Combinatorics, Geometry and Homology of Non-Crossing Partition Lattices for Finite Reflection Groups.

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Declaration

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Contents

Abstract

For W a finite, real, reflection group we can define a set of simple reflections. Each element $w \in W$ is characterised by its inversion set which is directly associated with the length function corresponding to the simple reflections. We can define a different length function in terms of **all** reflections of a finite reflection group. This leads to the definition of the partially ordered set of non-crossing partitions. The non-crossing partition lattice NCP_c is defined to be the set of elements of W that precede a fixed Coxeter element c. In [12] a geometric model $X(c)$ for NCP_c is constructed and extended to give the type-W associahedron, $\mu(AX(c))$, while in [9], $\mu(AX(c))$ is related to the type-W permutahedron. In [19], Reading also relates the type-W associahedron and permutahedron using the notion of Coxeter-sortable and reverse Coxeter-sortable elements in W.

In [9], the complex $\mu(AX(c))$ is defined by applying the linear operator $\mu = 2(I - c)^{-1}$ to $AX(c)$. In this thesis we first give an alternative elementary construction of $\mu(AX(c))$ using intersections of halfspaces. Next we show that the equivalence classes arising from the partition of W defined by $\mu(AX(c))$ are intervals in the weak order thus giving a new characterisation of Coxeter-sortable and reverse Coxeter-sortable elements. We also describe an algorithm for recursively computing the cardinality of these intervals. In [9], a bijection is constructed from elements of NCP_c to the facets of $\mu(AX(c))$. We calculate the size of the interval by reflecting these facets and expressing the result as a union of other facets. Finally, we construct a geometric basis for the homology of NCP_c by defining a homotopy equivalence between the proper part of the non-crossing partition lattice and the $(n-2)$-skeleton of $X(c)$. Using a general construction of a generic affine hyperplane for the central hyperplane arrangement defined by W, we relate this to the basis introduced by Bjorner and Wachs in [6] for the homology of the corresponding intersection lattice.

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Chapter 1

Introduction

For W a finite reflection group we can define a set of simple reflections. Each element $w \in W$ is characterised by its inversion set which is directly associated with the length function corresponding to the simple reflections. We can define a different length function in terms of **all** reflections of a finite reflection group. This leads to the definition of the partially ordered set of non-crossing partitions. A geometric model of this partially ordered set is given in [12] and is related to the generalised associahedron. Furthermore, the generalised associahedron is related to the corresponding permutahedron in [9].

This thesis involves aspects of the combinatorics, geometry and homology of non-crossing partition lattices for finite reflection groups. After reviewing background material, we give a specific construction of the complex $\mu(AX(c))$, a realisation of the generalised associahedron, using the intersections of halfspaces. The construction is modelled on the construction in [12].

We also give a new characterisation of Coxeter sortable and reverse Coxeter sortable elements which are defined by Reading in [20]. We use the identification of the associahedron, $\mu(AX(c))$, with the Cambrian fan to define an interval from a Coxeter sortable element to a reverse Coxeter sortable element.

In chapter 5, we describe an algorithm for recursively computing the size of the interval described above. The idea is to reflect the corresponding associahedron facet and express it as a union of other facets.

Finally, we define a geometric basis for the homology of the non-crossing partition lattices. We also construct an explicit embedding of this homology into the homology of the corresponding intersection lattice.

The layout of the thesis is as follows. We begin by reviewing background material about finite reflections groups and Coxeter groups using [7] and [15] as references. We then review details and results about posets and non-crossing partitions from [5] and [18]. We require descriptions and facts about the simplicial complexes $X(c)$, $AX(c)$, and $\mu(AX(c))$ from [12] and [9]. In chapter 3, we give the construction of the complex $\mu(AX(c))$. Chaper 4 deals with the interval. Chapter 5 describes the algorithm and in chapter 6 we construct the basis for the homology of the non-crossing partition lattices and the embedding described above.

Chapter 2

Preliminaries

2.1 Finite Reflection Groups

We begin in this chapter by introducing some background material. Firstly, we will review definitions and results about finite reflection groups followed by details about Coxeter groups.

2.1.1 Root Systems

Let V be a real n-dimensional euclidean space with inner product $(\cdot,\cdot)$ which in later sections we will write as the dot product. A *hyperplane* H in V is defined for a non-zero vector $\alpha \in V$ as $H = \{x \in V \mid (x,\alpha) = c\}$ for some constant c. In the special case where $c = 0$, H is an $(n-1)$-dimensional subspace of V and is called a *linear* hyperplane. We will refer to a hyperplane as *affine* if it does not pass through the origin ($c \neq 0$). The reflection in the linear hyperplane H is the linear transformation r_α on V which sends α to its negative while fixing H pointwise. The formula for reflecting a general vector $\beta \in V$ in H is

$$r_\alpha(\beta) = \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha.$$

From this formula we can see that $r_\alpha(\alpha) = -\alpha$ and for $\beta \in H$, $r_\alpha(\beta) = \beta$. The vector α is called a normal to the hyperplane H. A *finite reflection group* W is a finite group generated by reflections.

For Φ a finite set of non-zero vectors in V, we say that Φ is a *root system* for the finite reflection group W if the following two conditions hold

$$\Phi \cap \mathbf{R}\alpha = \{\alpha, -\alpha\} \text{ for all } \alpha \in \Phi,$$

$$r_\alpha(\Phi) = \Phi \text{ for all } \alpha \in \Phi.$$

A finite hyperplane arrangement $\mathcal{A}$ is a finite set of hyperplanes in V. The arrangement $\mathcal{A}$ is called a central hyperplane arrangement if all the hyperplanes in the arrangement pass through the origin. Otherwise, it is an affine arrangement. A central hyperplane arrangement $\mathcal{A}$ is essential if $\bigcap_{H\in\mathcal{A}} H = \{0\}$. We refer to the connected components of $V \setminus \mathcal{A}$ as regions. If $\mathcal{A}$ is the central hyperplane arrangement for a finite group W, we fix a region C of the hyperplane arrangement called the fundamental chamber and refer to the set of inward unit normals of C as the set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. C is always a cone on a spherical simplex by theorem in section 1.12 of [15]. We say that n is the *rank* of the finite reflection group W. If $\{\beta_1, \ldots, \beta_n\}$ is the dual basis to $\{\alpha_1, \ldots, \alpha_n\}$, then C is the positive cone on the set $\{\beta_1, \ldots, \beta_n\}$. A positive root is a linear combination of the elements of Π with non-negative coefficients. Likewise, a negative root is a linear combination of the elements of Π with non-positive coefficients. We denote the set of positive roots by Φ^+ and the set of negative roots by Φ^-. The set of roots Φ is the disjoint union of by Φ^+ and Φ^- (theorem 3 of section 1.6, chapter VI of [7]).

