n} w_\alpha(j) z^{-j} \right) &= 0 \\
1 - \beta\alpha \sum_{j=0}^{n} w(j) \frac{\alpha^n - 1}{\alpha - 1} &= 0 \\
1 + \beta \sum_{j=1}^{n} w(j)\alpha^j &= \alpha^{-1}.
\end{aligned}$$

Rewriting the equation in terms of the Final Value Theorem yields

$$\lim_{z\to 1} \widetilde{\delta_\alpha}(z)(1 - z^{-1}) \left(\frac{z}{1 - z^{-1}} - \frac{\beta\alpha \sum_{j=0}^{n} w(j) z^{-j}}{1 - z^{-1}} \right) = \frac{1}{1 + \beta\alpha \sum_{j=0}^{n} j\alpha^j w(j)}$$

Thus

$$\lim_{n\to\infty} \delta(n) = \frac{1}{1 + \beta\alpha \sum_{j=0}^{n} j\alpha^j w(j)}.$$

As $\alpha^n \delta(n) \to \delta^*$ as $n \to \infty$ then $\lim_{n\to\infty} \alpha^n r(n) = (1-\alpha)^{-1}\delta^* := R^*$ where R^* is given by (2.2.7). Therefore $\lim_{n\to\infty} \alpha^n r(n) = \frac{1}{(1-\alpha)}\delta^* =: R^*$, where $R^* > 0$ is given by (2.2.7). □

2.3 Cumulative Returns In The Linear Equation

We consider the linear stochastic difference equation for $n \geq 0$

$$Y(n+1) = Y(n) + \beta \left\{ \sum_{j=0}^{N_1-1} w_1(j)Y(n-j) - \sum_{j=0}^{N_2-1} w_2(j)Y(n-j) \right\} + \xi(n+1), \tag{2.3.1a}$$

$$Y(n) = \phi(n), \quad n \leq 0, \tag{2.3.1b}$$

where $\xi = \{\xi(n) : n \in \mathbb{N}\}$ is a is a sequence of independent, identically distributed variables obeying

$$\mathbb{E}[\xi(n)] = 0, \ \sigma^2 := \mathbb{E}[\xi(n)^2] \text{ for all } n \in \mathbb{N}. \tag{2.3.2}$$

Proposition 2.3.1. *Let r be the solution of* (2.2.2) *and let Y be the solution of* (2.3.1) *and y obey* (2.3.7) *then*

$$Y(n) = \sum_{j=1}^{n} r(n-j)\xi(j) + y(n). \tag{2.3.3}$$

Proof. Define $Z(n) = Y(n) - y(n)$ for all $n \geq -N$ where $Z(n) = 0$ for all $n \leq 0$, then $Z(n+1) = Y(n+1) - y(n+1)$. Using the fact that Y obeys (2.3.1) gives

$$\begin{aligned}
Z(n+1) = Y(n) &+ \beta \left(\sum_{j=0}^{N_1-1} w_1(j)Y(n-j) - \sum_{j=0}^{N_2-1} w_2(j)Y(n-j) \right) + \xi(n+1) \\
&- y(n) - \beta \left(\sum_{j=0}^{N_1-1} w_1(j)y(n-j) - \sum_{j=0}^{N_2-1} w_2(j)y(n-j) \right) \\
&= Y(n) - y(n) + \beta \sum_{j=0}^{N_1-1} w_1(j)(Y(n-j) - y(n-j)) \\
&- \beta \sum_{j=0}^{N_2-1} w_2(j)(Y(n-j) - y(n-j)) + \xi(n+1) \\
&= Z(n) + \beta(\sum_{j=0}^{N_1-1} w_1(j)Z(n-j) - \sum_{j=0}^{N_2-1} w_2(j)Z(n-j)) + \xi(n+1).
\end{aligned}$$

Therefore by Lemma 2.7.2

$$Z(n) = \sum_{j=0}^{n-1} r(n-1-j)\xi(j+1) = \sum_{j=1}^{n} r(n-j)\xi(j).$$

From (2.3.8) we know that $y(n) = r(n)\phi(0) + \beta(\phi_0 * r)(n)$ for $n \geq 1$. Then for $n \geq 1$
$Z(n) = \sum_{j=1}^{n} r(n-j)\xi(j)$ and $Y(n) = z(n) + y(n)$ as required. □

Remark 2.3.1. This propostion shows that the resolvent $Y(n)$ can be expressed in terms of the deterministic equation and the variation of constants formula.

2.3.1 Correlated random walk obeying the law of the iterated logarithm

If (2.2.3) holds, then Y behaves asymptotically as a random walk. For instance, if ξ obeys (2.3.2) the process S given by $S(n) = \sum_{j=1}^{n} \xi(j)$ is a random walk and obeys *the Law of the Iterated Logarithm*:

$$\limsup_{n\to\infty} \frac{S(n)}{\sqrt{2n\log\log n}} = -\liminf_{n\to\infty} \frac{S(n)}{\sqrt{2n\log\log n}} = \sigma, \quad \text{a.s.}$$

Theorem 2.3.1. *Let $\beta > 0$, N_1 and N_2 be positive integers with $N_1 > N_2$, w_1 and w_2 obey* (2.2.1)*, and β obey* (2.2.3)*. If ξ obeys* (2.3.2)*, and Y obeys* (2.3.1)*, then*

$$\limsup_{n\to\infty} \frac{Y(n)}{\sqrt{2n\log\log n}} = \frac{|\sigma|}{1-\beta\left(\sum_{j=0}^{N_2-1} jw_2(j) - \sum_{j=0}^{N_1-1} jw_1(j)\right)}, \quad \textit{a.s.} \tag{2.3.4}$$

$$\liminf_{n\to\infty} \frac{Y(n)}{\sqrt{2n\log\log n}} = -\frac{|\sigma|}{1-\beta\left(\sum_{j=0}^{N_2-1} jw_2(j) - \sum_{j=0}^{N_1-1} jw_1(j)\right)}, \quad \textit{a.s.} \tag{2.3.5}$$

Remark 2.3.2. In the above formulas the right–hand side limit is greater than $|\sigma|$, the traders cause excess volatility in the market. The size of the excess increases as the key parameter, that is

$$\beta\left(\sum_{j=0}^{N_2-1} jw_2(j) - \sum_{j=0}^{N_1-1} jw_1(j)\right),$$

tends to 1. The reason for this is more easily ilustrated by the following example.

Example 2.3.1. A common trading strategy is to compare the arithmetic average of prices or returns over 200 days (long run average) with that over 30 days (short run average). We generalise this here to a comparison of arithmetic averages over N_1 and N_2 days, where $N_1 > N_2$. To do this, we let

$$w_1(j) = \frac{1}{N_1}, \text{ for } j = 0, \cdots, N_1 - 1$$

and

$$w_2(j) = \frac{1}{N_2}, \text{ for } j = 0, \cdots, N_2 - 1.$$

It can be easily checked that w_1 and w_2 obey all the hypotheses of Theorem 2.3.1. Then the key parameter as defined above turns out to be

$$\beta\left(\sum_{j=0}^{N_2-1} jw_2(j) - \sum_{j=0}^{N_1-1} jw_1(j)\right) = \frac{\beta}{2}\left(N_2 - N_1\right).$$

If this value is less than one then the volatility is amplified by a factor of $\frac{1}{1-\frac{\beta}{2}(N_2-N_1)}$ (with respect to a market with feedback traders). Firstly, this factor increases as β increases, which means the market becomes more volatile the more aggresive the traders. Secondly, the factor increases as $N_2 - N_1$ increases, and it increases very rapidly as $\beta(N_2 - N_1)$ approaches 2. Therefore the market becomes arbitrarily volatile relative to a market free of feedback traders as we get ever closer to this boundary in (β, N_2, N_1) parameter space. $N_2 - N_1$ becomes large when the traders take very long–run moving averages with very short–run moving averages.

We consider another example in which traders compare the returns N_2 units of time ago with those N_1 units of time ago, where again $N_2 > N_1$. In this case X obeys

$$X(n+1) = X(n) + \beta\left(X(n-(N_1-1)) - X(n-(N_2-1))\right) + \xi(n+1),$$

and we identify w_1 and w_2 as follows:

$$w_1(n) = \begin{cases} 1, & n = N_1 - 1 \\ 0, & \text{otherwise}, \dots, \end{cases}$$

$$w_2(n) = \begin{cases} 1, & n = N_2 - 1 \\ 0, & \text{otherwise}, \dots, \end{cases}$$

Then the key parameter is defined as $\beta\,(N_2 - N_1)$. Again the volatility increases as β increases and as $N_2 - N_1$ increases. However, this market is less stable than that with arithmetic averages because moving averages smooth the effect of a large of the process at a given instant over the entire period sampled by the investors.

Remark 2.3.3. For this proof we write the resolvent $Y(n)$ in terms of the deterministic equation and the variation of constants formula. For the variation of constants formula we add on and take away the limit of the characteristic equation (in the stable case) leaving us with two terms. By combining Lemma 1.2.1 with the Borel–Cantelli lemma and by employing Lemma 2.7.1 we show for the first of these terms that its limit tends to zero. The limit of the second term is equal to the limit of the resolvent multiplied by the absolute value of the standard deviation of the news process. Once stochastic effects have been taken into account, it is this term that dominates the contribution from the deterministic equation, and therefore yields the limit for resolvent $Y(n)$.

Proof. For $n \geq 1$, $Y(n) = y(n) + \sum_{j=1}^{n} r(n-j)\xi(j)$, where

$$y(n+1) = y(n) + \beta\left(\sum_{j=0}^{N_1-1} w_1(j)y(n-j) - \sum_{j=0}^{N_2-1} w_2(j)y(n-j)\right) \quad n \geq 0,$$

$y(n) = \phi(n), n \leq 0$. Define $\Delta^*(n) = r^* - r(n)$ and $U(n) = \sum_{j=1}^{n} \Delta(n-j)\xi(j)$, we get

$$Y(n) = y(n) - U(n) + r^* \sum_{j=1}^{n} \xi(j) \qquad n \geq 1.$$

By (2.2.3), $r(n) \to r^*$, and so $\lim_{n\to\infty} y(n)$ exists. By the law of the iterated logarithm, we need to show $\lim_{n\to\infty} |U(n)|/\sqrt{2n\log\log n} = 0$ a.s. By Lemma 2.2.1, we have $\Delta(n) \geq 0$ and $\sum_{n=0}^{\infty} \Delta(n) = \sum_{n=1}^{\infty} n\delta(n)$. By (2.2.9) and the fact that $w(n) = 0$ for $n \geq N_2 - 1$, we see that δ is the summable solution of a finite lag linear difference equation, and therefore $|\delta(n)| \leq C\nu^n$ for some $\nu \in (0,1)$. Thus, $\Delta \in \ell^1(\mathbb{N},\mathbb{R})$.

Let $b(x) = \sqrt{x}$, $x \geq 0$. Then $b : [0,\infty) \to [0,\infty)$ is increasing and $b^{-1}(x) = x^2$. If ξ is a random variable with the same distribution as $\xi(n)$, by Lemma 1.2.1 we have

$$\sum_{n=1}^{\infty} \mathbb{P}[|\xi(n)| > \sqrt{n}] \leq \mathbb{E}[b^{-1}(|\xi|)] = \mathbb{E}[\xi^2] < \infty.$$

By the Borel–Cantelli lemma, $\limsup_{n\to\infty} |\xi(n)|/\sqrt{n} \leq 1$, a.s. which implies that $\lim_{n\to\infty} |\xi(n)|/\sqrt{2n\log\log n} = 0$ a.s. Thus, there is an a.s. event Ω^* such that for all $\omega \in \Omega^*$, and all $\varepsilon > 0$, there is $C(\varepsilon,\omega) > 0$ such that $|\xi(n,\omega)| < C(\varepsilon,\omega) + \varepsilon\sqrt{2n\log\log(n+e^e)} =: \gamma(n,\omega)$ for $n \in \mathbb{N}$. Since $\Delta \in \ell^1(\mathbb{N};\mathbb{R})$, by Lemma 2.7.1 we have

$$\limsup_{n\to\infty} \frac{|U(n,\omega)|}{\gamma(n,\omega)} \leq \limsup_{n\to\infty} \frac{\sum_{j=1}^{n} |\Delta(n-j)|\gamma(\omega,j)}{\gamma(n,\omega)} = \sum_{j=0}^{\infty} |\Delta(j)|;$$

thus $\limsup_{n\to\infty} |U(n,\omega)|/\sqrt{2n\log\log n} < \varepsilon \sum_{j=0}^{\infty} |\Delta(j)|$, hence the result. □

When (2.2.3) holds, it can further be shown that $y - U$ is an asymptotically stationary ARMA process, so Y is the sum of an asymptotically stationary (and mean reverting) process and a random walk. ARMA (autoregressive moving average) processes are used widely in financial econometrics (see e.g., [22]). Notice also that the limit on the righthand side of (2.3.4) is greater than $|\sigma|$; this shows that the large fluctuations of X are greater than those of the random walk S, which represents a market without the presence of feedback traders.

2.3.2 Presence of bubbles and crashes

When β obeys (2.2.5), we now prove $\alpha^n Y(n) \to Y^*$ as $n \to \infty$ where $\alpha \in (0,1)$ and Y^* is a random variable which is explicitly given in terms of ξ. Hence $Y(n)$ tends to $\pm\infty$ according to the sign of Y^*. But before we state the theorem the following result is needed for its proof:

Lemma 2.3.1. *For a continuous local Martingale M, the sets* $\{\langle M\rangle(\infty) < \infty\}$ *and* $\{\lim_{t\to\infty} M(t)\ exists\}$ *are almost–surely equal. Furthermore,* $\limsup_{t\to\infty} M(t) = \infty$ *and* $\liminf_{t\to\infty} M(t) = -\infty$ *almost surely on the set* $\{\langle M\rangle(\infty) = \infty\}$.

Theorem 2.3.2. *Let* $\beta > 0$*,* N_1 *and* N_2 *be positive integers with* $N_1 > N_2$*,* w_1 *and* w_2 *obey* (2.2.1)*, and* β *obey* (2.2.5)*. Suppose also that* $\alpha \in (0,1)$ *is given by* (2.2.6)*, and* R^* *by* (2.2.7)*. If* ξ *obeys* (2.3.2)*, and* Y *obeys* (2.3.1)*, then*

$$\lim_{n\to\infty} \alpha^n Y(n) = R^* \left(L(\phi) + \sum_{j=1}^{\infty} \alpha^j \xi(j) \right), \quad a.s.$$

where

$$L(\phi) = \phi(0) + \beta \sum_{j=0}^{N_1-2} \alpha^j \sum_{l=j-N_1+1}^{-1} w_1(j-l)\phi(l) - \beta \sum_{j=0}^{N_2-2} \alpha^j \sum_{l=j-N_2+1}^{-1} w_2(j-l)\phi(l). \qquad (2.3.6)$$

Remark 2.3.4. In the proof we write the resolvent $Y(n)$ in terms of the deterministic equation and variation of constants formula. We then calculate the exponential rate of growth of the deterministic equation via z-transforms. We add on and take away the exponential rate of growth of the deterministic equation from the variation of constant formula. By employing the martingale convergence theorem to this term we are able to compute its exponential rate of growth thus deriving the result.

Proof. Let y be given by

$$y(n+1) - y(n) = \beta \left(\sum_{k=0}^{N_1-1} w_1(k) y(n-k) - \sum_{k=0}^{N_2-1} w_2(k) y(n-k) \right). \qquad (2.3.7)$$

Taking the z- transform yields,

$$z\left(\widetilde{y}(z) - y(0)\right) = \beta \left(\sum_{n=0}^{\infty} z^{-n} \sum_{k=0}^{N_1-1} w_1(k) y(n-k) - \sum_{n=0}^{\infty} z^{-n} \sum_{k=0}^{N_2-1} w_2(k) y(n-k) \right).$$

Now

$$\begin{aligned}
\sum_{n=0}^{\infty} z^{-n} \sum_{k=0}^{N_1-1} w_1(k) y(n-k) &= \sum_{n=0}^{\infty} \sum_{k=0}^{N_1-1} w_1(k) z^{-k} y(n-k) z^{-(n-k)} \\
&= \sum_{k=0}^{N_1-1} w_1(k) z^{-k} \sum_{l=-k}^{\infty} z^{-l} y(l) \\
&= \sum_{k=0}^{N_1-1} w_1(k) z^{-k} \left(\sum_{l=-k}^{-1} z^{-l} y(l) + \sum_{l=0}^{\infty} z^{-l} y(l) \right) \\
&= \widetilde{y}(z) \sum_{k=0}^{N_1-1} w_1(k) z^{-k} + \sum_{k=0}^{N_1-1} w_1(k) z^{-k} \sum_{l=-k}^{-1} z^{-l} \phi(l).
\end{aligned}$$

Letting $m = l + k$, then

$$\sum_{k=0}^{N_1-1} w_1(k) z^{-k} \sum_{l=-k}^{-1} z^{-l} \phi(l) = \sum_{k=0}^{N_1-1} w_1(k) \sum_{m=0}^{k-1} z^{-m} \phi(m-k).$$

Thus,

$$z\left(\widetilde{y}(z)-y(0)\right)=\widetilde{y}(z)+\beta\left(\widetilde{y}(z)\sum_{k=0}^{N_1-1}w_1(k)z^{-k}+\sum_{k=0}^{N_1-1}w_1(k)\sum_{m=0}^{k-1}z^{-m}\phi(m-k)\right)$$
$$-\beta\left(\widetilde{y}(z)\sum_{k=0}^{N_2-1}w_2(k)z^{-k}-\sum_{k=0}^{N_2-1}w_2(k)\sum_{m=0}^{k-1}z^{-m}\phi(m-k)\right).$$

Define $\phi_0(n)=\phi_1(n)-\phi_2(n)$ and

$$\phi_m(n)=\begin{cases}\sum_{l=n-N_m+1}^{-1}w_m(n-l)\phi(l), & n=0,1,\dots,N_m-2\\ 0, & n=N_m-1,N_m,\dots,\end{cases}$$

then

$$\widetilde{y}(z)\left(z-1-\beta\sum_{k=0}^{N_m-1}w(k)z^{-k}\right)=\phi(0)+\beta\left(\sum_{m=0}^{j-1}z^{-m}\phi_0(n)\right).$$

Using the fact that

$$\widetilde{r}(z)=\frac{1}{z-1-\beta\sum_{k=0}^{N_m-1}z^{-k}w(k)}$$

and getting the inverse of the z-transform of the above yields

$$y(n)=r(n)\phi(0)+\beta(\phi_0*r)(n)\quad n\geq 1. \tag{2.3.8}$$

Then $y(n)=r(n)\phi(0)+\beta\sum_{j=0}^{N_2-2}r(n-j)\phi_0(j)$, $n\geq N_2-1$. As $\lim_{n\to\infty}r(n)\alpha^n=R^*$,

$$\lim_{n\to\infty}\alpha^n y(n)$$
$$=R^*\beta\left(\sum_{j=0}^{N_1-2}\alpha^j\sum_{l=j-N_1+1}^{-1}w_1(j-l)\phi(l)-\sum_{j=0}^{N_2-2}\alpha^j\sum_{l=j-N_2+1}^{-1}w_2(j-l)\phi(l)\right)$$
$$+R^*\phi(0).$$

Next, for $n\geq 1$, it is know that $Y(n)=y(n)+\sum_{j=1}^{n}r(n-j)\xi(j)$ thus

$$\alpha^n Y(n)=\alpha^n y(n)+\sum_{j=1}^{n}(\alpha^{n-j}r(n-j)-R^*)\alpha^j\xi(j)+R^*\sum_{j=1}^{n}\alpha^j\xi(j). \tag{2.3.9}$$

Let $M(n)=R^*\sum_{j=1}^{n}\alpha^j\xi(j)$. Since $\alpha\in(0,1)$, M is martingale with finite quadratic variation, so by the martingale convergence theorem $\lim_{n\to\infty}M(n)$ is finite a.s. By (2.3.2), $\mathbb{E}\sum_{j=1}^{n}\alpha^j|\xi(j)|\leq\alpha\sigma/(1-\alpha)$, so $\xi_\alpha(n):=\alpha^n\xi(n)\in\ell^1(\mathbb{N};\mathbb{R})$, a.s. As $r_1(n):=\alpha^n r(n)-R^*\to 0$, $(r_1*\xi_\alpha)(n)\to 0$ as $n\to\infty$. We end by letting $n\to\infty$ in (2.3.9). □

2.3.3 Bubble dynamics

We rewrite (2.3.9) as

$$\lim_{n\to\infty}\alpha^n Y(n)=R^*\left(L(\phi)+\sum_{j=1}^{\infty}\alpha^j\xi(j)\right)=:\Gamma(\phi),\quad \text{a.s.} \tag{2.3.10}$$

where the constant R^* is given by (2.2.7) and $L(\phi)$ is given by (2.3.6). We say that the market experiences a *bubble* if $\Gamma=\Gamma(\phi)>0$ and a *crash* if $\Gamma=\Gamma(\phi)<0$, because in the former case $Y(n)\to\infty$ as $n\to\infty$ at an exponential rate, while in the latter $Y(n)\to-\infty$ as $n\to\infty$. We remark that $\Gamma(\phi)\neq 0$ a.s. because Γ is normally distributed with non–zero variance. Therefore only bubbles or crashes can occur when (2.2.5) holds. In the next theorem, we analyse the dependence of the probability of a crash or bubble according to the behaviour of the initial returns ϕ on the interval set of times $\{-N_2+1,\cdots,0\}$.

Theorem 2.3.3. *Suppose that ξ obeys* (2.3.2)*. Suppose also that $\beta > 0$, N_1 and N_2 are positive integers with $N_1 > N_2$, w_1 and w_2 obey* (2.2.1)*, and β obeys* (2.2.5)*. Let Y be the solution of* (2.3.1)*.*

(i) If ϕ is constant, then $\mathbb{P}[\Gamma(\phi) > 0] = 1/2$.

(ii) Let $Y(\phi_1)$ be the solution of (2.3.1) *with initial condition ϕ_1 and $Y(\phi_2)$ be the solution of* (2.3.1) *with initial condition ϕ_2. If $\phi_1 - \phi_2$ is constant then*

$$\mathbb{P}[\Gamma(\phi_1) > 0] = \mathbb{P}[\Gamma(\phi_2) > 0].$$

(iii) Let ϕ be such that $\mathbb{P}[\Gamma(\phi) > 0] > 1/2$. Then $c \mapsto \mathbb{P}[\Gamma(c\phi) > 0]$ is increasing and moreover

$$\lim_{c\to\infty} \mathbb{P}[\Gamma(c\phi) > 0] = 1, \quad \lim_{c\to-\infty} \mathbb{P}[\Gamma(c\phi) > 0] = 0. \tag{2.3.11}$$

(iv) Let ϕ be such that $\mathbb{P}[\Gamma(\phi) > 0] < 1/2$. Then $c \mapsto \mathbb{P}[\Gamma(c\phi) > 0]$ is decreasing and moreover

$$\lim_{c\to\infty} \mathbb{P}[\Gamma(c\phi) > 0] = 0, \quad \lim_{c\to-\infty} \mathbb{P}[\Gamma(c\phi) > 0] = 1. \tag{2.3.12}$$

(v) If ϕ is non–decreasing with $\phi(0) > \phi(-1)$, then $\mathbb{P}[\Gamma(\phi) > 0] > 1/2$. Moreover $c \mapsto \mathbb{P}[\Gamma(c\phi) > 0]$ is increasing and obeys (2.3.11)*.*

(vi) If ϕ is non–increasing with $\phi(0) < \phi(-1)$, then $\mathbb{P}[\Gamma(\phi) > 0] < 1/2$. Moreover $c \mapsto \mathbb{P}[\Gamma(c\phi) > 0]$ is decreasing and obeys (2.3.12)*.*

Before giving the proof we interpret the conclusions (i)–(vi) of the theorem. Part (i) implies that if there is no trend in the returns on the interval set of times $\{-N_2 + 1, \cdots, 0\}$, then the market is equally likely to enter a bubble or a crash. This is sensible because the traders are not able to detect a trend in the market which might influence their decisions in one direction or another. Part (ii) suggests that it is the patterns of the recent returns which influences the probability of a bubble rather than whether the returns are high or low; this is emphasised by parts (v) and (vi) which show that if there is an initial upward trend in the returns then the speculators are more likely to extrapolate this rising trend, causing a bubble to occur. On the other hand if there is an initial downward trend in the returns, the speculators are more likely to trade in a manner that causes this trend to be extrapolated downwards, leading to a crash. The initial poor performance of the asset convinces positive feedback traders that informed traders believe the asset will perform poorly in future, so they sell (or short sell) the stock. This then forces prices lower, encouraging further selling, and the result of this downward spiral is a crash. The second part of the conclusion of parts (v) and (vi) echo the conclusion of parts (iii) and (iv) of the theorem. Part (iii) suggests that if there is a trend in the initial returns which makes the probability of a bubble more likely than that of a crash, an amplified version of that trend would make a bubble even more likely to occur, with greater amplifying factors leading to greater probabilities of a bubble. This suggests that when the traders receive stronger trending signals from the market (even if those signals are simply noise), they are more likely to make these trends self–fulfilling. Also, it can be seen that a mirror image of the trend which makes a bubble more likely is precisely what makes a crash more likely. Part (iv) shows that the situation for bubbles described in part (iii) holds symmetrically for crashes.

Theorem 2.3.3 concentrates on the impact of the initial returns on the probability of a bubble or crash. However, this probability also depends on the properties of the summation of ξ on the righthand side of (2.3.10). Because of this we can determine the impact of a sequence of "good news stories" about the asset at the time shortly after trading begins. Speaking very loosely, we can interpret this as a "majority" of the (infinitesimal) increments of ξ being positive. Since the summation on the righthand side of (2.3.10) diminishes exponentially as time increases, it is the sign of these "initial" increments of ξ that largely determines whether the summation assumes a positive or negative value. Therefore, initial good news about the stock tends to result in a positive value of the summation, while initial bad news about the stock tends to lead to a negative value of the summation. Therefore, if there is good initial news about the asset, the price of the stock tends to increase and the traders force the price higher by misperceiving this increase as arising from demand from informed speculators. As before, this induces further buying and the stock price undergoes a bubble. Similarly, initial bad news tends to precipitate a crash.

These remarks suggest that the mechanisms by which bubbles form in this model are consistent with the notion of mimetic contagion introduced by Orléan (cf. e.g., [65]). In mimetic contagion we may think of the market as comprising of two forms of traders, with new entrants choosing the trading strategy which tends to dominate at a given time. In the long-run, the proportion of traders in each category settles down to a value which is random but which depends quite strongly on what happens in the first trading periods. The similarities with mimetic contagion are as follows: in (2.3.10), the righthand side depends crucially on the market behaviour in the first few time periods; once a dominant trend becomes apparent, the trend following speculators will tend to extrapolate that trend; and the longrun behaviour (either a bubble or crash) is not known in advance.

Remark 2.3.5. To prove (i) - (iv) of the theorem we employ the property of the normal distribution to compute the value of the probability. The proof of (v) and (vi) are similar. For (v) we write y in terms of a function U and apply z-transforms to this "new" equation. Using the Final Value Theorem we show this limit is positive and hence calculate the probability of this event occurring.

Proof. To prove (i), notice that if $\phi(n) = R^*$ for all $n = -N_2+1, \cdots, 0$ then the solution y of (2.3.7) is $y(n,\phi) = R^*$ for all $n = -N_2+1, \cdots,$. Therefore $\lim_{n\to\infty} y(n,\phi)/\alpha^n = 0$ and so

$$\Gamma(\phi) = R^* \sum_{j=1}^{\infty} \alpha^j \xi(j) =: Z \tag{2.3.13}$$

where Z is normally distributed with zero mean and variance $R^{*2}\sum_{j=1}^{n}\alpha^{2j}$ and $\mathbb{P}[\Gamma(\phi) > 0] = \mathbb{P}[Z > 0] = 1/2$ as claimed.

For the proof of (ii), let $y(\phi_1)$ be the solution of (2.3.7) with initial condition ϕ_1 and $y(\phi_2)$ be the solution of (2.3.7) with initial condition ϕ_2. Let $z(n) := y(\phi_1, n) - y(\phi_2, n)$ for all $n = -N_2+1, \cdots,$. Then $z(n) = c$ for all $n = -N_2+1, \cdots, 0$ and $z(n+1) - z(n) = L(z_n)$ for $n \geq 0$. Therefore $z(n) = c$ for all $n = 0, \cdots$ and $y(\phi_1, n) - y(\phi_2, n) = c$ for all $n = -N_2+1, \cdots, 0$. If y is the solution of (2.3.7) we may define the operator L by

$$\lim_{n\to\infty} y(n,\phi)/\alpha^n = L(\phi) \tag{2.3.14}$$

where $L(\phi)$ is defined by (2.3.6) and $R^* > 0$. We note that R^* is independent of ϕ. Since $z(n)/\alpha^n \to 0$ as $n \to \infty$, we have

$$L(\phi_1) - L(\phi_2) = \lim_{n\to\infty} \frac{y(\phi_1, n)}{\alpha^n} - \lim_{n\to\infty} \frac{y(\phi_2, n)}{\alpha^n} = 0,$$

and $L(\phi_1) = L(\phi_2)$. Therefore $\Gamma(\phi_1) = \Gamma(\phi_2)$ and (ii) is proven.

We now prove (iii). By (3.6.2) and (2.3.6) we have

$$\Gamma(\phi) = L(\phi) + R^* \sum_{j=1}^{\infty} \alpha^j \xi(j) = L(\phi) + Z$$

where Z is defined by (2.3.13). If $\mathbb{P}[\Gamma(\phi) > 0] > 1/2$ we have

$$1/2 < \mathbb{P}[\Gamma(\phi) > 0] = \mathbb{P}[L(\phi) + Z > 0] = \mathbb{P}[Z > -L(\phi)] = 1 - \mathbb{P}[Z \leq -L(\phi)],$$

and $L(\phi) > 0$. Clearly $L(c\phi) = cL(\phi)$ for any $c \in \mathbb{R}$ and $\Gamma(c\phi) = cL(\phi) + Z$. As $L(\phi) > 0$ we have that $c \mapsto \mathbb{P}[\Gamma(c\phi) > 0]$ is increasing and

$$\lim_{c\to\infty} \mathbb{P}[\Gamma(c\phi) > 0] = 1, \qquad \lim_{c\to-\infty} \mathbb{P}[\Gamma(c\phi) > 0] = 0.$$

The proof of (iv) is similar. We now prove (v). Let y be defined by (2.3.7) where $y(n) = \phi(n)$ for all $n = -N_2+1, \cdots, 0$ and $y(n) = 0$ for all $n \leq -N_2$. Define $u(n) := y(n) - y(n-1)$ for $n \in \mathbb{Z}$. Then $u(n) = 0$ for all $n \leq -N_2$ and $u(n) = \phi(n) - \phi(n-1)$ for all $n = -N_2+1, \cdots, 0$. Let $n \geq 0$, then

$$u(n+1) = y(n+1) - y(n) = \beta \left(\sum_{j=0}^{N_1-1} w_1(j) y(n-j) - \sum_{j=0}^{N_2-1} w_2(j) y(n-j) \right).$$

Observe that $y(n) := \sum_{n=-\infty}^{\infty} u(n)$ where $n \in \mathbb{Z}$. Then for $n \geq 0$

$$u(n+1) = \beta \left(\sum_{j=0}^{N_1-1} w_1(j) \sum_{l=-\infty}^{\infty} u(l) - \sum_{j=0}^{N_2-1} w_2(j) \sum_{l=-\infty}^{\infty} u(l) \right).$$

Now

$$\begin{aligned}
\sum_{j=0}^{N_1-1} w_1(j) \sum_{l=-\infty}^{\infty} u(l) &= \sum_{l=-\infty}^{\infty} \left(\sum_{j=0}^{(n-l)\wedge(N_1-1)} w_1(j) \right) u(l) \\
&= \sum_{l=-\infty}^{-N_2} \left(\sum_{j=0}^{(N_1-1)\wedge(n-l)} w_1(j) \right) u(l) + \sum_{l=-N_2+1}^{-1} \left(\sum_{j=0}^{(N_1-1)\wedge(n-l)} w_1(j) \right) u(l) \\
&\quad + \sum_{l=0}^{\infty} \left(\sum_{j=0}^{(N_1-1)\wedge(n-l)} w_1(j) \right) u(l).
\end{aligned}$$

By definition $u(n) = 0$ for all $n \leq -N_2$ and consequently $\sum_{l=-\infty}^{-N_2} \left(\sum_{j=0}^{(N_1-1)\wedge(n-l)} w_1(j) \right) u(l) = 0$. Then for $n \geq 0$

$$\begin{aligned}
&\sum_{j=0}^{N_1-1} w_1(j) \sum_{l=-\infty}^{\infty} u(l) - \sum_{j=0}^{N_2-1} w_2(j) \sum_{l=-\infty}^{\infty} u(l) \\
&= \sum_{l=-N_2+1}^{-1} \left(\sum_{j=0}^{(N_1-1)\wedge(n-l)} w_1(j) \right) u(l) + \sum_{l=0}^{\infty} \left(\sum_{j=0}^{(N_1-1)\wedge(n-l)} w_1(j) \right) u(l) \\
&\quad - \sum_{l=-N_2+1}^{-1} \left(\sum_{j=0}^{(N_2-1)\wedge(n-l)} w_2(j) \right) u(l) - \sum_{l=0}^{\infty} \left(\sum_{j=0}^{(N_2-1)\wedge(n-l)} w_2(j) \right) u(l).
\end{aligned}$$

We define $w_1(j) = 0$ for all $j = N_1 \cdots, N_2 - 1$ and let $w(n) = \sum_{j=0}^{(N_1-1)\wedge n} w_1(j) - \sum_{j=0}^{(N_2-1)\wedge n} w_2(j)$ for all $n \geq 0$. Then for $n \geq 0$

$$\sum_{j=0}^{N_1-1} w_1(j) \sum_{l=-\infty}^{\infty} u(l) - \sum_{j=0}^{N_2-1} w_2(j) \sum_{l=-\infty}^{\infty} u(l) = \sum_{l=-N_2+1}^{-1} w(n-l)u(l) + \sum_{l=0}^{n} w(n-l)u(l)$$

and

$$\begin{aligned}
u(n+1) &= F(n) + \beta \sum_{l=0}^{n} w(n-l)u(l), \quad n \geq 0; \\
u(n) &= \phi(n) - \phi(n-1), \quad n = -N_2 + 1, \dots, 0; \\
u(n) &= 0, \quad n \leq -N_2
\end{aligned}$$

where $F(n) = \beta \sum_{l=-N_2+1}^{-1} w(n-l)u(l)$ for $n \geq 0$. We have already noted that $w(n) \geq 0$ for all $n \in \{0, \dots, N_2 - 2\}$ and that $w(n) = 0$ for all $n \geq N_2 - 1$. Since ϕ is non–decreasing, it follows that $u(n) \geq 0$ for all $n \leq -1$ and that $u(0) = \phi(0) - \phi(-1) > 0$ by hypothesis. Therefore for all $n \geq 0$, $F(n) \geq 0$. Since w_1 and w_2 obey (2.2.1) and (2.2.5), there exists $\alpha \in (0, 1)$ which is the unique solution of (2.2.6). Define $\lambda = 1/\alpha > 1$, so that $\lambda > 1$ is given by

$$\beta \sum_{j=0}^{\infty} w(j)\lambda^{-(j+1)} = 1.$$

Applying z–transforms to

$$u(n+1) = F(n) + \beta \sum_{l=0}^{n} w(n-l)u(l), \quad n \geq 0,$$

yields

$$\widetilde{u}(z) - u(0) = z^{-1}\widetilde{F}(z) + \beta z^{-1}\widetilde{w}(z)\widetilde{u}(z).$$

Hence

$$\widetilde{u}(z) = \frac{u(0) + z^{-1}\widetilde{F}(z)}{1 - \beta z^{-1}\widetilde{w}(z)}.$$

Define $u_\lambda(n) = u(n)/\lambda^n$ for $n \in \mathbb{Z}$. Then

$$\widetilde{u}(z) = \sum_{n=0}^{\infty}\left(\frac{z}{\lambda}\right)^{-n} u_\lambda(n) = \widetilde{u_\lambda}(z/\lambda).$$

and

$$\widetilde{u_\lambda}(z) = \frac{u(0) + \lambda^{-1}z^{-1}\widetilde{F}(\lambda z)}{1 - \beta\lambda^{-1}z^{-1}\sum_{n=0}^{\infty}(\lambda z)^{-n}w(n)}.$$

Notice by the definition of $\lambda > 1$ that the denominator tends to zero as $z \to 1$. If the limit

$$l_1 := \lim_{z\to 1} \frac{1 - z^{-1}}{1 - \beta\lambda^{-1}z^{-1}\sum_{n=0}^{\infty}(\lambda z)^{-n}w(n)}$$

exists and is finite, by the final value theorem for z–transforms we have

$$\lim_{n\to\infty}\frac{u(n)}{\lambda^n} = \lim_{n\to\infty} u_\lambda(n) = l_1\left(u(0) + \lambda^{-1}\widetilde{F}(\lambda)\right) = l_1\left(u(0) + \lambda^{-1}\sum_{n=0}^{\infty}\lambda^{-n}F(n)\right).$$

Since $u(0) > 0$ and $F(n) \geq 0$ for all $n \geq 0$, if $l_1 > 0$, then the limit is positive, and we have shown that $\mathbb{P}[\Gamma(\phi) > 0] > 1/2$. Using L'Hôpital's rule we obtain

$$\begin{aligned}
l_1 &= \lim_{z\to 1}\frac{1 - z^{-1}}{1 - \beta\sum_{n=0}^{\infty}\lambda^{-(n+1)}z^{-n-1}w(n)} \\
&= \lim_{z\to 1}\frac{z^{-2}}{\beta\sum_{n=0}^{\infty}\lambda^{-(n+1)}(n+1)z^{-n-2}w(n)} \\
&= \frac{1}{1 + \beta\sum_{n=0}^{\infty}\lambda^{-(n+1)}nw(n)}.
\end{aligned}$$

Since $W(n) \geq 0$, and β and λ are positive the limit l_1 is clearly positive and finite as required. □

2.4 Asymptotic behaviour of the autocovariance function in the Linear Equation

In this section we prove that the autocovariance function in the linear equation is positive. This means once a trend appears in the returns then this trend will persist. It is the continuation of this trend which underlines the bubble/crash dynamics of the system. However, it can also be shown that this correlation decays at an (exact and real) exponential rate. Therefore, although the market is inefficient (due to the presence of correlation) it does not retain a "long memory" of past price trends.

2.4.1 Asymptotic behaviour of δ

In the following Lemma we prove that the rate of exponential decay of $\delta(n)$ is finite and constant where $\delta(n) = r(n) - r(n-1)$. This result is required for the main proof of this section. Throughout this section we assume that

$$\sum_{j=0}^{n} w_1(j) > \sum_{j=0}^{n} w_2(j) \quad \text{for some } n \geq 0. \tag{2.4.1}$$

Lemma 2.4.1. *If* (2.2.3) *holds there exists* $\alpha_0 \in (0,1)$ *defined by*

$$\alpha_0 = 1 + \beta \left(\sum_{k=0}^{N_1-1} \alpha_0^{-k} w_1(k) - \sum_{k=0}^{N_2-1} \alpha_0^{-k} w_2(k) \right) \tag{2.4.2}$$

such that $\lim_{n\to\infty} \delta(n)/\alpha_0^n = \delta_0$*, where* $\delta_0 > 0$ *is given by*

$$\delta_0 = \frac{1}{1 + \alpha_0^{-1}\beta \sum_{k=1}^{N_2-1} k\alpha_0^{-k} \left(\sum_{j=0}^{k} w_1(j) - w_2(j) \right)}. \tag{2.4.3}$$

Proof. We assume $N_1 > 1$. By (2.2.1a), for $n \geq N_2 - 1$ we have $w(n) = 0$. For $0 \leq n \leq N_1 - 1$, as $N_1 < N_2$, we have $w(n) = \sum_{j=0}^{n} [w_1(j) - w_2(j)]$, and so by (2.2.1b), $w(n) \geq 0$ for $n = 0, 1, \dots, N_1 - 1$. Finally for $N_1 \leq n \leq N_2 - 2$, we have

$$w(n) = 1 - \sum_{j=0}^{n} w_2(j),$$

and so $w(n) \geq 0$ for $n = N_1, \dots, N_2 - 2$. Hence $w(n) \geq 0$ for all $n \geq 0$.

The condition that $\beta \sum_{n=0}^{\infty} w(n) < 1$ implies there is a unique $\alpha_0 \in (0,1)$ such that if we define $w_\alpha(n) = \alpha_0^{-n} w(n)$ we have

$$\beta\alpha_0^{-1} \sum_{n=0}^{\infty} w_{\alpha_0}(n) = 1.$$

We now show that such an α must satisfy (2.4.2). Since $w(n) = 0$ for all $n \geq N_2 - 1$ we have

$$\beta\alpha_0^{-1} \sum_{n=0}^{N_2-1} \alpha_0^{-n} w(n) = 1.$$

Using the fact that

$$\sum_{n=j}^{N_2-1} \alpha_0^{-(n-j)} = \frac{1 - (\alpha_0^{-1})^{N_2-j}}{1 - \alpha_0^{-1}},$$

and also that $\sum_{j=0}^{N_2-1} w_1(j) - w_2(j) = 0$ we have

$$\begin{aligned}
1 &= \beta\alpha_0^{-1} \sum_{n=0}^{N_2-1} \alpha_0^{-n} w(n) = \beta\alpha_0^{-1} \sum_{n=0}^{N_2-1} \alpha_0^{-n} \sum_{j=0}^{n\wedge(N_2-1)} \{w_1(j) - w_2(j)\} \\
&= \beta\alpha_0^{-1} \sum_{n=0}^{N_2-1} \alpha_0^{-n} \sum_{j=0}^{n} \{w_1(j) - w_2(j)\} \\
&= \beta\alpha_0^{-1} \sum_{j=0}^{N_2-1} \left(\sum_{n=j}^{N_2-1} \alpha_0^{-(n-j)} \right) \alpha_0^{-j} (w_1(j) - w_2(j)) \\
&= \beta\alpha_0^{-1} \frac{1}{1-\alpha_0^{-1}} \sum_{j=0}^{N_2-1} (1 - (\alpha_0^{-1})^{N_2-j}) \alpha_0^{-j} (w_1(j) - w_2(j)) \\
&= \beta\alpha_0^{-1} \frac{1}{1-\alpha_0^{-1}} \sum_{j=0}^{N_2-1} \alpha_0^{-j} (w_1(j) - w_2(j)) - (\alpha_0^{-1})^{N_2} \frac{\beta\alpha_0^{-1}}{1-\alpha_0^{-1}} \sum_{j=0}^{N_2-1} (w_1(j) - w_2(j)) \\
&= \beta\alpha_0^{-1} \frac{1}{1-\alpha_0^{-1}} \sum_{j=0}^{N_2-1} \alpha_0^{-j} (w_1(j) - w_2(j)) \\
&= \beta\alpha_0^{-1} \frac{1}{1-\alpha_0^{-1}} \left\{ \sum_{j=0}^{N_1-1} \alpha_0^{-j} w_1(j) - \sum_{j=0}^{N_2-1} \alpha_0^{-j} w_2(j) \right\}.
\end{aligned}$$

By rearranging this we see that $\alpha_0 \in (0,1)$ obeys (2.4.2).

Define $\delta_{\alpha_0}(n) = \delta(n)/\alpha_0^n$ for $n \geq 0$. Then by dividing across (2.2.9) by α_0^{n+1}, we get

$$\delta_{\alpha_0}(n+1) = \beta\alpha_0^{-1}\sum_{j=0}^{n} w_{\alpha_0}(j)\delta_{\alpha_0}(n-j), \quad n \geq 0; \quad \delta_{\alpha_0}(0) = 1.$$

By taking the z-transform we have

$$\lim_{n\to\infty} \delta_{\alpha_0}(n) = \frac{1}{1+\beta\alpha_0^{-1}\sum_{j=0}^{\infty} j w_{\alpha_0}(j)}$$

where

$$\beta\alpha_0^{-1}\sum_{j=0}^{\infty} w_{\alpha_0}(j) = 1$$

so as $w_{\alpha_0}(n) = 0$ for all $n \geq N_2 - 1$ we have

$$\lim_{n\to\infty} \delta(n)/\alpha_0^n = \frac{1}{1+\beta\alpha_0^{-1}\sum_{k=1}^{N_2-1} k\alpha_0^{-k}\left\{\sum_{j=0}^{k} w_1(j) - w_2(j)\right\}} =: \delta_0 > 0.$$

Hence $\delta(n)/\alpha_0^n \to \delta_0 > 0$ as $n \to \infty$. $\square$

2.4.2 Asymptotic behaviour of the autocovariance function

In this subsection, we analyse the patterns in the θ–returns, where $\theta > 0$ in the situation where the stability condition (2.2.3) holds. The θ–returns are simply the percentage gains or losses made by investing over a time period of θ units, and are denoted at time n by $Y_\theta(n)$. Let $\theta \geq 1$ and $\Delta \geq 0$ be integers. Extend r to $-\infty, \ldots, -N_2 - 1$ by setting $r(n) = 0$ for $n \leq -N_2 - 1$. If Y is the process given by (2.3.1) we define the process $Y_\theta = \{Y_\theta(n) : n \geq \theta + 1\}$ by

$$Y_\theta(n) := Y(n) - Y(n-\theta), \quad n \geq \theta + 1. \tag{2.4.4}$$

Let us also introduce the sequences r_θ and y_θ by

$$r_\theta(n) = r(n) - r(n-\theta), \quad n \geq 0, \tag{2.4.5}$$

and

$$y_\theta(n) = y(n) - y(n-\theta), \quad n \geq \theta + 1, \tag{2.4.6}$$

where r and y are the sequences given by (2.2.2) and (2.3.7) respectively. If y is the solution of (2.3.7) then the solution Y of (2.3.1) obeys

$$Y(n) = y(n) + \sum_{j=1}^{n} r(n-j)\xi(j), \quad n \geq 1,$$

so we have for all $n \geq \theta + 1$ the identity

$$Y_\theta(n) = y_\theta(n) + \sum_{j=1}^{n} r_\theta(n-j)\xi(j), \quad n \geq \theta + 1. \tag{2.4.7}$$

Theorem 2.4.1. *Let $\beta > 0$ and $N_2 > N_1 \geq 1$ be integers. Let $\theta \geq 1$, $\Delta \geq 0$ be integers and suppose that w_1 and w_2 obey* (2.2.1) *and that β, w_1 and w_2 obey* (2.2.3). *Suppose also the sequence of random variables ξ obeys* (2.3.2) *and let Y be the solution of* (2.3.1) *and Y_θ is the process defined by* (2.4.7). *Suppose that r_θ is given by* (2.4.5). *Then*

(i)

$$Cov(Y_\theta(n), Y_\theta(n+\Delta)) \geq 0, \quad \textit{for all } n \geq \theta + 1. \tag{2.4.8}$$

(ii) For every $\Delta \geq 0$ the limit

$$c_\theta(\Delta) := \lim_{n\to\infty} Cov(Y_\theta(n), Y_\theta(n+\Delta)) = \sigma^2 \sum_{l=0}^{\infty} r_\theta(l) r_\theta(l+\Delta) \tag{2.4.9}$$

exists and is finite.

(iii) There exists a unique $\alpha_0 \in (0,1)$ which obeys (2.4.2) *such that*

$$\lim_{\Delta\to\infty} c_\theta(\Delta)\alpha_0^{-\Delta} = \frac{\sigma^2(1-\alpha_0^\theta)(1-\alpha_0^{-\theta})}{(1-\alpha_0)(1-\alpha_0^{-1})} \times \frac{1}{1-\beta\alpha_0\sum_{j=0}^{N_2-1}\alpha_0^j w(j)} \cdot \frac{1}{1+\alpha_0^{-1}\beta\sum_{k=1}^{N_2-1} k\alpha_0^{-k}\sum_{j=0}^{k} w(j)} > 0. \tag{2.4.10}$$

We make some further observations and comments before the proof is given. An interesting conclusion of the theorem is that the θ–returns are positively autocorrelated. Therefore, even though the returns undergo iterated logarithm behaviour like standard Brownian motion, there is correlation between the increments of the process. The presence of a positive correlation means that trends in the returns have a tendency to persist. This is responsible for the fact that the largest fluctuations of the process Y are greater that those that would be seen if there were no trend–following speculators present. The correlation between returns of horizon length δ decays exponentially in the time lag Δ between successive observations, as $\Delta \to \infty$. Moreover, the exponent in the rate of decay is independent of Δ. Therefore, although the market is informationally inefficient because the future returns are correlated with past returns, the memory of recent events is discounted relatively quickly. This "short memory" is a consequence of the finite memory trading strategies employed by agents. The autocovariance function is positive because $\delta(n)$ is positive and the limit of the autocovariance function exists and is finite because $r_\theta(n)$ is bounded. Finally the exponential rate of decay of the correlation function is calculated by employing the result of lemma 2.4.1.

Remark 2.4.1. The proof of part (i) is straightforward and no outline is given. Part (ii) is proven by using the fact that r_θ is bounded and part (iii) is proven by employing Lemma 2.4.1.

Proof. By (2.4.7), (2.3.2) and the fact that y is deterministic we have $\mathbb{E}[Y_\theta(n)] = y_\theta(n)$ for $n \geq \theta+1$. Therefore for $n \geq \theta+1$ we have

$$\begin{aligned}\text{Cov}(Y_\theta(n), Y_\theta(n+\Delta)) =& \mathbb{E}\left[\sum_{j=1}^{n} r_\theta(n-j)\xi(j) \cdot \sum_{j=1}^{n+\Delta} r_\theta(n+\Delta-j)\xi(j)\right] \\ =& \sum_{j=1}^{n}\sum_{l=1}^{n+\Delta} r_\theta(n-j) r_\theta(n+\Delta-j)\mathbb{E}\left[\xi(j)\xi(l)\right].\end{aligned}$$

As $\Delta \geq 0$ and $\mathbb{E}[\xi(j)\xi(l)] = 0$ when $j = l$ where ξ obeys (2.3.2) we have

$$\text{Cov}(Y_\theta(n), Y_\theta(n+\Delta)) = \sigma^2 \sum_{j=1}^{n} r_\theta(n-j) r_\theta(n+\Delta-j) = \sigma^2 \sum_{l=0}^{n-1} r_\theta(l) r_\theta(l+\Delta).$$

Since we have extended $r(n)$ to all negative values of n, we can consider $\delta(n) = 0$ for all $n \leq 0$ or equivalently $\delta(n) = r(n) - r(n-1)$ for all $n \in \mathbb{Z}$. Hence we notice that

$$r_\theta(n) = \delta(n) + \delta(n-1) + \ldots + \delta(n-\theta+1). \tag{2.4.11}$$

Let $\Delta \geq 1$. Since $\delta(n) \geq 0$ for all $n \in \mathbb{Z}$ we have $r_\theta(n) \geq 0$ for all $n \geq 0$ and $\theta \geq 1$ as so we have that (2.4.8) holds for all $n \geq \theta+1$.

To prove part (ii) note from (2.4.11) that for each fixed $\theta \geq 1$, since $\delta(n) \to 0$ as $n \to \infty$ we have $r_\theta(n) \to 0$ as $n \to \infty$. Also as $\delta \in \ell^1(\mathbb{N};\mathbb{R}^+)$ we have $r_\theta \in \ell^1(\mathbb{N};\mathbb{R}^+)$ for each fixed $\theta \geq 1$. Thus for each $\Delta \geq 0$ we can consider the limit

$$c_\theta(\Delta) := \lim_{n\to\infty} \text{Cov}(Y_\theta(n), Y_\theta(n+\Delta)) = \sigma^2 \sum_{l=0}^{\infty} r_\theta(l) r_\theta(l+\Delta).$$

This limit is finite, because r_θ is bounded and $r_\theta \in \ell^1(\mathbb{N};\mathbb{R}^+)$; this proves (2.4.9).

It remains to prove (2.4.10). We have shown in Lemma 2.4.1 that if (2.2.3) holds there exists $\alpha_0 \in (0,1)$ defined by

$$\alpha_0 = 1 + \beta\left(\sum_{k=0}^{N_1-1} \alpha_0^{-k} w_1(k) - \sum_{k=0}^{N_2-1} \alpha_0^{-k} w_2(k)\right)$$

such that $\lim_{n\to\infty} \alpha_0^{-n}\delta(n) = \delta_0$, where $\delta_0 > 0$ is given by

$$\delta_0 = \frac{1}{1 + \alpha_0^{-1}\beta\alpha \sum_{j=1}^{N_2-1} k\alpha_0^{-k}\left(\sum_{j=0}^{k} w_1(j) - w_2(j)\right)},$$

Therefore by (2.4.11) we have

$$\lim_{n\to\infty} \frac{r_\theta(n)}{\alpha_0^n} = \delta_0 \frac{1-\alpha_0^{-\theta}}{1-\alpha_0^{-1}}. \tag{2.4.12}$$

Since $r_\theta \in \ell^1(\mathbb{N};\mathbb{R}^+)$ and $\alpha_0 \in (0,1)$, we have that

$$\sum_{l=0}^{\infty} r_\theta(l)\alpha_0^l < +\infty.$$

Now

$$c_\theta(\Delta)/\alpha_0^\Delta = \sigma^2 \sum_{l=0}^{\infty} r_\theta(l)\alpha_0^l\left(\alpha_0^{-(\Delta+l)} r_\theta(l+\Delta) - \delta_0\frac{1-\alpha_0^{-\theta}}{1-\alpha_0^{-1}}\right) + \sigma^2\sum_{l=0}^{\infty} r_\theta(l)\alpha_0^l \cdot \delta_0 \frac{1-\alpha_0^{-\theta}}{1-\alpha_0^{-1}}.$$

so as r_θ is summable, by (2.4.12) we have

$$\lim_{\Delta\to\infty} c_\theta(\Delta)/\alpha_0^\Delta = \sigma^2 \sum_{l=0}^{\infty} r_\theta(l)\alpha_0^l \cdot \delta_0\frac{1-\alpha_0^{-\theta}}{1-\alpha_0^{-1}}. \tag{2.4.13}$$

Observe that $r_\theta(0) = r(0) - r(-\theta) = 1$ for $\theta \geq 1$ so the limit on the right hand side of (2.4.13) is positive and moreover finite by the finiteness of $\sum_{l=0}^{\infty} r_\theta(l)\alpha_0^l$. Given that $\delta_0 > 0$ obeys (2.4.3), this formula agrees with (2.4.10).

To compute $\sum_{n=0}^{\infty} r_\theta(n)\alpha_0^n$ notice first from (2.4.11) that

$$r_\theta(n) = \sum_{j=0}^{\theta-1} \delta(n-j)$$

so

$$\begin{aligned}\sum_{n=0}^{\infty} r_\theta(n)\alpha_0^n &= \sum_{n=0}^{\infty}\alpha_0^n\sum_{j=0}^{\theta-1}\delta(n-j) = \sum_{j=0}^{\theta-1}\alpha_0^j\sum_{n=0}^{\infty}\alpha_0^{n-j}\delta(n-j)\\ &= \sum_{j=0}^{\theta-1}\alpha_0^j\sum_{n=j}^{\infty}\alpha_0^{n-j}\delta(n-j) = \sum_{j=0}^{\theta-1}\alpha_0^j\cdot\sum_{l=0}^{\infty}\alpha_0^l\delta(l) = \frac{1-\alpha_0^\theta}{1-\alpha_0}\sum_{n=0}^{\infty}\alpha_0^n\delta(n).\end{aligned}$$

Therefore by (2.4.13) we have

$$\lim_{\Delta\to\infty} c_\theta(\Delta)/\alpha_0^\Delta = \sigma^2\frac{1-\alpha_0^\theta}{1-\alpha_0}\sum_{n=0}^{\infty}\alpha_0^n\delta(n)\cdot\delta_0\frac{1-\alpha_0^{-\theta}}{1-\alpha_0^{-1}}. \tag{2.4.14}$$

It remains to determine $S := \sum_{n=0}^{\infty}\alpha_0^n\delta(n)$. To do this, we multiply across (2.2.9) by α_0^{n+1} for each $n \geq 0$; thus by summing over all n we get

$$\sum_{n=0}^{\infty}\alpha_0^{n+1}\delta(n+1) = \beta\alpha_0\sum_{n=0}^{\infty}\sum_{k=0}^{n}\alpha_0^{n-k}w(n-k)\alpha_0^k\delta(k) = \beta\alpha_0\sum_{j=0}^{\infty}\alpha_0^j w(j)\cdot\sum_{k=0}^{\infty}\alpha_0^k\delta(k).$$

Now

$$S - 1 = \sum_{j=0}^{\infty} \alpha_0^j \delta(j) - 1 = \sum_{n=0}^{\infty} \alpha_0^{n+1} \delta(n+1) + \alpha_0^0 \delta(0) - 1 = \sum_{n=0}^{\infty} \alpha_0^{n+1} \delta(n+1).$$

Therefore we have

$$S - 1 = \beta\alpha_0 \sum_{j=0}^{\infty} \alpha_0^j w(j) \cdot \sum_{k=0}^{\infty} \alpha_0^k \delta(k) = \beta\alpha_0 \sum_{j=0}^{\infty} \alpha_0^j w(j) \cdot S = \beta\alpha_0 \sum_{j=0}^{N_2-1} \alpha_0^j w(j) \cdot S.$$

Hence

$$\sum_{n=0}^{\infty} \alpha_0^n \delta(n) = S = \frac{1}{1 - \beta\alpha_0 \sum_{j=0}^{N_2-1} \alpha_0^j w(j)}. \tag{2.4.15}$$

Combining (2.4.14) and (2.4.15) we obtain

$$\lim_{\Delta\to\infty} c_\theta(\Delta)/\alpha_0^\Delta = \sigma^2 \frac{1-\alpha_0^\theta}{1-\alpha_0} \cdot \delta_0 \frac{1-\alpha_0^{-\theta}}{1-\alpha_0^{-1}} \cdot \frac{1}{1 - \beta\alpha_0 \sum_{j=0}^{N_2-1} \alpha_0^j w(j)},$$

as required. □

2.5 Cumulative Returns In The Nonlinear Equation

It is convenient in market models which seek to aggregate the behaviour of individual agents to assume that agents' demands are log linear in the price, and therefore linear in the returns. However, this assumption derives from a particular appetite to risk, and it is certainly possible to consider different demand functions. In this section (and also in Chapter 4) we suppose that the linear response in excess demand to the returns is essentially preserved when the returns are large positive or negative, but not for moderate levels of the returns. Therefore the traders have essentially log–linear demand when the market, in their opinion, is far from equilibrium. We allow for diverse attitudes towards risk by permitting the response of demand to returns to vary nonlinearly, allowing much greater flexibility among the investors when returns are closer to equilibrium levels. This has the impact of allowing investors in our model to be very responsive or very insensitive to changes in the returns when the market is relatively quiet. We do not base our models of investor behaviour on the existence of utility functions: however, allowing g to be almost any nonlinear function for a wide (though finite) range of returns is equivalent in the the utility framework to allowing both risk seeking and risk aversion of widely varying degrees among the investors. For reasons of modelling flexibility, and to test the robustness of the model to changes in hypotheses, we therefore study the following nonlinear stochastic difference equation for $n \in \mathbb{N}$

$$X(n+1) = X(n) + \sum_{j=0}^{N_1-1} w_1(j) g(X(n-j)) - \sum_{j=0}^{N_2-1} w_2(j) g(X(n-j)) + \xi(n+1), \tag{2.5.1a}$$

$$X(n) = \phi(n), \quad n = -N_2+1, \dots, 0. \tag{2.5.1b}$$

$g : \mathbb{R} \to \mathbb{R}$ is presumed have the following properties

$$g \in C(\mathbb{R};\mathbb{R}), \lim_{x\to\infty} \frac{g(x)}{x} = \lim_{x\to-\infty} \frac{g(x)}{x} = \beta \quad \text{for some } \beta \geq 0. \tag{2.5.2}$$

The preservation of the log–linear response to the returns is embodied by the second part of hypothesis (2.5.2). In practice, we might expect g to be non–decreasing, but this hypothesis is not needed in our proofs. We show that despite the presence of the non-linearity of g the returns undergo dynamics which are consistent with a correlated SBM or a bubble or crash characterised by exponential growing returns.

2.5.1 Law of the iterated logarithm for nonlinear model

We now show that if the conditions of Theorem 2.3.1 hold, the a.s. partial extrema of the solution of (2.5.1) grow exactly as those of the solution of (2.3.1). Moreover, the distance between these solutions is asymptotically negligible relative to the size of partial extrema, which are themselves consistent with the extrema of a random walk.

Theorem 2.5.1. *Let $\beta > 0$, N_1 and N_2 be positive integers with $N_1 > N_2$, w_1 and w_2 obey* (2.2.1)*, β obey* (2.2.3)*, and g obey* (2.5.2)*. If ξ obeys* (2.3.2)*, and Y obeys* (2.3.1)*, then the solution of* (2.5.1) *obeys*

$$\lim_{n\to\infty} \frac{|X(n) - Y(n)|}{\sqrt{2n\log\log n}} = 0, \tag{2.5.3}$$

$$\limsup_{n\to\infty} \frac{X(n)}{\sqrt{2n\log\log n}} = \frac{|\sigma|}{1 - \beta\left(\sum_{j=0}^{N_2-1} jw_2(j) - \sum_{j=0}^{N_1-1} jw_1(j)\right)}, \quad a.s. \tag{2.5.4}$$

$$\liminf_{n\to\infty} \frac{X(n)}{\sqrt{2n\log\log n}} = -\frac{|\sigma|}{1 - \beta\left(\sum_{j=0}^{N_2-1} jw_2(j) - \sum_{j=0}^{N_1-1} jw_1(j)\right)}, \quad a.s. \tag{2.5.5}$$

Remark 2.5.1. The proof of this theorem hinges on writing the nonlinear resolvent $X(n)$ in terms of the linear resolvent $Y(n)$ and another function $Z(n)$. Firstly we show that $Z(n)$ is bounded and secondly we show that its limit tends to zero. This enables us to conclude that the limit of $X(n)$ equals the limit of $Y(n)$.

Proof. Define $Z(n) = X(n) - Y(n)$ then

$$\begin{aligned} Z(n+1) - Z(n) = &\sum_{j=0}^{N_1-1} w_1(j)\left(g(X(n-j)) - \beta Y(n-j)\right) \\ &- \sum_{j=0}^{N_2-1} w_2(j)\left(g(X(n-j)) - \beta Y(n-j)\right). \end{aligned}$$

Using that $G(n+1) = \sum_{j=0}^{N_1-1} w_1(j)\gamma(X(n-j)) - \sum_{j=0}^{N_2-1} w_2(j)\gamma(X(n-j))$ and $\gamma(x) = g(x) - \beta x$ then

$$Z(n+1) - Z(n) = G(n+1) + \beta\left(\sum_{j=0}^{N_1-1} w_1(n-j)Z(j) - \sum_{j=0}^{N_2-1} w_2(n-j)Z(j)\right).$$

For $n \geq 1$,

$$\begin{aligned} Z(n) &= \sum_{j=0}^{n-1} r(n-1-j)G(j+1) \\ &= \sum_{j=0}^{n-1} r(n-1-j)\sum_{k=0}^{N_1-1} w_1(k)\gamma(X(j-k)) \\ &\quad - \sum_{j=0}^{n-1} r(n-1-j)\sum_{k=0}^{N_2-1} w_2(k)\gamma(X(j-k)). \end{aligned}$$

For $n \geq 2$, $n \geq N_1$ and letting $l = j - k$,

$$\begin{aligned} &\sum_{j=0}^{n-1} r(n-1-j)\sum_{k=0}^{N_1-1} w_1(k)\gamma(X(j-k)) \\ &= \sum_{l=-N_1+1}^{n-1} \gamma(X(l)) \sum_{k=0\vee -l}^{(N_1-1)\wedge(n-1-l)} w_1(k)r(n-1-l-k) \\ &= \sum_{l=-N_1+1}^{-1} \gamma(X(l)) \sum_{k=0\vee -l}^{(N_1-1)\wedge(n-1-l)} w_1(k)r(n-1-l-k) \\ &\quad + \sum_{l=0}^{n-1} \gamma(X(l)) \sum_{k=0\vee -l}^{(N_1-1)\wedge(n-1-l)} w_1(k)r(n-1-l-k). \end{aligned}$$

Let $m = n - l - 1$ and define

$$T_1(n) = \sum_{l=-N_1+1}^{-1} \gamma(X(l)) \sum_{k=0\vee -l}^{(m)\wedge(N_1-1)} w_1(k)r(m-k),$$

then

$$\begin{aligned}
&\sum_{j=0}^{n-1} r(n-1-j) \sum_{m=0}^{N_1-1} w_1(m)\gamma(X(j-m)) \\
&= T_1(n) + \frac{1}{\beta}\sum_{m=0}^{n-1} \gamma(X(n-m-1)) \sum_{k=0}^{(N_1-1)\wedge m} \beta w_1(k)r(m-k) \\
&= T_1(n) + \sum_{m=0}^{N_1-2} \gamma(X(n-m-1)) \sum_{k=0}^{(N_1-1)\wedge m} w_1(k)r(m-k) \\
&\qquad + \frac{1}{\beta}\sum_{m=N_1-1}^{n-1} \gamma(X(n-m-1)) \sum_{k=0}^{N_1-1} \beta w_1(k)r(m-k).
\end{aligned}$$

Define $S_1(n) = \sum_{m=0}^{N_1-2} \gamma(X(n-m-1)) \sum_{k=0}^{(N_1-1)\wedge m} w_1(k)r(m-k)$, then

$$= T_1(n) + S_1(n) + \frac{1}{\beta}\sum_{m=N_1-1}^{n-1} \gamma(X(n-m-1)) \sum_{k=0}^{N_1-1} \beta w_1(k)r(m-k).$$

Introducing analogous functions T_2 and S_2, and define $f_1(n) = T_1(n) - T_2(n)$, and $f_2(n) = S_1(n) - S_2(n)$. Using that $\frac{1}{\beta}\delta(m+1) = \sum_{k=0}^{N} w(k)r(m-k)$, where $w(k) = w_1(k) - w_2(k)$

$$\begin{aligned}
Z(n) = f_1(n) &+ \sum_{m=N_1-1}^{N_2-2} \gamma(X(n-m-1)) \sum_{k=0}^{N_1-1} w_1(k)r(m-k) \\
&\qquad + f_2(n) + \frac{1}{\beta}\sum_{m=N_2-1}^{n-1} \gamma(X(n-m-1))\delta(m+1).
\end{aligned}$$

Noting that $S_j(n) = \sum_{m=0}^{N_j-2} \gamma(X(n-m-1)) \sum_{k=0}^{m} w_j(k)r(m-k)$, we have that

$$|f_2(n)| \le \sum_{m=0}^{N_1-2} |\gamma(X(n-m-1))| C_1(m) + \sum_{m=0}^{N_2-2} |\gamma(X(n-m-1))| C_2(m).$$

Letting $C_3 := \sum_{k=0}^{N_1-1} w_1(k)r(m-k)$ then

$$\begin{aligned}
|Z(n)| &\le |f_1(n)| + \sum_{m=N_1-1}^{N_2-2} |\gamma(X(n-m-1))| C_3(m) \\
&\qquad + |f_2(n)| + \frac{1}{\beta}\sum_{m=N_2-1}^{n-1} |\gamma(X(n-m-1))|\delta(m+1) \\
&\le |f_1(n)| + \sum_{m=0}^{n-1} |\gamma(X(n-m-1))|\kappa(m),
\end{aligned}$$

where $C_4 := C_1 + C_2 + C_3$, $\kappa(m) := C_4(m)$, $m \le N_2 - 2$ and $\kappa(m) := \delta(m+1)/\beta$, $m \ge N_2 - 1$. By (2.5.2), for each $\varepsilon > 0$ there is $L(\varepsilon) > 0$ such that $|\gamma(x)| \le L(\varepsilon) + \varepsilon|x|$, $x \in \mathbb{R}$. Defining $f_2(n) = |f_1(n)| + L(\varepsilon)\sum_{l=0}^{n} \kappa(l)$, we get

$$|Z(n)| \le f_2(n) + \varepsilon\sum_{l=0}^{n} \kappa(l)|Y(n-1-l)| + \varepsilon\sum_{l=0}^{n} \kappa(l)|Z(n-1-l)|.$$

Since $\delta \in \ell^1(\mathbb{N};\mathbb{R}^+)$, κ is summable. Moreover, as $\lim_{n\to\infty} f_1(n)$, $\lim_{n\to\infty} f_2(n)$ exists. Defining $f_3(n) := f_2(n) + \max_{j=1,\dots,N_2-1}|Z(j)|$, we have

$$|Z(n)| \le f_3(n) + \varepsilon\sum_{j=0}^{n-1}\kappa(n-1-j)|Y(j)| + \varepsilon\sum_{j=0}^{n-1}\kappa(n-1-j)|Z(j)| \quad \forall \quad n \ge 0.$$

Fix $\varepsilon > 0$ so that $\varepsilon\sum_{n=0}^{\infty}\kappa(n) < 1/2$. Define ρ by $\rho(0) = 1$, $\rho(n+1) = \varepsilon(\kappa * \rho)(n)$, $n \in \mathbb{N}$, and z by

$$z(n+1) = f_3(n+1) + \varepsilon(\kappa * |Y|)(n) + \varepsilon(\kappa * z)(n) \quad \text{for} \quad n \in \mathbb{N},$$

and where $z(0) = 0$. Therefore $|Z(n)| \le z(n)$ and

$$z(n) = \sum_{j=1}^{n}\rho(n-j)\left(f_3(j) + \varepsilon\sum_{k=0}^{j-1}\kappa(j-1-k)|Y(k)|\right).$$

As $\rho \in \ell^1(\mathbb{N};(0,\infty))$, there is an f_4 obeying $\lim_{n\to\infty} f_4(n) = 0$ and $|Z(n)| \le f_4(n) + \varepsilon\sum_{k=0}^{n-1}(\rho*\kappa)(n-k-1)|Y(k)|$. Therefore, from 2.3.4 and Lemma 2.7.1 it follows that

$$\limsup_{n\to\infty}\frac{|Z(n)|}{\sqrt{2n\log\log n}} \le \varepsilon c'\sum_{k=0}^{\infty}(\rho*\kappa)(k) = \varepsilon c'\sum_{k=0}^{\infty}\kappa(k)\frac{1}{1-\varepsilon\sum_{k=0}^{\infty}\kappa(k)},$$

where $c' > 0$ is the righthand side of 2.3.4. Since ε can be taken as small as required, and the last inequality holds pathwise, we have (2.5.3). (2.5.4) is an immediate consequence of (2.5.3) and Theorem 2.3.1. □

2.5.2 Presence of bubbles and crashes in nonlinear model

Our next result shows that if the conditions of Theorem 2.5.1 hold and g obeys not only $g(x) = \beta x + \text{o}(x)$ as $|x| \to \infty$ but *a fortiori* $g(x) = \beta x + \text{O}(|x|^\nu)$ as $|x| \to \infty$ for some $\nu \in (0,1)$, then the a.s. rate of growth of the solution of (2.5.1) is exactly that of the solution of (2.3.1) in the sense that both $\alpha^n Y(n)$ and $\alpha^n X(n)$ tend to finite limits a.s. The proof of this result is inspired by an argument in [10].

Theorem 2.5.2. *Let $\beta > 0$, N_1 and N_2 be positive integers with $N_2 > N_1$, w_1 and w_2 obey* (2.2.1)*, and β obeys* (2.2.5)*. Suppose also that $\alpha \in (0,1)$ is given by* (2.2.6)*.If g obeys* (2.5.2)*, ξ obeys* (2.3.2)*, and X obeys* (2.5.1)*, then $\limsup_{n\to\infty}\log|X(n)|/n \le \log(1/\alpha)$, a.s. If, moreover,*

$$\text{There exists } \beta \ge 0 \text{ and } \nu \in (0,1) \text{ such that } \limsup_{|x|\to\infty}\left|\frac{g(x)-\beta x}{|x|^\nu}\right| < +\infty \tag{2.5.6}$$

then $\lim_{n\to\infty}\alpha^n X(n)$ exists and is finite a.s.

Remark 2.5.2. For this proof we write the resolvent $X(n)$ interms of the linear deterministic equation and varaiation of constants formula. Using this information we show that $X(n)$ is bounded and summable. From this we can show that $\limsup_{n\to\infty}\log|X(n)|/n \le \log(1/\alpha)$ almost surely. Next we add on and take away the exponential rate of growth of the characteristic equation from the modified form of $X(n)$. Using this new equation we show that the limit of the exponential rate of growth exists and is finite.

Proof. Now

$$X(n+1) = X(n) + \sum_{j=0}^{N_1-1}w_1(j)g(X(n-j)) - \sum_{j=0}^{N_2-1}w_2(j)g(X(n-j)) + \xi(n+1).$$

But $g(X) = \gamma(X) + \beta X$,

$$\begin{aligned}X(n+1) = X(n) &+ \beta\sum_{j=0}^{N_1-1}w_1(j)X(n-j) - \beta\sum_{j=0}^{N_2-1}w_2(j)X(n-j)\\ &+ \sum_{j=0}^{N_1-1}w_1(j)\gamma(X(n-j)) - \sum_{j=0}^{N_2-1}w_2(j)\gamma(X(n-j)) + \xi(n+1).\end{aligned}$$

Define

$$F(n+1) = \sum_{j=0}^{N_1-1} w_1(j)\gamma(X(n-j)) - \sum_{j=0}^{N_2-1} w_2(j)\gamma(X(n-j)),$$

then

$$X(n+1) = X(n) + \beta \sum_{j=0}^{N_1-1} w_1(j)X(n-j) - \beta \sum_{j=0}^{N_2-1} w_2(j)X(n-j) + F(n+1) + \xi(n+1).$$

If y is the solution of $y(n+1) = y(n) + \beta \left(\sum_{j=0}^{N_1-1} w_1(j)y(n-j) - \sum_{j=0}^{N_2-1} w_2(j)y(n-j) \right)$ we have

$$X(n) = y(n) + \sum_{j=1}^{n} r(n-j)[F(j) + \xi(j)] \quad n \geq 1.$$

Let $\eta > 0$ and multiply both sides of the equation by $(\alpha/(1+\eta))^n$, for $n \geq 1$

$$(\alpha/(1+\eta))^n X(n) = (\alpha/(1+\eta))^n y(n) + (\alpha/(1+\eta))^n \sum_{j=1}^{n} r(n-j)[F(j) + \xi(j)].$$

Define $\widetilde{X}(n) = (\alpha/(1+\eta))^n X(n)$, $\widetilde{y}(n) = (\alpha/(1+\eta)^n y(n)$, $\widetilde{r}(n) = (\alpha/(1+\eta))^n r(n)$ and $\widetilde{F}(n) = (\alpha/(1+\eta))^n F(n)$,

$$\widetilde{X}(n) = \widetilde{y}(n) + (1+\eta)^{-n} \sum_{j=1}^{n} r(n-j)\alpha^{n-j} \cdot \alpha^j \xi(j) + \sum_{j=1}^{n} \widetilde{r}(n-j)\widetilde{F}(j),$$

Let $x_1(n) = \widetilde{y}(n) + (1+\eta)^{-n} \sum_{j=1}^{n} r(n-j)\alpha^{n-j} \cdot \alpha^j \xi(j)$ then

$$\widetilde{X}(n) = x_1(n) + \sum_{j=1}^{n} \widetilde{r}(n-j)\widetilde{F}(j).$$

By Lemma 2.2.1 and Theorem 2.3.2, we have $\widetilde{y}, \widetilde{r} \in \ell^1(\mathbb{N};\mathbb{R})$. Moreover, as ξ_α is a.s. summable, $x_1 \in \ell^1(\mathbb{N};\mathbb{R})$ a.s. Now, for every $\varepsilon > 0$, with $w_+(n) = w_1(n) + w_2(n)$, and using the fact that $\max w_+(n) = 2$

$$|F(n+1)| \leq \sum_{j=0}^{N_1-1} w_1(j)(L(\varepsilon) + \varepsilon|X(n-j)|) + \sum_{j=0}^{N_2-1} w_2(j)(L(\varepsilon) + \varepsilon|X(n-j)|)$$
$$\leq 2L(\varepsilon) + \varepsilon \sum_{j=0}^{N_2-1} w_+(j)|X(n-j)|.$$

Hence

$$|\widetilde{F}(n+1)| \leq 2L(\varepsilon)(\alpha/(1+\eta))^{n+1} + \varepsilon(\alpha/(1+\eta)) \sum_{j=0}^{N_2-1} \widetilde{w}_+(j)|\widetilde{X}(n-j)|,$$

where $\widetilde{w}_+(n) = (\alpha/(1+\eta)^n w_+(n)$. Then

$$|\widetilde{X}(n)| \leq |x_1(n)| + 2L(\varepsilon) \sum_{j=1}^{n} |\widetilde{r}(n-j)|(\alpha/(1+\eta))^j$$
$$+ \varepsilon(\alpha/(1+\eta)) \sum_{j=1}^{n} |\widetilde{r}(n-j)| \sum_{l=0}^{N_2-1} \widetilde{w}_+(l)|\widetilde{X}(j-1-l)|.$$

Let

$$x_2(n) := |x_1(n)| + 2L(\varepsilon)\sum_{j=1}^{n} |\widetilde{r}(n-j)|(\alpha/(1+\eta))^j,$$

thus

$$|\widetilde{X}(n)| \leq x_2(n) + \varepsilon(\alpha/(1+\eta))\sum_{j=1}^{n} |\widetilde{r}(n-j)| \sum_{l=0}^{N_2-1} \widetilde{w}_+(l)|\widetilde{X}(j-1-l)|.$$

Let $k = j - l$,

$$|\widetilde{X}(n)| \leq x_2(n) + \varepsilon\alpha \sum_{k=-N_2+2}^{n} \sum_{j=1\vee k}^{k+N_2-1} |\widetilde{r}(n-j)|\widetilde{w}_+(j-k)|\widetilde{X}(k-1)|.$$

Splitting the double summation

$$|\widetilde{X}(n)| \leq x_2(n) + \varepsilon\alpha \sum_{k=-N_2+2}^{0} \sum_{j=1}^{k+N_2-1} |\widetilde{r}(n-j)|\widetilde{w}_+(j-k)|\widetilde{\phi}_0(k-1)| \\ + \varepsilon\alpha\sum_{k=1}^{n}\sum_{j=0}^{N_2-1} |\widetilde{r}(n-j)|\widetilde{w}_+(j-k)|\widetilde{X}(k-1)|.$$

Define $x_3(n) := x_2(n) + \varepsilon\alpha\sum_{k=-N_2+2}^{0}\sum_{j=1}^{k+N_2-1} |\widetilde{r}(n-j)|\widetilde{w}_+(j-k)|\widetilde{\phi}_0(k-1)|$ and $r_*(n) = \sum_{l=0}^{N_2-1} |\widetilde{r}(n-l)|\widetilde{w}_+(l)$,

$$|\widetilde{X}(n)| \leq x_3(n) + \varepsilon\alpha\sum_{k=1}^{n} r_*(n-k)|\widetilde{X}(k-1)|.$$

Since x_3 is summable, $\widetilde{X}$ is summable provided $\varepsilon\alpha\sum_{k=1}^{\infty}\widetilde{r}_*(k) < 1$. Since the sum is independent of $\varepsilon > 0$, for every $\eta > 0$, there is a $\varepsilon(\eta) > 0$ such that $\widetilde{X}$ is summable. Hence for each $\eta > 0$, $X(n)(\alpha/(1+\eta))^n \to 0$ as $n \to \infty$, and so $\limsup_{n\to\infty} \log|X(n)|/n \leq \log(1/\alpha)$ on each sample path in an a.s. event. Therefore the first assertion holds. To prove that $\alpha^n X(n)$ tends to a finite limit, we write

$$\alpha^n X(n) = \alpha^n y(n) + \sum_{j=1}^{n}(\alpha^{n-j} r(n-j) - R^*)(\alpha^j\xi(j) + \alpha^j F(j)) \\ + \sum_{j=1}^{n} R^*\alpha^j\xi(j) + \sum_{j=1}^{n} R^*\alpha^j F(j).$$

Since $r_\alpha(n) \to R^*$, and $\xi_\alpha \in \ell^1(\mathbb{N};\mathbb{R})$, we see that the righthand side tends to a finite limit once $F_\alpha \in \ell^1(\mathbb{N};\mathbb{R})$. Note that (2.5.6) implies that there exist $c_0 > 0$ and $c_1 > 0$ such that $|\gamma(x)| \leq c_0 + c_1|x|^\mu$. Fix $\eta \in (0, (1/\alpha)^{1/\mu-1} - 1)$. Thus

$$\alpha^{n+1}|F(n+1)| \leq 2c_0\alpha^{n+1} + c_1\sum_{j=0}^{N_2-1} w_+(j)|\widetilde{X}(n-j)|^\mu(\alpha/(1+\eta))^{-\mu(n-j)}\alpha^{n+1} \\ = c_1\alpha\sum_{j=0}^{N_2-1} w_+(j)(\alpha/(1+\eta))^{\mu j}|\widetilde{X}(n-j)|^\mu(\alpha^{1-\mu}/(1+\eta)^{-\mu})^n \\ + 2c_0\alpha^{n+1}.$$

Since $\widetilde{X}$ is bounded, there is a finite random variable C_5 such that

$$\alpha^{n+1}|F(n+1)| \leq 2c_0\alpha^{n+1} + C_5(\eta)\alpha'(\eta)^n,$$

where $\alpha'(\eta) := \alpha^{1-\mu}/(1+\eta)^{-\mu} \in (0,1)$. Hence $F_\alpha \in \ell^1(\mathbb{N};\mathbb{R})$, and so $\alpha^n X(n)$ tends to an a.s. finite limit, a.s. □

2.6 Large Deviations Of The Incremental Returns

In what follows, we find it convenient to introduce for $\Delta \in \mathbb{N}$ the process ξ_Δ given by

$$\xi_\Delta(n+1) = \sum_{j=n-\Delta+1}^{n} \xi(j+1). \tag{2.6.1}$$

We presume in this section that each variate in the process ξ has polynomially decaying distribution function F.

$$\text{There exists } \mu > 0, c_+, c_- > 0 \text{ such that } \lim_{x\to\infty}(1-F(x))x^\mu = c_+, \quad \lim_{x\to\infty} F(-x)|x|^\mu = c_-. \tag{2.6.2}$$

Since the ξ's are independent, we may apply the Borel–Cantelli lemma to establish the size of the largest fluctuations of the processes ξ and ξ_Δ.

