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Krylov Subspaces from Bilinear Representations of Nonlinear systems

Research paper

Abstract –

Purpose: The paper is aimed at the development of novel model reduction techniques for nonlinear systems. This is of paramount importance for the efficient simulation of state-of-the-art dynamical systems as arise in all aspects of engineering.

Methodology/Approach: The analysis is based on the bilinear and polynomial representation of nonlinear systems and the exact solution of the bilinear system in terms of Volterra series. Two sets of Krylov subspaces are identified which are the most essential for capturing the input-output behaviour of the system and therefore preserved in the reduced model.

Findings: The paper proposes two novel model-reduction strategies for nonlinear systems. The first involves the development, in a novel manner as compared to previous approaches, of a reduced-order model from a bilinear representation of the system while the second involves a reducing a polynomial approximation using Krylov subspaces derived from a related bilinear representation. Both techniques are shown to be effective through the evidence of a standard test example.

Research limitations/implications: The proposed methodology is applicable to so-called weakly nonlinear systems where both the bilinear and polynomial representations are valid.

Practical implications: From a circuits and systems viewpoint, systems involving micromachined devices or systems involving mixed technologies necessitate the development of reduced-order nonlinear models. From a control systems viewpoint, the design of controllers for nonlinear systems is greatly facilitated by nonlinear model reduction strategies. The suggested methods lead to an improvement in the accuracy of nonlinear model reduction.

Originality/value of paper: The proposed novel approaches for model reduction are particularly beneficial for the design of controllers for nonlinear systems and for the design of Radio-Frequency Integrated Circuits.

Keywords: Bilinear system, Model Reduction, System simulation

1. INTRODUCTION

With the growing complexity and dimensionality of state-of-the-art dynamical systems as arise in all aspects of engineering, model reduction is becoming a vital aspect of modern system simulation. While model reduction techniques for linear systems are well studied (e.g. (Antoulas, 2003; Antoulas *et al.*, 2001; Chiprout *et al.*, 1995; Feldmann *et al.*, 1995; Gallivan *et al.*, 1994; Gallivan *et al.*, 1999; Odabasoglu *et al.*, 1998; Phillips *et al.*, 2003; Silveira *et al.*, 1995; Shi *et al.*, 2003) and references therein) especially in the context of interconnect and package modelling, the study of nonlinear model reduction strategies has received considerably less attention. However, from a circuits and systems viewpoint, systems involving micromachined devices or systems involving mixed technologies necessitate the development of reduced-order nonlinear models. From a control systems viewpoint, the design of controllers for nonlinear systems is greatly facilitated by nonlinear model reduction strategies. Applications for effective nonlinear control design abound in engineering from the control of chemical process systems to the control of aeronautical and electrical power systems. Hence, the development of model reduction methods for nonlinear systems is of paramount importance to the general engineering community. Unfortunately, the study of nonlinear systems is much more complicated since their solutions (when they exist) can be of a quite complex nature (not unique, singular, chaotic etc.). Therefore, the development of suitable reduced-order modelling techniques represents a formidable challenge. Some recent work in this field is presented in (Bai, 2002; Bai *et al.*, 2002; Balakrishnan *et al.*, 2001; Chen *et al.*, 2000; Condon *et al.*, 2004; Condon *et al.*, 2005; Dong *et al.*, 2003; Gunupudi *et al.*, 1999; Hahn *et al.*, 2002; Lall *et al.*, 2002; Phillips, 2000; Phillips, 2003; Rewieński *et al.*, 2003; Rowley, 2004).

The present contribution proposes two novel techniques involving Krylov subspaces for model reduction of weakly nonlinear systems. These are systems in which the nonlinearities may be approximated by low-order polynomials.

The particular choice of *Krylov subspace* model reduction stems from the success of the Krylov paradigm in linear model reduction. Firstly for linear systems, the choice of a projection matrix is straightforward resulting from the moment-matching properties of the transfer function of the system. Secondly, the computation the projection matrix is straightforward involving only the solution of linear equations or matrix products. Furthermore, the Krylov approach enables the efficient formation and simulation of a reduced order model in that the reduced order model has the same form as the original system but is of much lower dimensionality. However, the development of Krylov approaches for nonlinear model reduction is not quite so straightforward. Consider the following nonlinear system:

$$\begin{aligned}\dot{x}(t) &= f(x(t)) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}\tag{1}$$

where $f : \mathfrak{R}^n \rightarrow \mathfrak{R}^n$ is a non-linear function with the initial condition $x(0)=x_0$ and $u(t), y(t) \in \mathfrak{R}^1$. $B, C \in \mathfrak{R}^n$ are constant vectors (C is a vector-row and B is a vector-column). To directly translate the projection scheme developed for linear model reduction would involve the determination of a projection matrix V which is orthogonal, $V^T V = I$, such that

$$\begin{aligned}\dot{x}_r(t) &= V^T f(Vx_r(t)) + \hat{B}u(t) \\ \hat{y}(t) &= CVx_r(t)\end{aligned}\tag{2}$$

where $x(t) = Vx_r(t)$ and $x_r(t)$ is the reduced state space. $\hat{B} = V^T B$ and the aim is that $\hat{y} \approx y$ where \hat{y} is the output of the reduced model. However, to date, there has been

no universal approach proposed for the determination of V . Furthermore, as outlined in (Phillips, 2003), the interpretation of (2) as a reduced-order model for a nonlinear system is dubious. Since f is a nonlinear function, it is not, in general, possible to pass V through the parentheses in (2) and thus computation of the nonlinear function, f , is unavoidable. Since the computation of such a function is often the major determining factor in the overall system simulation time, a reduction in the size of the state-space if achieved in this manner may not produce the desired effect as regards a significant reduction in computation time.

Thus, for nonlinear systems, some compromises have to be made if Krylov approaches are to be utilised. For this contribution, the compromise comprises either a restriction on the type of nonlinear system under consideration or an approximation of the nonlinear equations describing the system behaviour. In particular, two categories of nonlinear system representation will be considered – bilinear system representations and polynomial system representations. A bilinear system is one which is linear in state, linear in control but not linear jointly. Bilinear systems frequently arise naturally in engineering, for example, nuclear fission, chemical and biological models and ecological models (Mohler, 1973; Baillieul, 1995). However, even when the system itself is not naturally bilinear, the bilinear representation offers a superior representation to a linear model, naturally incorporating the higher-order polynomial terms in the series expansion of the nonlinear function, f .