We will now describe in detail two particular examples of root systems. We will refer to these throughout the thesis.

Example 2.1.1. The root system of type A_n is the set of $n(n+1)$ vectors $e_i - e_j$ in $\mathbf{R}^{n+1}$ with $i \neq j$. The simple roots are $\alpha_i = e_{i+1} - e_i$ for $i = 1, \ldots, n$ and the positive roots are $e_i - e_j$ for $1 \leq j < i \leq n+1$. The corresponding finite reflection group $W(A_n)$ acts on $\mathbf{R}^{n+1}$ by permuting the standard basis $\{e_1, e_2, \ldots, e_{n+1}\}$. Hence $W(A_n)$ can be identified with the group of $(n+1) \times (n+1)$ permutation matrices and with the symmetry group of the $(n+1)$-simplex.

Figure 2.1.1 shows the root system A_2 which consists of the set of vectors $e_i - e_j \in \mathbf{R}^3$ with $i \neq j$.

Example 2.1.2. The root system of type C_n is the set of $2n^2$ vectors in $\mathbf{R}^n$ which are either of the form $\pm e_i$ or $\pm e_i \pm e_j$ for $i \neq j$. The vectors e_1 and $e_{i+1} - e_i$ for $1 \leq i \leq n-1$

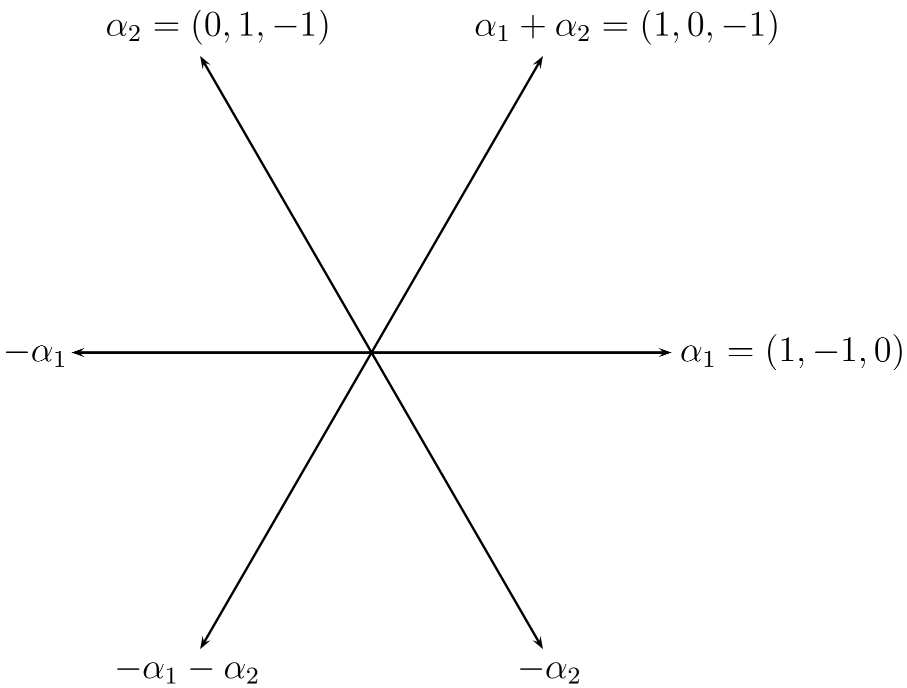


Figure 2.1.1: Root System

form the set of simple roots. The positive roots are e_i for $1 \leq i \leq n$ and $e_i \pm e_j$ for $1 \leq j < i \leq n$. The group $W(C_n)$ acts on the set $\{e_1, \ldots, e_n, -e_1, \ldots, -e_n\}$. $W(C_n)$ can be identified with the group of signed permutation matrices and with the symmetry group of the n-dimensional cube.

The finite reflection groups are completely classified [15]. The other finite reflection groups are the dihedral groups in dimension 2, the family of groups $W(D_n)$, a special index 2 subgroup of $W(C_n)$ and the sporadic examples corresponding to the root systems H_3, H_4, F_4, E_6, E_7 and E_8.

2.1.2 Coxeter Groups

A *Coxeter system* is a pair (W, S) with W a group and a subset $S \subset W$ a generating set of elements. The elements of the subset S are subject to relations of the form

$$(rr')^{m(r,r')} = 1$$

where $m(r', r)$ are positive integers with $m(r, r) = 1$ and $m(r, r') = m(r', r) \geq 2$ for $r \neq r'$. We will refer to such a group W as a Coxeter group. Every finite reflection group is a finite Coxeter group (theorem in section 1.9 of [15]) with the set S as the set of simple reflections. If Π is the set of simple roots in a root system Φ, then each $\alpha_i \in \Pi$ has an associated simple reflection r_i. The product of the n elements of S in any order is a *Coxeter element* c of W. All such elements c have the same order, which is called the *Coxeter number* of W and is denoted by h. We let T denote the set of *all reflections* in W. It will be important to note that the set of reflections is closed under conjugation. This is because

$$r_\beta r_\alpha (r_\beta)^{-1} = r_\beta r_\alpha r_\beta = r_\gamma$$

where $\gamma = r_\beta(\alpha)$. A Coxeter diagram is a graph with vertex set S and an edge from s_i to s_j if $m(s_i, s_j) > 2$. Since the Coxeter diagram for any finite Coxeter group is a tree (Theorem 1 of Section 4.2, Chapter VI of [7]), it is possible to partition the simple roots into two orthnormal sets $S_1 = \{\alpha_1, \dots, \alpha_s\}$ and $S_2 = \{\alpha_{s+1}, \dots, \alpha_n\}$. This induces a partition on the set of simple reflections into two sets whose elements commute with each other. The resulting Coxeter element $c = r_1 \dots r_s r_{s+1} \dots r_n$ is called a *bipartite* Coxeter element.