Lemma 2.6.1. *Suppose that ξ obeys* (2.3.2) *and* (2.6.2)*.*

(i) Let $\gamma_- : \mathbb{N} \to (0,\infty)$ be a sequence such that

$$\sum_{n=1}^{\infty} \gamma_-^{-\mu}(n) = \infty. \tag{2.6.3}$$

Then $\limsup_{n\to\infty} |\xi(n)|/\gamma_-(n) = \infty$, a.s. and for each $\Delta \in \mathbb{N}$, if ξ_Δ is the process defined in (2.6.1)*, we have $\limsup_{n\to\infty} |\xi_\Delta(n\Delta)|/\gamma_-(n) = \infty$, a.s.*

(ii) Let $\gamma_+ : \mathbb{N} \to (0,\infty)$ be a sequence such that

$$\sum_{n=1}^{\infty} \gamma_+^{-\mu}(n) < +\infty \tag{2.6.4}$$

Then $\limsup_{n\to\infty} |\xi(n)|/\gamma_+(n) = 0$, a.s. and for each $\Delta \in \mathbb{N}$, if ξ_Δ is the process defined in (2.6.1)*, we have $\limsup_{n\to\infty} |\xi_\Delta(n)|/\gamma_+(n) = 0$, a.s.*

Proof. Apart from the claim that $\limsup_{n\to\infty} |\xi_\Delta(n\Delta)|/\gamma_-(n) = \infty$, a.s., the other claims are straightforward consequences of the Borel–Cantelli lemma and (2.6.2), (2.6.3) and (2.6.4). We prove the remaining claim. Let $\Delta \in \mathbb{N}$ and define

$$\zeta_\Delta(n) := \xi_\Delta(n\Delta) = \sum_{j=(n-1)\Delta+1}^{n\Delta} \xi(j+1).$$

Then, $(\zeta_\Delta(n))_{n\geq 1}$ is a sequence of independently and identically distributed random variables. Since F is the distribution function of $\xi(\cdot)$ and the ξ's are independent, for any $c > 0$ we have

$$\mathbb{P}[|\zeta_\Delta(n)| > c\gamma_-(n)] = 1 - F^{(\star\Delta)}(c\gamma_-(n)) + 1 - F_-^{(\star\Delta)}(c\gamma_-(n))$$

where F_- is the distribution function of $-\xi(\cdot)$, and $F^{(\star\Delta)}$, $F_-^{(\star\Delta)}$ are the Δ–fold convolutions of F and F_-, respectively. By (2.6.2), $(1-F(x))x^\mu \to c_+$ and $(1-F_-(x))x^\mu = F(-x)x^\mu \to c_-$ as $x \to \infty$. Thus the right tail $1-F$ (resp. $1-F_-$) of the distribution F (resp. F_-) is *regularly varying at infinity* (see below for a definition), implying that

$$\lim_{x\to\infty} \frac{1-F^{(\star\Delta)}(x)}{1-F(x)} = \Delta, \quad \lim_{x\to\infty} \frac{1-F_-^{(\star\Delta)}(x)}{1-F_-(x)} = \Delta,$$

by appealing to e.g., Feller [34, Chapter VIII.8, p.279]. Since $\gamma_-(n) \to \infty$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{1-F^{(\star\Delta)}(c\gamma_-(n))}{\gamma_-(n)^{-\mu}} = \lim_{n\to\infty} \frac{1-F^{(\star\Delta)}(c\gamma_-(n))}{1-F(c\gamma_-(n))} \cdot \frac{1-F(c\gamma_-(n))}{(c\gamma_-(n))^{-\mu}} \cdot c^{-\mu} = \Delta c_+ c^{-\mu}.$$

Similarly $\lim_{n\to\infty}(1-F_-^{(\star\Delta)}(c\gamma_-(n))/\gamma_-(n)^{-\mu} = \Delta c_- c^{-\mu}$. Hence

$$\mathbb{P}[|\zeta_\Delta(n)| > c\gamma_-(n)]/\gamma_-(n)^{-\mu} \to \Delta c_+ c^{-\mu} + \Delta c_- c^{-\mu} \quad \text{as} \quad n\to\infty,$$

and so, by (2.6.3), $\sum_{n=1}^\infty \mathbb{P}[|\zeta_\Delta(n)| > c\gamma_-(n)] = \infty$ for all $c > 0$. Therefore, by the Borel–Cantelli lemma and the independence of $(\zeta_\Delta(n))_{n\geq 1}$, we have that $\limsup_{n\to\infty} |\zeta_\Delta(n)|/\gamma_-(n) \geq c$ a.s. Letting $c\to\infty$ through the integers gives $\limsup_{n\to\infty} |\zeta_\Delta(n)|/\gamma_-(n) = \infty$ a.s., which is the required result. □

It is possible for $m \in \mathbb{N}$ to choose functions $\gamma_{-,m}$ and $\gamma_{+,m}$ which obey these properties, for example

$$\gamma_{-,1}(n) = [n\log n]^{1/\alpha}, \ldots \gamma_{-,m}(n) = \left(n\prod_{j=1}^{m}\log_j n\right)^{1/\alpha} \tag{2.6.5a}$$

$$\gamma_{+,1}(n) = [n(\log n)^{1+\epsilon}]^{1/\alpha}, \ldots, \gamma_{+,m}(n) = \left(n\prod_{j=1}^{m-1}\log_j n \cdot (\log_m n)^{1+\epsilon}\right)^{1/\alpha}, \tag{2.6.5b}$$

where $\epsilon > 0$ is arbitrary, and we have used for $x > 0$ and $j \in \mathbb{N}$ the recursive notation $\log_j$ to signify the iterated composition of the natural logarithm function, according to $\log_1 x := \log x$, $\log_j x = \log(\log_{j-1} x)$ for $j \geq 2$.

2.6.1 Large fluctuations of the incremental returns in the linear model

Under the condition (2.6.2) on ξ, we determine for $\Delta \in \mathbb{N}$ the size of the largest fluctuations of the Δ–*returns process* $Y_\Delta = \{Y_\Delta(n) : n \geq -N_2 + 1 + \Delta\}$ where and Y is the process defined by (2.3.1) and

$$Y_\Delta(n) = Y(n) - Y(n-\Delta). \tag{2.6.6}$$

Before stating our main result on the rate of growth of the a.s. partial maxima of Y_Δ, we recall (see e.g., Feller [34, Chapter VIII]) that $h : [0,\infty) \to \mathbb{R}$ is *regularly varying at infinity (with index $\eta \in \mathbb{R}$)* if for all $\lambda > 0$ we have $\lim_{x\to\infty} h(\lambda x)/h(x) = \lambda^\eta$. Furthermore, if h is regularly varying, then $\lim_{x\to\infty} h(x-1)/h(x) = 1$.

Theorem 2.6.1. *Let $\beta > 0$, N_1 and N_2 be positive integers with $N_2 > N_1$, w_1 and w_2 obey* (2.2.1)*, β obey* (2.2.3)*. Let ξ obey* (2.3.2) *and* (2.6.2)*, and Y_Δ obey* (2.6.6)*.*

(i) If γ_- is regularly varying at infinity and obeys (2.6.3)*, then for every $\Delta \in \mathbb{N}$, $\limsup_{n\to\infty} |Y_\Delta(n)|/\gamma_-(n) = \infty$ a.s.*

(ii) If γ_+ is regularly varying at infinity and obeys (2.6.4)*, then for every $\Delta \in \mathbb{N}$, $\limsup_{n\to\infty} |Y_\Delta(n)|/\gamma_+(n) = 0$ a.s.*

Remark 2.6.1. To begin the proof we rewrite $Y_\Delta(n)$ using Lemma 2.6.2. Combining part (i) of Lemma 2.6.1 with a contradiction argument we prove part (i) of the theorem. For the second part of the proof we write $Y_\Delta(n)$ in terms of the determinstic equation and an alternative equation (which is defined in the proof) and by employing Lemma 2.7.1 we prove part (ii).

We notice that the examples of functions γ_+ and γ_- in (2.6.5) which obey (2.6.4) and (2.6.3) are both regularly varying at infinity with index $1/\mu > 0$. These sequences show that it is possible to determine the rate of growth of the a.s. partial maxima of Y_Δ to within an arbitrary iterated logarithmic factor. Moreover, the a.s. upper and lower bounds on the rate of growth of the partial maxima are exactly the same as those which apply to the innovation (or "news") process ξ. The key to the proof of Theorem 2.6.1 is the development of a linear difference equation for the Δ–increment, where $\Delta \in \mathbb{N}$. Let $N \in \mathbb{N}$ and $a = \{a(j) : j = 0, 1, \ldots, N-1\}$ be a real sequence. We consider

$$V(n+1) = V(n) + \sum_{j=0}^{N-1} a(k)V(n-k) + \xi(n+1), \quad n \geq 0. \tag{2.6.7}$$

and its Δ–increment $V_\Delta(n) = V(n) - V(n-\Delta)$.

Lemma 2.6.2. *Suppose a obeys $\sum_{k=0}^{N-1} a(k) = 0$. Define $A(n) = \sum_{j=0}^{N-2} a(k)$, $n = 0, \dots, N-2$. If ξ_Δ is defined by* (2.6.1)*, and V by* (2.6.7)*, then V_Δ given by $V_\Delta(n) = V(n) - V(n-\Delta)$ obeys $V_\Delta(n+1) = \sum_{k=0}^{N-2} A(k)V_\Delta(n-k) + \xi_\Delta(n+1)$.*

This is a special case of Lemma 2.6.3, which is stated and proved in the next section.

Proof of Theorem 2.6.1. We note that Lemma 2.6.2 applies to Y_Δ, where $N = N_2$ and

$$a(j) = \begin{cases} \beta(w_1(j) - w_2(j)), & j = 0, \dots, N_1 - 1 \\ -\beta w_2(j), & j = N_1, \dots, N_2 - 1, \end{cases}$$

We let $A(j) = \sum_{k=0}^{j} a(k)$ for $j = 0, \dots, N-2$. We prove part (i) of the theorem first. Suppose that the event B_Δ defined by $B_\Delta = \{\omega : \limsup_{n\to\infty} |Y_\Delta(n,\omega)|/\gamma_-(n) < \infty\}$ has positive probability. By part (i) of Lemma 2.6.1 and the fact that γ_- is regularly varying and obeys (2.6.3) we have

$$\limsup_{n\to\infty} \frac{|\xi_\Delta(n\Delta)|}{\gamma_-(n\Delta)} = \limsup_{n\to\infty} \frac{|\xi_\Delta(n\Delta)|}{\gamma_-(n)} \cdot \frac{\gamma_-(n)}{\gamma_-(n\Delta)} = \infty, \quad \text{a.s.} \tag{2.6.8}$$

Since Y_Δ obeys

$$Y_\Delta(n+1) = \sum_{k=0}^{N-2} A(k)Y_\Delta(n-k) + \xi_\Delta(n+1) \tag{2.6.9}$$

which rearranges to give $\xi_\Delta(n+1) = Y_\Delta(n+1) - \sum_{k=0}^{N-2} A(k)Y_\Delta(n-k)$, we have

$$\frac{|\xi_\Delta(n\Delta)|}{\gamma_-(n\Delta)} \le \frac{|Y_\Delta(n\Delta)|}{\gamma_-(n\Delta)} + \sum_{k=0}^{N-2} |A(k)| \frac{|Y_\Delta(n\Delta - 1 - k)|}{\gamma_-(n\Delta - 1 - k)} \cdot \frac{\gamma_-(n\Delta - 1 - k)}{\gamma_-(n\Delta)}.$$

For $\omega \in B_\Delta$ there exists a finite $C(\Delta,\omega) := \limsup_{n\to\infty} |Y_\Delta(n,\omega)|/\gamma_-(n)$. Since γ_- is regularly varying, for each $j \in \mathbb{N}$ we have $\gamma(n-j)/\gamma(n) \to 1$ as $n \to \infty$. Therefore

$$\limsup_{n\to\infty} \frac{|\xi_\Delta(n\Delta)|}{\gamma_-(n\Delta)} \le C(\Delta) + \sum_{k=0}^{N-2} |A(k)|C(\Delta) < \infty,$$

on B_Δ, which contradicts (2.6.8). Therefore the complement of B_Δ is an almost sure event and part (i) is proven. To prove part (ii), define ϱ by

$$\varrho(n+1) = \sum_{j=0}^{N-2} A(j)\varrho(n-j) \quad \text{for} \quad n \ge 0$$

with $\varrho(0) = 1$ and $\varrho(n) = 0$ for $n = -N+1, \dots, -1$. With a and A defined in terms of w_1 and w_2 above, the condition (2.2.3) implies that $\varrho \in \ell^1(\mathbb{N}, (0,\infty))$. By (2.6.9), we have

$$Y_\Delta(n) = y_\Delta(n) + \sum_{j=1}^{n} \varrho(n-j)\xi_\Delta(j) \quad \text{for} \quad n \ge 1,$$

where y_Δ is a deterministic sequence which, on account of $\varrho(n) \to 0$ as $n \to \infty$ obeys $y_\Delta(n) \to 0$ as $n \to \infty$. Since γ_+ obeys (2.6.4), we have $\gamma_+(n) \to \infty$ as $n \to \infty$. Therefore, as γ_+ is regularly varying at infinity, there exists an increasing γ_* such that $\gamma_+(n)/\gamma_*(n) \to 1$ as $n \to \infty$. Since γ_* is also regularly varying at infinity, it follows that $\gamma_*(n-1)/\gamma(n) \to 1$ as $n \to \infty$. Therefore, by Lemma 2.7.1, and the fact that $\xi_\Delta(n)/\gamma_+(n) \to 0$ as $n \to \infty$ a.s., we have that $(\varrho * \xi_\Delta)(n)/\gamma_+(n) \to 0$ as $n \to \infty$ a.s., and so $Y_\Delta(n)/\gamma_+(n) \to 0$ as $n \to \infty$ a.s., as required. □

2.6.2 Large fluctuations of the incremental returns in the nonlinear model

Under the condition (2.6.2) on ξ, we now determine for $\Delta \in \mathbb{N}$ the size of the largest fluctuations of the Δ*–returns process* $X_\Delta = \{X_\Delta(n) : n \geq -N_2 + 1 + \Delta\}$ where

$$X_\Delta(n) = X(n) - X(n - \Delta) \tag{2.6.10}$$

and X is the process defined by (2.5.1). In this case, in addition to (2.5.2), we require that the function g obeys

$$\text{There exists } K > 0 \text{ such that } |g(x) - g(y)| \leq K|x - y|, \text{ for all } x, y \in \mathbb{R}. \tag{2.6.11a}$$

$$\lim_{\delta\to\infty} \sup_{x,y\in\mathbb{R}:|x-y|\geq\delta} \left| \frac{g(x) - g(y)}{x - y} - \beta \right| = 0. \tag{2.6.11b}$$

Theorem 2.6.2. *Let $\beta > 0$, N_1 and N_2 be positive integers with $N_1 > N_2$, w_1 and w_2 obey* (2.2.1)*, β obey* (2.2.3)*. Suppose that g obeys* (2.5.2) *and* (2.6.11)*. Let ξ obey* (2.3.2) *and* (2.6.2)*, and X_Δ obey* (2.6.10)*.*

(i) If γ_- is regularly varying at infinity and obeys (2.6.3)*, then for every $\Delta \in \mathbb{N}$, $\limsup_{n\to\infty} |X_\Delta(n)|/\gamma_-(n) = \infty$ a.s.*

(ii) If γ_+ is regularly varying at infinity and obeys (2.6.4)*, then for every $\Delta \in \mathbb{N}$, $\limsup_{n\to\infty} |X_\Delta(n)|/\gamma_+(n) = 0$ a.s.*

Remark 2.6.2. Part (i) of the theorem is proved by contradiction. To prove part (ii) we let $X_\Delta(n) = Y_\Delta(n) + Z_\Delta(n)$ as with the previous nonlinear theorem . We show that $Z_\Delta(n)$ is bounded and hence $\limsup_{n\to\infty} |Z_\Delta(n)|/\gamma_+(n) = 0$ almost surely. As this is also the case for $Y_\Delta(n)$ then $\limsup_{n\to\infty} |X_\Delta(n)|/\gamma_+(n) = 0$ almost surely.

As in the linear case, the a.s. upper and lower bounds on the rate of growth of the partial maxima of X_Δ are exactly the same as those which apply to the innovation (or "news") process ξ. Also, we notice that the same functions γ_- and γ_+ are lower and upper bounds on the rate of growth of the large fluctuations regardless of the value of Δ. The key to this proof is the development of a difference equation for the Δ–increment, where $\Delta \in \mathbb{N}$. Let $N \in \mathbb{N}$ and $a = \{a(j) : j = 0, 1, \ldots, N - 1\}$ be a real sequence. We consider

$$W(n + 1) = W(n) + \sum_{k=0}^{N-1} a(k)g(W(n - k)) + \xi(n + 1), \quad n \geq 0, \tag{2.6.12}$$

$$W(n) = \psi(n), \quad n \leq 0, \tag{2.6.13}$$

and its Δ–increment $W_\Delta(n) = W(n) - W(n - \Delta)$.

Lemma 2.6.3. *Suppose that a is a sequence obeying $\sum_{k=0}^{N-1} a(k) = 0$. Define $A(n) = \sum_{j=0}^{N-2} a(k)$ for $n = 0, \ldots, N - 2$. If ξ_Δ is the process defined by* (2.6.1)*, and W the process defined by then W_Δ defined by $W_\Delta(n) = W(n) - W(n - \Delta)$ obeys*

$$W_\Delta(n+1) = \sum_{k=0}^{N-2} A(k) \{g(W(n - k)) - g(W(n - \Delta - k))\} + \xi_\Delta(n+1), \quad n \geq N+\Delta-1. \tag{2.6.14}$$

Proof. Let $n' \geq N - 1$, then,

$$\begin{aligned}
\sum_{n=0}^{n'} W(n + 1) &= \sum_{n=0}^{n'} W(n) + \sum_{n=0}^{n'} \sum_{k=0}^{N-1} a(k)g(W(n - k)) + \sum_{n=0}^{n'} \xi(n + 1) \\
W(n' + 1) - W(0) &= \sum_{k=0}^{N-1} a(k) \sum_{n=0}^{n'} g(W(n - k)) + \sum_{n=0}^{n'} \xi(n + 1) \\
&= \sum_{k=0}^{N-1} a(k) \sum_{l=-k}^{n'-k} g(W(l)) + \sum_{n=0}^{n'} \xi(n + 1) \\
&= \sum_{k=0}^{N-1} a(k) \sum_{l=-k}^{-1} g(\phi(l)) + \sum_{k=0}^{N-1} a(k) \sum_{l=0}^{n'-k} g(W(l)) + \sum_{n=0}^{n'} \xi(n + 1).
\end{aligned}$$

Letting $I = \sum_{k=0}^{N-1} a(k) \sum_{l=-k}^{-1} g(\psi(l)) + W(0)$,

$$W(n'+1) = \sum_{k=0}^{N-1} a(k) \sum_{l=0}^{n'-k} g(W(l)) + I + \sum_{j=0}^{n'} \xi(j+1), \quad n' \geq N-1.$$

Therefore for $n \geq N + \Delta - 1$, we have that

$$\begin{aligned} W_\Delta(n+1) &= W(n+1) - W(n-\Delta+1) \\ &= \sum_{k=0}^{N-1} a(k) \sum_{l=0}^{n-k} g(W(l)) + I + \sum_{j=0}^{n} \xi(j+1) \\ &\quad - \left(\sum_{k=0}^{N-1} a(k) \sum_{l=0}^{n-k-\Delta} g(W(l)) + I + \sum_{j=0}^{n-\Delta} \xi(j+1) \right) \\ &= \sum_{k=0}^{N-1} a(k) \sum_{l=n-\Delta-k+1}^{n-k} g(W(l)) + \xi_\Delta(n+1) \\ &= \sum_{l=n-N-\Delta+2}^{n} \sum_{k=0\vee(n-l-\Delta+1)}^{(N-1)\wedge(n-l)} a(k) g(W(l)) + \xi_\Delta(n+1). \end{aligned}$$

Consider now the case when $\Delta \geq N$.Letting $l = n - j$, then the first term on the righthand side of the last member in the above identity is

$$\begin{aligned} &\sum_{j=0}^{N+\Delta-2} \sum_{k=0\vee(j-\Delta+1)}^{(N-1)\wedge j} a(k) g(W(n-j)) \\ &= \sum_{j=0}^{N-2} \sum_{k=0}^{j} a(k) g(W(n-j)) + \sum_{j=N-1}^{\Delta-1} \sum_{k=0}^{N-1} a(k) g(W(n-j)) \\ &\qquad + \sum_{j=\Delta}^{N+\Delta-2} \sum_{k=j-\Delta+1}^{N-1} a(k) g(W(n-j)). \end{aligned}$$

The middle term is zero as $\sum_{k=0}^{N-1} a(k) = 0$. Since $A(j) = \sum_{k=0}^{j} a(k)$ and $\sum_{k=j-\Delta+1}^{N-1} a(k) = \sum_{k=0}^{N-1} a(k) - \sum_{k=0}^{j-\Delta} a(k) = -A(j-\Delta)$, we have

$$\begin{aligned} &\sum_{j=0}^{N+\Delta-2} \sum_{k=0\vee(j-\Delta+1)}^{(N-1)\wedge j} a(k) g(W(n-j)) \\ &= \sum_{j=0}^{N-2} A(j) g(W(n-j)) - \sum_{j=\Delta}^{N+\Delta-2} A(j-\Delta) g(W(n-j)) \\ &= \sum_{j=0}^{N-2} A(j) \left(g(W(n-j)) - g(W(n-j-\Delta)) \right). \end{aligned}$$

This is the first term on the righthand side of (2.6.14), and hence proves (2.6.14) in the case where $\Delta \geq N$. When $N > \Delta$, we have

$$\begin{aligned} &\sum_{j=0}^{N+\Delta-2} \sum_{k=0\vee(j-\Delta+1)}^{(N-1)\wedge j} a(k) g(W(n-j)) \\ &= \sum_{j=0}^{\Delta-1} \sum_{k=0}^{j} a(k) g(W(n-j)) + \sum_{j=\Delta}^{N-1} \sum_{k=j-\Delta+1}^{j} a(k) g(W(n-j)) \\ &\qquad + \sum_{j=N}^{N+\Delta-2} \sum_{k=j-\Delta+1}^{N-1} a(k) g(W(n-j)), \end{aligned}$$

and so

$$\sum_{j=0}^{N+\Delta-2}\sum_{k=0\vee(j-\Delta+1)}^{(N-1)\wedge j} a(k)g(W(n-j))$$
$$= \sum_{j=0}^{\Delta-1} A(j)g(W(n-j)) + \sum_{j=\Delta}^{N-1}(A(j)-A(j-\Delta))g(W(n-j))$$
$$- \sum_{j=N}^{N+\Delta-2} A(j-\Delta)g(W(n-j))$$
$$= \sum_{j=0}^{N-1} A(j)g(W(n-j)) - \sum_{j=\Delta}^{N+\Delta-2} A(j-\Delta)g(W(n-j))$$
$$= \sum_{j=0}^{N-1} A(j)g(W(n-j)) - \sum_{j=0}^{N-2} A(j)g(W(n-j-\Delta)).$$

As $A(j) = 0$ for $j > N-2$ then the above is equal to the first term on the righthand side of (2.6.14), and hence proves (2.6.14) in the case where $\Delta < N$. Since the case $\Delta \geq N$ has already been dealt with, the proof is complete. □

To prove lemma 2.6.2 we substitute $W(n+1) = W(n) + \sum_{k=0}^{N-1} a(k)g(W(n-k)) + \xi(n+1)$ in the above proof for $V(n+1) = V(n) + \sum_{k=0}^{N-1} a(k)V(n-k) + \xi(n+1)$ and follow line for line.

Proof of Theorem 2.6.2. The result of Lemma 2.6.3 applies to the process X in (2.5.1) and the Δ–increment X_Δ. With $N := N_2$, a defined by $a(j) = w_1(j) - w_2(j)$ for $j = 0,\ldots,N_1-1$, $a(j) = -w_2(j)$ for $j = N_1,\ldots,N_2-1$ and $A(j) = \sum_{k=0}^{j} a(k)$ for $j = 0,\ldots,N-2$, we have

$$X_\Delta(n+1) = \sum_{j=0}^{N-2} A(j)\left(g(X(n-j)) - g(X(n-\Delta-j))\right) + \xi_\Delta(n+1), \quad n \geq N+\Delta-1. \quad (2.6.15)$$

To prove part (i), we rearrange (2.6.15) and use (2.6.11a) to get

$$|\xi_\Delta(n+1)| \leq |X_\Delta(n+1)| + K\sum_{j=0}^{N-2} |A(j)||X_\Delta(n-j)|.$$

we have

$$\frac{|\xi_\Delta(n\Delta)|}{\gamma_-(n\Delta)} \leq \frac{|X_\Delta(n\Delta)|}{\gamma_-(n\Delta)} + K\sum_{j=0}^{N-2}|A(j)|\frac{|X_\Delta(n\Delta-1-j)|}{\gamma_-(n\Delta-1-j)} \cdot \frac{\gamma_-(n\Delta-1-j)}{\gamma_-(n\Delta)}.$$

For $\omega \in B_\Delta$, there exists a finite $C(\Delta,\omega) := \limsup_{n\to\infty} |X_\Delta(n,\omega)|/\gamma_-(n)$. Since γ_- is regularly varying, for each $j \in \mathbb{N}$ we have $\gamma(n-j)/\gamma(n) \to 1$ as $n \to \infty$. Therefore

$$\limsup_{n\to\infty} \frac{|\xi_\Delta(n\Delta)|}{\gamma_-(n\Delta)} \leq C(\Delta) + K\sum_{k=0}^{N-2}|A(k)|C(\Delta) < \infty,$$

on B_Δ, which contradicts (2.6.8). Therefore the complement of B_Δ is an almost sure event, and part (i) is proven. To prove part (ii) we use that $g(X) = \beta X + \gamma(X)$ and we notice that (2.6.11b) implies

$$\lim_{\delta\to\infty} \sup_{x,y\in\mathbb{R}:|x-y|\geq\delta} \left|\frac{\gamma(x)-\gamma(y)}{x-y}\right| = 0.$$

Therefore, for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that $|\gamma(x)-\gamma(y)| < \varepsilon|x-y|$ once $|x-y| > \delta(\varepsilon)$. Using (2.6.11a), we find that $|\gamma(x)-\gamma(y)| < (|\beta|+K)\delta(\varepsilon)$ for all $|x-y| \leq \delta(\varepsilon)$. With $L^*(\varepsilon) = (|\beta|+K)\delta(\varepsilon)$, we have $|\gamma(x)-\gamma(y)| \le L^*(\varepsilon)+\varepsilon|x-y|$ for all $x, y \in \mathbb{R}$. Therefore, for $n \ge N+\Delta-1$, from (2.6.15) we have

$$\begin{aligned} X_\Delta(n+1) &= \sum_{j=0}^{N-2} A(j)\left(g(X(n-j)) - g(X(n-\Delta-1))\right) + \xi_\Delta(n+1) \\ &= \sum_{j=0}^{N-2} A(j)\left(g(X_\Delta(n-j))\right) + \xi_\Delta(n+1) \\ &= \beta \sum_{j=0}^{N-2} A(j)\left(\beta X_\Delta(n-j) + \gamma(X_\Delta(n-j))\right) + \xi_\Delta(n+1) \\ &= \sum_{j=0}^{N-2} \beta A(j) X_\Delta(n-j) + F(n+1) + \xi_\Delta(n+1), \quad n \ge N+\Delta-1, \end{aligned}$$

where $F(n+1) = \sum_{j=0}^{N-2} A(j)\left(\gamma(X(n-j)) - \gamma(X(n-\Delta-j))\right)$. Hence for every $\varepsilon > 0$ we have

$$|F(n+1)| \le \sum_{j=0}^{N-2} |A(j)| \left(L^*(\varepsilon) + \varepsilon |X_\Delta(n-j)|\right).$$

Define $Z_\Delta(n) = X_\Delta(n) - Y_\Delta(n)$ where Y_Δ is given by $Y_\Delta(n) = \sum_{j=0}^{N-2} \beta A(j) Y_\Delta(n-j) + \xi_\Delta(n+1)$, and so Y_Δ is the process in Theorem 2.6.1. The condition (2.2.3) therefore implies that $\lim_{n\to\infty} Y_\Delta(n)/\gamma_+(n) = 0$ a.s. Moreover,

$$\begin{aligned} &Z_\Delta(n+1) = \\ &\sum_{k=0}^{N-2} A(k) g(X_\Delta(n-k)) + \xi_\Delta(n+1) - \beta \sum_{k=0}^{N-2} A(k) Y_\Delta(n-k) - \xi_\Delta(n+1) \\ &= \sum_{k=0}^{N-2} A(k)\left(g(X_\Delta(n-k)) - \beta Y_\Delta(n-k)\right) \\ &= \sum_{k=0}^{N-2} A(k)\left(\beta X_\Delta(n-k) - \beta Y_\Delta(n-k)\right) + \sum_{k=0}^{N-2} A(k)\gamma(X_\Delta(n-k)) \\ &= \sum_{j=0}^{N-2} \beta A(j) Z_\Delta(n-j) + F(n+1), \quad n \ge N+\Delta-1. \end{aligned}$$

The fact that $X_\Delta = Z_\Delta + Y_\Delta$ implies that

$$|F(n)| \le L^*(\varepsilon) \sum_{j=0}^{N-2} |A(j)| + \varepsilon \sum_{j=0}^{N-2} |A(j)||Y_\Delta(n-1-j)| + \varepsilon \sum_{j=0}^{N-2} |A(j)||Z_\Delta(n-1-j)|.$$

Introducing ϱ_β as the solution of $\varrho_\beta(n+1) = \sum_{j=0}^{N-2} \beta A(j) \varrho_\beta(n-j)$, $n \ge 0$, with $\varrho_\beta(0) = 1$, $\varrho_\beta(n) = 0$ for $n = -N+2, \ldots, -1$, then

$$Z_\Delta(n) = z_\Delta(n) + \sum_{j=N+\Delta-1}^{n} \varrho_\beta(n-j) F(j) \quad \text{for} \quad n \ge N+\Delta-1.$$

The condition (2.2.3) implies that ϱ_β is summable (it is nothing other than δ in Lemma 2.2.1), so $z_\Delta(n) \to 0$ as $n \to \infty$. Since $Y_\Delta(n)/\gamma_+(n) \to 0$ as $n \to \infty$ a.s., we have that

$$|Z_\Delta(n)| \le u_\Delta(n) + \varepsilon \sum_{j=N+\Delta-1}^{n} \varrho_\beta(n-j) \sum_{k=0}^{N-2} |A(k)||Z_\Delta(j-1-k)|, \quad n \ge N+\Delta-1,$$

where

$$u_\Delta(n) =$$
$$z_\Delta(n) + \sum_{j=N+\Delta-1}^{n} \varrho_\beta(n-j) \left(L^*(\varepsilon) \sum_{k=0}^{N-2} |A(k)| + \varepsilon \sum_{k=0}^{N-2} |A(k)||Y_\Delta(n-1-k)| \right),$$

and $u_\Delta(n)/\gamma_+(n) \to 0$ as $n \to \infty$ a.s. By taking $\varepsilon > 0$ sufficiently small and observing that $\varrho_\beta \in \ell^1(\mathbb{N}, \mathbb{R})$, an argument similar to that used at the end of the proof of Theorem 2.5.1 shows that $Z_\Delta(n)/\gamma_+(n) \to 0$ as $n \to \infty$ a.s., and hence that $X_\Delta = Y_\Delta + Z_\Delta$ obeys $X_\Delta(n)/\gamma_+(n) \to 0$ as $n \to \infty$ a.s. $\square$

2.7 Supporting Lemmas

The following Lemma is used in the proof of Theorem 2.3.1, Theorem 2.5.1, Theorem 2.6.1 and Theorem 2.6.2. It enables us to find the growth rate of a moving average of a slowly increasing function.

Lemma 2.7.1. *Let γ be positive and increasing with $\gamma(n-N)/\gamma(n) \to 1$, as $n \to \infty$, for all $N \in \mathbb{N}$. If $k = \{k(n) : n \in \mathbb{N}\}$ is non–negative with $\sum_{n=0}^{\infty} k(n) \in (0, \infty)$, then $\lim_{n\to\infty} (k * \gamma)(n)/\gamma(n) = \sum_{n=0}^{\infty} k(n)$.*

Proof. Without loss of generality, let $\sum_{n=0}^{\infty} k(n) = 1$. For every $\varepsilon > 0$ there is $N > 0$ such that $\sum_{j=N+1}^{\infty} k(j) < \varepsilon/2$. For $n \geq N+1$, we have

$$\begin{aligned}
\frac{(k*\gamma)(n)}{\gamma(n)} - \sum_{j=0}^{n} k(j) &= \sum_{j=0}^{n} \left(\frac{k(j)\gamma(n-j)}{\gamma(n)} \right) - \sum_{j=0}^{N} k(j) \\
&= \sum_{j=0}^{N} \frac{k(j)\gamma(n-j)}{\gamma(n)} - \sum_{j=0}^{N} k(j) + \sum_{j=N+1}^{n} \frac{k(j)\gamma(n-j)}{\gamma(n)} - \sum_{j=0}^{n} k(j) \\
&= \sum_{j=0}^{N} k(j) \left(\frac{\gamma(n-j)}{\gamma(n)} - 1 \right) + \sum_{j=N+1}^{n} k(j) \left(\frac{\gamma(n-j)}{\gamma(n)} - 1 \right).
\end{aligned}$$

Now as γ is an increasing sequence,

$$\begin{aligned}
\left| \sum_{j=0}^{N} k(j) \frac{\gamma(n-j)}{\gamma(n)} - 1 \right| &\leq \sum_{j=0}^{N} k(j) \left| \frac{\gamma(n-j)}{\gamma(n)} - 1 \right| \\
&\leq \max_{0 \leq j \leq N} \left| \frac{\gamma(n-j)}{\gamma(n)} - 1 \right| \\
&= \max_{0 \leq j \leq N} \left| \frac{\gamma(n-j) - \gamma(n)}{\gamma(n)} \right| \\
&= \max_{0 \leq j \leq N} \left(1 - \frac{\gamma(n-j)}{\gamma(n)} \right) \\
&= 1 - \frac{\gamma(n-N)}{\gamma(n)}.
\end{aligned}$$

Also

$$\begin{aligned}
\left| \sum_{j=N+1}^{n} k(j) \left(\frac{\gamma(n-j)}{\gamma(n)} - 1 \right) \right| &\leq \sum_{j=N+1}^{n} k(j) \left| \frac{\gamma(n-j)}{\gamma(n)} - 1 \right| \\
&\leq \sum_{j=N+1}^{n} k(j) \left(\frac{\gamma(n-j)}{\gamma(n)} - 1 \right) \\
&\leq 2 \sum_{j=N+1}^{n} k(j) \leq 2 \sum_{j=N+1}^{\infty} k(j).
\end{aligned}$$

Thus

$$\left|\sum_{j=0}^{n}\frac{k(j)\gamma(n-j)}{\gamma(n)}-\sum_{j=0}^{n}k(j)\right|\leq\left(1-\frac{\gamma(n-N)}{\gamma(n)}\right)+2\sum_{j=N+1}^{\infty}k(j).$$

Using $\gamma(n-N)/\gamma(n)\to 1$, and then letting $\varepsilon\to 0$ yields the result. □

The following Lemma shows that the difference equation as defined below satisfies the variation of constants formula. This result is used in the proof of the Law of the Iterated Logarithm.

Lemma 2.7.2. *Define* $Z(n+1)-Z(n)=G(n+1)+\beta\left(\sum_{j=0}^{N_1-1}w_1(j)Z(n-j)-\sum_{j=0}^{N_2-1}w_2(j)Z(n-j)\right)$*; Then*

$$Z(n)=\sum_{j=0}^{n-1}r(n-1-j)G(j+1),\quad n\geq 1 \tag{2.7.1}$$

Proof. For $n\geq 1$, suppose that $Z(n)=\sum_{j=0}^{n-1}r(n-1-j)G(j+1)$ then

$$Z(n+1)-Z(n)=G(n+1)+\beta\left(\sum_{j=0}^{N_1-1}w_1(j)Z(n-j)-\sum_{j=0}^{N_2-1}w_2(j)Z(n-j)\right),$$

yields

$$\sum_{j=0}^{n}r(n-j)G(j+1)-\sum_{j=0}^{n-1}r(n-1-j)G(j+1)=G(n+1)$$
$$+\beta\left(\sum_{k=0}^{N_1-1}w_1(k)\sum_{j=0}^{n-1-k}r(n-k-1-j)G(j+1)-\sum_{k=0}^{N_2-1}w_2(k)\sum_{j=0}^{n-1-k}r(n-k-1-j)G(j+1)\right).$$

Now

$$\sum_{j=0}^{n}r(n-j)G(j+1)-\sum_{j=0}^{n-1}r(n-1-j)G(j+1)=G(n+1)$$
$$+\sum_{j=0}^{n-1}\left(r(n-j)-r(n-1-j)\right)G(j+1).$$

According to equation (2.2.2)

$$r(n-j)-r(n-j-1)=\beta\left(\sum_{k=0}^{N_1-1}w_1(k)r(n-j-k-1)-\sum_{k=0}^{N_2-1}w_2(k)r(n-j-k-1)\right),$$

then

$$G(n+1)+\sum_{j=0}^{n-1}\left(r(n-j)-r(n-1-j)\right)G(j+1)=G(n+1)$$
$$+\sum_{j=0}^{n-1}\beta\left(\sum_{k=0}^{N_1-1}w_1(k)r(n-j-k)-\sum_{k=0}^{N_2-1}w_2(k)r(n-j-k)\right).$$

As both sides of the equation are equal we have (2.7.1) as required. □

In Theorem 2.5.1 we write the nonlinear resolvent $X(n)$ in terms of the linear resolvent $Y(n)$ and another resolvent $Z(n)$. The following Lemma shows $Z(n)$ is bounded.

Lemma 2.7.3. *Suppose Z and z are sequences defined by*

$$Z(n) \le F(n) + \sum_{j=0}^{n-1} a(n-1-j)Z(j), \quad n \ge 1 \tag{2.7.2}$$

$$z(n) = F(n) + \sum_{j=0}^{n-1} a(n-1-j)z(j), \quad n \ge 1 \tag{2.7.3}$$

where $Z(0) \le z(0)$ *and* $z(n) \ge 0$*. Then* $z(n) \ge Z(n)$*.*

Proof. Define $\Delta(n) = z(n) - Z(n)$, $n \ge 0$. Then $\Delta(0) \ge 0$.

$$\begin{aligned}\Delta(n) &= z(n) - Z(n) \\ &\ge \sum_{j=0}^{n-1} a(n-1-j)\,(z(j) - Z(j)) \\ &= \sum_{j=0}^{n-1} a(n-1-j)\Delta(j).\end{aligned}$$

Suppose $\Delta(j) \ge 0$ for all $j = 0, \cdots, n-l$. This is true for $n = 1$. But by the previous equation $\Delta(n) \ge 0$. Hence the result is true by induction. □

Chapter 3

An Affine Stochastic Functional Differential Equation model of an Inefficient Financial Market

3.1 Introduction

In the previous chapter we examined in detail the long–run behaviour of a discrete–time inefficient financial market in which the returns were governed by the demand of trend–following speculators. In particular we showed that the market can either exhibit a kind of (positively) correlated random walk or experience a bubble or crash, characterised by exponential growth in the returns. It is therefore interesting to ask whether these results are peculiar to discrete–time models or whether an analogous modelling of the speculators' behaviour in continuous–time would lead to similar dynamics in the returns. In this chapter we answer this question in the affirmative. Roughly speaking it is shown that the returns follow a kind of positively correlated SBM or undergo exponential growth. The condition under which these two possibilities arise are directly comparable to the mathematical conditions and financial interpretation under which these occur in the discrete–time model. In other words the results of the model are not only robust to the absence of exact linearity of the speculators demand function but also to the time scale on which trading takes place. Taken together with evidence form the following two chapters we see that the presence of trend–following speculators tends to lead to excess volatility or to bubbles or crashes independently of the precise model used.

To capture this mathematically we model the returns using a SFDE. Trading is now assumed to take place continuously as opposed to occurring at fixed (and uniformly spaced) points in time.Once again the returns respond to imbalances in the demand of speculators. This demand, as in the previous chapter depends on the difference between a short–run and long–run weighted average of the cumulative returns over the previous τ units of time. This planned ex-ante demand is supplemented by expost and unplanned demand which depends on "news" which reaches the speculators. In the time–honored fashion of continuous time modelling this cumulative news process is modelled by a scalar SBM. We suppose that the price adjustment at time t for a market with N traders is given by

$$dY(t) = \alpha \sum_{j=1}^{N} \beta_j \left(\int_{[-\theta_j,0]} Y(t+u) s_j(du) - \int_{[-\tau_j,0]} Y(t+u) l_j(du) \right) dt + \sigma\, dB(t). \tag{3.1.1}$$

Here s_j and l_j are finite measures, representing the short– and long–run weights that trader j uses to form their demand schedule. $\beta_j > 0$, $\alpha > 0$ and σ are constants. This is equivalent to the linear stochastic functional differential equation

$$Y(t) = \psi(0) + \int_0^t L(Y_s)\, ds + \int_0^t \sigma\, dB(s), \quad t \geq 0, \tag{3.1.2a}$$

$$Y(t) = \phi(t), \quad t \in [-\tau, 0]. \tag{3.1.2b}$$

where $L : C[-\tau, 0] \to \mathbb{R}$ is a linear functional with $\tau = \max_{j=1,\dots,N} \max(\tau_j, \theta_j)$, and

$$L(\phi) = \int_{[-\tau,0]} \phi(s) \nu(ds), \quad \phi \in C([-\tau, 0]; \mathbb{R}).$$

The measure $\nu \in M[-\tau, 0]$ inherits properties from the weights s_j and l_j and the constants β_j and α. These special properties influence the almost sure asymptotic behaviour as $t \to \infty$ of solutions of (3.1.1). Roughly speaking, we show that the market either follows a correlated Brownian motion or experiences a crash or bubble. Therefore, the presence of feedback traders produces more complicated or extreme price dynamics than would be present in a corresponding efficient market model in which the driving semimartingale is a continuous Gaussian process with *independent* increments.

In common with chapter 2 there are two main and analogous findings; Firstly, if the trend–following speculators do not react very aggressively to differences between the short–run and long–run returns, then the rate of growth of the partial maxima of the solution is the same as that of a standard Brownian motion. Therefore, to a first approximation, the market appears efficient. However, the size of these largest fluctuations is greater in the presence of trend following speculators than in their absence, where the market only reacts to "news". Hence the presence of these speculators tends to increase market volatility as well as causing positive (though exponentially decaying) correlation in the returns. The main result in this direction is part (a) of Theorem 3.3.1 and mirrors the corresponding results in discrete time in chapter 2. Secondly, when the trend–following speculators behave aggressively, the returns will tend to plus or minus infinity exponentially fast (see part (b) in Theorem 3.3.1). Again this is a direct analogue of results in chapter 2; moreover the same causes of bubble or crash can be identified. There are some distinction between the work in this chapter and that in chapter 2. Since we do not consider a non–Gaussian news process we do not obtain results on the large fluctuations of the returns which correspond to those in section titled "Large Deviations Of The Incremental Returns" in chapter 2. Furthermore we do not extend results to nonlinear equations in continuous time in this chapter. However results analogous to those proven in section titled "Cumulative Returns In The Nonlinear Equation" in chapter 2 are developed and proved in chapter 4.

In terms of financial economics, this chapter is an extension of previous work by Appleby & Swords [12] and Appleby, Swords and Rodkina [11], which considers corresponding discrete time equations and in which discrete–time analogues of Theorem 3.3.1 are proven. This chapter covers some special cases of results proven in Riedle [66] and for more general affine stochastic functional equations in which the structure of the Liapunov spectrum is extensively investigated. Here the scalar structure of the equation, and positivity and monotonicity of the underlying resolvent enable us to prove complementary results. Another related paper on the Liapunov spectrum for *linear* SFDEs is [63]. The analysis is also inspired by recent work of Appleby, Reynolds and Devin [5, 6] which studies affine stochastic Volterra equations that have non–equilibrium and random limits. A common theme with these papers and the current work is the fact that the characteristic equation of the underlying deterministic resolvent has zero as one of its solutions. An interesting recent paper which concentrates on non–equilibrium limits in deterministic functional differential equations is [26].

We have chosen to model the speculators' behaviour in this chapter using finite measures rather than through fixed delays or continuous averages of past returns. This affords some modelling advantages. It allows us to capture a very wide variety of moving average–type strategies within the same model. Here, we can consider a market comprising of agents who compare (i) current returns with a continuously computed moving average of historical returns, (ii) continuous short–term and long–term weighted averages of returns, (iii) corresponding weighted averages using only a finite number of times in the short and long–term averages and (iv) any combination of these strategies. Not only does this allow for a general and flexible model of feedback trading, it enables us to do so using a compact and unified notation which simplifies analysis and aids interpretation of economic results. Apart from notational advantages, we have the important implication that the manner in which traders compute moving averages is unimportant in the form of the ultimate dynamics. This is important in any mathematical model in economics, as model assumptions are unlikely to be satisfied in reality, rendering general models which are robust to changes in the assumptions particularly desirable.

This chapter has the following structure; Section 2 gives notation and supporting results; Section 3 states the main mathematical results of the chapter; while Section 4 shows how the hypotheses of these results are satisfied in the financial model. The interpretation of the results to the financial model are also explored in Section 4, along with a variety of concrete examples of moving average trading strategies which involve both continuous and discrete weights of past returns. The rest of the chapter is devoted to proofs which mimic those of chapter 2.

3.2 Preliminaries

We first turn our attention to the deterministic delay equation underlying the stochastic differential equation (3.1.2). Let $\widetilde{x}(z)$ denote the Laplace transform of x. For a fixed constant $\tau \geq 0$ we consider the deterministic linear delay differential equation

$$\begin{aligned} y'(t) &= \int_{[-\tau,0]} y(t+u)\,\nu(du) \quad \text{for } t \geq 0, \\ y(t) &= \phi(t) \quad \text{for } t \in [-\tau, 0], \end{aligned} \tag{3.2.1}$$

for a measure $\nu \in M = M[-\tau, 0]$, the space of signed Borel measures on $[-\tau, 0]$ with the total variation norm $\|\cdot\|_{TV}$ which is defined as $||a||_{TV} = \sup_{j=1}^{N} |a(t_j) - a(t_{j-1})|$ where the supremum is taken over all N and over all sets of points $t_j \in J$ such that $t_i < t_j$ for $i < j$. The initial function ϕ is assumed to be in the space $C[-\tau, 0] := \{\phi : [-\tau, 0] \to \mathbb{R} : \text{continuous}\}$. A function $y : [-\tau, \infty) \to \mathbb{R}$ is called a *solution* of (3.2.1) if y is continuous on $[-\tau, \infty)$, its restriction to $[0, \infty)$ is continuously differentiable, and y satisfies the first and second identity of (3.2.1) for all $t \geq 0$ and $t \in [-\tau, 0]$, respectively. It is well known that for every $\phi \in C[-\tau, 0]$ the problem (3.2.1) admits a unique solution $y = y(\cdot, \phi)$.

The *fundamental solution* or *resolvent* of (3.2.1) is the unique locally absolutely continuous function $r : [0, \infty) \to \mathbb{R}$ which satisfies

$$r(t) = 1 + \int_0^t \int_{[\max\{-\tau,-s\},0]} r(s+u)\,\nu(du)\,ds \quad \text{for } t \geq 0. \tag{3.2.2}$$

It plays a role which is analogous to the fundamental system in linear ordinary differential equations and the Green function in partial differential equations. Formally, it is the solution of (3.2.1) corresponding to the initial function $\phi = 1_{\{0\}}$. For later convenience we set $r(t) = 0$ for $t \in [-\tau, 0)$.

The solution $y(\cdot, \phi)$ of (3.2.1) for an arbitrary initial segment ϕ exists, is unique, and can be represented as

$$y(t, \phi) = \phi(0)r(t) + \int_{[-\tau,0]} \int_s^0 r(t+s-u)\phi(u)\,du\,\nu(ds) \quad \text{for } t \geq 0, \tag{3.2.3}$$

cf. lemma 3.9.3. Define the function $h : \mathbb{C} \to \mathbb{C}$ by

$$h(\lambda) = \lambda - \int_{[-\tau,0]} e^{\lambda s}\,\nu(ds). \tag{3.2.4}$$

Define also the set

$$\Lambda = \{\lambda \in \mathbb{C} : h(\lambda) = 0\}. \tag{3.2.5}$$

The function h is analytic, and so the elements of Λ are isolated. Define

$$v_0(\nu) := \sup\{\text{Re}\,(\lambda) :\ h(\lambda) = 0\}, \tag{3.2.6}$$

where $\text{Re}\,(z)$ denotes the real part of a complex number z. Furthermore, the cardinality of $\Lambda' := \Lambda \cap \{\text{Re}\,(\lambda) = v_0(\nu)\}$ is finite. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ we have

$$e^{-v_0(\nu)t} r(t) = \sum_{\lambda_j \in \Lambda'} \{p_j(t)\cos(\text{Im}\,(\lambda_j)t) + q_j(t)\sin(\text{Im}\,(\lambda_j)t)\} + o(e^{-\varepsilon t}), \quad t \to \infty, \tag{3.2.7}$$

where p_j and q_j are polynomials of degree $m_j - 1$, with m_j being the multiplicity of the zero $\lambda_j \in \Lambda'$ of h, and $\text{Im}\,(z)$ denoting the imaginary part of a complex number z. This is a simple restatement of Diekmann et al [30, Thm. 5.4].

Let us introduce equivalent notation for (3.2.1). For a function $y : [-\tau, \infty) \to \mathbb{R}$ we define the *segment of* y at time $t \geq 0$ by the function

$$y_t : [-\tau, 0] \to \mathbb{R}, \qquad y_t(u) := y(t+u).$$

If we equip the space $C[-\tau, 0]$ of continuous functions with the supremum norm Riesz' representation theorem guarantees that every continuous functional $L : C[-\tau, 0] \to \mathbb{R}$ is of the form

$$L(\psi) = \int_{[-\tau,0]} \psi(u)\,\nu(du)$$

for a measure $\nu \in M[-\tau, 0]$. Hence, we will write (3.2.1) in the form

$$y'(t) = L(y_t) \quad \text{for } t \geq 0, \qquad y_0 = \phi$$

and assume L to be a continuous and linear functional on $C[-\tau, 0]$.

Let us fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}(t))_{t\geq 0}$ satisfying the usual conditions and let $(B(t) : t \geq 0)$ be a standard Brownian motion on this space. We study the following stochastic differential equation with time delay:

$$\begin{aligned} dY(t) &= L(Y_t)\,dt + \sigma\,dB(t) \quad \text{for } t \geq 0, \\ Y(t) &= \phi(t) \quad \text{for } t \in [-\tau, 0], \end{aligned} \tag{3.2.8}$$

where L is a continuous and linear functional on $C[-\tau, 0]$ for a constant $\tau \geq 0$ and $\sigma \geq 0$.

For every $\phi \in C[-\tau, 0]$ there exists a unique, adapted strong solution $(Y(t, \phi) : t \geq -\tau)$ with finite second moments of (3.2.8) (cf., e.g., Mao [49]). The dependence of the solutions on the initial condition ϕ is neglected in our notation in what follows; that is, we will write $y(t) = y(t, \phi)$ and $Y(t) = Y(t, \phi)$ for the solutions of (3.2.1) and (3.2.8), respectively.

By lemma 3.9.2 the solution $(Y(t) : t \geq -\tau)$ of (3.2.8) obeys a variation of constants formula

$$Y(t) = \begin{cases} y(t) + \int_0^t r(t-s)\sigma\,dB(s), & t \geq 0, \\ \phi(t), & t \in [-\tau, 0], \end{cases} \tag{3.2.9}$$

where r is the fundamental solution of (3.2.1). This result mimics that of proposition 2.3.1 in chapter 2.

3.3 Main Theorems

If we assume that there is only one $\lambda \in \mathbb{C}$ with $\text{Re}\,(\lambda) = v_0(\nu)$, i.e. $\Lambda' = \{\lambda\}$ then it follows that λ is real-valued. If we assume furthermore that λ is a simple zero of h the representation (3.2.7) implies that there exists $\epsilon_0 > 0$ such that

$$r(t)e^{-v_0(\nu)t} = c + o(e^{-\varepsilon t}) \qquad \text{for all } \epsilon \in (0, \epsilon_0), \tag{3.3.1}$$

and moreover c obeys

$$c = \frac{1}{1 - \int_{[-\tau,0]} s e^{v_0(\nu)s}\nu(ds)}. \tag{3.3.2}$$

The formula for c can be determined by contour integration; see e.g., Chapter 7 of Gripenberg et al. [35]. The assumption that λ is a simple zero of h guarantees that c is well-defined because the denominator of c equals $h'(\lambda)$, i.e. is non-zero.

Theorem 3.3.1. *Suppose that r obeys* (3.3.1). *Then the solution Y of* (3.2.8) *satisfies*

(a) *if $v_0(\nu) = 0$, then*

$$\limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2t\log\log t}} = \sigma c \quad a.s.$$
$$\liminf_{t\to\infty} \frac{Y(t)}{\sqrt{2t\log\log t}} = -\sigma c \quad a.s.$$

(b) *if $v_0(\nu) > 0$, then a.s.*

$$\lim_{t\to\infty} e^{-v_0(\nu)t}Y(t) = c\left(\phi(0) + \int_{[-\tau,0]}\int_s^0 e^{v_0(\nu)(s-u)}\phi(u)\,du\,\nu(ds) + \sigma\int_0^\infty e^{-v_0(\nu)s}\,dB(s)\right).$$

In both cases, the constant c is given by (3.3.2).

Remark 3.3.1. The case $v_0(\nu) < 0$ is discussed in [36]. It turns out in this case that all solutions converge weakly to a stationary distribution. The trading strategies of the speculators in our market model forces the measure ν to obey $v_0(\nu) \geq 0$. Thus the situation where $v_0(\nu) < 0$ has no economical interpretation and will therefore not be considered.

Remark 3.3.2. The analogue of this Theorem can be found in chapter 2. Part (a) corresponds to Theorem 2.3.1 and part (b) corresponds to Theorem 2.3.2. The underlying idea of the proof is the same as that of chapter 2.

We notice that if $\sigma \neq 0$, then the random variable on the righthand side in (b) of Theorem 3.3.1 is normally distributed with non–zero variance. Thus, there is a positive probability that the limit is positive, and a positive probability that the limit is negative.

Given a measure ν it is often a rather delicate issue to determine the value of $v_0(\nu)$. In the following result we give sufficient conditions for this for a subclass of $M[-\tau, 0]$ which will cover the economic modelling later.

Theorem 3.3.2. *Suppose that* $0 \neq \nu \in M[-\tau, 0]$ *obeys*

$$\nu([-t, 0]) \geq 0 \quad \textit{for all } t \in [0, \tau], \tag{3.3.3}$$

$$\nu([-\tau, 0]) = 0. \tag{3.3.4}$$

(i) If

$$m(\nu) := \int_{[-\tau,0]} s\, \nu(ds) > 1 \tag{3.3.5}$$

then h *has a simple zero at* $\lambda = v_0(\nu) > 0$ *and all other zeros* λ *of* h *obey* $Re\,(\lambda) < v_0(\nu)$.

(ii) If

$$m(\nu) := \int_{[-\tau,0]} s\, \nu(ds) < 1 \tag{3.3.6}$$

then h *has a simple zero at* $\lambda = v_0(\nu) = 0$ *and all other zeros* λ *of* h *obey* $Re\,(\lambda) < v_0(\nu)$.

Remark 3.3.3. The condition given by 3.3.3 is required for exponential growth and 3.3.4 is required for the Law of the Iterated Logarithm. Equations 3.3.5 and 3.3.6 define the stability condition for an unstable and a stable market. The analogue of these two conditions in chapter 2 are 2.2.5 and 2.2.3 respectively.

Remark 3.3.4. For this proof we introduce the function $P(\lambda)$ which is written in terms of the characteristic equation. To prove the result for the unstable case we apply Lebesgue's Theorem which implies there exists a unique $\lambda_0 > 0$ so that $P(\lambda_0) = 1$. We show that $\lambda_0 >$ is simple by differentiation and then show that $\lambda_1 < \lambda_0$. An outline of the proof in the stable case is omitted as it is straightforward.

3.4 Applications to Financial Markets

3.4.1 Economic modelling

We now consider equation (3.2.8) in the context of a market model. Suppose that there are N traders in the economy, who determine their demand based on the cumulative returns Y on an asset. The trading strategy of the j-th agent at time t is as follows: he considers a short–run moving average of the cumulative returns price over the last θ_j units of time

$$\int_{[-\theta_j,0]} Y(t+u)\, s_j(du)$$

for a measure $s_j \in M[-\theta_j, 0]$ and also calculates a long–run average of cumulative returns over the last $\tau_j \geq \theta_j$ units of time

$$\int_{[-\tau_j,0]} Y(t+u)\, l_j(du)$$

for a measure $l_j \in M[-\tau_j, 0]$. The measures s_j and l_j reflect the weights the agent puts on the different past values. In order to make the short-run and long-run comparable the measures s_j and l_j are chosen such that

$$s_j([-\theta_j, 0]) = l_j([-\tau_j, 0]). \tag{3.4.1}$$

We extend s_j to $M[-\tau_j, 0]$ by setting $s_j(I) = 0$ for any Borel set $I \subseteq [-\tau_j, \theta_j)$. These averages can be distinguished as being "short–run" and "long–run" by hypothesising that the short–run average always allocates at least as much weight to the most recent t time units of returns as the long–run average does. Mathematically, this means that

$$\int_{[-t,0]} s_j(du) \geq \int_{[-t,0]} l_j(du), \quad t \in [0, \tau_j,]. \tag{3.4.2}$$

The averages are distinguishable by presuming that $s_j \neq l_j$.

Trader j then has planned demand at time t which depends upon the strength of the signal received from the market, the signal being stronger the greater the difference between the short–run and long run–average. We assume in the sequel that the trader buys the asset if the short–run average exceeds the long–run average and that he sells the asset if the short–run average lies below the long run average. The converse situation can be analysed analogously. The planned excess demand of trader j at time t is

$$\beta_j \left(\int_{[-\theta_j,0]} Y(t+u)\, s_j(du) - \int_{[-\tau_j,0]} Y(t+u)\, l_j(du) \right)$$

where $\beta_j \geq 0$. Therefore, the overall planned excess demand of all traders is

$$\sum_{j=1}^{N} \beta_j \left(\int_{[-\theta_j,0]} Y(t+u)\, s_j(du) - \int_{[-\tau_j,0]} Y(t+u)\, l_j(du) \right).$$

The constants β_j model the different influence of each trader on the total excess demand. Speculators react to other random stimuli— "news"— which are independent of past returns. The increments of this news are independent, so if the stimulus is a continuous process, this may be thought of as adding a further $\sigma(B(t_2) - B(t_1))$ to the traders' excess demand over the interval $[t_1, t_2]$ where B is a one–dimensional Brownian motion and $\sigma \geq 0$.

Finally, we suppose that returns increase when there is excess demand (resp. fall when there is excess supply), with the rise (resp. fall) being larger the greater the excess demand (resp. supply). One way to capture this is to suppose that the evolution of the returns is described by

$$dY(t) \quad = \quad \sum_{j=1}^{N} \beta_j \left(\int_{[-\theta_j,0]} Y(t+u)\, s_j(du) - \int_{[-\tau_j,0]} Y(t+u)\, l_j(du) \right) dt \ + \ \sigma\, dB(t). \tag{3.4.3}$$

We extend all measures s_j and l_j to the interval $[-\tau, 0]$ where $\tau = \max\{\tau_1, \dots, \tau_N\}$ by setting them zero outside their support. By introducing the measure $\nu \in M[-\tau, 0]$ defined by

$$\nu(du) := \sum_{j=1}^{N} \beta_j (s_j - l_j)(du) \tag{3.4.4}$$

and the linear functional L defined by

$$L : C[-\tau, 0] \to \mathbb{R}, \qquad L\phi = \int_{[-\tau,0]} Y(t+u)\, \nu(du)$$

we can rewrite equation (3.4.3) as

$$dY(t) = L(Y_t)\, dt + \sigma\, dB(t) \qquad \text{for all } t \geq 0.$$

Note, that under the conditions (3.4.1) and (3.4.2) on the measures s_j and l_j the measure ν satisfies the conditions in Theorem 3.3.2.

The evolution of the price of the risky asset $(S(t) : t \geq 0)$ is now given by

$$dS(t) = \mu S(t)\, dt + S(t)\, dY(t), \quad t \geq 0; \qquad S(0) = s_0 > 0. \tag{3.4.5}$$

We can think of μ as the non-random interest rate in the model and consider $\mathbb{P}$ as the equivalent risk-neutral measure. Applying Itô's formula shows as in the standard Black-Scholes model that the asset S can be represented by

$$S(t) = S(0) \exp\left(Y(t) + (\mu - \tfrac{1}{2}\sigma^2)t\right) \qquad \text{for all } t \geq 0. \tag{3.4.6}$$

In the case when the feedback traders are absent, i.e. $\beta_j = 0$ for all $j = 1, \dots, N$, we have $dY(t) = \sigma\, dB(t)$, in which case S is Geometric Brownian motion, evolving according to

$$dS(t) = \mu S(t)\, dt + \sigma S(t)\, dB(t), \quad t \geq 0; \qquad S(0) = s_0 > 0.$$

3.4.2 Economic interpretation of main results

Before considering specific examples of types of moving average strategies of the traders, we make some general comments about the economic implications of these results.

One of the most important consequence of our model is that the presence of trend-following agents makes the market more volatile. Suppose momentarily that the Brownian motion B is extended on all R. If ν obeys the conditions in Theorem 3.3.2 and $m(\nu) < 1$ then it can be shown that

$$Y(t) = \sigma c^* B(t) + \sigma \int_{-\infty}^{t} (r(t-s) - c^*)\, dB(s)$$

is a solution of the first equality in (3.2.8), which means that this solution of (3.2.8) can be written as the sum of a Brownian motion plus a stationary Gaussian process. The implication for the financial model is that the driving semimartingale Y is composed of a process with independent and stationary increments, plus a correlated process. It is this correlated process which is responsible for short–term trends that can arise in the price and this makes the market informationally inefficient. Also, as $c^* > 1$, we see that the largest fluctuations of Y are greater than would occur if in the standard Black-Scholes model with $\nu = 0$, i.e. $\beta_j = 0$ for all $j = 1, \ldots, N$. Therefore, the presence of trend–following speculators also make the market more volatile.

If $m(\nu) < 1$ then combining Theorem 3.3.1 and Theorem 3.3.2 implies

$$\limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2t \log\log t}} = \frac{\sigma}{1 - \int_{[-\tau,0]} u\,\nu(du)}, \quad \text{a.s.}$$

which because of (4.3.9) yields

$$\limsup_{t\to\infty} \frac{\log S(t) - (\mu - \frac{1}{2}\sigma^2)t}{\sqrt{2t \log\log t}} = \frac{\sigma}{1 - \int_{[-\tau,0]} u\,\nu(du)}, \quad \text{a.s.}$$

with a similar result available for the liminf. Therefore the process S experiences larger fluctuations from the trend rate of growth than it experiences in the absence of the trend chasing speculators. In other words, the presence of the trend chasing speculators makes the market more risky and leads to greater fluctuations. Moreover, the fluctuations increase in size with

$$\begin{aligned}\int_{[-\tau,0]} u\,\nu(du) &= \sum_{j=1}^{N} \beta_j \left(\int_{[-\theta_j,0]} u s_j(du) - \int_{[-\tau_j,0]} u l_j(du) \right) \\ &=: \sum_{j=1}^{N} m(s_j) - m(l_j).\end{aligned}$$

We now investigate what factors increase this quantity. Note, that in the most common situation when the measures s_j and l_j are non-negative then the quantities $m(s_j)$ and $m(l_j)$ are negative. The larger the absolute quantity $|m(s_j)|$, the greater the weight that trader j gives to recent returns when computing their short–run moving average. Therefore, $m(s_j)$ is a weight of the effective length of the "short–run" memory of trader j. In a similar manner, the quantity $m(l_j)$ is a weight of the effective length of the "long–run" memory of trader j. The greater the difference $m(s_j) - m(l_j)$ between these times, the larger the value of $m(\nu)$ and the more unstable that the market becomes. It may be seen that a large value of $m(s_j) - m(l_j)$ arises, for example, when trader j bases their short–run average on returns over a very short time–horizon, but whose long–run average gives significant weight to returns from the relatively distant past. This strategy can obviously introduce significant feedback from the distant past, so causing trends from the returns in the past to persist for long periods of time, which will tend to cause excess volatility. As we shortly see, it can even lead to the formation of bubbles or crashes as well.

To take a simple example: if all traders make their decisions based only on a comparison of returns θ periods ago with returns τ periods ago, where $\theta < \tau$, then we have $m(s_j) = \theta$ and $m(l_j) = \tau$, and so bubbles form if $(\theta - \tau)\sum_{j=1}^{N} \beta_j > 1$ while we have a correlated Brownian motion market if $(\theta - \tau)\sum_{j=1}^{N} \beta_j < 1$.

A large value of β_j corresponds to aggressive or confident speculative behaviour. The planned excess demand of trader j is β_j times the difference between the short–run and long–run weighted averages of returns. Therefore, for larger β_j, a smaller signal from the market is required to produce a given response from trader j. Hence, as an increase in β_j also increases $m(\nu)$, aggressive or overconfident trend chasing strategies will tend to increase the market volatility, as switches from an advancing to a declining market are amplified by the trend chasing strategy, causing increased volatility and greater extreme fluctuations in the price.

Finally, the greater the value of α, the more responsive is the price to changes in demand and the greater the value of $m(\nu)$.

Summarising these effects, we see that aggressive responses from traders, giving significant weight to the returns in the more distant past and responsiveness in the price to changes in demand, will all tend to destabilise the market. In fact, when these effects are so pronounced that $m(\nu) > 1$ we have that $\lim_{t\to\infty} e^{-v_0(\nu)t}Y(t) =: \Gamma$ exists and is almost surely non–zero, and it assumes a positive and negative value with positive probability. In this case, the aggressive response and long memory of the traders is sufficient to force the market into a bubble (when $\Gamma > 0$) or a crash (when $\Gamma < 0$). Therefore, the more aggressive the responses from traders and the greater the weight that they allocate to returns in the more distant past, the more readily the market leaves the correlated Brownian motion regime ($m(\nu) < 1$), and enters the bubble or crash regime ($m(\nu) > 1$).

3.4.3 Bubble dynamics

In the case when $\nu \in M([-\tau, 0], \mathbb{R})$ obeys (3.3.3) and (3.3.4) and is such that (3.3.5), then there is a unique $v_0(\nu) > 0$ such that a.s.

$$\lim_{t\to\infty} e^{-v_0(\nu)t}Y(t)$$
$$= c\left(\phi(0) + \int_{[-\tau,0]}\int_s^0 e^{v_0(\nu)(s-u)}\phi(u)\,du\,\nu(ds) + \sigma\int_0^\infty e^{-v_0(\nu)s}\,dB(s)\right) =: \Gamma(\phi). \quad (3.4.7)$$

where the constant c is given by (3.3.2). We say that the market experiences a *bubble* if $\Gamma = \Gamma(\phi) > 0$ and a *crash* if $\Gamma = \Gamma(\phi) < 0$, because in the former case $Y(t) \to \infty$ as $t \to \infty$ at an exponential rate, while in the latter $Y(t) \to -\infty$ as $t \to \infty$. We remark that $\Gamma(\phi) \neq 0$ a.s. because Γ is normally distributed with non–zero variance. Therefore only bubbles or crashes can occur when ν obeys (3.3.3), (3.3.4) and (3.3.5). In the next theorem, we analyse the dependence of the probability of a crash or bubble according to the behaviour of the initial returns ϕ on the interval $[-\tau, 0]$.

Theorem 3.4.1. *Suppose that $\nu \in M([-\tau, 0], \mathbb{R})$ obeys* (3.3.3) *and* (3.3.4) *and is such that* (3.3.5)*. Let Y be the solution of* (3.1.2a) *with initial condition $\phi \in C([-\tau, 0], \mathbb{R})$.*

(i) If ϕ is constant, then $\mathbb{P}[\Gamma(\phi) > 0] = 1/2$.

(ii) Let $Y(\phi_1)$ be the solution of (3.1.2a) *with initial condition ϕ_1 and $Y(\phi_2)$ be the solution of* (3.1.2a) *with initial condition ϕ_2. If $\phi_1 - \phi_2$ is constant then*

$$\mathbb{P}[\Gamma(\phi_1) > 0] = \mathbb{P}[\Gamma(\phi_2) > 0].$$

(iii) Let $\phi \in C([-\tau, 0], \mathbb{R})$ be such that $\mathbb{P}[\Gamma(\phi) > 0] > 1/2$. Then $\alpha \mapsto \mathbb{P}[\Gamma(\alpha\phi) > 0]$ is increasing and moreover

$$\lim_{\alpha\to\infty} \mathbb{P}[\Gamma(\alpha\phi) > 0] = 1, \qquad \lim_{\alpha\to-\infty} \mathbb{P}[\Gamma(\alpha\phi) > 0] = 0. \quad (3.4.8)$$

(iv) Let $\phi \in C([-\tau, 0], \mathbb{R})$ be such that $\mathbb{P}[\Gamma(\phi) > 0] < 1/2$. Then $\alpha \mapsto \mathbb{P}[\Gamma(\alpha\phi) > 0]$ is decreasing and moreover

$$\lim_{\alpha\to\infty} \mathbb{P}[\Gamma(\alpha\phi) > 0] = 0, \qquad \lim_{\alpha\to-\infty} \mathbb{P}[\Gamma(\alpha\phi) > 0] = 1. \quad (3.4.9)$$

(v) If $\phi \in C^1([-\tau, 0], \mathbb{R})$ is increasing with $\phi'(0) > 0$, then $\mathbb{P}[\Gamma(\phi) > 0] > 1/2$. Moreover $\alpha \mapsto \mathbb{P}[\Gamma(\alpha\phi) > 0]$ is increasing and obeys (3.4.8)*.*

(vi) If $\phi \in C^1([-\tau,0],\mathbb{R})$ is decreasing with $\phi'(0) < 0$, then $\mathbb{P}[\Gamma(\phi) > 0] < 1/2$. Moreover $\alpha \mapsto \mathbb{P}[\Gamma(\alpha\phi) > 0]$ is decreasing and obeys (3.4.9).

Remark 3.4.1. This Theorem and its interpretation is an analogue of Theorem 2.3.3 in chapter 2. As the proofs are very similar no outline is given.

3.4.4 Positive and exponentially fading autocovariance in the returns

In this subsection, we analyse the patterns in the δ–returns, where $\delta > 0$ in the situation where the measure ν obeys (3.3.3) (3.3.4) and the stability condition (3.3.6). The δ–returns are simply the percentage gains or losses made by investing over a time period of δ units, and are denoted at time t by $Y_\delta(t)$. Accordingly we define the process Y_δ by

$$Y_\delta(t) := Y(t) - Y(t-\delta), \quad t \geq \delta. \tag{3.4.10}$$

It is convenient to extend r to $(-\infty,-\tau)$ by setting $r(t) = 0$ for $t \in (-\infty,-\tau)$, and to introduce the function r_δ defined by

$$r_\delta(t) = r(t) - r(t-\delta), \quad t \geq 0. \tag{3.4.11}$$

Since $Y(t) = y(t) + \int_0^t r(t-s)\sigma\,dB(s)$ for $t \geq 0$, we have the identity

$$Y_\delta(t) = y(t) - y(t-\delta) + \int_0^t r_\delta(t-s)\sigma\,dB(s), \quad t \geq \delta. \tag{3.4.12}$$

The next theorem shows firstly that Y_δ is an asymptotically stationary process for each $\delta > 0$. Secondly, we show that it is positively autocorrelated at all time horizons. Thirdly, it is shown that the autocorrelation of δ returns separated by Δ time units, decays at an exact exponential rate in Δ as $\Delta \to \infty$. We state the result.

Theorem 3.4.2. *Let $\delta > 0$, $\Delta \geq 0$. Suppose that $\nu \in M([-\tau,0],\mathbb{R})$ obeys* (3.3.3) *and* (3.3.4) *and* (3.3.6). *Let Y be the solution of* (3.2.8) *with initial condition $\phi \in C([-\tau,0],\mathbb{R})$. Suppose that r is given by* (3.2.2), *r_δ by* (3.4.11)*, and Y_δ by* (3.4.12). *Then*

(i)

$$\text{Cov}(Y_\delta(t), Y_\delta(t+\Delta)) > 0, \quad \text{for all } t \geq \delta. \tag{3.4.13}$$

(ii) For every $\Delta \geq 0$ the limit

$$c_\delta(\Delta) := \lim_{t\to\infty} \text{Cov}(Y_\delta(t), Y_\delta(t+\Delta)) = \sigma^2 \int_0^\infty r_\delta(u) r_\delta(u+\Delta)\,du \tag{3.4.14}$$

exists and is finite. Moreover, $\lim_{\Delta\to\infty} c_\delta(\Delta) = 0$ and $c_\delta \in L^1(0,\infty)$ for each $\delta > 0$.

(iii) If ν is such that

$$\text{Leb}\{t \in [-\tau,0] : \nu([t,0]) > 0\} > 0,$$

there exists a unique $\lambda > 0$ such that $-\lambda \in \Lambda$ and $c_\delta(\Delta)$ obeys

$$\lim_{\Delta\to\infty} c_\delta(\Delta)e^{\lambda\Delta} = \frac{\sigma^2(1-e^{\lambda\delta})}{1-\int_{[-\tau,0]} ue^{-\lambda u}\nu(du)} \left(\int_0^\delta r(u)e^{-\lambda u}\,du + \frac{(1-e^{-\lambda\delta})\int_{[-\tau,0]} e^{\lambda u}\nu(du)}{\lambda(\lambda - \int_{[-\tau,0]} e^{\lambda u}\nu(du))} \right), \tag{3.4.15}$$

with the limit being finite and positive.

Remark 3.4.2. This Theorem is an analogue of Theorem 2.4.1 in chapter 2 and the proof is very similar.

Once again the proof is postponed to the end. We make some further observations and comments. An interesting conclusion of the theorem is that the δ–returns are positively autocorrelated. Therefore, even though the returns undergo iterated logarithm behaviour like standard Brownian motion, there is correlation between the increments of the process. The presence of a positive correlation means that trends in the returns have a tendency to persist. This is responsible for the fact that the largest fluctuations of the process Y are greater that those that would be seen if there were no trend–following speculators present. The correlation between returns of horizon length δ decays exponentially in the time lag Δ between successive observations, as $\Delta \to \infty$. Moreover, the exponent in the rate of decay is independent of Δ. Therefore, although the market is informationally inefficient in the sense of Fama because the future returns are correlated with past returns, the memory of recent events is discounted relatively quickly. This "short memory" is a consequence of the finite memory trading strategies employed by agents.