While some Krylov subspace based approaches have been proposed for nonlinear model reduction e.g. (Bai, 2002; Bai *et al.*, 2002; Phillips, 2000; Phillips, 2003) employing bilinear and polynomial representations, the current work employs them in

a rather different manner to that previously presented. Full details of the new approaches and their position relative to existing methods will be detailed in subsequent sections. Section 2 will present the first approach and the second is detailed in Section 3. An illustrative and standard example (Bai *et al.*, 2002; Chen *et al.*, 2000; Dong *et al.*, 2003; Phillips, 2000; Phillips, 2003; Rewieński *et al.*, 2003) is given in Section 4, which confirms the efficacy of the proposed approaches. It should be noted that this example is not intended to be a practical application of the techniques. It is chosen to enable ease of comparison of the proposed techniques with existing approaches and to confirm the theoretical proposals put forth throughout the paper.

2. BILINEAR APPROXIMATION OF WEAKLY NONLINEAR SYSTEMS

Consider again the nonlinear system in (1). For the ensuing analysis, it is assumed that the system (1) is weakly nonlinear with an asymptotically stable equilibrium point as described in (Bai *et al.*, 2002; Chen *et al.*, 2000; Dong *et al.*, 2003; Phillips, 2000; Phillips, 2003). Without loss of generality, it is assumed that $x = 0$ is the stable equilibrium point of the system i.e. $f(0) = 0$. Under this assumption $f(x)$ can be expanded in a generalised Taylor's series about the equilibrium point $x = 0$:

$$f(x) = A_1 x^{(1)} + A_2 x^{(2)} + A_3 x^{(3)} + \dots \quad (3)$$

where $x^{(1)} = x$, $x^{(2)} = x \otimes x$, $x^{(3)} = x \otimes x \otimes x$, etc. and \otimes denotes the Kronecker product. Stability of the system implies that all the eigenvalues of A_1 have negative real parts.

The systems under consideration will be assumed to be weakly nonlinear. Hence, the condition that each term in the Taylor's expansion is small compared to the previous

one will be taken to hold. Consequently, the system in (1) can be approximated by the well-known bilinear representation (Carleman bilinearization) of (1) (Phillips, 2000; Phillips, 2003):

$$\begin{aligned}\dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \hat{N}\hat{x}(t)u(t) + \hat{B}u(t) \\ y(t) &= \hat{C}\hat{x}(t)\end{aligned}\quad (4)$$

where

$$\hat{x}(t) = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \end{bmatrix} \quad (5)$$

and $\hat{A}, \hat{N}, \hat{B}$ and \hat{C} are constant matrices:

$$\hat{A} = \begin{bmatrix} A_1 & A_2 & \dots & & \\ & A_{21} & A_{22} & & \\ & & A_{31} & A_{32} & \\ & & & \ddots & \\ & & & & \ddots \end{bmatrix} \quad \hat{N} = \begin{bmatrix} 0 & 0 & \dots & & \\ B_{20} & 0 & 0 & \dots & \\ & B_{30} & 0 & & \\ & & & \ddots & \\ & & & & \ddots \end{bmatrix} \quad \hat{B} = \begin{bmatrix} B \\ 0 \\ \vdots \end{bmatrix} \quad \hat{C} = [C \quad 0 \quad \dots].$$

The matrices A_i are defined from the Taylor's series expansion in (3) and

$$A_{ji} = A_i \otimes I \otimes \dots \otimes I + I \otimes A_i \otimes \dots \otimes I + \dots + I \otimes I \otimes \dots \otimes A_i,$$

$$B_{j0} = B \otimes I \otimes \dots \otimes I + I \otimes B \otimes \dots \otimes I + \dots + I \otimes I \otimes \dots \otimes B$$

where I is the $n \times n$ identity matrix and there are j terms in each sum. Thus \hat{A} , \hat{N} are square matrices of dimension $n + n^2 + \dots + n^K$; \hat{x} , \hat{B} , \hat{C} are vectors with $n + n^2 + \dots + n^K$ components if K terms in the Taylor's series expansion are taken into account.

If A_I is stable then \hat{A} is also stable, since each of its diagonal blocks is stable (see e.g. (Condon *et al.*, 2005) for details). This however does not guarantee that the bilinear system (4) is BIBO (bounded input-bounded output) stable. A sufficient condition for stability on the interval $[0, \tau]$ is (Condon *et al.*, 2005)

$$|u(t)| \leq K_B \quad (6)$$

for all $t \in [0, \tau]$ where $\hat{A} + \lambda \hat{N}$ is stable for all $\lambda \in [-K_B, K_B]$. (Since \hat{A} is stable, $\hat{A} + \lambda \hat{N}$ is stable in the vicinity of $\lambda = 0$, i.e. there exists an interval $[-K_B, K_B]$, such that $\hat{A} + \lambda \hat{N}$ is stable for all $\lambda \in [-K_B, K_B]$).

As stated in the introduction, the rationale for employing the bilinear representation is that it allows higher-order terms to be explicitly incorporated in the subsequent model reduction technique and hence is superior to employing a linear representation. However, for practical purposes, the matrix $\hat{x}(t)$ requires truncation. For the present work, \hat{x} is taken as:

$$\hat{x}(t) = \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix}$$

which corresponds to taking into account the quadratic terms in (3).

Consequently,

$$\hat{A} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_{21} \end{bmatrix}, \quad \hat{N} = \begin{bmatrix} 0 & 0 \\ N & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \hat{C} = [C \quad 0] \quad (7)$$

where A_1 and A_2 are the matrices in (3), $A_{21} = A_1 \otimes I + I \otimes A_1$, $N = B \otimes I + I \otimes B$ where I is the $n \times n$ identity matrix. Thus, \hat{A} and \hat{N} are square matrices of dimension $n + n^2$; \hat{x} , \hat{B} , \hat{C} are vectors with $n + n^2$ components.