The hyperplane arrangement for a finite reflection group is called a Coxeter arrangement. Recall that we fix a fundamental chamber C with inward normals $\alpha_1, \dots, \alpha_n$. The regions of the Coxeter arrangement are in one-to-one correspondance with the elements of W because the Coxeter group is generated by the fundamental reflections. C corresponds to the identity element. The intersection of the arrangement with the unit sphere is the *simplicial permutahedron* of type W.

For each $w \in W$, let $l(w)$ denote the least number k such that w can be written as a word of length k in the alphabet S. We define the *weak order* for elements $u, w \in W$ by

$$u \leq_R w \Leftrightarrow l(w) = l(u) + l(u^{-1}w).$$

Associated with each w is an *inversion set* $\mathrm{Inv}(w)$ where $\mathrm{Inv}(w) = \{t \in T \mid l(tw) <_R l(w)\}$.

It is proved in proposition 1.2 of [13] that $u \leq_R w$ if and only if $\mathrm{Inv}(u) \subset \mathrm{Inv}(w)$. An element of W is uniquely determined by its inversion set. We say the set $\mathcal{I} \subset \Phi^+$ is *closed* if, for each pair ρ_1 and ρ_2 of positive roots in $\mathcal{I}$ with the property that $\rho_1 + \rho_2$ is a root, then $\rho_1 + \rho_2$ is also an element of $\mathcal{I}$. If a set $\mathcal{I}$ is closed and its complement in Φ^+ is closed, then $\mathcal{I}$ is an inversion set of a unique element $w \in W$ [17].

We now give two examples of Coxeter groups which we will refer to throughout the thesis and which are related to the root systems in examples 2.1.1 and 2.1.2 respectively.

Example 2.1.3. Recall that $W(A_n)$ can be identified with the symmetry group of the $(n+1)$-simplex and hence with the symmetric group S_{n+1} of permutations of the set $[n+1] := \{1, \ldots, n+1\}$. The underlying vector space, on which W acts, is the hyperplane in $\mathbf{R}^{n+1}$ given by $x_1 + x_2 + \cdots + x_{n+1} = 0$. The cycle $(i_1, \ldots, i_k)$ will denote the permutation sending e_{i_1} to e_{i_2}, e_{i_2} to e_{i_3}, ..., $e_{i_{k-1}}$ to e_{i_k} and e_{i_k} to e_{i_1}. In particular the reflection (i, j) will denote the transposition interchanging e_i and e_j. The simple reflections are given by $(i, i+1)$ for $1 \leq i \leq n$. For convenience of notation, we will omit the commas in cycle notation for small values of n where no confusion can arise.

In the particular case where $n = 3$, a set of simple roots is given by

$$\alpha_1 = 1/\sqrt{2}(1, -1, 0, 0), \;\; \alpha_2 = 1/\sqrt{2}(0, 0, 1, -1), \;\; \alpha_3 = 1/\sqrt{2}(0, 1, -1, 0).$$

The simple reflections are $(12), (23)$ and (34). Then the Coxeter element $c = (12)(34)(23) = (1243)$ so that $c(x, y, z, w) = (z, x, w, y)$. This Coxeter element c is bipartite with $S_1 = \{(12), (34)\}$ and $S_2 = \{(23)\}$.

Example 2.1.4. Recall that $W(C_n)$ is the symmetry group of the n-cube. We use a variation of the notation used in [10] for elements of $W(C_n)$. We will denote reflection in the hyperplane $x_i = 0$ by $[i]$. For instance, the reflection $[1]$ in C_3 is such that $[1](x, y, z) = (-x, y, z)$. For $i \neq j$ reflection in the hyperplane $x_i = x_j$ will be denoted by one of the expressions $(i,j),(j,i),(\bar{i},\bar{j})$ or $(\bar{j},\bar{i})$. Similarly we will denote reflection in the hyperplane $x_i = -x_j$ by one of the expressions $(i,\bar{j}),(\bar{i},j),(j,\bar{i})$ or $(\bar{j},i)$. If the cycle $c = (i_1,\dots,i_k)$ is disjoint from the cycle $\bar{c} = (-i_1,\dots,-i_k)$ then we write $(i_1,\dots,i_k)$ to refer to the cycle $(i_1,\dots,i_k)(-i_1,\dots,-i_k)$. If $c = \bar{c} = (i_1,\dots,i_k,-i_1,\dots,-i_k)$ then we write $c = [i_1,\dots,i_k]$.

For $n = 3$ a set of simple roots is given by

$$\alpha_1 = (1,0,0),\ \ \alpha_2 = (1/\sqrt{2})(0,-1,1),\ \ \alpha_3 = (1/\sqrt{2})(-1,1,0).$$

The corresponding simple reflections are $[1],(23),(12)$. The set $T = \{[1],(12),(23),[2],[3],(13),(1\bar{2}),(1\bar{3}),(2\bar{3})\}$. We fix $c = [1](23)(12) = [132]$. Note that c is bipartite.

2.2 The Non-Crossing Partition Lattice

There is a lattice associated with the finite reflection group W called the non-crossing partition lattice. This lattice was first introduced by Kreweras for $W = A_n$ [16]. The set of non-crossing partitions for general W was proved to be a lattice in [4] and [12]. For any finite reflection group W, we refer to the lattice as the type W non-crossing partition lattice. In this section, we first review the definitions of posets and lattices and then introduce the set of non-crossing partitions.

2.2.1 Posets

Definition 2.2.1. A relation $\preceq$ on a set P is a partial order if it is reflexive, anti-symmetric and transitive. The order $\preceq$ is a total order if for any $x,y \in P$ either $x \preceq y$ or $y \preceq x$. A set which has a partial order defined on it is called a partially ordered set or poset.

Example 2.2.2. Figure 2.2.1 shows an example of a poset whose underlying set is the set of subsets of $\{x,y,z\}$. The subsets are ordered by containment.