3.4.5 Examples of investment strategies

In each of the following examples, we consider only one agent and his or her trading strategy. Because of that we neglect the parameters β_j in the model.

Current returns versus past returns

Suppose that the investor compares the current value of the cumulative returns Y with a continuous time weighted average over the last τ units. To put this in the form of the model considered in the beginning of this section, the current value of the cumulative returns is weighted by

$$s(du) := \alpha\delta_0(du)$$

for a constant $\alpha > 0$ and where δ_0 denotes the Dirac measure in 0. The cumulative returns in the long-run are weighted by

$$l(du) := f(u)\, du,$$

where f is a nonnegative function in $L^1([-\tau, 0])$ with $\|f\|_{L^1} = \alpha$ and for some $\tau > 0$. Then the measure

$$\nu(du) := s(du) - l(du) = \alpha\delta_0(du) - f(u)\, du$$

satisfies the conditions in Theorem 3.3.2 with the moment given by

$$m(\nu) = -\int_{-\tau}^{0} uf(u)\, du.$$

The linear functional L is of the form

$$L : C([-\tau, 0]) \to R, \qquad L(\phi) = \alpha\phi(0) - \int_{-\tau}^{0} \phi(u)f(u)\, du.$$

Thus, if $m(\nu) < 1$ then the cumulative returns obey

$$\limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2t \log\log t}} = \frac{\sigma}{1 + \int_{-\tau}^{0} uf(u)\, du} \quad \text{a.s.},$$
$$\liminf_{t\to\infty} \frac{Y(t)}{\sqrt{2t \log\log t}} = -\frac{\sigma}{1 + \int_{-\tau}^{0} uf(u)\, du} \quad \text{a.s.}$$

On the other hand if $m(\nu) > 1$ then there exists a unique $\lambda > 0$ such that a.s.

$$\lim_{t\to\infty} e^{-\lambda t} Y(t)$$
$$= \frac{1}{1 + \int_{-\tau}^{0} ue^{\lambda u} f(u)\, du} \left(\phi(0) - \int_{-\tau}^{0} f(s) \int_{s}^{0} e^{\lambda(s-u)}\phi(u)\, du\, ds + \int_{0}^{\infty} \sigma e^{-\lambda s}\, dB(s) \right).$$

Short run versus long run moving averages

Suppose that the investor compares a continuous time weighted average of the cumulative returns Y over the last θ units of time with a moving average over the last $\tau \geq \theta$ units of time. The short-run is weighted by a nonnegative function $f \in L^1([-\theta, 0])$ whereas the long-run by a nonnegative function $g \in L^1([-\tau, 0])$ with $\|f\|_{L^1} = \|g\|_{L^1} > 0$. We extend f to $[-\tau, 0]$ by setting $f(u) = 0$ for $u \in [-\tau, -\theta)$. If we suppose in addition that

$$\int_{-t}^{0} f(u)\,du \geq \int_{-t}^{0} g(u)\,du \qquad \text{for all } t \in [-\tau, 0],$$

then the measure $\nu(du) := (f(u) - g(u))\,du$ satisfies all the conditions in Theorem 3.3.2 with the moment

$$m(\nu) = \int_{-\tau}^{0} u(f(u) - g(u))\,du.$$

The linear functional L is given by

$$L : C([-\tau, 0]) \to R, \qquad L(\phi) = \int_{-\tau}^{0} \phi(u)(f(u) - g(u))\,du.$$

If we have $m(\nu) < 1$ then combining Theorem 3.3.1 and Theorem 3.3.2 yield

$$\limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2t \log\log t}} = \frac{\sigma}{1 - \int_{-\tau}^{0} u(f(u) - g(u))\,du} \quad \text{a.s.}$$
$$\liminf_{t\to\infty} \frac{Y(t)}{\sqrt{2t \log\log t}} = -\frac{\sigma}{1 - \int_{-\tau}^{0} u(f(u) - g(u))\,du} \quad \text{a.s.}$$

On the other hand, if $m(\nu) > 1$ then there exists a unique positive $\lambda > 0$ such that a.s.

$$\lim_{t\to\infty} e^{-\lambda t} Y(t)$$
$$= c\left(\phi(0) + \int_{\tau}^{0} (f(s) - g(s)) \int_{s}^{0} e^{\lambda(s-u)} \phi(u)\,du\,ds + \int_{0}^{\infty} \sigma e^{-\lambda s}\,dB(s)\right)$$

where

$$c = \frac{1}{1 - \int_{-\tau}^{0} s e^{\lambda s}(f(s) - g(s))\,ds}.$$

Discrete–time moving averages

Suppose that the investor compares a weighted average of the cumulative returns at m points in time over the last θ units of time with a weighted average of the cumulative returns at n points in time over the last τ units of time, where $\tau \geq \theta$. Let the cumulative returns in the short-run be observed at time points $-\theta := -\theta_m < \cdots < -\theta_1 \leq 0$ and in the long-run at time points $-\tau := -\tau_n < \cdots < -\tau_1 \leq 0$. Then the short-run observations are averaged according to a measure

$$s(du) := \sum_{j=1}^{m} \alpha_j \delta_{-\theta_j}(du)$$

for some weights $\alpha_j \geq 0$ and the long-run observations according to

$$l(du) := \sum_{j=1}^{n} \beta_j \delta_{-\tau_j}(du)$$

for some weights $\beta_j \geq 0$. If we assume that

$$\alpha_1 + \cdots + \alpha_m = \beta_1 + \cdots + \beta_n > 0,$$
$$\sum_{j=1}^{m} \alpha_j \chi_{[-t,0]}(-\theta_j) \geq \sum_{j=1}^{n} \beta_j \chi_{[-t,0]}(-\tau_j) \qquad \text{for all } t \in [0,\tau],$$

then the measure $\nu(du) := s(du) - l(du)$ satisfies all the conditions in Theorem 3.3.2 with the moment

$$m(\nu) = \sum_{j=1}^{n} \beta_j \tau_j - \sum_{j=1}^{m} \alpha_j \theta_j.$$

The linear functional L is given by

$$L : C([-\tau, 0]) \to R, \qquad L(\phi) = \sum_{j=1}^{m} \alpha_j \phi(-\theta_j) - \sum_{j=1}^{n} \beta_j \phi(-\tau_j).$$

If $m(\nu) < 1$ then the cumulative returns evolve according to

$$\limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2t \log\log t}} = \frac{\sigma}{1 - \sum_{j=1}^{n} \beta_j \tau_j + \sum_{j=1}^{m} \alpha_j \theta_j} \quad \text{a.s}$$
$$\liminf_{t\to\infty} \frac{Y(t)}{\sqrt{2t \log\log t}} = -\frac{\sigma}{1 - \sum_{j=1}^{n} \beta_j \tau_j + \sum_{j=1}^{m} \alpha_j \theta_j} \quad \text{a.s.}$$

On the other hand, if $m(\nu) > 1$ then there exists a unique positive $\lambda > 0$ such that a.s.

$$\lim_{t\to\infty} e^{-\lambda t} Y(t) = c\left(\phi(0) + \int_0^\infty \sigma e^{-\lambda s}\, dB(s)\right)$$
$$+ c\left(\sum_{j=1}^{m} \alpha_j \int_{-\theta_j}^{0} e^{-\lambda(\theta_j + u)} \phi(u)\, du - \sum_{j=1}^{n} \beta_j \int_{-\tau_j}^{0} e^{-\lambda(\tau_j+u)} \phi(u)\, du\right),$$

where

$$c = \frac{1}{1 - \sum_{j=1}^{n} \beta_j \tau_j e^{-\lambda \tau_j} + \sum_{j=1}^{m} \alpha_j \theta_j e^{-\lambda\theta_j}}.$$

3.5 Proof of Theorem 3.3.1

We start this section by proving a kind of law of the iterated logarithm for the Gaussian process $(Q(t) : t \geq 0)$ defined by

$$Q(t) := \int_0^t f(t-s)\, dB(s) \tag{3.5.1}$$

for a differentiable function $f \in L^2(\mathbb{R}^+)$ with $f' \in L^2(\mathbb{R}^+)$, ($f \in W^{2,1}([0,\infty))$). Instead of following the direct proof one can also use general results on the law of iterated logarithm for Gaussian process, see for example the monograph [46].

Lemma 3.5.1. *Suppose that $f \in L^2(\mathbb{R}^+)$ with $f' \in L^2(\mathbb{R}^+)$. Then the Gaussian process Q defined in* (3.5.1) *satisfies*

$$\limsup_{t\to\infty} \frac{Q(t)}{\sqrt{2\log t}} \leq \|f\|_{L^2}^{1/2} \quad \textit{a.s.}$$

Remark 3.5.1. For this proof we decompose equation 3.5.1 into three terms. By Lemma 3.9.4 we can show that the limit of the first term is zero. For the second term we calculate a bound on it's second moment. Then using the Borel–Cantelli Lemma we show that it's limit is less than one. By applying Mill's estimate and the Borel–Cantelli Lemma to the third term we prove the above result.

Lemma 3.5.2. *Suppose that B is a standard Brownian motion. Then for every $\epsilon \in (0,1)$ we have*

$$\lim_{n\to\infty} \sup_{n^\epsilon \le t \le (n+1)^\epsilon} \frac{|B(t)-B(n^\epsilon)|}{\sqrt{\epsilon \log n}} = 0 \quad a.s. \tag{3.5.2}$$

Remark 3.5.2. See Section 3.9 for the proof of the Lemma. As the proof is short no outline is given.

Proof. Because f is differentiable we obtain by partial integration and the stochastic Fubini's Theorem

$$\begin{aligned} Q(t) &= \int_0^t \left(f(0) + \int_0^{t-s} f'(u)\,du \right) dB(s) \\ &= \int_0^t \left(\int_0^v f'(v-s)\,dB(s) \right) dv + f(0)B(t). \end{aligned}$$

Thus, for arbitrary $\epsilon \in (0,1)$ and $n \in \mathbb{N}$ we can decompose $Q(t)$ according to

$$\begin{aligned} Q(t) &= Q(t) - Q(n^\epsilon) + Q(n^\epsilon) \\ &= f(0)(B(t)-B(n^\epsilon)) + \int_{n^\epsilon}^t \left(\int_0^v f'(v-s)\,dB(s) \right) dv + Q(n^\epsilon). \end{aligned} \tag{3.5.3}$$

We now analyse each term in (3.5.3). By Lemma 3.9.4, the first term obeys

$$\lim_{n\to\infty} \sup_{n^\epsilon \le t \le (n+1)^\epsilon} \frac{|f(0)||B(t)-B(n^\epsilon)|}{\sqrt{\epsilon \log n}} = 0 \quad \text{a.s.} \tag{3.5.4}$$

For estimating the second term in (3.5.3) choose $k \in \mathbb{N}$ such that $(1-\epsilon)2k > 1$ and let

$$U_n = \sup_{n^\epsilon \le t \le (n+1)^\epsilon} \left| \int_{n^\epsilon}^t \left(\int_0^v f'(v-s)\,dB(s) \right) dv \right|.$$

Applying Cauchy-Schwarz inequality implies

$$\begin{aligned} \mathbb{E}\left[{U_n}^{2k} \right] &= \mathbb{E}\left[\sup_{n^\epsilon \le t \le (n+1)^\epsilon} \left| \int_{n^\epsilon}^t \left(\int_0^v f'(v-s)\,dB(s) \right) dv \right|^{2k} \right] \\ &\le \mathbb{E}\left[\sup_{n^\epsilon \le t \le (n+1)^\epsilon} (t-n^\epsilon)^{2k-1} \left(\int_{n^\epsilon}^t \left| \left(\int_0^v f'(v-s)\,dB(s) \right) \right|^{2k} dv \right) \right] \\ &= ((n+1)^\epsilon - n^\epsilon)^{2k-1} \int_{n^\epsilon}^{(n+1)^\epsilon} \mathbb{E}\left[J(v)^{2k} \right] dv, \end{aligned}$$

where $J(v) = \int_0^v f'(v-s)\,dB(s)$ is normally distributed with zero mean and variance $\int_0^v f'(s)^2\,ds$. Since $f' \in L^2(0,\infty)$ we have $\mathbb{E}[J(v)^{2k}] = C_k \left(\int_0^v f'(s)^2\,ds \right)^k \le C_k \|f'\|_{L^2}^k$. Moreover for every n there exists $n_* \in [n, n+1]$ such that $(n+1)^\epsilon - n^\epsilon = \epsilon n_*^{\epsilon-1} \le \epsilon n^{\epsilon-1}$. Because $(1-\epsilon)2k > 1$, $\mathbb{E}[U_n^{2k}]$ is summable. By Chebyshev's Inequality, $\mathbb{P}[|U_n| \ge 1] \le \mathbb{E}[U_n^{2k}]$, and so the Borel–Cantelli Lemma implies

$$\limsup_{n\to\infty} \sup_{n^\epsilon \le t \le (n+1)^\epsilon} \left| \int_{n^\epsilon}^t \left(\int_0^v f'(v-s)\,dB(s) \right) dv \right| \le 1 \quad \text{a.s.} \tag{3.5.5}$$

For estimating the last term in (3.5.3) we define the standardized normal random variable

$$Z_{n^\epsilon} := Q(n^\epsilon) \left(\int_0^{n^\epsilon} f^2(s)\,ds \right)^{-1/2}.$$

For any $\theta > 1$ we get by Mill's estimate

$$\begin{aligned} \mathbb{P}\left[|Z_{n^\epsilon}| > \sqrt{2\theta \log(n^\epsilon)} \right] &\le \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\theta \log(n^\epsilon)}} \cdot e^{-\theta \log(n^\epsilon)} \\ &= \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\theta \log(n^\epsilon)}} \cdot \frac{1}{(n^\epsilon)^\theta}. \end{aligned}$$

Thus, choosing $\theta\epsilon > 1$, we get

$$\sum_{n=2}^{\infty} \mathbb{P}\left[|Z_{n^\epsilon}| > \sqrt{2\theta \log(n^\epsilon)}\right] < \infty.$$

According to the Borel–Cantelli lemma, there exists a random integer n_0 such that

$$\limsup_{n\to\infty} \frac{|Z_{n^\epsilon}|}{\sqrt{2\log(n^\epsilon)}} \le \sqrt{\theta} \quad \text{a.s.}$$

Letting $\theta \to \frac{1}{\epsilon}$ through the rational numbers, we get

$$\limsup_{t\to\infty} \frac{Q(n^\epsilon)}{\sqrt{2\log n^\epsilon}} \le \frac{1}{\sqrt{\epsilon}}\sqrt{\|f\|_{L^2}}. \tag{3.5.6}$$

Finally, (3.5.4), (3.5.5) and (3.5.6) allows us to conclude for the decomposition (3.5.3) that

$$\limsup_{n\to\infty} \sup_{n^\epsilon \le t \le (n+1)^\epsilon} \frac{|Q(t)|}{\sqrt{2\log t}} \le \frac{1}{\sqrt{\epsilon}}\sqrt{\|f\|_{L^2}} \quad \text{a.s.,}$$

which implies

$$\limsup_{t\to\infty} \frac{|Q(t)|}{\sqrt{2\log t}} \le \frac{1}{\sqrt{\epsilon}}\sqrt{\|f\|_{L^2}}, \quad \text{a.s.}$$

Finally, by letting $\epsilon \nearrow 1$ through the rational numbers, we get

$$\lim_{t\to\infty} \frac{|Q(t)|}{\sqrt{2\log t}} \le \sqrt{\|f\|_{L^2}} \quad \text{a.s.}$$

as required. □

Proof of (a) in Theorem 3.3.1. The solution Y of equation (3.2.8) can be decomposed in

$$\begin{aligned} Y(t) &= y(t) + \sigma \int_0^t r(t-s)\,dB(s) \\ &= y(t) + \sigma Q(t) + \sigma c B(t), \end{aligned}$$

where y is the solution of equation (3.2.1) and Q is defined by

$$Q(t) = \int_0^t (r(t-s) - c)\,dB(s).$$

We next notice that (3.2.3) and the fact that $r(t) \to c$ as $t \to \infty$ together imply that

$$\lim_{t\to\infty} y(t) = \phi(0)c + c\int_{[-\tau,0]}\int_s^0 \phi(u)\,du\,\nu(ds). \tag{3.5.7}$$

By combining the Law of the Iterated Logarithm for standard Brownian motion together Lemma 3.5.1 and (3.5.7), we find that

$$\limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2t\log\log t}} = \sigma c, \qquad \liminf_{t\to\infty} \frac{Y(t)}{\sqrt{2t\log\log t}} = -\sigma c \quad \text{a.s.,}$$

as required. □

To prove the second part of Theorem 3.3.1 we require another Lemma on a convolution Gaussian process, which is similar to the one considered in Lemma 3.5.1.

Lemma 3.5.3. *Define for $\lambda > 0$ and $k \in L^2(0,\infty)$ with $k' \in L^2(0,\infty)$ a Gaussian process $(K(t) : t \geq 0)$ by*

$$K(t) = \int_0^t k(t-s)e^{-\lambda s}\, dB(s).$$

Then $\lim_{t\to\infty} K(t) = 0$ a.s.

Remark 3.5.3. We begin the proof by splitting $K(t)$ into three terms. Applying the Cauchy–Schwarz inequality and taking suprema over $[a_n, a_{n+1}]$ we rewrite these terms in terms of the second moment. We then show that the second moments are finite. By employing Fubini's Theorem we can show that the $\lim_{t\to\infty} K(t) = 0$ almost surely.

Proof. Applying the stochastic Fubini's Theorem we obtain the representation

$$\begin{aligned} K(t) &= \int_0^t \left(k(0) + \int_0^{t-s} k'(u)\, du \right) e^{-\lambda s}\, dB(s) \\ &= k(0) \int_0^t e^{-\lambda s}\, dB(s) + \int_0^t \left(\int_0^{t-s} k'(u)\, du \right) e^{-\lambda s}\, dB(s) \\ &= k(0) \int_0^t e^{-\lambda s}\, dB(s) + \int_0^t \int_0^v k'(v-s) e^{-\lambda s}\, dB(s)\, dv. \end{aligned}$$

Thus, for an arbitrary increasing sequence $(a_n)_{n=0}^{\infty}$ and $t \in [a_n, a_{n+1})$ we have the identity

$$K(t) = K(a_n) + k(0) \int_{a_n}^t e^{-\lambda s}\, dB(s) + \int_{a_n}^t \int_0^v k'(v-s) e^{-\lambda s}\, dB(s)\, dv.$$

Then using the Cauchy–Schwarz inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ and taking suprema over $[a_n, a_{n+1}]$ result in

$$\begin{aligned} \mathbb{E}\left[\sup_{a_n \leq t \leq a_{n+1}} K(t)^2 \right] &\leq 3\mathbb{E}[K(a_n)^2] + 3k(0)^2 \mathbb{E}\left[\sup_{a_n \leq t \leq a_{n+1}} \left(\int_{a_n}^t e^{-\lambda s}\, dB(s) \right)^2 \right] \\ &\quad + 3(a_{n+1} - a_n) \int_{a_n}^{a_{n+1}} \mathbb{E}\left[\left(\int_0^v k'(v-s) e^{-\lambda s}\, dB(s) \right)^2 \right] dv. \end{aligned} \tag{3.5.8}$$

The function $t \mapsto \mathbb{E}[K^2(t)]$ is in $L^1(\mathbb{R}^+)$ because

$$\int_0^\infty \mathbb{E}[K^2(t)]\, dt = \int_0^\infty \int_0^t k^2(t-s) e^{-2\lambda s}\, ds\, dt = \frac{1}{2\lambda} \int_0^\infty k^2(u)\, du < \infty.$$

Thus, the integrability criterion for series implies that we can choose the sequence $(a_n)_{n=0}^{\infty}$ with $a_0 = 0$, $0 < a_{n+1} - a_n < 1$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} a_n = \infty$ such that

$$\sum_{n=1}^{\infty} \mathbb{E}[K(a_n)^2] < +\infty, \tag{3.5.9}$$

see [3, Lemma 3]. Doob's inequality implies

$$\sum_{n=0}^{\infty} \mathbb{E}\left[\sup_{a_n \leq t \leq a_{n+1}} \left(\int_{a_n}^t e^{-\lambda s}\, dB(s) \right)^2 \right] \leq \sum_{n=0}^{\infty} 4 \int_{a_n}^{a_{n+1}} e^{-2\lambda s}\, ds < \infty. \tag{3.5.10}$$

Applying Itô's isometry and letting $e_{2\lambda}(t) = e^{-2\lambda t}$ gives

$$\int_{a_n}^{a_{n+1}} \mathbb{E}\left[\left(\int_0^v k'(v-s) e^{-\lambda s}\, dB(s) \right)^2 \right] dv = \int_{a_n}^{a_{n+1}} (k'^2 * e_{2\lambda})(v)\, dv.$$

Since $k' \in L^2(0,\infty)$ and $e_{2\lambda} \in L^1(0,\infty)$ we have $k'^2 * e_{2\lambda} \in L^1(0,\infty)$, and so, by using the fact that $a_{n+1} - a_n < 1$, it follows that

$$\begin{aligned}\sum_{n=0}^{\infty}(a_{n+1}-a_n)\int_{a_n}^{a_{n+1}} \mathbb{E}\left[\left(\int_0^v k'(v-s)e^{-\lambda s}\,dB(s)\right)^2\right] dv \\ \leq \int_0^\infty (k'^2 * e_{2\lambda})(v)\,dv < \infty.\end{aligned} \tag{3.5.11}$$

Applying (3.5.9), (3.5.10) and (3.5.11) to the representation (3.5.8) gives

$$\sum_{n=0}^{\infty} \mathbb{E}\left[\sup_{a_n \le t \le a_{n+1}} K(t)^2\right] < \infty.$$

Fubini's Theorem implies

$$\sum_{n=0}^{\infty} \sup_{a_n \le t \le a_{n+1}} K(t)^2 < \infty \qquad \text{a.s.}$$

yielding $K(t) \to 0$ a.s. as $t \to \infty$. □

Proof of (b) in Theorem 3.3.1. Define

$$k(t) = e^{-v_0(\nu)t} r(t) - c, \tag{3.5.12}$$

$$K(t) = \int_0^t k(t-s)e^{-v_0(\nu)s}\,dB(s), \tag{3.5.13}$$

where $v_0(\nu) > 0$ is defined by (3.2.6) and c is defined by (3.3.1). By the variation of constants formula (3.2.9) we have

$$e^{-v_0(\nu)t}Y(t) = e^{-v_0(\nu)t}y(t) + \sigma c\int_0^t e^{-v_0(\nu)s}\,dB(s) + \sigma K(t). \tag{3.5.14}$$

The second term on the righthand side of (3.5.14) tends to the almost surely finite random variable $c\sigma\int_0^\infty e^{-v_0(\nu)s}\,dB(s)$ as $t \to \infty$ a.s., by the martingale convergence theorem.

By (3.3.1) the function k is in $L^2(0,\infty)$. In order to prove that k' is also in $L^2(0,\infty)$ note that

$$k'(t) = -v_0(\nu)e^{-v_0(\nu)t}r(t) + \int_{[-\tau,0]} e^{-v_0(\nu)(t+s)}r(t+s)e^{v_0(\nu)s}\,\nu(ds).$$

Because $v_0(\nu)$ is a zero of h we have

$$k'(t) = -v_0(\nu)(e^{-v_0(\nu)t}r(t) - c) + \int_{[-\tau,0]} (e^{-v_0(\nu)(t+s)}r(t+s) - c)e^{v_0(\nu)s}\,\nu(ds).$$

Hence, by (3.3.1), we have that $k' \in L^2(0,\infty)$ which enables us to apply Lemma 3.5.3 and to conclude $K(t) \to 0$ a.s. as $t \to \infty$.

For the first term in (3.5.14) the formula (3.2.3) yields

$$\begin{aligned} e^{-v_0(\nu)t}y(t) &= \phi(0)e^{-v_0(\nu)t}r(t) \\ &\quad + \int_{[-\tau,0]}\int_s^0 e^{-v_0(\nu)(t+s-u)}r(t+s-u)e^{v_0(\nu)(s-u)}\phi(u)\,du\,\nu(ds) \\ &\to \phi(0)c + \int_{[-\tau,0]}\int_s^0 ce^{v_0(u)(s-u)}\phi(u)\,du\,\nu(ds) \qquad \text{as } t \to \infty, \end{aligned}$$

which completes the proof. □

3.6 Proof of Theorem 3.4.1

To prove (i), notice that if $\phi(t) = c$ for all $t \in [-\tau, 0]$, then the solution y of (3.2.1) is $y(t,\phi) = c$ for all $t \geq -\tau$. Therefore $\lim_{t\to\infty} y(t,\phi)/e^{v_0(\nu t)} = 0$, and so

$$\Gamma(\phi) = c\sigma \int_0^\infty e^{-v_0(\nu)s}\, dB(s) =: Z, \tag{3.6.1}$$

where Z is normally distributed with zero mean and variance $c^2\sigma^2/(2v_0(\nu))$. Therefore $\mathbb{P}[\Gamma(\phi) > 0] = \mathbb{P}[Z > 0] = 1/2$ as claimed.

For the proof of (ii), let $y(\phi_1)$ be the solution of (3.2.1) with initial condition ϕ_1 and $y(\phi_2)$ be the solution of (3.2.1) with initial condition ϕ_2. Let $z(t) := y(\phi_1, t) - y(\phi_2, t)$ for all $t \geq -\tau$. Then $z(t) = c$ for all $t \in [-\tau, 0]$ and $z'(t) = L(z_t)$ for $t \geq 0$. Therefore $z(t) = c$ for all $t \geq 0$, or $y(\phi_1, t) - y(\phi_2, t) = c$ for all $t \geq -\tau$. If y is the solution of (3.2.1) we may define the operator $L_1 : C([-\tau, 0], \mathbb{R}) \to \mathbb{R}$ by

$$\lim_{t\to\infty} y(t,\phi)/e^{v_0(\nu t)} = L_1(\phi) \tag{3.6.2}$$

where $L_1(\phi)$ is given by the formula

$$L_1(\phi) = c\left(\phi(0) + \int_{[-\tau,0]} \int_s^0 e^{v_0(\nu)(s-u)}\phi(u)\, du\, \nu(ds)\right), \tag{3.6.3}$$

with $c > 0$ given by (3.3.2). We note that c is independent of ϕ. Since $z(t)/e^{v_0(\nu t)} \to 0$ as $t \to \infty$, we have

$$L_1(\phi_1) - L_1(\phi_2) = \lim_{t\to\infty} \frac{y(\phi_1,t)}{e^{v_0(\nu t)}} - \lim_{t\to\infty} \frac{y(\phi_2,t)}{e^{v_0(\nu t)}} = 0,$$

so $L_1(\phi_1) = L_1(\phi_2)$. Therefore $\Gamma(\phi_1) = \Gamma(\phi_2)$ and (ii) is proven.

We now prove (iii). By (3.6.2) and (3.6.3) we have

$$\Gamma(\phi) = L_1(\phi) + c\sigma \int_0^\infty e^{-v_0(\nu)s}\, dB(s) = L_1(\phi) + Z,$$

where Z is defined by (3.6.1). Therefore if $\mathbb{P}[\Gamma(\phi) > 0] > 1/2$, we have

$$1/2 < \mathbb{P}[\Gamma(\phi) > 0] = \mathbb{P}[L_1(\phi) + Z > 0] = \mathbb{P}[Z > -L_1(\phi)] = 1 - \mathbb{P}[Z \leq -L_1(\phi)],$$

so $\mathbb{P}[Z \leq -L_1(\phi)] < 1/2$, which implies $L_1(\phi) > 0$. Clearly $L_1(\alpha\phi) = \alpha L_1(\phi)$ for any $\alpha \in \mathbb{R}$, and so $\Gamma(\alpha\phi) = \alpha L_1(\phi) + Z$. Now $\mathbb{P}[\Gamma(\alpha\phi) > 0] = \mathbb{P}[Z > -\alpha L_1(\phi)]$, so as $L_1(\phi) > 0$ we have that $\alpha \mapsto \mathbb{P}[\Gamma(\alpha\phi) > 0]$ is increasing, with $\lim_{\alpha\to\infty} \mathbb{P}[\Gamma(\alpha\phi_1) > 0] = 1$ and $\lim_{\alpha\to-\infty} \mathbb{P}[\Gamma(\alpha\phi_1) > 0] = 0$. The proof of (iv) is similar.

We prove (v). Let y be the solution of (3.2.1). Extend ν to $(-\infty, -\tau]$ by setting $\nu(E) = 0$ for every Borel set $E \subset (-\infty, -\tau)$. Next consider $\nu_+(E) = \nu(-E)$ for every $E \subseteq [0, \infty)$. Also extend ϕ to $[-\infty, -\tau)$ by setting $\phi(t) = 0$ for all $t < -\tau$. Then y satisfies

$$y'(t) = \int_{[-\tau,0]} y(t+s)\nu(ds) = \int_{(-\infty,0]} y(t+s)\nu(ds) = \int_{[0,\infty)} y(t-s)\nu_+(ds),$$

so with $F_1(t) = \int_{[t,\infty)} \phi(t-s)\nu_+(ds)$ for $t \geq 0$ we have

$$y'(t) = \int_{[0,t]} y(t-s)\nu_+(ds) + F_1(t), \quad t > 0, \quad y(0) = \phi(0).$$

Define N by

$$N(t) = \int_{[0,t]} \nu_+(ds), \quad t \geq 0, \tag{3.6.4}$$

and F by

$$F(t) = \int_{[0,t]} \phi(0)\nu_+(ds) + \int_{[0,\infty)} \phi(t-s)\nu_+(ds), \quad t \geq 0. \tag{3.6.5}$$

Since $y \in C^1(0,\infty)$ we have

$$\begin{aligned}\int_{[0,t]} y(t-s)\nu_+(ds) &= \int_{[0,t]} \nu_+(ds)\left(\phi(0) + \int_0^{t-s} y'(u)\,du\right)\\ &= N(t)\phi(0) + \int_{[0,t]} \nu_+(ds)\int_0^{t-s} y'(u)\,du\\ &= N(t)\phi(0) + \int_{[0,t]}\int_0^{t-s} \nu_+(ds)y'(u)\,du\\ &= N(t)\phi(0) + \int_0^t \int_{[0,t-u]} \nu_+(ds)y'(u)\,du\\ &= N(t)\phi(0) + \int_0^t N(t-s)y'(u)\,du.\end{aligned}$$

Therefore with F defined by (3.6.5) we get

$$y'(t) = \int_{[0,t]} N(t-u)y'(u)\,du + F(t), \quad t > 0. \tag{3.6.6}$$

For $t \geq \tau$ since (3.3.3) holds and $\nu_+(E) = 0$ for all Borel sets $E \subset [\tau,\infty)$ we have

$$F(t) = \int_{[0,\tau]} \phi(0)\cdot\nu_+(ds) + \int_{(\tau,t]} \phi(0)\cdot\nu_+(ds) + \int_{[t,\infty)} \phi(t-s)\nu_+(ds) = 0.$$

For $0 \leq t \leq \tau$ define $N_1(t) = \int_{[0,t)} \nu_+(ds)$ with $N_1(0) = 0$. Then

$$\begin{aligned}F(t) &= \int_{t-\tau}^0 N_1(t-u)\phi'(u)\,du\\ &= \int_t^\tau N_1(s)\phi'(t-s)\,ds\end{aligned}$$

Since $N_1(t) \geq 0$ for all $t \geq 0$ and $F(t) \geq 0$ for all $t \geq 0$ we have from (3.6.6) that $y'(t) \geq 0$ for all $t \geq 0$. Since $m(\nu) > 1$ there exists a unique $\lambda > 0$ such that $\int_0^\tau N_1(s)e^{-\lambda s}\,ds = 1$; in fact $v_0(\nu) = \lambda > 0$. Therefore we have

$$y'(t)e^{-\lambda t} = F(t)e^{-\lambda t} + \int_{[0,t]} N_1(s)e^{-\lambda s}e^{-\lambda(t-s)}y'(t-s)\,ds, \quad t \geq 0,$$

and so by the renewal theorem and the fact that $F(t) = 0$ and $N(t) = 0$ for $t \geq \tau$ we have

$$\lim_{t\to\infty} \frac{y'(t)}{e^{\lambda t}} = \frac{\int_0^\infty e^{-\lambda s}F(s)\,ds}{\int_0^\infty sN_1(s)e^{-\lambda s}\,ds} = \frac{\int_0^\tau e^{-\lambda s}F(s)\,ds}{\int_0^\tau sN_1(s)e^{-\lambda s}\,ds}.$$

From this we see that

$$\lim_{t\to\infty} \frac{y(t)}{e^{\lambda t}} = \frac{1}{\lambda}\frac{\int_0^\tau e^{-\lambda s}F(s)\,ds}{\int_0^\tau sN_1(s)e^{-\lambda s}\,ds}.$$

Therefore $\mathbb{P}[\Gamma(\phi) > 0] > 1/2$ provided $\int_0^\tau e^{-\lambda s}F(s)\,ds > 0$, which is true if F is positive on some subinterval of $[0,\tau]$.

Since $\phi'(0) > 0$ and $\phi' \in C([-\tau,0],\mathbb{R})$ there exists $\beta_2 \in (0,\tau)$ such that $\phi'(u) > 0$ for all $u \in (-\beta_2, 0] \in [-\tau, 0]$. Since $m(\nu) > 1$ we have $\int_0^\tau N_1(s)\,ds > 1$ and therefore there exists an interval $(\theta_1,\theta_2) \subset [0,\tau]$ such that $N_1(s) > 0$ for $s \in (\theta_1,\theta_2) \subset [0,\tau]$.

Let $t \in [0,\tau]$ be such that $0 \vee (\theta_1 - \beta_2) < t < \theta_1$. Define the interval $I_t := (t, \theta_2 \wedge (t+\beta_2)) \subset (t,\tau)$ and $J = (\theta_1,\tau)$. Since $t > \theta_1 - \beta_2$ we have $t + \beta_2 > \theta_1$, and, because $\theta_2 > \theta_1$, we have $\theta_2 \wedge (t+\beta_2) > \theta_1$. Therefore $I_t \cap J$ is a nontrivial open interval. Let $s \in I_t$. Then we have $\theta_2 > s > t$ and $s < t + \beta_2$, so $0 > t - s > -\beta_2$. Hence for all $s \in I_t$ we have $\phi'(t-s) > 0$, and so $\phi'(t-s) > 0$ for all $s \in I_t \cap J$. If $s \in J \cap I_t$, we have $s \in I_t$, $s > \theta_1$, and $s < \theta_2$. Hence for all $s \in I_t \cap J$ we have $N_1(s) > 0$. Therefore as

$N_1(s)\phi'(t-s) > 0$ for all $s \in I_t \cap J$, and each $t \in (0 \vee (\theta_1 - \beta_2), \theta_1)$, for each $t \in (0 \vee (\theta_1 - \beta_2), \theta_1)$ we have

$$F(t) = \int_{[t,\tau]} N_1(s)\phi'(t-s)\,ds \geq \int_{I_t \cap J} N_1(s)\phi'(t-s)\,ds > 0,$$

so $\int_0^\tau e^{-\lambda s} F(s)\,ds > 0$ which implies $\mathbb{P}[\Gamma(\phi) > 0] > 1/2$. The fact that $\alpha \mapsto \mathbb{P}[\Gamma(\alpha\phi) > 0]$ is increasing and that (3.4.8) hold is a consequence of part (iii).

The proof of part (iv) is similar to that of part (v). If ϕ is decreasing, and $y(\phi)$ is the solution of (3.2.1), we note that $y_- = -y(\phi)$ is the solution of (3.2.1) with initial condition $\phi_- = -\phi$. Since ϕ_- is increasing with $\phi_-'(0) > 0$, part (v) can now be applied to y_- to give the result.

3.7 Proof of Theorem 3.4.2

By (3.4.12) for $t \geq \delta$ we have

$$\text{Cov}(Y_\delta(t), Y_\delta(t+\Delta)) = \mathbb{E}\left[\int_0^t r_\delta(t-s)\sigma\,dB(s) \cdot \int_0^{t+\Delta} r_\delta(t+\Delta-s)\sigma\,dB(s)\right].$$

Hence as $\Delta \geq 0$

$$\text{Cov}(Y_\delta(t), Y_\delta(t+\Delta)) = \sigma^2 \int_0^t r_\delta(t-s) r_\delta(t+\Delta-s)\,ds = \sigma^2 \int_0^t r_\delta(u) r_\delta(u+\Delta)\,du.$$

Extend ν to $(-\infty, -\tau]$ by setting $\nu(E) = 0$ for every Borel set $E \subset (-\infty, -\tau)$. Next consider $\nu_+(E) = \nu(-E)$ for every $E \subseteq [0, \infty)$. Then r satisfies

$$r'(t) = \int_{[-\tau,0]} r(t+s)\nu(ds) = \int_{(-\infty,0]} r(t+s)\nu(ds) = \int_{[0,\infty)} r(t-s)\nu_+(ds),$$

so

$$r'(t) = \int_{[0,t]} r(t-s)\nu_+(ds), \quad t > 0, \quad r(0) = 1.$$

Let N be defined by (3.6.4). In the case when (3.3.3) and (3.3.4) hold we have that $N(t) \geq 0$ for all $t \geq 0$, $\lim_{t\to\infty} N(t) = 0$, and in particular $N(t) = 0$ for all $t \geq \tau$. Since $r \in C^1(0, \infty)$ we have $r(t-s) = 1 + \int_0^{t-s} r'(u)\,du$, then

$$r'(t) = \int_{[0,t]} \nu(ds) \left(1 + \int_0^{t-s} r'(u)\,du\right), \quad t > 0.$$

Notice that $r'(0) = \int_{[0,0]} \nu_+(\,ds) r(-s) = V_+(0) \geq 0$. By definition of N this gives

$$r'(t) = N(t) + \int_{[0,t]} \nu_+(ds) \int_0^{t-s} r'(u)\,du, \quad t > 0.$$

By Fubini's Theorem

$$\begin{aligned}
\int_{[0,t]} \nu_+(ds) \int_0^{t-s} r'(u)\,du &= \int_{[0,t]} \int_0^{t-s} \nu_+(ds) r'(u)\,du \\
&= \int_0^t \left(\int_{[0,t-s]} \nu_+(ds)\right) r'(u)\,du \\
&= \int_0^t N(t-u) r'(u)\,du.
\end{aligned}$$

Therefore

$$r'(t) = N(t) + \int_{[0,t]} N(s) r'(t-s)\,ds, \quad t > 0. \tag{3.7.1}$$

Since $N(t) \geq 0$, we have $r'(t) \geq 0$ for all $t \geq 0$. Let $\Delta > 0$. If r_δ is defined by (3.4.11), then $r_\delta(t) \geq 0$ for all $t \geq 0$. Since $r_\delta(0) > 0$, by continuity we have that (3.4.13) holds for all $t \geq \delta$.

If we suppose that $m(\nu) < 1$, then

$$\begin{aligned}
\int_0^\infty N(t)\,dt &= \int_0^\infty \int_{[0,t]} \nu_+(ds)\,dt \\
&= \int_0^\tau \int_{[0,t]} \nu_+(ds)\,dt + \int_\tau^\infty \left(\int_{[0,\tau]} \nu_+(ds) + \int_{[\tau,t]} \nu_+(ds) \right) dt \\
&= \int_0^\tau \int_{[0,t]} \nu_+(ds)\,dt = \int_{[0,\tau]} \int_s^\tau dt \cdot \nu_+(ds) \\
&= \int_{[0,\tau]} (\tau - s)\nu_+(ds) = \int_{[0,\tau]} -s\nu_+(ds) = \int_{[-\tau,0]} s\nu(ds) = m(\nu) < 1.
\end{aligned}$$

Therefore $r' \in L^1(0,\infty)$ and so $r'(t) \to 0$ as $t \to \infty$. Therefore for $t \geq \delta$ we have

$$r_\delta(t) = r(t) - r(t-\delta) = \int_{t-\delta}^t r'(s)\,ds.$$

Hence $r_\delta(t) \to 0$ as $t \to \infty$. Also for $T \geq 2\delta$ we have

$$\begin{aligned}
\int_\delta^T r_\delta(t)\,dt &= \int_\delta^T \int_{t-\delta}^t r'(s)\,ds\,dt = \int_0^T \int_{s\vee\delta}^{(s+\delta)\wedge T} dtr'(s)\,ds \\
&= \int_0^\delta \int_\delta^{s+\delta} dtr'(s)\,ds + \int_\delta^{T-\delta} \int_s^{s+\delta} dtr'(s)\,ds + \int_{T-\delta}^T \int_s^T dtr'(s)\,ds \\
&= \int_0^\delta sr'(s)\,ds + \int_\delta^{T-\delta} \delta r'(s)\,ds + \int_{T-\delta}^T (T-s)r'(s)\,ds \\
&\leq \delta \int_0^T r'(s)\,ds.
\end{aligned}$$

Therefore $r_\delta \in L^1(0,\infty)$ for each $\delta > 0$. For each $\Delta > 0$ we can consider the limit

$$c_\delta(\Delta) := \lim_{t\to\infty} \text{Cov}(Y_\delta(t), Y_\delta(t+\Delta)) = \sigma^2 \int_0^\infty r_\delta(u) r_\delta(u+\Delta)\,du.$$

This limit is finite, because r_δ is bounded and $r_\delta \in L^1(0,\infty)$, proving (3.4.14). Next we have $0 \leq c_\delta(\Delta) \leq \sigma^2 \int_0^\infty r_\delta(u)\,du \cdot \sup_{v\geq\Delta} r_\delta(v)$. Therefore $c_\delta(\Delta) \to 0$ as $\Delta \to \infty$. Also we have

$$\begin{aligned}
\int_0^\infty c_\delta(\Delta)\,d\Delta &= \sigma^2 \int_0^\infty \int_0^\infty r_\delta(u) r_\delta(u+\Delta)\,du\,d\Delta \\
&= \sigma^2 \int_0^\infty r_\delta(u) \left(\int_u^\infty r_\delta(v)\,dv \right) du \leq \sigma^2 \left(\int_0^\infty r_\delta(u)\,du \right)^2,
\end{aligned}$$

and so $c_\delta \in L^1(0,\infty)$.

It remains to prove (3.4.15). We note that because $r'(t) \geq 0$, $N(t) \geq 0$ and $\int_0^\infty N(s)\,ds = \int_0^\tau N(s)\,ds < 1$ and $N(t) = 0$ for all $t \geq \tau$ that there exists a unique $\lambda = \lambda(N) > 0$ such that

$$1 = \int_0^\tau N(s) e^{\lambda s}\,ds = \int_0^\infty N(s) e^{\lambda s}\,ds.$$

Since

$$\begin{aligned}
\int_0^\tau N(t) e^{\lambda t}\,dt &= \int_0^\tau \left(\int_{[0,t]} \nu_+(ds) \right) e^{\lambda t}\,dt = \int_{[0,\tau]} \int_s^\tau e^{\lambda t}\,dt\nu_+(ds) \\
&= \frac{1}{\lambda} \int_{[0,\tau]} \left(e^{\lambda\tau} - e^{\lambda s} \right) \nu_+(ds) \\
&= \frac{1}{\lambda} \left(\int_{[0,\tau]} e^{\lambda\tau} \nu_+(ds) - \int_{[0,\tau]} e^{\lambda s} \nu_+(ds) \right),
\end{aligned}$$

we have that $\lambda > 0$ obeys

$$\lambda = -\int_{[0,\tau]} e^{\lambda s}\nu_+(ds). \tag{3.7.2}$$

Now $h(-\lambda) = -\lambda - \int_{[-\tau,0]} e^{-\lambda s}\nu(ds) = -\lambda - \int_{[0,\tau]} e^{\lambda u}\nu_+(du) = 0$, so $-\lambda \in \Lambda$.

Moreover, with $r_{(\lambda)}(t) := r'(t)e^{\lambda t}$, $N_{(\lambda)}(t) := e^{\lambda t}N(t)$ we get

$$r_{(\lambda)}(t) = N_{(\lambda)}(t) + \int_0^t N_{(\lambda)}(t-s)r_{(\lambda)}(s)\,ds, \quad t \geq 0,$$

and so by the renewal theorem we have

$$\lim_{t\to\infty} r'(t)e^{\lambda t} = \lim_{t\to\infty} r_{(\lambda)}(t) = \frac{1}{\int_0^\infty tN_{(\lambda)}(t)\,dt} = \frac{1}{\int_0^\tau te^{\lambda t}\int_{[0,t]}\nu_+(ds)\,dt} =: R_\lambda.$$

We can simplify R_λ. Since $\int_s^\tau te^{\lambda t}\,dt = \tau e^{\lambda\tau}/\lambda - se^{\lambda s}/\lambda - (e^{\lambda\tau} - e^{\lambda s})/\lambda^2$, and using (3.3.4) and (3.7.2) we have

$$\begin{aligned}
\int_0^\tau te^{\lambda t}\int_{[0,t]}\nu_+(ds)\,dt &= \int_{[0,\tau]}\int_s^\tau te^{\lambda t}\,dt\nu_+(ds) = \int_{[0,\tau]}\left(\int_s^\tau te^{\lambda t}\,dt\right)\nu_+(ds) \\
&= \int_{[0,\tau]}\left\{\frac{1}{\lambda}\tau e^{\lambda\tau} - \frac{1}{\lambda}se^{\lambda s} - \frac{1}{\lambda^2}(e^{\lambda\tau} - e^{\lambda s})\right\}\nu_+(ds) \\
&= \int_{[0,\tau]}\left\{-\frac{1}{\lambda}se^{\lambda s} + \frac{1}{\lambda^2}e^{\lambda s}\right\}\nu_+(ds) \\
&= -\frac{1}{\lambda}\int_{[0,\tau]} se^{\lambda s}\nu_+(ds) - \frac{1}{\lambda}.
\end{aligned}$$

Therefore by the definition of ν_+ we get

$$\int_0^\infty tN_\lambda(t)\,dt = -\frac{1}{\lambda} + \frac{1}{\lambda}\int_{[-\tau,0]} ue^{-\lambda u}\nu(du),$$

so

$$R_\lambda = \frac{\lambda}{-1 + \int_{[-\tau,0]} ue^{-\lambda u}\nu(du)}. \tag{3.7.3}$$

Since

$$r_\delta(t)e^{\lambda t} = \int_{t-\delta}^t r'(s)e^{\lambda s}\cdot e^{\lambda(t-s)}\,ds = \int_{t-\delta}^t (r'(s)e^{\lambda s} - R_\lambda)\cdot e^{\lambda(t-s)}\,ds + R_\lambda\int_0^\delta e^{\lambda u}\,du,$$

we have

$$\lim_{t\to\infty} r_\delta(t)e^{\lambda t} = R_\lambda\int_0^\delta e^{\lambda u}\,du.$$

Now

$$c_\delta(\Delta)e^{\lambda\Delta} = \sigma^2\int_0^\infty r_\delta(u)e^{-\lambda u}r_\delta(u+\Delta)e^{\lambda(\Delta+u)}\,du,$$

so as r_δ is in L^1 and $\lambda > 0$ we have

$$\lim_{\Delta\to\infty} c_\delta(\Delta)e^{\lambda\Delta} = \sigma^2 R_\lambda\int_0^\delta e^{\lambda u}\,du\int_0^\infty r_\delta(u)e^{-\lambda u}\,du. \tag{3.7.4}$$

The righthand side of (3.7.4) is positive. Finally, we write it in terms of data. By (3.7.3) we have

$$\sigma^2 R_\lambda\int_0^\delta e^{\lambda u}\,du = \sigma^2\frac{e^{\lambda\delta} - 1}{-1 + \int_{[-\tau,0]} ue^{-\lambda u}\nu(du)}.$$

To compute $\int_0^\infty r_\delta(u)e^{-\lambda u}\,du$ notice first that

$$\int_0^\infty r_\delta(u)e^{-\lambda u}\,du = \int_0^\delta r(u)e^{-\lambda u}\,du + \int_\delta^\infty \int_{u-\delta}^u r'(s)\,dse^{-\lambda u}\,du.$$

Hence

$$\int_0^\infty r_\delta(u)e^{-\lambda u}\,du = \int_0^\delta r(u)e^{-\lambda u}\,du - \frac{e^{-\lambda\delta}-1}{\lambda}\int_0^\infty r'(s)e^{-\lambda s}\,ds. \tag{3.7.5}$$

We evaluate the second integral on the righthand side. By (3.7.1) we have

$$\int_0^\infty r'(s)e^{-\lambda s}\,ds = \frac{\int_0^\infty N(s)e^{-\lambda s}\,ds}{1-\int_0^\infty N(s)e^{-\lambda s}\,ds} = \frac{\int_0^\tau N(s)e^{-\lambda s}\,ds}{1-\int_0^\tau N(s)e^{-\lambda s}\,ds}.$$

Next we have

$$\begin{aligned}\int_0^\tau N(t)e^{-\lambda t}\,dt &= \int_0^\tau \left(\int_{[0,t]} \nu_+(ds)\right) e^{-\lambda t}\,dt = \int_{[0,\tau]}\left(\int_s^\tau e^{-\lambda t}\,dt\right)\nu_+(ds)\\ &= \int_{[0,\tau]}\left(\frac{1}{-\lambda}e^{-\lambda\tau} - \frac{1}{-\lambda}e^{-\lambda s}\right)\nu_+(ds) = \frac{1}{\lambda}\int_{[0,\tau]} e^{-\lambda s}\nu_+(ds).\end{aligned}$$

Hence

$$\int_0^\tau N(t)e^{-\lambda t}\,dt = \frac{1}{\lambda}\int_{[0,\tau]} e^{-\lambda s}\nu_+(ds) = \frac{1}{\lambda}\int_{[-\tau,0]} e^{\lambda u}\nu(du),$$

and so

$$\int_0^\infty r'(s)e^{-\lambda s}\,ds = \frac{\int_{[-\tau,0]} e^{\lambda u}\nu(du)}{\lambda - \int_{[-\tau,0]} e^{\lambda u}\nu(du)}. \tag{3.7.6}$$

By (3.7.5) and (3.7.6) we have

$$\int_0^\infty r_\delta(u)e^{-\lambda u}\,du = \int_0^\delta r(u)e^{-\lambda u}\,du - \frac{e^{-\lambda\delta}-1}{\lambda}\frac{\int_{[-\tau,0]} e^{\lambda u}\nu(du)}{\lambda - \int_{[-\tau,0]} e^{\lambda u}\nu(du)}.$$

Hence

$$\begin{aligned}&\lim_{\Delta\to\infty} c_\delta(\Delta)e^{\lambda\Delta}\\ &\qquad= \frac{\sigma^2(1-e^{\lambda\delta})}{1-\int_{[-\tau,0]} ue^{-\lambda u}\nu(du)}\left(\int_0^\delta r(u)e^{-\lambda u}\,du + \frac{1-e^{-\lambda\delta}}{\lambda}\frac{\int_{[-\tau,0]} e^{\lambda u}\nu(du)}{\lambda - \int_{[-\tau,0]} e^{\lambda u}\nu(du)}\right),\end{aligned}$$

as required.

3.8 Proof of Theorem 3.3.2

To prove Theorem 3.3.2, it is convenient to introduce the function

$$P:\mathbb{C}\to\mathbb{C}, \qquad P(\lambda) = \int_0^\tau e^{-\lambda t}\int_{[-t,0]} \nu(\,ds)\,dt \tag{3.8.1}$$

and the function

$$N:[0,\tau]\to\mathbb{R}, \qquad N(t) = \nu([-t,0]). \tag{3.8.2}$$

Fubini's theorem and $\nu([-\tau,0])=0$ yield

$$P(\lambda) = \int_{[-\tau,0]}\left(\int_{-s}^\tau e^{-\lambda t}\,dt\right)\nu(\,ds) = \frac{1}{\lambda}\int_{[-\tau,0]} e^{\lambda s}\nu(\,ds) = -\frac{h(\lambda)}{\lambda} + 1 \tag{3.8.3}$$

for $\lambda \neq 0$. Therefore, for $\lambda \neq 0$ we have that $P(\lambda) = 1$ if and only if $h(\lambda) = 0$. For $\lambda = 0$ Fubini's theorem yields

$$P(0) = \int_{[-\tau,0]} \int_{-s}^{\tau} dt\, \nu(ds) = \int_{[-\tau,0]} s\, \nu(ds) = m(\nu). \tag{3.8.4}$$

Proof of (i): Because of (3.8.4) we have $P(0) > 1$ and due to Lebesgue's theorem we see $P(\lambda) \to 0$ as $\lambda \to \infty$. Hence there exists a unique $\lambda_0 > 0$ such that $P(\lambda_0) = 1$ and so a unique $\lambda_0 > 0$ such that $h(\lambda_0) = 0$.

To see that this root λ_0 is simple we differentiate P and deduce by (3.3.3)

$$P'(\lambda) = -\int_0^{\tau} te^{-\lambda t} \int_{[-t,0]} \nu(\,ds)\, dt < 0 \qquad \text{for all } \lambda \geq 0.$$

On the other hand differentiating P by using representation (3.8.3) results in $P'(\lambda) = -h'(\lambda)/\lambda + h(\lambda)/\lambda^2$ for $\lambda > 0$. Since $P'(\lambda_0) < 0$, and $P'(\lambda_0) = -h'(\lambda_0)/\lambda_0$, we have $h'(\lambda_0) > 0$.

Suppose there exists $\lambda_2 \in \mathbb{R}$ such that $h(\lambda_0 + i\lambda_2) = 0$. Then $P(\lambda_0 + i\lambda_2) = 1$ yields

$$1 = \int_0^{\tau} e^{-\lambda_0 t} \cos(\lambda_2 t) N(t)\, dt, \qquad 0 = \int_0^{\tau} e^{-\lambda_0 t} \sin(\lambda_2 t) N(t)\, dt. \tag{3.8.5}$$

Since $h(\lambda_0) = 0$, we have $P(\lambda_0) = 1$, or $1 = \int_0^{\tau} e^{-\lambda_0 t} N(t)\, dt$. Using this and the first equality in (3.8.5) gives

$$\int_0^{\tau} e^{-\lambda_0 t}(1 - \cos(\lambda_2 t)) N(t)\, dt = 0. \tag{3.8.6}$$

But because N is non-negative and not vanishing everywhere this yields $\lambda_2 = 0$.

Since we have already shown that there are no other zeros of h on the line $\mathrm{Re}\,(\lambda) = \lambda_0$, it is enough to show that $\lambda_1 \leq \lambda_0$ for all $\lambda_1 \in \mathbb{R}$ with $h(\lambda_1 + i\lambda_2) = 0$ for some $\lambda_2 \in \mathbb{R}$. Because $P(\lambda_1 + i\lambda_2) = 1$ we have

$$1 = \mathrm{Re}\,(P(\lambda_1 + i\lambda_2)) = \int_0^{\tau} e^{-\lambda_1 t} \cos(\lambda_2 t) N(t)\, dt \leq \int_0^{\tau} e^{-\lambda_1 t} N(t)\, dt = P(\lambda_1).$$

Since P is decreasing on $\mathbb{R}$ and $P(\lambda_0) = 1$, we must have $\lambda_1 \leq \lambda_0$, which, by the above remark, proves part (i).

Proof of (ii): The assumption $\nu([-\tau, 0]) = 0$ implies that h has a root in 0. It is simple since $h'(0) = 1 - \int_{[-\tau,0]} s\nu(\,ds) > 0$, using $m(\nu) < 1$.

Suppose there exists $\lambda_2 \neq 0$ such that $h(i\lambda_2) = 0$. Then (3.8.3) implies $P(i\lambda_2) = 1$ which results in

$$1 = \int_0^{\tau} \cos(\lambda_2 t) N(t)\, dt, \qquad 0 = \int_0^{\tau} \sin(\lambda_2 t) N(t)\, dt. \tag{3.8.7}$$

On the other hand (3.8.4) yields

$$\int_0^{\tau} N(t)\, dt = P(0) = m(\nu) < 1.$$

Consequently, by using the first equality in (3.8.7) we get

$$\int_0^{\tau} N(t)\big(1 - \cos(\lambda_2 t)\big)\, dt < 0,$$

which contradicts $N \geq 0$. Hence $h(i\lambda_2) \neq 0$ for all $\lambda_2 \neq 0$.

The same argument as in (i) shows that for all other roots $\lambda_1 + i\lambda_2$ of h we have $\lambda_1 < 0$.

3.9 Supporting Lemmas

In this section, Lemma 3.9.1, 3.9.2 and 3.9.3 are interrelated. Each one simplifies the form of the solution of either the stochastic or deterministic differential equation supplied by a previous result.

Lemma 3.9.1. *Let $Y(t)$ and $y(t)$ be solutions of the processes defined by* (3.2.8) *and* (3.2.1) *respectively. Then the solution $(Y(t)\,:\,t\geq -\tau)$ obeys the variation of constants formula*

$$Y(t)=\begin{cases} y(t)+\int_0^t r(t-s)\sigma\,dB(s), & t\geq 0,\\ \phi(t), & t\in[-\tau,0],\end{cases} \tag{3.9.1}$$

where r is the fundamental solution of (3.2.1).

Remark 3.9.1. As the proof is straightforward no outline is given.

Proof. Let $Z(t)=Y(t)-y(t)$, then

$$\begin{aligned} dZ(t)&=dY(t)-dy(t)\\ &=\int_{[-\tau,0]}Y(t+s)\nu(\,ds)\,dt+\sigma\,dB(t)-\int_{[-\tau,0]}y(t+s)\nu(\,ds)\,dt.\end{aligned}$$

Hence

$$dZ(t)=\int_{[-\tau,0]}Z(t+s)\nu(\,ds)\,dt+\sigma\,dB(t),\quad t\geq 0, \tag{3.9.2}$$

where $Z(t)=0$ for all values of $t\in[-\tau,0]$. Consider the process

$$U(t)=\begin{cases} \sigma\int_0^t r(t-s)\,dB(s), & t\geq 0\\ 0, & t<0.\end{cases}$$

We will show that U satisfies the equation for Z. Now by the stochastic Fubini's theorem

$$\begin{aligned} U(t)&=\sigma\int_0^t r(t-s)\,dB(s)\\ &=\sigma\int_0^t\left(1+\int_0^{t-s}r'(u)\,du\right)dB(s)\\ &=\sigma B(t)+\sigma\int_0^t\int_s^t r'(v-s)\,dv\,dB(s)\\ &=\sigma B(t)+\sigma\int_0^t\int_0^v r'(v-s)\,dB(s)\,dv.\end{aligned}$$

Therefore

$$dU(t)=\sigma\,dB(t)+\sigma\int_0^t r'(t-s)\,dB(s)\,dt,\qquad t>0.$$

Now for $t\geq\tau$,

$$\begin{aligned} \int_{[-\tau,0]}U(t+s)\nu(\,ds)&=\sigma\int_{s\in[-\tau,0]}\nu(\,ds)\int_{u=0}^{t+s}r(t+s-u)\,dB(u)\\ &=\sigma\int_0^t\left(\int_{[u-t,0]\wedge[-\tau,0]}r(t+s-u)\nu(\,ds)\right)dB(u).\end{aligned}$$

For $-\tau<u-t<0$, define

$$I(u,t)=\int_{[u-t,0]}r(t+s-u)\nu(\,ds).$$

Then

$$\begin{aligned} I(u,t)&=\int_{[-\tau,0]}r(t-u+s)\nu(\,ds)-\int_{[-\tau,u-t]}r(t-u+s)\nu(\,ds)\\ &=r'(t-u),\end{aligned}$$

and for $u - t \leq -\tau$,

$$\begin{aligned} I(u,t) = &\int_{[-\tau,0]} r(t-u+s)\nu(\,ds) \\ =& r'(t-u). \end{aligned}$$

Thus

$$\int_{[-\tau,0]} U(t+s)\nu(\,ds) = \sigma \int_0^t r'(t-u)\,dB(u), \quad t \geq \tau.$$

Now for $t \in [0,\tau]$ we have

$$\begin{aligned} \int_{[-\tau,0]} U(t+s)\nu(\,ds) &= \int_{[-\tau,-t)} U(t+s)\nu(\,ds) + \int_{(-t,0]} U(t+s)\nu(\,ds) \\ &= \int_{[-t,0]} U(t+s)\nu(\,ds) \\ &= \int_{[-t,0]} \nu(\,ds)\sigma \int_{[0,t+s]} r(t+s-u)\,dB(u) \\ &= \sigma \int_0^t \int_{[-t,0]n[u-t,0]} r(t+s-u)\nu(\,ds)\,dB(u) \\ &= \sigma \int_0^t \int_{[u-t,0]} r(t+s-u)\nu(\,ds)\,dB(s) \\ &= \sigma \int_0^t r'(t-u)\,dB(u). \end{aligned}$$

Thus for $t \geq 0$,

$$\int_{[-\tau,0]} U(t+s)\nu(\,ds) = \sigma \int_0^t r'(t-u)\,dB(u).$$

Hence for $t \geq 0$

$$dU(t) = \sigma\,dB(t) + \int_{[-\tau,0]} U(t+s)\nu(\,ds)\,dt,$$

where $U(t) = 0$ for $t \in [-\tau, 0]$. But the solution of 3.9.2 with initial condition $Z(t) = 0$ for all $t \in [-\tau, 0]$ is unique, so $Z(t) = U(t)$ for all $t \geq -\tau$ and hence $Y(t)$ obeys the variation of constants formulas 3.9.1. □

Lemma 3.9.2. *Let y be the solution of* (3.2.1)*. Then*

$$y(t) = r(t)\phi(0) + \int_0^t r(t-s)\phi_2(s)\,ds, \qquad t \geq 0 \tag{3.9.3}$$

where r is the fundamental solution of (3.2.1) *and*

$$\phi_2(t) = \begin{cases} \int_{[-\tau,0]} \phi_1(s+t)\nu(\,ds), & \textit{for all} \quad t \in [0,\tau] \\ 0, & otherwise, \end{cases}$$

Remark 3.9.2. As the proof is straightforward no outline is given.

Proof. Now $y'(t) = \int_{[-\tau,0]} y(t+s)v(\,ds)$ where $t > 0$. We compute its Laplace Transform is $\widetilde{y}$ in what follows. First, we integrate to get

$$\int_0^\infty e^{-\lambda t} y'(t)\,dt = \int_0^\infty e^{-\lambda t} \int_{[-\tau,0]} y(t+s)\nu(\,ds)\,dt.$$

Hence

$$\begin{aligned}\lambda\widetilde{y}(\lambda)-\phi(0) &= \int_{[-\tau,0]} e^{\lambda s}\nu(\,ds)\int_0^\infty e^{-\lambda(t+s)}y(t+s)\,dt\\ &= \int_{[-\tau,0]} e^{\lambda s}\nu(\,ds)\int_s^\infty e^{-\lambda u}y(u)\,du\\ &= \int_{[-\tau,0]} e^{\lambda s}\nu(\,ds)\left(\int_0^\infty e^{-\lambda u}y(u)\,du-\int_0^s e^{-\lambda u}y(u)\,du\right)\\ &= \int_{[-\tau,0]} e^{\lambda s}\nu(\,ds)\widetilde{y}(\lambda)+\int_{[-\tau,0]} e^{\lambda s}\nu(\,ds)\int_s^0 e^{-\lambda u}\phi(u)\,du.\end{aligned}$$

Then

$$\left(\lambda-\int_{[-\tau,0]} e^{\lambda s}\nu(\,ds)\right)\widetilde{y}(\lambda)=\phi(0)+\int_{[-\tau,0]} e^{\lambda s}\nu(\,ds)\int_s^0 e^{-\lambda u}\phi(u)\,du.$$

If r is the differential resolvent we have $\left(\lambda-\int_{[-\tau,0]} e^{\lambda s}\nu(\,ds)\right)\widetilde{r}(\lambda)=1$, so

$$\widetilde{y}(\lambda)=\phi(0)\widetilde{r}(\lambda)+\widetilde{r}(\lambda)\left(\int_{[-\tau,0]} e^{\lambda s}\nu(\,ds)\int_s^0 e^{-\lambda u}\phi(u)\,du\right).$$

We need to take inverse transforms to establish (3.9.3). To do so we write the second term on the right hand side as a product of transforms. To this end, we write

$$\begin{aligned}\int_{[-\tau,0]} e^{\lambda s}\nu(\,ds)\int_s^0 e^{-\lambda u}\phi(u)\,du &= \int_{[-\tau,0]}\nu(\,ds)\int_s^0 e^{-\lambda(u-s)}\phi(u)\,du\\ &= \int_{[-\tau,0]}\nu(\,ds)\int_0^{-s} e^{-\lambda w}\phi(s+w)\,dw.\end{aligned}$$

Now define

$$\phi_1(t)=\begin{cases}\phi(t), & \text{for all } t\in[-\tau,0],\\ 0, & t>0.\end{cases}$$

Then

$$\begin{aligned}&\int_{[-\tau,0]}\nu(\,ds)\int_0^{-s} e^{-\lambda w}\phi(s+w)\,dw=\\ &-\int_{[-\tau,0]}\nu(\,ds)\int_{-s}^\infty e^{-\lambda w}\phi(s+w)\,dw+\int_{[-\tau,0]}\nu(\,ds)\int_0^\infty e^{-\lambda w}\phi(s+w)\,dw\\ &=\int_{[-\tau,0]}\nu(\,ds)\int_0^\infty e^{-\lambda w}\phi_1(w+s)\,dw\\ &=\int_0^\infty e^{-\lambda w}\int_{[-\tau,0]}\nu(\,ds)\phi_1(w+s)\,dw\\ &=\widetilde{\phi}_2(\lambda).\end{aligned}$$

Thus $\widetilde{y}(\lambda)=\widetilde{r}(\lambda)\phi(0)+\widetilde{r}(\lambda)\widetilde{\phi}_2(\lambda)$ which implies

$$y(t)=r(t)\phi(0)+\int_0^t r(t-s)\phi_2(s)\,ds,\quad t\geq 0.$$

□

Finally we rewrite the formula in (3.9.3) in a form which simply depends on ϕ.

Lemma 3.9.3. *Let y be the solution of* (3.2.1). *Then*

$$y(t) = r(t)\phi(0) + \int_{[-\tau,0]} \int_s^0 r(t+s-u)\phi(u)\,du\,\nu(\,ds), \quad t \geq 0. \tag{3.9.4}$$

where r is the fundamental solution of (3.2.1).

Remark 3.9.3. As the proof is straightforward no outline is given.

Proof. By Lemma 3.9.2

$$\begin{aligned} y(t) &= r(t)\phi(0) + \int_0^t r(t-s)\phi_2(s)\,ds \\ &= r(t)\phi(0) + \int_0^t r(t-s)\int_{[-\tau,0]} \phi_1(s+u)\nu(\,du)\,ds. \end{aligned}$$

Now

$$\begin{aligned} \int_0^t r(t-s)\int_{[-\tau,0]} \phi_1(s+u)\nu(\,du)\,ds &= \int_{[-\tau,0]}\int_0^t r(t-s)\phi_1(s+u)\,ds\,\nu(\,du) \\ &= \int_{[-\tau,0]}\int_s^{t+s} r(t+s-w)\phi_1(w)\,dw\,\nu(\,ds) \\ &= \int_{[-\tau,0]}\int_s^0 r(t+s-u)\phi_1(u)\,du\nu(\,ds) \\ &+ \int_{[-\tau,0]}\int_0^{t+s} r(t+s-u)\phi_1(u)\,du\,\nu(\,ds). \end{aligned}$$

If $t \geq \tau$, then for $s \in [-\tau, 0]$,

$$\begin{aligned} &\int_{[-\tau,0]}\left(\int_s^0 r(t+s-u)\phi_1(u)\,du + \int_0^{t+s} r(t+s-u)\phi_1(u)\,du\right)\nu(\,ds) \\ &\qquad = \int_{[-\tau,0]}\int_s^0 r(t+s-u)\phi(u)\,du\,\nu(\,ds). \end{aligned}$$

This confirms (3.9.4), in the case $t \geq \tau$. If $0 \leq t < \tau$ then,

$$\begin{aligned} &\int_{[-\tau,0]}\int_s^{t+s} r(t+s-u)\phi_1(u)\,du\,\nu(\,ds) = \\ &\int_{[-\tau,-t]}\int_s^{t+s} r(t+s-u)\phi_1(u)\,du\,\nu(\,ds) + \int_{[-t,0]}\int_s^{t+s} r(t+s-u)\phi_1(u)\,du\,\nu(\,ds). \end{aligned}$$

We express the first term to get

$$\begin{aligned} &\int_{[-\tau,-t]}\int_s^{t+s} r(t+s-u)\phi_1(u)\,du\,\nu(\,ds) \\ &= \int_{[-\tau,-t]}\int_s^0 r(t+s-u)\phi(u)\,du\,\nu(\,ds) - \int_{[-\tau,-t]}\int_{t+s}^0 r(t+s-u)\phi(u)\,du\,\nu(\,ds) \\ &= \int_{[-\tau,-t]}\int_s^0 r(t+s-u)\phi(u)\,du\,\nu(\,ds), \end{aligned}$$

and then the second, yielding

$$\begin{aligned} &\int_{[-t,0]}\int_s^{t+s} r(t+s-u)\phi_1(u)\,du\,\nu(\,ds) \\ &= \int_{[-t,0]}\int_s^0 r(t+s-u)\phi_1(u)\,du\,\nu(\,ds) + \int_{[-t,0]}\int_0^{t+s} r(t+s-u)\phi_1(u)\,du\,\nu(\,ds) \\ &= \int_{[-t,0]}\int_s^0 r(t+s-u)\phi(u)\,du\,\nu(\,ds). \end{aligned}$$

Therefore if $t \in [0,\tau]$ we again get

$$\int_0^t r(t-s)\int_{[-\tau,0]}\phi_1(s+u)\nu(\,du)\,ds = \int_{[-\tau,0]}\int_s^0 r(t+s-u)\phi(u)\,du\,\nu(\,ds)$$

and hence obtain (3.9.4) in this case, as required.

$$y(t) = r(t)\phi(0) + \int_{[-\tau,0]}\int_s^0 r(t+s-u)\phi(u)\,du\,\nu(\,ds),$$

as required. □

Lemma 3.9.4. *Suppose that B is a standard Brownian motion. Then for every $\epsilon \in (0,1)$ we have*

$$\lim_{n\to\infty}\sup_{n^\epsilon \le t\le (n+1)^\epsilon}\frac{|B(t)-B(n^\epsilon)|}{\sqrt{\epsilon\log n}} = 0 \quad a.s. \tag{3.9.5}$$

Proof. Using some properties of Brownian motion we obtain

$$\begin{aligned}
&\mathbb{P}[\sup_{n^\epsilon\le t\le(n+1)^\epsilon}|B(t)-B(n^\epsilon)|>1]\\
&=\mathbb{P}[\{\sup_{n^\epsilon\le t\le(n+1)^\epsilon}(B(t)-B(n^\epsilon))>1\}\cup\{\inf_{n^\epsilon\le t\le(n+1)^\epsilon}(B(t)-B(n^\epsilon))<-1\}]\\
&\le 2\mathbb{P}[\sup_{n^\epsilon\le t\le(n+1)^\epsilon}(B(t)-B(n^\epsilon))>1]\\
&=2\mathbb{P}[\sup_{0\le t\le(n+1)^\epsilon-n^\epsilon}B(t)>1]\\
&=2\mathbb{P}[|B((n+1)^\epsilon-n^\epsilon)|>1]\\
&=2\mathbb{P}\left[|Z_\epsilon(n)|>x_n\right]\le\frac{4}{\sqrt{2\pi}}\frac{1}{x_n}e^{-x_n^2/2},
\end{aligned}$$

where $x_n := 1/\sqrt{(n+1)^\epsilon - n^\epsilon}$ and we have used the fact that

$$Z_\epsilon(n) := \frac{B((n+1)^\epsilon-n^\epsilon)}{\sqrt{(n+1)^\epsilon-n^\epsilon}},$$

is $\mathcal{N}(0,1)$ distributed. By the mean value theorem $x_n = \sqrt{\epsilon}^{-1}n_*^{(1-\epsilon)/2}$ for some $n_* \in [n, n+1]$. Therefore $\sqrt{\epsilon}^{-1}n^{(1-\epsilon)/2} \le x_n$, and so

$$\mathbb{P}[\sup_{n^\epsilon\le t\le(n+1)^\epsilon}|B(t)-B(n^\epsilon)|>1]\le\frac{4}{\sqrt{2\pi}}\frac{1}{\sqrt{\epsilon}^{-1}n^{(1-\epsilon)/2}}e^{-\frac{1}{2\epsilon}n^{(1-\epsilon)}}.$$

Therefore

$$\sum_{n=1}^\infty\mathbb{P}[\sup_{n^\epsilon\le t\le(n+1)^\epsilon}|B(t)-B(n^\epsilon)|>1]<+\infty,$$

and so by the Borel–Cantelli lemma there exists an almost sure event Ω^* and for each $\omega\in\Omega^*$ there exists $n_0 = n_0(\omega)\in\mathbb{N}$ such that for all $n > n_0$, we have $\sup_{n^\epsilon\le t\le(n+1)^\epsilon}|B(t)-B(n^\epsilon)|\le 1$. This gives (3.9.5). □

Chapter 4

A Nonlinear Stochastic Functional Differential Equation model of an Inefficient Financial Market

4.1 Introduction

As with the previous chapter, we present a stochastic functional differential equation model of an inefficient financial market. Once again the model is informationally inefficient, in the sense that past movements of the stock price have an influence on future movements. Following the model of the previous chapter we suppose that the price adjustment at time t for a market with N traders is given by

$$dX(t) = \sum_{j=1}^{N} \beta_j \left(\int_{[-\theta_j,0]} g(X(t+u))s_j(du) - \int_{[-\tau_j,0]} g(X(t+u))l_j(du) \right) dt + \sigma\, dB(t). \quad (4.1.1)$$

Here s_j and l_j are finite measures, representing the short– and long–run weights that trader j uses to form their demand schedule. $\beta_j > 0$ and σ are constants. This is equivalent to the nonlinear stochastic functional differential equation

$$X(t) = \psi(0) + \int_0^t \left\{ \int_{[-\tau,0]} g(X(s+u))\nu(du) \right\} ds + \int_0^t \sigma\, dB(s), \quad t \geq 0, \quad (4.1.2a)$$

$$X(t) = \phi(t), \quad t \in [-\tau, 0], \quad (4.1.2b)$$

where $\tau = \max_{j=1,\dots,N} \max(\tau_j, \theta_j)$. The measure $\nu \in M[-\tau, 0]$ inherits properties from the weights s_j and l_j, and the constants β_j and α. These special properties influence the almost sure asymptotic behaviour as $t \to \infty$ of solutions of (4.1.1).

The distinguishing feature of this model is the presence of the nonlinear function g in the place of the linear function $x \mapsto \beta x$ in the equation (3.1.1). As in chapter 2, this nonlinear function allows us to capture differing attitudes to risk among the traders when the returns do not depart too far from equilibrium values.

Despite the presence of the nonlinearity g we can still show that the returns undergo long run dynamics consistent with either a correlated standard Brownian motion, or a bubble or crash characterised by exponentially growing returns. These results are directly comparable to the corresponding results for nonlinear discrete–time equation in chapter 2 and to continuous–time affine equations in chapter 3. The economic interpretation of these results and the conditions under which they apply carries over without any significant amendment.

The chapter has the following structure; Section 2 gives notation and supporting results; Section 3 states the main mathematical results of the paper, while Section 4 shows how the hypotheses of these results are satisfied in the financial model. The interpretation of the results to the financial model are also explored in Section 4 and the rest of the chapter is devoted to proofs.

4.2 Preliminaries

Please refer to chapter 3 section 3.2, from (3.2.1) to (3.2.1) and (3.2.9), for an outline of various equations. Let us fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}(t))_{t\geq 0}$ satisfying the usual conditions and let $(B(t) : t \geq 0)$ be a standard Brownian motion on this space. We study the following stochastic differential equation with time delay:

$$\begin{aligned} dX(t) &= \int_{[-\tau,0]} g(X(t+s))\nu(ds)\, dt + \sigma\, dB(t) \quad \text{for } t \geq 0, \\ X(t) &= \phi(t) \quad \text{for } t \in [-\tau, 0], \end{aligned} \quad (4.2.1)$$

for a constant $\tau \geq 0$ and $\sigma \geq 0$. We presume that g obeys

$$g \in C(\mathbb{R};\mathbb{R}) \text{ is locally Lipschitz continuous} \tag{4.2.2a}$$

$$\lim_{x\to\infty} \frac{g(x)}{x} = 1, \quad \lim_{x\to-\infty} \frac{g(x)}{x} = 1. \tag{4.2.2b}$$

Then for every $\phi \in C[-\tau,0]$ there exists a unique, adapted strong solution $(X(t,\phi) : t \geq -\tau)$ with finite second moments of (4.2.3) (cf., e.g., Mao [49]).

It is convenient to introduce an associated affine stochastic differential with time delay

$$\begin{aligned} dY(t) &= \int_{[-\tau,0]} \nu(ds)Y(t+s)\,dt + \sigma\,dB(t) \quad \text{for } t \geq 0, \\ Y(t) &= \phi(t) \quad \text{for } t \in [-\tau,0]. \end{aligned} \tag{4.2.3}$$

For every $\phi \in C[-\tau,0]$ there exists a unique, adapted strong solution $(Y(t,\phi) : t \geq -\tau)$ with finite second moments of (4.2.3) (cf., e.g., Mao [49]). The dependence of the solutions on the initial condition ϕ is neglected in our notation in what follows; that is, we will write $X(t) = X(t,\phi)$ and $Y(t) = Y(t,\phi)$ for the solutions of (4.2.1) and (4.2.3) respectively.

4.3 Statement and Discussion of Main Mathematical Results

In advance of stating our main results concerning the nonlinear equation (4.2.1), first recall the result concerning the linear equation (3.2.8). We can now state a nonlinear version of Theorem 3.3.1, which is the main mathematical result of the chapter. Roughly speaking, because (3.2.8) is a "linearisation" of (4.2.1) at infinity, the asymptotic behaviour of the solution of (4.2.1) follows that of (3.2.8) according as to whether $v_0(\nu)$ is zero or positive.