Now for the purposes of developing the new Krylov approach, consider, initially, the case of a bilinear system subject to a constant input \bar{u} .

$$\begin{aligned} \dot{\hat{x}}(t) &= (\hat{A} + \bar{u} \hat{N}) \hat{x}(t) + \hat{B} \bar{u} \\ y(t) &= \hat{C} \hat{x}(t) \end{aligned} \quad (8)$$

This obviously results in the linear system (8) and thus the system, (8), possesses all the advantageous properties pertaining to linear systems. Bearing this observation in mind, for the general case where $u(t) \neq \text{const}$, it is proposed to introduce a parameter κ which depends on $u(t)$: $\kappa = \kappa[u]$ and $-K_B \leq \kappa[u] \leq K_B$ reflecting the fact that $|u(t)| \leq K_B$ where K_B is a constant bound as given in (6). It is assumed that κ is a functional on u (i.e. a parameter that does not depend on t but does depend on the behaviour of $u(t)$ for $t \in [0, \tau]$ and on the specific choice of interval, $[0, \tau]$). The dependence of κ on $u(t)$ is important. This enables the Krylov subspaces of the system (8) to vary with $u(t)$ which results in a superior model reduction technique for a bilinear representation. Now, in order to define $\kappa[u]$, consider a rescaling of the input $u \rightarrow \gamma u$ where γ is a constant and γu is sufficiently small. This transforms the bilinear system into another bilinear system with source $u(t)$ as follows:

$$\begin{aligned}\hat{x}(t) &= \hat{A}\hat{x}(t) + \gamma\hat{N}\hat{x}(t)u(t) + (\gamma\hat{B})u(t) \\ y(t) &= \hat{C}\hat{x}(t)\end{aligned}\tag{9}$$

A rescaling of \hat{B} does not affect the determination of Krylov spaces.

However, the term $\gamma\hat{N}\hat{x}u$ in (9) indicates that it is necessary to impose the validity of the following property of κ :

$$\kappa[\gamma u] = \gamma\kappa[u]\tag{10}$$

i.e. $\kappa[u]$ must be a linear functional on u . Therefore, for the general case of a non-constant input, it is reasonable to define $\kappa[u]$ as follows:

$$\kappa[u] = \frac{1}{\tau} \int_0^\tau u(t) dt.\tag{11}$$

The definition in (11) obviously obeys (10). This choice of κ is extensively used in the study of different types of bilinear systems; see e.g. (Baillieul, 1995). Since for

bilinear systems the input-output mapping (e.g. see (20) below) depends nonlinearly on the input, then the validity of approximation (11) for a particular system can be only confirmed from computer simulations.

The proposition then is to employ the following linear system that is related to the original bilinear system to extract a projection subspace for the bilinear system with a $\kappa[u]$ value determined from (11) or by some other means:

$$\begin{aligned}\hat{x}(t) &= (\hat{A} + \kappa\hat{N})\hat{x}(t) + \hat{B}u(t) \\ y(t) &= \hat{C}\hat{x}(t)\end{aligned}\tag{12}$$

When the system in (12) is represented in the frequency domain, the input and output are related by the following transfer function:

$$H(s) = \hat{C}(s\hat{I} - \hat{A}_\kappa)^{-1}\hat{B}\tag{13}$$

where $\hat{A}_\kappa = \hat{A} + \kappa\hat{N}$ and \hat{I} is the corresponding $n + n^2$ dimensional identity matrix.

$H(s)$ may be expanded about a selected expansion point s_0 as follows:

$$\begin{aligned}H(s) &= \hat{C}\left((s - s_0)\hat{I} - (\hat{A}_\kappa - s_0\hat{I})\right)^{-1}\hat{B} \equiv \hat{C}\left((s - s_0)\hat{I} - \hat{A}_{\kappa,s_0}\right)^{-1}\hat{B} = \\ &= \sum_{p=0}^{\infty} m_p (s - s_0)^p\end{aligned}\tag{14}$$

where

$$\hat{A}_{\kappa,s_0} \equiv \hat{A}_\kappa - s_0\hat{I} = \hat{A} + \kappa\hat{N} - s_0\hat{I}\tag{15}$$

and m_p (termed moments) are:

$$m_p = -\hat{C}\hat{A}_{\kappa,s_0}^{-(p+1)}\hat{B}\tag{16}$$

The choice of expansion point s_0 is an issue that arises not only in relation to nonlinear reduction but also in Linear system theory. In general, the choice depends on which part of the spectrum of \hat{A}_κ that requires emphasis - it is desirable that the selected part is that which is the most significant for the input-output mapping- e.g. see the discussion in (Gallivan *et al.*, 1994). Many authors report significant improvement in relation to Krylov spaces when several expansion points are employed- e.g. (Chiprout *et al.*, 1995; Gallivan *et al.*, 1994; Phillips *et al.*, 2003; Shi *et al.*, 2003). Some selection strategies for the expansion points are proposed in (Chiprout *et al.*, 1995).

For model reduction purposes, what is of interest is the relationship between the moments and the formation of suitable projection subspaces. All of the moments (16) may be written as scalar products between the following left and right Krylov spaces e.g. (Bai *et al.*, 2002):

$$\begin{aligned} \hat{K}_{R1}(\hat{A}_{\kappa, s_0}^{-1}, \hat{B}) &\equiv \text{span}\{\hat{B}, \hat{A}_{\kappa, s_0}^{-1} \hat{B}, \dots, \hat{A}_{\kappa, s_0}^{-(n+n^2-1)} \hat{B}\}, \\ \hat{K}_{L1}(\hat{C}, \hat{A}_{\kappa, s_0}^{-1}) &\equiv \text{span}\{\hat{C}, \hat{C} \hat{A}_{\kappa, s_0}^{-1}, \dots, \hat{C} \hat{A}_{\kappa, s_0}^{-(n+n^2-1)}\}. \end{aligned} \quad (17)$$

(The notation is adopted throughout that Krylov subspaces of the form \hat{K} have dimension $n + n^2 \dots$ while Krylov spaces of the form K have dimension n).