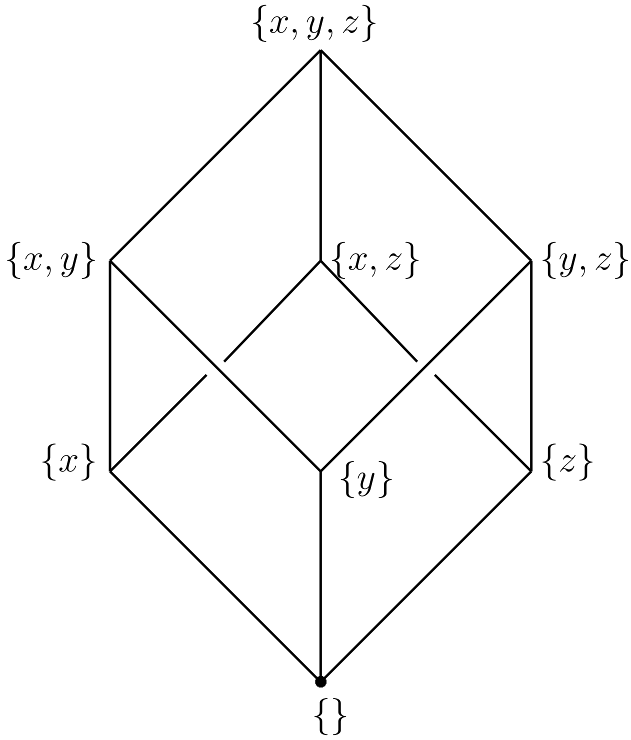


Figure 2.2.1: Poset of subsets of $\{x, y, z\}$

The poset P is said to be bounded if it has a top element usually denoted $\hat{1}$ (meaning $x \preceq \hat{1}$ for all $x \in P$) and a bottom element usually denoted $\hat{0}$ (meaning $\hat{0} \preceq x$ for all $x \in P$). The *proper part* of a bounded poset P is denoted by $\bar{P}$ and defined to be $\bar{P} = P \setminus \{\hat{0}, \hat{1}\}$.

For a poset P, let $|P|$ denote the *simplicial complex* associated to it. $|P|$ is a simplicial complex whose vertices are the elements of the poset P and whose simplices are the non-empty finite chains in P. We say that the poset P is contractible if the simplicial complex $|P|$ is contractible.

For Δ a simplicial complex, let $P(\Delta)$ denote the *poset of simplices* in Δ ordered by inclusion. The *barycentric subdivision* of the simplicial complex Δ is denoted $sd(\Delta)$, where $sd(\Delta)$ is the simplicial complex $|P(\Delta)|$. The simplicial complexes Δ and $sd(\Delta)$ are homeomorphic. An example of a simplicial complex, the poset associated to it and the barycentric subdivision is shown in figure 2.2.2.

The simplicial complex of a bounded poset P is the cone (from $\hat{1}$) on the cone (from $\hat{0}$)

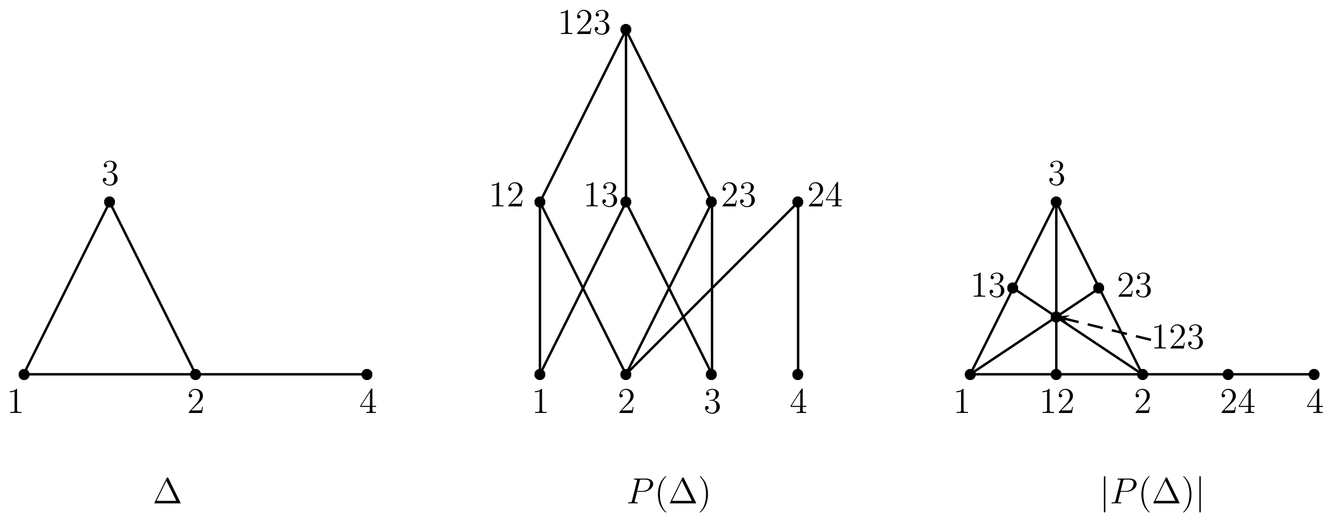


Figure 2.2.2: Barycentric subdivision of a simplicial complex

on the proper part $\bar{P}$. Usually the proper part is topologically more interesting as we will see in chapter 6.

Definition 2.2.3. Consider any partially ordered set $(P, \preceq)$. An element y of the poset $\mathcal{P}$ is an upper bound of $x_1, x_2 \in \mathcal{P}$ if $x_1 \preceq y$ and $x_2 \preceq y$. The element y is a least upper bound or lub of x_1, x_2 if it is an upper bound such that $y \preceq z$ for any upper bound z of x_1, x_2. Analogously, $y \in \mathcal{P}$ is called a lower bound of x_1, x_2 if $x_1 \succeq y$ and $x_2 \succeq y$. We say y is a greatest lower bound or glb if it is a lower bound such that $y \succeq z$ for any lower bound z of x_1, x_2. $\mathcal{P}$ is called a *lattice* if the glb and lub exist for all pairs of elements in $\mathcal{P}$. We can define binary operations on $\mathcal{P}$ where $\vee$ is called the join operation and $\wedge$ is called the meet operation such that:

- $x \vee y = lub(x, y)$
- $x \wedge y = glb(x, y)$

Example 2.2.4. The intersection lattice $L(\mathcal{A})$ for a central hyperplane arrangement $\mathcal{A}$ is an example of a lattice and one to which we will refer later. $L(\mathcal{A})$ is the lattice of non-empty intersections of $\mathcal{A}$, ordered by reverse inclusion. The minimal element $\hat{0}$ is V.