Theorem 4.3.1. *Let $\sigma \neq 0$. Let $g \in C(\mathbb{R};\mathbb{R})$ satisfy* (4.2.2)*, and let $\nu \in M[-\tau,0]$. Suppose that r obeys* (3.3.1)*. Then the solution X of* (4.2.1) *satisfies*

(i) If $v_0(\nu) = 0$, then

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = |\sigma|c, \quad \liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = -|\sigma|c, \quad \textit{a.s.}$$

(ii) If $v_0(\nu) > 0$, then

$$\limsup_{t\to\infty} \frac{1}{t}\log|X(t)| \leq v_0(\nu), \quad \textit{a.s.}$$

If, moreover there exists a non–decreasing continuous $\gamma_0 : [0,\infty) \to (0,\infty)$ such that

$$|g(x) - x| \leq \gamma_0(|x|), \quad x \in \mathbb{R}; \quad \int_1^\infty \frac{\gamma_0(x)}{x^2}\,dx < +\infty, \tag{4.3.1}$$

then

$$\lim_{t\to\infty} e^{-v_0(\nu)t}X(t) = \Lambda, \quad \textit{a.s.} \tag{4.3.2}$$

where

$$\Lambda = c\left(\phi(0) + \int_{[-\tau,0]}\int_s^0 e^{v_0(\nu)(s-u)}g(\phi(u))\,du\,\nu(ds) + \sigma\int_0^\infty e^{-v_0(\nu)s}\,dB(s)\right) + cv_0(\nu)\int_0^\infty e^{-v_0(\nu)s}\{g(X(s)) - X(s)\}\,ds. \tag{4.3.3}$$

In both cases, the constant c is given by (3.3.2) *and Λ is finite.*

Remark 4.3.1. The function $\gamma_0(x) = \max_{0\le s\le |x|} |g(s) - s|$ is non-decreasing, continuous positive and obeys the first part of Theorem 4.3.1. Therefore if

$$\int_1^\infty \frac{\max_{0\le s\le |x|} |g(s) - s|}{x^2}\, dx < \infty$$

and $v_0(\nu) > 0$, then (4.3.2) and (4.3.3) hold.

Remark 4.3.2. The analogue of this Theorem can be found in chapter 2. Part (a) corresponds to Theorem 2.5.1 and part (b) corresponds to Theorem 2.5.2. The proof of this Theorem is similar to the proof of the Theorems mentioned in chapter 2 and for this reason no outline of the proof is given.

The next result shows that in the case when $v_0(\nu) > 0$ the condition (4.3.1) on g is difficult to relax without losing the property that X is asymptotic to $e^{v_0(\nu)t}$ as $t \to \infty$.

Theorem 4.3.2. *Let $\sigma \neq 0$. Let $g \in C(\mathbb{R};\mathbb{R})$ satisfy* (4.2.2)*, and let $\nu \in M[-\tau, 0]$. Suppose that r obeys* (3.3.1) *with $v_0(\nu) > 0$. Suppose there exists $x_1 > 0$ such that γ is monotone on $(-\infty, -x_1] \cup [x_1, \infty)$, and that the solution X of* (4.2.1) *obeys* (4.3.2) *where Λ is a finite $\mathcal{F}^B(\infty)$–measurable random variable which assumes positive values on an event of positive probability, and negative values on a event of positive probability. Then Λ is given by* (4.3.3)*, and*

$$\int_1^\infty \frac{\gamma(x)}{x^2}\, dx \text{ and } \int_{-\infty}^{-1} \frac{\gamma(x)}{x^2}\, dx \text{ are finite.} \tag{4.3.4}$$

The result also shows that the existence of a limit in (4.3.2) forces that limit to be given by (4.3.3).

An explicit equivalence between the size of $g(x) - x$ and the existence of the limit in (4.3.2) can be obtained by imposing some extra monotonicity and structure on g.

Theorem 4.3.3. *Let $\sigma \neq 0$. Let $g \in C(\mathbb{R};\mathbb{R})$ satisfy* (4.2.2)*, and let $\nu \in M[-\tau, 0]$. Suppose that r obeys* (3.3.1) *with $v_0(\nu) > 0$. Suppose that $x \mapsto g(x) - x$ is odd and monotone on $\mathbb{R}$. Let X be the solution of* (4.2.1).

(i) If

$$\int_1^\infty \frac{|g(x) - x|}{x^2}\, dx < +\infty, \tag{4.3.5}$$

then there is a finite $\mathcal{F}^B(\infty)$–measurable random variable Λ such that X obeys (4.3.2)*, and Λ is given by* (4.3.3).

(ii) Suppose there is a finite $\mathcal{F}^B(\infty)$–measurable random variable Λ positive values on an event of positive probability, and negative values on a event of positive probability, and that X obeys (4.3.2)*. Then Λ is given by* (4.3.3) *and g obeys* (4.3.5).

Proof. Let $\gamma(x) = g(x) - x$ for $x \in \mathbb{R}$. Then γ is monotone and odd. Define $\gamma_0(x) = |\gamma(x)|$ for $x \in \mathbb{R}$. Then

$$\gamma_0(-x) = |\gamma(-x)| = |-\gamma(x)| = |\gamma(x)| = \gamma_0(x).$$

Then $|g(x) - x| = \gamma_0(x) = \gamma_0(|x|)$ for $x \in \mathbb{R}$. (4.3.5) implies

$$\int_1^\infty \frac{\gamma_0(x)}{x^2}\, dx = \int_1^\infty \frac{|g(x) - x|}{x^2}\, dx < +\infty.$$

By part (ii) of Theorem 4.3.1 we have that X obeys (4.3.2) and that Λ in (4.3.2) obeys (4.3.3). This proves part (i) of the result.

To prove part (ii), since γ is monotone on $\mathbb{R}$ it follows from Theorem 4.3.2 that Λ is given by (4.3.3) and that γ obeys (4.3.4). The latter conclusion implies that

$$\int_1^\infty \frac{g(x) - x}{x^2}\, dx \quad \text{and} \quad \int_{-\infty}^{-1} \frac{g(x) - x}{x^2}\, dx \quad \text{are finite.}$$

Since $x \mapsto g(x) - x$ is odd, we may rewrite the second integral as

$$\int_{-\infty}^{-1} \frac{g(x) - x}{x^2}\, dx = \int_1^\infty \frac{g(-u) - (-u)}{u^2}\, du = \int_1^\infty \frac{u - g(u)}{u^2}\, du.$$

Therefore both $\int_1^\infty (g(x)-x)/x^2\,dx$ and $\int_1^\infty (x-g(x))/x^2\,dx$ are finite.

Since $x \mapsto g(x)-x$ is odd, we have that $0 = g(0)-0$. If $x \mapsto g(x)-x$ is monotone non–decreasing, then $g(x)-x \geq 0$ for all $x \geq 0$ and so

$$\int_1^\infty \frac{|g(x)-x|}{x^2}\,dx = \int_1^\infty \frac{g(x)-x}{x^2}\,dx < +\infty,$$

and the proof of part (ii) is complete in this case. If, on the other hand, $x \mapsto g(x)-x$ is monotone non–increasing, then $g(x)-x \leq 0$ for all $x \geq 0$ and so

$$\int_1^\infty \frac{|g(x)-x|}{x^2}\,dx = \int_1^\infty \frac{x-g(x)}{x^2}\,dx < +\infty.$$

and part (ii) has been established in this case also. □

Given a measure ν it is often a rather delicate issue to determine the value of $v_0(\nu)$, but Theorem 3.3.2 gives sufficient conditions for this for a subclass of $M[-\tau,0]$ which will cover the economic modelling later. Theorem 3.3.2 guarantees that when ν obeys (3.3.3) and (3.3.4), then $m(\nu) < 1$ implies part (a) of Theorem 4.3.1, while $m(\nu) > 1$ implies part (b).

The following auxiliary lemma is required in the proof of Theorem 4.3.1; its proof is deferred to the final section.

Lemma 4.3.1. *Let ϑ be positive and increasing with $\vartheta(t-T)/\vartheta(t) \to 1$, as $t \to \infty$, for all $T \geq 0$. If κ is non–negative with $\int_0^\infty \kappa(s)\,ds \in (0,\infty)$, then*

$$\lim_{t\to\infty} \frac{1}{\vartheta(t)} \int_0^t \kappa(t-s)\vartheta(s)\,ds = \int_0^\infty \kappa(s)\,ds.$$

Remark 4.3.3. This Lemma is an analogue of Lemma 2.7.1 in chapter 2.

4.3.1 Economic modelling

Equation (4.2.1) application to the financial market mimics that of chapter 3. The only difference is the trading strategy of the j-th agent at time t is as follows: he considers a short–run moving average of a near linear transform the cumulative returns over the last θ_j units of time

$$\int_{[-\theta_j,0]} g(X(t+u))\,s_j(du)$$

for a measure $s_j \in M[-\theta_j,0]$ and also calculates a corresponding long–run average of cumulative returns over the last $\tau_j \geq \theta_j$ units of time

$$\int_{[-\tau_j,0]} g(X(t+u))\,l_j(du)$$

for a measure $l_j \in M[-\tau_j,0]$. So the planned excess demand of trader j at time t is

$$\beta_j \left(\int_{[-\theta_j,0]} g(X(t+u))\,s_j(du) - \int_{[-\tau_j,0]} g(X(t+u))\,l_j(du) \right)$$

where $\beta_j \geq 0$. Therefore, the overall planned excess demand of all traders is

$$\sum_{j=1}^N \beta_j \left(\int_{[-\theta_j,0]} g(X(t+u))\,s_j(du) - \int_{[-\tau_j,0]} g(X(t+u))\,l_j(du) \right),$$

and the returns are described by

$$dX(t) = \sum_{j=1}^N \beta_j \left(\int_{[-\theta_j,0]} g(X(t+u))\,s_j(du) - \int_{[-\tau_j,0]} g(X(t+u))\,l_j(du) \right) dt\sigma\,dB(t). \qquad (4.3.6)$$

We extend all measures s_j and l_j to the interval $[-\tau, 0]$ where $\tau = \max\{\tau_1, \dots, \tau_N\}$ by setting them zero outside their support. Again introducing the measure $\nu \in M[-\tau, 0]$ defined by

$$\nu(du) := \sum_{j=1}^{N} \beta_j (s_j - l_j)(du) \tag{4.3.7}$$

we can rewrite equation (4.3.6) as

$$dX(t) = \int_{[-\tau,0]} \nu(ds) g(X(t+s))\, dt + \sigma\, dB(t) \qquad \text{for all } t \geq 0,$$

which is the form of (4.2.1). The evolution of the price of the risky asset $(S(t) : t \geq 0)$ is now given by

$$dS(t) = \mu S(t)\, dt + S(t)\, dX(t), \quad t \geq 0; \qquad S(0) = s_0 > 0. \tag{4.3.8}$$

We can think of μ as the non-random interest rate in the model and consider $\mathbb{P}$ as the equivalent risk-neutral measure. Applying Itô's formula shows as in the standard Black-Scholes model that the price of the asset S can be represented by

$$S(t) = S(0) \exp\left(X(t) + (\mu - \tfrac{1}{2}\sigma^2)t\right) \qquad \text{for all } t \geq 0. \tag{4.3.9}$$

In the case when the feedback traders are absent, i.e. $\beta_j = 0$ for all $j = 1, \dots, N$, we have $dX(t) = \sigma\, dB(t)$, in which case S is Geometric Brownian motion, evolving according to

$$dS(t) = \mu S(t)\, dt + \sigma S(t)\, dB(t), \quad t \geq 0; \qquad S(0) = s_0 > 0.$$

4.4 Proofs

4.4.1 Proof of Theorem 4.3.1, part (i)

Let Y be a solution of (4.2.3) with $Y(t) = \phi(t)$, $t \in [-\tau, 0]$. Define $Z(t) = X(t) - Y(t)$, $t \geq -\tau$. Then $Z \in C^1(0, \infty)$ obeys

$$Z'(t) = \int_{[-\tau,0]} g(X(t+s))\nu(ds)\, dt - \int_{[-\tau,0]} Y(t+s)\nu(ds), \quad t > 0,$$

with $Z(t) = 0$ for $t \in [-\tau, 0]$. Define $\gamma(x) = g(x) - x$, $x \in \mathbb{R}$. Then

$$Z'(t) = \int_{[-\tau,0]} Z(t+s)\nu(ds) + \int_{[-\tau,0]} \gamma(X(t+s))\nu(ds)\, ds, \quad t > 0.$$

Hence by variation of constants and Fubini's Theorem, we get

$$\begin{aligned} Z(t) &= \int_0^t r(t-s) \int_{[-\tau,0]} \gamma(X(s+u))\nu(du)\, ds \\ &= \int_{[-\tau,0]} \left\{ \int_0^t r(t-s)\gamma(X(s+u))\, ds \right\} \nu(du). \end{aligned}$$

Therefore for $t \geq \tau$, and using the fact that $X(v) = \phi(v)$ for $v \in [-\tau, 0]$, we have

$$Z(t) = F_1(t) + \int_{[-\tau,0]} \left\{ \int_{-u}^t r(t-s)\gamma(X(s+u))\, ds \right\} \nu(du), \tag{4.4.1}$$

where

$$F_1(t) = \int_{[-\tau,0]} \left\{ \int_0^{-u} r(t-s)\gamma(\phi(s+u))\, ds \right\} \nu(du), \quad t \geq \tau. \tag{4.4.2}$$

Notice that as $r(t) \to c$ as $t \to \infty$ implies that $F_1(t) \to F^*$ as $t \to \infty$.

Now, by Fubini's Theorem, and (3.2.2), we get

$$\begin{aligned}
&\int_{[-\tau,0]} \left\{ \int_{-u}^{t} r(t-s)\gamma(X(s+u))\,ds \right\} \nu(du) \\
&= \int_{[-\tau,0]} \left\{ \int_{0}^{t+u} r(t+u-v)\gamma(X(v))\,dv \right\} \nu(du) \\
&= \int_0^t \left\{ \int_{[v-t,0],u\geq -\tau} r(t+u-v)\nu(du) \right\} \gamma(X(v))\,dv \\
&= \int_0^{t-\tau} \left\{ \int_{[-\tau,0]} r(t+u-v)\nu(du) \right\} \gamma(X(v))\,dv \\
&\qquad + \int_{t-\tau}^{t} \left\{ \int_{[v-t,0]} r(t+u-v)\nu(du) \right\} \gamma(X(v))\,dv \\
&= \int_0^{t-\tau} r'(t-v)\gamma(X(v))\,dv + \int_{t-\tau}^{t} \left\{ \int_{[v-t,0]} r(t+u-v)\nu(du) \right\} \gamma(X(v))\,dv \\
&= \int_0^{t-\tau} r'(t-v)\gamma(X(v))\,dv + \int_{t-\tau}^{t} u_0(t-v)\gamma(X(v))\,dv,
\end{aligned}$$

where we have defined $u_0(t) = \int_{[-t,0]} \nu(du)r(t+u)$, $t \in [0,\tau]$. Inserting this into (4.4.1), we obtain

$$Z(t) = F_1(t) + \int_0^t u(t-v)\gamma(X(v))\,dv, \quad t \geq \tau,$$

where we have defined

$$u(t) = \begin{cases} r'(t), & t > \tau, \\ u_0(t), & t \in [0,\tau]. \end{cases}$$

Note that $r - c \in L^1(0,\infty)$ implies that $r' \in L^1(0,\infty)$, so $u \in L^1(0,\infty)$. Property (4.2.2b) implies that for every $\varepsilon > 0$ such that $\varepsilon \int_0^\infty |u(s)|\,ds < 1/2$, there exists a $\Lambda(\varepsilon) > 0$ such that

$$|\gamma(x)| \leq \Lambda(\varepsilon) + \varepsilon|x|, \quad \text{for all } x \in \mathbb{R}.$$

Therefore for all $t \geq \tau$,

$$\begin{aligned}
|Z(t)| &\leq |F_1(t)| + \Lambda(\varepsilon)\int_0^t |u(t-v)|\,dv + \varepsilon \int_0^t |u(t-v)||X(v)|\,dv \\
&\leq F_2(t) + \varepsilon \int_0^t |u(t-v)||Z(v)|\,dv,
\end{aligned}$$

where $F_2(t) = |F_1(t)| + \Lambda(\varepsilon)\int_0^\infty |u(v)|\,dv + \varepsilon \int_0^t |u(t-v)||Y(v)|\,dv$ for $t \geq \tau$. Now define

$$F(t) = \begin{cases} \max_{t\in[0,\tau]} |Z(s)|, & t \in [0,\tau), \\ F_2(t), & t \geq \tau, \end{cases}$$

Then

$$|Z(t)| \leq F(t) + \varepsilon \int_0^t |u(t-v)||Z(v)|\,dv, \quad t \geq 0. \tag{4.4.3}$$

Now, we determine the asymptotic behaviour of Z. To do this we need to find the asymptotic behaviour of F. By Theorem 3.3.1, Y obeys

$$\limsup_{t\to\infty} \frac{|Y(t)|}{\sqrt{2t\log\log t}} = c|\sigma|, \quad \text{a.s.}$$

Fix $\omega \in \Omega^*$, the almost sure event on which the last statement holds. Then by Lemma 4.3.1, we have

$$\limsup_{t\to\infty} \frac{F(t,\omega)}{\sqrt{2t\log\log t}} \leq \varepsilon c|\sigma| \int_0^\infty |u(s)|\,ds. \tag{4.4.4}$$

Next note by (4.4.3) that the solution $U(\cdot,\omega)$ of

$$U(t,\omega) = F(t,\omega) + \varepsilon \int_0^t |u(t-v)|U(v,\omega)\,dv, \quad t \geq 0,$$

obeys $|Z(t,\omega)| \leq U(t,\omega)$ for all $t \geq 0$. Defining ρ by $\rho(t) = \varepsilon|u(t)| + \int_0^t \varepsilon|u(t-s)|\rho(s)\,ds$, $t \geq 0$, we have

$$U(t,\omega) = F(t,\omega) + \int_0^t \rho(t-s)F(s,\omega)\,ds, \quad t \geq 0.$$

Since $\varepsilon\int_0^\infty |u(s)|\,ds < 1/2$, we have that $\rho \geq 0$ is in $L^1(0,\infty)$ and obeys

$$1 + \int_0^\infty \rho(s)\,ds = 1 + \varepsilon\frac{\int_0^\infty |u(t)|\,dt}{1-\varepsilon\int_0^\infty |u(t)|\,dt} = \frac{1}{1-\varepsilon\int_0^\infty |u(t)|\,dt}.$$

Therefore, by Lemma 4.3.1, we obtain

$$\limsup_{t\to\infty} \frac{U(t,\omega)}{\sqrt{2t\log\log t}} \leq \varepsilon c|\sigma| \int_0^\infty |u(s)|\,ds \left(1 + \int_0^\infty \rho(s)\,ds\right),$$

so

$$\limsup_{t\to\infty} \frac{|Z(t,\omega)|}{\sqrt{2t\log\log t}} \leq \varepsilon c|\sigma| \int_0^\infty |u(s)|\,ds \cdot \frac{1}{1-\varepsilon\int_0^\infty |u(t)|\,dt}.$$

Therefore, we may let $\varepsilon \to 0$ to get

$$\lim_{t\to\infty} \frac{Z(t,\omega)}{\sqrt{2t\log\log t}} = 0.$$

Since this holds for each ω in an almost sure event and $Z = X - Y$, we have that

$$\limsup_{t\to\infty} \frac{|X(t)-Y(t)|}{\sqrt{2t\log\log t}} = 0, \quad \text{a.s.}$$

Therefore, by applying the result of part (a) of Theorem 3.3.1, we see that the assertion of part (i) of the Theorem 4.3.1 must hold.

4.4.2 Proof of Theorem 4.3.1, part (ii)

Let $Z(t) = X(t) - Y(t)$, $t \geq 0$ where Y is the solution of (4.2.3) with $Y(t) = X(t) = \phi(t)$ for $t \in [-\tau, 0]$. With $\gamma(x) = g(x) - x$, F_1 defined by

$$F_1(t) = \int_{[-\tau,0]} \left\{\int_0^{-u} r(t-s)\gamma(\phi(s+u))\,ds\right\} \nu(du), \quad t \geq \tau,$$

and u is defined by

$$u(t) = \begin{cases} r'(t), & t > \tau, \\ \int_{[-t,0]} r(t+u)\nu(du), & t \in [0,\tau], \end{cases}$$

we obtain

$$Z(t) = F_1(t) + \int_0^t u(t-v)\gamma(X(v))\,dv, \quad t \geq \tau,$$

Hence with $F(t) = F_1(t) + Y(t)$, we have

$$X(t) = F(t) + \int_0^t u(t-s)\gamma(X(s))\,ds, \quad t \geq \tau.$$

By Theorem 3.3.1, we have that $\lim_{t\to\infty} e^{-v_0(\nu)t}Y(t) =: \Gamma_0$ a.s. and by hypothesis we have $\lim_{t\to\infty} r(t)e^{-v_0(\nu)t} = c$, where c is given by (3.3.2) and $v_0(\nu)$ is real, simple zero of the characteristic equation. Next as

$$e^{-v_0(\nu)t}F_1(t) = \int_{[-\tau,0]} \left\{ \int_0^{-u} (e^{-v_0(\nu)(t-s)}r(t-s) - c)e^{-v_0(\nu)s}\gamma(\phi(s+u)) \right\} \nu(du) + \int_{[-\tau,0]} \left\{ \int_0^{-u} ce^{-v_0(\nu)s}\gamma(\phi(s+u))\, ds \right\} \nu(du),$$

we obtain

$$\lim_{t\to\infty} e^{-v_0(\nu)t}F_1(t) = \int_{[-\tau,0]} \left\{ \int_0^{-u} ce^{-v_0(\nu)s}\gamma(\phi(s+u))\, ds \right\} \nu(du) =: \Gamma_1.$$

Define $\widetilde{X}(t) = e^{-v_0(\nu)t}X(t)$ for $t \geq -\tau$ and for $t \geq 0$ define $\widetilde{u}(t) = e^{-v_0(\nu)t}u(t)$ and $\widetilde{F}(t) = e^{-v_0(\nu)t}F(t)$. Then we have $\lim_{t\to\infty} \widetilde{F}(t) = \Gamma_0 + \Gamma_1$. Also

$$\begin{aligned}
\lim_{t\to\infty} \widetilde{u}(t) = \lim_{t\to\infty} e^{-v_0(\nu)t}r'(t) &= \lim_{t\to\infty} \int_{[-\tau,0]} \nu(ds)e^{v_0(\nu)s}e^{-v_0(\nu)(t+s)}r(t+s) \\
&= c\int_{[-\tau,0]} \nu(ds)e^{v_0(\nu)s} = cv_0(\nu).
\end{aligned}$$

By the definition of $\widetilde{X}$, $\widetilde{u}$ etc., we have

$$\widetilde{X}(t) = \widetilde{F}(t) + \int_0^t \widetilde{u}(t-s)e^{-v_0(\nu)s}\gamma(e^{v_0(\nu)s}\widetilde{X}(s))\, ds, \quad t \geq \tau. \tag{4.4.5}$$

Now, let $X_\epsilon(t) = \widetilde{X}(t)e^{-(v_0(\nu)+\epsilon)t}$ etc, so that

$$X_\epsilon(t) = F_\epsilon(t) + \int_0^t u_\epsilon(t-s)e^{-(v_0(\nu)+\epsilon)s}\gamma(e^{(v_0(\nu)+\epsilon)s}X_\epsilon(s))\, ds, \quad t \geq \tau.$$

For every $\eta > 0$ there exists $L(\eta) > 0$ such that $|\gamma(x)| \leq L(\eta) + \eta|x|$. Choose $\eta > 0$ sufficiently small so that $\eta\int_0^\infty e^{-\epsilon s}|\widetilde{u}(s)|\, ds < 1$. Hence for $t \geq \tau$ we have

$$\begin{aligned}
|X_\epsilon(t)| &\leq |F_\epsilon(t)| + \int_0^t |u_\epsilon(t-s)| \left\{ L(\eta)e^{-(v_0(\nu)+\epsilon)s} + \eta|X_\epsilon(s)| \right\}\, ds \\
&\leq |F_\epsilon(t)| + \eta\int_0^t |u_\epsilon(t-s)||X_\epsilon(s)|\, ds + L(\eta)\int_0^t |u_\epsilon(t-s)|e^{-(v_0(\nu)+\epsilon)s}\, ds \\
&=: G_{\epsilon,\eta}(t) + \eta\int_0^t |u_\epsilon(t-s)||X_\epsilon(s)|\, ds.
\end{aligned}$$

Since $u_\epsilon \in L^1(0,\infty)$ and $F_\epsilon \in L^1(0,\infty)$ it follows that $G_{\epsilon,\eta} \in L^1(0,\infty)$. Similarly, $u_\epsilon(t) \to 0$ and $F_\epsilon(t) \to 0$ as $t \to \infty$, and hence $G_{\epsilon,\eta}(t) \to 0$ as $t \to \infty$. Since $G_{\epsilon,\eta} \in L^1(0,\infty)$ and $\eta\int_0^\infty e^{-\epsilon s}|\widetilde{u}(s)|\, ds < 1$, we have $X_\epsilon \in L^1(0,\infty)$ a.s. Thus, as $u_\epsilon(t) \to 0$ as $t \to \infty$, we have that $X_\epsilon(t) \to 0$ as $t \to \infty$ a.s. We have thus shown that there exist a.s. events Ω_ϵ such that

$$\lim_{t\to\infty} e^{-(v_0(\nu)+\epsilon)t}X(t) = 0, \quad \text{on } \Omega_\epsilon.$$

We hold $\epsilon > 0$ fixed temporarily. The limit implies that for all $\omega \in \Omega_\epsilon$ there exists $T(\epsilon,\omega)$ so that $|X(t,\omega)| < e^{(v_0(\nu)+\epsilon)t}$ for all $t > T(\epsilon,\omega)$. Hence

$$\frac{1}{t}\log|X(t)| \leq v_0(\nu) + \epsilon, \quad t > T(\epsilon,\omega),$$

and so

$$\limsup_{t\to\infty} \frac{1}{t}\log|X(t)| \leq v_0(\nu) + \epsilon, \quad \text{a.s. on } \Omega_\epsilon.$$

Now, let $\Omega^* = \cap_{\epsilon\in\mathbb{Q}\cap(0,1)}\Omega_\epsilon$. Then Ω^* is an almost sure event and we have

$$\limsup_{t\to\infty}\frac{1}{t}\log|X(t)| \le v_0(\nu), \quad \text{a.s. on } \Omega^*,$$

as required.

It remains to prove the existence of the limit $\lim_{t\to\infty} e^{-v_0(\nu)t}X(t)$ under the hypothesis (4.3.1) on g. We begin by making some observations. Since $\widetilde{F}$ is continuous and $\widetilde{F}(t)\to\Gamma_0+\Gamma_1$ as $t\to\infty$, we may define $f_1=\sup_{t\ge 0}|\widetilde{F}(t)|\in(0,\infty)$. Since $\widetilde{u}$ is continuous and $\widetilde{u}(t)\to cv_0(\nu)$ as $t\to\infty$, we may define $u_1=\sup_{t\ge0}|\widetilde{u}(t)|\in(0,\infty)$. Thus

$$|\widetilde{X}(t)| \le f_1 + \int_0^t u_1 e^{-v_0(\nu)s}|\gamma(e^{v_0(\nu)s}\widetilde{X}(s))|\,ds, \quad t\ge\tau.$$

Since $|\gamma(x)|\le\gamma_0(|x|)$ for $x\in\mathbb{R}$ we have

$$|\widetilde{X}(t)| \le f_1 + u_1\int_0^t e^{-v_0(\nu)s}\gamma_0(e^{v_0(\nu)s}|\widetilde{X}(s)|)\,ds, \quad t\ge\tau.$$

Define $f_2>f_1$ such that

$$\int_{f_2}^\infty \frac{1}{s^2}\gamma_0(s)\,ds\cdot\frac{u_1}{v_0(\nu)} < \frac{1}{2}. \tag{4.4.6}$$

Define $I(t)=\int_0^t e^{-v_0(\nu)s}\gamma_0(e^{v_0(\nu)s}|\widetilde{X}(s)|)\,ds$ for $t\ge0$. Since γ_0 is continuous and positive, I is non–decreasing and differentiable, and I obeys $|\widetilde{X}(t)|\le f_1+u_1I(t)$ for $t\ge\tau$ and

$$I'(t)=e^{-v_0(\nu)t}\gamma_0(e^{v_0(\nu)t}|\widetilde{X}(t)|).$$

Therefore as γ_0 is non–decreasing for $t\ge\tau$ we have

$$e^{v_0(\nu)t}I'(t)=\gamma_0(e^{v_0(\nu)t}|\widetilde{X}(t)|)\le\gamma_0(e^{v_0(\nu)t}(f_1+u_1I(t))).$$

Hence for any $t\ge\tau$ we have

$$I(t)\le I(\tau)+\int_\tau^t e^{-v_0(\nu)s}\gamma_0(e^{v_0(\nu)s}(f_1+u_1I(s)))\,ds.$$

Since I is non–decreasing and γ_0 is non–decreasing, with $y^*:=f_2+u_1I(t)>f_1+u_1I(t)$ we have

$$\begin{aligned}I(t)&\le I(\tau)+\int_\tau^t e^{-v_0(\nu)s}\gamma_0(e^{v_0(\nu)s}(f_1+u_1I(s)))\,ds\\&\le I(\tau)+\int_\tau^t e^{-v_0(\nu)s}\gamma_0(y^*e^{v_0(\nu)s})\,ds.\end{aligned}$$

Now, as $u\mapsto\gamma_0(u)/u^2$ is integrable and (4.4.6) holds, we have

$$\begin{aligned}\int_\tau^t e^{-v_0(\nu)s}\gamma_0(y^*e^{v_0(\nu)s})\,ds&=\int_{y^*e^{v_0(\nu)\tau}}^{y^*e^{v_0(\nu)t}}\frac{y^*}{w}\gamma_0(w)\frac{1}{v_0(\nu)}\frac{1}{w}\,dw\\&\le\int_{y^*}^\infty\frac{\gamma_0(w)}{w^2}\,dw\cdot\frac{y^*}{v_0(\nu)}\\&\le\int_{f_2}^\infty\frac{\gamma_0(w)}{w^2}\,dw\cdot\frac{f_2+u_1I(t)}{v_0(\nu)}\\&\le\int_{f_2}^\infty\frac{\gamma_0(w)}{w^2}\,dw\cdot\frac{f_2}{v_0(\nu)}+\frac{1}{2}I(t),\end{aligned}$$

where we have also used the facts that $y^*>f_2$ and $v_0(\nu)>0$ to extend the limits of integration. Thus for any $t\ge\tau$ we have

$$I(t)\le I(\tau)+\int_{f_2}^\infty\frac{\gamma_0(w)}{w^2}\,dw\cdot\frac{f_2}{v_0(\nu)}+\frac{1}{2}I(t).$$

Therefore

$$I(t) \le 2I(\tau) + 2\int_{f_2}^{\infty} \frac{\gamma_0(w)}{w^2}\,dw \cdot \frac{f_2}{v_0(\nu)}, \quad t \ge \tau.$$

Thus I is bounded, and therefore it follows that there exists a finite $\bar{x} > 0$ such that $|\widetilde{X}(t)| \le \bar{x}$ for all $t \ge 0$. Define $J(t) = e^{-v_0(\nu)t}\gamma(e^{v_0(\nu)t}\widetilde{X}(t))$. This yields

$$\widetilde{X}(t) = \widetilde{F}(t) + \int_0^t \widetilde{u}(t-s)J(s)\,ds, \quad t \ge \tau.$$

Since $\widetilde{u}(t) \to cv_0(\nu)$ as $t \to \infty$, if we can show that $J \in L^1(0,\infty)$, then

$$\lim_{t\to\infty} e^{-v_0 t}X(t) = \Gamma_0 + \Gamma_1 + cv_0(\nu)\int_0^{\infty} J(s)\,ds, \quad \text{a.s.}$$

which establishes the existence of the required limit. To show that J is integrable, we simply note that as $|\widetilde{X}(t)| \le \bar{x}$ and γ_0 is non–decreasing, we can obtain

$$\begin{aligned}\int_0^{\infty} |J(t)|\,dt &= \int_0^{\infty} e^{-v_0(\nu)t}|\gamma(e^{v_0(\nu)t}\widetilde{X}(t))|\,dt \le \int_0^{\infty} e^{-v_0(\nu)t}\gamma_0(e^{v_0(\nu)t}|\widetilde{X}(t)|)\,dt \\ &\le \int_0^{\infty} e^{-v_0(\nu)t}\gamma_0(e^{v_0(\nu)t}\bar{x})\,dt \\ &= \int_{\bar{x}}^{\infty} \frac{\bar{x}}{w}\gamma_0(w)\frac{1}{v_0(\nu)}\frac{1}{w}\,dw < +\infty,\end{aligned}$$

as required. Therefore

$$\begin{aligned}&\lim_{t\to\infty} e^{-v_0 t}X(t) \\ &\quad = c\left(\phi(0) + \int_{[-\tau,0]}\int_s^0 e^{v_0(\nu)(s-u)}\phi(u)\,du\,\nu(ds) + \sigma\int_0^{\infty} e^{-v_0(\nu)s}\,dB(s)\right) \\ &\qquad + c\int_{[-\tau,0]}\left\{\int_s^0 e^{v_0(\nu)(s-u)}\gamma(\phi(u))\,du\right\}\nu(ds) + cv_0(\nu)\int_0^{\infty} e^{-v_0(\nu)s}\gamma(X(s))\,ds.\end{aligned}$$

Since $g(x) = x + \gamma(x)$, we get (4.3.2) as required.

4.4.3 Proof of Theorem 4.3.2

Suppose that $X(t)e^{-v_0(\nu)t} \to \Lambda \ne 0$ as $t \to \infty$. Since

$$\widetilde{X}(t) = \widetilde{F}(t) + \int_0^t \widetilde{u}(t-s)e^{-v_0(\nu)s}\gamma(e^{v_0(\nu)s}\widetilde{X}(s))\,ds, \quad t \ge \tau,$$

it follows that we must have

$$\lim_{t\to\infty}\int_0^t \widetilde{u}(t-s)e^{-v_0(\nu)s}\gamma(X(s))\,ds \quad \text{exists.}$$

Now

$$\widetilde{u}(t) - cv_0(\nu) = \int_{[-\tau,0]} e^{v_0(\nu)s}(r(t+s)e^{-v_0(\nu)(t+s)} - c)\nu(ds)$$

so by (3.3.1), $\widetilde{u} - cv_0(\nu)$ is integrable. Hence

$$\lim_{t\to\infty}\left\{\int_0^t (\widetilde{u}(t-s) - cv_0(\nu))\,e^{-v_0(\nu)s}\gamma(X(s))\,ds + cv_0(\nu)\int_0^t e^{-v_0(\nu)s}\gamma(X(s))\,ds\right\}$$

exists. Now $\gamma(x)/x = g(x)/x - 1 \to 0$ as $|x| \to \infty$. Therefore, for every $\varepsilon > 0$ there is a $\Lambda(\varepsilon) > 0$ such that for all $t \geq 0$ we have

$$e^{-v_0(\nu)t}|\gamma(X(t))| \leq e^{-v_0(\nu)t}\Lambda(\varepsilon) + \varepsilon e^{-v_0(\nu)t}|X(t)|.$$

Hence as $X(t)e^{-v_0(\nu)t} \to \Lambda$ as $t \to \infty$, and $v_0(\nu) > 0$ we have

$$\limsup_{t\to\infty} e^{-v_0(\nu)t}|\gamma(X(t))| \leq \varepsilon \limsup_{t\to\infty} e^{-v_0(\nu)t}|X(t)| = \varepsilon\Lambda.$$

Since $\varepsilon > 0$ is arbitrary, we see that $e^{-v_0(\nu)t}\gamma(X(t)) \to 0$ as $t \to \infty$. Hence

$$\lim_{t\to\infty} \int_0^t (\widetilde{u}(t-s) - cv_0(\nu))\, e^{-v_0(\nu)s}\gamma(X(s))\, ds = 0,$$

and therefore as $cv_0(\nu) > 0$ it follows that

$$\lim_{t\to\infty} \int_0^t e^{-v_0(\nu)s}\gamma(X(s))\, ds \quad \text{exists and is finite.}$$

Suppose that $\Lambda > 0$. Then there exists $T_1 > 0$ such that $3\Lambda/2 > X(t)e^{-v_0(\nu)t} > \Lambda/2$ for all $t > T_1$, so that $3\Lambda/2e^{v_0(\nu)t} > X(t) > \Lambda/2e^{v_0(\nu)t}$ for all $t > T_1$. Clearly there exists $T_2 \geq T_1$ such that $\Lambda/2e^{v_0(\nu)T_2} \geq x_1$. Without loss of generality, we may take γ non–decreasing on $[x_1, \infty)$. Then we have

$$\begin{aligned} C := \int_0^\infty e^{-v_0(\nu)s}\gamma(X(s))\, ds &= \int_0^{T_2} e^{-v_0(\nu)s}\gamma(X(s))\, ds + \int_{T_2}^\infty e^{-v_0(\nu)s}\gamma(X(s))\, ds \\ &\geq \int_0^{T_2} e^{-v_0(\nu)s}\gamma(X(s))\, ds + \int_{T_2}^\infty e^{-v_0(\nu)s}\gamma(\frac{\Lambda}{2}e^{v_0(\nu)s})\, ds. \end{aligned}$$

Also we have

$$C \leq \int_0^{T_2} e^{-v_0(\nu)s}\gamma(X(s))\, ds + \int_{T_2}^\infty e^{-v_0(\nu)s}\gamma(\frac{3\Lambda}{2}e^{v_0(\nu)s})\, ds.$$

Hence there exists a finite C_1 such that

$$C_1 \geq \int_{T_2}^\infty e^{-v_0(\nu)s}\gamma(\frac{\Lambda}{2}e^{v_0(\nu)s})\, ds = \int_{\Lambda/2e^{v_0(\nu)T_2}}^\infty \frac{\Lambda/2}{u}\gamma(u)\frac{1}{v_0(\nu)}\frac{1}{u}\, du.$$

Therefore there is a finite C_1' such that

$$C_1' \geq \int_{\Lambda/2e^{v_0(\nu)T_2}}^\infty \frac{\gamma(u)}{u^2}\, du.$$

Similarly, we can deduce that there is a finite C_2 such that

$$C_2' \leq \int_{3\Lambda/2e^{v_0(\nu)T_2}}^\infty \frac{\gamma(u)}{u^2}\, du,$$

which implies that $\int_1^\infty \gamma(u)/u^2\, du$ is finite.

Suppose that $\Lambda < 0$. Then there exists $T_3 > 0$ such that $3\Lambda/2 < X(t)e^{-v_0(\nu)t} < \Lambda/2$ for all $t > T_3$, so that $\Lambda/2e^{v_0(\nu)t} > X(t) > 3\Lambda/2e^{v_0(\nu)t}$ for all $t > T_3$. Clearly there exists $T_4 \geq T_3$ such that $\Lambda/2e^{v_0(\nu)T_4} \leq -x_1$. Without loss of generality, we may take γ to be non–increasing on $(-\infty, -x_1]$. This implies that $\gamma(\Lambda/2e^{v_0(\nu)t}) < \gamma(X(t)) < \gamma(3\Lambda/2e^{v_0(\nu)t})$ for $t \geq T_4$, so we get

$$\begin{aligned} C := \int_0^\infty e^{-v_0(\nu)s}\gamma(X(s))\, ds &= \int_0^{T_4} e^{-v_0(\nu)s}\gamma(X(s))\, ds + \int_{T_4}^\infty e^{-v_0(\nu)s}\gamma(X(s))\, ds \\ &\geq \int_0^{T_4} e^{-v_0(\nu)s}\gamma(X(s))\, ds + \int_{T_4}^\infty e^{-v_0(\nu)s}\gamma(\frac{\Lambda}{2}e^{v_0(\nu)s})\, ds. \end{aligned}$$

Also we have

$$C \leq \int_0^{T_4} e^{-v_0(\nu)s}\gamma(X(s))\,ds + \int_{T_4}^{\infty} e^{-v_0(\nu)s}\gamma(\frac{3\Lambda}{2}e^{v_0(\nu)s})\,ds.$$

Hence there exists a finite C_1 such that

$$C_1 \geq \int_{T_4}^{\infty} e^{-v_0(\nu)s}\gamma(\frac{\Lambda}{2}e^{v_0(\nu)s})\,ds = \int_{-\infty}^{\Lambda/2e^{v_0(\nu)T_4}} \frac{-\Lambda/2}{u}\gamma(u)\frac{1}{v_0(\nu)}\frac{1}{u}\,du.$$

Therefore there is a finite C_1' such that

$$C_1' \geq \int_{-\infty}^{\Lambda/2e^{v_0(\nu)T_4}} \frac{\gamma(u)}{u^2}\,du.$$

Similarly, we can deduce that there is a finite C_2' such that

$$C_2' \leq \int_{-\infty}^{3\Lambda/2e^{v_0(\nu)T_4}} \frac{\gamma(u)}{u^2}\,du,$$

which implies that $\int_{-\infty}^{-1}\gamma(u)/u^2\,du$ is finite.

4.4.4 Proof of Lemma 4.3.1

Without loss of generality, let $\int_0^\infty \kappa(s)\,ds = 1$. For every $\varepsilon > 0$ there is $T = T(\varepsilon) > 0$ such that $\int_T^\infty \kappa(s)\,ds < \varepsilon$. For $t \geq T$, we have

$$\begin{aligned}\frac{(\kappa * \vartheta)(t)}{\vartheta(t)} - \int_0^t \kappa(s)\,ds = \int_0^T \kappa(s)\left(\frac{\vartheta(t-s)}{\vartheta(t)} - 1\right)\,ds \\ + \int_T^t \kappa(s)\left(\frac{\vartheta(t-s)}{\vartheta(t)} - 1\right)\,ds.\end{aligned}$$

Now, as $\int_0^\infty \kappa(s)\,ds = 1$, and ϑ is an increasing function

$$\left|\int_0^T \kappa(s)\left(\frac{\vartheta(t-s)}{\vartheta(t)} - 1\right)\,ds\right| \leq 1 - \frac{\vartheta(t-T)}{\vartheta(t)}.$$

Moreover, as κ is non–negative and ϑ increasing, we have

$$\left|\int_T^t \kappa(s)\left(\frac{\vartheta(t-s)}{\vartheta(t)} - 1\right)\,ds\right| = \int_T^t \kappa(s)\left(1 - \frac{\vartheta(t-s)}{\vartheta(t)}\right)\,ds \leq \int_T^\infty \kappa(s)\,ds.$$

Thus

$$\begin{aligned}\left|\int_0^t \frac{\kappa(s)\vartheta(t-s)}{\vartheta(t)}\,ds - \int_0^t \kappa(s)\,ds\right| &\leq 1 - \frac{\vartheta(t-T)}{\vartheta(t)} + \int_T^\infty \kappa(s)\,ds \\ &< 1 + \frac{\vartheta(t-T)}{\vartheta(t)} + \varepsilon.\end{aligned}$$

Using $\vartheta(t-T)/\vartheta(t) \to 1$ as $t \to \infty$, and then letting $\varepsilon \to 0$ yields the result.

Chapter 5

Market Models from Max-Type Stochastic Functional Equations

5.1 Introduction

This chapter introduces a class of stochastic functional difference equations whose structure is motivated by three ubiquitous forms of heuristic investment strategy in financial markets: the comparison of current prices with a reference level; trading on noise (or the latest news); and trading based on a comparison of the local maximum of prices with the current price. It is the presence of the last category of speculative behaviour which makes it reasonable to incorporate a maximum functional of the process on the right hand of the stochastic difference equation. Accordingly the equations studied are stochastic functional difference equations in which the maximum of the solution over the last N time units appears on the righthand side. Roughly speaking, we show that the market experiences either fluctuations or undergoes dynamics consistent with a crash or bubble.

Earlier in the thesis we have considered moving averages a feature of many of these stochastic functional equation models. A number of papers using these equations or difference equations to model risky asset price dynamics include [1, 2, 14, 12, 11, 21, 38]. These moving averages lead to stochastic Volterra equations or to equations using linear functionals of past prices or returns, such as those studied in earlier chapters. These equations are quite tractable analytically but do not allow for the inclusion of agents who use the maximum of the last several periods of returns as a trading indicator. If we wish to include such agents in our model, it will first be necessary to understand and deduce some properties of stochastic functional difference equations with maxima. A body of literature on deterministic equations with maxima has begun to mature in the last ten years, building on original work on the stability of functional difference inequalities with maxima found in [37]. The main result in this direction is referred to as *Halanay's inequality*. Current research on deterministic functional differential equations with maxima covers results on existence, oscillation and asymptotic behaviour. A selection of important and representative recent papers is [41, 47, 40]. On the other hand, Halanay's inequality has been employed in numerical analysis [19], and in the numerical analysis of stochastic functional differential equations [15, 17, 18] in particular. Despite the analysis in [15, 17, 18], it seems that very limited information about stochastic functional difference equations with maxima has appeared in the mathematical literature. For this type of stochastic difference equation, it is usually not possible to express the solution explicitly in terms of an underlying deterministic difference equation. However, by employing a constructive comparison technique similar to that developed in the study of almost sure asymptotic behaviour of SFDEs in [4, 9], we find it possible to determine quite sharp estimates on the rate of growth of both the partial maxima and of the solutions themselves in the recurrent and transient cases respectively.

We describe how the results in this chapter can be considered in terms of financial economics. Our model is informationally inefficient, in the sense that past movements of the stock price have an influence on future movements. We assume that there is trading at intervals of one time unit (a day, say) with prices fixed in the intervening period. The inefficiency stems from the presence of a class of trend–following speculators, whose demand for the asset depends on the difference between the current level of the daily returns and the maximum of the daily returns over the past N time units. We assume that there are another class of traders who compare returns with a reference level, and that traders can also respond to "news", represented in the model as a source of independent and identically distributed random variables independent of the returns.

By considering the excess demand of traders, we are led to analyse a stochastic difference equation of the form

$$X(n+1) = X(n) + \alpha X(n) + \beta \max_{n-N\le j\le n} X(j) + \xi(n+1), \quad n \ge 0, \tag{5.1.1}$$

where ξ is the "news". Here $X(n)$ represents the daily returns, shifted by a constant, and α and $\beta > 0$ are constants which incorporate the trading behaviour of the various classes of speculator. These special properties influence the almost sure asymptotic behaviour as $n \to \infty$ of solutions of (5.1.1). We are able to identify two comprehensive and non—overlapping regions in (α, β) parameter space in which the equation has either transient or recurrent solutions. In the transient case we determine the growth rate of solutions, while in the recurrent case we determine the size of the largest fluctuations of solutions. Roughly speaking, we show that the returns exhibit dynamics consistent with the daily returns being stationary, or the market experiences a crash or bubble.

The recurrent case ($1+\alpha+\beta<1$), which results from the presence of sufficient negative feedback from the instantaneous term, may be interpreted as a conventional fluctuating market and enable us to show that the daily returns are governed by fluctuations consistent with a stationary process. Such recurrent behaviour arises if the technical speculators do not react very aggressively to differences between the maximum return and current return, in which case the rate of growth of the partial maxima of the solution is the same as that of the noise term. These partial maxima measure the asymptotic rate of growth of the largest fluctuations. Therefore, the trading does not itself produce very excess volatility, and so to a first approximation the market appears efficient. However, upper bounds on the size of these largest fluctuations is greater in the presence of trend following speculators than in their absence, where the market only reacts to "news". Hence the presence of these speculators tends to increase market volatility as well as causing correlation in the returns. Representative results in this direction are Theorems 5.2.2, 5.2.3 and 5.2.5, in which regularly varying, polynomial and thin–tailed distribution functions are tackled, respectively.

The transient case ($1+\alpha+\beta>1$) yields a mathematical realisation of a runaway stock market bubble or crash. This occurs if the reference traders are predominantly positive feedback traders, and the daily returns will tend to plus or minus infinity exponentially fast (see Theorem 5.2.1). The manner in which these dynamics form is consistent with the phenomenon of mimetic contagion [65].

In terms of financial economics, the chapter is an extension of previous work by Appleby & Swords [12] and Appleby, Swords and Rodkina [11], which consider discrete time equations in which speculators use moving averages rather than maxima to determine their trading strategies, and of Appleby and Wu [13] which considers related equations in continuous time. In each of [12, 11, 13] analogues of Theorem 5.2.1 and e.g., Theorem 5.2.2 are proven. This is important in any mathematical model in economics, as model assumptions are unlikely to be satisfied in reality, rendering general models which are robust to changes in the assumptions particularly desirable. The common feature with results in this chapter and those in [12, 11] is that an excess of positive feedback trading leads to a market bubble or crash, while the presence of heavy tailed noise leads to large fluctuations in the incremental returns consistent with the returns coming from a heavy–tailed stationary distribution. In this chapter, we extend results in [11] and actually show that the large fluctuations in the daily returns are equivalent to the same kind of thick tailed noise term ξ. Moreover, unlike the situation in [12, 11], we are unable to avail of a variation of constants formula, or a linearisation of an underlying linear Volterra difference equation owing to the presence of the maximum functional on the righthand side of (5.1.1). The results of this chapter are in many cases discrete–time analogues of those in [13]. However, due to variety of types of tail behaviour of the discrete stochastic sequences ξ (as opposed to the Gaussian distribution of increments of Brownian motion), we can admit larger fluctuations in the returns in e.g., Theorem 5.2.2.

The chapter has the following structure; Section 2 gives the mathematical model of the market, and reduces the equation for the returns to the simplified form (5.1.1); Section 3 gives notation and gives statements of and discussion about the main mathematical results of the chapter; Section 4 explores the interpretation of the results to the financial model. The rest of the chapter is devoted to proofs.

5.1.1 The Economic Model

We suppose that there are M_1 *reference level* traders and M_2 *technical* traders. We assume that these traders do not change their investment strategies over time and have infinite lives. We may interpret this latter assumption as allowing for the replacement of a trader with a finite lifetime by another with the same investment strategy. Trading takes place at times $1, 2, 3, \ldots$; for simplicity, we think of these times as representing the start of the first, second, third etc. trading day. The (daily) return over the time interval $[n, n+1)$ is $R(n)$.

Reference traders believe that daily returns should either (a) revert towards a mean level, or (b) will depart from that level. The latter case reflects the idea that if the returns are currently at a high (resp. low) level this is a signal of higher (resp. lower) returns to come and so it is advantageous to buy (resp. sell) in advance of the increase (resp. decrease) in prices. The mean level r_l chosen is idiosyncratic to the l–th trader and the planned excess demand is proportional to the deviation of the return from the reference level. Therefore, there is $a_l \in \mathbb{R}$ such that the planned excess demand of reference trader $l = 1, \ldots, N_1$ just before trading at time $n+1$ is $a_l(R(n)-r_l)$. The planned excess demand of all reference traders before trading at time $n+1$ is therefore

$$\sum_{l=1}^{M_1} a_l(R(n)-r_l).$$

Some reference traders are *contrarians*: such traders buy if the daily return is below the reference level and sell if it is above this level. This type of trader is modelled by setting $a_l < 0$. Other traders are *positive feedback traders*: such traders buy if the daily return is above the reference level, and sell if it is below the reference level. This type of trader is modelled by setting $a_l > 0$.

Technical traders believe that patterns in the returns are significant and should be traced. We suppose that there are M_2 such traders and consider for concreteness the j-th trader. He believes that if the current daily return $R(n)$ is *significantly* (meaning more than a tolerance of τ_j units) below the maximum daily return over the past N days, then this is a signal that the market will advance. Trader j has planned excess demand before trading at time $n+1$ proportional to the strength of the signal

$$\max_{n-N\leq k\leq n} R(k)-R(n)-\tau_j.$$

The tolerance τ_j is idiosyncratic to the trader. However, we assume for mathematical convenience that all traders have the same length of memory, N days. The planned excess demand of trader j before trading at time $n+1$ is therefore

$$b_j\left(\max_{n-N\leq k\leq n} R(k)-R(n)-\tau_j\right)$$

where $b_j > 0$ means that a positive signal leads to buying, while a negative signal leads to selling at time $n+1$.

We note that if $R(n)=\max_{n-N\leq j\leq n} R(j)$, then the signal is negative and trader j sells. Therefore, if the market is experiencing a very strong day relative to the recent past, the trader sells, expecting a reversal of the market in the near future. Therefore the planned excess demand of all technical traders before trading at time $n+1$ is therefore

$$\sum_{j=1}^{M_2} b_j\left(\max_{n-N\leq k\leq n} R(k)-R(n)-\tau_j\right).$$

Speculators react to other random stimuli— "news"— which is independent of past returns. This "news" comes in the form of a signal of strength $\xi'(n+1)$, arriving just before trading at time $n+1$. For speculator $j=1,\ldots,M_1+M_2$ this leads to unplanned excess demand $\varsigma_j\xi(n+1)$. We let $\xi(n+1)=\sum_{j=1}^{M_1+M_2}\varsigma_j\cdot\xi(n+1)$.

We suppose that the daily return will be greater (resp. less) tomorrow than today if there is excess demand (resp. supply), with the rise (resp. fall) being larger the greater the excess demand (resp. supply). Hence, the price adjustment at time $n+1$ for a market with M_1 reference traders and M_2 technical traders is given by

$$R(n+1)-R(n)=\sum_{l=1}^{M_1} a_l(R(n)-r_l)+\sum_{j=1}^{M_2} b_j\left(\max_{n-N\leq k\leq n} R(k)-R(n)-\tau_j\right) + \xi(n+1), \quad n\geq 0. \quad (5.1.2)$$

We now show how to reduce (5.1.2) to the equation (5.1.1). Suppose that

$$\sum_{l=1}^{M_1} a_l \neq 0. \quad (5.1.3)$$

Define

$$\alpha'=\sum_{l=1}^{M_1} a_l, \quad r^*=\frac{1}{\alpha'}\left(\sum_{l=1}^{M_1} a_l r_l+\sum_{j=1}^{M_2} b_j\tau_j\right), \quad (5.1.4)$$

and $X(n) = R(n) - r^*$. Then $R(n) = X(n) + r^*$ and so

$$X(n+1) - X(n) = \sum_{l=1}^{M_1} a_l(X(n) + r^* - r_l) + \sum_{j=1}^{M_2} b_j \left(\max_{n-N\le k\le n} X(k) - X(n) - \tau_j \right) + \xi(n+1), \quad n \ge 0.$$

Therefore, with $\beta := \sum_{j=1}^{M_2} b_j > 0$, we have

$$X(n+1) = X(n) + (\alpha' - \beta)\, X(n) + \beta \max_{n-N\le k\le n} X(k) + \left\{ r^*\alpha' - \sum_{l=1}^{M_1} a_l r_l - \sum_{j=1}^{M_2} b_j \tau_j \right\} + \xi(n+1), \quad n \ge 0.$$

By using the definition $\alpha := \alpha' - \beta$ and (5.1.4) we get

$$X(n+1) = X(n) + \alpha X(n) + \beta \max_{n-N\le k\le n} X(k) + \xi(n+1), \quad n \ge 0. \tag{5.1.5}$$

Note that $\beta > 0$. It is seen that (5.1.5) is nothing other than the equation (5.1.1). We examine the mathematical properties of (5.1.5) in the next sections, returning in Section 4 to interpret these properties in the context of the economic model.

5.2 Statement of the Problem and Discussion of Main Results

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability triple and suppose that $\xi = \{\xi(n) : n \ge 1\}$ is a sequence of random variables such that

$$\xi(n) \text{ is a sequence of i.i.d. random variables with distribution function } F. \tag{5.2.1}$$

Let $N \in \mathbb{N}$, and let the following hypothesis stand throughout the chapter

$$1 + \alpha > 0, \quad \beta > 0. \tag{5.2.2}$$

Let $X = \{X(n) : n \ge -N\}$ be the solution of

$$X(n+1) = X(n) + \alpha X(n) + \beta \max_{n-N\le j\le n} X(j) + \xi(n+1), \quad n \ge 0 \tag{5.2.3a}$$

$$X(n) = \psi(n), \quad n \in \{-N, -N+1, \dots, 0\}. \tag{5.2.3b}$$

If $\mathcal{G}(n) = \sigma\{\xi(j) : 1 \le j \le n\}$ is the natural filtration generated by the process $\xi = \{\xi(n) : n \ge 1\}$, then there is a unique $\mathcal{G}(n)$–adapted solution X of (5.2.3). We employ the conventional Landau big O and little o notation throughout the chapter.

The asymptotic estimates on the solution of (5.2.3) given in the chapter are consequences of the following comparison results. The first deals with the case when the solution grows exponentially; the second when the solution fluctuates unboundedly.

The following proposition is required for the proof of Theroem 5.2.1.

Proposition 5.2.1. *Let ξ be a process obeying* (5.2.1). *Let $\beta > 0$, $1 + \alpha > 0$ and $1 + \alpha + \beta > 1$. Let X be the solution of* (5.2.3). *Let $C := 1 + \max_{j=1,\cdots,N+1} |X(j)|$. Then*

$$X_+(n) = (1+\alpha+\beta)^{n-1} \left(C + \sum_{j=0}^{n-1} (1+\alpha+\beta)^{-j} \left(|\xi(j+1)| + 1 \right) \right), \quad n \ge 1, \tag{5.2.4}$$

and we have

$$X_+(n+1) = X_+(n) + \alpha X_+(n) + \beta \max_{n-N\le j\le n} X_+(j) + |\xi(n+1)| + 1, \quad n \ge N+1. \tag{5.2.5}$$

Moreover $|X(n)| < X_+(n)$ for $n \ge 1$.

Proposition 5.2.2. *Let ξ be a process obeying* (5.2.1)*. Let $\beta > 0$, $1+\alpha > 0$ and $1+\alpha+\beta < 1$. Let X be the solution of* (5.2.3)*. Suppose that γ is a positive, increasing sequence with $\gamma(n) \to \infty$ as $n \to \infty$ such that*

$$\limsup_{n\to\infty} \frac{|\xi(n+1,\omega)|}{\gamma(n)} \le c(\omega). \tag{5.2.6}$$

Then

$$\limsup_{n\to\infty} \frac{|X(n,\omega)|}{\gamma(n)} \le \frac{c(\omega)}{1-(1+\alpha+\beta)}.$$

We are now in a position to state the main results of the chapter.

5.2.1 Exponential growth when $1+\alpha+\beta > 1$

In the case when $1+\alpha+\beta > 1$, the solution of (5.2.3) exhibits exact geometric asymptotic growth to $\pm\infty$.

Theorem 5.2.1. *Let ξ be a process obeying* (5.2.1) *and for which $\int_{-\infty}^{\infty} |x|\,dF(x) < +\infty$, and X be the $\mathcal{G}(n)$–adapted solution of* (5.2.3)*. Suppose α, β obey* (5.2.2) *and $1+\alpha+\beta > 1$. Then there exists a $\mathcal{G}(\infty)$–measurable finite random variable Λ such that*

$$\lim_{n\to\infty} (1+\alpha+\beta)^{-n} X(n) = \Lambda, \quad \textit{a.s.} \tag{5.2.7}$$

where

$$\Lambda = X(0) + \beta \sum_{j=0}^{\infty} (1+\alpha+\beta)^{-(1+j)} \left(\max_{j-N\le l\le j} X(l) - X(j) \right) + \sum_{l=1}^{\infty} (1+\alpha+\beta)^{-l}\, \xi(l). \tag{5.2.8}$$

Remark 5.2.1. For the proof of this Theorem we define the equation of the resolvent $X(n)$ by three terms. We know that the first two terms have limits (possibly infinite). By extending the definition of the last term and by employing the Borel–Cantelli Lemma we also show that this term has a limit. We combine proposition 5.2.1 with the Borel-Cantelli Lemma to prove the three limits are finite.

An explicit formula for Λ in (5.2.7) is not available, although a formula for Λ depends on a functional of X and ξ is established given by (5.2.8). Perusal of the formula for Λ reveals that $\Lambda \ge X(0) + \sum_{l=1}^{\infty}(1+\alpha+\beta)^{-l}\xi(l)$. Therefore $\mathbb{P}[\Lambda > 0] > 0$. If we temporarily emphasise the dependence on ψ, by writing $\Lambda = \Lambda(\psi)$ we see that $\lim_{\psi(0)\to\infty} \mathbb{P}[\Lambda(\psi) > 0] = 1$. Therefore, an increasingly large initial condition increases the probability that $X(n) \to \infty$ as $n \to \infty$ rather than $X(n) \to -\infty$. Moreover, $X(n) \to +\infty$ as $n \to \infty$ is the favoured limit in the case when each ξ has symmetric distribution (with expectation zero): when $\psi(0) = 0$, we have $\Lambda \ge \sum_{l=1}^{\infty}(1+\alpha+\beta)^{-l}\xi(l)$, and so $\mathbb{P}[\Gamma > 0] \ge 1/2$. These comments are of particular interest from the perspective of financial modelling as we will see in the next Section.

The result is also a discrete–time analogue of Theorem 1 in [13] for a related continuous–time equation. We will supply other analogues; so we introduce the continuous time process here. Let $\sigma \ne 0$ be a real number, let $\tau > 0$ and suppose that $\psi \in C([-\tau, 0]; (0,\infty))$ be a deterministic function. Suppose B is a standard one–dimensional Brownian motion with natural filtration $\mathcal{F}^B$. We consider the stochastic functional differential equation of Itô type

$$Y(t) = \psi(0) + \int_0^t \left(aY(s) + b \sup_{s-\tau\le u\le s} Y(u) \right) ds + \int_0^t \sigma\, dB(s), \quad t \ge 0; \tag{5.2.9a}$$

$$Y(t) = \psi(t), \quad t \in [-\tau, 0]. \tag{5.2.9b}$$

Under these conditions, (5.2.9) has a unique global strong solution (cf. e.g. [49]). A usual differential shorthand for (5.2.9) is

$$dY(t) = \left(aY(t) + b \sup_{t-\tau\le s\le t} Y(s) \right) dt + \sigma\, dB(t), \tag{5.2.10}$$

We presume $b > 0$. If also $a+b > 0$, then the unique continuous adapted process Y which satisfies (5.2.9) is such that there exists an almost surely finite $\mathcal{F}^B(\infty)$–measurable random variable Γ such that

$$\lim_{t\to\infty} Y(t) e^{-(a+b)t} = \Gamma, \quad \text{a.s.} \tag{5.2.11}$$

where

$$\Gamma = Y(0) + \int_0^\infty be^{-(a+b)s}\left(\sup_{s-\tau\le u\le s} Y(u) - Y(s)\right) ds + \int_0^\infty e^{-(a+b)s}\sigma\, dB(s).$$

5.2.2 Growth of large fluctuations when $1+\alpha+\beta<1$

We now consider the asymptotic behaviour in the other case when $1+\alpha+\beta \in (0,1)$; we remark that the case $1+\alpha+\beta\le 0$ is ruled out by the hypothesis (5.2.2).

We first consider the case when the tails of the distribution function F are "heavy" in the sense that their tails decay at a polynomial rate as $|x|\to\infty$. One way of characterising this is to assume that there is a number $\mu>0$ such that

$$\int_{-\infty}^\infty |x|^{\mu-\varepsilon}\, dF(x) < +\infty, \quad \int_{-\infty}^\infty |x|^{\mu+\varepsilon}\, dF(x) = +\infty, \quad \text{for all } \varepsilon>0;$$

This property is enjoyed by any distribution function F whose left tail $\Phi_-(x) := F(-x)$ and right tail $\Phi_+(x) = 1 - F(x)$ decay whose density decays according to $f(x) = \mathrm{O}(|x|^{-(\mu+1)})$ as $|x|\to\infty$ but $1/f(x) = \mathrm{o}\left(|x|^{\mu+1}\right)$ as $|x|\to\infty$.

We also need to use the notion of regular variation at infinity for certain sequences. We say that a sequence γ is regularly varying at infinity with index η if

$$\lim_{n\to\infty}\frac{\gamma([\lambda n])}{\gamma(n)} = \lambda^\eta \text{ for all } \lambda>0.$$

Theorem 5.2.2. *Let ξ be a process obeying* (5.2.1)*, and X be the solution of* (5.2.3)*. Let α,β obey* (5.2.2) *and $1+\alpha+\beta<1$. Suppose that γ_+ and γ_- are increasing functions regularly varying with index $\rho>0$ at infinity. Then the following are equivalent:*

(I)

$$\int_{-\infty}^\infty \gamma_+^{-1}(|x|)\, dF(x) < +\infty \quad \int_{-\infty}^\infty \gamma_-^{-1}(|x|)\, dF(x) = +\infty; \tag{5.2.12}$$

(II) The process ξ obeys

$$\limsup_{n\to\infty}\frac{|\xi(n+1)|}{\gamma_+(n)} = 0, \quad \limsup_{n\to\infty}\frac{|\xi(n+1)|}{\gamma_-(n)} = \infty, \quad \textit{a.s.} \tag{5.2.13}$$

(III) The process X obeys

$$\limsup_{n\to\infty}\frac{|X(n)|}{\gamma_+(n)} = 0, \quad \limsup_{n\to\infty}\frac{|X(n)|}{\gamma_-(n)} = \infty, \quad \textit{a.s.} \tag{5.2.14}$$

Remark 5.2.2. To prove (III) implies (II) we use Lemma 5.4.1. To prove (II) implies (I) we combine Borel–Cantelli Lemma with Lemma 1.2.1. To prove (I) implies (III) we combine Lemma 1.2.1 with the Borel–Cantelli Lemma.

We have the following Corollary in the case when $\gamma_\pm(n) = n^{\frac{1}{\mu}\pm\varepsilon}$.

Theorem 5.2.3. *Let ξ be a process obeying* (5.2.1)*, and X be the solution of* (5.2.3)*. Let α,β obey* (5.2.2) *and $1+\alpha+\beta<1$. Then the following are equivalent:*

(I) There exists $\mu>0$ such that

$$\int_{-\infty}^\infty |x|^{\mu-\varepsilon}\, dF(x) < +\infty, \quad \int_{-\infty}^\infty |x|^{\mu+\varepsilon}\, dF(x) = +\infty, \quad \textit{for all } \varepsilon>0; \tag{5.2.15}$$

(II) There exists $\mu>0$ such that

$$\limsup_{n\to\infty}\frac{\log|\xi(n+1)|}{\log n} = \frac{1}{\mu}, \quad \textit{a.s.} \tag{5.2.16}$$

(III) There exists $\mu > 0$ such that

$$\limsup_{n\to\infty} \frac{\log|X(n)|}{\log n} = \frac{1}{\mu}, \quad \textit{a.s.} \tag{5.2.17}$$

Finally, a result is available in the case when the noise is thin tailed in the sense that

$$\limsup_{n\to\infty} \frac{|\xi(n+1)|}{\gamma(n)} = 1, \quad \text{a.s.} \tag{5.2.18}$$

for some $\gamma \in \text{RV}_\infty(0)$ which is increasing and obeys $\gamma(n) \to \infty$ as $n \to \infty$. As we now see, one consequence of (5.2.18) is that F is a distribution for which all moments are finite, viz.,

$$\int_{-\infty}^{\infty} |x|^M \, dF(x) < +\infty, \quad \text{for all } M > 0. \tag{5.2.19}$$

To prove (5.2.19), we note that it will be shown in Theorem 5.2.4 that (5.2.18) and $\gamma \in \text{RV}_\infty(0)$ implies the existence of $K_1, K_2 \in (0, \infty)$ such that

$$\int_{-\infty}^{\infty} \gamma^{-1}(|x|/K_1)\, dF(x) = +\infty, \quad \int_{-\infty}^{\infty} \gamma^{-1}(|x|/K_2)\, dF(x) < +\infty; \tag{5.2.20}$$

Since $\gamma \in \text{RV}_\infty(0)$, we have $\log\gamma(x)/\log x \to 0$ as $x \to \infty$. Since γ is increasing and $\gamma(x) \to \infty$ as $x \to \infty$, we have $\log y/\log\gamma^{-1}(y) \to 0$ as $y \to \infty$, which implies $\log\gamma^{-1}(x)/\log x \to \infty$ as $x \to \infty$, and so for every $M > 0$ there is $x_1(M) > 0$ such that $\gamma^{-1}(x) > x^M$, $x > x_1(M)$. Thus $\gamma^{-1}(x/K_2) > x^M/(K_2)^M$, $x > K_2 x(M) =: x(M)$. Therefore, it follows that

$$\begin{aligned}
&\int_{-\infty}^{\infty} |x|^M \, dF(x) \\
&= \int_{-x(M)}^{x(M)} |x|^M \, dF(x) + K_2^M \int_{-\infty}^{-x(M)} \frac{|x|^M}{K_2^M} \, dF(x) + K_2^M \int_{x(M)}^{\infty} \frac{|x|^M}{K_2^M} \, dF(x) \\
&\le \int_{-x(M)}^{x(M)} |x|^M \, dF(x) + K_2^M \left(\int_{-\infty}^{-x(M)} + \int_{x(M)}^{\infty} \right) \gamma^{-1}(|x|/K_2)\, dF(x) \\
&\le \int_{-x(M)}^{x(M)} |x|^M \, dF(x) + K_2^M \int_{-\infty}^{\infty} \gamma^{-1}(|x|/K_2)\, dF(x),
\end{aligned}$$

which is finite for every $M > 0$ by (5.2.20), proving (5.2.19).

Theorem 5.2.4. *Let ξ be a process obeying* (5.2.1)*, and X be the solution of* (5.2.3)*. Suppose that α, β obey* (5.2.2) *and $1+\alpha+\beta < 1$. Suppose that γ is an increasing function in $RV_\infty(0)$ such that ξ and γ obey* (5.2.18)*. Then*

$$\frac{1}{1+1+\alpha+\beta} \le \limsup_{n\to\infty} \frac{|X(n)|}{\gamma(n)} \le \frac{1}{1-(1+\alpha+\beta)}, \quad \textit{a.s.} \tag{5.2.21}$$

Remark 5.2.3. This Theorem is proved by contradiction.

Example 5.2.1. If $\xi(n)$ is a sequence of independent and identically distributed normal random variables with mean zero and variance σ^2, then

$$\limsup_{n\to\infty} \frac{|\xi(n)|}{\sqrt{2\log n}} = |\sigma| \qquad \text{a.s.}$$

Hence by Theorem 5.2.4, if $1+\alpha > 0$, $\beta > 0$ and $1+\alpha+\beta < 1$, we have

$$\frac{\sigma}{1+1+\alpha+\beta} \le \limsup_{n\to\infty} \frac{|X(n)|}{\sqrt{2\log n}} \le \frac{\sigma}{1-(1+\alpha+\beta)}, \quad \text{a.s.}$$

This example proves a discrete–time analogue of a result for the stochastic functional differential equation (5.2.10) in the case where $a+b < 0$. In Theorem 2 of [13] it is shown that for the process Y obeying (5.2.10) that there exist deterministic $C_1 > 0$, $C_2 > 0$ such that

$$C_1 \le \limsup_{t\to\infty} \frac{|Y(t)|}{\sqrt{2\log t}} \le C_2, \quad \text{a.s.}$$

We can show that not only does the slow growth of the large fluctuations of $|\xi|$ imply the same essential rate of growth of the fluctuations of $|X|$, but that both are also equivalent to the thin tailed condition (5.2.20). The following theorem captures this analogue of Theorem 5.2.2.

Theorem 5.2.5. *Let ξ be a process obeying* (5.2.1)*, and X be the solution of* (5.2.3)*. Suppose that α,β obey* (5.2.2) *and $1+\alpha+\beta<1$. Suppose that γ is an increasing function in $RV_\infty(0)$. Then the following are equivalent:*

(I) There exist $K_1, K_2 \in (0,\infty)$ such that

$$\int_{-\infty}^{\infty} \gamma^{-1}(|x|/K_1)\, dF(x) = +\infty, \quad \int_{-\infty}^{\infty} \gamma^{-1}(|x|/K_2)\, dF(x) < +\infty; \tag{5.2.22}$$

(II) There exists deterministic $0 < c_1 \le c_2 < +\infty$ such that the sequence ξ obeys

$$c_1 \le \limsup_{n\to\infty} \frac{|\xi(n+1)|}{\gamma(n)} \le c_2, \quad \textit{a.s.}; \tag{5.2.23}$$

(III) There exist deterministic $0 < C_1 \le C_2 < +\infty$ such the process X obeys

$$C_1 \le \limsup_{n\to\infty} \frac{|X(n)|}{\gamma(n)} \le C_2, \quad \textit{a.s.} \tag{5.2.24}$$

Remark 5.2.4. Employing the method of the proof of Theorem 5.2.4 we show that (II) implies (III). Employing proposition 5.2.1 we show (III) implies (II). Combining Borel–Cantelli Lemma with Lemma 1.2.1 we prove (II) implies (I). Combining Lemma 1.2.1 and (5.2.22) with the Borel–Cantelli Lemma we show (I) implies (III).

Remark 5.2.5. If we use the notation (I)(K_1, K_2) to be equivalent to (5.2.22), (II)(c_1, c_2) to be equivalent to the statement (5.2.23) and the notation (III)(C_1, C_2) to be equivalent to (5.2.24), then (II)(c_1, c_2) implies (III)$(c_1/(1+1+\alpha+\beta), c_2/(1-(1+\alpha+\beta)))$, and (III)$(C_1, C_2)$ implies (II)$(C_1(1-(1+\alpha+\beta)), C_2(1+(1+\alpha+\beta)))$. Moreover, (I)$(K_1, K_2)$ implies (II)(K_1, K_2) and (II)(c_1, c_2) implies (I)$(c_1(1-\varepsilon), c_2(1+\varepsilon))$ for every $\varepsilon \in (0,1)$.

Example 5.2.2. If there exist $0 < C_1 < C_2 < \infty$ such that

$$C_1 \le \limsup_{n\to\infty} \frac{|X(n)|}{\sqrt{2\log n}} \le C_2, \quad \text{a.s.}$$

by Theorem 5.2.5 we then have

$$C_1(1+(1+\alpha+\beta)) \le \limsup_{n\to\infty} \frac{|\xi(n)|}{\sqrt{2\log n}} \le C_2(1-(1+\alpha+\beta)), \quad \text{a.s.}$$

Now, as we identify $\gamma(x) = \sqrt{2\log x}$, we have $\gamma^{-1}(x) = \exp(x^2/2)$, by the above comment for each $\varepsilon \in (0,1)$ we have

$$\int_{-\infty}^{\infty} \exp\left(\frac{x^2}{2C_1^2(1+(1+\alpha+\beta))^2(1-\varepsilon)^2}\right) dF(x) = +\infty,$$
$$\int_{-\infty}^{\infty} \exp\left(\frac{x^2}{2C_2^2(1-(1+\alpha+\beta))^2(1+\varepsilon)^2}\right) dF(x) < +\infty.$$

5.3 Interpretation of Main Results to the Economic Model

Since $b_j > 0$ for each $j = 1, \dots, M_2$ we have $\beta > 0$ as required by (5.2.2). The requirement that $1+\alpha = 1+\alpha' - \beta > 0$ is equivalent to

$$1+\alpha > 0 \quad \text{if and only if} \quad 1 + \sum_{l=1}^{M_1} a_l - \sum_{j=1}^{M_2} b_j > 0. \tag{5.3.1}$$

We note that the key parameter $1+\alpha+\beta$ is given by $1+\alpha+\beta = 1+\alpha' = 1+\sum_{l=1}^{M_1} a_l$. Therefore we have

$$1+\alpha+\beta > 1 \quad \text{if and only if} \quad \sum_{l=1}^{M_1} a_l > 0. \tag{5.3.2}$$

The condition, $\sum_{j=1}^{M_2} b_j - \sum_{l=1}^{M_1} a_l < 1$, is required as a standing assumption to ensure that the results in Section 3 hold. The condition holds if the individual agents do not act very strongly at any time to the signals provided by their trading rules which are based on discrepancies between current returns and indicators of returns (reference levels in the case of the reference traders and the maximum of the last N returns in the case of the technical traders). At any time step, the agents will react sufficiently moderately to ensure that $\sum_{j=1}^{M_2} b_j - \sum_{l=1}^{M_1} a_l < 1$, provided the time gap between trades is sufficiently small.

This can be seen explicitly by considering the continuous–time stochastic functional differential equation

$$dr(t) = \left\{ \sum_{l=1}^{M_1} \alpha_l (r(t) - r_l) + \sum_{j=1}^{M_2} \beta_j \left(\max_{t-\tau \le s \le t} r(s) - r(t) - \tau_j \right) \right\} dt + \sigma\, dB(t), \quad t \ge 0,$$

where B is a standard one–dimensional Brownian motion. This equation is guaranteed to possess a strong unique adapted continuous solution. If we take a uniform Euler–Maruyama discretisation with time step $h > 0$, and letting $R(n)$ be the approximation to $r(nh)$, the evolution can be approximated by

$$R(n+1) - R(n) = \sum_{l=1}^{M_1} h\alpha_l (R(n) - r_l) + \sum_{j=1}^{M_2} h\beta_j \left(\max_{t-N \le s \le t} r(s) - r(t) - \tau_j \right) + \xi(n+1)$$

where N is the nearest integer to τ/h and ξ is a sequence of zero mean independent normal random variables with variance $\sigma^2 h$. This equation is in the form (5.1.2). In this case, if $h > 0$ is sufficiently small, we have

$$h \left(\sum_{j=1}^{M_2} \beta_j - \sum_{l=1}^{M_1} \alpha_l \right) < 1.$$

Together with the condition $\sum_{j=1}^{M_2} \beta_j > 0$, this ensures that the standing hypothesis (5.2.2) holds.

If the feedback traders are, on the whole, of positive feedback type (in which case $\sum_{l=1}^{M_1} a_l > 0$), by Theorem 5.2.1 the market experiences a bubble or crash according to

$$\lim_{t\to\infty} (1 + \sum_{l=1}^{M_1} a_l)^{-n} R(n) = R(0) - \frac{1}{\sum_{l=1}^{M_1} a_l} \left(\sum_{l=1}^{M_1} a_l r_l + \sum_{j=1}^{M_2} b_j \tau_j \right) + \sum_{j=1}^{M_2} b_j \cdot \sum_{j=0}^{\infty} \left(1 + \sum_{l=1}^{M_1} a_l \right)^{-1-j} \left(\max_{j-N \le l \le j} R(l) - R(j) \right) + \sum_{l=1}^{\infty} \left(1 + \sum_{j=1}^{M_1} a_j \right)^{-l} \xi(l), \tag{5.3.3}$$

almost surely. It should be noted that the presence of the technical traders neither prevents nor promotes the creation of this runaway event.

Examining the limit on the righthand side of (5.3.3), we see that it tends to infinity a.s. as $R(0) \to \infty$. Therefore, the larger the initial daily return, the greater the probability that $R(n) \to \infty$ as $n \to \infty$. This explains at least in part the manner in which this bubble forms; if initially the stock performs well, this encourages positive feedback traders to take this good performance as a signal that informed investors believe the stock will do well in future, so they buy the stock. This then forces prices up further, encouraging further buying. This upward spiral continues and a bubble ensues.

Conversely, if the initial value of $R(0)$ is negative, but $|R(0)|$ is large, this tends to make the limit on the righthand side of (5.3.3) negative and makes the event $\{R(n) \to -\infty \text{ as } n \to \infty\}$ more probable. In this situation, this helps to explains the crash dynamics: an initially poor performance by the stock convinces positive feedback traders that informed traders believe the stock will perform poorly in future, so they sell (or short sell) the stock. This then forces prices lower, encouraging further selling, and the result of this downward spiral is a crash. In both cases, and when $a_l > 0$ for each l, we see that it is the level of the market $R(0)$ relative to the "weighted consensus" return $r_c := \sum_{l=1}^{M_1} a_l r_l / \sum_{l=1}^{M_1} a_l$ of the reference traders that is particularly important; the greater the difference $R(0) - r_c$, the more probable a bubble.