The projection subspace is formed by taking the first k vectors of each Krylov space. To avoid ill-conditioning in the reduction matrices, the two bases can be made biorthogonal, i.e. $WV = I_k$ where the vector-columns of V are from \hat{K}_{R1} , the vector-rows of W are from \hat{K}_{L1} and I_k is the $k \times k$ identity matrix.

The state vector $\hat{x}(t)$ may then be approximated by the ‘reduced’ k -dimensional state vector $x_r(t)$ i.e. $\hat{x}(t) \approx Vx_r(t)$. The resultant reduced bilinear system is therefore:

$$\begin{aligned}\dot{x}_r(t) &= A_r x_r(t) + N_r x_r(t)u(t) + B_r u(t) \\ y(t) &= C_r x_r(t)\end{aligned}\tag{18}$$

where $A_r = W\hat{A}V$, $N_r = W\hat{N}V$, $B_r = W\hat{B}$ and $C_r = \hat{C}V$.

Thus, by employing the system in (12) it is possible to define suitable Krylov spaces that may subsequently be used for determining a reduced-order bilinear system. The success of employing such subspaces will be evident from the results in Section 4.

3. QUADRATIC APPROXIMATION OF WEAKLY NONLINEAR SYSTEMS

The second technique proposed for model reduction of weakly nonlinear systems is based on a polynomial approximation of the given system (1). However, for ease of explanation, the current work will be restricted to a quadratic approximation which is as follows:

$$\begin{aligned}\dot{x}(t) &= A_1 x + A_2 x \otimes x + Bu(t) \\ y(t) &= Cx(t)\end{aligned}\tag{19}$$

Particular observations in relation to a corresponding bilinear approximation of the nonlinear system can provide some insight into the construction of a suitable Krylov space for model reduction of the quadratic representation. Therefore, the first part of this section will again focus on the bilinear representation. Consider the solution to the bilinear system in (4) (D'Alessandro, *et al.*, 1974):

$$\hat{y}(t) = y_x(t) + y_u(t) + y_{xu}(t)\tag{20}$$

where $y_x(t) = \hat{C}e^{\hat{A}t}\hat{x}_0$,

$$y_u(t) = \sum_{i=1}^{\infty} \frac{1}{i!} \int_0^t \dots \int_0^t w_i(t_1, \dots, t_i) \left[\prod_{k=1}^i u(t-t_k) \right] dt_1 \dots dt_i$$

$$y_{xu}(t) = \sum_{i=1}^{\infty} \frac{1}{i!} \int_0^t \dots \int_0^t z_i(t_1, \dots, t_i) \hat{N}e^{\hat{A}(t-t_1)} \hat{x}_0 \left[\prod_{k=1}^i u(t-t_k) \right] dt_1 \dots dt_i$$

$$w_i(t_1, \dots, t_i) = \sum_{\wp_i} v_i(t_1, \dots, t_i)$$

$$v_1(t_1) = \hat{C} e^{\hat{A} t_1} \hat{B},$$

$$v_i(t_1, \dots, t_i) = \hat{C} e^{\hat{A} t_i} \hat{N} e^{\hat{A}(t_{i-1}-t_i)} \dots \hat{N} e^{\hat{A}(t_1-t_2)} \hat{B} \left[\prod_{k=0}^{i-2} \theta(t_{k+1} - t_{k+2}) \right],$$

\wp_i is the set of permutations of $\{t_1, \dots, t_i\}$ and θ is the unit step function. The expression for z_i can be obtained from those for v_i with \hat{B} replaced by the identity matrix I .

The kernels $\hat{C} e^{\hat{A} t_i} \hat{N} e^{\hat{A}(t_{i-1}-t_i)} \dots \hat{N} e^{\hat{A}(t_1-t_2)} \hat{B}$ of this solution naturally lead to the multi-dimensional transfer functions of the form (Bai *et al.*, 2002):

$$H_i(s_1, s_2, \dots, s_i) = \hat{C} (s_1 \hat{I} - \hat{A})^{-1} \hat{N} \dots \hat{N} (s_1 \hat{I} - \hat{A})^{-1} \hat{B} \quad (21)$$

The coefficients in a power series expansion of H_i (about s_0) are the following multi-moments:

$$m_i(l_1, l_2, \dots, l_i) = (-1)^i \hat{C} \hat{A}_{s_0}^{-l_i} \hat{N} \dots \hat{N} \hat{A}_{s_0}^{-l_2} \hat{N} \hat{A}_{s_0}^{-l_1} \hat{B} \quad (22)$$

where l_i are nonnegative integers and $\hat{A}_{s_0} = \hat{A} - s_0 \hat{I}$. Since the selection of s_0 depends on the spectrum of \hat{A} and not on the input, the most suitable value can be selected using a test-input.

The matrices involved in (22) have the following noteworthy structure:

$$\hat{A}_{s_0}^{-1} = \begin{bmatrix} A_{1,s_0}^{-1} & -A_{1,s_0}^{-1} A_2 A_{21,s_0}^{-1} \\ 0 & A_{21,s_0}^{-1} \end{bmatrix} \quad (23)$$

$$\hat{A}_{s_0}^{-l} = \begin{bmatrix} A_{1,s_0}^{-l} & -\sum_{l_1+l_2=l} A_{1,s_0}^{-l_1} A_2 A_{21,s_0}^{-l_2} \\ 0 & A_{21,s_0}^{-l} \end{bmatrix} \quad (24)$$

where $A_{1,s_0} = A_1 - s_0 I$ and $A_{21,s_0} = A_{21} - s_0 I \otimes I$.