The intersection lattice for the hyperplane arrangement in example 2.1.1 is a lattice (as is shown in figure 2.2.3) because the arrangement is central.

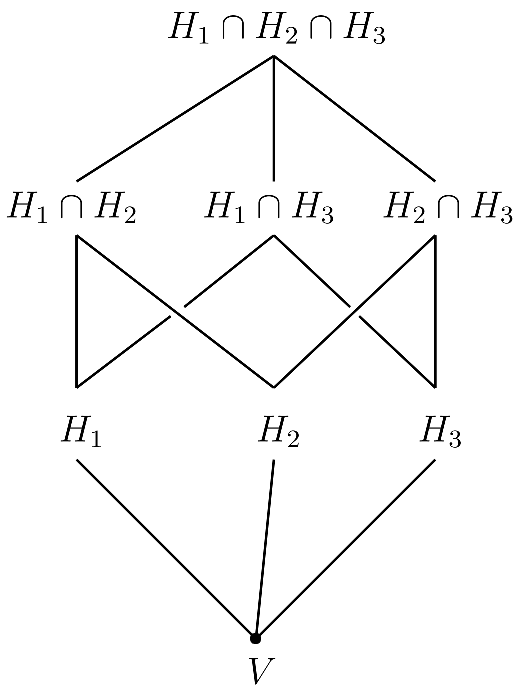


Figure 2.2.3: $L(\mathcal{A})$ for a central hyperplane arrangement $\mathcal{A}$

If $\mathcal{A}$ is not a central hyperplane arrangement, $L(\mathcal{A})$ is not a lattice. An example is shown in figure 2.2.4. The least upper bound does not exist for all pairs of elements, so $L(\mathcal{A})$ is not a lattice.

2.2.2 Non-Crossing Partitions

In order to introduce the concept of a non-crossing partition, we must first define a second length function on W. For $w \in W$, let $\ell(w)$ denote the smallest k such that w can be written as a product of k reflections from T. We warn the reader that this length function is distinct from the one defined earlier in terms of the simple reflections. Define the partial order $\preceq$ on W by declaring for $u, w \in W$:

$$u \preceq w \quad \Leftrightarrow \quad \ell(w) = \ell(u) + \ell(u^{-1}w). \tag{2.2.1}$$

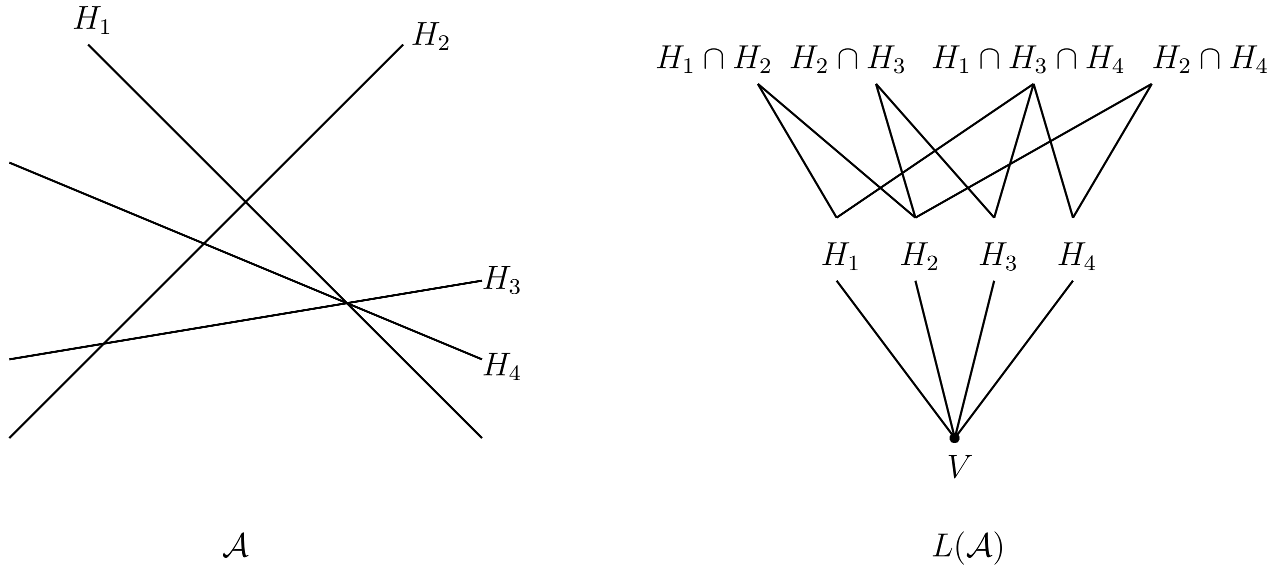


Figure 2.2.4: $L(\mathcal{A})$ for a hyperplane arrangement $\mathcal{A}$ that is not central

We will refer to this partial order as the total reflection length order.

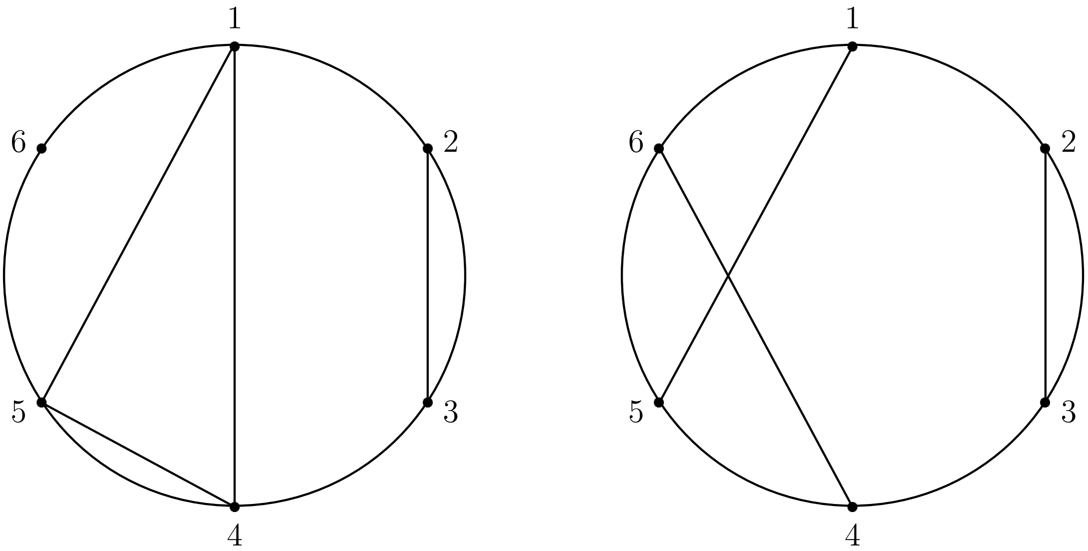
Example 2.2.5. Let $W = W(C_6)$ we let $c = [123456]$ and consider the element $u = [1456]$. We can write $[1456] = [1](14)(45)(56)$ so therefore $\ell(u) = 4$. The element $v = (14\bar{5}\bar{6})$ can be written as a product of elements in T as follows $(14\bar{5}\bar{6}) = (14)(4\bar{5})(5\bar{6})$ so therefore $\ell(v) = 3$. Also $v^{-1}u = (1\bar{6}\bar{5}4)[1456] = (1)[4](5)(6)$ which has length 1, so therefore $v \preceq u$.