We next ask what is the impact of a sequence of "good news stories" about the stock at the time shortly after trading begins. We can interpret this as a majority of noise terms $\xi(n)$ being positive for small n. Since the non-random part of the summand in the martingale term on the righthand side of (5.3.3) diminishes rapidly as time increases, it is the sign of these "initial" values of ξ that largely determines whether the sum assumes a positive or negative value. Therefore, initial good news about the stock tends to result in a positive value of this sum, while initial bad news about the stock tends to lead to a negative value of the sum. Therefore, if there is good initial news about the stock, the price of the stock tends to increase and the positive feedback traders force the price higher by misperceiving this increase as arising from demand from informed speculators. As before, this induces further buying and the stock price undergoes a bubble. Similarly, initial bad news tends to precipitate a crash.

Finally, consider the penultimate term on the righthand side of (5.3.3). Firstly, we note that it is always positive and that the main contribution to the overall value is from the time shortly after trading begins. This contribution is relatively large if the returns are relatively low, because in this case $\max_{j-N\leq l\leq j} R(l)$ will tend to strictly exceed $R(j)$. If the returns are running below their maximum during the initial period of trading, the technical traders will tend to force the returns upwards; this trend will then be extrapolated by the positive feedback traders, increasing the probability of a bubble. On the other hand, the contribution of the penultimate term in (5.3.3) is smaller if the returns are generally increasing and therefore at or close to their N–day running maximum. However, in this case, the contributions of the first two terms on the righthand side are quite likely to be positive; so the additional bubble–promoting impact of the penultimate term, although modest, is likely to be unimportant. Hence, the penultimate term tends to have its greatest bubble–promoting impact when other bubble–promoting factors (such as strong initial returns relative to the reference levels of the feedback traders, or a sequence of good news stories about the stock) are not so strong. Therefore, it seems that the technical traders can also "seed" a bubble in a market which is naturally prone to generate a bubble. Hence, the interaction of such traders with the positive feedback traders can make bubbles more likely.

These remarks suggest that the mechanisms by which bubbles form in this model are consistent with the notion of mimetic contagion introduced by Orléan (cf. e.g., [65]). In mimetic contagion we may think of the market as comprising of two forms of traders, with new entrants choosing the trading strategy which tends to dominate at a given time. In the long–run, the proportion of traders in each category settles down to a value which is random but which depends quite strongly on what happens in the first trading periods. The similarities with mimetic contagion are as follows: in (5.3.3), the righthand side depends crucially on the market behaviour in the first few time periods; once a dominant trend becomes apparent, the positive feedback traders will tend to extrapolate that trend; and the long–run behaviour (either a bubble or crash) is not known in advance.

If the feedback traders are, on the whole, of negative feedback type (in which case $\sum_{l=1}^{M_1} a_l < 0$), the market experiences large fluctuations whose size is intimately connected with the distribution of independently and identically distributed news variates. For example, by Theorem 5.2.3 the tails of the distribution function F of the "news" variates decay polynomially and satisfy

$$\int_{-\infty}^{\infty} |x|^{\mu-\varepsilon}\, dF(x) < +\infty, \quad \int_{-\infty}^{\infty} |x|^{\mu+\varepsilon}\, dF(x) = +\infty, \quad \text{for all } \varepsilon > 0,$$

for some $\mu > 0$ if and only if

$$\limsup_{n\to\infty} \frac{\log |R(n)|}{\log n} = \frac{1}{\mu}, \quad \text{a.s.}$$

5.4 Proof of Theorems

5.4.1 Proof of Proposition 5.2.1

For $n \geq N+1$, X_+ defined by 5.2.4 is increasing and

$$\max_{n-N\leq j\leq n} X_+(j) = X_+(n).$$

Then for $n \geq N+1$

$$\begin{aligned}
&X_+(n+1) - (1+\alpha+\beta)X_+(n) - |\xi(n+1)| - 1 \\
&\quad = (1+\alpha+\beta)^n \left(C + \sum_{j=0}^{n} (1+\alpha+\beta)^{-j} \left(|\xi(j+1)|+1\right) \right) - 1 - |\xi(n+1)| \\
&\qquad - (1+\alpha+\beta)\,(1+\alpha+\beta)^{n-1} \left(C + \sum_{j=0}^{n-1} (1+\alpha+\beta)^{-j} \left(|\xi(j+1)|+1\right) \right) \\
&\qquad = 1 - 1 + |\xi(n+1)| - |\xi(n+1)| \\
&\qquad = 0.
\end{aligned}$$

We next prove that $X_+(n) > |X(n)|$ for $n = 1, \dots, N+1$. This holds for instance if

$$\min_{j=1,\cdots N+1} X_+(j) > \max_{j=1,\cdots N+1} |X(j)|,$$

as

$$\begin{aligned}
\min_{j=1,\cdots,N+1} X_+(j) &= X_+(1) \\
&= C + (1+\alpha+\beta)^0 \left(|\xi(1)|+1\right) \\
&= C + |\xi(1)| + 1 \\
&\geq 1 + \max_{j=1,\cdots,N+1} |X(j)| + 1 > \max_{j=1,\cdots,N+1} |X(j)|.
\end{aligned}$$

Finally, we must show that $X_+(n) > |X(n)|$ for all $n > N+1$. Suppose to the contrary that there is a minimal $N_1 > N+1$ such that $X_+(N_1+1) \leq |X(N_1+1)|$, so that $X_+(n) > |X(n)|$ for all $n \leq N_1$. Then, as $1+\alpha > 0$, and $\beta > 0$, for $n \geq 0$, we have

$$\begin{aligned}
|X(n+1)| &= \left| (1+\alpha)X(n) + \beta \max_{n-N \leq j \leq n} X(j) + \xi(n+1) \right| \\
&\leq (1+\alpha)|X(n)| + \beta \max_{n-N \leq j \leq n} |X(j)| + |\xi(n+1)| \\
&< |X(n)| + \alpha|X(n)| + \beta \max_{n-N \leq j \leq n} |X(j)| + 1 + |\xi(n+1)|.
\end{aligned}$$

Hence

$$\begin{aligned}
|X(N_1+1)| &< |X(N_1)| + \alpha|X(N_1)| + \beta \max_{N_1-N \leq j \leq N_1} |X(j)| + 1 + |\xi(N_1+1)| \\
&< X_+(N_1)(1+\alpha) + \beta \max_{N_1-N \leq j \leq N_1} X_+(j) + 1 + |\xi(N_1+1)| \\
&= X_+(N_1+1) \leq |X(N_1+1)|,
\end{aligned}$$

a contradiction. Therefore we must have that $X_+(n) > |X(n)|$ for all $n > N+1$, and thus that $X_+(n) > |X(n)|$ for all $n \geq 1$, as claimed.

5.4.2 Proof of Proposition 5.2.2

Since $1+\alpha > 0$ and $\beta > 0$ we have

$$|X(n+1)| \leq (1+\alpha)|X(n)| + \beta \max_{n-N \leq j \leq n} |X(j)| + |\xi(n+1)|, \quad n \geq 0.$$

By (5.2.6), it follows that for every $\varepsilon > 0$ there is an $N_1(\varepsilon) \in \mathbb{N}$ such that $|\xi(n+1,\omega)| \leq c(\omega)(1+\varepsilon)\gamma(n)$ for $n \geq N_1(\varepsilon)$. Define

$$c_1(\omega,\varepsilon) = \frac{c(\omega)(1+\varepsilon)}{1-(1+\alpha+\beta)}, \quad c_2(\omega,\varepsilon) = 1 + \max_{N_1(\varepsilon)-N \leq j \leq N_1(\varepsilon)} |X(j,\omega)| > 0.$$

Let $X_+(n,\omega) = c_2(\omega,\varepsilon) + c_1(\omega,\varepsilon)\gamma(n)$ for $n \geq N_1(\varepsilon) - N$. Then for $n = N_1(\varepsilon) - N, \dots, N_1(\varepsilon)$ we have

$$X_+(n,\omega) \geq c_2(\omega) = 1 + \max_{N_1(\varepsilon)-N \leq j \leq N_1(\varepsilon)} |X(j,\omega)| > |X(n,\omega)|.$$

For $n \geq N_1(\varepsilon)$, since $1+\alpha+\beta < 1$, $c_2 > 0$ and γ is increasing we have

$$\begin{aligned}
&X_+(n+1) - (1+\alpha)X_+(n) - \beta \max_{n-N\leq j\leq n} X_+(j) - |\xi(n+1)| \\
&= c_2 + c_1\gamma(n+1) - (1+\alpha)(c_2 + c_1\gamma(n)) - \beta(c_2 + c_1\gamma(n)) - |\xi(n+1)| \\
&= c_2(1 - (1+\alpha+\beta)) + c_1\gamma(n+1) - (1+\alpha+\beta)c_1\gamma(n) - |\xi(n+1)| \\
&> c_1\gamma(n) - (1+\alpha+\beta)c_1\gamma(n) - c(1+\varepsilon)\gamma(n) \\
&= \gamma(n)\left(c_1(1-(1+\alpha+\beta) - c(1+\varepsilon)\right) = 0.
\end{aligned}$$

Thus

$$X_+(n+1) > (1+\alpha)X_+(n) + \beta \max_{n-N\leq j\leq n} X_+(j) + |\xi(n+1)|, \quad n \geq N_1(\varepsilon).$$

By the argument of Proposition 5.2.1, it follows that $|X(n,\omega)| \leq X_+(n,\omega)$ for $n \geq N_1(\varepsilon,\omega)$. Thus

$$\limsup_{n\to\infty} \frac{|X(n,\omega)|}{\gamma(n)} \leq \limsup_{n\to\infty} \frac{c_2(\omega,\varepsilon) + c_1(\omega,\varepsilon)\gamma(n)}{\gamma(n)} = c_1(\varepsilon,\omega) = \frac{c(\omega)(1+\varepsilon)}{1-(1+\alpha+\beta)}.$$

Letting $\varepsilon \to 0$ gives the result.

5.4.3 Proof of Theorem 5.2.1

We first rewrite (5.2.3) according to

$$X(n+1) = (1+\alpha+\beta)X(n) + \beta\left(\max_{n-N\leq j\leq n} X(j) - X(n)\right) + \xi(n+1), \quad n \geq 0.$$

Define $H(n) = \beta\left(\max_{n-N\leq j\leq n} X(j) - X(n)\right)$ for $n \geq 1$. Note that $H(n) \geq 0$ for all $n \geq 1$. Then

$$X(n+1) = (1+\alpha+\beta)X(n) + H(n) + \xi(n+1), \quad n \geq 0.$$

Multiplying both sides by $\sum_{j=0}^{n-1}(1+\alpha+\beta)^{n-1-j}$ yields

$$\begin{aligned}
X(n) = (1+\alpha+\beta)^n X(0) + \sum_{j=0}^{n-1}(1+\alpha+\beta)^{n-1-j}H(j) \\
+ \sum_{j=0}^{n-1}(1+\alpha+\beta)^{n-1-j}\xi(j+1), \quad n \geq 1.
\end{aligned}$$

Therefore, for $n \geq 1$ we have

$$(1+\alpha+\beta)^{-n}X(n) = X(0) + \sum_{j=0}^{n-1}(1+\alpha+\beta)^{-1-j}H(j) + \sum_{l=1}^{n}(1+\alpha+\beta)^{-l}\xi(l).$$

Since $H(n) \geq 0$ and $1+\alpha+\beta > 1$, the first two terms on the righthand side have a limit (possibly infinite) as $n \to \infty$. As to the last term, consider the (possibly infinite) extended random variable

$$S_\infty = \sum_{l=1}^{\infty}(1+\alpha+\beta)^{-l}\xi(l).$$

Since each ξ has the same distribution function F, with $\mu_1 := \int_{-\infty}^{\infty}|x|\,dF(x) < +\infty$, we have

$$\begin{aligned}
\mathbb{E}[|S_\infty|] &\leq \sum_{l=1}^{\infty}(1+\alpha+\beta)^{-l}\mathbb{E}|\xi(l)| \\
&\leq \mu_1 \frac{1}{1+\alpha+\beta}\sum_{j=0}^{\infty}(1+\alpha+\beta)^{-j} = \mu_1\frac{1}{1+\alpha+\beta}\frac{1}{1-(1+\alpha+\beta)^{-1}}.
\end{aligned}$$

Therefore S_∞ is an almost surely finite $\mathcal{G}_\infty$–measurable random variable. Hence

$$\begin{aligned}\mathbb{E}\left|S_\infty - \sum_{l=1}^{n}(1+\alpha+\beta)^{-l}\xi(l)\right| &\leq \sum_{l=n+1}^{\infty}(1+\alpha+\beta)^{-l}\mathbb{E}|\xi(l)| \\ &= \mu_1(1+\alpha+\beta)^{-(n+1)}\sum_{l=n+1}^{\infty}(1+\alpha+\beta)^{n+1-l} \\ &= \frac{\mu_1(1+\alpha+\beta)^{-n}}{(1+\alpha+\beta)-1}.\end{aligned}$$

By Chebyshev's inequality and the summability of the righthand side, the Borel–Cantelli Lemma implies that

$$\lim_{n\to\infty}\sum_{l=1}^{n}(1+\alpha+\beta)^{-l}\xi(l) = S_\infty, \quad \text{a.s.}$$

Therefore we have that (5.2.7) holds, with

$$\Lambda := X(0) + \beta\sum_{j=0}^{\infty}(1+\alpha+\beta)^{-1-j}\left(\max_{j-N\leq l\leq j}X(l) - X(j)\right) + \sum_{l=1}^{\infty}(1+\alpha+\beta)^{-l}\xi(l)$$

possibly infinite. We note that Λ is $\mathcal{G}_\infty$ measurable. If we can show that

$$\limsup_{n\to\infty}|X(n)|(1+\alpha+\beta)^{-n} < +\infty, \quad \text{a.s.}, \tag{5.4.1}$$

then it is guaranteed that Λ is finite a.s., and the result is proven. To establish (5.4.1), we note by Proposition 5.2.1 that

$$(1+\alpha+\beta)^{-(n-1)}|X(n)| \leq C + \sum_{j=0}^{n-1}(1+\alpha+\beta)^{-j}\left(|\xi(j+1)|+1\right), \quad n\geq 1,$$

where $C := 1 + \max_{j=1,\cdots,N+1}|X(j)|$. Now, with

$$\tilde{S}_\infty = \sum_{j=0}^{\infty}(1+\alpha+\beta)^{-j}\left(|\xi(j+1)|+1\right),$$

we have that $\tilde{S}_\infty > 0$ and $\mathbb{E}[\tilde{S}_\infty] < +\infty$. Therefore

$$\mathbb{E}\left|\tilde{S}_\infty - \sum_{j=0}^{n-1}(1+\alpha+\beta)^{-j}\left(|\xi(j+1)|+1\right)\right| = (\mu_1+1)\sum_{j=n}^{\infty}(1+\alpha+\beta)^{-j},$$

by Chebyshev's inequality and the summability of the righthand side the Borel-Cantelli Lemma implies that $\lim_{n\to\infty}\sum_{j=0}^{n-1}(1+\alpha+\beta)^{-j}\left(|\xi(j+1)|+1\right) = \tilde{S}_\infty$, a.s. Therefore

$$\limsup_{n\to\infty}(1+\alpha+\beta)^{-(n-1)}|X(n)| \leq C + \tilde{S}_\infty < +\infty, \quad \text{a.s.},$$

proving (5.4.1).

5.4.4 Proof of Theorem 5.2.2

We start by showing that under the hypotheses of Theorem 5.2.2 that (5.2.13) and (5.2.14) are equivalent. In the proofs in the rest of the paper, we sometimes use the fact that a regularly varying sequence or function γ has the property that

$$\lim_{n\to\infty}\frac{\gamma(n-M)}{\gamma(n)} = 1, \quad \text{for each } M\in\mathbb{N}. \tag{5.4.2}$$

Lemma 5.4.1. *Let ξ be a process obeying* (5.2.1)*, and X be the solution of* (5.2.3)*. Let α, β obey* (5.2.2) *and $1+\alpha+\beta < 1$. Suppose that γ_+ and γ_- are increasing functions in $RV_\infty(\mu)$ for $\mu > 0$. Then the following are equivalent:*

(I) The process ξ obeys (5.2.13)*;*

(II) The process X obeys (5.2.14).

Remark 5.4.1. This lemma is proved by contradiction.

Proof. We first prove that the second statement of (5.2.13) implies the second in (5.2.14). Suppose that

$$\limsup_{n\to\infty} \frac{|\xi(n+1)|}{\gamma_-(n)} = \infty, \quad \text{a.s.}$$

Let the event on which this is true be Ω^*. We want to prove that

$$\limsup_{n\to\infty} \frac{|X(n)|}{\gamma_-(n)} = \infty, \quad \text{a.s.}$$

Suppose to the contrary that there exists an event A with $\mathbb{P}[A] > 0$ such that

$$A = \left\{\omega : \limsup_{n\to\infty} \frac{|X(n,\omega)|}{\gamma_-(n)} < \infty\right\}.$$

Now define $A^* = A \cap \Omega^*$, so that $\mathbb{P}[A^*] = \mathbb{P}[A] > 0$. Thus for $\omega \in A^*$, we have

$$\limsup_{n\to\infty} \frac{|\xi(n+1,\omega)|}{\gamma_-(n)} \leq \limsup_{n\to\infty} \frac{|X(n,\omega)|}{\gamma_-(n)} + |1+\alpha| \limsup_{n\to\infty} \frac{|X(n,\omega)|}{\gamma_-(n)} + \beta \max_{n-N\leq j\leq n} \frac{|X(j,\omega)|}{\gamma_-(n)} < \infty,$$

which forces a contradiction. Hence $\mathbb{P}[A] = 0$, or

$$\limsup_{n\to\infty} \frac{|X(n)|}{\gamma_-(n)} = \infty, \quad \text{a.s.}$$

as required.

We now prove that the first statement in (5.2.14) implies the first statement in (5.2.13). We have by hypothesis that

$$\limsup_{n\to\infty} \frac{|X(n,\omega)|}{\gamma_+(n)} = 0.$$

Let the event on which this holds be called A such that

$$A = \left\{\omega : \limsup_{n\to\infty} \frac{|\xi(n+1,\omega)|}{\gamma_+(n)} > 0\right\}$$

Consider $A^* = \Omega^* \cap A$. Then $\mathbb{P}[A^*] > 0$ and we have that

$$\limsup_{n\to\infty} \frac{|\xi(n+1,\omega)|}{\gamma_+(n)} \leq \limsup_{n\to\infty} \frac{|X(n,\omega)|}{\gamma_+(n)} + |1+\alpha| \limsup_{n\to\infty} \frac{|X(n,\omega)|}{\gamma_+(n)} + \beta \max_{n-N\leq j\leq n} \frac{|X(j,\omega)|}{\gamma_+(n)} = 0,$$

which forces a contradiction. Hence $\mathbb{P}[A] = 0$ or

$$\limsup_{n\to\infty} \frac{|\xi(n+1,\omega)|}{\gamma_+(n)} = 0, \quad \text{a.s.}$$

To prove that the first statement in (5.2.13) implies the first statement in (5.2.14), we start be remarking that there exists an a.s. event Ω^* such that for all $\omega \in \Omega^*$, we have

$$\limsup_{n\to\infty} \frac{|\xi(n+1,\omega)|}{\gamma_+(n)} = 0.$$

Hence for every $\varepsilon > 0$ we have

$$\limsup_{n\to\infty} \frac{|\xi(n+1,\omega)|}{\gamma_+(n)} \leq \varepsilon.$$

Setting $c(\omega) = \varepsilon$ in (5.2.6) we obtain by Proposition 5.2.2 that for all $\omega \in \Omega^*$ we have

$$\limsup_{n\to\infty} \frac{|X(n,\omega)|}{\gamma_+(n)} \leq \frac{\varepsilon}{1-(1+\alpha+\beta)}.$$

Letting $\varepsilon \to 0$ yields

$$\limsup_{n\to\infty} \frac{|X(n,\omega)|}{\gamma_+(n)} = 0, \quad \text{for all } \omega \in \Omega^*,$$

and so

$$\limsup_{n\to\infty} \frac{X(n)}{\gamma_+(n)} = 0, \quad \text{a.s.},$$

as required.

We now prove that the second statement in (5.2.14) implies the second statement of (5.2.13). We have by hypothesis that

$$\limsup_{n\to\infty} \frac{|X(n)|}{\gamma_-(n)} = \infty, \quad \text{a.s.}$$

Let the event on which this holds be called Ω^*. Suppose that there exists an event A with $\mathbb{P}[A] > 0$ such that

$$A = \left\{\omega : \limsup_{n\to\infty} \frac{|\xi(n+1,\omega)|}{\gamma_-(n)} < +\infty\right\}.$$

Consider $A^* = A \cap \Omega^*$. Then $\mathbb{P}[A^*] > 0$ and for $\omega \in A^*$ we have

$$\limsup_{n\to\infty} \frac{|\xi(n+1,\omega)|}{\gamma_-(n)} =: c(\omega) < +\infty.$$

By Proposition 5.2.2, we have that

$$\limsup_{n\to\infty} \frac{|X(n,\omega)|}{\gamma_-(n)} \leq \frac{c(\omega)}{1-(1+\alpha+\beta)} < +\infty \quad \text{for } \omega \in A^*.$$

This gives a contradiction. Hence $\mathbb{P}[A] = 0$, and so

$$\limsup_{n\to\infty} \frac{|X(n)|}{\gamma_-(n)} = \infty, \quad \text{a.s.}$$

as claimed. □

5.4.5 Proof of Theorem 5.2.2

We show first that (III) implies (II) and that (II) implies (I). Lemma 5.4.1 implies

$$\limsup_{n\to\infty} \frac{|\xi(n+1)|}{\gamma_-(n)} = \infty, \quad \text{a.s.} \tag{5.4.3}$$

and

$$\limsup_{n\to\infty} \frac{|\xi(n+1)|}{\gamma_+(n)} = 0, \quad \text{a.s.} \tag{5.4.4}$$

Together (5.4.3) and (5.4.4) imply (II).

If (II) (i.e., (5.2.13)) holds, then (5.4.3) and (5.4.4) hold. Since $(\xi(n))_{n\geq 0}$ are independent random variables, (5.4.3) and the Borel–Cantelli lemma implies that

$$\sum_{n=1}^{\infty} \mathbb{P}[|\xi(n+1)| \geq M_1\gamma_-(n)] = +\infty, \quad \text{for every } M_1 > 0,$$

while the independence and (5.4.4) implies

$$\sum_{n=1}^{\infty} \mathbb{P}[|\xi(n+1)| > M_2\gamma_+(n)] < +\infty \quad \text{for every } M_2 > 0.$$

Let ζ be a random variable with the same distribution as $\xi(n)$ for all $n \geq 1$. Since ξ obeys (5.2.1), we have

$$\sum_{n=1}^{\infty} \mathbb{P}[|\zeta| \geq M_1\gamma_-(n)] = +\infty, \quad \sum_{n=1}^{\infty} \mathbb{P}[|\zeta| > M_2\gamma_+(n)] < +\infty.$$

Since γ_- is an increasing function, the function $\Gamma_{M,-}(x) = M\gamma_-(x)$ has inverse given by $\Gamma_{M,-}^{-1}(x) = \gamma_-^{-1}(x/M)$. By lemma 1.2.1

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbb{P}[|\zeta| \geq M_1\gamma_-(n)] &\leq \mathbb{E}[\Gamma_{M_1,-}^{-1}(|\zeta|)] \\ &= \mathbb{E}[\gamma_-^{-1}(|\zeta|/M_1)] = \int_{-\infty}^{\infty} \gamma_-^{-1}(|x|/M_1)\, dF(x).\end{aligned}$$

Since γ_- is in $\text{RV}_\infty(\mu)$, it follows that γ_-^{-1} is in $\text{RV}_\infty(1/\mu)$, and so it follows that

$$\int_{-\infty}^{\infty} \gamma_-^{-1}(|x|)\, dF(x) = +\infty.$$

Define $\Gamma_{M,+}(x) = M\gamma_+(x)$, which has inverse given by $\Gamma_{M,+}^{-1}(x) = \gamma_+^{-1}(x/M)$. On the other hand, Lemma 1.2.1 also implies for all $M_2 > 0$ that

$$\int_{-\infty}^{\infty} \gamma_+^{-1}(|x|/M_2)\, dF(x) = \mathbb{E}[\Gamma_{M_2,+}^{-1}|\zeta|] \leq \sum_{n=0}^{\infty} \mathbb{P}[|\zeta| > M_2\gamma_+(n)] < +\infty,$$

so $\int_{-\infty}^{\infty} \gamma_+^{-1}(|x|/M_2)\, dF(x) < +\infty$. Therefore we have (5.2.12). Hence (II) implies (I).

We now show that (I) implies (III). Let $M_1 > 0$. Then by Lemma 1.2.1 and (5.2.12), we have

$$\sum_{n=0}^{\infty} \mathbb{P}[|\zeta| > M_1\gamma_-(n)] \geq \mathbb{E}[\gamma_-^{-1}(|\zeta|/M_1)] = \int_{-\infty}^{\infty} \gamma_-^{-1}(|x|/M_1)\, dF(x) = +\infty,$$

using the fact that $\gamma_-^{-1}(x/M_1)/\gamma_-^{-1}(x) \to (1/M_1)^{1/\mu}$ as $x \to \infty$. Since $\xi(n)$ has the same distribution as ζ, we have

$$\sum_{n=0}^{\infty} \mathbb{P}[|\xi(n+1)| > M_1\gamma_-(n)] = +\infty. \tag{5.4.5}$$

Therefore by the Borel–Cantelli lemma, we have

$$\limsup_{n\to\infty} \frac{|\xi(n+1)|}{\gamma_-(n)} \geq M_1, \quad \text{a.s. on } \Omega_{M_1,+},$$

where $\Omega_{M_1,+}$ is an almost sure event. Let $\Omega_+ = \cap_{M_1\in\mathbb{N}}\Omega_{M_1,+}$. Then Ω_+ is an almost sure event, and we have

$$\limsup_{n\to\infty} \frac{|\xi(n+1)|}{\gamma_-(n)} = \infty, \quad \text{a.s. on } \Omega_+.$$

Lemma 5.4.1 now implies

$$\limsup_{n\to\infty} \frac{|X(n)|}{\gamma_-(n)} = \infty, \quad \text{a.s. on } \Omega_+. \tag{5.4.6}$$

Let $M_2 > 0$. Similarly, by Lemma 1.2.1 and (5.2.12), we have

$$\sum_{n=1}^{\infty} \mathbb{P}[|\zeta| \geq M_2\gamma_+(n)] \leq \mathbb{E}[\gamma_+^{-1}(|\zeta|/M_2)] < +\infty.$$

Since $\xi(n)$ has the same distribution as ζ, we have

$$\sum_{n=1}^{\infty} \mathbb{P}[|\xi(n+1)| \geq M_2\gamma_+(n)] < +\infty. \tag{5.4.7}$$

Therefore by the Borel–Cantelli lemma, we have

$$\limsup_{n\to\infty} \frac{|\xi(n+1)|}{\gamma_+(n)} \leq M_2, \quad \text{a.s. on } \Omega^*_{M_2,-},$$

where $\Omega^*_{M_2,-}$ is an almost sure event. Let $\Omega_- = \cap_{M_2\in(0,1)\cap\mathbb{Q}}\Omega^*_{M_2,-}$. Then Ω_- is an almost sure event, and we have

$$\limsup_{n\to\infty} \frac{|\xi(n+1)|}{\gamma_+(n)} = 0, \quad \text{a.s. on } \Omega_-.$$

Theorem 5.4.1 then implies

$$\limsup_{n\to\infty} \frac{|X(n)|}{\gamma_+(n)} = 0, \quad \text{a.s. on } \Omega_-. \tag{5.4.8}$$

Together (5.4.6) and (5.4.8) imply (III).

5.4.6 Proof of Theorem 5.2.4

To establish the lower bound in (5.2.21), suppose that there is an event A such that

$$A = \left\{\omega : \limsup_{n\to\infty} \frac{|X(n,\omega)|}{\gamma(n)} = C_2(\omega) < \frac{1}{1+1+\alpha+\beta}\right\}$$

and $\mathbb{P}[A] > 0$. Let

$$\Omega^* = \left\{\omega : \limsup_{n\to\infty} \frac{|\xi(n+1,\omega)|}{\gamma(n)} = 1\right\}.$$

Then $\mathbb{P}[\Omega^*] = 1$. Let $A^* = A \cap \Omega^*$. Then $\mathbb{P}[A^*] = \mathbb{P}[A] > 0$. Let $\omega \in A^*$ then

$$\limsup_{n\to\infty} \frac{|X(n,\omega)|}{\gamma(n)} = C_2(\omega), \qquad \limsup_{n\to\infty} \frac{|\xi(n+1,\omega)|}{\gamma(n)} = 1. \tag{5.4.9}$$

By definition

$$X(n+1,\omega) = (1+\alpha)X(n,\omega) + \beta \max_{n-N\leq j\leq n} X(j,\omega) + \xi(n+1,\omega).$$

Then

$$|\xi(n+1,\omega)| \leq |X(n+1,\omega)| + |1+\alpha||X(n,\omega)| + \beta \max_{n-N\leq j\leq n} |X(j,\omega)|.$$

Since γ is regularly varying and obeys (5.4.2) we have that

$$\limsup_{n\to\infty} \frac{|X(n+1,\omega)|}{\gamma(n)} = \limsup_{n\to\infty} \frac{|X(n+1,\omega)|}{\gamma(n+1)} \cdot \frac{\gamma(n+1)}{\gamma(n)} = C_2(\omega). \tag{5.4.10}$$

Also

$$\frac{\max_{n-N\le j\le n}|X(j,\omega)|}{\gamma(n)} = \max_{n-N\le j\le n}\frac{|X(j,\omega)|}{\gamma(n)}$$
$$= \max_{n-N\le j\le n}\frac{|X(j,\omega)|}{\gamma(j)}\cdot\frac{\gamma(j)}{\gamma(n)}.$$

So by (5.4.2) we have

$$\limsup_{n\to\infty}\frac{\max_{n-N\le j\le n}|X(j,\omega)|}{\gamma(n)} \le C_2(\omega). \tag{5.4.11}$$

Thus

$$\limsup_{n\to\infty}\frac{|\xi(n+1,\omega)|}{\gamma(n)} \le C_2(\omega)+(1+\alpha)C_2(\omega)+\beta C_2(\omega) < 1, \tag{5.4.12}$$

which contradicts (5.4.9) and the assumption that $\mathbb{P}[A]>0$. Hence $\mathbb{P}[A]=0$ and so $\mathbb{P}[\bar{A}]=1$ or

$$\limsup_{n\to\infty}\frac{|X(n)|}{\gamma(n)} \ge \frac{1}{1+(1+\alpha+\beta)},\quad \text{a.s.}$$

which is the lower estimate in (5.2.21).

To prove the upper estimate in (5.2.21), we simply apply Proposition 5.2.2 with $c(\omega)=1$ in (5.2.6).

5.4.7 Proof of Theorem 5.2.5

By employing the method of proof of Theorem 5.2.4, we can show that

$$c_1 \le \limsup_{n\to\infty}\frac{|\xi(n+1)|}{\gamma(n)} \le c_2,\quad \text{a.s.} \tag{5.4.13}$$

implies

$$\frac{c_1}{1+(1+\alpha+\beta)} \le \limsup_{n\to\infty}\frac{|X(n)|}{\gamma(n)} \le \frac{c_2}{1-(1+\alpha+\beta)},\quad \text{a.s.}$$

Thus (II) implies (III). If (III) holds, then

$$\limsup_{n\to\infty}\frac{|\xi(n+1,\omega)|}{\gamma(n)} \le \limsup_{n\to\infty}\frac{|X(n+1,\omega)|}{\gamma(n)}$$
$$+(1+\alpha)\limsup_{n\to\infty}\frac{|X(n,\omega)|}{\gamma(n)}+\beta\limsup_{n\to\infty}\max_{n-N\le j\le n}\frac{|X(j,\omega)|}{\gamma(j)}\cdot\frac{\gamma(j)}{\gamma(n)},$$

so

$$\limsup_{n\to\infty}\frac{|\xi(n+1,\omega)|}{\gamma(n)} \le C_2+(1+\alpha)C_2+\beta C_2,$$

as required. Finally, suppose that there is an event A

$$A=\{\omega:\limsup_{n\to\infty}\frac{|\xi(n+1,\omega)|}{\gamma(n)} =: c_1(\omega) < C_1(1-(1+\alpha+\beta))\},$$

where $\mathbb{P}[A]>0$. Then by Proposition 5.2.2, we have for each $\omega\in A$ that

$$\limsup_{n\to\infty}\frac{|X(n,\omega)|}{\gamma(n)} \le \frac{c_1(\omega)}{1-(1+\alpha+\beta)} < C_1.$$

Therefore for all $\omega\in A^*$ we have

$$C_1 \le \limsup_{n\to\infty}\frac{|X(n,\omega)|}{\gamma(n)} < C_1,$$

a contradiction. Hence (III) implies (II).

We now show that (II) implies (I). If (II) holds then (5.4.13) holds. Since $(\xi(n))_{n\geq 0}$ are independent random variables and for all values $\omega \in \Omega^*$, there exists $N(\omega, \epsilon) \in \mathbb{N}$ such that

$$\frac{|\xi(n+1,\omega)|}{\gamma(n)} \leq c_2(1+\epsilon) \quad \text{for all } n > N(\omega, \epsilon).$$

Then

$$\mathbb{P}\left[\frac{|\xi(n+1)|}{\gamma(n)} > c_2(1+\epsilon) \text{ i.o.}\right] = 0$$

and by the Borel–Cantelli lemma

$$\sum_{n=1}^{\infty} \mathbb{P}[|\xi(n+1)| \geq c_2(1+\epsilon)\gamma(n)] < +\infty, \quad \text{for every } \epsilon > 0.$$

Similarly

$$\mathbb{P}\left[\frac{|\xi(n+1)|}{\gamma(n)} > c_1(1-\epsilon) \text{ i.o.}\right] = 1,$$

so then for every $\epsilon \in (0,1)$

$$\sum_{n=1}^{\infty} \mathbb{P}[|\xi(n+1)| \geq c_1(1-\epsilon)\gamma(n)] = +\infty, \quad \text{for every } \epsilon \in (0,1).$$

Let ζ be a random variable with the same distribution as $\xi(n)$ for all $n \geq 1$. Since ξ obeys (5.2.1), we have

$$\sum_{n=1}^{\infty} \mathbb{P}[|\zeta| \geq c_1(1-\epsilon)\gamma(n)] = +\infty, \quad \sum_{n=1}^{\infty} \mathbb{P}[|\zeta| > c_2(1+\epsilon)\gamma(n)] < +\infty$$

and since γ is an increasing function, the functions $\Gamma_2(x) = c_2(1+\epsilon)\gamma(x)$ and $\Gamma_1(x) = c_1(1-\epsilon)\gamma(x)$ have inverses given by $\Gamma_2^{-1}(x) = \gamma^{-1}(x/c_2(1+\epsilon))$ and $\Gamma_1^{-1}(x) = \gamma^{-1}(x/c_1(1-\epsilon))$. By Lemma 1.2.1 we have

$$\sum_{n=1}^{\infty} \mathbb{P}[|\zeta| \geq c_1(1-\epsilon)\gamma(n)] \leq \mathbb{E}[\Gamma_1^{-1}(|\zeta|)]$$
$$= \mathbb{E}\left[\gamma^{-1}\left(\frac{|\zeta|}{c_1(1-\epsilon)}\right)\right] = \int_{-\infty}^{\infty} \gamma^{-1}\left(\frac{|x|}{c_1(1-\epsilon)}\right) dF(x).$$

Thus

$$\int_{-\infty}^{\infty} \gamma^{-1}(|x|/c_1(1-\epsilon))\, dF(x) = +\infty \quad \text{for every } \epsilon \in (0,1).$$

On the other hand, Lemma 1.2.1 also implies for all $\epsilon \in (0,1)$ that

$$\int_{-\infty}^{\infty} \gamma^{-1}(|x|/c_2(1+\epsilon))\, dF(x) = \mathbb{E}[\Gamma_2^{-1}|\zeta|] \leq \sum_{n=0}^{\infty} \mathbb{P}[|\zeta| > c_2(1+\epsilon)\gamma(n)] < +\infty,$$

so $\int_{-\infty}^{\infty} \gamma^{-1}(|x|/c_2(1+\epsilon))\, dF(x) < +\infty$ for all $\epsilon \in (0,1)$. Therefore we have (5.2.22). Hence (II) implies (I). We now show that (I) implies (III). By Lemma 1.2.1 and (5.2.22), we have

$$\sum_{n=0}^{\infty} \mathbb{P}[|\zeta| > K_1\gamma(n)] \geq \mathbb{E}[\gamma^{-1}(|\zeta|/K_1)] = \int_{-\infty}^{\infty} \gamma^{-1}(|x|/K_1)\, dF(x) = +\infty.$$

Since $\xi(n)$ has the same distribution as ζ, we have

$$\sum_{n=0}^{\infty} \mathbb{P}[|\xi(n+1)| > K_1\gamma(n)] = +\infty.$$

Therefore by the Borel–Cantelli lemma, we have

$$\limsup_{n\to\infty} \frac{|\xi(n+1)|}{\gamma(n)} \geq K_1, \quad \text{a.s. on } \Omega_1,$$

where Ω_1 is an almost sure event. Similarly, by Lemma 1.2.1 and (5.2.22), we have

$$\sum_{n=1}^{\infty} \mathbb{P}[|\zeta| \geq K_2\gamma(n)] \leq \mathbb{E}[\gamma^{-1}(|\zeta|/K_2)] < +\infty.$$

Since $\xi(n)$ has the same distribution as ζ, we have

$$\sum_{n=1}^{\infty} \mathbb{P}[|\xi(n+1)| \geq K_2\gamma(n)] < +\infty.$$

Therefore by the Borel–Cantelli lemma, we have

$$\limsup_{n\to\infty} \frac{|\xi(n+1)|}{\gamma(n)} \leq K_2, \quad \text{a.s. on } \Omega_2.$$

Together we get that

$$K_1 \leq \limsup_{n\to\infty} \frac{|\xi(n+1)|}{\gamma(n)} \leq K_2, \quad \text{a.s. on } \Omega_1 \cap \Omega_2,$$

which is (II), and this implies (III).

Chapter 6

Spurious Long–Run Behaviour from Numerical Methods for SFDEs with Continuous Weight Functions

6.1 Introduction

In this chapter we show that the long–time behaviour of the SFDE with *discrete* weights

$$dX(t) = \left(\sum_{j=1}^{m} \alpha_j g(X(t-\theta_j)) - \sum_{j=1}^{p} \beta_j g(X(t-\tau_j)) \right) dt + \sigma\, dB(t), \quad t \geq 0 \tag{6.1.1}$$

can be reproduced by using a standard Euler method with a sufficiently small but uniform step size. However, a similar simple method does not suffice to reproduce the asymptotic behaviour of the SFDE

$$dX(t) = \left(\int_0^{\tau_1} w_1(s) g(X(t-s))\, ds - \int_0^{\tau_2} w_2(s) g(X(t-s))\, ds \right) dt + \sigma\, dB(t), \quad t \geq 0. \tag{6.1.2}$$

in which the weight functions w_1 and w_2 are *continuous*. By considering specific simulations, we show that problems arise both for a naive explicit discretisation and also for a modified discretisation which attempts to fix the solution of the characteristic equation associated with the linear deterministic equation underlying (6.1.2) at zero. For an equation with a particular choice of weights w_1 and w_2, we show that these two methods are unreliable. However, a third method of discretising (6.1.2) for these choices of w_1 and w_2, when applied to a deterministic version of (6.1.2), seems to give asymptotic behaviour which is consistent with the continuous time solution provided that h is chosen sufficiently small. This lays the ground for work in the final chapter on a comprehensive numerical method which gives qualitatively satisfactory results for the asymptotic behaviour of (6.1.2) for any choice of continuous weight functions, while also controlling the error of the approximation on any compact interval.

The chapter has the following structure; Section 2 applies the uniform Euler scheme to a SFDE with discrete weights; Section 3 discusses whether such a scheme will reproduce the long–time behaviour of the SFDE with continuously distributed weights; Section 4 applies both the native and modified Euler to the SFDE with continuously distributed weights.

6.2 Euler Scheme for Discrete Weights

In this chapter our analysis will focus mainly on the case where the speculators use continuously distributed moving averages rather than discrete ones. It will be seen that in the former case we will need some kind of specialised numerical method to mimic correctly the almost sure asymptotic behaviour of the continuous time equation.

In this section we show that we can preserve the asymptotic behaviour if we use a standard Euler–Maruyama method in the case that the speculators use discrete moving averages. We consider the most general form of discrete weights introduced in the examples in Chapter 4. Our results say (roughly) that if we take a fixed step size sufficiently small, then we can recover the basic type of almost sure asymptotic behaviour, and if we let the step size get smaller and smaller, we can estimate key growth parameters with arbitrary accuracy.

We recall the details of the discrete moving average investment strategy outlined in Section 3.4.5. Suppose the investor compares a weighted average of the cumulative returns at m points in time over the last θ units of time with a weighted average of the cumulative returns at p points in time over the last τ units of time, where $\tau \geq \theta$. Let the cumulative returns in the short-run be observed at time points $-\theta := -\theta_m < \cdots < -\theta_1 \leq 0$ and in the long-run at time points $-\tau := -\tau_p < \cdots < -\tau_1 \leq 0$. A weight $\alpha_j \geq 0$ is attached to the observation at the time θ_j, while a weight $\beta_j \geq 0$ is associated with the observation at the time τ_j. We assume that the weights and observation times obey

$$\alpha_1 + \cdots + \alpha_m = \beta_1 + \cdots + \beta_p > 0, \tag{6.2.1}$$

$$\sum_{j=1}^{m} \alpha_j \chi_{[-t,0]}(-\theta_j) \geq \sum_{j=1}^{p} \beta_j \chi_{[-t,0]}(-\tau_j) \qquad \text{for all } t \in [0,\tau], \tag{6.2.2}$$

then the measure

$$\nu(du) := \sum_{j=1}^{m} \alpha_j \delta_{-\theta_j}(du) - \sum_{j=1}^{p} \beta_j \delta_{-\tau_j}(du)$$

satisfies all the conditions in Theorem 3.3.2 with the moment

$$m(\nu) = \sum_{j=1}^{p} \beta_j \tau_j - \sum_{j=1}^{m} \alpha_j \theta_j.$$

As previously, we assume that g obeys

$$g : \mathbb{R} \to \mathbb{R} \text{ is locally Lipschitz continuous}, \qquad \lim_{|x|\to\infty} \frac{g(x)}{x} = 1. \tag{6.2.3}$$

Let $\phi \in C([-\tau,0],\mathbb{R})$. The continuous time functional differential equation is

$$dX(t) = \left(\sum_{j=1}^{m} \alpha_j g(X(t-\theta_j)) - \sum_{j=1}^{p} \beta_j g(X(t-\tau_j)) \right) dt + \sigma \, dB(t), \quad t \geq 0 \tag{6.2.4}$$

where $X(t) = \phi(t)$ for $t \in [-\tau, 0]$. If $m(\nu) < 1$ then the cumulative returns evolve according to

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = \frac{\sigma}{1 - \sum_{j=1}^{p} \beta_j \tau_j + \sum_{j=1}^{m} \alpha_j \theta_j} \quad \text{a.s}$$

$$\liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = -\frac{\sigma}{1 - \sum_{j=1}^{p} \beta_j \tau_j + \sum_{j=1}^{m} \alpha_j \theta_j} \quad \text{a.s.}$$

On the other hand, if $m(\nu) > 1$ then there exists a unique positive $\lambda > 0$ such that a.s.

$$\lim_{t\to\infty} e^{-\lambda t} X(t) = c\left(\phi(0) + \int_0^\infty \sigma e^{-\lambda s} \, dB(s)\right) + c\left(\sum_{j=1}^{m} \alpha_j \int_{-\theta_j}^{0} e^{-\lambda(\theta_j+u)} \phi(u)\, du - \sum_{j=1}^{p} \beta_j \int_{-\tau_j}^{0} e^{-\lambda(\tau_j+u)} \phi(u)\, du \right),$$

where

$$c = \frac{1}{1 - \sum_{j=1}^{p} \beta_j \tau_j e^{-\lambda\tau_j} + \sum_{j=1}^{m} \alpha_j \theta_j e^{-\lambda\theta_j}}. \tag{6.2.5}$$

It turns out that the asymptotic behaviour of (6.2.4) is preserved by a uniform Euler discretisation of (6.2.4) once the uniform step size $h > 0$ is sufficiently small. We demonstrate this fact over the next few pages. First define $\tau_0 = 0$ and $\theta_0 = 0$. Let

$$\mathcal{T} = \{t \in \mathbb{R} : t = \tau_j \text{ for some } j = 0,\ldots,p \text{ or } t = \theta_k \text{ for some } k = 0,\ldots,m\}. \tag{6.2.6}$$

Define $d(\mathcal{T}) = \min\{|x-y| : x,y \in \mathcal{T},\, x \neq y\}$. Clearly $d(\mathcal{T}) > 0$. Let $h \in (0, d(\mathcal{T}))$ and define

$$\mathcal{T}(h) = \{n \in \mathbb{N} : n = [\tau_j/h] \text{ for some } j = 0,\ldots,p \text{ or } n = [\theta_k/h] \text{ for some k=0,\ldots,m}\}.$$

Define the integers $M_j(h) = [\theta_j/h]$ for $j = 1,\ldots,m$ and $N_j(h) = [\tau_j/h]$ and $j = 1,\ldots,p$. The fact that $h < d(\mathcal{T})$ implies that the order of the non–negative integers $(N_j(h))_{j=1,\ldots,p}$, $(M_j(h))_{j=1,\ldots,m}$ preserves the order of the real sequences $(\tau_j)_{j=1,\ldots,p}$ $(\theta_j)_{j=1,\ldots,m}$ in the sense that

$$\tau_j < \theta_k \text{ for some } j,k \text{ implies } N_j(h) < M_k(h)$$
$$\tau_j > \theta_k \text{ for some } j,k \text{ implies } N_j(h) > M_k(h)$$
$$\tau_j = \theta_k \text{ for some } j,k \text{ implies } N_j(h) = M_k(h),$$

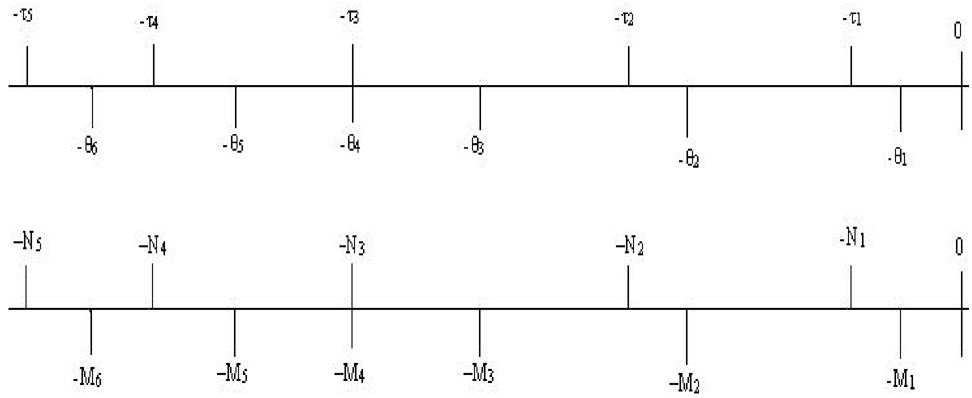

Figure 6.1: θ , M short–run and τ, N long–run time delays

and the fact that the sequences $(\theta_j)_{j=1,\dots,m}$ and $(\tau_j)_{j=1,\dots,p}$ are increasing implies that the sequences $(M_j(h))_{j=1,\dots,m}$ and $(N_j(h))_{j=1,\dots,p}$ are increasing.

First we note form the graphs that $\theta_i \geq \tau_i$ and $M_i \geq N_i$ as required by (6.2.2). If we let $j = 1$ and $k = 2$ then $\tau_1 < \theta_2$ and $N_1(h) > M_2(h)$. On the other hand if we set $j = k = 2$, then $\tau_2 > \theta_2$ and $N_2(h) > M_2(h)$. Final if $j = 3$ and $k = 4$ then $\tau_3 = \theta_4$ and $N_3(h) = M_4(h)$ as required.

We now consider the stochastic difference equation

$$X_h(n+1) = X_h(n) + h\left(\sum_{j=1}^{m}\alpha_j g(X_h(n-M_j(h))) - \sum_{j=1}^{p}\beta_j g(X_h(n-N_j(h)))\right) + \sigma\sqrt{h}\xi(n+1), \quad n \geq 0 \quad (6.2.7)$$

where $X_h(n) = \phi(nh)$ for $n = -N_p(h), \dots, 0$.

We define $w_{1,h}(n)$ for $n = 0, \dots, M_m(h)$. If $n \in \{0, \dots, M_m(h)\}$ is such that there is a $j \in \{1, \dots, m\}$ for which $M_j(h) = n$, define $w_{1,h}(n) = \alpha_j h$; otherwise let $w_{1,h}(n) = 0$. We similarly define $w_{2,h}(n)$ for $n = 0, \dots, N_p(h)$. If $n \in \{0, \dots, N_p(h)\}$ is such that there is a $j \in \{1, \dots, p\}$ for which $N_j(h) = n$, define $w_{2,h}(n) = \beta_j h$; otherwise let $w_{2,h}(n) = 0$. This means that (6.2.7) can be rewritten as

$$X_h(n+1) = X_h(n) + \sum_{j=0}^{M_m(h)} w_{1,h}(j) g(X_h(n-j)) - \sum_{j=0}^{N_p(h)} w_{2,h}(j) g(X_h(n-j)) + \sigma\sqrt{h}\xi(n+1), \quad n \geq 0$$

Since each α_j and β_j is positive, we have

$$w_{1,h}(n) \geq 0, \quad n = 0, \dots, M_m(h) \quad w_{2,h}(n) \geq 0, \quad n = 0, \dots, N_p(h).$$

The condition (6.2.1) implies that

$$\sum_{j=0}^{M_m(h)} w_{1,h}(j) = \sum_{j=0}^{N_p(h)} w_{2,h}(j) > 0.$$

Finally, the condition (6.2.2) guarantees that

$$\sum_{j=0}^{n} w_{1,h}(j) \geq \sum_{j=0}^{n} w_{2,h}(j), \quad n = 0, \dots, M_m(h).$$

Therefore, if

$$\sum_{j=0}^{N_p(h)} jw_{2,h}(j) - \sum_{j=0}^{M_m(h)} jw_{1,h}(j) < 1,$$

we have

$$\limsup_{n\to\infty} \frac{X_h(n)}{\sqrt{2(nh)\log\log(nh)}} = \frac{\sigma}{1 - \sum_{j=0}^{N_p(h)} jw_{2,h}(j) + \sum_{j=0}^{M_m(h)} jw_{1,h}(j)}, \quad \text{a.s.}$$
$$\liminf_{n\to\infty} \frac{X_h(n)}{\sqrt{2(nh)\log\log(nh)}} = -\frac{\sigma}{1 - \sum_{j=0}^{N_p(h)} jw_{2,h}(j) + \sum_{j=0}^{M_m(h)} jw_{1,h}(j)}, \quad \text{a.s.}$$

while if

$$\sum_{j=0}^{N_p(h)} jw_{2,h}(j) - \sum_{j=0}^{M_m(h)} jw_{1,h}(j) > 1,$$

we have

$$\lim_{n\to\infty} \frac{X_h(n)}{\alpha(h)^n} \text{ exists and is finite a.s.}$$

where $\alpha(h) > 1$ is the unique solution of

$$\alpha(h) = 1 + h\left(\sum_{j=1}^{m} \alpha_j \alpha(h)^{-[\theta_j/h]} - \sum_{j=1}^{p} \beta_j \alpha(h)^{-[\tau_j/h]}\right).$$

Noting that

$$\sum_{j=0}^{M_m(h)} jw_{1,h}(j) = \sum_{j=1}^{m} \alpha_j h M_j(h), \quad \sum_{j=0}^{N_p(h)} jw_{2,h}(j) = \sum_{j=1}^{p} \beta_j h N_j(h).$$

The discrete analogue of $m(\nu)$ is:

$$m_h(\nu) := \sum_{j=1}^{p} \beta_j h N_j - \sum_{j=1}^{m} \alpha_j h M_j = \sum_{j=1}^{p} \beta_j h[\tau_j/h] - \sum_{j=1}^{m} \alpha_j h[\theta_j/h].$$

Therefore

$$-(m_h(\nu) - m(\nu)) = -\left(\sum_{j=1}^{p} \beta_j(\tau_j - h[\tau_j/h]) + \sum_{j=1}^{m} \alpha_j(\theta_j - h[\theta_j/h])\right).$$

which implies

$$\begin{aligned}
|m_h(\nu) - m(\nu)| &\le \sum_{j=1}^{p} \beta_j|\tau_j - h[\tau_j/h]| + \sum_{j=1}^{m} \alpha_j|\theta_j - h[\theta_j/h]|, \\
&\le \sum_{j=1}^{p} h\beta_j|\tau_j/h - [\tau_j/h]| + \sum_{j=1}^{m} h\alpha_j|\theta_j/h - [\theta_j/h]|, \\
&\le h\left(\sum_{j=1}^{p} \beta_j + \sum_{j=1}^{m} \alpha_j\right).
\end{aligned}$$

If $m(\nu) < 1$, and $h < h_1$ where

$$h_1\left(\sum_{j=1}^{p} \beta_j + \sum_{j=1}^{m} \alpha_j\right) = 1 - m(\nu), \tag{6.2.8}$$

then

$$m_h(\nu) = m_h(\nu) - m(\nu) + m(\nu) \leq h\left(\sum_{j=1}^{p}\beta_j + \sum_{j=1}^{m}\alpha_j\right) + m(\nu) < 1.$$

On the other hand if $m(\nu) > 1$ and $h < h_2$ where

$$h_2\left(\sum_{j=1}^{p}\beta_j + \sum_{j=1}^{m}\alpha_j\right) = m(\nu) - 1, \tag{6.2.9}$$

then

$$m_h(\nu) = m_h(\nu) - m(\nu) + m(\nu) \geq -h\left(\sum_{j=1}^{p}\beta_j + \sum_{j=1}^{m}\alpha_j\right) + m(\nu) > 1.$$

Theorem 6.2.1. *Suppose that* (6.2.2), (6.2.1) *hold and that* $m(\nu) < 1$. *Suppose* g *obeys* (6.2.3). *Let* $\mathcal{T}$ *be given by* (6.2.6). *Let* $0 < h < d(\mathcal{T}) \wedge h_1$, *where* $h_1 > 0$ *is defined by* (6.2.8). *If* X_h *is the solution of* (6.2.7), *we have*

$$\limsup_{n\to\infty} \frac{X_h(n)}{\sqrt{2(nh)\log\log(nh)}} = \frac{\sigma}{1 - \sum_{j=1}^{p}\beta_j h[\tau_j/h] + \sum_{j=1}^{m}\alpha_j h[\theta_j/h]}, \quad \textit{a.s.} \tag{6.2.10}$$

$$\liminf_{n\to\infty} \frac{X_h(n)}{\sqrt{2(nh)\log\log(nh)}} = -\frac{\sigma}{1 - \sum_{j=1}^{p}\beta_j h[\tau_j/h] + \sum_{j=1}^{m}\alpha_j h[\theta_j/h]}, \quad \textit{a.s.} \tag{6.2.11}$$

while the solution X *of* (6.2.4) *obeys*

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log(t)}} = \frac{\sigma}{1 - \sum_{j=1}^{p}\beta_j\tau_j + \sum_{j=1}^{m}\alpha_j\theta_j}, \quad \textit{a.s.} \tag{6.2.12}$$

$$\liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log(t)}} = -\frac{\sigma}{1 - \sum_{j=1}^{p}\beta_j\tau_j + \sum_{j=1}^{m}\alpha_j\theta_j}, \quad \textit{a.s.} \tag{6.2.13}$$

We notice that the asymptotic behaviour of the solution X_h of the discrete scheme (6.2.7) enjoys the same almost sure iterated logarithm rate of growth of the large fluctuations as the solution X of (6.2.4), provided that the step size h is less than some critical size h^* which can be determined from data associated with the original equation (6.2.4). Moreover, it can be seen that as $h \to 0^+$, the size of the limiting constants on the righthand side of (6.2.10) and (6.2.11) converges to the limiting constants on the righthand side of (6.2.12) and (6.2.13).

We now consider what happens when the moment $m(\nu) > 1$ for the continuous equation.

Theorem 6.2.2. *Suppose that* (6.2.2), (6.2.1) *hold and that* $m(\nu) > 1$. *Suppose* g *obeys* (6.2.3). *Suppose also that there exists a non–decreasing continuous* $\gamma_0 : [0,\infty) \to (0,\infty)$ *such that*

$$|g(x) - x| \leq \gamma_0(|x|), \quad x \in \mathbb{R}; \quad \int_1^\infty \frac{\gamma_0(x)}{x^2}\,dx < +\infty. \tag{6.2.14}$$

Let $\mathcal{T}$ *be given by* (6.2.6). *Let* $0 < h < d(\mathcal{T}) \wedge h_2$, *where* $h_2 > 0$ *is defined by* (6.2.9). *If* X_h *is the solution of* (6.2.7), *there exists a unique* $\lambda(h) > 0$ *satisfying*

$$\frac{e^{\lambda(h)h} - 1}{h} = \sum_{j=1}^{m}\alpha_j e^{-\lambda(h)[\theta_j/h]h} - \sum_{j=1}^{p}\beta_j e^{-\lambda(h)[\tau_j/h]h} \tag{6.2.15}$$

such that there exists an almost surely finite random variable $\Lambda(h)$ *such that*

$$\lim_{n\to\infty} \frac{X_h(n)}{e^{\lambda(h)nh}} = \Lambda(h), \quad \textit{a.s.} \tag{6.2.16}$$

If X *is the solution of* (6.2.4), *there exists a unique positive* $\lambda > 0$ *satisfying*

$$\lambda = \sum_{j=1}^{m}\alpha_j e^{-\lambda\theta_j} - \sum_{j=1}^{p}\beta_j e^{-\lambda\tau_j} \tag{6.2.17}$$

such that there exists an almost surely finite random variable Λ such that

$$\lim_{t\to\infty} e^{-\lambda t} X(t) = \Lambda, \quad a.s. \tag{6.2.18}$$

Once again the asymptotic behaviour of the solution X_h of the discrete scheme (6.2.7) enjoys exact almost sure real exponential growth, just as the solution X of (6.2.4) does, provided that the step size h is less than some critical size h^*. As before, the critical step size h^* can be determined from data associated with the original equation (6.2.4). Finally, it can be seen from (6.2.15) and (6.2.17) that as $h \to 0^+$, the exponential rate of growth $\lambda(h) > 0$ of the solution of (6.2.7) converges to the exponential rate of growth $\lambda > 0$ of the solution X of the continuous equation (6.2.4).

6.3 Euler Scheme for Continuous Weights

In this section we consider whether a uniform Euler discretisation will reproduce the long–time behaviour of the SFDE in the case when the weights attached to values of the process in the past are continuously distributed. This is in contrast to the equations considered in the last section, in which a discrete weighted average of past values are taken. In broad terms, by means of a mixture of analysis and numerical experiments, we show that an Euler scheme which is implemented directly will not mimic the asymptotic behaviour of the SFDE, despite the fact that this scheme enables us to control with arbitrary accuracy the mean square error on any compact interval, given sufficient computational effort. Furthermore, we are able to detect the presence of these spurious computational features even in absence of noise or of nonlinear terms in the space variable. Consequently, in our discussion in this section, we tend to study the properties of discretisations of the underlying linear deterministic equation.

More specifically, we show the following:

(i) The Euler scheme applied directly to the SFDE will in general not accurately reproduce the presence of the zero solution of the characteristic equation of the underlying linear deterministic equation; this will lead to spurious asymptotic behaviour in the case when this zero is the solution of the characteristic equation with largest real part.

(ii) It is possible to modify the standard uniform Euler method in (i) in such a manner that the underlying discrete linear resolvent has characteristic equation with a unit solution, which now correctly mimics the presence of the zero solution of the characteristic equation of the underlying linear continuous equation. However, numerical experiments seem to suggest that errors arising from round–off, truncation (or both) in the computer implementation do not hold the solution of the discrete characteristic equation with largest modulus at unity. This leads once again to spurious long run asymptotic behaviour of the discrete time solution. Moreover, it is not clear that these problems can be alleviated easily by reducing the step size.

(iii) The Euler scheme in (ii) can be further modified in such a way to remove the unit solution of the characteristic equation entirely. However, this changes the structure of the Euler scheme to one more reminiscent of a Volterra summation equation with finite memory. Numerical experiments in this case confirm that this adjustment leads to more satisfactory long–run discrete dynamics. However, the positivity of a sequence depending on the weights is not assured by this method. This is an undesirable feature from a modelling perspective because this positivity is instrumental in causing the presence of a positive correlation in the returns at all time horizons, a feature of the economic model which is responsible for the excess volatility and bubbles. This is unsatisfactory because these properties are among the advantageous characteristics of the dynamics. Moreover, in the absence of such positivity, it is more difficult to conduct an analysis of the long–run discrete dynamics of the scheme.

The findings of (i)–(iii) tend to suggest that we should approach the question of long–term simulation of equations with continuous weights by means of standard Euler methods with extreme caution. However, the satisfactory performance of the algorithm in (iii) suggests that it might be feasible to rehabilitate such schemes by exploiting the presence of the zero solution of the characteristic equation at the outset and developing an Euler scheme in which the associated unit solution of the discrete characteristic equation has been removed. The analysis and experiments also suggest that it would be sensible from both theoretical and modelling perspectives to impose a positivity restriction on an analogue of the sequence which depends on the weights in (iii), provided that this adjustment neither incurs excessive error nor introduces a systematic bias into the long–run simulations. A major part of the remainder of the thesis is devoted to showing that these issues can be successfully addressed.

In this section, the Euler methods employed are all explicit. However, there is no reason to exclude a priori the use of implicit methods on the drift term. We have not considered implicit methods for these simulations because the contribution from the implicit term in the drift will be of small order, since it arises from a continuously distributed weighted average. The evidence of (i)–(iii) suggests that it is not the presence or absence of such small terms which are of main importance, since the correction to force the unit solution of the characteristic equation at unity does not fix the underlying numerical instability when the solution is simulated on a computer. Rather, the problem seems to stem from the presence of this unit or near unit solution in the discrete scheme in the first place. Thus, it appears from (iii) that the removal of this unit solution of the discrete characteristic equation greatly improves the performance of the simulation and because explicit schemes are more readily implemented, our priority is to remove this unit root above all other considerations; then, if an explicit scheme proves satisfactory, we need not consider implicit schemes on the grounds of simplicity.

In the next section, we distil the evidence gained from implementing standard Euler methods to the case where the weights are continuous. Our experience suggests that we seek a method which has the following properties:

- The method converts the zero solution of the continuous linear resolvent to a unit solution of the discrete linear resolvent and then removes it. This suggests that the final form of the difference scheme might be a Volterra summation equation with finite memory.
- The method preserves positivity properties of the original continuous time equation, aiding both analysis of the method and preserving in discrete time salient economic properties of returns' dynamics.
- The method is *robust* in the sense that it preserves the essential type of the long run dynamics (i.e., either iterated logarithm large fluctuations, or real exponential explosion of solutions) regardless of the underlying model parameters, once the step size is chosen less than some critical level.
- The method recovers key growth parameters (e.g., exponential growth rate or normalising constant for iterated logarithm growth) with arbitrary accuracy for sufficiently small and uniform step size.
- The method can still be used for numerical analysis on finite time intervals with mean square error tending to zero with an explicitly computable bound as the step size tends to zero.

It transpires that we can develop an alternative numerical method which is based on discretising a random continuous time Volterra equation satisfied by the solution of the continuous time SFDE. Despite this nonstandard approach, it is nonetheless interesting to note that the form of the Volterra summation equation satisfied by the discrete equation is very similar to that satisfied by the Euler scheme in (ii) above. Therefore, this nonstandard approach is in practice closely related to the standard Euler method but incorporates some special features which improve reliability and exploit the particular structure of the SFDE.

6.4 Standard Euler Scheme and Modifications

Let $\tau_2 > \tau_1 > 0$, and define $\tau = \max(\tau_1, \tau_2) = \tau_2$. Let w_1 and w_2 be continuous, non–negative functions on $[0, \tau_1]$ and $[0, \tau_2]$ respectively, and suppose that

$$\int_0^{\tau_1} w_1(s)\, ds = \int_0^{\tau_2} w_2(s)\, ds = 1, \tag{6.4.1}$$

as well as

$$\int_0^t w_1(s)\, ds \geq \int_0^t w_2(s)\, ds, \quad t \in [0, \tau_1]. \tag{6.4.2}$$

We presume moreover that w_1 and w_2 are not identically equal, so that continuity ensures that there is a subinterval of $[0, \tau_1]$ interval on which the inequality (6.4.2) is strict. Let $\psi \in C([-\tau, 0]; \mathbb{R})$, $\sigma \neq 0$, and g

be a locally Lipschitz continuous function that obeys

$$\lim_{|x|\to\infty} \frac{g(x)}{x} = \beta$$

for some $\beta > 0$. Let B be a standard one–dimensional Brownian motion. Then there is a unique continuous adapted process X which obeys

$$dX(t) = \left(\int_0^{\tau_1} w_1(s) g(X(t-s))\, ds - \int_0^{\tau_2} w_2(s) g(X(t-s))\, ds \right) dt \\ + \sigma\, dB(t), \quad t \geq 0. \quad (6.4.3)$$

The long–time behaviour of solutions of (6.4.3) has been determined in Chapter 4 . Specifically, we show that if

$$\beta \left(\int_0^{\tau_2} s w_2(s)\, ds - \int_0^{\tau_1} s w_1(s)\, ds \right) < 1 \quad (6.4.4)$$

then

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t \log\log t}} = \frac{\sigma}{1 - \beta(\int_0^{\tau_2} s w_2(s)\, ds - \int_0^{\tau_1} s w_1(s))}, \quad \text{a.s.}$$
$$\liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t \log\log t}} = -\frac{\sigma}{1 - \beta(\int_0^{\tau_2} s w_2(s)\, ds - \int_0^{\tau_1} s w_1(s))}, \quad \text{a.s.}$$

while under the condition that g obeys

$$\int_1^{\infty} \frac{\max_{0\leq |s| \leq x} |g(s) - s|}{x^2}\, dx < +\infty$$

and

$$\beta \left(\int_0^{\tau_2} s w_2(s)\, ds - \int_0^{\tau_1} s w_1(s)\, ds \right) > 1 \quad (6.4.5)$$

we have that

$$\lim_{t\to\infty} \frac{X(t)}{e^{\lambda t}} \quad \text{exists and is a.s. finite}$$

where λ is given by the unique positive solution of the characteristic equation

$$\lambda = \beta \left(\int_0^{\tau_1} w_1(s) e^{-\lambda s}\, ds - \int_0^{\tau_2} w_2(s) e^{-\lambda s}\, ds \right). \quad (6.4.6)$$

In order to determine the essential problems associated with the behaviour of discretised versions of (6.4.3), we focus on the equation in which g is linear, the noise term is absent (i.e., $\sigma = 0$ and the initial function ψ is zero on $[-\tau, 0)$ and obeys $\psi(0) = 1$. When we have determined potential problems and solutions related to the discretisation of the underlying deterministic problem, we will show in the next chapter that the appropriate numerical methods for the fully nonlinear and stochastic equation resolves all the problems outlined.

The resulting simplified and deterministic equation is

$$x'(t) = \beta \left(\int_0^{\tau_1} w_1(s) x(t-s)\, ds - \int_0^{\tau_2} w_2(s) x(t-s)\, ds \right), \quad t > 0; \quad (6.4.7)$$
$$x(0) = 1, \quad x(t) = 0, \quad t \in [-\tau, 0). \quad (6.4.8)$$

We note that in the case when (6.4.4) holds, $x(t)$ tends to a nonzero finite limit as $t \to \infty$, while in the case when (6.4.5) holds, $x(t)$ tends to infinity at the rate $e^{\lambda t}$ as $t \to \infty$ where $\lambda > 0$ is the unique positive solution of (6.4.6). We wish any numerical method to reproduce this qualitative behaviour.

6.4.1 Naive explicit Euler scheme

Suppose that $h > 0$ and that we allow, for $n \in \mathbb{N}$, $x_h(n)$ to be the approximation to $x(t)$ at time $t = nh$. Define the integers

$$N_1 = \lfloor \tau_1/h \rfloor, \quad N_2 = \lfloor \tau_2/h \rfloor \tag{6.4.9}$$

in such a way that $N_2 > N_1 > 1$. This can be arranged for $h > 0$ sufficiently small. We define $x_h(n)$ by

$$\begin{aligned} x_h(n+1) &= x_h(n) \\ &\quad + \beta h \left(\sum_{j=0}^{N_1-1} w_1(jh)x_h(n-j)h - \sum_{j=0}^{N_2-1} w_2(jh)x_h(n-j)h \right), \quad n \geq 0; \end{aligned} \tag{6.4.10a}$$

$$x_h(0) = 1, \quad x_h(n) = 0, \quad n \in [-N_2+1, 0). \tag{6.4.10b}$$

Examination of the characteristic equation associated with (6.4.10) shows that the characteristic equation has a solution at unity if and only if

$$\sum_{j=0}^{N_1-1} w_1(jh) = \sum_{j=0}^{N_2-1} w_2(jh). \tag{6.4.11}$$

This is not automatically implied by (6.4.1). Therefore, the zero eigenvalue of the characteristic equation (6.4.6) is not necessarily transformed exactly to a unit solution of the characteristic equation associated with (6.4.10). In the case when (6.4.4) holds, but (6.4.11) does not, we cannot have that $x_h(n)$ tends to a positive limit as $n \to \infty$, and therefore the asymptotic behaviour of the solution of (6.4.7) cannot be reproduced in this case.

Simulations confirm that this problem arises in the case when (6.4.11) fails to hold, and the simple Euler method (6.4.10) is employed.

Example 6.4.1. We consider the case when $\tau_1 = 1$, $\tau_2 = 2$

$$w_1(t) = C_1 e^{-t}, \quad t \in [0,1], \quad w_2(t) = C_2 e^{-t}, \quad t \in [0,2]. \tag{6.4.12}$$

We pick $h = 0.001$ and therefore $N_1 = 1000$, $N_2 = 2000$. In this case in order that w_1 and w_2 obey (6.4.1) we require $C_1 = 1/(1-e^{-1})$ and $C_2 = 1/(1-e^{-2})$. Noting that $\int_0^t se^{-s}\,ds = 1 - e^{-t} - te^{-t}$, we have

$$\begin{aligned} \int_0^{\tau_2} sw_2(s)\,ds - \int_0^{\tau_1} sw_1(s)\,ds &= C_2 \int_0^2 se^{-s}\,ds - C_1 \int_0^1 se^{-s}\,ds, \\ &= \frac{1-3e^{-2}}{1-e^{-2}} - \frac{1-2e^{-1}}{1-e^{-1}} = \frac{1}{e+1}. \end{aligned}$$

In the first simulation, we have $\beta = 1.2$, in which case

$$\beta \left(\int_0^{\tau_2} sw_2(s)\,ds - \int_0^{\tau_1} sw_1(s)\,ds \right) = \frac{1.2}{e+1} < 1.$$

In the second simulation, we have $\beta = 2e^2/(e-1)$, in which case

$$\beta \left(\int_0^{\tau_2} sw_2(s)\,ds - \int_0^{\tau_1} sw_1(s)\,ds \right) = \frac{2e^2}{(e-1)(e+1)} = 2\frac{e^2}{e^2-1} > 1.$$

In this situation, we note with $\lambda = 1$ that

$$
\begin{aligned}
&\lambda - \beta\left(\int_0^{\tau_1} w_1(s)e^{-\lambda s}\,ds - \int_0^{\tau_2} w_2(s)e^{-\lambda s}\,ds\right)\\
&= 1 - 2\frac{e}{e-1}e\left(C_1\int_0^1 e^{-2s}\,ds - C_2\int_0^2 e^{-2s}\,ds\right),\\
&= 1 - 2\frac{e}{e-1}e\left(\frac{C_1}{2}(-e^{-2}+1) - \frac{C_2}{2}\left(-e^{-4}+1\right)\right),\\
&= 1 - \frac{e}{e-1}e\left(\frac{1-e^{-2}}{1-e^{-1}} - \frac{1-e^{-4}}{1-e^{-2}}\right),\\
&= 1 - \frac{e}{e-1}e\left(\frac{(e-1)(e+1)}{e(e-1)} - \frac{(e^2-1)(e^2+1)}{e^2(e^2-1)}\right),\\
&= 1 - \frac{e}{e-1}e\frac{1}{e}\left(1-\frac{1}{e}\right) = 1 - \frac{e}{e-1}\left(\frac{e-1}{e}\right) = 0.
\end{aligned}
$$

Therefore, when $\beta = 2e^2/(e-1)$, we have that

$$\lim_{t\to\infty}\frac{x(t)}{e^t} \quad \text{is positive and finite.}$$

Our analysis predicts in the case when $h = 0.1$ and $\beta = 1.2$ that $x_h(n)$ does not tend to the correct limit as confirmed by the figure on the left. On the other hand when $\beta = \frac{2e^2}{1-e}$ and $h = 0.1$ that equation does not display exponential growth as indicated by the diagram on the right.

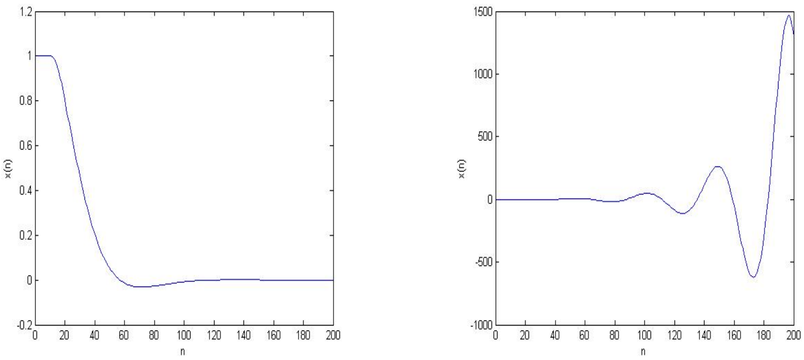

Figure 6.2: Stable and Exponential growth of Naive Euler

6.4.2 Weighted Euler scheme

In order to rectify the problem that the characteristic equation associated with (6.4.11) has a unit solution, we weight the terms in the approximation of the drift in (6.4.10) by the discrete sums of w_1 and w_2 in (6.4.11).

Once again we suppose that $x_h(n)$ is the approximation to $x(t)$ at time $t = nh$. Define the integers N_1 and N_2 as in (6.4.9) so that once again $N_2 > N_1 > 1$. Define

$$W_1(h) = \sum_{j=0}^{N_1-1} w_1(jh)h, \quad W_2(h) = \sum_{j=0}^{N_2-1} w_2(jh)h. \tag{6.4.13}$$

We define $x_h(n)$ for $n \geq 0$ by

$$x_h(n+1) = x_h(n) + \beta h \left(\frac{1}{W_1(h)} \sum_{j=0}^{N_1-1} w_1(jh) x_h(n-j) h - \frac{1}{W_2(h)} \sum_{j=0}^{N_2-1} w_2(jh) x_h(n-j) h \right), \quad (6.4.14)$$

with initial conditions

$$x_h(0) = 1, \quad x_h(n) = 0, \quad n \in [-N_2+1, 0).$$

In contrast with (6.4.10), the characteristic equation associated with (6.4.14) always has unity as a solution.

Suppose that (6.4.4) holds, so that the solution of (6.4.7) is asymptotically constant. Since the solution of the characteristic equation (6.4.6) with largest real part has been mapped to the unit solution of the characteristic equation of (6.4.14), for sufficiently small $h > 0$, we might expect the solution of the equation (6.4.14) to reproduce the asymptotic behaviour of (6.4.7). However, for small step size h, the number of terms in each of the sums W_1 and W_2 as well as in the sums in (6.4.14) are of order $1/h$, and the increasingly large number of terms (each of which makes a smaller and smaller contribution as h decreases) increases the possibility of roundoff or truncation error, so that in practice the unit solution of the characteristic equation associated with (6.4.14) does not always give rise to an asymptotically constant solution of (6.4.14). This supposition is given credence by the result of the simulation below.

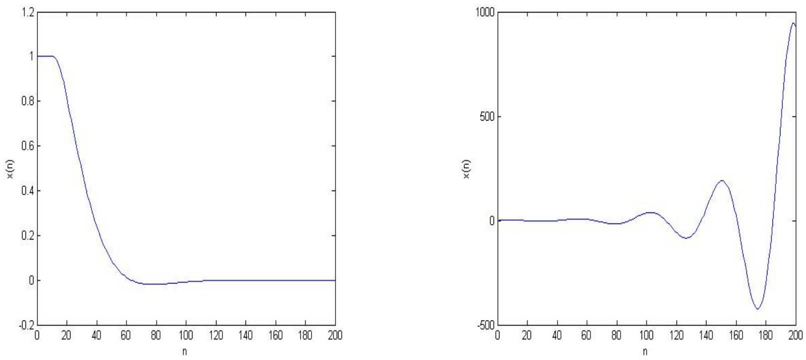

Figure 6.3: Stable and Exponential growth of Weighted Euler

For the above graphs we have used the exponential weights as defined by (6.4.12). Our analysis predicts in the case when $h = 0.1$ and $\beta = 1.2$ that $x_h(n)$ does not tend to the correct limit as confirmed by the figure on the left. Also when $\beta = \frac{2e^2}{1-e}$ and $h = 0.1$ the graph on the right does not display exponential growth as would be expected.

6.4.3 Removal of the unit solution of the characteristic equation

The results of the last two subsections suggests that numerical stability of the long run behaviour can only be achieved by removing unit solution of the characteristic equation associated with the discrete scheme. We start with equation (6.4.14) because its characteristic equation has such a unit solution.