As a result of this structure and that of \hat{B} and \hat{C} (see Section 2 (7)), the kernels of degree 1 may be written as:

$$m_1(l) = -\hat{C}\hat{A}_{s_0}^{-l}\hat{B} = -CA_{1,s_0}^{-l}B \quad (25)$$

This indicates that they can therefore be represented as scalar products between vectors of the following n -dimensional (**not $n + n^2$ -dimensional**) Krylov spaces:

$$\begin{aligned} K_{R,1}(A_{1,s_0}^{-1}, B) &\equiv \text{span}\{B, A_{1,s_0}^{-1}B, \dots, A_{1,s_0}^{-(n-1)}B\}, \\ K_{L,1}(C, A_{1,s_0}^{-1}) &\equiv \text{span}\{C, CA_{1,s_0}^{-1}, \dots, CA_{1,s_0}^{-(n-1)}\}. \end{aligned} \quad (26)$$

Similarly, kernels of degree 2 may also be represented as scalar products between vectors of n -dimensional Krylov spaces:

$$m_2(p, l) = \hat{C}\hat{A}_{s_0}^{-l}\hat{N}\hat{A}_{s_0}^{-p}\hat{B} = -\sum_{l_1+l_2=l} CA_{1,s_0}^{-l_1}A_2A_{21,s_0}^{-l_2}NA_{1,s_0}^{-p}B \quad (27)$$

The combined Krylov spaces are as follows:

$$\begin{aligned} K_{R,2} &= K_{R,1}(A_{1,s_0}^{-1}, A_2 \cdot K_{R,1}(A_{21,s_0}^{-1}, N \cdot K_{R,1}(A_{1,s_0}^{-1}, B))) \\ K_{L,2} &= K_{L,1}(K_{L,1}(K_{L,1}(C, A_{1,s_0}^{-1}) \cdot A_2, A_{21,s_0}) \cdot N, A_{1,s_0}^{-1}) \\ K_R &= K_{R,1} \cup K_{R,2} \\ K_L &= K_{L,1} \cup K_{L,2} \\ K_R &\equiv \bigcup_{l_1, l_2 \geq 0, p > 0} \text{span}\{A_{1,s_0}^{-p}B; A_{1,s_0}^{-l_1}A_2A_{21,s_0}^{-l_2}NA_{1,s_0}^{-p}B\} \\ K_L &\equiv \bigcup_{l_1, l_2 \geq 0, p > 0} \text{span}\{CA_{1,s_0}^{-l_1}; CA_{1,s_0}^{-l_1}A_2A_{21,s_0}^{-l_2}NA_{1,s_0}^{-p}\}. \end{aligned} \quad (28)$$

Thus, the crucial observation is that n -dimensional Krylov spaces suffice for matching both degree 1 and degree 2 kernels (and possibly some of the degree 3 kernels) for a bilinear system.

Since the bilinear system of (4)-(7) was formed on the basis of inclusion only of quadratic terms in the Taylor's series approximation of the original system, it is

proposed that the Krylov spaces in (28) can be used for reduction of the n -dimensional quadratic system in (19). Taking the relevant k -dimensional (where k is the order of the reduced system) left and right subspaces, biorthogonal bases and projection matrices V and W can be constructed as described in Section 2. The reduced quadratic system is then:

$$\begin{aligned}\dot{x}_r(t) &= A_{1,r}x_r(t) + A_{2,r}x_r \otimes x_r + B_ru(t) \\ y(t) &= C_rx_r(t)\end{aligned}\tag{29}$$

where x_r is k -dimensional, $A_{1,r} = WA_1V$ (of dimension $k \times k$), $A_{2,r} = WA_2(V \otimes V)$ (of dimension $k \times k^2$), $B_r = WB$ (of dimension k), $C_r = CV$ (of dimension k).

and $x(t) \approx Vx_r(t)$.

4. AN ILLUSTRATIVE EXAMPLE

The circuit employed is the nonlinear RC ladder shown in Fig. 1 (frequently employed as a test circuit for model reduction techniques (Bai *et al.*, 2002; Chen *et al.*, 2000; Dong *et al.*, 2003; Phillips, 2000; Phillips, 2003; Rewieński *et al.*, 2003)). The nonlinear resistors (a diode in parallel with a unit resistor) have the constitutive relation $i(v) = (e^{40v} - 1) + v$ and the capacitors have unit capacitance.

The input to the system is a current source $u(t)$ entering node 1 and the output is the voltage taken at node 1. The number of nodes in the system is $n=30$.

The system is initially approximated by a bilinear system of order $30+30^2=930$. The time interval chosen for consideration is $t \in [0,1]$, i.e. $\tau = 1$, the input is $u(t) = e^{-t}$ and

hence the parameter κ in (11) and (12) is evaluated as $\kappa = \int_0^1 e^{-t} dt = 1 - e^{-1} = 0.6321$.

The reduction process for the bilinear system of order 930 to a bilinear system of order 3 is implemented utilising the Krylov spaces defined by (17) (Method 1). In order to compare various results, the root-mean squared error is calculated between the outputs of the bilinear model (4) of order 930 and the reduced-order models (18). Table 1 shows the results achieved with $\kappa = 0.6321$ for a selection of expansion points s_0 . (Obviously, tests were carried out for a much larger range of expansion points and what is given in Table 1 corresponds only to a suitable selection). The variation in the results clearly shows the importance of judicious choice of expansion point. The best result is achieved with an expansion point of $s_0 = 2.7$ with the corresponding rms error equal to 1.1×10^{-5} . To confirm the validity and superiority of this new approach (Method 1), results are also shown for the case where $\kappa = 0$. This corresponds to the standard linear approximation of a bilinear representation. With $\kappa = 0$ the rms error is $\sim 10^{-3}$ for a large range of values of s_0 around 3.0 (where the rms values are at their lowest). Phillips (2003) also proposes determining a projection basis for bilinear systems based on a Krylov approach. He chooses an initial basis V_1 from $\hat{K}_{L1} = \hat{K}_{R1} = \hat{K}(\hat{A}^{-1}, \hat{B})$ such that first-order kernels are matched up to terms in $s^{q_1} - \hat{A}^{-m} \hat{B} \in \text{span}(V_1) \quad m < q_1$. He then proceeds to determine a basis V_2 from $\hat{K}_{L2} = \hat{K}_{R2} = \hat{K}(\hat{A}^{-1}, \hat{N} \hat{K}_{R1}(\hat{A}^{-1}, \hat{B}))$ and finally constructs $V \in R^{n \times k}$ (k is the order of the reduced system) from $\hat{K}_L = \hat{K}_R = \hat{K}_{L1} \cup \hat{K}_{L2}$.