Definition 2.2.6. The set NCP_c of non-crossing partitions is the set of elements of W that precede c in the total reflection length order (2.2.1).

The set NCP_c is a poset with respect to this partial order. In fact, (by [12] for example), NCP_c forms a lattice and is called the non-crossing partition lattice. The lattice NCP_c is the interval $[1, c]$ in the partial order (2.2.1).

The sets NCP_c for the particular cases $W(A_n)$ and $W(C_n)$ and for appropriate choice of c have simple descriptions which we now give. A partition of a set S is a disjoint set of subsets whose union is all of S. These subsets are usually called blocks. A non-crossing partition is a partition of the vertices of a regular $(n+1)$-gon labelled by the set $[n+1]$ so that the convex hulls of its blocks are pairwise disjoint. Equivalently, a non-crossing partition of the set $[n+1]$ is a partition such that for $a < b < c < d$, if a and c belong to a block B_1 and b and d belong to a block B_2 then $B_1 = B_2$. It is shown in [8] that the set of permutations σ satisfying $\sigma \preceq (1, 2, \ldots, n+1) = c$ are precisely the permutations whose orbits are non-crossing partitions and whose cycles are oriented in the obvious way. Thus, the type $W(A_n)$ non-crossing partitions coincide with the classical non-crossing partitions. The picture on the left of figure 2.2.5 in the example below illustrates such a partition.

Example 2.2.7. In figure 2.2.5 we show an example of a non-crossing partition and a crossing partition of the set $\{1, \ldots, 6\}$.

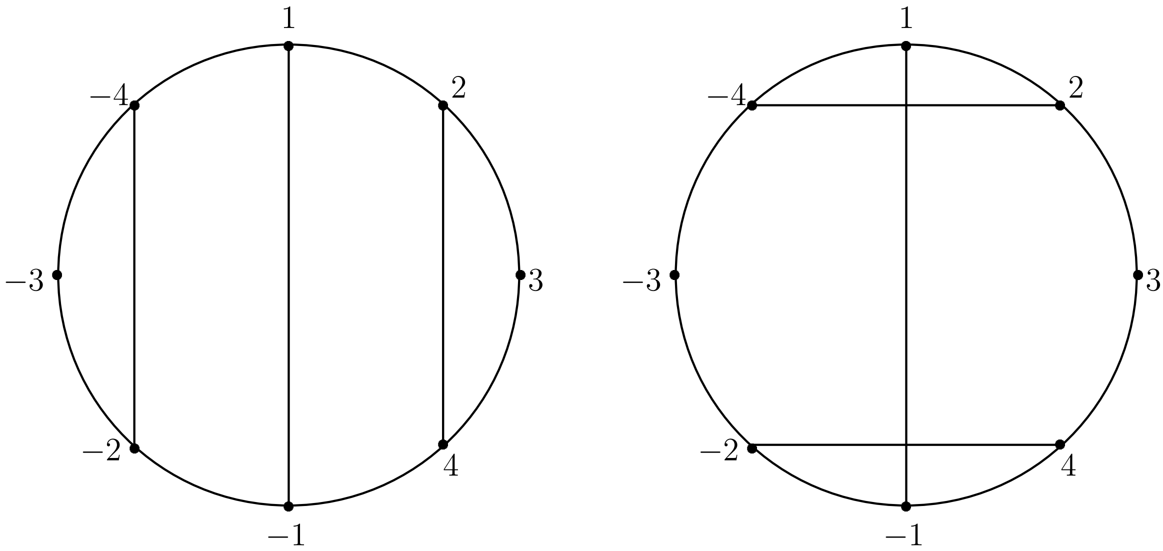


$(23)(145) \preceq (123456)$ $\qquad$ $(15)(23)(46) \npreceq (123456)$

Figure 2.2.5: A non-crossing partition and a crossing partition of the set $\{1, \ldots, 6\}$

For the case $W(C_n)$, we can arrange the numbers $1, \ldots, n, -1, \ldots, -n$ around a $2n$-gon and, as in the $W(A_n)$ case, form the convex hulls of the blocks. A partition of the set $\{1, \ldots, n, -1, \ldots, -n\}$ is non-crossing if it is symmetric with respect to a half-turn and the blocks do not cross, as is illustrated in figure 2.2.6.

The fact that these non-crossing partitions correspond to the non-crossing partition of type $W(C_n)$ is shown in [10] and [4]. We give a formula for the number of non-crossing partitions for groups of type A_n and C_n.