We have that $x_h(n)$ is given by (6.4.14) viz.,

$$x_h(n+1) = x_h(n) + h \left(\frac{1}{W_1(h)} \sum_{j=0}^{N_1-1} w_1(jh) g(x_h(n-j)) h - \frac{1}{W_2(h)} \sum_{j=0}^{N_2-1} w_2(jh) g(x_h(n-j)) h \right) + \sigma\sqrt{h}\xi(n+1), \quad n \geq 0,$$

with initial conditions $x_h(n) = \psi_h(n)$ for $n \in [-N_2+1, 0)$. We let $x_h(0) = \psi(0)$. With W_1 and W_2 defined by (6.4.13), define

$$W_h(j) = \begin{cases} \frac{w_1(jh)}{W_1(h)} - \frac{w_2(jh)}{W_2(h)}, & j \in \{0, \dots, N_1-1\}, \\ -\frac{w_2(jh)}{W_2(h)}, & j \in \{N_1, \dots, N_2-1\}, \end{cases} \tag{6.4.15}$$

With $N = \max(N_1, N_2) = N_2 \geq 2$, we have

$$x_h(n+1) = x_h(n) + h\sum_{j=0}^{N-1} W_h(j) g(x_h(n-j))h + \sigma\sqrt{h}\xi(n+1), \quad n \geq 0.$$

Moreover, we have

$$\begin{aligned}\sum_{j=0}^{N-1} W_h(j) &= \sum_{j=0}^{N_1-1}\left(\frac{w_1(jh)}{W_1(h)} - \frac{w_2(jh)}{W_2(h)}\right) - \sum_{j=N_1}^{N_2-1}\frac{w_2(jh)}{W_2(h)}, \\ &= 1 - \sum_{j=0}^{N_1-1}\frac{w_2(jh)}{W_2(h)} - \sum_{j=N_1}^{N_2-1}\frac{w_2(jh)}{W_2(h)} = 1 - \sum_{j=0}^{N_2-1}\frac{w_2(jh)}{W_2(h)} = 0.\end{aligned}$$

Next we get

$$\sum_{j=0}^{n}(x_h(j+1) - x_h(j)) = \sum_{j=0}^{n} h\sum_{l=0}^{N-1} W_h(l) g(x_h(j-l))h + \sigma\sqrt{h}\sum_{j=0}^{n}\xi(j+1), \quad n \geq 0.$$

Therefore for $n \geq 0$, we have

$$\begin{aligned}x_h(n+1) - \psi(0) &= \sum_{l=0}^{N-1} hW_h(l)\sum_{j=0}^{n} g(x_h(j-l))h + \sigma\sqrt{h}\sum_{j=0}^{n}\xi(j+1), \\ &= \sum_{l=0}^{N-1} hW_h(l)\sum_{k=-l}^{n-l} g(x_h(k))h + \sigma\sqrt{h}\sum_{j=0}^{n}\xi(j+1), \\ &= \sum_{k=-N+1}^{n} g(x_h(k))h \sum_{l=0\vee -k}^{(n-k)\wedge(N-1)} hW_h(l) + \sigma\sqrt{h}\sum_{j=0}^{n}\xi(j+1), \\ &= \sum_{k=-N+1}^{-1} g(\psi_h(k))h \sum_{l=-k}^{(n-k)\wedge(N-1)} hW_h(l) \\ &\quad + \sum_{k=0}^{n} g(x_h(k))h \sum_{l=0}^{(n-k)\wedge(N-1)} hW_h(l) + \sigma\sqrt{h}\sum_{j=0}^{n}\xi(j+1).\end{aligned}$$

Therefore with

$$I(\psi_h, n) := \psi(0) + \sum_{k=-N+1}^{-1} g(\psi_h(k))h \sum_{l=-k}^{(n-k)\wedge(N-1)} hW_h(l), \quad n \geq 0, \tag{6.4.16}$$

we have

$$x_h(n+1) = I(\psi_h, n) + \sum_{k=0}^{n} g(x_h(k))h \sum_{l=0}^{(n-k)\wedge(N-1)} hW_h(l) + \sigma\sqrt{h}\sum_{j=0}^{n}\xi(j+1).$$

Define

$$W_h^*(j) = \sum_{l=0}^{j} hW_h(l), \quad j = 0, \dots, N-2. \tag{6.4.17}$$

Then we have for $1 \le N-1 \le n$

$$\begin{aligned}
x_h(n+1) &= I(\psi_h, n) + \sum_{k=0}^{n-N+1} g(x_h(k))h \sum_{l=0}^{(n-k)\wedge(N-1)} hW_h(l) \\
&\quad + \sum_{k=n-N+2}^{n} g(x_h(k))h \sum_{l=0}^{(n-k)\wedge(N-1)} hW_h(l) + \sigma\sqrt{h}\sum_{j=0}^{n} \xi(j+1), \\
&= I(\psi_h, n) + \sum_{k=0}^{n-N+1} g(x_h(k))h \sum_{l=0}^{N-1} hW_h(l) \\
&\quad + \sum_{k=n-N+2}^{n} g(x_h(k))h \sum_{l=0}^{n-k} hW_h(l) + \sigma\sqrt{h}\sum_{j=0}^{n} \xi(j+1), \\
&= I(\psi_h, n) + \sum_{k=n-N+2}^{n} g(x_h(k))h \sum_{l=0}^{n-k} hW_h(l) + \sigma\sqrt{h}\sum_{j=0}^{n} \xi(j+1), \\
&= I(\psi_h, n) + \sum_{j=0}^{N-2} g(x_h(n-j))hW_h^*(j) + \sigma\sqrt{h}\sum_{j=0}^{n} \xi(j+1).
\end{aligned}$$

For $n = 0, \dots, N-2$ we have

$$\begin{aligned}
x_h(n+1) &= I(\psi_h, n) + \sum_{j=0}^{n} g(x_h(n-j))h \sum_{l=0}^{j\wedge(N-1)} hW_h(l) + \sigma\sqrt{h}\sum_{j=0}^{n} \xi(j+1), \\
&= I(\psi_h, n) + \sum_{j=0}^{n} g(x_h(n-j))hW_h^*(j) + \sigma\sqrt{h}\sum_{j=0}^{n} \xi(j+1).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
x_h(n+1) &= I(\psi_h, n) + \sum_{j=0}^{N-2} g(x_h(n-j))hW_h^*(j) \\
&\qquad + \sigma\sqrt{h}\sum_{j=0}^{n} \xi(j+1), \quad n \ge N-1 \ge 1, \\
x_h(n+1) &= I(\psi_h, n) + \sum_{j=0}^{n} g(x_h(n-j))hW_h^*(j) \\
&\qquad + \sigma\sqrt{h}\sum_{j=0}^{n} \xi(j+1), \quad 0 \le n \le N-2.
\end{aligned}$$

Notice from (6.4.16) that $I(\psi_h, n)$ is equal to

$$I(\psi_h, n) = \psi(0) + \sum_{j=1}^{N-1} g(\psi_h(-j))h \sum_{l=j}^{(n+j)\wedge(N-1)} hW_h(l),$$

so that for $n \ge N-2$, we have

$$I(\psi_h, n) = \psi(0) + \sum_{j=1}^{N-1} g(\psi_h(-j))h \sum_{l=j}^{N-1} hW_h(l) =: I^*(\psi_h). \tag{6.4.18}$$

Hence $I(\psi_h, n)$ is constant for all $n \ge N-2$. This implies that

$$x_h(n+1) = I^*(\psi_h) + \sum_{j=0}^{N-2} g(x_h(n-j))hW_h^*(j) + \sigma\sqrt{h}\sum_{j=0}^{n} \xi(j+1), \quad 1 \le N-1 \le n, \tag{6.4.19}$$

and

$$x_h(n+1) = I(\psi_h, n) + \sum_{j=0}^{n} g(x_h(n-j))hW_h^*(j) + \sigma\sqrt{h}\sum_{j=0}^{n}\xi(j+1), \quad 0 \le n \le N-2. \quad (6.4.20)$$

We will observe in the next chapter that another discretisation of (6.4.3) gives rise to a Volterra summation equation of the form (6.4.19) and (6.4.20) for the discrete approximation x_h of X. In that method, we endow an analogue of the sequence W_h^* with a positivity property which does not change on compact intervals the order of the mean square of the sup norm of the approximation error. This positivity also enables us to recover the asymptotic behaviour of the continuous equation (6.4.3) for sufficiently small step size h irrespective of whether the stability condition (6.4.4) or the instability condition (6.4.5) holds.

In a following example, the discretisation (6.4.19), (6.4.20) in the case of the exponential weights defined in (6.4.12) gives rise to positive weights $W_h^*(j)$ for all $j \in \{0, \dots, N-2\}$. This positivity allows us to use analysis applied to discrete–time equations in previous chapters to determine the asymptotic behaviour of x_h, and to show that it is consistent with the asymptotic behaviour of the continuous time solution X of (6.4.3). This in part motivates the imposition of positivity on the analogue of the weights in the next chapter.

In the linear deterministic case when $g(x) = \beta x$, $\psi_h(n) = 0$ for $n < 0$, $\psi(0) = 1$ and $\sigma = 0$, we have from (6.4.16) that $I(\psi_h, n) = 1$ for all $n \ge 0$ and so

$$x_h(n+1) = 1 + \sum_{j=0}^{N-2} \beta x_h(n-j)hW_h^*(j), \quad 1 \le N-1 \le n,$$
$$x_h(n+1) = 1 + \sum_{j=0}^{n} \beta x_h(n-j)hW_h^*(j), \quad 0 \le n \le N-2.$$

Therefore

$$x_h(n+1) = 1 + \sum_{j=0}^{n\wedge(N-2)} \beta x_h(n-j)hW_h^*(j), \quad n \ge 0. \quad (6.4.21)$$

It is sometimes convenient for asymptotic analysis to extend $W_h^*(j) := 0$ for $N-1 \le j$. Then for $N-1 \le n$ (6.4.21) reads

$$\begin{aligned} x_h(n+1) &= 1 + \sum_{j=0}^{n\wedge(N-2)} \beta x_h(n-j)hW_h^*(j), \\ &= 1 + \sum_{j=0}^{N-2} \beta x_h(n-j)hW_h^*(j), \\ &= 1 + \sum_{j=0}^{N-2} \beta x_h(n-j)hW_h^*(j) + \sum_{j=N-1}^{n} \beta x_h(n-j)hW_h^*(j), \\ &= 1 + \sum_{j=0}^{n} \beta x_h(n-j)hW_h^*(j). \end{aligned}$$

Therefore, by extending W_h^* to $\mathbb{N} \setminus \{0, 1, \cdots, N-2\}$ as above, (6.4.21) is equivalent to

$$x_h(n+1) = 1 + \sum_{j=0}^{n} \beta x_h(n-j)hW_h^*(j), \quad n \ge 0. \quad (6.4.22)$$

Example 6.4.2. Since

$$W_1(h) = \sum_{j=0}^{N_1-1} C_1 e^{-jh}h, \quad W_2(h) = \sum_{j=0}^{N_2-1} C_2 e^{-jh}h.$$

we have

$$W_h(j) = \begin{cases} \left(\frac{1}{\sum_{l=0}^{N_1-1} e^{-lh}h} - \frac{1}{\sum_{l=0}^{N_2-1} e^{-lh}h}\right) e^{-jh}, & j \in \{0,\dots,N_1-1\}, \\ -\frac{e^{-jh}}{\sum_{l=0}^{N_2-1} e^{-lh}h}, & j \in \{N_1,\dots,N_2-1\}, \end{cases}$$

Therefore for $j \in \{0,\dots,N_1-1\}$ we have

$$W_h^*(j) = \sum_{l=0}^{j} hW_h(l) = \left(\frac{1}{\sum_{m=0}^{N_1-1} e^{-mh}} - \frac{1}{\sum_{m=0}^{N_2-1} e^{-mh}}\right) \sum_{l=0}^{j} e^{-lh} > 0.$$

On the other hand for $j \in \{N_1,\dots,N_2-2\}$ we have

$$\begin{aligned} W_h^*(j) &= \sum_{l=0}^{N_1-1} hW_h(l) + \sum_{l=N_1}^{j} hW_h(l), \\ &= \left(\frac{1}{\sum_{m=0}^{N_1-1} e^{-mh}} - \frac{1}{\sum_{m=0}^{N_2-1} e^{-mh}}\right) \sum_{l=0}^{N_1-1} e^{-lh} - \frac{1}{\sum_{m=0}^{N_2-1} e^{-mh}} \sum_{l=N_1}^{j} e^{-lh}, \\ &= 1 - \frac{\sum_{l=0}^{N_1-1} e^{-lh}}{\sum_{m=0}^{N_2-1} e^{-mh}} - \frac{\sum_{l=N_1}^{j} e^{-lh}}{\sum_{m=0}^{N_2-1} e^{-mh}} = 1 - \frac{\sum_{l=0}^{j} e^{-lh}}{\sum_{m=0}^{N_2-1} e^{-mh}} > 0. \end{aligned}$$

Therefore $W_h^*(j) > 0$ for all $j \in \{0,\dots,N-2\}$.

We now consider the long–run dynamics of $\sum_{j=0}^{N-2} hW_h^*(j)$ for these choices of w_1, w_2 and h. When we do simulations for this example it turns out that positivity is always ensured. In fact as the figure below shows the simulated W_h^* even has the property that it is increasing on $\{0,\dots,N_1-1\}$ and decreasing on $\{N_1,\dots,N_2-2\}$ just as the formula indicates.

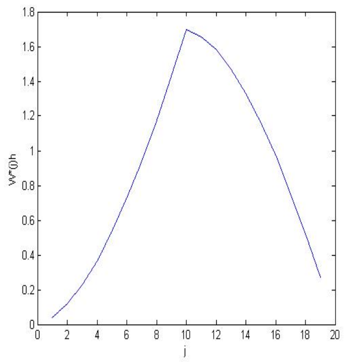

Figure 6.4: $\sum_{j=0}^{N-2} hW_h^*(j)$

Suppose that

$$\beta \sum_{j=0}^{N-2} W_h^*(j)h < 1. \tag{6.4.23}$$

Then we can use arguments of Chapter 2 to deduce that

$$\lim_{n\to\infty} x_h(n) = \frac{1}{1 - \beta\sum_{j=0}^{N-2} W_h^*(j)h}.$$

Simulations seem to confirm that this limit is attained under the condition (6.4.23). In fact, since we can show that

$$\lim_{h\to 0^+} \sum_{j=0}^{N-2} W_h^*(j)h = \frac{\int_0^{\tau_2} se^{-s}\,ds}{\int_0^{\tau_2} e^{-s}\,ds} - \frac{\int_0^{\tau_1} se^{-s}\,ds}{\int_0^{\tau_1} e^{-s}\,ds}, \tag{6.4.24}$$

it follows that the resolvent (which is the solution of (6.4.21)) approaches a limit as $n \to \infty$ that itself approaches a limit as $h \to 0^+$. Moreover, this latter limit is exactly that attained by the linear differential resolvent (6.4.7). Therefore, it can be seen that by increasing the computational effort (by reducing h), we obtain a better approximation of the asymptotic behaviour of the continuous time equation.

Suppose on the other hand that

$$\beta \sum_{j=0}^{N-2} W_h^*(j)h > 1. \tag{6.4.25}$$

Then we can use arguments of Chapter 4 to deduce that there is a unique $\alpha = \alpha(h) > 1$ such that

$$\beta \sum_{j=0}^{N-2} W_h^*(j)\alpha(h)^{-(j+1)}h = 1,$$

and that $\alpha(h)$ determines the asymptotic behaviour in the sense that

$$\lim_{n\to\infty} \frac{x_h(n)}{\alpha(h)^n} \quad \text{exists and is positive and finite.}$$

This is also confirmed by simulations in the case that (6.4.25) holds. In the next chapter, we will show that if we write $\log \alpha(h)/h = \lambda(h)$, then not only can the rate of growth of (6.4.21) be written as

$$\lim_{n\to\infty} \frac{x_h(n)}{e^{\lambda(h)nh}} \quad \text{which exists and is positive and finite,}$$

but

$$\lim_{h\to 0^+} \lambda(h) = \lambda > 0 \tag{6.4.26}$$

where λ is the unique positive solution of (6.4.6). Therefore, the rate of real exponential growth (or Liapunov exponent) predicted by the discrete scheme (6.4.21) converges (as the step size $h \to 0^+$) to the real exponential rate of growth of the linear differential resolvent x which solves (6.4.7), and obeys

$$\lim_{t\to\infty} \frac{x(t)}{e^{\lambda t}} \quad \text{exists and is finite.}$$

Simulations also seem to confirm (6.4.26); in the case where $\beta = 2e^2/(e-1)$, $\lambda = 1$. When we have taken $h = 0.001$, we find that $\hat{\lambda}(h) = 1.05$ after $5,000$ iterations and when we take $h = 0.0001$ we have $\hat{\lambda}(h) = 1.08$ after $30,000$ iterations where we have approximated the Liapunov exponent $\hat{\lambda}$ by

$$\hat{\lambda}(h) = \frac{1}{n_{\text{max}}h} \log x_h(n_{\text{max}})$$

where $n_{\text{max}} > N$ is the maximum number of iterations.

We now confirm that (6.4.24). We first compute

$$
\begin{aligned}
\sum_{j=0}^{N-2} W_h^*(j)h &= \sum_{j=0}^{N_1-1} W_h^*(j)h + \sum_{j=N_1}^{N_2-1} W_h^*(j)h, \\
&= \left(\frac{1}{\sum_{m=0}^{N_1-1} e^{-mh}} - \frac{1}{\sum_{m=0}^{N_2-1} e^{-mh}} \right) \sum_{j=0}^{N_1-1} \sum_{l=0}^{j} e^{-lh}h \\
&\quad + \sum_{j=N_1}^{N_2-1} \left(1 - \frac{\sum_{l=0}^{j} e^{-lh}}{\sum_{m=0}^{N_2-1} e^{-mh}} \right) h, \\
&= \left(\frac{1}{\sum_{m=0}^{N_1-1} he^{-mh}} - \frac{1}{\sum_{m=0}^{N_2-1} he^{-mh}} \right) \sum_{l=0}^{N_1-1} e^{-lh}h(N_1 - l)h \\
&\quad + \frac{1}{\sum_{m=0}^{N_2-1} e^{-mh}h} \sum_{j=N_1}^{N_2-1} \left(\sum_{l=0}^{N_2-1} e^{-lh} - \sum_{l=0}^{j} e^{-lh} \right) h^2, \\
&= \left(\frac{1}{\sum_{m=0}^{N_1-1} he^{-mh}} - \frac{1}{\sum_{m=0}^{N_2-1} he^{-mh}} \right) \sum_{l=0}^{N_1-1} e^{-lh}h(N_1h - lh) \\
&\quad + \frac{1}{\sum_{m=0}^{N_2-1} e^{-mh}h} \sum_{j=N_1}^{N_2-1} \sum_{l=j+1}^{N_2-1} e^{-lh}h^2, \\
&= \left(\frac{1}{\sum_{m=0}^{N_1-1} he^{-mh}} - \frac{1}{\sum_{m=0}^{N_2-1} he^{-mh}} \right) \sum_{l=0}^{N_1-1} e^{-lh}h(N_1h - lh) \\
&\quad + \frac{1}{\sum_{m=0}^{N_2-1} e^{-mh}h} \sum_{l=N_1+1}^{N_2-1} e^{-lh}h(lh - N_1h).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lim_{h\to 0^+} \sum_{j=0}^{N-2} W_h^*(j)h &= \left(\frac{1}{\int_0^{\tau_1} e^{-s}\,ds} - \frac{1}{\int_0^{\tau_2} e^{-s}\,ds} \right) \lim_{h\to 0^+} \sum_{l=0}^{N_1-1} e^{-lh}(\tau_1 - lh)h \\
&\quad + \frac{1}{\int_0^{\tau_2} e^{-s}\,ds} \lim_{h\to 0^+} \sum_{l=N_1+1}^{N_2-1} e^{-lh}h(lh - \tau_1).
\end{aligned}
$$

Next we have

$$
\sum_{l=0}^{N_1-1} e^{-lh}(\tau_1 - hl)h = \tau_1 \sum_{l=0}^{N_1-1} e^{-lh}h - \sum_{l=0}^{N_1-1} e^{-lh}lhh \to \tau_1 \int_0^{\tau_1} e^{-s}\,ds - \int_0^{\tau_1} e^{-s}s\,ds,
$$

and

$$
\begin{aligned}
\sum_{l=N_1+1}^{N_2-1} e^{-lh}h(lh - \tau_1) &= -\tau_1 \sum_{l=N_1+1}^{N_2-1} e^{-lh}h + \sum_{l=N_1+1}^{N_2-1} e^{-lh}hlh \\
&\to -\tau_1 \int_{\tau_1}^{\tau_2} e^{-s}\,ds + \int_{\tau_1}^{\tau_2} e^{-s}s\,ds.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\lim_{h\to 0^+} \sum_{j=0}^{N-2} W_h^*(j)h &= \left(\frac{1}{\int_0^{\tau_1} e^{-s}\,ds} - \frac{1}{\int_0^{\tau_2} e^{-s}\,ds} \right) \left(\tau_1 \int_0^{\tau_1} e^{-s}\,ds - \int_0^{\tau_1} e^{-s}s\,ds \right) \\
&\quad + \frac{1}{\int_0^{\tau_2} e^{-s}\,ds} \left(-\tau_1 \int_{\tau_1}^{\tau_2} e^{-s}\,ds + \int_{\tau_1}^{\tau_2} e^{-s}s\,ds \right).
\end{aligned}
$$

Hence

$$\lim_{h\to 0^+}\sum_{j=0}^{N-2} W_h^*(j)h = \frac{\tau_1\int_0^{\tau_1} e^{-s}\,ds}{\int_0^{\tau_1} e^{-s}\,ds} - \frac{\tau_1\int_0^{\tau_1} e^{-s}\,ds}{\int_0^{\tau_2} e^{-s}\,ds} - \frac{\int_0^{\tau_1} se^{-s}\,ds}{\int_0^{\tau_1} e^{-s}\,ds} + \frac{\int_0^{\tau_1} e^{-s}s\,ds}{\int_0^{\tau_2} e^{-s}\,ds}$$
$$- \frac{\tau_1\int_{\tau_1}^{\tau_2} e^{-s}\,ds}{\int_0^{\tau_2} e^{-s}\,ds} + \frac{\int_{\tau_1}^{\tau_2} e^{-s}s\,ds}{\int_0^{\tau_2} e^{-s}\,ds}.$$

Therefore we get

$$\lim_{h\to 0^+}\sum_{j=0}^{N-2} W_h^*(j)h = \frac{\int_0^{\tau_2} se^{-s}\,ds}{\int_0^{\tau_2} e^{-s}\,ds} - \frac{\int_0^{\tau_1} se^{-s}\,ds}{\int_0^{\tau_1} e^{-s}\,ds}.$$

Our analysis predicts in the case when $h = 0.1$ and $\beta = 1.2$ that $x_h(n) \to \frac{1}{1-\beta\sum_0^N hW_h^*(j)} = 1.4765153710910428$ and this seems to be confirmed by the figure below on the left hand side. On the other hand when $\beta = \frac{2e^2}{e-1}$ and $h = 0.1$ we expect that $x_h(n) \to Che^{\lambda(h)nh}$ as $n \to \infty$, where $\lambda(h) \to 1$ as $h \to 0^+$. Since we have chosen h relatively small we expect that $\lim_{n\to\infty}\frac{1}{nh}\log x_h(n) = \lambda(h)$ to be close to 1 which is consistent with the figure on the right which give $\lambda(0.0) = 1.03$.

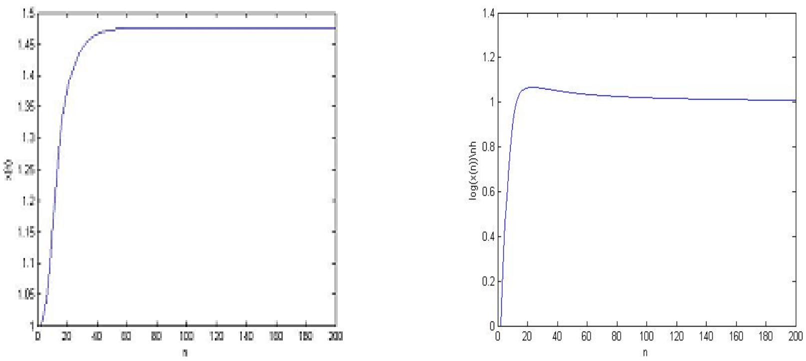

Figure 6.5: Stable Growth and Liapunov Exponent of Modified Euler

Chapter 7

Asymptotically Consistent Long–Run Behaviour from Numerical Methods for SFDEs with Continuous Weight Functions

7.1 Introduction

In this chapter we show that it is possible to perform a $h > 0$ uniform discretisation of the stochastic differential equation with memory given by

$$dX(t) = \left(\int_0^{\tau_1} w_1(s)g(X(t-s))\,ds - \int_0^{\tau_2} w_2(s)g(X(t-s))\,ds\right)dt + \sigma dB(t), \quad t \geq 0 \qquad (7.1.1)$$

in such a way that the asymptotic behaviour of the discretisation $(\widehat{X}_h(n))_{n\geq 0}$ captures that of the solution of (7.1.1). Here w_1 and w_2 are positive continuous functions with unit integrals over their domains of definition and which obey

$$\int_0^t w_1(s)\,ds \geq \int_0^t w_2(s)\,ds, \quad t \in [0, \tau_1]$$

with $w_1 \neq w_2$. Also g is asymptotically linear at infinity in the sense that there exists $\beta \geq 0$ such that

$$\lim_{x\to\infty} \frac{g(x)}{x} = \lim_{x\to-\infty} \frac{g(x)}{x} = \beta.$$

In the case when a stability condition holds, it can be shown that for all step sizes h sufficiently small that the discretisation is recurrent on $\mathbb{R}$ just as is the continuous process X, and enjoys the same iterated logarithmic large fluctuations as X, namely

$$\limsup_{n\to\infty} \frac{\widehat{X}_h(n)}{\sqrt{2(nh)\log\log(nh)}} = A(h) = -\liminf_{n\to\infty} \frac{\widehat{X}_h(n)}{\sqrt{2(nh)\log\log(nh)}}, \quad \text{a.s.}$$

Moreover, as $h \to 0^+$, the constant $A(h) \to A > 0$, which is exactly the normalising constant that appears in X

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = A = -\liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}}, \quad \text{a.s.}$$

Therefore, by increasing the computational effort the asymptotic behaviour of $\widehat{X}_h$ better approximates that of X.

In the case when the stability condition does not hold and an additional condition on $g(x)$ is imposed which restricts the degree of its asymptotic departure from βx as $|x| \to \infty$, we show that, provided h is sufficiently small, $\widehat{X}_h$ inherits the exact a.s. exponential rate of growth of X

$$\lim_{n\to\infty} \frac{\widehat{X}_h(n)}{e^{\lambda(h)nh}} = L(h), \quad \text{a.s.}$$

where $\lambda(h) \to \lambda^* > 0$ as $h \to 0^+$, and λ^* is the almost sure deterministic rate of growth of the solution X of (7.1.1) i.e.,

$$\lim_{t\to\infty} \frac{X(t)}{e^{\lambda^* t}} = L, \quad \text{a.s.}$$

The method used to discretise the problem is interesting because it can be shown that simple Euler discretisations of (7.1.1) do *not* preserve the asymptotic behaviour, at least in the case when the stability condition holds. Moreover, the discretisation also preserves the positivity and exponentially fading memory present in the autocovariance function of increments of the process X. In the final chapter we even show that the discretisation method outlined here obeys

$$\lim_{h\to 0^+} \mathbb{E}\left[\max_{0\le t\le T} |\bar{X}_h(t) - X(t)|^2\right] = 0, \quad \text{for any } T > 0, \tag{7.1.2}$$

where $\bar{X}_h$ is a piecewise constant process defined on $[0,T]$ for which $\bar{X}_h(t) = \widehat{X}_h([t/h])$ for $t \ge 0$. The condition (7.1.2) is enjoyed by conventional Euler–Maruyama methods for stochastic functional differential equations, so although our discretisation method is designed to reproduce asymptotic behaviour, it is sufficiently robust to also control the mean square of the error on any compact interval $[0,T]$.

7.1.1 Connection with dynamic consistency

The work in this chapter is in the theme of analysis suggested by Mickens on the *dynamic consistency* of finite difference methods for differential equations. Roughly speaking, the discretisation of a differential equation exhibits dynamic consistency if the original equation (or its solution) has a property P and the discretised equation (or its solution) also has property P. It is assumed that the system is convergent. In relation to our equation this refers to the Law of the Iterated Logarithm and bubble,crash dynamics. Some surveys of this work can be found in [56, 57, 59, 58]. Results pertaining to deterministic differential equations with delay are [60] and [44].

The emphasis in the work of Mickens and his school is, in the first instance, to recover property P in their discretisation: error control is considered secondary, although still of importance. This is our perspective here, as standard methods will guarantee strong convergence of a uniform mesh E-M scheme to the true solution: a new discretisation is only developed in order to recover the asymptotic behaviour of the continuous equation.

Our work differs from that of Mickens in some regards: he highlights, for instance, the importance of making non–standard (sometimes non–uniform) discretisations and of using implicit or semi–implicit methods. It turns out that we do not need to be so sophisticated here. Rather, our method hinges on rewriting the differential equation as an integral equation and using properties of the equation to explicitly remove certain terms from the discretisation.

Results in the spirit of dynamic consistency abound for stochastic differential equations without delay. However, there seems to be less literature for SDDEs. An example in which dynamic consistency is demonstrated is in Appleby and Kelly [8, 7], in which it is shown that correct discretisation of SDDEs with vanishing delay requires a non–uniform mesh and, in common with our analysis here, reformulates the differential equation before discretising it.

7.2 Conversion to Integral Equation

As in the previous chapter, suppose that $\tau := \tau_2 > \tau_1 > 0$ and that

$$w_1 \in C([0,\tau_1];[0,\infty)), \quad w_2 \in C([0,\tau_2];[0,\infty)) \tag{7.2.1}$$

We also request that

$$\int_0^{\tau_1} w_1(s)\,ds = 1, \quad \int_0^{\tau_2} w_2(s)\,ds = 1 \tag{7.2.2}$$

and that

$$\int_0^t w_1(s)\,ds \ge \int_0^t w_2(s)\,ds, \quad t\in[0,\tau_1]. \tag{7.2.3}$$

Assume that the function $g:\mathbb{R}\to\mathbb{R}$ obeys

$$g \text{ is locally Lipschitz continuous,} \tag{7.2.4}$$

and that there exists $\beta \ge 0$ such that

$$\lim_{x\to\infty}\frac{g(x)}{x} = \lim_{x\to-\infty}\frac{g(x)}{x} = \beta. \tag{7.2.5}$$

Let $\sigma \neq 0$ and B be a standard one–dimensional Brownian motion. Let $\psi \in C([-\tau, 0]; \mathbb{R})$. Then there is a unique continuous adapted process X which satisfies

$$dX(t) = \left(\int_0^{\tau_1} w_1(s) g(X(t-s))\, ds - \int_0^{\tau_2} w_2(s) g(X(t-s))\, ds \right) dt \tag{7.2.6a}$$
$$+\sigma dB(t), \quad t \geq 0,$$
$$X(t) = \psi(t), \qquad t \in [-\tau, 0]. \tag{7.2.6b}$$

We rewrite (7.2.6) as a Volterra integral equation. In doing so, we find it convenient to introduce some auxiliary functions. Define W_1, W_2 and W by

$$W_i(t) = \int_0^{t \wedge \tau_i} w_i(s)\, ds, \quad t \geq 0 \quad i = 1, 2, \quad W(t) := W_1(t) - W_2(t), \quad t \geq 0. \tag{7.2.7}$$

We also introduce the functions I_1 and I_2 which depend on the function ψ

$$I_i(\psi, t) = \int_{-\tau_i}^0 \left(\int_{-s}^{\tau_i \wedge (t-s)} w_i(u)\, du \right) g(\psi(s))\, ds, \quad t \geq 0, \quad i = 1, 2, \tag{7.2.8}$$

and the constants

$$I_i^*(\psi) = \int_{-\tau_i}^0 \left(\int_{-s}^{\tau_i} w_i(u)\, du \right) g(\psi(s))\, ds, \quad i = 1, 2. \tag{7.2.9}$$

Lemma 7.2.1. *Suppose that w_1 and w_2 obey* (7.2.1), (7.2.2) *and that g obeys* (7.2.4) *and* (7.2.5). *Then there is a unique continuous adapted process X which obeys* (7.2.6).

(i) $I_i(\psi, t) = I_i^*$, where $t \geq \tau_i$ and

(ii) If W is given by (7.2.7) *and I_i by* (7.2.8) *then X obeys*

$$X(t) = \psi(0) + I_1(\psi, t) - I_2(\psi, t) + \int_0^t W(s) g(X(t-s))\, ds + \sigma B(t), \quad t \geq 0,$$
$$X(t) = \psi(0) + I_1^*(\psi) - I_2^*(\psi) + \int_0^{\tau_2} W(s) g(X(t-s))\, ds + \sigma B(t), \quad t \geq \tau_2.$$

Remark 7.2.1. No outline of the proof is given because although the calculations are long they are straightforward.

7.3 Construction of the Numerical Scheme

We now discretise the Volterra integral equation just derived. Since w_1 and w_2 are continuous and defined on compact intervals, both possess moduli of continuity. More precisely there exist functions $\delta_1 : [0, \infty) \to [0, \infty)$ and $\delta_2 : [0, \infty) \to [0, \infty)$ such that $\delta_i(0) = 0$ and $\lim_{h \to 0^+} \delta_i(h) = 0$ for $i = 1, 2$ and

$$\max_{|t-s| \leq h, s, t \in [0, \tau_i]} |w_i(t) - w_i(s)| \leq \delta_i(h) \qquad \text{for all } h \in [0, \tau_i], \quad i = 1, 2. \tag{7.3.1}$$

We pick $h > 0$ so small that we may define $N_2 = N_2(h) \in \mathbb{N}$, $N_2 \geq 2$ such that

$$N_2 \leq \tau_2 / h < 1 + N_2. \tag{7.3.2}$$

Extend $w_1(t) = 0$ for $t \in [\tau_1, \tau_2]$. We now define the sequence $\widehat{W}_h$ parameterised $h > 0$ by

$$\widehat{W}_h(0) = \tau_2 (\delta_1(h) + \delta_2(h)) \tag{7.3.3a}$$
$$\widehat{W}_h(j) = \tau_2 (\delta_1(h) + \delta_2(h)) + \sum_{l=0}^{j-1} w_1(lh) h - \sum_{l=0}^{j-1} w_2(lh) h, \; j = 1, \cdots, N_2(h) \tag{7.3.3b}$$
$$\widehat{W}_h(N_2 + 1) = 0. \tag{7.3.3c}$$

It is implicit here that $\widehat{W}_h(n)$ is an approximation to $W(nh)$. However, in order to recover the positivity of W in the approximation, we have added a correction term to the naive approximation

$$\widehat{W}_{\text{naive}}(n) := \sum_{l=0}^{n-1} w_1(lh)h - \sum_{l=0}^{n-1} w_2(lh)h.$$

The following lemma shows that this can be achieved in such a way that any resulting biasses or errors introduced by the correction can be controlled. Also in the lemma, we record some estimates on the approximation of $\widehat{W}$ to W.

Lemma 7.3.1. *Let $h > 0$, and suppose that $\tau_2 > \tau_1$. Let w_i have modulus of continuity δ_i given by* (7.3.1). *Suppose that $N_2 = N_2(h)$ obeys* (7.3.2). *Define $\widehat{W}_h$ by* (7.3.3).

(i) For $j = 0, \dots, N_2$, $\widehat{W}_h(j) \geq 0$.

(ii) With $\overline{w}_i = \sup_{t\in[0,\tau_i]} w_i(t)$, we have

$$\left|\sum_{j=0}^{N_2} \widehat{W}_h(j)h - \left(\int_0^{\tau_2} sw_2(s)\,ds - \int_0^{\tau_1} sw_1(s)\,ds\right)\right| \leq \frac{3}{2}\tau_2^2\left(\delta_1(h) + \delta_2(h)\right) + h\left\{4 + \tau_2\left(\overline{w}_1 + \overline{w}_2\right) + 2\tau_2\left(\delta_1(h) + \delta_2(h)\right)\right\} := \eta(h). \quad (7.3.4)$$

(iii) If

$$\beta\left(\int_0^{\tau_2} sw_2(s)\,ds - \int_0^{\tau_1} sw_1(s)\,ds\right) < 1, \quad (7.3.5)$$

then there is $h_1 > 0$ which obeys

$$\beta\eta(h_1) = 1 - \beta\left(\int_0^{\tau_2} sw_2(s)\,ds - \int_0^{\tau_1} sw_1(s)\,ds\right), \quad (7.3.6)$$

such that for $h < h_1$ we have $\beta\sum_{j=0}^{N_2}\widehat{W}_h(j)h < 1$.

(iv) If

$$\beta\left(\int_0^{\tau_2} sw_2(s)\,ds - \int_0^{\tau_1} sw_1(s)\,ds\right) > 1, \quad (7.3.7)$$

then there is $h_2 > 0$ which obeys

$$\beta\eta(h_2) = \beta\left(\int_0^{\tau_2} sw_2(s)\,ds - \int_0^{\tau_1} sw_1(s)\,ds\right) - 1, \quad (7.3.8)$$

such that for $h < h_2$ we have $\beta\sum_{j=0}^{N_2}\widehat{W}_h(j)h > 1$.

Remark 7.3.1. Part (iii) of the above lemma is saying that if the stability parameter is less than one in the continuous case then this parameter will also hold for the discretised equation for small h. In other words the asymptotic behaviour is preserved for small h. The same holds for part (iv).

It is standard that

$$\lim_{h\to 0^+}\left|\sum_{j=0}^{N_2} \widehat{W}_h(j)h - \left(\int_0^{\tau_2} sw_2(s)\,ds - \int_0^{\tau_1} sw_1(s)\,ds\right)\right| = 0$$

thus ensuring the existence of h_1 and h_2 in parts (iii) and (iv). However, it is convenient to have an explicit bound on

$$\left|\sum_{j=0}^{N_2} \widehat{W}_h(j)h - \left(\int_0^{\tau_2} sw_2(s)\,ds - \int_0^{\tau_1} sw_1(s)\,ds\right)\right|$$

in terms of h so that we can determine a bound on a value of h which will guarantee that the asymptotic behaviour of the continuous–time process X given by (7.2.6) and its discrete approximation $\widehat{X}_h$. In the case when w_1 and w_2 are Lipschitz continuous there exists $K_3 > 0$ such that $|w_i(t) - w_i(s)| \leq K_3|t-s|$ for all $0 \leq s \leq t \leq \tau_i$. Therefore we can choose δ_i in such a way that $\delta_i(h) \leq K_3 h$ for $i = 1, 2$. Then we have

$$\eta(h) \leq h\left\{4 + 3\tau_2^2 K_3 + \tau_2\left(\overline{w}_1 + \overline{w}_2\right) + 4\tau_2 K_3 h\right\}, \tag{7.3.9}$$

From this estimate and parts (iii) and (iv) of Lemma 7.3.1 we can deduce an explicit estimate on the critical values of h (h_1 if (7.3.5) holds and h_2 if (7.3.7) holds) because the righthand side is a quadratic in h. Indeed it can even be estimated by a linear function in h, because the requirement that $N_2 \geq 2$ and (7.3.2) forces $h \leq \tau_2/2$.

Remark 7.3.2. Part (i) of the Lemma is proved by using the properties of the moduli of continuity. We prove part (ii) by deriving an equation for $\sum_{j=0}^{N_2} \widehat{W}_h(j)h - \int_0^{\tau_2} W(s)\,ds$ which comprises of five terms. We simplify these five terms further to obtain the desired result. Parts (iii) and (iv) are proved by exploiting the continuity and the monotonicity of the moduli of continuity.

We are now in a position to discretise X. Let $\widehat{W}_h$ be defined by (7.3.3). Suppose that

$$(\xi(n))_{n\geq 1} \text{ is a sequence of i.i.d.} N(0,1) \text{ random variables.} \tag{7.3.10}$$

We suppose for $n \geq 0$ that $\widehat{X}(n)$ is an approximation for $X(nh)$. Suppose we approximate $I_i(\psi, nh)$ by $I_i^*(\psi, n)$. Define $(\widehat{X}_h(n))_{n\geq 0}$ by

$$\widehat{X}_h(n+1) = \psi(0) + I_1^*(\psi, n) - I_2^*(\psi, n) + \sum_{j=0}^{n} \widehat{W}_h(j) g(\widehat{X}_h(n-j))h \tag{7.3.11a}$$

$$+ \sigma\sqrt{h}\sum_{j=0}^{n} \xi(j+1), \qquad n = 0, \cdots, N_2 - 1,$$

$$\widehat{X}_h(n+1) = \psi(0) + I_1^*(\psi) - I_2^*(\psi) + \sum_{j=0}^{N_2} \widehat{W}_h(j) g(\widehat{X}_h(n-j))h \tag{7.3.11b}$$

$$+ \sigma\sqrt{h}\sum_{j=0}^{n} \xi(j+1), \qquad n \geq N_2,$$

where $\widehat{X}_h(0) = \psi(0)$.

7.4 Statement and Discussion of Main Results for Discrete Equation

It is convenient to introduce the discrete resolvent $\widehat{r}$ defined by

$$\widehat{r}(n+1) = \beta\sum_{j=0}^{N_2} \widehat{W}_h(j) h \widehat{r}(n-j), \quad n \geq 0, \tag{7.4.1a}$$

$$\widehat{r}(0) = 1; \qquad \widehat{r}(n) = 0, \qquad n = -N_2, -N_2+1, \ldots, -1. \tag{7.4.1b}$$

We can prove the following result concerning asymptotic behaviour of solutions of (7.3.11) under the condition (7.3.5).

Theorem 7.4.1. *Let $h > 0$, $\tau_2 > \tau_1 > 0$ and define N_2 by* (7.3.2)*. Suppose that w_1 and w_2 obey* (7.2.1)*,* (7.2.3) *and* (7.2.3)*, and that g obeys* (7.2.4) *and* (7.2.5)*. Suppose $\beta > 0$ is such that* (7.3.5) *holds. Let $h_1 > 0$ be defined by* (7.3.6) *and suppose that $h < h_1$. Suppose that $\widehat{X}_h$ is defined by* (7.3.11)*. Then*

$$\limsup_{n\to\infty} \frac{\widehat{X}_h(n)}{\sqrt{2(nh)\log_2(nh)}} = \frac{|\sigma|}{1 - \beta\sum_{j=0}^{N_2}\widehat{W}_h(j)h} =: A(h), \quad \textit{a.s.}$$

$$\liminf_{n\to\infty} \frac{\widehat{X}_h(n)}{\sqrt{2(nh)\log_2(nh)}} = -\frac{|\sigma|}{1 - \beta\sum_{j=0}^{N_2}\widehat{W}_h(j)h}, \quad \textit{a.s.}$$

Moreover

$$\lim_{h\to 0^+} A(h) = \frac{\sigma}{1-\beta\left(\int_0^{\tau_2} sw_2(s)\,ds - \int_0^{\tau_1} sw_1(s)\,ds\right)}.$$

Remark 7.4.1. First we prove the result for the linear equation. Then we rewrite $\widehat{X}_h(n) = \widehat{Z}_h(n) + \widehat{Y}_h(n)$ where $\widehat{Z}_h(n)$ is a bounded function whose limit tends to zero. Subsequently the limit of the nonlinear equation is equal to the limit of the linear equation.

This result is consistent with the asymptotic behaviour of the solution X of (7.2.6); Not only does Theorem 7.4.1 show that $\widehat{X}_h$ mimics the almost sure iterated logarithmic rate of growth of the partial maxima of X, but that the normalising constant $A(h)$ converges as $h \to 0^+$ to the normalising constant on the righthand side of the continuous process X. The relevant continuous–time result appears in Chapter 4, and is restated here for ease of comparison.

Theorem 7.4.2. *Let $\tau_2 > \tau_1 > 0$. Suppose that w_1 and w_2 obey* (7.2.1), (7.2.3) *and* (7.2.3)*, and that g obeys* (7.2.4) *and* (7.2.5)*. Suppose $\beta > 0$ is such that* (7.3.5) *holds. Then the solution X of* (7.2.6) *satisfies*

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = \frac{|\sigma|}{1-\beta\left(\int_0^{\tau_2} sw_2(s)\,ds - \int_0^{\tau_1} sw_1(s)\,ds\right)}, \quad a.s.$$
$$\liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = -\frac{|\sigma|}{1-\beta\left(\int_0^{\tau_2} sw_2(s)\,ds - \int_0^{\tau_1} sw_1(s)\,ds\right)}, \quad a.s.$$

In the case when β obeys (7.3.7), we have that $\widehat{X}_h$ grows at an exact a.s. exponentially rate; moreover the rate at which $\widehat{X}_h$ grows converges to a limit as $h \to 0^+$

Theorem 7.4.3. *Let $h > 0$, $\tau_2 > \tau_1 > 0$ and define N_2 by* (7.3.2)*. Suppose that w_1 and w_2 obey* (7.2.1), (7.2.3) *and* (7.2.3)*, and that g obeys* (7.2.4) *and* (7.2.5) *and also*

$$\int_1^{\infty} \frac{\max_{|s|\le x} |g(s) - \beta s|}{x^2}\,dx < +\infty. \tag{7.4.2}$$

Suppose $\beta > 0$ is such that (7.3.7) *holds. Let $h_2 > 0$ be defined by* (7.3.8) *and suppose that $h < h_2$. Suppose that $\widehat{X}_h$ is defined by* (7.3.11)*. Then there exists a $\lambda(h) > 0$ and a finite $\mathcal{F}^{\xi}(\infty)$–measurable random variable $L^*(h)$ such that*

$$\lim_{n\to\infty} \frac{\widehat{X}_h(n)}{e^{\lambda(h)nh}} = L^*(h), \quad a.s. \tag{7.4.3}$$

Moreover $\lambda(h) > 0$ obeys

$$\lim_{h\to 0^+} \lambda(h) = \lambda^*, \tag{7.4.4}$$

where $\lambda^* > 0$ is the unique positive solution of

$$\lambda^* = \beta\left(\int_0^{\tau_1} w_1(s)e^{-\lambda^* s}\,ds - \int_0^{\tau_2} e^{-\lambda^* s} w_2(s)\,ds\right). \tag{7.4.5}$$

The asymptotic behaviour of $\widehat{X}_h$ is consistent with that the solution X of (7.2.6), in that X grows at an exact a.s. exponential rate provided the nonlinearity g satisfies (7.4.2) and that the discrete exponential a.s. rate of growth $\lambda(h)$ of $\widehat{X}_h$ converges to the almost sure exponential rate of growth $\lambda^* > 0$ of X. Once again, the corresponding continuous–time result is proven in Chapter 4, and is restated here to aid comparison.

Theorem 7.4.4. *Let $\tau_2 > \tau_1 > 0$. Suppose that w_1 and w_2 obey* (7.2.1), (7.2.3) *and* (7.2.3)*, and that g obeys* (7.2.4) *and* (7.2.5) *and also* (7.4.2)*. Suppose $\beta > 0$ is such that* (7.3.7) *holds. Then the solution X of* (7.2.6) *satisfies*

$$\lim_{t\to\infty} \frac{X(t)}{e^{\lambda^* t}} = L, \quad a.s. \tag{7.4.6}$$

where $\lambda^* > 0$ is the unique positive solution of (7.4.5) *and L is an a.s. finite and $\mathcal{F}^B(\infty)$–measurable random variable.*

Remark 7.4.2. To prove Theorem 7.4.3 we first show that $|\widehat{X}_h|$ is bounded by the function x^*. We then prove that the limit of x^* is finite and subsequently the limit of $|\widehat{X}_h|$ is finite.

7.5 Proofs of Supporting Results

7.5.1 Proof of Lemma 7.2.1

Written in integral form (7.2.6a) is

$$X(t) = \psi(0) + \int_0^t \int_0^{\tau_1} w_1(u) g(X(s-u))\,du\,ds - \int_0^t \int_0^{\tau_2} w_2(u) g(X(s-u))\,du\,ds \\ + \sigma B(t), \quad t \ge 0.$$

Let $t \ge \tau_i$,

$$\begin{aligned}\int_0^t \int_0^{\tau_i} w_i(u) g(X(s-u))\,du\,ds &= \int_0^t \int_{s-\tau_i}^{s} w_i(s-v) g(X(v))\,dv\,ds \\ &= \int_0^{\tau_i} \int_{s-\tau_i}^{s} w_i(s-v) g(X(v))\,dv\,ds + \int_{\tau_i}^t \int_{s-\tau_i}^{s} w_i(s-v) g(X(v))\,dv\,ds.\end{aligned}$$

Now

$$\begin{aligned}&\int_{\tau_i}^t \int_{s-\tau_i}^{s} w_i(s-v) g(X(v))\,dv\,ds \\ &= \int_0^t \int_{v\vee\tau_i}^{(v+\tau_i)\wedge t} w_i(s-v)\,ds\, g(X(v))\,dv \\ &= \int_0^t \left(\int_{0\vee(\tau_i-v)}^{\tau_i\wedge(t-v)} w_i(u)\,du \right) g(X(v))\,dv \\ &= \int_0^{\tau_i} \left(\int_{0\vee(\tau_i-v)}^{\tau_i\wedge(t-v)} w_i(u)\,du \right) g(X(v))\,dv + \int_{\tau_i}^t \left(\int_{0\vee(\tau_i-v)}^{\tau_i\wedge(t-v)} w_i(u)\,du \right) g(X(v))\,dv \\ &= \int_0^{\tau_i} \left(\int_{\tau_i-v}^{\tau_i\wedge(t-v)} w_i(u)\,du \right) g(X(v))\,dv + \int_{\tau_i}^t \left(\int_{0}^{\tau_i\wedge(t-v)} w_i(u)\,du \right) g(X(v))\,dv.\end{aligned}$$

On the other hand

$$\begin{aligned}&\int_0^{\tau_i} \int_{s-\tau_i}^{s} w_i(s-v)\, g(X(v))\,dv\,ds \\ &= \int_{-\tau_i}^{\tau_i} \int_{v\vee 0}^{(v+\tau_i)\wedge\tau_i} w_i(s-v)\,ds g(X(v))\,dv \\ &= \int_{-\tau_i}^{0} \left(\int_{0}^{(v+\tau_i)\wedge\tau_i} w_i(s-v)\,ds \right) g(\psi(v))\,dv \\ &\qquad + \int_{0}^{\tau_i} \left(\int_{v}^{(v+\tau_i)\wedge\tau_i} w_i(s-v)\,ds \right) g(X(v))\,dv \\ &= \int_{-\tau_i}^{0} \left(\int_{0}^{v+\tau_i} w_i(s-v)\,ds \right) g(\psi(v))\,dv + \int_{0}^{\tau_i} \left(\int_{v}^{\tau_i} w_i(s-v)\,ds \right) g(X(v))\,dv \\ &= \int_{-\tau_i}^{0} \left(\int_{-v}^{\tau_i} w_i(u)\,du \right) g(\psi(v))\,dv + \int_{0}^{\tau_i} \left(\int_{0}^{\tau_i-v} w_i(u)\,du \right) g(X(v))\,dv.\end{aligned}$$

Therefore for $t \geq \tau_i$

$$\int_0^t \int_0^{\tau_i} w_i(u) g(X(s-u))\,du\,ds = I_i^*(\psi) + \int_0^{\tau_i} \left(\int_0^{\tau_i \wedge (t-v)} w_i(u)\,du \right) g(X(v))\,dv$$
$$+ \int_{\tau_i}^t \left(\int_0^{\tau_i \wedge (t-v)} w_i(u)\,du \right) g(X(v))\,dv$$
$$= I_i^*(\psi) + \int_0^t W_i(t-v) g(X(v))\,dv$$
$$= I_i^*(\psi) + \int_0^t W_i(s) g(X(t-s))\,ds.$$

Let $t \in [0, \tau_i]$, then

$$\int_0^t \int_{s-\tau_i}^s w_i(s-v) g(X(v))\,dv\,ds$$
$$= \int_{-\tau_i}^t \left(\int_{v\vee 0}^{(v+\tau_i)\wedge t} w_i(s-v)\,ds \right) g(X(v))\,dv$$
$$= \int_{-\tau_i}^0 \left(\int_{v\vee 0}^{(v+\tau_i)\wedge t} w_i(s-v)\,ds \right) g(\psi(v))\,dv$$
$$+ \int_0^t \left(\int_{v\vee 0}^{(v+\tau_i)\wedge t} w_i(s-v)\,ds \right) g(X(v))\,dv$$
$$= \int_{-\tau_i}^0 \left(\int_0^{(v+\tau_i)\wedge t} w_i(s-v)\,ds \right) g(\psi(v))\,dv + \int_0^t \left(\int_v^t w_i(s-v)\,ds \right) g(X(v))\,dv$$
$$= \int_{-\tau_i}^0 \left(\int_{-v}^{(t-v)\wedge \tau_i} w_i(u)\,du \right) g(\psi(v))\,dv + \int_0^t \left(\int_0^{t-v} w_i(u)\,du \right) g(X(v))\,dv$$
$$= I_i(\psi, t) + \int_0^t \left(\int_0^{t-v} w_i(u)\,du \right) g(X(v))\,dv,$$

because $t - v \leq t \leq \tau_i$ for $v \in [0, t]$ and $t \leq \tau_i$, then

$$\int_0^t \int_{s-\tau_i}^s w_i(s-v) g(X(v))\,dv\,ds = I_i(\psi, t) + \int_0^t W_i(s) g(X(t-s))\,ds.$$

For $t \geq \tau_i$, $t - s \geq \tau_i$, if $s \in [-\tau_i, 0]$, hence $\tau_i \wedge (t-s) = \tau_i$, so $I_i(\psi, t) = I_i^*(\psi)$. For $t \geq 0$

$$\int_0^t \int_0^{\tau_i} w_i(u) g(X(s-u))\,du\,ds = I_i^*(\psi, t) + \int_0^t W_i(s) g(X(t-s))\,ds.$$

Therefore

$$X(t) = \psi(0) + I_1(\psi, t) + \int_0^t W_1(s) g(X(t-s))\,ds - I_2(\psi, t)$$
$$- \int_0^t W_2(s) g(X(t-s))\,ds + \sigma B(t)$$

which proves (ii). Let $t \geq \tau_2$, then

$$X(t) = \psi(0) + I_1^*(\psi) - I_2^*(\psi) + \int_0^t \left(W_1(s) - W_2(s)\right) g(X(t-s))\,ds + \sigma B(t)$$
$$= \psi(0) + I_1^*(\psi) - I_2^*(\psi) + \int_0^{\tau_2} \left(W_1(s) - W_2(s)\right) g(X(t-s))\,ds$$
$$+ \sigma\, B(t) + \int_{\tau_2}^t \left(W_1(s) - W_2(s)\right) g(X(t-s))\,ds.$$

For $t \geq \tau_2$, $W_i(t) = \int_0^{t\wedge\tau_i} w_i(s)\,ds = \int_0^{\tau_i} w_i(s)\,ds = 1$, then for $t \geq \tau$

$$X(t) = \psi(0) + I_1^*(\psi) - I_2^*(\psi) + \int_0^{\tau_2} (W_1(s) - W_2(s))\, g(X(t-s))\,ds + \sigma B(t),$$

which completes the proof.

7.5.2 Proof of Lemma 7.3.1

Clearly $\widehat{W}_h(0) \geq 0$, and $\widehat{W}_h(N_2+1) = 0$ by (7.3.3c). For $j = 1, \ldots, N_2$ we have $0 \leq j-1 \leq N_2 - 1$. Therefore $0 \leq (j-1)h \leq (N_2-1)h \leq N_2 h \leq \tau_2$ and both terms on the right hand side of (7.3.3b) are well–defined. For each $j = 1, \ldots, N_2$ we have $jh \leq N_2 h \leq \tau_2$, so for $i = 1, 2$

$$\begin{aligned}\left|\int_0^{jh} w_i(s)\,ds - \sum_{l=0}^{j-1} w_i(lh)h\right| &= \left|\sum_{l=0}^{j-1}\left(\int_{lh}^{(l+1)h} w_i(s)\,ds - \int_{lh}^{(l+1)h} w_i(lh)\,ds\right)\right| \\ &\leq \sum_{l=0}^{j-1}\int_{lh}^{(l+1)h} |w_i(s) - w_i(lh)|\,ds \\ &\leq \sum_{l=0}^{j-1} h\delta_i(h) \leq \sum_{l=0}^{N_2-1} h\delta_i(h) = \delta_i(h)hN_2 \leq \delta_i(h)\tau_2.\end{aligned}$$

Therefore

$$\left|\int_0^{jh} w_1(s)\,ds - \sum_{l=0}^{j-1} w_1(lh)h\right| \leq \delta_1(h)\tau_2, \qquad j = 1, \ldots, N_2, \tag{7.5.1}$$

$$\left|\int_0^{jh} w_2(s)\,ds - \sum_{l=0}^{j-1} w_2(lh)h\right| \leq \delta_2(h)\tau_2, \qquad j = 1, \ldots, N_2. \tag{7.5.2}$$

Hence we have

$$\begin{aligned}\sum_{l=0}^{j-1} w_1(lh)h - \sum_{l=0}^{j-1} w_2(lh)h &\geq \int_0^{jh} w_1(s)\,ds - \delta_1(h)\tau_2 - \int_0^{jh} w_2(s)\,ds - \delta_2(h)\tau_2 \\ &= \int_0^{jh} w_1(s)\,ds - \int_0^{jh} w_2(s)\,ds - \tau_2(\delta_1(h) + \delta_2(h)).\end{aligned}$$

Therefore for $j = 1, \ldots, N_2$ as $jh \in [0, N_2 h] \subseteq [0, \tau_2]$ by (7.3.3b) and (7.2.3) we have

$$\widehat{W}_h(j) \geq \int_0^{jh} w_1(s)\,ds - \int_0^{jh} w_2(s)\,ds \geq 0,$$

as required. To prove part (ii), define $w(t) := w_1(t) - w_2(t)$ for $t \in [0, \tau]$, and note from (7.3.3) that

$$\begin{aligned}&\sum_{j=0}^{N_2} \widehat{W}_h(j)h - \int_0^{\tau_2} W(s)\,ds \\ &= \sum_{j=0}^{N_2} h\tau_2(\delta_1(h) + \delta_2(h)) + \sum_{j=1}^{N_2-1} h \sum_{l=0}^{j-1} w(lh)h - \sum_{j=1}^{N_2-1}\int_{jh}^{(j+1)h} W(s)\,ds \\ &\qquad + h\sum_{l=0}^{N_2-1} w(lh)h - \int_0^h W(s)\,ds - \int_{N_2 h}^{\tau_2} W(s)\,ds \\ &= (N_2+1)h\tau_2(\delta_1(h) + \delta_2(h)) + \sum_{j=1}^{N_2-1}\int_{jh}^{(j+1)h}\left(\sum_{l=0}^{j-1} w(lh)h - W(s)\right)ds \\ &\qquad + h\sum_{l=0}^{N_2-1} w(lh)h - \int_0^h W(s)\,ds - \int_{N_2 h}^{\tau_2} W(s)\,ds.\end{aligned}$$

Now we estimate the first term and the last three terms on the righthand side. By (7.3.2) we have

$$(N_2+1)h\tau_2\left(\delta_1(h)+\delta_2(h)\right) \le (\tau_2+h)\,\tau_2\left(\delta_1(h)+\delta_2(h)\right). \tag{7.5.3}$$

Next by (7.5.1), (7.5.2), (7.3.2) and (7.2.2) we have

$$\begin{aligned}\left|\sum_{l=0}^{N_2-1} w(lh)h\right| &= \left|\sum_{l=0}^{N_2-1} w_1(lh)h - \sum_{l=0}^{N_2-1} w_2(lh)h\right| \\ &\le \sum_{l=0}^{N_2-1} w_1(lh)h + \sum_{l=0}^{N_2-1} w_2(lh)h \\ &\le \int_0^{N_2h} w_1(s)\,ds + \tau_2\delta_1(h) + \int_0^{N_2h} w_2(s)\,ds + \tau_2\delta_1(h) \\ &\le 2+\tau_2(\delta_1(h)+\delta_2(h)).\end{aligned} \tag{7.5.4}$$

Now $W(t) = W_1(t) - W_2(t) = \int_0^{t\wedge\tau_1} w_1(s)\,ds - \int_0^{t\wedge\tau_2} w_2(s)\,ds$ so $W(t) \le 1$. Hence

$$\int_0^h W(s)\,ds \le h. \tag{7.5.5}$$

Also as N_2 obeys (7.3.2) we have

$$\int_{N_2h}^{\tau_2} W(s)\,ds \le \tau_2 - N_2h < (1+N_2)h - N_2h = h. \tag{7.5.6}$$

Therefore by (7.5.3), (7.5.4), (7.5.5) and (7.5.6) we have

$$\begin{aligned}\left|\sum_{j=0}^{N_2} \widehat{W}_h(j)h - \int_0^{\tau_2} W(s)\,ds\right| &\le (\tau_2+h)\tau_2\left(\delta_1(h)+\delta_2(h)\right) \\ &+ h\left(2+\tau_2(\delta_1(h)+\delta_2(h))\right) + 2h + \left|\sum_{j=1}^{N_2-1}\int_{jh}^{(j+1)h}\left(\sum_{l=0}^{j-1} w(lh)h - \int_0^s w(u)\,du\right)ds\right|.\end{aligned} \tag{7.5.7}$$

We estimate the last term on the righthand side of (7.5.7). For $s \in [jh,(j+1)h]$

$$\begin{aligned}&\sum_{l=0}^{j-1} w(lh)h - \int_0^s w(u)\,du \\ &= \sum_{l=0}^{j-1}\int_{lh}^{(l+1)h} w(lh)\,du - \sum_{l=0}^{j-1}\int_{lh}^{(l+1)h} w(u)\,du - \int_{jh}^s w(u)\,du \\ &= \sum_{l=0}^{j-1}\int_{lh}^{(l+1)h}\left(w(lh)-w(u)\right)du - \int_{jh}^s w(u)\,du.\end{aligned}$$

Hence for $s \in [jh,(j+1)h]$

$$\left|\sum_{l=0}^{j-1} w(lh)h - \int_0^s w(u)\,du\right| \le \sum_{l=0}^{j-1}\int_{lh}^{(l+1)h} |w(lh)-w(u)|\,du + \int_{jh}^s |w(u)|\,du.$$

Now for $u \in [lh,(l+1)h]$ by (7.3.1) we have

$$|w(lh)-w(u)| \le |w_1(lh)-w_1(u)| + |w_2(lh)-w_2(u)| \le \delta_1(h)+\delta_2(h).$$

Therefore for $s \in [jh,(j+1)h]$

$$\begin{aligned}\left|\sum_{l=0}^{j-1} w(lh)h - \int_0^s w(u)\,du\right| &\le jh\left(\delta_1(h)+\delta_2(h)\right) + \int_{jh}^s w_1(u)\,du + \int_{jh}^s w_2(u)\,du \\ &\le jh\left(\delta_1(h)+\delta_2(h)\right) + \left(\overline{w}_1+\overline{w}_2\right)h,\end{aligned}$$

where we have defined $\overline{w}_i = \sup_{t\in[0,\tau_i]} w_i(t)$. Therefore

$$\begin{aligned}
&\left|\sum_{j=1}^{N_2-1}\int_{jh}^{(j+1)h}\left(\sum_{l=0}^{j-1} w(lh)h - \int_0^s w(u)\,du\right)\,ds\right| \\
&\qquad\le \sum_{j=1}^{N_2-1}\int_{jh}^{(j+1)h}\left|\sum_{l=0}^{j-1} w(lh)h - \int_0^s w(u)\,du\right|\,ds \\
&\qquad\le \sum_{j=1}^{N_2-1}\int_{jh}^{(j+1)h} jh\left(\delta_1(h)+\delta_2(h)\right) + (\overline{w_1}+\overline{w_2})h\,ds \\
&\qquad\le \sum_{j=1}^{N_2-1}\left\{jh^2\left(\delta_1(h)+\delta_2(h)\right) + \left(\overline{w}_1+\overline{w}_2\right)h^2\right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left|\sum_{j=1}^{N_2-1}\int_{jh}^{(j+1)h}\left(\sum_{l=0}^{j-1} w(lh)h - \int_0^s w(u)\,du\right)\,ds\right| \\
&\qquad\qquad\le h^2\left(\delta_1(h)+\delta_2(h)\right)\sum_{j=1}^{N_2-1} j + (N_2-1)h\left(\overline{w}_1+\overline{w}_2\right)h. \quad (7.5.8)
\end{aligned}$$

Thus by (7.5.7) and (7.5.8) we have

$$\begin{aligned}
&\left|\sum_{j=0}^{N_2}\widehat{W}_h(j)h - \int_0^{\tau_2} W(s)\,ds\right| \\
&\le (\tau_2+h)\tau_2\left(\delta_1(h)+\delta_2(h)\right) + h\left(4+\tau_2(\delta_1(h)+\delta_2(h))\right) \\
&\quad + h^2\left(\delta_1(h)+\delta_2(h)\right)\frac{N_2(N_2-1)}{2} + (N_2-1)h\left(\overline{w}_1+\overline{w}_2\right)h \\
&= (\tau_2^2+h\tau_2)\left(\delta_1(h)+\delta_2(h)\right) + 4h + \tau_2 h(\delta_1(h)+\delta_2(h)) \\
&\quad + \left(\delta_1(h)+\delta_2(h)\right)\frac{N_2h(N_2h-h)}{2} + (N_2h-h)\left(\overline{w}_1+\overline{w}_2\right)h \\
&\le \left(\frac{3}{2}\tau_2^2 + 2h\tau_2\right)\left(\delta_1(h)+\delta_2(h)\right) + 4h + \tau_2\left(\overline{w}_1+\overline{w}_2\right)h.
\end{aligned}$$

Therefore we have (7.3.4). Because we can deduce the identity

$$\begin{aligned}
\int_0^{\tau_2} W(s)\,ds &= \int_0^{\tau_2}\int_0^{s\wedge\tau_1} w_1(u)\,du\,ds - \int_0^{\tau_2}\int_0^{s} w_2(u)\,du\,ds \\
&= \int_0^{\tau_1}\int_u^{\tau_1} w_1(u)\,ds\,du + \int_{\tau_1}^{\tau_2} 1\,ds - \int_0^{\tau_2}\int_u^{\tau_2} w_2(u)\,ds\,du \\
&= \int_0^{\tau_1}(\tau_1-u)w_1(u)\,du + \tau_2 - \tau_1 - \int_0^{\tau_2}(\tau_2-u)w_2(u)\,du \\
&= \int_0^{\tau_2} uw_2(u)\,du - \int_0^{\tau_1} uw_1(u)\,du,
\end{aligned}$$

using the fact that $\int_0^{\tau_i} w_i(u)\,du = 1$, together with the continuity of w_1 and w_2 and the fact that $w_1(t) = 0$ for $t \in (\tau_1, \tau_2]$. Suppose that (7.3.5) holds. We note that the moduli of continuity are non–decreasing. Therefore $h \mapsto \eta(h)$ is an increasing function. Since δ_i is continuous, we have that η is continuous and we also have $\eta(0+) = 0$. Therefore there exists $h_1 > 0$ which obeys (7.3.6). Hence for $h < h_1$, by (7.3.6) we have

$$\beta \sum_{j=0}^{N_2} \widehat{W}_h(j)h \le \beta \left(\int_0^{\tau_2} s w_2(s)\, ds - \int_0^{\tau_1} s w_1(s)\, ds \right) + \beta\eta(h)$$
$$< \beta \left(\int_0^{\tau_2} s w_2(s)\, ds - \int_0^{\tau_1} s w_1(s)\, ds \right) + \beta\eta(h_1) = 1.$$

This proves part (iii). To prove part (iv), suppose that (7.3.7) holds. Once again by the continuity and monotonicity of the moduli of continuity δ_1 and δ_2 there exists $h_2 > 0$ such that h_2 obeys (7.3.8). Since $h \mapsto \eta(h)$ is an increasing function, by (7.3.8) for $h < h_2$ we have

$$\beta \sum_{j=0}^{N_2} \widehat{W}_h(j)h \ge \beta \left(\int_0^{\tau_2} s w_2(s)\, ds - \int_0^{\tau_1} s w_1(s)\, ds \right) - \beta\eta(h)$$
$$> \beta \left(\int_0^{\tau_2} s w_2(s)\, ds - \int_0^{\tau_1} s w_1(s)\, ds \right) + \beta\eta(h_1) = 1,$$

which proves part (iv).

7.6 Proof of Theorem 7.4.1

The asymptotic analysis of the nonlinear equation (7.3.11) is facilitated by studying an auxiliary linear equation. To this end we define $c = \psi(0) + \widehat{I}_1^*(\psi) - \widehat{I}_2^*(\psi)$

7.6.1 Linear stochastic equation

$$\widehat{Y}(n+1) = c + \sum_{j=0}^{N_2} \beta \widehat{W}_h(j) h \widehat{Y}(n-j) + \sum_{j=0}^{n} \xi_h(j+1), \qquad n \ge N_2, \tag{7.6.1a}$$
$$\widehat{Y}(n) = \widehat{X}_h(n), \qquad n = 0, \dots, N_2, \tag{7.6.1b}$$

where $\xi_h(j+1) = \sigma\sqrt{h}\xi(j+1)$ for $j \ge 0$.