With this approach the second order regular kernels of the reduced model and the original bilinear system match up to terms in $s^{q_1} s^{q_2} - \hat{A}^{-l} \hat{N} \hat{A}^{-m} \hat{B} = V \hat{A}_r^{-l} \hat{N}_r \hat{A}_r^{-m} \hat{B}_r \quad m < q_1 \quad l < q_2$ where the subscript r denotes matrices of the reduced-order model. Implementation of this approach in conjunction with

biorthogonalisation yields an rms error of 2.4×10^{-4} . (Since biorthogonalisation was employed with the proposed new methods, it was employed with the approach of Phillips (2003) so as to compare like-with-like). Biorthogonalisation is known to be more efficient in multimoment matching than the use of one-sided Krylov subspaces (Bai *et al.*, 2002). As evidenced by these results, the new method proposed leads to equally precise results. Fig. 2 shows that the reduced-order model is effective for a variety of inputs although κ and s_0 were determined with an exponential input as the ‘test’ input.

The second approach is that suggested in Section 3 whereby the system is approximated by a quadratic system (19) of order 30. The reduction to a $k = 3$ dimensional quadratic system is implemented by utilising the following Krylov subspaces (based on (28)):

$$\begin{aligned} \text{span}\{V\} &= \{B, A_{1,s_0}^{-1} B, A_{1,s_0}^{-2} B\}, \\ \text{span}\{W^T\} &= \{C, CA_{1,s_0}^{-1}, CA_{1,s_0}^{-1} A_2 A_{21,s_0}^{-1} N A_{1,s_0}^{-1}\}^T. \end{aligned} \quad (30)$$

and the chosen expansion point is $s_0 = 2.3$. This particular choice of subspaces and expansion point is found to yield the best results. The subspace selection is based on matching the first four of the first-order kernels as the first-order kernels contribute most to the response of the system. The elements of $\text{span}\{V\}$ are from $K_{R,1}(A_{1,s_0}^{-1}, B)$ in (28). The first two elements of $\text{span}\{W^T\}$ are selected from $K_{L,1}(C, A_{1,s_0}^{-1})$ and the third element is selected from $K_{L,2}$. The root-mean squared error is calculated between the outputs of the quadratic model (19) and the reduced-order model (29). The rms error achieved is 1.0×10^{-4} highlighting the efficacy of this second novel

approach. The rms values are for the case of an exponential input to the system (Fig 3A), but the reduced model gives excellent results for other inputs-Fig 3B-D.

If a projection basis is chosen based solely on a linear approximation of the system, the rms error is 7.6×10^{-4} . It should be noted that choosing a basis based on a linear approximation of the system is the standard approach for reducing a quadratic model (Chen *et al.*, 2000) (and also indicated in (Phillips,2003)). These results clearly indicate that the proposed approach results in high precision reduced-order models.

5. CONCLUSIONS

Two novel methods involving Krylov subspaces for model reduction of weakly nonlinear systems have been proposed. The first method is based on bilinearisation of the system and utilizing the Krylov basis for a related linear system (12). The second approach involves using Krylov bases identified for a bilinear representation of the system for subsequent reduction of a polynomial approximation of the system.

The first method (that proposed in Section 2) has the advantage that the choice of Krylov bases is straightforward as is the case for all linear systems. It consists of the selecting the first k vectors of the set given in (17). The drawback with the approach is that which is common to all techniques that involve working directly with bilinear systems and is that the size of bilinear system is large i.e. $n^2 + n$ if only quadratic terms are taken into account. Another point to note is that an extra parameter is introduced into the system, κ (11). Taking the average value of a test input over the interval of interest for the determination of this parameter is a fairly standard practice for a large class of systems. Obviously, for an input such as $u(t) = \sin \omega t$ over a large time interval, $\tau \rightarrow \infty$, clearly the average value will be zero and the system is approximated with a linear approximation of the bilinear representation. However,

for the cases of inputs with a non-zero average over a finite interval such as constant (step) inputs; decaying exponential, then utilising (12) is advantageous. The paper also indicates the necessity of the judicious choice of expansion point, s_0 , in model reduction methods involving series expansions for both linear and nonlinear systems.

The advantage of the method proposed in Section 3 is that there is no need to work directly with the large bilinear system. All that is required is to employ the Krylov spaces resulting from a bilinear representation of the system. Furthermore, there is no new parameter introduced into the modelling process. However, once again the choice of expansion point is a factor that must be taken into account in the application of reduction technique. Also, the selection of the vectors from the general Krylov spaces (28) needs to be made judiciously.

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Captions

Table 1: Results for nonlinear ladder

Fig. 1. Nonlinear circuit (exponential nonlinearity)

Fig. 2. Reduction with method 1 (dash-dotted line) of the original model (solid line).

The full order bilinear model is presented with dotted line.

A: exponential input $u(t)=exp(-t)$;

B: unit step input;

C: sinusoidal input, $u(t) = \frac{1}{2} \sin(4\pi t)$;

D: $u(t) = \frac{1}{2} \cos(2\pi t)$.

Fig. 3. Reduction with Krylov space (30) (dash-dotted line) of the original model (solid line). The full order quadratic approximation of the model (i.e. the nonlinear function (3) with the first two terms in the expansion) is presented with dotted line.

A: exponential input $u(t)=exp(-t)$;

B: unit step input;

C: sinusoidal input, $u(t) = \frac{1}{2} \sin(4\pi t)$;

D: $u(t) = \frac{1}{2} \cos(2\pi t)$.