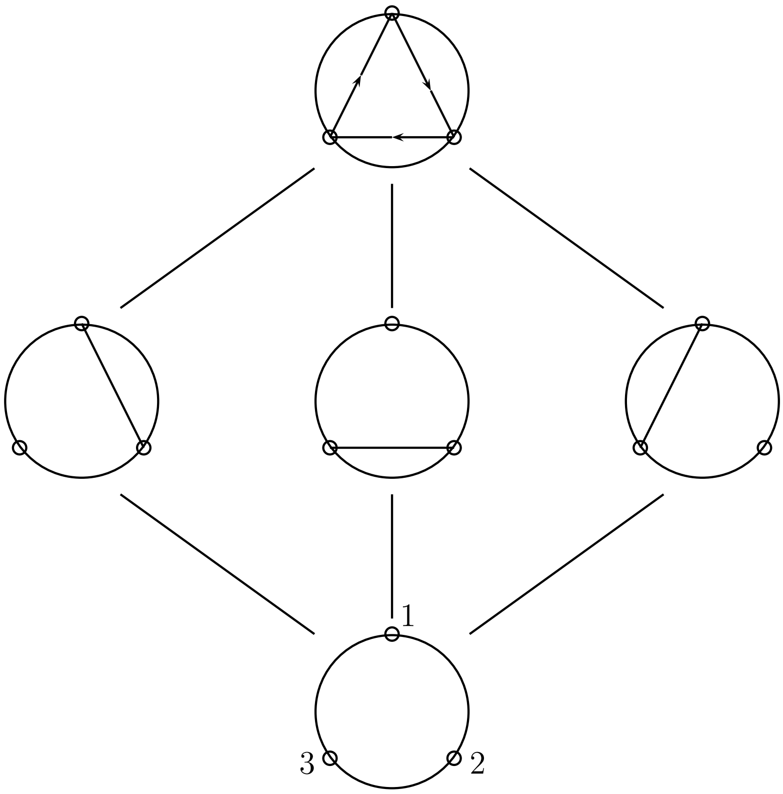


$[1](24) \preceq [1234]$ $\quad$ $[1](2\bar{4}) \not\preceq [1234]$

Figure 2.2.6: A non-crossing partition and a crossing partition for $W(C_4)$

A_n	$\frac{1}{n+2}\binom{2n+2}{n+1}$
C_n	$\binom{2n}{n}$

Example 2.2.8. The non-crossing partition lattice for $W(A_2)$ is drawn below. In this case, the non-crossing partition lattice is equivalent to the intersection lattice. However, this is not the case for $W(A_3)$. For $c = (1234)$, the reflection $(13)(24)$ is crossing and is therefore not in the non-crossing partition lattice, although the corresponding subspace lies in the intersection lattice.



2.3 $X(c), AX(c)$ and the Operator μ

In this section, we review results about the simplicial complex $X(c)$ which is defined in [12]. The complex $X(c)$ is a geometric model for NCP_c. It is extended to the complex $AX(c)$ in [9]. A linear operator $\mu = 2(I-c)^{-1}$ is defined in [12] and applied to $AX(c)$ to define the simplicial complex denoted by $\mu(AX(c))$. We review these complexes and their properties here.

2.3.1 The simplicial complexes $X(c)$, $EX(c)$ and $AX(c)$

We fix the fundamental chamber C with inward unit normals $\alpha_1, \ldots, \alpha_n$ and let $r_1, \ldots, r_n$ be the corresponding reflections. We fix the bipartite Coxeter element of W given by $c = r_1 r_2 \ldots r_n$. As in [12] we define a total order on roots by $\rho_i = r_1 \ldots r_{i-1}\alpha_i$ where the α's and r's are defined cyclically mod n. We will say that $\rho_i < \rho_j$ if $i < j$ when referring to this order. Using the fact that c is bipartite, Steinberg shows in [22] that the positive roots relative to the fundamental chamber are $\{\rho_1, \rho_2, \ldots, \rho_{nh/2}\}$ where h is the order of the Coxeter element in W. Let T denote the reflection set of W. This consists of the set of reflections $r(\rho_i)$ where ρ_i is a positive root and $r(\rho_i)$ is the reflection in the hyperplane $\rho_i^\perp$ orthogonal to ρ_i.

In [11] there are particular subspaces of $\mathbf{R}^n$ whose definitions and properties we will require. If $O(n)$ denotes the orthogonal group, then for $A \in O(n)$ we associate the subspaces $M(A)$ and $F(A)$ to A where $M(A) = im(A - I)$ and $F(A) = ker(A - I)$. We refer to $M(A)$ as the moved space of A and to $F(A)$ as the fixed space of A. In [11] it is shown that $M(A) = F(A)^\perp$.

Notation 2.3.1. Throughout the thesis, we will denote by $\rho(r)$ the positive root associated to the reflection r.

We will have occasion to define spherical simplices and cones using non-zero vectors. We note that any set $S = \{v_1, \dots, v_k\}$ of linearly independant vectors determines a positive cone, denoted cone(S), by

$$\text{cone}(S) = \{x \in \mathbf{R}^n \mid x = \sum a_i v_i, a_i \geq 0\}$$

and that this intersects the unit sphere in a spherical simplex. In the case $k = 1$, the cone is called a *ray* and the simplex is just a single vertex.

The simplicial complex $X(c)$ has the set of positive roots $\{\rho_1, \rho_2, \dots, \rho_{nh/2}\}$ as vertex set. An *edge* joins ρ_i to ρ_j if $i < j$ and $r(\rho_i)r(\rho_j) \preceq c^{-1}$. The vertices $\{\rho_{i_1}, \dots, \rho_{i_k}\}$ form a $(k-1)$-*simplex* if they are pairwise joined by edges. It is shown in [12] that the set $\{\rho_{i_1}, \dots, \rho_{i_k}\}$ forms a $(k-1)$-simplex if and only if

$$\rho_1 \leq \rho_{i_1} < \rho_{i_2} < \cdots < \rho_{i_k} \leq \rho_{nh/2} \quad \text{and} \quad \ell[r(\rho_{i_1}) \dots r(\rho_{i_k})c] = n - k.$$

For each $w \preceq c$ there is a subcomplex $X(w)$ of $X(c)$ associated to it. Let $\Gamma_w = \{\tau_1, \tau_2, \dots, \tau_t\}$ be the set of positive roots whose reflections precede w in the partial order. This set is $\Phi^+ \cap M(w)$. The subcomplex $X(w)$ is the collection of simplices of $X(c)$ whose vertices lie in Γ_w. It is shown in [12] that $X(w)$ is a simplex on the simple system determined by Γ_w.