Lemma 7.6.1. *Let $h > 0$, $\tau_2 > \tau_1 > 0$ and define N_2 by* (7.3.2). *Suppose that w_1 and w_2 obey* (7.2.1), (7.2.3) *and* (7.2.3). *Suppose $\beta > 0$ is such that* (7.3.5) *holds. Let $h_1 > 0$ be defined by* (7.3.6) *and suppose that $h < h_1$. Suppose that $\widehat{Y}$ is defined by* (7.6.1). *Then*

$$\limsup_{n\to\infty} \frac{\widehat{Y}(n)}{\sqrt{2(nh)\log_2(nh)}} = \frac{|\sigma|}{1 - \beta\sum_{j=0}^{N_2} \widehat{W}_h(j)h} =: A(h), \quad \textit{a.s.}$$
$$\liminf_{n\to\infty} \frac{\widehat{Y}(n)}{\sqrt{2(nh)\log_2(nh)}} = -A(h), \quad \textit{a.s.}$$

Moreover
$$\lim_{h\to 0^+} A(h) = \frac{\sigma}{1 - \beta\left(\int_0^{\tau_2} s w_2(s)\, ds - \int_0^{\tau_1} s w_1(s)\, ds\right)}.$$

Proof. Define $F(n+1) = c + \sum_{j=0}^{n} \xi_h(j+1)$ where $n \ge N_2$. Then $\widehat{Y}(n) = \widehat{X}_h(n)$ for $n = 0, \dots, N_2$ and

$$\widehat{Y}(n+1) = \sum_{j=0}^{N_2} \beta \widehat{W}_h(j) h \widehat{Y}(n-j) + F(n+1), \quad n \ge N_2. \tag{7.6.2}$$

Define also $\widehat{y}$ by

$$\widehat{y}(n+1) = \sum_{j=0}^{N_2} \beta \widehat{W}_h(j) h \widehat{y}(n-j), \quad n \ge N_2; \quad \widehat{y}(n) = \widehat{X}_h(n), \quad n = 0, 1, \dots, N_2. \tag{7.6.3}$$

Define $U(n) := \widehat{Y}(n) - \widehat{y}(n)$ for $n \geq 0$. Then $U(n) = 0$ for $n = 0, 1, \ldots, N_2$ and

$$U(n+1) = \sum_{j=0}^{N_2} \beta \widehat{W}_h(j) h U(n-j) + F(n+1), \quad n \geq N_2.$$

For $n \geq N_2 + 1$ by the variation of constants formula we have $U(n) = \sum_{j=N_2}^{n-1} \widehat{r}(n-j-1)F(j+1)$. Then for $n \geq N_2 + 1$ we have

$$\begin{aligned}
\widehat{Y}(n) &= \widehat{y}(n) + \sum_{j=N_2}^{n-1} \widehat{r}(n-j-1) \left(c + \sum_{l=0}^{j} \xi_h(l+1) \right) \\
&= \widehat{y}(n) + c \sum_{j=N_2}^{n-1} \widehat{r}(n-j-1) + \sum_{j=N_2}^{n-1} \widehat{r}(n-j-1) \sum_{l=0}^{j} \xi_h(l+1).
\end{aligned}$$

Define

$$\widehat{R}(n) := \sum_{j=0}^{n} \widehat{r}(j), \quad n \geq 0, \tag{7.6.4}$$

and

$$y_1(n) = \widehat{y}(n) + c \sum_{j=N_2}^{n-1} \widehat{r}(n-j-1), \quad n \geq N_2 + 1.$$

Therefore for $n \geq N_2 + 1$, we obtain

$$\begin{aligned}
\widehat{Y}(n) &= y_1(n) + \sum_{j=N_2}^{n-1} \widehat{r}(n-j-1) \sum_{l=0}^{j} \xi_h(l+1) \\
&= y_1(n) + \sum_{l=0}^{n-1} \left(\sum_{j=l \vee N_2}^{n-1} \widehat{r}(n-j-1) \right) \xi_h(l+1) \\
&= y_1(n) + \sum_{l=0}^{N_2-1} \left(\sum_{j=l \vee N_2}^{n-1} \widehat{r}(n-j-1) \right) \xi_h(l+1) \\
&\qquad + \sum_{l=N_2}^{n-1} \left(\sum_{j=l \vee N_2}^{n-1} \widehat{r}(n-j-1) \right) \xi_h(l+1) \\
&= y_1(n) + \sum_{l=0}^{N_2-1} \left(\sum_{k=0}^{n-N_2-1} \widehat{r}(k) \right) \xi_h(l+1) + \sum_{l=N_2}^{n-1} \sum_{k=0}^{n-l-1} \widehat{r}(k) \xi_h(l+1).
\end{aligned}$$

Define for $n \geq N_2 + 1$

$$y_2(n) := \widehat{y}(n) + c \sum_{j=N_2}^{n-1} \widehat{r}(n-j-1) + \sum_{l=0}^{N_2-1} \left(\sum_{k=0}^{n-N_2-1} \widehat{r}(k) \right) \xi_h(l+1),$$

so

$$y_2(n) = \widehat{y}(n) + c \sum_{l=0}^{n-N_2-1} \widehat{r}(l) + \sum_{l=0}^{N_2-1} \left(\sum_{k=0}^{n-N_2-1} \widehat{r}(k) \right) \xi_h(l+1), \quad n \geq N_2 + 1. \tag{7.6.5}$$

Therefore

$$\widehat{Y}(n) = y_2(n) + \sum_{l=N_2}^{n-1} \widehat{R}(n-l-1) \xi_h(l+1), \quad n \geq N_2 + 1. \tag{7.6.6}$$

Since $\beta \sum_{j=0}^{N_2} \widehat{W}_h(j) h < 1$, we have

$$\lim_{n \to \infty} \widehat{R}(n) =: R^* = \frac{1}{1 - \beta \sum_{j=0}^{N_2} \widehat{W}_h(j) h}.$$

and $\widehat{y}(n) \to L$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} y_2(n) = L + cR^* + R^* \sum_{l=0}^{N_2-1} \xi_h(l+1) =: L_2.$$

From (7.6.6) we have for $n \geq N_2 + 1$

$$\widehat{Y}(n) = y_2(n) + R^* \sum_{l=N_2}^{n-1} \xi_h(l+1) - \sum_{l=N_2}^{n-1} (R^* - \widehat{R}(n-l-1))\xi_h(l+1). \tag{7.6.7}$$

Since ξ_h are independently and identically distributed with zero mean and variance $\sigma^2 h$, by the Law of the Iterated Logarithm we have

$$\limsup_{n\to\infty} \frac{\sum_{l=N_2}^{n-1} \xi_h(l+1)}{\sqrt{2n\log_2 n}} = |\sigma|\sqrt{h}, \quad \liminf_{n\to\infty} \frac{\sum_{l=N_2}^{n-1} \xi_h(l+1)}{\sqrt{2n\log_2 n}} = -|\sigma|\sqrt{h}, \quad \text{a.s.} \tag{7.6.8}$$

Define $V(n) = \sum_{l=N_2}^{n-1}(R^* - \widehat{R}(n-l-1))\xi_h(l+1)$. Let $b(x) = \sqrt{x}$, $x \geq 0$. Then $b : [0,\infty) \to [0,\infty)$ is increasing and $b^{-1}(x) = x^2$. If ξ is a random variable with the same distribution as $\xi_h(n)$, by Corollary 4.1.3 in [28], we have

$$\sum_{n=1}^{\infty} \mathbb{P}[|\xi_h(n)| > \sqrt{n}] \leq \mathbb{E}[b^{-1}(|\xi|)] = \mathbb{E}[\xi^2] < \infty.$$

By the Borel–Cantelli lemma, $\limsup_{n\to\infty} |\xi_h(n)|/\sqrt{n} \leq 1$, a.s. which implies that $\lim_{n\to\infty} |\xi_h(n)|/\sqrt{2n\log\log n} = 0$ a.s. Thus, there is an a.s. event Ω^* such that for all $\omega \in \Omega^*$, and all $\varepsilon > 0$, there is $C(\varepsilon,\omega) > 0$ such that

$$|\xi_h(n,\omega)| < C(\varepsilon,\omega) + \varepsilon\sqrt{2n\log\log(n+e^e)} =: \gamma(n,\omega), \quad n \in \mathbb{N}.$$

By $\beta\sum_{j=0}^{N_2} j\widehat{W}_h(j)h < 1$, $R^* - \widehat{R} \in \ell^1(\mathbb{N};\mathbb{R})$, so we have

$$\limsup_{n\to\infty} \frac{|V(n,\omega)|}{\gamma(n,\omega)} \leq \limsup_{n\to\infty} \frac{\sum_{j=N_2}^{n-1} |R^* - R(n-j-1)|\gamma(\omega,j)}{\gamma(n,\omega)} = \sum_{j=0}^{\infty} |R^* - R(j)|;$$

thus $\limsup_{n\to\infty} |V(n,\omega)|/\sqrt{2n\log\log n} < \varepsilon\sum_{j=0}^{\infty} |R^* - R(j)|$, hence

$$\lim_{n\to\infty} \frac{\sum_{l=N_2}^{n-1}(R^* - \widehat{R}(n-l-1))\xi_h(l+1)}{\sqrt{2n\log\log n}} = 0, \quad \text{a.s.} \tag{7.6.9}$$

Therefore by (7.6.7), (7.6.8), and (7.6.9) we have

$$\limsup_{n\to\infty} \frac{\widehat{Y}(n)}{\sqrt{2n\log_2 n}} = \frac{|\sigma|\sqrt{h}}{1 - \beta\sum_{j=0}^{N_2} \widehat{W}(j)h} = -\liminf_{n\to\infty} \frac{\widehat{Y}(n)}{\sqrt{2n\log_2 n}} \quad \text{a.s}$$

and therefore

$$\limsup_{n\to\infty} \frac{\widehat{Y}(n)}{\sqrt{2(nh)\log_2(nh)}} = \frac{|\sigma|}{1 - \beta\sum_{j=0}^{N_2} h\widehat{W}_h(j)} = -\liminf_{n\to\infty} \frac{\widehat{Y}(n)}{\sqrt{2(nh)\log_2(nh)}},$$

almost surely, as required. □

7.6.2 Proof of Theorem 7.4.1

Define

$$\gamma(x) = g(x) - \beta x, \quad \text{for all } x \in \mathbb{R}, \tag{7.6.10}$$

and let $c = \psi(0) + \widehat{I}_1^*(\psi) - \widehat{I}_2^*(\psi)$. Then by (7.3.11b) we have

$$\widehat{X}_h(n+1) = c + \sum_{j=0}^{N_2} \widehat{W}_h(j) h g(\widehat{X}_h(n-j)) + \sum_{j=0}^{n} \xi_h(j+1), \qquad n \geq N_2,$$

where $\xi_h(j+1) := \sigma\sqrt{h}\xi(j+1)$ for $j \geq 0$. Define $\widehat{Z}(n) = \widehat{X}_h(n) - \widehat{Y}(n)$ for $n \geq 0$. Then $\widehat{Z}(n) = 0$ for $n \in \{0, 1, \cdots, N_2\}$. For $n \geq N_2$,

$$\begin{aligned}\widehat{Z}(n+1) &= \sum_{j=0}^{N_2} \widehat{W}_h(j) h \left(g(\widehat{X}_h(n-j)) - \beta \widehat{Y}(n-j) \right) \\ &= \sum_{j=0}^{N_2} \widehat{W}_h(j) h \left(g(\widehat{X}_h(n-j)) - \beta \widehat{X}_h(n-j) \right) + \sum_{j=0}^{N_2} \beta \widehat{W}_h(j) h \widehat{Z}(n-j) \\ &= G(n+1) + \sum_{j=0}^{N_2} \beta \widehat{W}_h(j) h \widehat{Z}(n-j),\end{aligned}$$

where $G(n+1) := \sum_{j=0}^{N_2} \widehat{W}_h(j) h \gamma(\widehat{X}_h(n-j))$. Therefore we have

$$\widehat{Z}(n+1) = G(n+1) + \sum_{j=0}^{N_2} \beta \widehat{W}_h(j) h \widehat{Z}(n-j),\ n \geq N_2, \quad \widehat{Z}(n) = 0,\ n = 0, 1, \ldots, N_2.$$

Define $Z_-(n) = \widehat{Z}(n + N_2)$ for $n \geq -N_2$ and $G_-(n) = G(n + N_2)$ for $n \geq 0$, then

$$\begin{aligned} Z_-(n+1) = G_-(n+1) + \sum_{j=0}^{N_2} \beta \widehat{W}_h(j) h Z_-(n-j), \qquad n \geq 0 \\ Z_-(n) = 0, \qquad n = -1, -2, \ldots, -N_2. \end{aligned}$$

Now for $n \geq 1$ we have $Z_-(n) = \sum_{j=0}^{n-1} \widehat{r}(n-1-j) G_-(j+1)$ and so $\widehat{Z}(n + N_2) = \sum_{j=0}^{n-1} \widehat{r}(n-1-j) G(j+1+N_2)$ for $n \geq 1$. Therefore for $m \geq N_2 + 1$ we have

$$\widehat{Z}(m) = \sum_{j=0}^{m-N_2-1} \widehat{r}(m - N_2 - 1 - j) G(j+1+N_2) = \sum_{l=N_2}^{m-1} \widehat{r}(m-l-1) G(l+1).$$

For $n \geq N_2 + 1$

$$\begin{aligned}
\widehat{Z}(n) &= \sum_{j=N_2}^{n-1} \widehat{r}(n-j-1)G(j+1) \\
&= \sum_{j=N_2}^{n-1} \widehat{r}(n-j-1) \sum_{l=0}^{N_2} \widehat{W}_h(l)h\gamma(\widehat{X}_h(j-l)) \\
&= \sum_{j=N_2}^{n-1} \widehat{r}(n-j-1) \sum_{k=j-N_2}^{j} \widehat{W}_h(j-k)h\gamma(\widehat{X}_h(k)) \\
&= \sum_{k=0}^{n-1} \left(\sum_{j=k\vee N_2}^{(N_2+k)\wedge(n-1)} \widehat{r}(n-j-1)\widehat{W}_h(j-k)h \right) \gamma(\widehat{X}_h(k)) \\
&= \sum_{k=0}^{N_2-1} \left(\sum_{j=N_2}^{(N_2+k)\wedge(n-1)} \widehat{r}(n-j-1)\widehat{W}_h(j-k)h \right) \gamma(\widehat{X}_h(k)) \\
&\qquad + \sum_{k=N_2}^{n-1} \left(\sum_{j=k}^{(N_2+k)\wedge(n-1)} \widehat{r}(n-j-1)\widehat{W}_h(j-k)h \right) \gamma(\widehat{X}_h(k)) \\
&= f_1(n) + \sum_{k=N_2}^{n-1} \left(\sum_{j=k}^{(N_2+k)\wedge(n-1)} \widehat{r}(n-j-1)\widehat{W}_h(j-k)h \right) \gamma(\widehat{X}_h(k)),
\end{aligned}$$

where

$$f_1(n) := \sum_{k=0}^{N_2-1} \left(\sum_{j=N_2}^{(N_2+k)\wedge(n-1)} \widehat{r}(n-j-1)\widehat{W}_h(j-k)h \right) \gamma(\widehat{X}_h(k)) \tag{7.6.11}$$

obeys $f_1(n) \to 0$ as $n \to \infty$. Next let $n \geq 2N_2 + 1$. Then

$$\begin{aligned}
&\sum_{k=N_2}^{n-1} \left(\sum_{j=k}^{(N_2+k)\wedge(n-1)} \widehat{r}(n-j-1)\widehat{W}_h(j-k)h \right) \gamma(\widehat{X}_h(k)) \\
&\qquad = \sum_{k=N_2}^{n-1-N_2} \left(\sum_{j=k}^{N_2+k} \widehat{r}(n-j-1)\widehat{W}_h(j-k)h \right) \gamma(\widehat{X}_h(k)) \\
&\qquad\quad + \sum_{k=n-N_2}^{n-1} \left(\sum_{j=k}^{n-1} \widehat{r}(n-j-1)\widehat{W}_h(j-k)h \right) \gamma(\widehat{X}_h(k)).
\end{aligned}$$

Define $\eta(j) = \sum_{l=0}^{j-1} \widehat{r}(j-l-1)\widehat{W}_h(l)h$ for $1 \leq j \leq N_2$. Then for $j \in \{1, \dots, N_2\}$ we have

$$\begin{aligned}
\eta(j) &= \sum_{l=0}^{j-1} \widehat{r}(j-l-1)\widehat{W}_h(l)h \\
&= \sum_{l=0}^{N_2} \widehat{r}(j-l-1)\widehat{W}_h(l)h - \sum_{l=j}^{N_2} \widehat{r}(j-l-1)\widehat{W}_h(l)h \\
&= \frac{1}{\beta} \sum_{l=0}^{N_2} \beta\widehat{r}(j-l-1)\widehat{W}_h(l)h = \frac{1}{\beta}\widehat{r}(j),
\end{aligned}$$

where we used (7.4.1a) at the last step and (7.4.1b) at the first. Therefore

$$\sum_{j=k}^{n-1} \widehat{r}(n-j-1)\widehat{W}_h(j-k)h = \sum_{l=0}^{n-k-1} \widehat{r}(n-k-l-1)\widehat{W}_h(l)h = \eta(n-k).$$

Consider the first term on the righthand side, where $k \in \{N_2, \dots, n-1-N_2\}$. Then

$$\sum_{j=k}^{N_2+k} \widehat{r}(n-j-1)\widehat{W}_h(j-k)h = \frac{1}{\beta}\sum_{l=0}^{N_2} \widehat{r}(n-k-1-l)\beta\widehat{W}_h(l)h.$$

Since $k \le n-1-N_2$, we have $n-k-1 \ge n-1-n+1+N_2 = N_2 > 0$. Hence

$$\sum_{j=k}^{N_2+k} \widehat{r}(n-j-1)\widehat{W}_h(j-k)h = \frac{1}{\beta}\widehat{r}(n-k) := \eta(n-k)$$

where we have defined

$$\eta(j) = \frac{1}{\beta}\widehat{r}(j), \quad j \ge 1. \tag{7.6.12}$$

For $n \ge 2N_2+1$ by (7.6.12) we have

$$\begin{aligned}\sum_{k=N_2}^{n-1} &\left(\sum_{j=k}^{(N_2+k)\wedge(n-1)} \widehat{r}(n-j-1)\widehat{W}_h(j-k)h\right)\gamma(\widehat{X}_h(k)) \\ &= \sum_{k=N_2}^{n-1-N_2} \eta(n-k)\gamma(\widehat{X}_h(k)) + \sum_{k=n-N_2}^{n-1} \eta(n-k)\gamma(\widehat{X}_h(k)).\end{aligned}$$

Thus

$$\widehat{Z}(n) = f_1(n) + \sum_{k=N_2}^{n-1} \eta(n-k)\gamma(\widehat{X}_h(k)), \quad n \ge 2N_2+1, \tag{7.6.13}$$

and because $\widehat{r}$ is summable, we have $\eta \in L^1(\mathbb{N};\mathbb{R})$ and also $\eta(k) \ge 0$ for all $k \ge 1$. Let $\epsilon > 0$ be so small that $\epsilon\sum_{l=1}^{\infty}\eta(l) < \frac{1}{2}$. Also by (7.2.5), for every $\epsilon > 0$ there exists $L(\epsilon) > 0$ such that $|\gamma(x)| \le L(\epsilon) + \epsilon|x|$ for all $x \ge 0$. Therefore for $n \ge 2N_2+1$,

$$\begin{aligned}|\widehat{Z}(n)| &\le |f_1(n)| + \sum_{k=N_2}^{n-1}\eta(n-k)\left(L(\epsilon) + \epsilon|\widehat{X}_h(k)|\right) \\ &= |f_1(n)| + L(\epsilon)\sum_{k=N_2}^{n-1}\eta(n-k) + \epsilon\sum_{k=N_2}^{n-1}\eta(n-k)|\widehat{Y}(k)| \\ &\qquad + \epsilon\sum_{k=N_2}^{n-1}\eta(n-k)|\widehat{Z}(k)| \\ &\le f_2 + \epsilon\sum_{k=N_2}^{n-1}\eta(n-k)|\widehat{Y}(k)| + \epsilon\sum_{k=N_2}^{n-1}\eta(n-k)|\widehat{Z}(k)| \\ &\le f_2 + \epsilon\sum_{k=0}^{n-1}\eta(n-k)|\widehat{Y}(k)| + \epsilon\sum_{k=0}^{n-1}\eta(n-k)|\widehat{Z}(k)|,\end{aligned}$$

where $f_2 \ge 0$ is defined so that $|f_1(n)| + L(\epsilon)\sum_{k=N_2}^{n-1}\eta(n-k) \le f_2$ for all $n \ge 2N_2+1$. Let $f_3 = \max_{0\le j\le 2N_2+1}|\widehat{Z}(j)|$ and $f_4 := f_2 + f_3$. Then for $n \ge 1$

$$|\widehat{Z}(n)| \le f_4 + \epsilon\sum_{k=0}^{n-1}\eta(n-k)|\widehat{Y}(k)| + \epsilon\sum_{k=0}^{n-1}\eta(n-k)|\widehat{Z}(k)|.$$

Define $f_5(n) := f_4 + \epsilon\sum_{k=0}^{n-1}\eta(n-k)|\widehat{Y}(k)|$ for $n \ge 1$. Therefore

$$\limsup_{n\to\infty}\frac{f_5(n)}{\sqrt{2n\log_2(n)}} \le \epsilon\sum_{k=1}^{\infty}\eta(k)\frac{|\sigma|\sqrt{h}}{1-\sum_{j=0}^{N_2}\beta\widehat{W}_h(j)h} =: \epsilon K(h), \quad \text{a.s.} \tag{7.6.14}$$

Define $\eta_-(n) := \eta(n+1)$, $n \geq 0$. Then for $n \geq 0$ we have

$$|\widehat{Z}(n+1)| \leq f_5(n+1) + \epsilon \sum_{k=0}^{n} \eta(n+1-k)|\widehat{Z}(k)| = f_5(n+1) + \epsilon \sum_{k=0}^{n} \eta_-(n-k)|\widehat{Z}(k)|.$$

Define $(\overline{Z}_\epsilon(n))_{n\geq 0}$ so that $\overline{Z}_\epsilon(0) = |\widehat{Z}(0)|$ and define

$$\overline{Z}_\epsilon(n+1) = f_5(n+1) + \epsilon \sum_{k=0}^{n} \eta_-(n-k)\overline{Z}_\epsilon(k), \quad n \geq 0.$$

Then $|\widehat{Z}(n)| \leq \overline{Z}_\epsilon(n)$ for $n \geq 1$. Clearly $\sum_{j=0}^{\infty} \eta_-(j) = \sum_{j=1}^{\infty} \eta(j) < \infty$, and $\sum_{j=0}^{\infty} \epsilon\eta_-(j) < \frac{1}{2}$. Let $(\rho_\epsilon(n))_{n\geq 0}$ be defined by $\rho_\epsilon(0) = 1$ and

$$\rho_\epsilon(n+1) = \epsilon\eta_-(n+1) + \epsilon \sum_{k=0}^{n} \eta_-(n-k)\rho_\epsilon(k), \quad n \geq 0.$$

Note that $\sum_{n=0}^{\infty} \rho_\epsilon(n)$ is finite, because $\sum_{j=0}^{\infty} \epsilon\eta_-(j) < 1/2$. In fact we have

$$\sum_{n=0}^{\infty} \rho_\epsilon(n) = 1 + \sum_{n=1}^{\infty} \rho_\epsilon(n) = 1 + \epsilon \sum_{n=1}^{\infty} \eta_-(n) + \epsilon \sum_{n=0}^{\infty} \eta_-(n) \sum_{n=0}^{\infty} \rho_\epsilon(n).$$

Therefore

$$\sum_{n=0}^{\infty} \rho_\epsilon(n) = \frac{1 + \epsilon \sum_{n=1}^{\infty} \eta_-(n)}{1 - \epsilon \sum_{n=0}^{\infty} \eta_-(n)},$$

and $\overline{Z}_\epsilon(n) = f_5(n) + \sum_{k=0}^{n} \rho_\epsilon(n-k) f_5(k)$ for $n \geq 0$. Then by (7.6.14) we have

$$\begin{aligned}
\limsup_{n\to\infty} \frac{|\overline{Z}_\epsilon(n)|}{\sqrt{2n \log_2 n}} &\leq \epsilon K(h) + \sum_{k=0}^{\infty} \rho_\epsilon(k) \epsilon K(h) \\
&= \epsilon K(h) \left(1 + \frac{1 + \epsilon \sum_{n=1}^{\infty} \eta_-(n)}{1 - \epsilon \sum_{n=0}^{\infty} \eta_-(n)}\right) \leq \epsilon K_1(h),
\end{aligned}$$

for all $\epsilon \in (0, \epsilon(h))$ so small that $\sum_{j=1}^{\infty} \epsilon\eta_-(j) < 1/2$. Therefore we have

$$\limsup_{n\to\infty} \frac{|\widehat{Z}(n)|}{\sqrt{2n \log_2 n}} \leq \limsup_{n\to\infty} \frac{|\overline{Z}_\epsilon(n)|}{\sqrt{2n \log_2 n}} \leq \epsilon K_1(h).$$

Therefore as $\epsilon > 0$ is chosen arbitrarily, we have that

$$\limsup_{n\to\infty} \frac{|\widehat{Z}(n)|}{\sqrt{2(nh) \log_2(nh)}} = 0$$

on each sample path. Thus

$$\limsup_{n\to\infty} \frac{|\widehat{X}_h(n) - \widehat{Y}(n)|}{\sqrt{2(nh) \log_2(nh)}} = 0, \quad \text{a.s.}$$

Combining this with Lemma 7.6.1, we obtain

$$\limsup_{n\to\infty} \frac{\widehat{X}_h(n)}{\sqrt{2(nh) \log_2(nh)}} = \frac{|\sigma|}{1 - \beta \sum_{j=0}^{N_2} h\widehat{W}_h(j)} = -\liminf_{n\to\infty} \frac{\widehat{X}_h(n)}{\sqrt{2(nh) \log_2(nh)}},$$

almost surely, proving the first part of the result. The second part follows from Lemma 7.3.1.

7.7 Proof of Theorem 7.4.3

We start with a preliminary lemma which tells us that the exponential rate of growth of the discretised equation mirrors that of the continuous equation. Also as the step size tends to zero the rate of growths in the two systems will concur.

Lemma 7.7.1. *Let $h > 0$, $\tau_2 > \tau_1 > 0$ and define N_2 by* (7.3.2). *Suppose that w_1 and w_2 obey* (7.2.1), (7.2.3) *and* (7.2.3). *Suppose that $\beta > 0$ is such that* (7.3.7) *holds. Suppose also that $h < h_2$ where $h_2 > 0$ is defined by* (7.3.8).

(i) There exists a $\lambda^* > 0$ such that

$$\beta \int_0^{\tau_2} W(s) e^{-\lambda^* s}\, ds = 1. \tag{7.7.1}$$

(ii) There exists a unique $\alpha(h) > 1$ such that

$$\alpha(h) = \sum_{j=0}^{N_2} \beta \widehat{W}_h(j) h \alpha(h)^{-j}, \tag{7.7.2}$$

and therefore $\widehat{r}$ defined by (7.4.1) *obeys*

$$\lim_{n\to\infty} \frac{\widehat{r}(n)}{\alpha(h)^n} = R^*(h) > 0. \tag{7.7.3}$$

(iii) With $\alpha(h)$ given by (7.7.2)*, define*

$$\lambda(h) = \frac{1}{h} \log \alpha(h) > 0. \tag{7.7.4}$$

Then

$$\lim_{h\to 0^+} \lambda(h) = \lambda^*, \tag{7.7.5}$$

where $\lambda^* > 0$ is given by (7.7.1).

Remark 7.7.1. The proofs of (i) and (ii) are short and straightforward and no outline is given. To prove (iii) we establish the bound $|I(\lambda) - F_h(\lambda)| < \beta C_2(h)$. Then we use properties of the functions I and F combined with the bound to show that $|\lambda(h) - \lambda^*| < \epsilon$. Hence the required result.

Proof. (i) has already been proven. For (ii) define

$$G_h(\alpha) = \beta \sum_{j=0}^{N_2(h)} \widehat{W}_h(j) h \alpha^{-(j+1)}, \quad \alpha \geq 1.$$

Recall that $\widehat{W}_h(j) \geq 0$ for all $j = 0, \ldots, N_2$ and in particular that $\widehat{W}_h(0) > 0$. Then (a) $G_h(1) > 1$ because $\beta \sum_{j=0}^{N_2(h)} \widehat{W}_h(j) h > 1$; (b) $\alpha \mapsto G_h(\alpha)$ is decreasing and continuous in α; (c) $\lim_{\alpha\to\infty} G_h(\alpha) = 0$. Therefore there exists a unique $\alpha(h) > 1$ such that $G_h(\alpha(h)) = 1$ which gives (7.7.2). (iii) If $\lambda(h)$ is given by (7.7.4) we have $\alpha(h) = e^{(\lambda(h)h)}$. Define

$$F_h(\lambda) = \beta \sum_{j=0}^{N_2(h)} \widehat{W}_h(j) h e^{-\lambda h(j+1)}, \quad \lambda \geq 0.$$

Then $F_h(\lambda(h)) = 1$; note also that $\lambda \mapsto F_h(\lambda)$ is decreasing and continuous. Define

$$\begin{aligned} C_2(h) &= 2h + \sum_{j=0}^{N_2(h)-1} \int_{jh}^{(j+1)h} \left| W(s) - \overline{W}_h(s) \right| ds \\ &= 2h + \int_0^{N_2(h)h} \left| W(s) - \overline{W}_h(s) \right| ds, \end{aligned}$$

where $\overline{W}_h(s) := \widehat{W}_h(j)$ for $jh \le s < (j+1)h$. Then $C_2(h) \to 0$, as $h \to 0^+$. Define

$$I(\lambda) = \beta \int_0^{\tau_2} W(s) e^{-\lambda s}\, ds.$$

Then (a) $I(\lambda^*) = 1$, (b) $\lambda \mapsto I(\lambda)$ is decreasing. Let $\epsilon \in (0, \lambda^*)$ be fixed. Then as $I(\lambda^* - \epsilon) > 1$ and $I(\lambda^* + \epsilon) < 1$, there is $h_2(\epsilon) > 0$ such that $\beta C_2(h) < I(\lambda^* - \epsilon) - 1$ where $h < h_2(\epsilon)$ and $h_3(\epsilon) > 0$ such that $\beta C_2(h) < 1 - I(\lambda^* + \epsilon)$, $h < h_3(\epsilon)$. Define also $h^*(\epsilon) = \min(h_2(\epsilon), h_3(\epsilon), h_2)$. Therefore for $h < h^*(\epsilon)$ we have

$$F_h(\lambda(h)) = 1; \quad \beta C_2(h) < I(\lambda^* - \epsilon) - 1; \quad \beta C_2(h) < 1 - I(\lambda^* + \epsilon). \tag{7.7.6}$$

$$\begin{aligned}
I(\lambda) - F_h(\lambda) &= \beta \int_0^{\tau_2} W(s) e^{-\lambda s}\, ds - \beta \sum_{j=0}^{N_2(h)} \widehat{W}_h(j) h e^{-\lambda(j+1)h} \\
&= \beta \left(\int_0^{N_2 h} W(s) e^{-\lambda s}\, ds - \sum_{j=0}^{N_2-1} \int_{jh}^{(j+1)h} \widehat{W}_h(j) e^{-\lambda(j+1)h}\, ds \right) \\
&\qquad + \beta \int_{N_2 h}^{\tau_2} W(s) e^{-\lambda s}\, ds \\
&= \beta \sum_{j=0}^{N_2-1} \int_{jh}^{(j+1)h} \left(W(s) e^{-\lambda s} - \overline{W}_h(s) e^{-\lambda(j+1)h} \right) ds \\
&\qquad + \beta \int_{N_2 h}^{\tau_2} W(s) e^{-\lambda s}\, ds \\
&= \beta \sum_{j=0}^{N_2-1} \int_{jh}^{(j+1)h} \left(e^{-\lambda(j+1)h} \left(W(s) - \overline{W}_h(s) \right) + W(s) \left(e^{-\lambda s} - e^{-\lambda(j+1)h} \right) \right) ds \\
&\qquad + \beta \int_{N_2 h}^{\tau_2} W(s) e^{-\lambda s}\, ds.
\end{aligned}$$

Thus as $\lambda \ge 0$ and $W(s) \le 1$ we have

$$\begin{aligned}
&|I(\lambda) - F_h(\lambda)| \\
&\le \beta \sum_{j=0}^{N_2-1} e^{-\lambda(j+1)h} \int_{jh}^{(j+1)h} \left| W(s) - \overline{W}_h(s) \right| ds \\
&\qquad + \beta \sum_{j=0}^{N_2-1} \int_{jh}^{(j+1)h} \left(e^{-\lambda s} - e^{-\lambda(j+1)h} \right) ds + \beta h e^{-\lambda N_2 h} \\
&\le \beta \sum_{j=0}^{N_2-1} \int_{jh}^{(j+1)h} \left| W(s) - \overline{W}_h(s) \right| ds + \beta \sum_{j=0}^{N_2-1} h \left(e^{-\lambda jh} - e^{-\lambda(j+1)h} \right) + \beta h \\
&= \beta \sum_{j=0}^{N_2-1} \int_{jh}^{(j+1)h} \left| W(s) - \overline{W}_h(s) \right| ds + \beta h \sum_{j=0}^{N_2-1} e^{-\lambda jh} \left(1 - e^{-\lambda h} \right) + \beta h \\
&= \beta \sum_{j=0}^{N_2-1} \int_{jh}^{(j+1)h} \left| W(s) - \overline{W}_h(s) \right| ds + \beta h \left(1 - e^{-\lambda h} \right) \frac{1 - e^{-\lambda h N_2}}{1 - e^{-\lambda h}} + \beta h \\
&\le 2\beta h + \beta \sum_{j=0}^{N_2-1} \int_{jh}^{(j+1)h} \left| W(s) - \overline{W}_h(s) \right| ds \\
&=: \beta C_2(h).
\end{aligned}$$

In the case when $\lambda = 0$ the second sum on the righthand side is zero and the estimate holds. Therefore for all $\lambda \geq 0$ and $h > 0$ we have

$$|I(\lambda) - F_h(\lambda)| \leq \beta C_2(h).$$

Now let $h < h^*(\epsilon)$. Then $F_h(\lambda^* + \epsilon) - I(\lambda^* + \epsilon) \leq \beta C_2(h)$ and by (7.7.6)

$$F_h(\lambda^* + \epsilon) \leq \beta C_2(h) + I(\lambda^* + \epsilon) < 1. \tag{7.7.7}$$

Also for $h < h^*(\epsilon)$ we have $F_h(\lambda^* - \epsilon) - I(\lambda^* - \epsilon) \geq -\beta C_2(h)$ and by (7.7.6)

$$F_h(\lambda^* - \epsilon) \geq I(\lambda^* + \epsilon) - \beta C_2(h) > 1 + \beta C_2(h) - \beta C_2(h) = 1. \tag{7.7.8}$$

Therefore for all $\epsilon \in (0, \lambda^*)$ there exists $h^*(\epsilon) > 0$ such that for all $h < h^*(\epsilon)$

$$F_h(\lambda^* - \epsilon) > 1 = F_h(\lambda(h)) = 1 > F_h(\lambda^* + \epsilon).$$

Since $\lambda \mapsto F_h(\lambda)$ is decreasing, $\lambda^* - \epsilon < \lambda(h) < \lambda^* + \epsilon$ or $|\lambda(h) - \lambda^*| < \epsilon$. Thus for all $\epsilon \in (0, \lambda^*)$ there is $h^*(\epsilon) > 0$ so that for all $h < h^*(\epsilon)$ we have $|\lambda(h) - \lambda| < \epsilon$. But this is equivalent to (7.7.5). □

Lemma 7.7.2. *Let $h > 0$, $\tau_2 > \tau_1 > 0$ and define N_2 by* (7.3.2). *Suppose that w_1 and w_2 obey* (7.2.1), (7.2.3) *and* (7.2.3). *Suppose that $\beta > 0$ is such that* (7.3.7) *holds. Suppose also that $h < h_2$ where $h_2 > 0$ is defined by* (7.3.8). *Then there exists a unique $\alpha(h) > 1$ obeying* (7.7.2) *and therefore $\widehat{r}$ defined by* (7.4.1) *obeys* (7.7.3) *for some $R^*(h) > 0$. Moreover the solution $\widehat{y}$ of* (7.6.3) *obeys*

$$\lim_{n\to\infty} \frac{\widehat{y}(n)}{\alpha^n} = R^*(h)\alpha(h)^{-N_2}\Bigg(\widehat{X}_h(N_2) \\ + \sum_{j=0}^{N_2-1} \alpha(h)^{-(1+j)} \sum_{l=j+1}^{N_2} \beta h \widehat{W}_h(l)\widehat{X}_h(j-l+N_2)\Bigg).$$

Remark 7.7.2. In this lemma we calculate an explicit formula for the exponential rate of growth of the linear deterministic equation. Proof of the lemma is standard.

Proof. Define $y_-(n) = \widehat{y}(n+N_2)$ for $n \geq -N_2$, $x_-(n) = \widehat{X}_h(n+N_2)$ for $n = -N_2, \ldots, 0$ and extend $x_-(n)$ for $n \leq -N_2 - 1$ by $x_-(n) = 0$ for $n \leq -N_2 - 1$. Define $v(j) = \beta\widehat{W}_h(j)h$ for $j = 0, \ldots, N_2$ and $v(j) = 0$ for $j \geq N_2 + 1$. Therefore $y_-(n) = x_-(n)$ for $n = -N_2, \ldots, 0$ and by (7.6.3) we have

$$y_-(n+1) = \sum_{j=0}^{N_2} v(j)y_-(n-j) = \sum_{j=0}^{\infty} v(j)y_-(n-j), \quad n \geq 0.$$

Taking $z-$transforms yields

$$\widetilde{y_-}(z) = \sum_{n=0}^{\infty} z^{-(n+1)}y_-(n+1) + y_-(0) = y_-(0) + z^{-1}\sum_{n=0}^{\infty} z^{-n}\sum_{j=0}^{\infty} v(j)y_-(n-j),$$

so with $a(n) := \sum_{j=n+1}^{\infty} v(j)x_-(n-j)$ for $n \geq 0$ we have

$$z\widetilde{y_-}(z) = zy_-(0) + \sum_{n=0}^{\infty} z^{-n}\sum_{j=0}^{n} v(j)y_-(n-j) + \sum_{n=0}^{\infty} z^{-n}\sum_{j=n+1}^{\infty} v(j)x_-(n-j)$$
$$= zy_-(0) + \widetilde{v}(z)\widetilde{y_-}(z) + \sum_{n=0}^{\infty} z^{-n}a(n) = zy_-(0) + \widetilde{v}(z)\widetilde{y_-}(z) + \widetilde{a}(z).$$

Hence $\widetilde{y_-}(z)(z - \widetilde{v}(z)) = zy_-(0) + \widetilde{a}(z)$. On the other hand $\widetilde{\widehat{r}}(z)(z - \widetilde{v}(z)) = z$, so $z\widetilde{y_-}(z) = \widetilde{y_-}(z)\widetilde{\widehat{r}}(z)(z - \widetilde{v}(z)) = z\widetilde{\widehat{r}}(z)y_-(0) + \widetilde{\widehat{r}}(z)\widetilde{a}(z)$, which implies for $z \neq 0$ that

$$\widetilde{y_-}(z) = \widetilde{\widehat{r}}(z)y_-(0) + z^{-1}\widetilde{\widehat{r}}(z)\widetilde{a}(z).$$

Therefore with $b(n) := (r * a)(n-1)$ for $n \geq 1$ and $b(0) := 0$, we have

$$\widetilde{b}(z) = \sum_{n=1}^{\infty} b(n)z^{-n} = z^{-1}\sum_{n=1}^{\infty}(r*a)(n-1)z^{-(n-1)} = z^{-1}\sum_{l=0}^{\infty}(r*a)(l)z^{-l},$$

so $\widetilde{b}(z) = z^{-1}\widetilde{\widehat{r}}(z)\widetilde{a}(z)$. Therefore we have $y_-(n) = \widehat{r}(n)y_-(0) + b(n)$ for $n \geq 0$, or

$$y_-(n) = \widehat{r}(n)y_-(0) + (\widehat{r}*a)(n-1) = \widehat{r}(n)y_-(0) + \sum_{j=0}^{n-1}\widehat{r}(n-1-j)a(j), \quad n \geq 1.$$

Since $v(j) = 0$ for $j \geq N_2 + 1$ for $n \geq N_2$ we have $a(n) = 0$. For $n \in \{0, \ldots, N_2 - 1\}$ we get

$$a(n) = \sum_{j=n+1}^{N_2} v(j)x_-(n-j) + \sum_{j=N_2+1}^{\infty} v(j)x_-(n-j) = \sum_{j=n+1}^{N_2} v(j)x_-(n-j)$$

Therefore for $n \geq N_2 + 1$ we have

$$\begin{aligned}\widehat{y}(n+N_2) = y_-(n) &= \widehat{r}(n)y_-(0) + \sum_{j=0}^{N_2-1}\widehat{r}(n-1-j)a(j) + \sum_{j=N_2}^{n-1}\widehat{r}(n-1-j)a(j)\\ &= \widehat{r}(n)\widehat{y}(N_2) + \sum_{j=0}^{N_2-1}\widehat{r}(n-1-j)a(j).\end{aligned}$$

Hence for $n \geq 2N_2 + 1$ we have $\widehat{y}(n) = \widehat{r}(n-N_2)\widehat{y}(N_2) + \sum_{j=0}^{N_2-1}\widehat{r}(n-N_2-1-j)a(j)$ which yields the limit

$$\begin{aligned}\lim_{n\to\infty}\frac{\widehat{y}(n)}{\alpha(h)^n} &= \lim_{n\to\infty}\frac{\widehat{r}(n-N_2)}{\alpha(h)^{n-N_2}}\alpha(h)^{-N_2}\widehat{y}(N_2)\\ &\qquad + \lim_{n\to\infty}\sum_{j=0}^{N_2-1}\frac{\widehat{r}(n-N_2-1-j)}{\alpha(h)^{n-N_2-1-j}}\alpha(h)^{-N_2-1-j}a(j)\\ &= R^*(h)\alpha(h)^{-N_2}\left(\widehat{y}(N_2) + \sum_{j=0}^{N_2-1}\alpha(h)^{-(1+j)}a(j)\right)\\ &= R^*(h)\alpha(h)^{-N_2}\left(\widehat{y}(N_2) + \sum_{j=0}^{N_2-1}\alpha(h)^{-(1+j)}\sum_{l=j+1}^{N_2}v(l)x_-(j-l)\right)\\ &= R^*(h)\alpha(h)^{-N_2}\Bigg(\widehat{X}_h(N_2)\\ &\qquad + \sum_{j=0}^{N_2-1}\alpha(h)^{-(1+j)}\sum_{l=j+1}^{N_2}\beta h\widehat{W}_h(l)\widehat{X}_h(j-l+N_2)\Bigg),\end{aligned}$$

as required. ☐

7.7.1 Proof of Theorem 7.4.3

We start by determining the asymptotic behaviour of $\widehat{Y}$ defined by (7.6.1), recalling the formulae

$$y_2(n) = \widehat{y}(n) + c\sum_{l=0}^{n-N_2-1}\widehat{r}(l) + \sum_{l=0}^{N_2-1}\left(\sum_{k=0}^{n-N_2-1}\widehat{r}(k)\right)\xi_h(l+1), \quad n \geq N_2+1,$$

and

$$\widehat{Y}(n) = y_2(n) + \sum_{l=N_2}^{n-1}\widehat{R}(n-l-1)\xi_h(l+1), \quad n \geq N_2+1.$$

Under the hypothesis that $h < h_2$ we have that $\widehat{W}_h(j) \geq 0$ for all $j \geq 0$ and that $\beta \sum_{j=0}^{N_2} \widehat{W}_h(j)h > 1$. Therefore by part (ii) of Lemma 7.7.1, there exists $\alpha = \alpha(h) > 1$ which obeys (7.7.2) and $R^*(h) > 0$ such that $\widehat{r}$ obeys (7.7.3). Therefore, with $\widehat{R}$ defined by (7.6.4) we have

$$\begin{aligned}\lim_{n\to\infty} \frac{\widehat{R}(n)}{\alpha(h)^n} &= \lim_{n\to\infty} \frac{\sum_{j=0}^n \widehat{r}(j)}{\alpha(h)^n} = \lim_{n\to\infty} \sum_{j=0}^n \frac{\widehat{r}(j)}{\alpha(h)^j} \cdot \alpha(h)^{-(n-j)} = R^*(h) \sum_{j=0}^\infty \alpha(h)^{-j} \\ &= R^*(h) \frac{1}{1-\alpha(h)^{-1}}.\end{aligned}$$

Next for $n \geq N_2 + 1$ we have

$$\frac{y_2(n)}{\alpha(h)^n} = \frac{\widehat{y}(n)}{\alpha(h)^n} + \frac{R(n-N_2-1)}{\alpha(h)^n} \left(c + \sum_{l=0}^{N_2-1} \xi_h(l+1) \right).$$

By Lemma 7.7.2 there exists a finite y^* such that $\widehat{y}$ defined by (7.6.3) obeys

$$\lim_{n\to\infty} \widehat{y}(n)/\alpha(h)^n =: y^*. \tag{7.7.9}$$

Since $\widehat{Y}$ obeys (7.7.9), we have

$$\lim_{n\to\infty} \frac{y_2(n)}{\alpha(h)^n} = y^* + R^*(h) \frac{1}{1-\alpha(h)^{-1}} \frac{1}{\alpha(h)^{N_2+1}} \left(c + \sum_{l=0}^{N_2-1} \xi_h(l+1) \right) =: y_2^*.$$

Therefore as

$$\begin{aligned}\frac{\widehat{Y}(n)}{\alpha(h)^n} = \frac{y_2(n)}{\alpha(h)^n} &+ \sum_{l=0}^{n-1} \frac{\widehat{R}(n-l-1)}{\alpha(h)^{n-l-1}} \frac{1}{\alpha(h)^{l+1}} \xi_h(l+1) \\ &- \sum_{l=0}^{N_2-1} \frac{\widehat{R}(n-l-1)}{\alpha(h)^{n-l-1}} \frac{1}{\alpha(h)^{l+1}} \xi_h(l+1),\end{aligned}$$

we have

$$\begin{aligned}\lim_{n\to\infty} \frac{\widehat{Y}(n)}{\alpha(h)^n} &= y_2^* + R^*(h) \frac{1}{1-\alpha(h)^{-1}} \sum_{l=N_2}^\infty \alpha(h)^{-(l+1)} \xi_h(l+1) \\ &= y^* + \frac{R^*(h)}{1-\alpha(h)^{-1}} \left(\frac{1}{\alpha(h)^{N_2+1}} \left(c + \sum_{l=0}^{N_2-1} \xi_h(l+1) \right) \right) \\ &\quad + \frac{R^*(h)}{1-\alpha(h)^{-1}} \left(\sum_{l=N_2}^\infty \alpha(h)^{-(l+1)} \xi_h(l+1) \right) =: Y^*.\end{aligned}$$

We now turn our attention to the nonlinear equation (7.3.11). Let γ be defined by (7.6.10) and define γ_0 by $\gamma_0(x) := \max_{|s|\leq x} |\gamma(s)|$ where $x \geq 0$. Then

$$|\gamma(y)| \leq \max_{|s|\leq|y|} |\gamma(s)| = \gamma_0(|y|), \quad \text{for all } y \in \mathbb{R} \tag{7.7.10}$$

and γ_0 is non–decreasing. Therefore (7.4.2) implies

$$\int_1^\infty \frac{\gamma_0(x)}{x^2}\,dx = \int_1^\infty \frac{\max_{|s|\leq x} |\gamma(s)|}{x^2}\,dx = \int_1^\infty \frac{\max_{|s|\leq x} |g(s)-\beta s|}{x^2}\,dx < +\infty. \tag{7.7.11}$$

For $n \geq 2N_2 + 1$, by (7.6.13) and (7.6.12) we have

$$\begin{aligned}\widehat{Z}(n) &= f_1(n) + \sum_{k=N_2}^{n-1} \eta(n-k)\gamma(\widehat{X}_h(k)) \\ &= f_1(n) + \sum_{k=N_2}^{n-1} \frac{1}{\beta} \widehat{r}(n-k)\gamma(\widehat{X}_h(k)).\end{aligned}$$

Define $F_1(n) := \widehat{Y}(n) + f_1(n)$ then for $n \geq 2N_2 + 1$,

$$\widehat{X}_h(n) = F_1(n) + \sum_{k=N_2}^{n-1} \frac{1}{\beta}\widehat{r}(n-k)\gamma(\widehat{X}_h(k)).$$

If $n - 1 \geq 2(N_2 - 1)$ then by (7.6.11) we have

$$f_1(n) = \sum_{k=0}^{N_2-1} \left(\sum_{j=N_2}^{N_2+k} \widehat{r}(n-j-1)\widehat{W}_h(j-k)h \right) \gamma(\widehat{X}_h(k)).$$

Therefore as $\widehat{r}$ obeys (7.7.3) we have

$$\begin{aligned}\lim_{n\to\infty} \frac{f_1(n)}{\alpha^n} &= \lim_{n\to\infty} \sum_{k=0}^{N_2-1} \left(\sum_{j=N_2}^{N_2+k} \frac{\widehat{r}(n-j-1)}{\alpha^{n-j-1}} \alpha^{-(j+1)} \widehat{W}_h(j-k)h \right) \gamma(\widehat{X}_h(k)) \\ &= \sum_{k=0}^{N_2-1} \left(\sum_{j=N_2}^{N_2+k} R^*(h)\alpha^{-(j+1)}\widehat{W}_h(j-k)h \right) \gamma(\widehat{X}_h(k)) =: f_1^*.\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \frac{F_1(n)}{\alpha(h)^n} = f_1^* + Y^* =: F_1^*. \tag{7.7.12}$$

Let $\widetilde{X}_h(n) = \widehat{X}_h(n)/\alpha^n$ and $\widetilde{F}_1(n) = F_1(n)/\alpha^n$. Then for $n \geq 2N_1 + 1$,

$$\widetilde{X}_h(n) = \widetilde{F}_1(n) + \sum_{k=N_2}^{n-1} \frac{1}{\beta} \frac{\widehat{r}(n-k)}{\alpha^{n-k}} \frac{\gamma(\widehat{X}_h(k))}{\alpha^k}.$$

Therefore with $\tilde{r}(n) := \widehat{r}(n)/\alpha^n$ we get

$$\widetilde{X}_h(n) = \widetilde{F}_1(n) + \frac{1}{\beta} \sum_{k=N_2}^{n-1} \widetilde{r}(n-k)\alpha^{-k}\gamma(\alpha^k \widetilde{X}_h(k)).$$

Since $\tilde{r}$, $\widetilde{F}_1$ are bounded sequences and (7.7.10) holds, there exists $u^* > 0$ and $F_1^* > 0$ such that

$$|\widetilde{X}_h(n)| \leq F_1^* + u^* \sum_{j=0}^{n-1} \alpha^{-j}\gamma_0 \left(\alpha^j |\widetilde{X}_h(j)| \right), \quad n \geq 2N_2 + 1. \tag{7.7.13}$$

Define $I(n) = \sum_{j=0}^{n-1} \alpha^{-j}\gamma_0(\alpha^j|\widetilde{X}_h(j)|)$, when $n \geq 1$. For $n \geq 2(N_1 + 1)$

$$\begin{aligned}I(n) - I(n-1) &= \sum_{j=0}^{n-1} \alpha^{-j}\gamma_0(\alpha^j|\widetilde{X}_h(j)|) - \sum_{j=0}^{n-2} \alpha^{-j}\gamma_0 \left(\alpha^j |\widetilde{X}_h(j)| \right) \\ &= \alpha^{-(n-1)}\gamma_0 \left(\alpha^{n-1}|\widetilde{X}_h(n-1)| \right) \\ &\leq \alpha^{-(n-1)}\gamma_0 \left(\alpha^{n-1} \left(F_1^* + u^* I(n-1) \right) \right),\end{aligned}$$

where we have used the monotonicity of γ_0 at the last step. By (7.7.11) we have that $x \mapsto \gamma_0(x)/x^2$ is in $L^1(1,\infty)$. Therefore as $\alpha > 1$ we may define $F_2^* > F_1^*$ such that

$$\int_{F_2^*}^{\infty} \frac{1}{x^2}\gamma_0(x)\,dx \cdot \frac{u^*\alpha^2}{\alpha - 1} < \frac{1}{2}.$$

For $n \geq 2N_2 + 2$,

$$\sum_{j=2N_2+2}^{n} (I(j) - I(j-1)) \leq \sum_{j=2N_2+2}^{n} \alpha^{-(j-1)}\gamma_0 \left(\alpha^{j-1} \left(F_1^* + u^* I(j-1) \right) \right),$$

so

$$\begin{aligned}
I(n) &\leq I(2N_2+1) + \sum_{j=2N_2+2}^{n} \alpha^{-(j-1)}\gamma_0\left(\alpha^{j-1}\left(F_1^* + u^* I(j-1)\right)\right) \\
&\leq I(2N_2+1) + \sum_{j=2N_2+2}^{n} \alpha^{-(j-1)}\gamma_0\left(\alpha^{j-1}\left(F_2^* + u^* I(n-1)\right)\right) \\
&\leq I(2N_2+1) + \sum_{j=2N_2+2}^{n} \alpha^{-(j-1)}\gamma_0\left(\alpha^{j-1}\left(F_2^* + u^* I(n)\right)\right).
\end{aligned}$$

Define $y^* = F_2^* + u^* I(n)$. Then for $n \geq 2N_1 + 1$ and $\alpha > 1$

$$\begin{aligned}
I(n) &\leq I(2N_2+1) + \sum_{j=2N_2+2}^{n} \alpha^{-(j-1)}\gamma_0\left(\alpha^{j-1} y^*\right) \\
&= I(2N_1+1) + \left(\sum_{j=2N_2+2}^{n} \frac{\gamma_0\left(\alpha^{j-1} y^*\right)}{y^* \alpha^{j-1}}\right)\left(F_2^* + u^* I(n)\right).
\end{aligned}$$

As $x \mapsto \gamma_0(x)$ is non–decreasing

$$\begin{aligned}
\int_{y^*\alpha^{m+1}}^{\infty} \frac{\gamma_0(x)}{x^2}\,dx &= \sum_{j=m+2}^{\infty} \int_{y^*\alpha^{j-1}}^{y^*\alpha^j} \frac{\gamma_0(x)}{x^2}\,dx \\
&\geq \sum_{j=m+2}^{\infty} \frac{\gamma_0\left(y^*\alpha^{j-1}\right)}{y^{*2}\alpha^{2j}}\left(y^*\alpha^j - y^*\alpha^{j-1}\right) \\
&= \sum_{j=m+2}^{\infty} \frac{\gamma_0\left(y^*\alpha^{j-1}\right)}{y^*\alpha^{2j}\alpha^{-1}}\left(\alpha^{j-1}(\alpha-1)\alpha^{-1}\right) \\
&= \sum_{j=m+2}^{\infty} \frac{\gamma_0(y^*\alpha^{j-1})}{y^*\alpha^{j-1}} \frac{\alpha-1}{\alpha^2}.
\end{aligned}$$

Thus

$$\int_{y^*\alpha^{2N_2+1}}^{\infty} \frac{\gamma_0(x)}{x^2}\,dx \geq \frac{\alpha-1}{\alpha^2} \sum_{j=2N_2+2}^{\infty} \frac{\gamma_0\left(y^*\alpha^{j-1}\right)}{y^*\alpha^{j-1}}.$$

So for $n \geq 2N_1 + 2$,

$$\begin{aligned}
I(n) &\leq I(2N_1+1) + \frac{\alpha^2}{\alpha-1}\int_{y^*\alpha^{2N_2+1}}^{\infty} \frac{\gamma_0(x)}{x^2}\,dx\,\left(F_2^* + u^* I(n)\right) \\
&\leq I(2N_1+1) + \int_{y^*}^{\infty} \frac{\gamma_0(x)}{x^2}\,dx \frac{\alpha^2}{\alpha-1}\left(F_2^* + u^* I(n)\right) \\
&\leq I(2N_1+1) + \int_{F_2^*}^{\infty} \frac{\gamma_0(x)}{x^2}\,dx\left(\frac{\alpha^2 F_2^*}{\alpha-1} + \frac{\alpha^2 u^*}{\alpha-1} I(n)\right) \\
&= I(2N_1+1) + \int_{F_2^*}^{\infty} \frac{\gamma_0(x)}{x^2}\,dx \cdot \frac{\alpha^2 F_2^*}{\alpha-1} + \int_{F_2^*}^{\infty} \frac{\gamma_0(x)}{x^2}\,dx \frac{\alpha^2 u^*}{\alpha-1} I(n) \\
&\leq I(2N_1+1) + \frac{F_2^*}{u^*} \cdot \frac{1}{2} + \frac{1}{2} I(n).
\end{aligned}$$

Thus for $n \geq 2N_1 + 2$,

$$I(n) \leq 2I(2N_1+1) + \frac{F_2^*}{u^*}.$$

Hence I is bounded. By the definition of I and (7.7.13), this implies $|\widetilde{X}_h|$ is bounded; therefore there exists an almost surely finite random variable $x^* > 0$ such that $|\widetilde{X}_h(n)| \leq x^*$ for all $n \geq 0$ a.s. Now for

$n \geq 2N_1 + 1$ we have

$$\widetilde{X}_h(n) = \widetilde{F}_1(n) + \frac{1}{\beta} \sum_{k=N_2}^{n-1} \widetilde{r}(n-k)\alpha^{-k}\gamma\left(\alpha^k \widetilde{X}_h(k)\right).$$

By (7.7.12), and the fact that $\tilde{r}(n) = \widehat{r}(n)/\alpha^n$ tends to a finite limit as $n \to \infty$, we have that $\widetilde{X}_h(n)$ tends to a finite limit as $n \to \infty$ provided

$$\sum_{k=N_2}^{\infty} \alpha^{-k} \left|\gamma\left(\alpha^k \widetilde{X}_h(k)\right)\right| < +\infty, \quad \text{a.s,} \tag{7.7.14}$$

but as $|\widetilde{X}_h(k)| \leq x^*$ for all $k \geq 0$, by (7.7.10), and γ_0 is non–decreasing, we have

$$\sum_{k=N_2}^{n} \alpha^{-k} \left|\gamma(\alpha^k \widetilde{X}_h(k))\right| \leq \sum_{k=N_2}^{n} \alpha^{-2}\gamma_0\left(\alpha^k |\widetilde{X}_h(k)|\right) \leq \sum_{k=N_2}^{n} \alpha^{-k}\gamma_0\left(\alpha^k x^*\right)$$
$$\leq \sum_{k=0}^{n} \frac{\gamma_0\left(\alpha^k x^*\right)}{\alpha^k x^*} \cdot x^*.$$

By (7.7.11) and the fact that γ_0 is non–decreasing, we have

$$+\infty > \int_{x^*}^{\infty} \frac{\gamma_0(x)}{x^2}\,dx = \sum_{k=0}^{\infty} \int_{x^*\alpha^k}^{x^*\alpha^{k+1}} \frac{\gamma_0(x)}{x^2}\,dx \geq \sum_{k=0}^{\infty} \frac{\gamma_0(x^*\alpha^k)}{{x^*}^2\alpha^{2k}} \cdot x^*\alpha^k(\alpha-1)$$
$$= (\alpha-1)\sum_{k=0}^{\infty} \frac{\gamma_0(x^*\alpha^k)}{x^*\alpha^k},$$

which proves

$$\sum_{k=N_2}^{n} \alpha^{-k} \left|\gamma(\alpha^k \widetilde{X}_h(k))\right| \leq \sum_{k=0}^{n} \frac{\gamma_0\left(\alpha^k x^*\right)}{\alpha^k x^*} \cdot x^*$$
$$\leq \frac{1}{\alpha-1} \int_{x^*}^{\infty} \frac{\gamma_0(x)}{x^2}\,dx < +\infty,$$

which proves (7.7.14). We have shown that there exists a finite $L^*(h)$ such that

$$\lim_{n\to\infty} \frac{\widehat{X}_h(n)}{\alpha(h)^n} = \lim_{n\to\infty} \widetilde{X}_h(n) = L^*(h), \quad \text{a.s.}$$

By the definition of $\lambda(h) > 0$ in (7.7.4) we have (7.4.3). By Lemma 7.7.1 it follows that $\lambda(h) \to \lambda^* > 0$ where λ^* obeys (7.7.1). But such a λ^* is also the unique solution of (7.4.5) by the definition of W.

Chapter 8

Convergence of Euler Scheme for an Asymptotically Consistent Numerical Methods for SFDEs with Continuous Weight Functions

8.1 Introduction

In the previous chapter, we developed a discrete process X_h as the solution of a Volterra summation equation, which reproduced the main asymptotic features of the stochastic delay differential equation

$$dX(t) = \left(\int_0^{\tau_1} w_1(s)g(X(t-s))\,ds - \int_0^{\tau_2} w_2(s)g(X(t-s))\,ds\right)dt + \sigma\,dB(t),\; t \geq 0 \qquad (8.1.1)$$

provided that the step size h is chosen sufficiently small. The construction of the process X_h was motivated by the fact Euler discretisations of (8.1.1) do not faithfully reproduce the almost sure asymptotic behaviour of X. However, the results on X_h concentrate on the *pathwise asymptotic* behaviour of X_h rather than *moment error* on *finite intervals*. It is known that Euler discretisations obey

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|X(t)-\overline{X}_h(t)|^2\right] \leq C(h,T), \quad C(h,T)\to 0 \text{ as } h\to 0^+ \text{ for each } T>0. \qquad (8.1.2)$$

where $\overline{X}_h$ is an extension of X_h to continuous time. It is therefore reasonable to ask: can we similarly extend the solution of our asymptotically consistent scheme X_h to continuous time in such a way that we can show the continuous time extension $\overline{X}_h$ obeys (8.1.2)? In this chapter we show that the answer is "yes", and provide an estimate on $C(h,T)$ which depends on problem data. Moreover, we can show that it obeys $C(h,T)\to 0$ as $h\to 0$ for each fixed $T>0$. In fact, in the case when each of w_1, w_2 and ψ is Hölder continuous of order $1/2$ or greater, and g is globally Lipschitz continuous, we can show that the error is of the from

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|X(t)-\overline{X}_h(t)|^2\right] \leq c_1 h(1+c_2 e^{c_3 T}), \quad \text{for all } h<h^*,$$

where $c_1, c_2, c_3 > 0$ are constants independent of h and T.

8.1.1 Discussion of Literature

The problem of constructing satisfactory numerical methods for stochastic functional and delay differential equations has been investigated vigorously over the last ten years, and many fundamental results have been obtained. One of the most important issues is to establish that any approximation converges (in a suitable sense) over finite intervals to the true solution, to determine a rate at which this convergence takes place, and to find an upper bound on the error incurred. These are the questions addressed in this chapter for the non–standard discretisation we have chosen to employ. In order to demonstrate how our results compare with the literature, we briefly review some of the known results.

One of the early works by Küchler and Platen [45] which develops discrete time approximations for solutions of SDDEs with fixed time delays which converge in a strong sense. Other early work on the numerical approximation of hereditary equations was considered by Tudor [67]. An explicit one–step Euler–Maruyama method for SFDEs with discrete delays was considered in Baker and Buckwar [16], and strong convergence was obtained. Interestingly, this error depends on the index of Holder continuity of the initial function, as in one of our main results. An extension of the results in [16] to SDDEs with variable delay, including interpolation between meshpoints, was conducted by Mao and Sabanis [55], and to general SFDEs by Mao [50]. The paper [50] also extends these results to equations with both locally and globally Lipschitz continuous functionals. The order of strong convergence and Holder continuity condition on the initial function are the same as those found and required, respectively, in our analysis. The strong convergence order is improved to unity in a Milstein scheme in Hu, Mohammed and Yan [39] where there are discrete delays. The error is bounded according to

$$\sup_{-\tau \le t \le T} \mathbb{E}[|X(t) - X_h(t)|^2] \le ch^2$$

given initial data which is Holder–continuous with exponent $1/2$, rather than Mao's estimate of

$$\mathbb{E}\left[\sup_{-\tau \le t \le T} \mathbb{E}[|X(t) - X_h(t)|^2\right] \le ch.$$

A relationship between p–th order mean–square consistency and p–th order mean square convergence for very general SFDEs is established in Buckwar [23]; in particular, it is shown (under some natural and mild conditions) for that p–th order consistency implies p–th order convergence. Moreover, [23] considers drift implicit as well as explicit methods.

Weak convergence for general SFDEs is considered in Buckwar, Mohammed, Kuske and Shardlow [24]. Results on consistency of the methods are obtained in many of the above–mentioned papers also. Linear multistep methods are analysed in [25]. However, neither weak convergence nor consistency analysis are considered here, and we restrict our attention here to one–step methods.

Despite the special structure of the original continuous equation, our results shown that the strong convergence of the numerical solution, as well as the convergence rate and error estimate do *not* improve what is already known in the literature. However, this is not of primary concern, because standard methods do not seem to preserve the appropriate continuous time asymptotic behaviour. Thus, even though our convergence results might be crude, they show that the nonstandard method not only preserves asymptotic behaviour, but behaves acceptably in comparison to standard methods because it exhibits mean–square convergence of order $1/2$.

8.1.2 Limitations of results and future work

The method of discretisation employed in the last two chapters deals satisfactorily with the long–run dynamics, and recovers results with are at least comparable with the error analysis in the literature. However, it must be admitted that the error estimates are obtained with greater effort, and for a much more limited class of equations. Indeed, we have not attempted to employ implicit schemes, have not studied finite–dimensional equations, nor considered equations with non–constant diffusion coefficient. This is due to constraints of time and space: in any event, to do so would lead away from the main direction of the thesis.

Despite this, we feel for equations exhibiting the type of positivity properties seen here, that it may be possible to again reformulate the SDDE to a Volterra integral equation. Then it would be necessary to discretise in order to retain the dominant real solutions of the characteristic equation of the underlying differential resolvent. As regards non–constant noise, an analysis of the continuous–time equation would first be required to determine the asymptotic behaviour, and this to date is absent. An example of an SFDE for which the exact exponential behaviour of the underlying deterministic equation is preserved for unbounded noise is given in [13]. Given that we have been able to recover the asymptotic behaviour for constant noise given in [13] in Chapter 5, there is some reason to be confident that we could once again perform the asymptotic analysis in discrete–time.

8.2 Recapitulation of Results from Previous Chapter

As in the previous chapter, suppose that $\tau := \tau_2 > \tau_1 > 0$ and that

$$w_1 \in C([0, \tau_1]; [0, \infty)), \quad w_2 \in C([0, \tau_2]; [0, \infty)) \tag{8.2.1}$$

We also request that

$$\int_0^{\tau_1} w_1(s)\, ds = 1, \quad \int_0^{\tau_2} w_2(s)\, ds = 1 \tag{8.2.2}$$

and that

$$\int_0^t w_1(s)\, ds \ge \int_0^t w_2(s)\, ds, \quad t \in [0, \tau_1]. \tag{8.2.3}$$

Assume that the function $g : \mathbb{R} \to \mathbb{R}$ obeys

$$g \text{ is globally Lipschitz continuous with } |g(x) - g(y)| \le K_2|x - y| \text{ for all } x, y \in \mathbb{R}; \tag{8.2.4}$$
$$g \text{ is globally linearly bounded with } |g(x)| \le K_2(1 + |x|) \text{ for all } x \in \mathbb{R}, \tag{8.2.5}$$

and that there exists

$$\beta \ge 0 \text{ such that } \lim_{x\to\infty} \frac{g(x)}{x} = \lim_{x\to-\infty} \frac{g(x)}{x} = \beta. \tag{8.2.6}$$

Let $\sigma \ne 0$ and B be a standard one–dimensional Brownian motion. Let $\psi \in C([-\tau, 0]; \mathbb{R})$. Then there is a unique continuous adapted process X which satisfies

$$dX(t) = \left(\int_0^{\tau_1} w_1(s)g(X(t-s))\, ds - \int_0^{\tau_2} w_2(s)g(X(t-s))\, ds \right) dt \tag{8.2.7a}$$
$$+\sigma dB(t), \quad t \ge 0,$$
$$X(t) = \psi(t), \qquad t \in [-\tau, 0]. \tag{8.2.7b}$$

We rewrite (8.2.7) as a Volterra integral equation. In doing so, we find it convenient to introduce some auxiliary functions. Define W_1, W_2 and W by

$$W_i(t) = \int_0^{t\wedge\tau_i} w_i(s)\, ds, \quad t \ge 0 \quad i = 1, 2, \quad W(t) := W_1(t) - W_2(t), \quad t \ge 0. \tag{8.2.8}$$

We also introduce the functions I_1 and I_2 which depend on the function ψ

$$I_i(\psi, t) = \int_{-\tau_i}^{0} \left(\int_{-s}^{\tau_i\wedge(t-s)} w_i(u)\, du \right) g(\psi(s))\, ds, \quad t \ge 0, \quad i = 1, 2, \tag{8.2.9}$$

and the constants

$$I_i^*(\psi) = \int_{-\tau_i}^{0} \left(\int_{-s}^{\tau_i} w_i(u)\, du \right) g(\psi(s))\, ds, \quad i = 1, 2. \tag{8.2.10}$$

We have already shown under these hypotheses that X can be written as the solution of a Volterra integral equation.

Lemma 8.2.1. *Suppose that w_1 and w_2 obey* (8.2.1), (8.2.2) *and that g obeys* (8.2.4) *and* (8.2.6). *Then there is a unique continuous adapted process X which obeys* (8.2.7).

(i) $I_i(\psi, t) = I_i^*$, where $t \ge \tau_i$ and

(ii) If W is given by (8.2.8) *and I_i by* (8.2.9) *then X obeys*

$$X(t) = \psi(0) + I_1(\psi, t) - I_2(\psi, t) + \int_0^t W(s)g(X(t-s))\, ds + \sigma B(t), \quad t \ge 0,$$
$$X(t) = \psi(0) + I_1^*(\psi) - I_2^*(\psi) + \int_0^{\tau_2} W(s)g(X(t-s))\, ds + \sigma B(t), \quad t \ge \tau_2.$$

Since w_1 and w_2 are continuous and defined on compact intervals, both possess moduli of continuity. More precisely there exist functions $\delta_1 : [0, \infty) \to [0, \infty)$ and $\delta_2 : [0, \infty) \to [0, \infty)$ such that $\delta_i(0) = 0$ and $\lim_{h\to0^+} \delta_i(h) = 0$ for $i = 1, 2$ and

$$\max_{|t-s|\le h, s,t\in[0,\tau_i]} |w_i(t) - w_i(s)| \le \delta_i(h) \qquad \text{for all } h \in [0, \tau_i], \quad i = 1, 2. \tag{8.2.11}$$

Moreover each δ_i is non–decreasing. It is also useful to introduce the notation

$$\bar{w}_i = \max_{0\le t\le\tau_i} w_i(t), \quad i = 1, 2. \tag{8.2.12}$$

Since ψ is continuous and defined on a compact interval, it also possesses a modulus of continuity. More precisely, there exists a function $\delta_3 : [0,\infty) \to [0,\infty)$ such that $\delta_3(0) = 0$ and δ_3 is non–decreasing with $\lim_{h\to 0^+} \delta_3(h) = 0$ and

$$\max_{0<t-s\leq h; s,t\in[0,\tau]} |\psi(t) - \psi(s)| \leq \delta_3(h) \qquad \text{for all } h \in [0,\tau]. \tag{8.2.13}$$

Define also

$$G := \max_{-\tau\leq s\leq 0} |g(\psi(s))|. \tag{8.2.14}$$

We pick $h \in (0,\tau_1)$ so small that we may define $N_2 = N_2(h) \in \mathbb{N}$, $N_2 \geq 2$ such that

$$N_2 h \leq \tau_2 < (1 + N_2)h. \tag{8.2.15}$$

This automatically forces $h < \tau_2$. Extend $w_1(t) = 0$ for $t \in [\tau_1, \tau_2]$. We now define the sequence $\widehat{W}_h$ parameterised $h > 0$ by

$$\widehat{W}_h(0) = \tau_2 (\delta_1(h) + \delta_2(h)) \tag{8.2.16a}$$

$$\widehat{W}_h(j) = \tau_2 (\delta_1(h) + \delta_2(h)) + \sum_{l=0}^{j-1} w_1(lh)h - \sum_{l=0}^{j-1} w_2(lh)h,\ j = 1, \cdots, N_2(h) \tag{8.2.16b}$$

$$\widehat{W}_h(N_2 + 1) = 0. \tag{8.2.16c}$$

It is implicit here that $\widehat{W}_h(n)$ is an approximation to $W(nh)$. However, in order to recover the positivity of W in the approximation, we have added a correction term to the naive approximation

$$\widehat{W}_{\text{naive}}(n) := \sum_{l=0}^{n-1} w_1(lh)h - \sum_{l=0}^{n-1} w_2(lh)h.$$

The following lemma shows that this can be achieved in such a way that any resulting biasses or errors introduced by the correction can be controlled. Also in the lemma, we record some estimates on the approximation of $\widehat{W}$ to W.

Lemma 8.2.2. *Let $h > 0$, and suppose that $\tau_2 > \tau_1$. Let w_i have modulus of continuity δ_i given by* (8.2.11)*. Suppose that $N_2 = N_2(h)$ obeys* (8.2.15)*. Define $\widehat{W}_h$ by* (8.2.16)*.*

(i) For $j = 0, \ldots, N_2$, $\widehat{W}_h(j) \geq 0$.

(ii) With $\overline{w}_i$ defined by (8.2.12)*, we have*

$$\left| \sum_{j=0}^{N_2} \widehat{W}_h(j)h - \left(\int_0^{\tau_2} s w_2(s)\,ds - \int_0^{\tau_1} s w_1(s)\,ds \right) \right|$$
$$\leq \frac{3}{2}\tau_2^2 (\delta_1(h) + \delta_2(h)) + h\{4 + \tau_2(\overline{w}_1 + \overline{w}_2) + 2\tau_2(\delta_1(h) + \delta_2(h))\} := \eta(h). \tag{8.2.17}$$

We recall the discretisation of X. Let $\widehat{W}_h$ be defined by (8.2.16). Suppose that

$$(\xi(n))_{n\geq 1} \text{ is a sequence of i.i.d. } N(0,1) \text{ random variables.} \tag{8.2.18}$$

We suppose for $n \geq 0$ that $\widehat{X}(n)$ is an approximation for $X(nh)$. Suppose we approximate $I_i(\psi, nh)$ by $\widehat{I}_i(\psi, n)$ and $I_i^*(\psi)$ by $\widehat{I}_i^*(\psi)$. Suitable formulae for $\widehat{I}_i^*$ and $\widehat{I}_i$ are given at the end, in (8.6.2). Define

$(\widehat{X}_h(n))_{n\geq 0}$ by

$$\widehat{X}_h(n+1) = \psi(0) + \widehat{I}_1(\psi,n) - \widehat{I}_2(\psi,n) + \sum_{j=0}^{n} \widehat{W}_h(j) g(\widehat{X}_h(n-j))h \tag{8.2.19a}$$

$$+ \sigma\sqrt{h}\sum_{j=0}^{n}\xi(j+1), \qquad n = 0,\cdots,N_2-1,$$

$$\widehat{X}_h(n+1) = \psi(0) + \widehat{I}_1^*(\psi) - \widehat{I}_2^*(\psi) + \sum_{j=0}^{N_2} \widehat{W}_h(j) g(\widehat{X}_h(n-j))h \tag{8.2.19b}$$

$$+ \sigma\sqrt{h}\sum_{j=0}^{n}\xi(j+1), \qquad n \geq N_2,$$

where $\widehat{X}_h(0) = \psi(0)$. The construction of $\widehat{I}_i^*$ and $\widehat{I}_i$ in (8.6.2) uses implicitly the identification $\widehat{X}_h(n) = \psi(nh)$ for $n = -N_2, \ldots, 0$.

8.3 Discussion and Statement of Main Result

The equations (8.2.19a) and (8.2.19b) define a sequence of random variables approximating the continuous–time process X which is the solution of (8.2.7) at a *sequence* of times. We would like however to be able to estimate the error between the true solution of (8.2.7) and its approximation at *all* times on any given interval $[0,T]$. To this end, we introduce a piecewise continuous interpolation $\overline{X}_h$ associated with a uniform step size $h > 0$. Our main result, which we now state, demonstrates that this interpolant converges to X as $h \to 0^+$ in the sense that (8.1.2) holds.

Moreover, an explicit estimate is known for $C(h,T)$ in (8.1.2) in terms of the data. Of course, such results have already been established for a wide range of general stochastic functional differential equations, so such a result does not add to the state of the art; the convergence rate of solutions is no faster than that of standard methods on any compact interval $[0,T]$. Indeed, as the scheme (8.2.19) exploits some idiosyncratic features of (8.2.7), and the proof of convergence exploits these features, it would seem that we have constructed a more complicated scheme, whose effectiveness is more difficult to establish, and which supplies a more conservative error bound than general, simpler existing methods. Therefore, the merit of establishing afresh such convergence result would seem to be severely limited.

We attempt now to justify our interest in such a result. We remember that we have already demonstrated by means of sample simulations that standard Euler–Maruyama methods for discretising (8.2.7) do not reliably reproduce the asymptotic behaviour of the solution of (8.2.7), even though the mean–square error in (8.1.2) can be made arbitrarily small on any interval $[0,T]$ by choosing $h > 0$ sufficiently small. Therefore, we may think of the method (8.2.19) as being more *robust* than standard Euler schemes: not only does it reproduce the asymptotic behaviour of (8.2.7) (a purpose for which it has been designed) in contrast to simple Euler schemes, but it also satisfies (8.1.2) (for which it has not been designed), albeit with greater error for a given amount of computational effort than an Euler scheme.

With X_h defined by (8.2.19), we define the piecewise continuous approximation $\overline{X}_h$ to X by

$$\overline{X}_h(t) = \widehat{X}_h\left(\left[\frac{t}{h}\right]\right), \quad t \geq 0. \tag{8.3.1}$$

Theorem 8.3.1. *Suppose that g obeys* (8.2.5)*,* (8.2.4) *that w_1 and w_2 are non–negative continuous functions which obey* (8.2.2) *and* (8.2.3)*. and that ψ is continuous. Let X be the unique continuous adapted process which obeys* (8.2.7)*. Let $T > 0$ and $h > 0$ be such that $h < \tau_1 \wedge T$. Let $\overline{X}_h$ be defined by* (8.3.1) *where $\widehat{X}_h$ is the solution of* (8.2.19)*. Then there exists a $h^* > 0$ independent of T such that there exists a function $C : [0,h^*) \times [0,\infty) : (h,T) \mapsto C(h,T)$ such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X(t) - \overline{X}_h(t)|^2\right] \leq C(h,T), \quad 0 < h < h^*, \tag{8.3.2}$$

where $C(h,T) \to 0$ as $h \to 0^+$ for each fixed $T > 0$. Furthermore, if w_1, w_2 and ψ are Hölder continuous of order $1/2$ or greater, there exist h and T independent positive constants c_1, c_2 and c_3 such that

$$\mathbb{E}\left[\sup_{0\le t\le T}|X(t)-\overline{X}_h(t)|^2\right] \le c_1 h(1+c_2 e^{c_3 T}), \quad h\in(0,h^*). \tag{8.3.3}$$

Remark 8.3.1. For equations with Lipschitz continuous coefficients with additive noise (such as (8.2.7)) standard uniform Euler methods with step size h gives an error $O(h^2)$ in the sense that

$$\max_{1\le n\le N_3}\mathbb{E}[|X(nh)-\hat{X}_h(n)|^2] \le Ch^2, \tag{8.3.4}$$

for some constant $C > 0$, which can depend on T, where N_3 is defined by (8.5.1). Our estimate in (8.3.3) is only $O(h)$, however. The reason for this is that the estimate in (8.3.4), which can be estimated with arbitrarily small error by Monte Carlo simulation, considers only the values of the error at the meshpoints. On the other hand, the error considered in (8.3.3) deals with the error over the whole interval. To derive an error estimate in this case, it is necessary to have estimates for the interpolation error over intervals of length h for both the true solution X and the continuous extension $\overline{X}_h$ of $\hat{X}_h$. However, the relevant interpolation error in the former case is of order h only, owing to the estimates of $|B(t)-B(nh)|$ for $t\in[nh,(n+1)h]$. The relevant estimate is

$$\mathbb{E}\left[\max_{nh\le t\le(n+1)h}|B(t)-B(nh)|^2\right] \le 4h,$$

and this error propagates into the corresponding error estimates for $|X(t)-X(nh)|$ on $[nh,(n+1)h]$. This "forcing term" in the Volterra difference inequality for the error on each interval leads to the $O(h)$ estimate in (8.3.3).

Remark 8.3.2. The construction of $h^* > 0$ in the proof of Theorem 8.3.1 is technical. Here is an estimate. Let $K_3 > 0$ be defined by

$$K_3 := 5K_2^2\left(\int_0^{\tau_2} W(s)\,ds + \eta^*\right), \tag{8.3.5}$$

where W is defined by (8.2.8), K_2 is the constant in (8.2.5) and (8.2.4), $\eta^* > 0$ is given by

$$\eta^* := \frac{3}{2}\tau_2^2\left(\delta_1(\tau_1)+\delta_2(\tau_2)\right) + h\left\{4+\tau_2\left(\overline{w}_1+\overline{w}_2\right)+2\tau_2\left(\delta_1(\tau_1)+\delta_2(\tau_2)\right)\right\}, \tag{8.3.6}$$

$\overline{w}_1, \overline{w}_2$ are given by (8.2.12), and δ_1, δ_2 by (8.2.11). Define C_4 by

$$C_4(h) = (\tau_2+h)\left(2\left(\delta_1(h)+\delta_2(h)\right)\tau_2 + h(\overline{w}_1+\overline{w}_2)\right), \quad \text{for } h\ge 0, \tag{8.3.7}$$

and C_3 by

$$C_3(h) := 2K_3 h + K_3 C_4(h) + 2K_3\tau_2(\delta_1(h)+\delta_2(h))h, \quad \text{for } h\ge 0. \tag{8.3.8}$$

Then we may choose $h^* > 0$ such that

$$C_3(h^*) \le 1/4. \tag{8.3.9}$$

Remark 8.3.3. We also give an estimate on $C(h)$ in (8.3.2). Let C_2 be defined by

$$C_2(T) = \left(\frac{1}{2}+\frac{33\sigma^2}{4\sqrt{2}K_2}+3\max_{-\tau\le s\le t}\psi(s)^2\right)e^{8\sqrt{2}K_2 T}, \quad T\ge 0. \tag{8.3.10}$$

Also define

$$C_1(T) = 12K_2^2\tau_1 + 12K_2^2\tau_1 C_2(2T) + 12\sigma^2, \quad T\ge 0, \tag{8.3.11}$$

where K_2 is the linear growth and Lipschitz constant of g from (8.2.5) and (8.2.4). With δ_3 defined by (8.2.13), δ_1 and δ_2 defined by (8.2.11), and G defined by (8.2.14), define

$$\varepsilon_i(h) = K_2\delta_3(h)\tau_i + 2G\tau_i\left(\tau_i\delta_i(h)+h\bar{w}_i\right) + hG + 2(2+\tau_i\delta_i(h))Gh, \quad i=1,2, \tag{8.3.12}$$

$$\varepsilon_i^*(h) = K_2\delta_3(h)\tau_i + G\tau_i\left(\tau_i\delta_i(h)+h\bar{w}_i\right) + Gh + (2+\tau_i\delta_i(h))Gh, \quad i=1,2. \tag{8.3.13}$$

Also define

$$\varepsilon(h) = \Bigg\{5C_1(T \vee \tau_2)h + 5\left(\varepsilon_1^*(h) + \varepsilon_2^*(h)\right)^2 \vee 5\left(\varepsilon_1(h) + \varepsilon_2(h)\right)^2 \tag{8.3.14}$$

$$+ \left(5K_2^2 + 5K_2^2 C_2(T \vee \tau_2)\right) C_4(h)^2 \Bigg\} \bigvee \Bigg\{12K_2^2 h^2 + 12\sigma^2 h + 12K_2^2 h^2 C_2(h)\Bigg\}$$

It turns out that we obtain the estimate for the mean square error with step h to be

$$C(h) = \varepsilon(h)\left\{1 + 2K_3\left(\int_0^{\tau_2} W(s)\,ds + \eta^*\right) e^{4K_3(T+\tau_1 \wedge T)}\right\},$$

where η^* is defined by (8.3.6) and K_3 by (8.3.5). Thus we have the error bound

$$\mathbb{E}\left[\max_{t\in[0,T]} \left|X(t) - \overline{X}_h(t)\right|^2\right] \le \varepsilon(h)\left\{1 + 2K_3\left(\int_0^{\tau_2} W(s)\,ds + \eta^*\right) e^{4K_3(T+\tau_1 \wedge T)}\right\}. \tag{8.3.15}$$

Remark 8.3.4. It can be seen that if w_1, w_2 and ψ are Hölder continuous of order greater than or equal to $1/2$, then the estimate (8.3.15) is of the form

$$\mathbb{E}\left[\max_{t\in[0,T]} \left|X(t) - \overline{X}_h(t)\right|^2\right] \le C_5(T)h,$$

where C_5 is independent of h. This is because in this case $\varepsilon(h)$ is order h for small h. To see this, the assumption on the Hölder continuity implies $|w_1(t)-w_1(s)| \le K_5|t-s|^{1/2}$, $|w_2(t)-w_2(s)| \le K_6|t-s|^{1/2}$, $|\psi(t) - \psi(s)| \le K_7|t-s|^{1/2}$. These facts force $\delta_i(h) \le K_8 h^{1/2}$ for $i = 1, 2, 3$, from which we can infer that $\varepsilon_i(h)$ and $\varepsilon_i^*(h)$ are order $\sqrt{h}$ for small h and $i = 1, 2$. The fact that $\delta_i(h) \le K_8 h^{1/2}$ for $i = 1, 2$ also implies that $C_4(h)$ is order h for small h. These estimates show that $\varepsilon(h)$ in (8.3.14) is order h, as required. We also note that C_5 cannot grow faster than exponentially in T. This follows from exponential growth bounds in T in the h–independent factor in (8.3.15), provided that C_1 and C_2 grow exponentially fast in T. We see that an exponential bound on C_2 implies an exponential bound on C_1. An exponential bound on C_1 is true by e.g., Theorem 5.4.1 in [52]. An estimate for the equation (8.2.7) is given in Lemma 8.4.1 below.

Remark 8.3.5. The proof of this theorem involves deriving six Lemmas. The six Lemmas calculate bounds: Lemma 8.4.1 calculates the bound on the second moment; Lemma 8.4.2 calculates a bound on the difference between the resolvent $X(t)$ and the resolvent evaluated at the mesh point $X(nh)$; Lemma 8.4.3 calculates a bound on the difference between the continuous and discrete weights; Lemma 8.5.1 calculates a bound for the moment error on the three intervals $[0, h]$, $[nh, (n + 1)h]$ for $T < \tau_2$, and $[nh, (n + 1)h]$ for $T > \tau_2$. Lemma 8.5.2 we calculate α^* which enables us to calculate the bound h^* on the mesh size h. Lemma 8.5.3 calculates a single bound for the moment error. Combining all this information together we are able to prove Theorem 8.3.1.

8.4 Preliminary Results

Lemma 8.4.1. *Let X be the solution of* (8.2.7) *and let $T > 0$. Then*

$$\mathbb{E}\left[\max_{t\in[-\tau_2,T]} |X(t)|^2\right] \le C_2(T), \tag{8.4.1}$$

where C_2 is defined in (8.3.10).