Table 1

κ	s_0	Rms error
0	0	1.3×10^{-2}
0	3.0 ± 1.0	$(6.5 \pm 0.2) \times 10^{-3}$
0.6321	0	1.7×10^{-2}
0.6321	2.4	2.9×10^{-4}
0.6321	2.5	1.9×10^{-4}
0.6321	2.6	1.0×10^{-4}
0.6321	2.7	1.1×10^{-5}
0.6321	2.8	6.4×10^{-5}
0.6321	2.9	1.3×10^{-4}
0.6321	3.0	2.0×10^{-4}

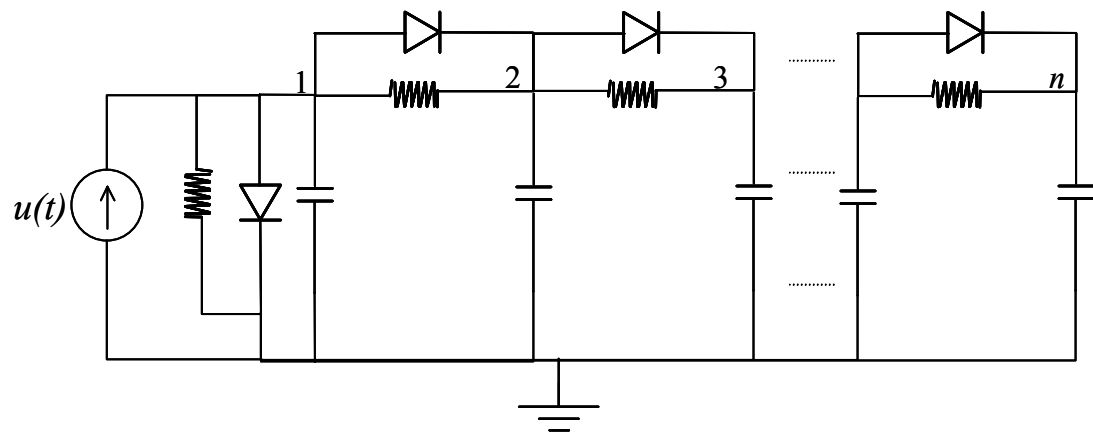


Fig 1.