Proof. The proof that (8.4.1) holds can be deduced by an argument similar to that of Theorem 5.4.1 in [52]. We give the revised argument in full. Let $\phi \in C([-\tau, 0]; \mathbb{R})$ and define f by

$$f(\phi) := \int_0^{\tau_1} w_1(s)g(\phi(-s))\,ds - \int_0^{\tau_2} w_2(s)g(\phi(-s))\,ds.$$

Then X obeys $dX(t) = f(X_t)\,dt + \sigma\,dB(t)$. Suppose temporarily there exists $K > 0$ such that $|f(\phi)|^2 \leq K(1 + \|\phi\|^2)$. Then for any $c > 0$ we have

$$\begin{aligned} X^2(t) &= X^2(0) + \int_0^t \left(2X(s)f(X_s) + \sigma^2\right)\,ds + \int_0^t 2\sigma X(s)\,dB(s) \\ &\leq X^2(0) + \int_0^t \left(cX^2(s) + \frac{1}{c}f^2(X_s) + \sigma^2\right)\,ds + \int_0^t 2\sigma X(s)\,dB(s) \\ &\leq X^2(0) + \int_0^t \left(cX^2(s) + \frac{K}{c}(1 + \|X_s\|^2) + \sigma^2\right)\,ds + 2\sigma\int_0^t X(s)\,dB(s). \end{aligned}$$

Hence we have

$$\begin{aligned} \mathbb{E}\left[\sup_{0\leq t\leq T} X^2(t)\right] \leq X^2(0) + \mathbb{E}\int_0^T \left(cX^2(s) + \frac{K}{c}(1 + \|X_s\|^2) + \sigma^2\right)\,ds \\ + 2|\sigma|\mathbb{E}\left[\sup_{0\leq t\leq T}\int_0^t X(s)\,dB(s)\right]. \end{aligned}$$

Now by the Burkholder Davis Gundy inequality for any $\alpha > 0$ and for any $b > 0$ we get

$$\begin{aligned} \mathbb{E}\left[\sup_{0\leq t\leq T}\int_0^t X(s)\,dB(s)\right] &\leq 4\mathbb{E}\left[\left(\left(\int_0^T X(s)^2\,ds\right)\right)^{1/2}\right] \\ &\leq 2\mathbb{E}\left[2\left(\sup_{0\leq t\leq T} X(s)^2\right)^{1/2}\left(\int_0^T 1\,ds\right)^{1/2}\right] \\ &\leq 2\mathbb{E}\left[\alpha\sup_{0\leq t\leq T} X(s)^2 + \frac{1}{\alpha}T\right]. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}\left[\sup_{0\leq t\leq T} X^2(t)\right] \leq X^2(0) + \mathbb{E}\int_0^T \left(cX^2(s) + \frac{K}{c}(1 + \|X_s\|^2) + \sigma^2 + \frac{4|\sigma|}{\alpha}\right)\,ds \\ + 4\alpha|\sigma|\mathbb{E}\left[\sup_{0\leq t\leq T} X(s)^2\right]. \end{aligned}$$

Therefore with $c = \sqrt{K}$, $b = \frac{1}{2\sqrt{K}}\left(\sqrt{K} + \sigma^2 + \frac{4|\sigma|}{\alpha}\right)$ and $1 - 4\alpha|\sigma| > 0$ we have

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\leq t\leq T} X^2(t)\right] \\ &\leq X^2(0) + \mathbb{E}\int_0^T \left(c\sup_{0\leq u\leq s} X(u)^2 + \frac{K}{c}\sup_{-\tau\leq u\leq s} X(u)^2 + \left(\frac{K}{c} + \sigma^2 + \frac{4|\sigma|}{\alpha}\right)\right)\,ds \\ &\qquad + 4|\sigma|\alpha\mathbb{E}\left[\sup_{0\leq t\leq T}|X(t)|^2\right] \\ &\leq X^2(0) + \mathbb{E}\int_0^T \left(\left(c + \frac{K}{c}\right)\sup_{-\tau\leq u\leq s} X(u)^2 + \left(\frac{K}{c} + \sigma^2 + \frac{4|\sigma|}{\alpha}\right)\right)\,ds \\ &\qquad + 4|\sigma|\alpha\mathbb{E}\left[\sup_{0\leq t\leq T}|X(t)|^2\right] \\ &\leq \frac{1}{1 - 4\alpha|\sigma|}X^2(0) + \frac{2\sqrt{K}}{1 - 4\alpha|\sigma|}\mathbb{E}\int_0^T \left(b + \sup_{-\tau\leq u\leq s}|X(u)|^2\right)\,ds. \end{aligned}$$

Also because

$$\mathbb{E}\left[\max_{-\tau\leq s\leq t} X(s)^2\right] \leq \max_{-\tau\leq s\leq t}\psi(s)^2 + \mathbb{E}\left[\max_{-\tau\leq s\leq t} X(s)^2\right],$$

we get

$$b+\mathbb{E}\left[\sup_{-\tau\le t\le T} X^2(t)\right] \le b+\max_{-\tau\le s\le t}\psi(s)^2+\frac{1}{1-4\alpha|\sigma|}\psi^2(0)$$
$$+\frac{2\sqrt{K}}{1-4\alpha|\sigma|}\int_0^T\left(b+\mathbb{E}\sup_{-\tau\le u\le s}|X(u)|^2\right)ds.$$

Let $x(t)=b+\mathbb{E}\left[\sup_{-\tau\le u\le t}|X(u)|^2\right]$ for $T\ge 0$. Then

$$x(T)\le b+\max_{-\tau\le s\le t}\psi(s)^2+\frac{1}{1-4\alpha|\sigma|}\psi^2(0)+\frac{2\sqrt{K}}{1-4\alpha|\sigma|}\int_0^T x(s)\,ds,\quad T\ge 0.$$

Hence by Gronwall's inequality, we have for $T\ge 0$

$$b+\mathbb{E}\left[\sup_{-\tau\le u\le t}|X(u)|^2\right]=x(T)\le\left(b+\max_{-\tau\le s\le t}\psi(s)^2+\frac{1}{1-4\alpha|\sigma|}\psi^2(0)\right)e^{\frac{2\sqrt{K}}{1-4\alpha|\sigma|}T},$$

so by fixing $\alpha=1/(8|\sigma|)$ we get $1-4|\sigma|\alpha=1/2$ and so

$$\begin{aligned}&\mathbb{E}\left[\sup_{-\tau\le u\le t}|X(u)|^2\right]\\&\le\left(\frac{1}{2\sqrt{K}}\left(\sqrt{K}+33\sigma^2\right)+\max_{-\tau\le s\le t}\psi(s)^2+2\psi^2(0)\right)e^{4\sqrt{K}T}\\&\le\left(\frac{1}{2}+\frac{33\sigma^2}{2\sqrt{K}}+3\max_{-\tau\le s\le t}\psi(s)^2\right)e^{4\sqrt{K}T}.\end{aligned}$$

It remains to estimate K. By (8.2.5), (8.2.1) and (8.2.2) we have

$$\begin{aligned}|f(\phi)|&\le\int_0^{\tau_1}w_1(s)K_2(1+|\phi(-s)|)\,ds+\int_0^{\tau_2}w_2(s)K_2(1+|\phi(-s)|\,ds\\&\le K_2\int_0^{\tau_1}w_1(s)|\phi(-s)|\,ds+2K_2+K_2\int_0^{\tau_2}w_2(s)|\phi(-s)|\,ds\\&\le K_2\|\phi\|\int_0^{\tau_1}w_1(s)\,ds+2K_2+K_2\|\phi\|\int_0^{\tau_2}w_2(s)\,ds\\&\le 2K_2(1+\|\phi\|).\end{aligned}$$

Therefore $|f(\phi)|^2\le 4K_2^2(1+\|\phi\|)^2\le 8K_2^2(1+\|\phi\|^2)$. Hence we may set $K=8K_2^2$; thus $\sqrt{K}=2\sqrt{2}K_2$. This together with the definition of C_2 from (8.3.10) gives the estimate (8.4.1) as required. □

Lemma 8.4.2. *Let X be the solution of* (8.2.7). *Let $T>0$. If $h\in(0,\tau_1\wedge T)$, we have*

$$\mathbb{E}\left[\max_{t\in[nh,(n+1)h]}|X(t)-X(nh)|^2\right]\le C_1(T)h,\qquad \textit{for all } n\le T/h. \tag{8.4.2}$$

where the increasing function $t\mapsto C_1(t)$ is defined by (8.3.11).

Remark 8.4.1. We begin this proof by directly calculating $|X(t)-X(nh)|$. Combining Doob's and Cauchy–Schwarz inequality we get an explicit estimate for the bound.

The estimate supplied by this Lemma is much less sharp than comparable results available in the literature. However, as it enables us to establish simpler estimates later on, and we have not attempted throughout to optimise the upper bound on the mean square error estimate, we do not press the argument here either. Part of our motivation here is to avoid analysing separately the cases where $T<\tau_2$ and $T\ge\tau_2$.

Proof of Lemma 8.4.2. We show that (8.4.1) implies (8.4.2). Let $t\in[nh,(n+1)h]$. Then

$$X(t)-X(nh)=\int_{nh}^t\left(\int_0^{\tau_1}w_1(u)g\left(X(s-u)\right)du-\int_0^{\tau_2}w_2(u)g\left(X(s-u)\right)du\right)ds$$
$$+\sigma(B(t)-B(nh)).$$

Therefore by (8.2.5)

$$
\begin{aligned}
&|X(t)-X(nh)| \\
&\leq K_2 \int_{nh}^{t} \left(\int_0^{\tau_1} w_1(u)(1+|X(s-u)|)\,du + \int_0^{\tau_2} w_2(u)(1+|X(s-u)|)\,du \right) ds \\
&\qquad\qquad + |\sigma||B(t)-B(nh)|,
\end{aligned}
$$

which by (8.2.2) yields

$$
\begin{aligned}
&\max_{t\in[nh,(n+1)h]} |X(t)-X(nh)| \\
&\leq K_2 \int_{nh}^{(n+1)h} \left(2 + \int_0^{\tau_2} (w_1(u)+w_2(u))\,|X(s-u)|\,du \right) ds \\
&\quad + |\sigma| \max_{nh\leq t\leq (n+1)h} \left| \int_{nh}^{t} dB(s) \right| \\
&\leq 2K_2 h + K_2 \int_{nh}^{(n+1)h} \int_0^{\tau_2} (w_1(u)+w_2(u))\,|X(s-u)|\,du\,ds \\
&\quad + |\sigma| \max_{nh\leq t\leq (n+1)h} \left| \int_{nh}^{t} dB(s) \right|.
\end{aligned}
$$

Using Doob's and Cauchy–Schwarz inequalities we get

$$
\begin{aligned}
&\mathbb{E}\left[\max_{t\in[nh,(n+1)h]} |X(t)-X(nh)|^2 \right] \\
&\leq 12K_2^2h^2 + 3K_2^2 \mathbb{E}\Bigg[\left(\int_{nh}^{(n+1)h} \left[\int_0^{\tau_2} (w_1(u)+w_2(u))\,|X(s-u)|\,du \right] ds \right)^2 \Bigg] \\
&\qquad + 3\sigma^2 \mathbb{E}\left[\max_{nh\leq t\leq (n+1)h} \left| \int_{nh}^{t} dB(s) \right|^2 \right] \\
&\leq 12K_2^2h^2 + 3K_2^2 h \mathbb{E}\Bigg[\int_{nh}^{(n+1)h} \left[\int_0^{\tau_2} (w_1(u)+w_2(u))\,|X(s-u)|\,du \right]^2 ds \Bigg] \\
&\qquad + 3\sigma^2 \mathbb{E}\left[\max_{nh\leq t\leq (n+1)h} \left| \int_{nh}^{t} dB(s) \right|^2 \right] \\
&\leq 12K_2^2h^2 + 3K_2^2 h \int_{nh}^{(n+1)h} 2 \int_0^{\tau_2} (w_1(u)+w_2(u)) \mathbb{E}\left[|X(s-u)|^2 \right] du\,ds \\
&\qquad + 12\sigma^2 h \\
&\leq 12K_2^2h^2 + 12K_2^2h^2 C_2((n+1)h) + 12\sigma^2 h \\
&\leq h\left(12K_2^2 h + 12K_2^2 h C_2(T+h) + 12\sigma^2 \right) \\
&\leq h\left(12K_2^2 \tau_1 + 12K_2^2 \tau_1 C_2(2T) + 12\sigma^2 \right),
\end{aligned}
$$

where we have used the fact that $h < T \wedge \tau_1$ and that C_2 is increasing. The definition of C_1 gives the result. □

Lemma 8.4.3. *Let W be defined by* (8.2.8). *Let $h \in (0,\tau_1)$ and N_2 be given by* (8.2.15). *Define $\overline{W}_h$ by*

$$
\overline{W}_h(t) = \widehat{W}_h\left(\left[\frac{t}{h} \right] \right), \quad t \geq 0, \tag{8.4.3}
$$

and $\eta^* > 0$ by (8.3.6). *Then*

(i)

$$
\int_0^{(N_2+1)h} \overline{W}_h(s)\,ds \leq \int_0^{\tau_2} W(s)\,ds + \eta^*, \tag{8.4.4}
$$

(ii) If C_4 is defined by (8.3.7) *then $C_4(h) \to 0$ as $h \to 0^+$ and*

$$\int_0^{(N_2+1)h} |\overline{W}_h(s) - W(s)|\, ds \le C_4(h). \tag{8.4.5}$$

Remark 8.4.2. This lemma provides some uniform bounds on the difference between W and $\overline{W}_h$. As this proof is short no outline is given.

Proof. Recalling that $\int_0^{\tau_2} W(s)\,ds = \int_0^{\tau_2} sw_2(s)\,ds - \int_0^{\tau_1} sw_1(s)\,ds$, and by using (8.4.3) and (8.2.17), we obtain

$$\int_0^{(N_2+1)h} \overline{W}_h(s)\,ds = \sum_{j=0}^{N_2} \int_{jh}^{(j+1)h} \overline{W}_h(s)\,ds = \sum_{j=0}^{N_2} \widehat{W}_h(j)h \le \int_0^{\tau_2} W(s)\,ds + \eta(h),$$

where η is defined as the righthand side of (8.2.17). Since the moduli of continuity of w_1 and w_2 are non–decreasing in their arguments, and w_1 and w_2 are restricted to $[0,\tau_1]$ and $[0,\tau_2]$ respectively, inspection of the formula for η reveals that $\eta(h) \le \eta^*$, where η^* is given by (8.3.6). Therefore (8.4.4) holds. To prove (8.4.5) we start by noting

$$\int_0^{(N_2+1)h} |W(s) - \overline{W}_h(s)|\,ds = \sum_{j=0}^{N_2} \int_{jh}^{(j+1)h} |W(s) - \widehat{W}_h(j)|\,ds.$$

We estimate $|W(s) - \widehat{W}_h(j)|$ for $s \in [jh,(j+1)h]$ and $j = 0,\cdots,N_2$. Recall from the extension of w_1 to $(\tau_1,\tau_2]$ and the definition $w(s) = w_1(s) - w_2(s)$ that $W(s) = \int_0^{s\wedge\tau_2} w(u)\,du$. It can be shown that

$$|W(s) - \widehat{W}_h(j)| \le 2(\delta_1(h) + \delta_2(h))\tau_2 + h(\overline{w}_1 + \overline{w}_2).$$

Thus

$$\int_{jh}^{(j+1)h} |W(s) - \widehat{W}_h(j)|\,ds \le h\left\{2\left(\delta_1(h) + \delta_2(h)\right)\tau_2 + h(\overline{w}_1 + \overline{w}_2)\right\},$$

and so

$$\begin{aligned}\int_0^{(N_2+1)h} |W(s) - \widehat{W}_h(j)|\,ds &\le h(N_2+1)\left(2\left(\delta_1(h) + \delta_2(h)\right)\tau_2 + h(\overline{w}_1 + \overline{w}_2)\right)\\ &\le (\tau_2 + h)\left(2\left(\delta_1(h) + \delta_2(h)\right)\tau_2 + h(\overline{w}_1 + \overline{w}_2)\right),\end{aligned}$$

which by the definition of $C_4(h)$ in (8.3.7), proves (8.4.5). The fact that $\delta_i(h) \to 0$ as $h \to 0^+$ shows that $C_4(h) \to 0$ as $h \to 0^+$. □

8.5 Proof of Theorem 8.3.1

The proof of Theorem 8.3.1 is the result of a sequence of lemmata. We first introduce the integer $N_3 = N_3(h)$ defined by

$$N_3 h \le T < (N_3+1)h. \tag{8.5.1}$$

Our first (and most important) lemma shows that the error $(A_n)_{n\ge 0}$ defined by

$$A_{n+1} = \mathbb{E}\left[\max_{t\in[nh,(n+1)h]} \left|X(t) - \overline{X}_h(t)\right|^2\right], \quad n = 0,\ldots,N_3 \tag{8.5.2}$$

obeys a linear Volterra difference inequality.

Lemma 8.5.1. *Let $h > 0$ such that $h < \tau_1 \wedge T$. Define*

$$\varepsilon_i(h) = \max_{n=1,\ldots,N_2} \left|I_i(\psi, nh) - \widehat{I}_i(\psi, n)\right|, \quad i = 1,2, \tag{8.5.3}$$

and

$$\varepsilon_i^*(h) = \left| I_i^*(\psi) - \widehat{I}_i^*(\psi) \right|, \quad i = 1, 2. \tag{8.5.4}$$

Let $\eta^* > 0$ *be given by* (8.3.6) *and* W *by* (8.2.8) *and define* $K_3 > 0$ *by* (8.3.5)*. Also define* $\varepsilon(h) > 0$ *by* (8.3.14)*, where* $C_4(h)$ *is defined by* (8.3.7)*, and* C_1 *and* C_2 *are defined by* (8.3.11) *and* (8.4.1) *respectively. Then* (A_n) *defined by* (8.5.2) *obeys*

$$A_1 \leq \varepsilon(h), \tag{8.5.5}$$

$$A_{n+1} \leq \varepsilon(h) + K_3 \sum_{j=0}^{n-1} \widehat{W}_h(j) h A_{n-j}, \quad n = 1, \ldots, N_2, \tag{8.5.6}$$

$$A_{n+1} \leq \varepsilon(h) + K_3 \sum_{j=0}^{N_2} \widehat{W}_h(j) h A_{n-j}, \quad n = N_2 + 1, \ldots, N_3. \tag{8.5.7}$$

We have assumed in the proof that $T > \tau_2$ in which case we can choose $N_3 > N_2$ such that $h > 0$.

Remark 8.5.1. We begin this proof by deriving two estimates for $\overline{X}_h((n+1)h)$ on the two intervals $n = 0, \cdots, N_2 - 1$ and $n \geq N_2$. Combining this information with the definition of $X(t)$ we compute the bound on the first interval i.e. 8.5.5. Next we switch out attention to the interval $[n, (n+1)h]$ when $T < \tau_2$. For $1, \cdots, N_2$ we calculate the formula for $\max_{t \in [nh, (n+1)h]} |X(t) - X_h(t)|$ and subsequently the bound for the second moment. We follow a similar procedure to calculate the bound on the third interval i.e. $[n, (n+1)h]$ where $T > \tau_2$

Proof. By the definition of (8.3.1) and (8.4.3) we have $\overline{X}_h(nh) = \widehat{X}_h([n]) = \widehat{X}_h(n)$ as well as $\overline{W}_h(nh) = \widehat{W}_h([n]) = \widehat{W}_h(n)$. For $n = 0, \ldots, N_2 - 1$ we have

$$\begin{aligned}
&\int_0^{(n+1)h} \overline{W}_h(s) g(\overline{X}_h((n+1)h - s))\, ds \\
&= \sum_{j=0}^{n} \int_{jh}^{(j+1)h} \overline{W}_h(s) g(\overline{X}_h((n+1)h - s))\, ds \\
&= \sum_{j=0}^{n} \widehat{W}_h(j) \int_{jh}^{(j+1)h} g(\overline{X}_h((n+1)h - s))\, ds \\
&= \sum_{j=0}^{n} \widehat{W}_h(j) \int_{(n-j)h}^{(n+1-j)h} g(\overline{X}_h(u))\, du \\
&= \sum_{j=0}^{n} \widehat{W}_h(j) h g(\widehat{X}_h(n-j)).
\end{aligned}$$

Therefore for $n = 0, \ldots, N_2 - 1$, by (8.2.19a), because $\overline{X}_h((n+1)h) = \widehat{X}_h(n+1)$ we have

$$\begin{aligned}
\overline{X}_h((n+1)h) = \psi(0) + \widehat{I}_1(\psi, n) - \widehat{I}_2(\psi, n) + \int_0^{(n+1)h} \overline{W}_h(s) g(\overline{X}_h((n+1)h - s))\, ds \\
+ \sigma B((n+1)h), \quad n = 0, \ldots, N_2 - 1.
\end{aligned} \tag{8.5.8}$$

Let $N_2 \in \mathbb{N}$. Then

$$\begin{aligned}
&\int_0^{(N_2+1)h} \overline{W}_h(s) g(\overline{X}_h((n+1)h-s))\, ds \\
&= \sum_{j=0}^{N_2} \int_{jh}^{(j+1)h} \overline{W}_h(s) g(\overline{X}_h((n+1)h-s))\, ds \\
&= \sum_{j=0}^{N_2} \widehat{W}_h(j) \int_{jh}^{(j+1)h} g(\overline{X}_h((n+1)h-s))\, ds \\
&= \sum_{j=0}^{N_2} \widehat{W}_h(j) \int_{(n-j)h}^{(n-j+1)h} g(\overline{X}_h(u))\, du \\
&= \sum_{j=0}^{N_2} \widehat{W}_h(j) g(\widehat{X}_h(n-j)) h.
\end{aligned}$$

Then for $n \geq N_2$, since $\overline{X}_h((n+1)h) = \widehat{X}_h(n+1)$, by (8.2.19b) we have

$$\begin{aligned}
\overline{X}_h((n+1)h) &= \psi(0) + \widehat{I}_1^*(\psi) - \widehat{I}_2^*(\psi) \\
&\quad + \int_0^{(N_2+1)h} \overline{W}_h(s) g(\overline{X}_h(n+1)h-s)\, ds + \sigma B((n+1)h), \quad n \geq N_2. \quad (8.5.9)
\end{aligned}$$

We start by estimating the error on the interval $[0,h]$. Let $t \in [0,h)$. Then

$$\begin{aligned}
X(t) - \overline{X}_h(t) &= X(t) - \widehat{X}_h(0) = X(t) - \psi(0) \\
&= \int_0^t \left(\int_0^{\tau_1} w_1(u) g(X(s-u))\, du - \int_0^{\tau_2} w_2(u) g(X(s-u))\, du \right) ds \\
&\qquad + \sigma B(t).
\end{aligned}$$

Therefore by (8.2.2) and (8.2.5)

$$\begin{aligned}
&\max_{0 \leq t \leq h} \left| X(t) - \overline{X}_h(t) \right| \\
&\leq \int_0^h \left(\int_0^{\tau_1} w_1(u) \left| g(X(s-u)) \right| du + \int_0^{\tau_2} w_2(u) \left| g(X(s-u)) \right| du \right) ds \\
&\qquad + |\sigma| \max_{0 \leq t \leq h} |B(t)| \\
&\leq \int_0^h \left(\int_0^{\tau_1} w_1(u) K_2 (1 + |X(s-u)|)\, du + \int_0^{\tau_2} w_2(u) K_2 (1 + |X(s-u)|)\, du \right) ds \\
&\qquad + |\sigma| \max_{0 \leq t \leq h} |B(t)| \\
&\leq K_2 \int_0^h \left(2 + \int_0^{\tau_2} (w_1(u) + w_2(u)) \left| X(s-u) \right| du \right) ds + |\sigma| \max_{0 \leq t \leq h} |B(t)| \\
&= 2K_2 h + K_2 \int_0^h \int_0^{\tau_2} (w_1(u) + w_2(u)) \left| X(s-u) \right| du\, ds + |\sigma| \max_{t \in [0,h]} \left| \int_0^t dB(s) \right|.
\end{aligned}$$

Then by (8.2.2) and (8.4.1) and using Doob's inequality we have

$$
\begin{aligned}
&\mathbb{E}\left[\max_{0\le t\le h}\left|X(t)-\overline{X}_h(t)\right|^2\right]\\
&\le 12K_2^2h^2+3K_2^2h\int_0^h\mathbb{E}\left[\int_0^{\tau_2}(w_1(u)+w_2(u))\,|X(s-u)|\,du\right]^2 ds\\
&\qquad\qquad +3\sigma^2\mathbb{E}\left[\max_{0\le t\le h}\left|\int_0^t dB(s)\right|^2\right]\\
&\le 12K_2^2h^2+12\sigma^2h+3K_2^2h\int_0^h 2\int_0^{\tau_2}(w_1(u)+w_2(u))\mathbb{E}\left[|X(s-u)|^2\right]du\,ds\\
&\le 12K_2^2h^2+12\sigma^2h+6K_2^2h\int_0^h 2C_2(h)\,ds,
\end{aligned}
$$

so by the definition of A_1 we have

$$
A_1=\mathbb{E}\left[\max_{0\le t\le h}\left|X(t)-\overline{X}_h(t)\right|^2\right]\le 12K_2^2h^2+12\sigma^2h+12K_2^2h^2C_2(h)\le\varepsilon(h), \tag{8.5.10}
$$

where we note the definition of $\varepsilon(h)$ in (8.3.14). This proves (8.5.5). Next we develop an estimate for the error on the interval $[nh,(n+1)h]$ for $1\le n\le N_2$. Let $n\in\{1,\cdots,N_2\}$. Then by (8.2.15) we have $h\le nh\le N_2h\le\tau_2$. Let $t\in[nh,(n+1)h)$. Then by (8.5.8) and Lemma 8.2.1 we have

$$
\begin{aligned}
&X(t)-\overline{X}_h(t)\\
&=X(t)-X(nh)+X(nh)-\widehat{X}_h(n)\\
&=X(t)-X(nh)\\
&\quad+\psi(0)+I_1(\psi,nh)-I_2(\psi,nh)+\int_0^{nh}W(s)g\left(X(nh-s)\right)ds+\sigma B(nh)\\
&\quad-\left(\psi(0)+\widehat{I}_1(\psi,n)-\widehat{I}_2(\psi,n)+\int_0^{nh}\overline{W}_h(s)g\left(\overline{X}_h(nh-s)\right)ds+\sigma B(nh)\right).
\end{aligned}
$$

Thus for $t\in[nh,(n+1)h)$ where $n\in\{1,\cdots,N_2\}$,

$$
\begin{aligned}
&X(t)-\overline{X}_h(t)\\
&=X(t)-X(nh)+I_1(\psi,nh)-\widehat{I}_1(\psi,n)-\left(I_2(\psi,nh)-\widehat{I}_2(\psi,n)\right)\\
&\qquad+\int_0^{nh}W(s)g\left(X(nh-s)\right)ds-\int_0^{nh}\overline{W}_h(s)g\left(\overline{X}_h(nh-s)\right)ds.
\end{aligned}\tag{8.5.11}
$$

We consider first the last two terms on the righthand side of (8.5.11). By (8.2.4) and (8.2.5) we have

$$\begin{aligned}
&\left|\int_0^{nh} W(s)g(X(nh-s)) - \overline{W}_h(s)g(\overline{X}_h(nh-s))\,ds\right| \\
&\le \left|\int_0^{nh} \big(W(s)-\overline{W}_h(s)\big)\,g(X(nh-s))\,ds\right| \\
&\qquad + \left|\int_0^{nh} \overline{W}_h(s)\,\big(g(X(nh-s)) - g(\overline{X}_h(nh-s))\big)\,ds\right| \\
&\le \int_0^{nh} \left|W(s)-\overline{W}_h(s)\right| |g(X(nh-s))|\,ds \\
&\qquad + K_2\int_0^{nh} \overline{W}_h(s)\left|X(nh-s)-\overline{X}_h(nh-s)\right|\,ds \\
&\le K_2\int_0^{nh} \left|W(s)-\overline{W}_h(s)\right| (1+|X(nh-s)|)\,ds \\
&\qquad\qquad + K_2\int_0^{nh} \overline{W}_h(s)\left|X(nh-s)-\overline{X}_h(nh-s)\right|\,ds.
\end{aligned}$$

Then

$$\begin{aligned}
&\left|\int_0^{nh} W(s)g(X(nh-s)) - \overline{W}_h(s)g(\overline{X}_h(nh-s))\,ds\right| \\
&\le K_2\int_0^{nh} \left|W(s)-\overline{W}_h(s)\right|\,ds + K_2\int_0^{nh} \left|W(s)-\overline{W}_h(s)\right| |X(nh-s)|\,ds \\
&\qquad + K_2\int_0^{nh} \overline{W}_h(s)\left|X(nh-s)-\overline{X}_h(nh-s)\right|\,ds. \qquad (8.5.12)
\end{aligned}$$

Recall the definition of ε_i given in (8.5.3). Then by (8.5.3), (8.5.11), and (8.5.12) for $1 \le n \le N_2$ we have

$$\begin{aligned}
&\max_{t\in[nh,(n+1)h]} \left|X(t)-\overline{X}_h(t)\right| \\
&\quad \le \max_{t\in[nh,(n+1)h]} |X(t)-X(nh)| + \varepsilon_1(h) + \varepsilon_2(h) \\
&\qquad + K_2\int_0^{nh} \left|W(s)-\overline{W}_h(s)\right|\,ds + K_2\int_0^{nh} \left|W(s)-\overline{W}_h(s)\right| |X(nh-s)|\,ds \\
&\qquad\quad + K_2\int_0^{nh} \overline{W}_h(s)\left|X(nh-s)-\overline{X}_h(nh-s)\right|\,ds.
\end{aligned}$$

Therefore as $(a_1+a_2+a_3+a_4+a_5)^2 \le 5(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2)$ then by (8.4.2)

$$\begin{aligned}
&\mathbb{E}\left[\max_{t\in[nh,(n+1)h]} \left|X(t)-\overline{X}_h(t)\right|^2\right] \\
&\le 5\mathbb{E}\left[\max_{t\in[nh,(n+1)h]} |X(t)-X(nh)|^2\right] + 5\left(\varepsilon_1(h)+\varepsilon_2(h)\right)^2 \\
&\quad + 5K_2^2\left(\int_0^{nh} \left|W(s)-\overline{W}_h(s)\right|\,ds\right)^2 \\
&\qquad + 5K_2^2\int_0^{nh} \left|W(s)-\overline{W}_h(s)\right|\,ds \int_0^{nh} \left|W(s)-\overline{W}_h(s)\right| \mathbb{E}|X(nh-s)^2|\,ds \\
&\qquad\quad + 5K_2^2\mathbb{E}\left[\int_0^{nh} \overline{W}_h(s)\left|X(nh-s)-\overline{X}_h(nh-s)\right|\,ds\right]^2.
\end{aligned}$$

The estimate of the first term on the righthand side is straightforward but noteworthy. Since $n \leq N_2$, the definition (8.2.15) implies $n \leq N_2 \leq \tau_2/h$. Also, $h < \tau_2$ by construction. Therefore by Lemma 8.4.2 (specifically by making the choice $T = \tau_2$ in (8.4.2)), we get

$$\mathbb{E}\left[\max_{t\in[nh,(n+1)h]} |X(t) - X(nh)|^2\right] \leq C_1(\tau_2)h.$$

Therefore for $n = 1, \dots, N_2$, by the definition of C_2 (viz., (8.4.1)), we have

$$\begin{aligned}
&\mathbb{E}\left[\max_{t\in[nh,(n+1)h]} \left|X(t) - \overline{X}_h(t)\right|^2\right] \\
&\leq 5C_1(\tau_2)h + 5\left(\varepsilon_1(h) + \varepsilon_2(h)\right)^2 \\
&\quad + \left(5K_2^2 + 5K_2^2 C_2(\tau_2)\right)\left(\int_0^{nh} \left|W(s) - \overline{W}_h(s)\right| ds\right)^2 \\
&\quad + 5K_2^2 \int_0^{nh} \overline{W}_h(s)\, ds \int_0^{nh} \overline{W}_h(s)\mathbb{E}\left[|X(nh-s) - \overline{X}_h(nh-s)|^2\right] ds. \qquad (8.5.13)
\end{aligned}$$

For $n = 0, \dots, N_2$ let A_{n+1} be given by (8.5.2). Then

$$\begin{aligned}
&\int_0^{nh} \overline{W}_h(s)\mathbb{E}\left[\left|X(nh-s) - \overline{X}_h(nh-s)\right|^2\right] ds \\
&\quad = \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} \overline{W}_h(s)\mathbb{E}\left[\left|X(nh-s) - \overline{X}_h(nh-s)\right|^2\right] ds \\
&\quad \leq \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} \overline{W}_h(s)\mathbb{E}\left[\max_{jh\leq u\leq(j+1)h} \left|X(nh-u) - \overline{X}_h(nh-u)\right|^2\right] ds \\
&\quad = \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} \overline{W}_h(s)\mathbb{E}\left[\max_{(n-j-1)h\leq v\leq(n-j)h} \left|X(v) - \overline{X}_h(v)\right|^2\right] ds \\
&\quad = \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} \overline{W}_h(s)A_{n-j}\, ds = \sum_{j=0}^{n-1} \widehat{W}_h(j)hA_{n-j}.
\end{aligned}$$

Using this estimate, (8.5.2) and (8.5.13) we obtain for $n = 1, \dots, N_2$

$$\begin{aligned}
A_{n+1} &\leq 5C_1(\tau_2)h + 5\left(\varepsilon_1(h) + \varepsilon_2(h)\right)^2 \\
&\quad + \left(5K_2^2 + 5K_2^2 C_2(\tau_2)\right)\left(\int_0^{nh} \left|W(s) - \overline{W}_h(s)\right| ds\right)^2 \\
&\quad + 5K_2^2 \int_0^{nh} \overline{W}_h(s)\, ds \cdot \sum_{j=0}^{n-1} \widehat{W}_h(j)hA_{n-j}, \quad n = 1, \dots, N_2. \qquad (8.5.14)
\end{aligned}$$

Finally, we deduce an estimate for the error on $[nh, (n+1)h]$ for $N_2+1 \leq n \leq N_3$. In this case by (8.2.15) we have $\tau_2 < (N_2+1)h \leq nh$. Let $t \in [nh, (n+1)h]$. Then by (8.5.9) and Lemma 8.2.1 we have

$$\begin{aligned}
&X(t) - \overline{X}_h(t) \\
&= X(t) - X(nh) + X(nh) - \widehat{X}_h(n) \\
&= X(t) - X(nh) + \psi(0) + I_1^*(\psi) - I_2^*(\psi) + \int_0^{\tau_2} W(s)g\left(X(nh-s)\right) ds + \sigma B(nh) \\
&\quad - \left(\psi(0) + \widehat{I}_1^*(\psi) - \widehat{I}_2^*(\psi) + \int_0^{(N_2+1)h} \overline{W}_h(s)g(\overline{X}_h(nh-s))\, ds + \sigma B(nh)\right) \\
&= X(t) - X(nh) + I_1^*(\psi) - \widehat{I}_1^*(\psi) - \left(I_2^*(\psi) - \widehat{I}_2^*(\psi)\right) \\
&\qquad + \int_0^{\tau_2} W(s)g(X(nh-s))\, ds - \int_0^{(N_2+1)h} \overline{W}_h(s)g(\overline{X}_h(nh-s))\, ds. \qquad (8.5.15)
\end{aligned}$$

Next as $W(t) = 0$ for $t \geq \tau_2$ and $\tau_2 < (N_2 + 1)h$ we have

$$\begin{aligned}
&\int_0^{\tau_2} W(s)g(X(nh-s))\,ds - \int_0^{(N_2+1)h} \overline{W}_h(s)g(\overline{X}_h(nh-s))\,ds \\
&= \int_0^{(N_2+1)h} \left(W(s)g(X(nh-s)) - \overline{W}_h(s)g(\overline{X}_h(nh-s))\right)\,ds \\
&= \int_0^{(N_2+1)h} \left(W(s) - \overline{W}_h(s)\right) g(X(nh-s))\,ds \\
&\qquad + \int_0^{(N_2+1)h} \overline{W}_h(s)\left(g(X(nh-s)) - g(\overline{X}_h(nh-s))\right)\,ds.
\end{aligned}$$

Therefore by (8.2.4) and (8.2.5) we have

$$\begin{aligned}
&\left|\int_0^{\tau_2} W(s)g(X(nh-s))\,ds - \int_0^{(N_2+1)h} \overline{W}_h(s)g(\overline{X}_h(nh-s))\,ds\right| \\
&\leq K_2 \int_0^{(N_2+1)h} \left|W(s) - \overline{W}_h(s)\right| (1 + |X(nh-s)|)\,ds \\
&\qquad + K_2 \int_0^{(N_2+1)h} \overline{W}_h(s)|X(nh-s) - \overline{X}(nh-s)|\,ds
\end{aligned}$$

so

$$\begin{aligned}
&\left|\int_0^{\tau_2} W(s)g(X(nh-s))\,ds - \int_0^{(N_2+1)h} \overline{W}_h(s)g(\overline{X}_h(nh-s))\,ds\right| \\
&\leq K_2 \int_0^{(N_2+1)h} \left|W(s) - \overline{W}_h(s)\right|\,ds \\
&\qquad + K_2 \int_0^{(N_2+1)h} \left|W(s) - \overline{W}_h(s)\right| |X(nh-s)|\,ds \\
&\qquad\qquad + K_2 \int_0^{(N_2+1)h} \overline{W}_h(s)|X(nh-s) - \overline{X}_h(nh-s)|\,ds. \qquad (8.5.16)
\end{aligned}$$

Recall the definition of $\varepsilon_i^*(h)$, $i = 1, 2$, from (8.5.4). Then for $n \geq N_2 + 1$, by (8.5.15), (8.5.16), and (8.5.4) we have

$$\begin{aligned}
&\max_{t\in[nh,(n+1)h]} \left|X(t) - \overline{X}_h(t)\right| \\
&\leq \max_{t\in[nh,(n+1)h]} |X(t) - X(nh)| + \varepsilon_1^*(h) + \varepsilon_2^*(h) \\
&\qquad + K_2 \int_0^{(N_2+1)h} \left|W(s) - \overline{W}_h(s)\right|\,ds \\
&\qquad\qquad + K_2 \int_0^{(N_2+1)h} \left|W(s) - \overline{W}_h(s)\right| |X(nh-s)|\,ds \\
&\qquad\qquad\qquad + K_2 \int_0^{(N_2+1)h} \overline{W}_h(s)\left|X(nh-s) - \overline{X}_h(nh-s)\right|\,ds.
\end{aligned}$$

Therefore as $(a_1+a_2+a_3+a_4+a_5)^2 \leq 5(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2)$ then by (8.4.2)

$$\begin{aligned}
&\mathbb{E}\left[\max_{t\in[nh,(n+1)h]}\left|X(t)-\overline{X}_h(t)\right|^2\right] \\
&\leq 5\mathbb{E}\left[\max_{t\in[nh,(n+1)h]}|X(t)-X(nh)|^2\right]+5\left(\varepsilon_1^*(h)+\varepsilon_2^*(h)\right)^2 \\
&+5K_2^2\left(\int_0^{(N_2+1)h}\left|W(s)-\overline{W}_h(s)\right|\,ds\right)^2 \\
&+5K_2^2\int_0^{(N_2+1)h}\left|W(s)-\overline{W}_h(s)\right|\,ds\int_0^{(N_2+1)h}\left|W(s)-\overline{W}_h(s)\right|\mathbb{E}|X(nh-s)^2|\,ds \\
&\quad+5K_2^2\mathbb{E}\left[\int_0^{(N_2+1)h}\overline{W}_h(s)\left|X(nh-s)-\overline{X}_h(nh-s)\right|\,ds\right]^2.
\end{aligned}$$

Since N_3 is defined by (8.5.1) and we have $n \leq N_3$, it follows that $n \leq T/h$. Also, $h < T$ by construction. Therefore by Lemma 8.4.2 (specifically (8.4.2)), we get

$$\mathbb{E}\left[\max_{t\in[nh,(n+1)h]}|X(t)-X(nh)|^2\right] \leq C_1(T)h.$$

Therefore for $n = 1, \ldots, N_2$, by the definition of C_2 (viz., (8.4.1)), we have

$$\begin{aligned}
&\mathbb{E}\left[\max_{t\in[nh,(n+1)h]}\left|X(t)-\overline{X}_h(t)\right|^2\right] \\
&\leq 5C_1(T)h+5\left(\varepsilon_1^*(h)+\varepsilon_2^*(h)\right)^2 \\
&\quad+\left(5K_2^2+5K_2^2C_2(T)\right)\left(\int_0^{(N_2+1)h}\left|W(s)-\overline{W}_h(s)\right|\,ds\right)^2 \\
&\quad+5K_2^2\int_0^{(N_2+1)h}\overline{W}_h(s)\,ds\int_0^{(N_2+1)h}\overline{W}_h(s)\mathbb{E}\left[|X(nh-s)-\overline{X}_h(nh-s)|^2\right]\,ds.
\end{aligned} \tag{8.5.17}$$

We estimate the last term on the righthand side of (8.5.17)

$$\begin{aligned}
&\int_0^{(N_2+1)h}\overline{W}_h(s)\mathbb{E}\left[\left|X(nh-s)-\overline{X}_h(nh-s)\right|^2\right]\,ds \\
&\qquad=\sum_{j=0}^{N_2}\int_{jh}^{(j+1)h}\overline{W}_h(s)\mathbb{E}\left[\left|X(nh-s)-\overline{X}_h(nh-s)\right|^2\right]\,ds \\
&\qquad\leq\sum_{j=0}^{N_2}\int_{jh}^{(j+1)h}\overline{W}_h(s)\mathbb{E}\left[\max_{jh\leq u\leq(j+1)h}\left|X(nh-u)-\overline{X}_h(nh-u)\right|^2\right]\,ds \\
&\qquad=\sum_{j=0}^{N_2}\int_{jh}^{(j+1)h}\overline{W}_h(s)\mathbb{E}\left[\max_{(n-j-1)h\leq v\leq(n-j)h}\left|X(v)-\overline{X}_h(v)\right|^2\right]\,ds \\
&\qquad=\sum_{j=0}^{N_2}\int_{jh}^{(j+1)h}\overline{W}_h(s)A_{n-j}\,ds=\sum_{j=0}^{N_2}\widehat{W}_h(j)hA_{n-j}.
\end{aligned}$$

Using this estimate, (8.5.2) and (8.5.17) we obtain for $n = N_2+1, \ldots, N_3$

$$\begin{aligned}
A_{n+1} &\leq 5C_1(T)h+5\left(\varepsilon_1^*(h)+\varepsilon_2^*(h)\right)^2 \\
&\quad+\left(5K_2^2+5K_2^2C_2(T)\right)\left(\int_0^{(N_2+1)h}\left|W(s)-\overline{W}_h(s)\right|\,ds\right)^2 \\
&\qquad+5K_2^2\int_0^{(N_2+1)h}\overline{W}_h(s)\,ds\cdot\sum_{j=0}^{N_2}\widehat{W}_h(j)hA_{n-j}, \quad n=N_2+1,\ldots,N_3.
\end{aligned} \tag{8.5.18}$$

We now simplify and consolidate the estimates (8.5.14) and (8.5.18). By (8.4.4) and (8.3.5) we have

$$K_3 = 5K_2^2 \left(\int_0^{\tau_2} W(s)\, ds + \eta^* \right) \geq 5K_2^2 \int_0^{(N_2+1)h} \overline{W}_h(s)\, ds. \tag{8.5.19}$$

By (8.3.14) and (8.4.5) we have

$$\begin{aligned} \varepsilon(h) &\geq 5C_1(T \vee \tau_2)h + 5\left(\varepsilon_1^*(h) + \varepsilon_2^*(h)\right)^2 \vee 5\left(\varepsilon_1(h) + \varepsilon_2(h)\right)^2 \\ &\qquad + \left(5K_2^2 + 5K_2^2 C_2(T \vee \tau_2)\right) C_4(h)^2 \\ &\geq 5C_1(T \vee \tau_2 + h)h + 5\left(\varepsilon_1^*(h) + \varepsilon_2^*(h)\right)^2 \vee 5\left(\varepsilon_1(h) + \varepsilon_2(h)\right)^2 \\ &\qquad + \left(5K_2^2 + 5K_2^2 C_2(T \vee \tau_2)\right) \left(\int_0^{(N_2+1)h} |\overline{W}_h(s) - W(s)|\, ds \right)^2, \end{aligned} \tag{8.5.20}$$

where $C_4(h)$ is defined by (8.3.7). Then by using the estimates (8.5.14), (8.5.18) together with (8.5.19) and (8.5.20) we obtain the estimates (8.5.6) and (8.5.7) as required. This completes the proof of the lemma. □

Our next lemma is technical and short. For this reason no explanation of the proof is given. Amongst other things, it enables us to estimate an upper bound on the step size i.e. h^* for which an explicit estimate can be given on the mean square error in (8.3.2). It also prepares an estimate which will be of importance in deriving an explicit upper bound on the errors A_n defined in (8.5.2).

Lemma 8.5.2. *Let* $\alpha^* > 0$ *be such that*

$$K_3 \int_0^{\tau_2} W(s) e^{-\alpha^* s}\, ds < \frac{1}{4}. \tag{8.5.21}$$

Define C_3 *by* (8.3.8)*, and let* $h^* > 0$ *be such that* (8.3.9) *holds. Let* $h < h^*$*. Then there exists* $\lambda = \lambda(h) > 1$ *such that*

$$\sum_{j=0}^{N_2} \widehat{W}_h(j) h K_3 \lambda(h)^{-(j+1)} \leq \frac{1}{2}. \tag{8.5.22}$$

Remark 8.5.2. We notice that $\alpha^* = 4K_3 > 0$ suffices in (8.5.21). Since $W(t) \leq 1$ for all $t \geq 0$ we have

$$K_3 \int_0^{\tau_2} W(s) e^{-\alpha^* s}\, ds \leq K_3 \int_0^{\tau_2} e^{-\alpha^* s}\, ds < K_3 \int_0^{\infty} e^{-\alpha^* s}\, ds = \frac{K_3}{\alpha^*} = \frac{1}{4},$$

as required.

Proof of Lemma 8.5.2. With $\alpha^* > 0$ defined as in (8.5.21), let $\lambda(h) = e^{\alpha^* h} > 1$. We have the identity

$$\begin{aligned} \int_0^{(N_2+1)h} \overline{W}_h(s) e^{-\alpha^* s}\, ds &= \sum_{j=0}^{N_2} \int_{jh}^{(j+1)h} \overline{W}_h(s) e^{-\alpha^* s}\, ds \\ &= \sum_{j=0}^{N_2} \widehat{W}_h(j) \int_{jh}^{(j+1)h} e^{-\alpha^* s}\, ds \\ &= \frac{1 - e^{-\alpha^* h}}{\alpha^* h} \sum_{j=0}^{N_2} \widehat{W}_h(j) h \left(e^{\alpha^* h}\right)^{-j} \\ &= \frac{1 - e^{-\alpha^* h}}{\alpha^* h} \sum_{j=0}^{N_2} \widehat{W}_h(j) h \lambda(h)^{-j}. \end{aligned}$$

Thus

$$\begin{aligned} \int_0^{(N_2+1)h} K_3 \overline{W}_h(s) e^{-\alpha^* s}\, ds &= \frac{1 - e^{-\alpha^* h}}{\alpha^* h} \lambda(h) \sum_{j=0}^{N_2} K_3 \widehat{W}_h(j) h \lambda(h)^{-(j+1)} \\ &= \frac{e^{\alpha^* h} - 1}{\alpha^* h} \sum_{j=0}^{N_2} K_3 \widehat{W}_h(j) h \lambda(h)^{-(j+1)}. \end{aligned}$$

Hence if

$$\int_0^{(N_2+1)h} K_3\overline{W}_h(s)e^{-\alpha^* s}\,ds \le \frac{1}{2}\frac{e^{\alpha^* h}-1}{\alpha^* h}, \tag{8.5.23}$$

then (8.5.22) holds, as required. Then

$$\begin{aligned}
&\int_0^{(N_2+1)h} K_3\overline{W}_h(s)e^{-\alpha^* s}\,ds - K_3\int_0^{\tau_2} W(s)e^{-\alpha^* s}\,ds \\
&= K_3\left(\int_0^{N_2h} \overline{W}_h(s)e^{-\alpha^* s}\,ds + \int_{N_2h}^{(N_2+1)h} \overline{W}_h(s)e^{-\alpha^* s}\,ds\right) \\
&\quad - K_3\left(\int_0^{N_2h} W(s)e^{-\alpha^* s}\,ds + \int_{N_2h}^{\tau_2} W(s)e^{-\alpha^* s}\,ds\right) \\
&= K_3\int_0^{N_2h} \left(\overline{W}_h(s) - W(s)\right)e^{-\alpha^* s}\,ds \\
&\qquad + K_3\int_{N_2h}^{(N_2+1)h} \overline{W}_h(s)e^{-\alpha^* s}\,ds - K_3\int_{N_2h}^{\tau_2} W(s)e^{-\alpha^* s}\,ds.
\end{aligned}$$

By (8.2.8) we have $W(t) \le 1$ for all $t \ge 0$; moreover by (8.2.3), $W(t) \ge 0$ for all $t \ge 0$. Thus as $\alpha^* > 0$ and the definition of $C_4(h)$, $C_3(h)$ and using the fact that $h < h^*$, we have

$$\begin{aligned}
&\left|\int_0^{(N_2+1)h} K_3\overline{W}_h(s)e^{-\alpha^* s}\,ds - K_3\int_0^{\tau_2} W(s)e^{-\alpha^* s}\,ds\right| \\
&\le K_3\int_0^{N_2h} \left|\overline{W}_h(s) - W(s)\right|\,ds + K_3\widehat{W}_h(N_2)h + K_3h \\
&\le K_3C_4(h) + K_3\widehat{W}_h(N_2)h + K_3h \\
&\le K_3C_4(h) + K_3|W(N_2h) - \widehat{W}_h(N_2)|h + K_3W(N_2h)h + K_3h \\
&\le 2K_3h + K_3C_4(h) + K_3|W(N_2h) - \widehat{W}_h(N_2)|h \\
&\le 2K_3h + K_3C_4(h) + 2K_3\tau_2(\delta_1(h) + \delta_2(h))h = C_3(h) \le \frac{1}{4}.
\end{aligned}$$

Therefore by (8.5.21) and the last estimate, we obtain

$$\int_0^{(N_2+1)h} K_3\overline{W}_h(s)e^{-\alpha^* s}\,ds \le K_3\int_0^{\tau_2} W(s)e^{-\alpha^* s}\,ds + C_3(h) < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}. \tag{8.5.24}$$

We now use this estimate to prove (8.5.23). Since $x \mapsto x/(e^x - 1)$ is decreasing on $(0, \infty)$, we have $\alpha^* h/(e^{\alpha^* h} - 1) < 1$, so by (8.5.24) we have

$$\frac{\alpha^* h}{e^{\alpha^* h}-1}\int_0^{(N_2+1)h} K_3\overline{W}_h(s)e^{-\alpha^* s}\,ds < \frac{1}{2}$$

which is (8.5.23). Hence (8.5.22) holds and the lemma is proven. □

The construction of α^* and $h^* > 0$ in Lemma 8.5.2 along with the inequality (8.5.22) enable us to derive an explicit exponential upper bound on the growth of A_n defined in (8.5.2). The form of this bound will eventually allow us to derive an estimate for the error in (8.3.2) which remains under control as $h \to 0^+$.

Lemma 8.5.3. *Let* $\alpha^* > 0$ *obey* (8.5.21)*. Let* h^* *be defined by* (8.3.9)*. Then for* $h < h^*$, $\lambda(h) = e^{\alpha^* h} > 1$ *satisfies* (8.5.22)*. Define*

$$S(h) = K_3\sum_{j=0}^{N_2}\widehat{W}_h(j)h. \tag{8.5.25}$$

If A_n *obeys* (8.5.5), (8.5.6)*, and* (8.5.7)*, then*

$$A_n \le \varepsilon(h)\left(1 + 2S(h)\lambda(h)^n\right), \quad n = 1, \ldots, N_3. \tag{8.5.26}$$

The estimate (8.5.6) holds in the case when $T \le \tau_2$. In that case we need only consider $n \le N_2 \le N_3$ in the proof below.

Remark 8.5.3. As the proof is short no outline is given.

Proof. For $n = 1$, the proof is trivial. Suppose that (8.5.26) is true for $m \le n$ i.e., suppose that

$$A_m \le \varepsilon(h)\left(1 + 2S(h)\lambda(h)^m\right), \quad \text{for all } 1 \le m \le n. \tag{8.5.27}$$

We now show that the same estimate holds for A_{n+1}. Let $n = 1, \ldots, N_2$. By (8.5.6) we get

$$\begin{aligned}
A_{n+1} &\le \varepsilon(h) + K_3 \sum_{j=0}^{n-1} \widehat{W}_h(j) h A_{n-j} = \varepsilon(h) + K_3 \sum_{j=1}^{n} \widehat{W}_h(n-j) h A_j \\
&\le \varepsilon(h) + \varepsilon(h) K_3 \sum_{j=1}^{n} \widehat{W}_h(n-j) h \left(1 + 2S(h)\lambda(h)^j\right) \\
&\le \varepsilon(h)\left(1 + K_3 \sum_{j=1}^{n} h\widehat{W}_h(n-j)\right) + \varepsilon(h) K_3 2S(h) \sum_{j=1}^{n} h\widehat{W}_h(n-j)\lambda(h)^j \\
&= \varepsilon(h)\left(1 + K_3 \sum_{l=0}^{n-1} \widehat{W}_h(l) h\right) + \varepsilon(h) 2S(h) \sum_{l=0}^{n-1} K_3 h\widehat{W}_h(l)\lambda(h)^{-(l+1)}\lambda(h)^{n+1} \\
&\le \varepsilon(h)\left(1 + K_3 \sum_{l=0}^{N_2} \widehat{W}_h(l) h\right) + \varepsilon(h) 2S(h) \sum_{l=0}^{N_2} K_3 h\widehat{W}_h(l)\lambda(h)^{-(l+1)}\lambda(h)^{n+1}.
\end{aligned}$$

Thus by (8.5.25) and (8.5.22) we get

$$\begin{aligned}
A_{n+1} &\le \varepsilon(h)\left(1 + S(h)\right) + 2\varepsilon(h)S(h) \sum_{l=0}^{N_2} K_3 h\widehat{W}_h(l)\lambda(h)^{-(l+1)}\lambda(h)^{n+1} \\
&\le \varepsilon(h)\left(1 + S(h)\right) + 2\varepsilon(h)S(h)\frac{1}{2}\lambda(h)^{n+1} \\
&= \varepsilon(h)\left(1 + S(h) + S(h)\lambda(h)^{n+1}\right) \\
&\le \varepsilon(h)\left(1 + S(h)\lambda(h)^{n+1} + S(h)\lambda(h)^{(n+1)}\right) \\
&= \varepsilon(h)\left(1 + 2S(h)\lambda(h)^{n+1}\right).
\end{aligned}$$

Hence $A_{n+1} \le \varepsilon(h)\left(1 + 2S(h)\lambda(h)^{n+1}\right)$, where $n \in \{1, \ldots, N_2\}$. Therefore (8.5.26) holds for all $n \le N_2 + 1$. Now let $n \ge N_2 + 1$ and suppose that (8.5.27) holds. Then by (8.5.7) we get

$$\begin{aligned}
A_{n+1} &\le \varepsilon(h) + K_3 \sum_{j=0}^{N_2} \widehat{W}_h(j) h A_{n-j} \\
&\le \varepsilon(h) + K_3 \sum_{j=0}^{N_2} \widehat{W}_h(j) h \varepsilon(h)\left(1 + 2S(h)\lambda(h)^{n-j}\right) \\
&= \varepsilon(h)\left(1 + K_3 \sum_{j=0}^{N_2} \widehat{W}_h(j) h\right) + K_3 \sum_{j=0}^{N_2} \widehat{W}_h(j) h 2S(h)\varepsilon(h)\lambda(h)^{-(j+1)}\lambda(h)^{n+1} \\
&\le \varepsilon(h)\left(1 + S(h)\right) + 2S(h)\varepsilon(h)\frac{1}{2}\lambda(h)^{n+1} \\
&= \varepsilon(h)\left(1 + S(h) + S(h)\lambda(h)^{n+1}\right) \\
&\le \varepsilon(h)\left(1 + S(h)\lambda(h)^{n+1} + S(h)\lambda(h)^{n+1}\right) \\
&= \varepsilon(h)\left(1 + 2S(h)\lambda(h)^{n+1}\right),
\end{aligned}$$

which proves the result. □

We are finally in a position to prove Theorem 8.3.1.

Proof of Theorem 8.3.1. For $h < h^*$, by Lemma 8.5.3, we have for $n = 0, \dots, N_3$ that

$$\mathbb{E}\left[\max_{t\in[nh,(n+1)h]} \left|X(t) - \overline{X}_h(t)\right|^2\right] = A_{n+1} \le \varepsilon(h)(1 + 2S(h)\lambda(h)^{n+1}).$$

Since $\lambda(h) = e^{\alpha^* h} > 1$ and $n \le N_3$ we have

$$\mathbb{E}\left[\max_{t\in[0,(n+1)h]} \left|X(t) - \overline{X}_h(t)\right|^2\right] \le \varepsilon(h)\left(1 + 2S(h)e^{\alpha^* h(n+1)}\right).$$

Since N_3 obeys $N_3 h \le T < (N_3 + 1)h$, we have

$$\begin{aligned}\mathbb{E}\left[\max_{t\in[0,T]} \left|X(t) - \overline{X}_h(t)\right|^2\right] &\le \mathbb{E}\left[\max_{t\in[0,(N_3+1)h]} \left|X(t) - \overline{X}_h(t)\right|^2\right] \\ &\le \varepsilon(h)\left(1 + 2S(h)e^{\alpha^* h(N_3+1)}\right) \\ &\le \varepsilon(h)\left(1 + 2S(h)e^{\alpha^*(T+h)}\right).\end{aligned}$$

Next, as $h < \tau_1$ and $S(h) = K_3 \sum_{j=0}^{N_2} \widehat{W}_h(j)h$, by (8.4.4), we have

$$S(h) \le K_3\left(\int_0^{\tau_2} W(s)\,ds + \eta^*\right),$$

where $\eta^* > 0$ is given by (8.3.6). Therefore as $h < \tau_1 \wedge T$, we have

$$\mathbb{E}\left[\max_{t\in[0,T]} \left|X(t) - \overline{X}_h(t)\right|^2\right] \le \varepsilon(h)\left\{1 + 2K_3\left(\int_0^{\tau_2} W(s)\,ds + \eta^*\right)e^{\alpha^*(T+\tau_1\wedge T)}\right\}.$$

We call the righthand side $C(h)$. This proves the estimate (8.3.2). We note that $C(h) \to 0$ as $h \to 0^+$ provided $\varepsilon(h) \to 0$ as $h \to 0^+$ where $\varepsilon(h)$ is defined by (8.3.14). Since $C_4(h) \to 0$ as $h \to 0^+$, we have that

$$\lim_{h\to 0^+} \varepsilon_i(h) = 0, \quad \lim_{h\to 0^+} \varepsilon_i^*(h) = 0 \quad \text{for } i = 1, 2 \tag{8.5.28}$$

implies $\varepsilon(h) \to 0$ as $h \to 0^+$. □

It remains to prove that (8.5.28) holds. In the next section we give discrete approximations to I_i and I_i^* for which this can be achieved.

8.6 Proof of (8.5.28)

We now give approximations to $I_i(\psi, \cdot)$ and $I_i^*(\psi)$ which are parameterised by h and which possess the property (8.5.28). First note that $I_i(\psi, \cdot)$ and $I_i^*(\psi)$ can be rewritten according to

$$I_i(\psi, t) = \int_0^{\tau_i} (W_i(t + u) - W_i(u))g(\psi(-u))\,du, \quad 0 \le t \le \tau_i, \tag{8.6.1a}$$

$$I_i^*(\psi) = \int_0^{\tau_i} (1 - W_i(u))g(\psi(-u))\,du, \tag{8.6.1b}$$

where $I_i(\psi, t) = I_i^*(\psi)$ for $t \ge \tau_i$. Consider the approximations

$$\widehat{I}_i(\psi, nh) = \sum_{j=0}^{N_i} \left\{\left(\widehat{W}_{h,i}(n + j) - \widehat{W}_{h,i}(j)\right)g(\psi(-jh))\right\}h, \quad 1 \le n \le N_i, \tag{8.6.2a}$$

$$\widehat{I}_i^*(\psi) = \sum_{j=0}^{N_i} \left(1 - \widehat{W}_{h,i}(j)\right)g(\psi(-jh))h, \tag{8.6.2b}$$

where

$$\widehat{W}_{h,i}(n) = 0, \qquad n = 0, \tag{8.6.3a}$$

$$\widehat{W}_{h,i}(n) = \sum_{l=0}^{n-1} w_i(lh)h, \qquad n = 1, \dots, N_i(h), \tag{8.6.3b}$$

$$\widehat{W}_{h,i}(n) = 1, \qquad n \geq N_i(h) + 1. \tag{8.6.3c}$$

Lemma 8.6.1. *Let $i = 1, 2$. Let $I_i(\psi, \cdot)$ be defined by* (8.6.2a) *where $\widehat{W}_{i,h}$ obeys* (8.6.3)*. Suppose that G is defined by* (8.2.14) *and δ_3 by* (8.2.13)*. Then*

$$|I_i(\psi, nh) - \widehat{I}_i(\psi, n)| \leq K_2\delta_3(h)\tau_i + 2G\tau_i\left(\tau_i\delta_i(h) + h\bar{w}_i\right) + hG + 2(2 + \tau_i\delta_i(h))Gh, \quad n = 1, \dots, N_i. \tag{8.6.4}$$

Therefore, by (8.5.3)*, we can define $\varepsilon_i(h)$ by* (8.3.12)*. Moreover*

$$\lim_{h\to 0^+} \varepsilon_i(h) = \lim_{h\to 0^+} \max_{h=1,\dots,N_2(h)} |I_i(\psi, nh) - \widehat{I}_i(nh, \psi)| = 0.$$

Remark 8.6.1. We begin the proof by calculating an explicit estimate for $|I_i(\psi, nh) - \widehat{I}_i(\psi, n)|$ which comprises of three terms. We then simplify the form of the first term. These calculations are not difficult but are long and take up the rest of the proof.

Proof. Let $1 \leq n \leq N_2$. Now as $nh \leq \tau_i$ by (8.6.2a) and (8.6.1a) we have

$$\begin{aligned}
I_i(\psi, nh) - \widehat{I}_i(\psi, n) &= \int_0^{\tau_i} (W_i(nh + u) - W_i(u))g(\psi(-u))\,du \\
&\quad - \sum_{j=0}^{N_i} \left\{\left(\widehat{W}_{h,i}(n+j) - \widehat{W}_{h,i}(j)\right) g(\psi(-jh))\right\} h \\
&= \sum_{j=0}^{N_i-1} \int_{jh}^{(j+1)h} (W_i(nh + u) - W_i(u))g(\psi(-u))\,du \\
&\quad + \int_{N_ih}^{\tau_i} (W_i(nh + u) - W_i(u))g(\psi(-u))\,du \\
&\quad - \sum_{j=0}^{N_i-1} \left\{\left(\widehat{W}_{h,i}(n+j) - \widehat{W}_{h,i}(j)\right) g(\psi(-jh))\right\} h \\
&\quad - \left\{\left(\widehat{W}_{h,i}(n+N_i) - \widehat{W}_{h,i}(N_i)\right) g(\psi(-N_ih))\right\} h.
\end{aligned}$$

Therefore as $W_i(t) \in [0, 1]$ for all $t \geq 0$, $\widehat{W}_{h,i}(j) = 1$ for all $j \geq N_i + 1$, and (8.2.14) holds, we have

$$\begin{aligned}
&|I_i(\psi, nh) - \widehat{I}_i(\psi, n)| \\
&\leq \sum_{j=0}^{N_i-1} \int_{jh}^{(j+1)h} \Big|(W_i(nh + u) - W_i(u))g(\psi(-u)) \\
&\quad - \left(\widehat{W}_{h,i}(n+j) - \widehat{W}_{h,i}(j)\right) g(\psi(-jh))\Big|\,du \\
&\quad + \int_{N_ih}^{\tau_i} |W_i(nh + u) - W_i(u))|G\,du + \left\{|\widehat{W}_{h,i}(n+N_i) - \widehat{W}_{h,i}(N_i)|G\right\} h \\
&\leq \sum_{j=0}^{N_i-1} \int_{jh}^{(j+1)h} \Big|(W_i(nh + u) - W_i(u))g(\psi(-u)) \\
&\quad - \left(\widehat{W}_{h,i}(n+j) - \widehat{W}_{h,i}(j)\right) g(\psi(-jh))\Big|\,du + hG + |1 - \widehat{W}_{h,i}(N_i)|Gh.
\end{aligned} \tag{8.6.5}$$

The rest of the proof is devoted to estimating the first term on the righthand side of (8.6.5). We first estimate the integrand. Let $j = 0 \dots N_i - 1$, $u \in [jh, (j+1)]$, $1 \le n \le N_i$. Then

$$
\begin{aligned}
&(W_i(nh+u) - W_i(u))g(\psi(-u)) - \left(\widehat{W}_{h,i}(n+j) - \widehat{W}_{h,i}(j)\right) g(\psi(-jh)) \\
&= (W_i(nh+u) - W_i(u)) \left\{g(\psi(-u)) - g(\psi(-jh))\right\} \\
&\quad + \left(W_i(nh+u) - W_i(u)) - (\widehat{W}_{h,i}(n+j) - \widehat{W}_{h,i}(j))\right) g(\psi(-jh)).
\end{aligned}
$$

Thus by (8.2.14), (8.2.4) and (8.2.13) we have

$$
\begin{aligned}
&|(W_i(nh+u) - W_i(u))g(\psi(-u)) - \left(\widehat{W}_{h,i}(n+j) - \widehat{W}_{h,i}(j)\right) g(\psi(-jh))| \\
&\le |W_i(nh+u) - W_i(u)| \, |g(\psi(-u)) - g(\psi(-jh))| \\
&\quad + \left| W_i(nh+u) - W_i(u)) - (\widehat{W}_{h,i}(n+j) - \widehat{W}_{h,i}(j)) \right| G \\
&\le K_2 |\psi(-u) - \psi(-jh)| \\
&\quad + |W_i(nh+u) - \widehat{W}_{h,i}(n+j)| G + |W_i(u) - \widehat{W}_{h,i}(j)| G \\
&\le K_2 \delta_3(h) + |W_i(nh+u) - \widehat{W}_{h,i}(n+j)| G + |W_i(u) - \widehat{W}_{h,i}(j)| G.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_{jh}^{(j+1)h} &|(W_i(nh+u) - W_i(u))g(\psi(-u)) - \left(\widehat{W}_{h,i}(n+j) - \widehat{W}_{h,i}(j)\right) g(\psi(-jh))| \, du \\
&\le K_2 \delta_3(h) h + G \int_{jh}^{(j+1)h} |W_i(nh+u) - \widehat{W}_{h,i}(n+j)| \, du \\
&\qquad + G \int_{jh}^{(j+1)h} |W_i(u) - \widehat{W}_{h,i}(j)| \, du.
\end{aligned}
$$

If $N_i - n + 1 \le j$, then $N_i + 1 \le j + n$, so $\widehat{W}_{h,i}(n+j) = 1$. Also $\tau_i < (N_i+1)h \le (n+j)h \le nh + u$. Therefore $W_i(nh+u) = 1$. Therefore

$$
\begin{aligned}
\sum_{j=0}^{N_i-1} \int_{jh}^{(j+1)h} &\Big| (W_i(nh+u) - W_i(u)) \, g(\psi(-u)) \\
&\qquad - \left(\widehat{W}_{h,i}(n+j) - \widehat{W}_{h,i}(j)\right) g(\psi(-jh)) \Big| \, du \\
&\le K_2 \delta_3(h) \tau_i + G \sum_{j=0}^{N_i-1} \int_{jh}^{(j+1)h} |W_i(nh+u) - \widehat{W}_{h,i}(n+j)| \, du \\
&\qquad + G \sum_{j=0}^{N_i-1} \int_{jh}^{(j+1)h} |W_i(u) - \widehat{W}_{h,i}(j)| \, du. \qquad (8.6.6)
\end{aligned}
$$

We now estimate the second term on the righthand side of (8.6.6). If $0 \le j \le N_i - n - 1$, then $\widehat{W}_{h,i}(n+j) = \sum_{l=0}^{n+j-1} w_i(lh)h$. Also $nh + u \le (n+j+1)h \le N_i h \le \tau_i$. Therefore $W_i(nh+u) = \int_0^{nh+u} w_i(s) \, ds$. Hence

$$
\begin{aligned}
&\sum_{j=0}^{N_i-1} \int_{jh}^{(j+1)h} |W_i(nh+u) - \widehat{W}_{h,i}(n+j)| \, du \\
&\le \sum_{j=0}^{N_i-1} \int_{jh}^{(j+1)h} \left| \int_0^{nh+u} w_i(s) \, ds - \sum_{l=0}^{n+j-1} w_i(lh)h \right| du.
\end{aligned}
$$

The integrand can be estimated by

$$\begin{aligned}
&\left|\int_0^{nh+u} w_i(s)\,ds - \sum_{l=0}^{n+j-1} w_i(lh)h\right| \\
&\le \left|\int_0^{nh+jh} w_i(s)\,ds - \sum_{l=0}^{n+j-1} w_i(lh)h\right| + \int_{(n+j)h}^{nh+u} w_i(s)\,ds \\
&\le \left|\sum_{l=0}^{n+j-1}\int_{lh}^{(l+1)h} w_i(s) - w_i(lh)\,ds\right| + h\bar{w}_i \\
&\le \sum_{l=0}^{n+j-1}\int_{lh}^{(l+1)h} |w_i(s) - w_i(lh)|\,ds + h\bar{w}_i \\
&\le \sum_{l=0}^{n+j-1} \delta_i(h)h + h\bar{w}_i = (n+j)\delta_i(h)h + h\bar{w}_i \\
&\le (N_i-1)\delta_i(h)h + h\bar{w}_i \le \tau_i\delta_i(h) + h\bar{w}_i.
\end{aligned}$$

Therefore

$$\begin{aligned}
&G\sum_{j=0}^{N_i-n-1}\int_{jh}^{(j+1)h} |W_i(nh+u) - \widehat{W}_{h,i}(n+j)|\,du \\
&\le G\sum_{j=0}^{N_i-n-1} h(\tau_i\delta_i(h) + h\bar{w}_i) = G(N_i-n)h(\tau_i\delta_i(h) + h\bar{w}_i) \\
&\le GN_ih(\tau_i\delta_i(h) + h\bar{w}_i),
\end{aligned}$$

so

$$G\sum_{j=0}^{N_i-n-1}\int_{jh}^{(j+1)h} |W_i(nh+u) - \widehat{W}_{h,i}(n+j)|\,du \\
\le G\tau_i\left(\tau_i\delta_i(h) + h\bar{w}_i\right), \quad n = 1,\dots,N_i. \quad (8.6.7)$$

This estimate holds for $n=0$ also. Inserting this estimate into (8.6.6) yields

$$\begin{aligned}
&\sum_{j=0}^{N_i-1}\int_{jh}^{(j+1)h}\Big|(W_i(nh+u) - W_i(u))\,g(\psi(-u)) \\
&\qquad - \left(\widehat{W}_{h,i}(n+j) - \widehat{W}_{h,i}(j)\right) g(\psi(-jh))\Big|\,du \\
&\le K_2\delta_3(h)\tau_i + G\int_{(N_i-n)h}^{(N_i-n+1)h} |W_i(nh+u) - \widehat{W}_{h,i}(N_i)|\,du + 2G\tau_i\left(\tau_i\delta_i(h) + h\bar{w}_i\right) \\
&\le K_2\delta_3(h)\tau_i + hG\max_{v\in[N_ih,(N_i+1)h]} |W_i(v) - \widehat{W}_{h,i}(N_i)| + 2G\tau_i\left(\tau_i\delta_i(h) + h\bar{w}_i\right).
\end{aligned}$$

Inserting this estimate into (8.6.5) we get

$$|I_i(\psi,nh) - \widehat{I}_i(\psi,n)| \le K_2\delta_3(h)\tau_i + 2G\tau_i\left(\tau_i\delta_i(h) + h\bar{w}_i\right) + hG + 2(1+\widehat{W}_{h,i}(N_i))Gh, \quad (8.6.8)$$

where we also use the bounds $|W_i(v) - \widehat{W}_{h,i}(N_i)| \le 1 + \widehat{W}_{h,i}(N_i)$ and $|1 - \widehat{W}_{h,i}(N_i)| \le 1 + \widehat{W}_{h,i}(N_i)$. Finally we estimate $\widehat{W}_{h,i}(N_i)$ purely in terms of h. Using (8.6.3b) we have

$$\begin{aligned}
|\widehat{W}_{h,i}(N_i) - W_i(N_ih)| &= \left|\int_0^{N_ih} w_i(s)\,ds - \sum_{l=0}^{N_i-1} w_i(lh)h\right| \\
&\le \sum_{l=0}^{N_i-1}\int_{lh}^{(l+1)h} |w_i(s) - w_i(lh)|\,ds \le \sum_{l=0}^{N_i-1}\delta_i(h)h = N_i\delta_i(h)h \le \tau_i\delta_i(h).
\end{aligned}$$

Thus $\widehat{W}_{h,i}(N_i) \le 1 + \tau_i\delta_i(h)$. Using this together with (8.6.8) gives the estimate (8.6.4) as required. □

Lemma 8.6.2. *Let $i = 1, 2$. Let $I_i^*(\psi)$ be defined by* (8.6.2a) *where $\widehat{W}_{i,h}$ obeys* (8.6.3)*. Suppose that G is defined by* (8.2.14) *and δ_3 by* (8.2.13)*. Then*

$$|I_i^*(\psi) - \widehat{I}_i^*(\psi)| \le K_2\delta_3(h)\tau_i + G\tau_i\left(\tau_i\delta_i(h) + h\bar{w}_i\right) + hG + (2 + \tau_i\delta_i(h))Gh. \tag{8.6.9}$$

Therefore, by (8.5.4)*, we can define $\varepsilon_i^*(h)$ by* (8.3.13)*. Moreover*

$$\lim_{h\to 0^+} \varepsilon_i^*(h) = \lim_{h\to 0^+} |I_i(\psi) - \widehat{I}_i^*(\psi)| = 0.$$

Remark 8.6.2. As proof is short no outline is given.

Proof. By (8.6.1b) and (8.6.2b), we have

$$\begin{aligned} & I_i^*(\psi) - \widehat{I}_i^*(\psi) \\ &= \sum_{j=0}^{N_i-1}\int_{jh}^{(j+1)h}(1 - W_i(u))g(\psi(-u))\,du + \int_{N_ih}^{\tau_i}(1 - W_i(u))g(\psi(-u))\,du \\ &\quad - \sum_{j=0}^{N_i-1}\left\{\left(1 - \widehat{W}_{h,i}(j)\right)g(\psi(-jh))\right\}h - \left\{\left(1 - \widehat{W}_{h,i}(N_i)\right)g(\psi(-N_ih))\right\}h. \end{aligned}$$

Therefore by (8.2.14), and the fact that $W_i(t) \in [0, 1]$ for all $t \ge 0$ we have,

$$\begin{aligned} & |I_i^*(\psi) - \widehat{I}_i^*(\psi)| \\ &\le \sum_{j=0}^{N_i-1}\int_{jh}^{(j+1)h}\left|(1 - W_i(u))g(\psi(-u)) - \left(1 - \widehat{W}_{h,i}(j)\right)g(\psi(-jh))\right|du \\ &\qquad + \int_{N_ih}^{\tau_i}|1 - W_i(u))|G\,du + \left\{|1 - \widehat{W}_{h,i}(N_i)|G\right\}h \\ &\le \sum_{j=0}^{N_i-1}\int_{jh}^{(j+1)h}\left|(1 - W_i(u))g(\psi(-u)) - \left(1 - \widehat{W}_{h,i}(j)\right)g(\psi(-jh))\right|du \\ &\quad + hG + |1 - \widehat{W}_{h,i}(N_i)|Gh. \end{aligned} \tag{8.6.10}$$

We now estimate the first term on the righthand side of (8.6.10). Let $j = 0\ldots N_i - 1$, $u \in [jh, (j + 1)]$, $1 \le n \le N_i$. Then

$$\begin{aligned} & (1 - W_i(u))g(\psi(-u)) - \left(1 - \widehat{W}_{h,i}(j)\right)g(\psi(-jh)) \\ &= (1 - W_i(u))\left\{g(\psi(-u)) - g(\psi(-jh))\right\} + \left(\widehat{W}_{h,i}(j) - W_i(u))\right)g(\psi(-jh)). \end{aligned}$$

Thus as $W_i(t) \in [0, 1]$ for all $t \ge 0$, g obeys (8.2.4), ψ obeys (8.2.13), and (8.2.14) holds, we get

$$\begin{aligned} & |(1 - W_i(u))g(\psi(-u)) - \left(1 - \widehat{W}_{h,i}(j)\right)g(\psi(-jh))| \\ &\le |1 - W_i(u)|\,|g(\psi(-u)) - g(\psi(-jh))| + \left|-W_i(u)) + \widehat{W}_{h,i}(j))\right|G \\ &\le K_2|\psi(-u) - \psi(-jh)| + |W_i(u) - \widehat{W}_{h,i}(j)|G \\ &\le K_2\delta_3(h) + |W_i(u) - \widehat{W}_{h,i}(j)|G. \end{aligned}$$

Therefore

$$\begin{aligned} & \sum_{j=0}^{N_i-1}\int_{jh}^{(j+1)h}\left|(1 - W_i(u))\,g(\psi(-u)) - \left(1 - \widehat{W}_{h,i}(j)\right)g(\psi(-jh))\right|du \\ &\qquad\qquad \le K_2\delta_3(h)\tau_i + G\sum_{j=0}^{N_i-1}\int_{jh}^{(j+1)h}|W_i(u) - \widehat{W}_{h,i}(j)|\,du. \end{aligned} \tag{8.6.11}$$

Recalling that the estimate (8.6.7) holds for $n = 0$, we can use (8.6.7) with $n = 0$ and (8.6.11) to get

$$\sum_{j=0}^{N_i-1} \int_{jh}^{(j+1)h} \left| (1 - W_i(u))\, g(\psi(-u)) - \left(1 - \widehat{W}_{h,i}(j)\right) g(\psi(-jh)) \right| du$$
$$\leq K_2 \delta_3(h) \tau_i + G\tau_i \left(\tau_i \delta_i(h) + h\bar{w}_i\right).$$

Therefore using this bound in conjunction with by (8.6.10) we get

$$|I_i^*(\psi) - \widehat{I}_i^*(\psi)| \leq K_2 \delta_3(h) \tau_i + G\tau_i \left(\tau_i \delta_i(h) + h\bar{w}_i\right) + hG + (1 + \widehat{W}_{h,i}(N_i))Gh. \tag{8.6.12}$$

Substituting the bound $\widehat{W}_{h,i}(N_i) \leq 1 + \tau_i \delta_i(h)$ into (8.6.12) gives (8.6.9) as required. □

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