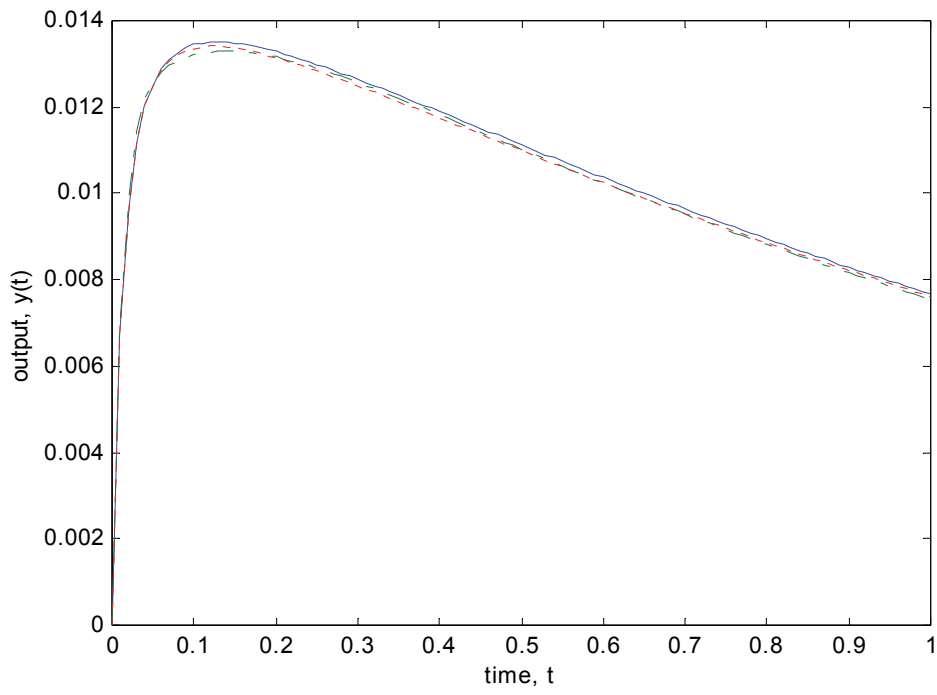


Fig. 2A

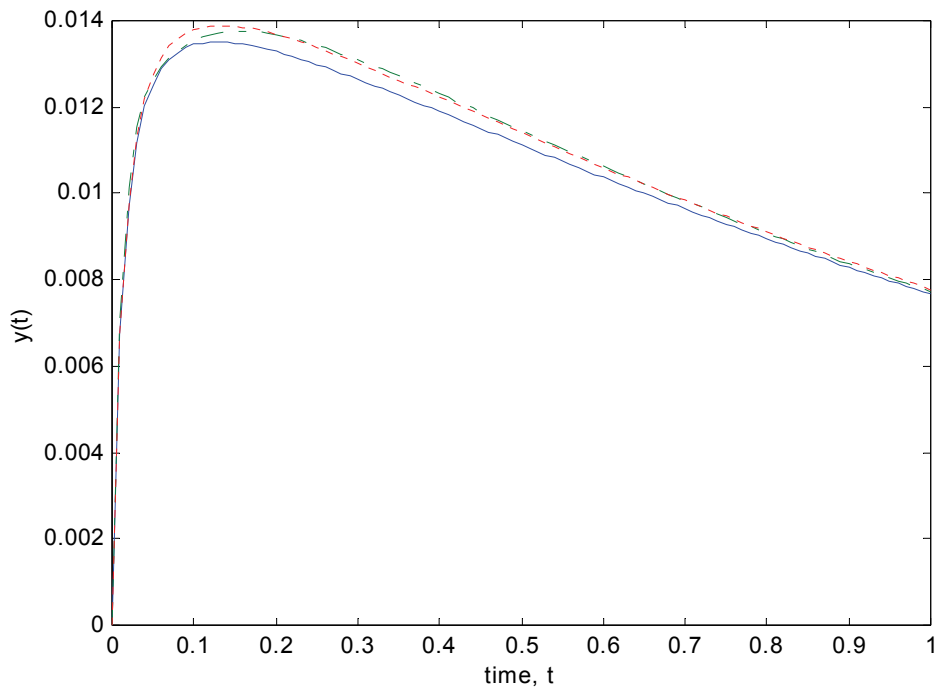


Fig 3A

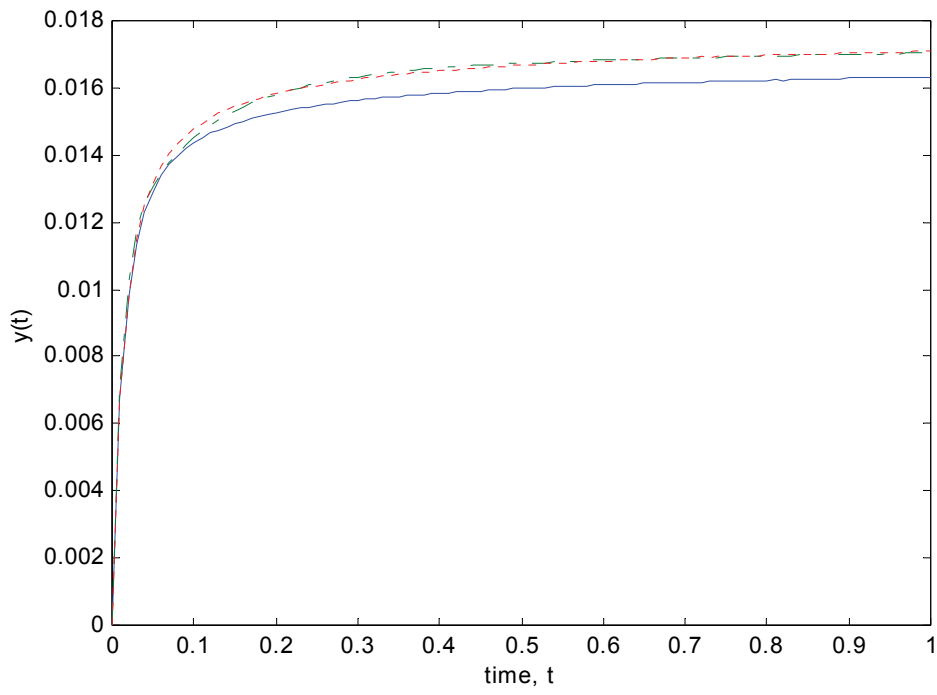


Fig 3B

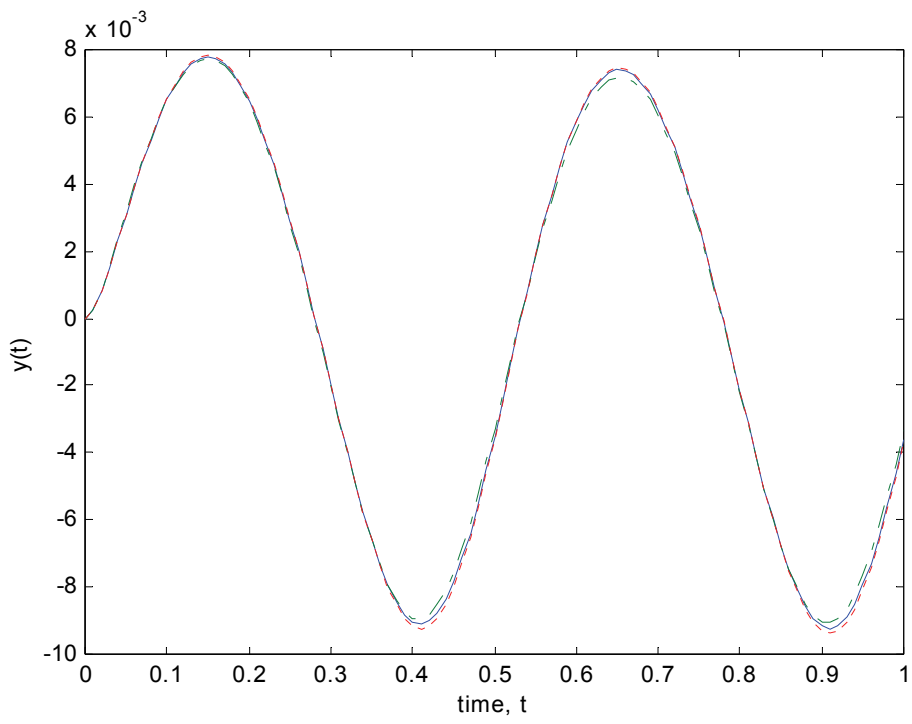


Fig 3C

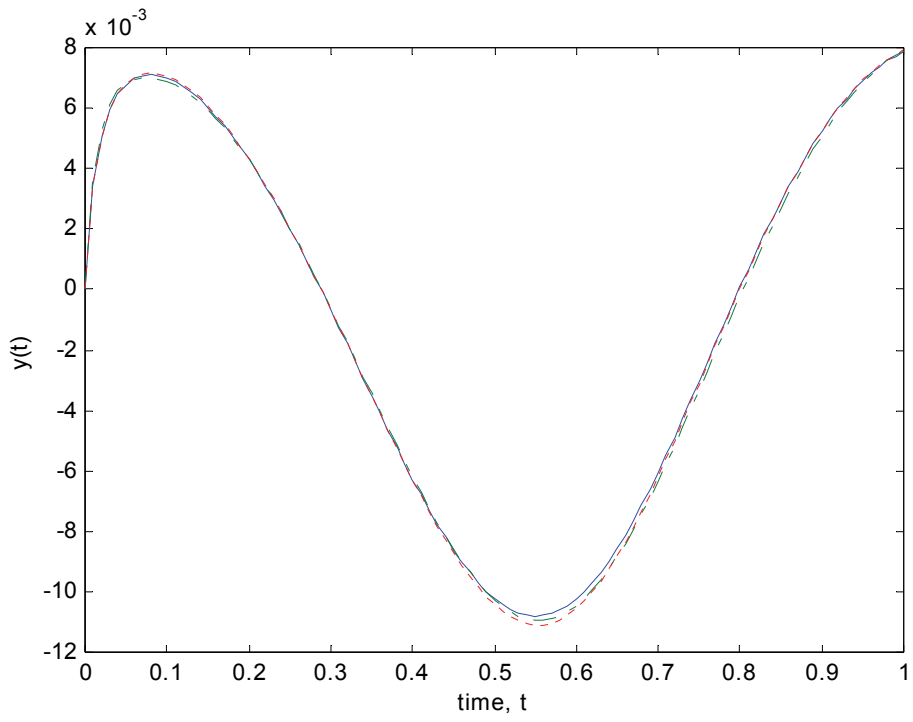


Fig. 3D