Example 2.3.2. Continuing with example 2.1.3 from subsection 2.1.2, let $W = W(A_3)$ and fix $c = (1243)$. Recall that the set of simple roots $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ where

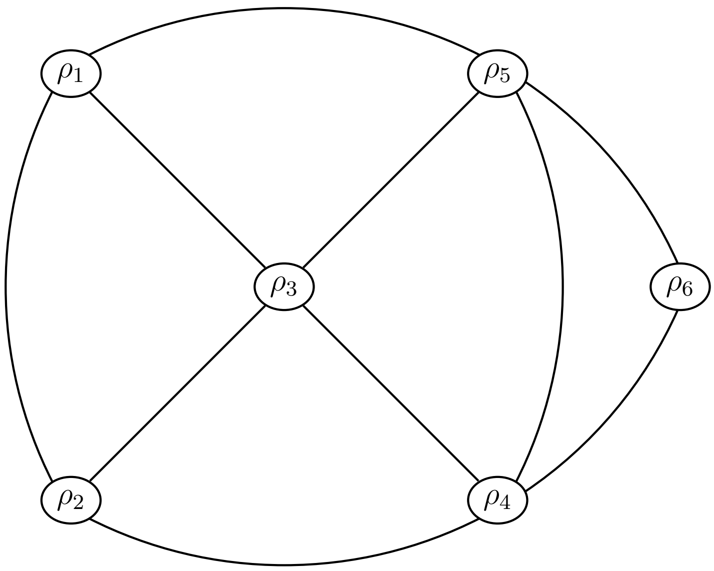
$$\alpha_1 = 1/\sqrt{2}(1,-1,0,0), \alpha_2 = 1/\sqrt{2}(0,0,1,-1), \alpha_3 = 1/\sqrt{2}(0,1,-1,0).$$

Then we can calculate the set of positive roots. There are $nh/2 = 6$ such roots.

$$\rho_1 = 1/\sqrt{2}(1,-1,0,0), \;\; \rho_2 = 1/\sqrt{2}(0,0,1,-1), \;\; \rho_3 = 1/\sqrt{2}(1,0,0,-1)$$

$$\rho_4 = 1/\sqrt{2}(0,1,0,-1), \;\; \rho_5 = 1/\sqrt{2}(1,0,-1,0), \;\; \rho_6 = 1/\sqrt{2}(0,1,-1,0)$$

Then $X(c)$ has vertices $\rho_1, \ldots, \rho_6$. It is illustrated below.



Example 2.3.3. As in example 2.1.4, let $W = W(C_3), \alpha_1 = (1,0,0), \alpha_2 = 1/\sqrt{2}(0,-1,1)$, $\alpha_3 = 1/\sqrt{2}(-1,1,0)$. The total order on roots induces an order on the set of reflections where $r(\rho_i) \leq r(\rho_j)$ if $\rho_i \leq \rho_j$. In this case the reflection order is

$$[1], (23), (1\bar{3}), [3], (1\bar{2}), (2\bar{3}), [2], (13), (12).$$

For $w = (12\bar{3})$, we can define the subcomplex $X(w)$ of $X(c)$ to be the collection of simplices with vertex set $\Gamma_w = \{\rho(1\bar{3}), \rho(2\bar{3}), \rho(12)\}$.

In [12], another larger simplicial complex $EX(c)$ is defined. It is an extension of the complex $X(c)$ and it is proved in theorem 8.2 of [12] that $EX(c)$ is a simplicial complex and $|EX(c)|$ is a sphere of dimension $n-1$. The vertex set of $EX(c)$ consists of the set of positive roots and the negatives of the simple roots. In theorem 8.3 of [12], it is proven that $EX(c)$ coincides with the generalised associahedron defined in [14] for crystallographic W.

The complex $AX(c)$ is the image of $EX(c)$ under an isometry. It has vertex set $\rho_1, \dots, \rho_{nh/2}, \dots, \rho_{nh/2+n}$ and it has a simplex on $\{\tau_1, \dots, \tau_k\}$ if

$$\rho_1 \leq \tau_1 < \tau_2 < \cdots < \tau_k \leq \rho_{nh/2+n} \quad \text{and} \quad \ell[r(\tau_1)\dots r(\tau_k)c] = n-k.$$

Since $AX(c)$ is an isometric copy of $EX(c)$, the geometric realisation of $AX(c)$ is an $(n-1)$-dimensional sphere and combinatorially it also is equivalent to the type W generalised associahedron.

2.3.2 The complex $\mu(AX(c))$

Definition 2.3.4. We will denote the dual basis to $\{\alpha_1, \dots, \alpha_n\}$ by $\{\beta_1, \dots, \beta_n\}$. This means that

$$\beta_i \cdot \alpha_j = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

In [12] the vectors μ_i for $1 \leq i \leq nh$ are defined to be $\mu_i = r_1 \dots r_{i-1}\beta_i$ where the β's and r's are defined cyclically mod n. There is therefore an order on the set of μ vectors. From Corollary 4.2 of [12], we know that $\mu(\rho_i) = \mu_i$ for any i.

Notation 2.3.5. For convenience of notation, we will use $\mu(r)$ to denote the vector $\mu(\rho(r))$.

The numbers $\mu_i \cdot \rho_j$ have some properties which we will refer to and use throughout this document. They are listed here.

1. $\mu_i \cdot \rho_i = 1$

2. $\mu_i \cdot \rho_j = -\mu_{j+n} \cdot \rho_i$ for all i and j

3. $\mu_i \cdot \rho_j \geq 0$ for $1 \leq i \leq j \leq nh/2$

4. $\mu_{i+k} \cdot \rho_i = 0$ for $1 \leq k \leq n-1$ and for all i

5. $\mu_j \cdot \rho_i \leq 0$ for $1 \leq i < j \leq nh/2$

We can construct a matrix with $\mu_i \cdot \rho_j$ the entry in ith row and jth column. The matrix has 1's down the diagonal, 0's for $(n-1)$ entries below the diagonal, non-positive numbers below the diagonal and non-negative above. We refer the reader to the matrix composed for example 2.3.6. However at this point the reader can ignore the last four rows of the matrix.

In [9] the operator $\mu = 2(I-c)^{-1}$ is applied to the simplicial complex $AX(c)$ and the result is a simplicial complex denoted $\mu(AX(c))$. We will denote by $\mu(X(c))$ the positive part of $\mu(AX(c))$. The complex $\mu(AX(c))$ has vertex set $\mu(\rho_1), \ldots, \mu(\rho_{nh/2}), \ldots, \mu(\rho_{nh/2+n})$ and a simplex on $\{\mu(\tau_1), \ldots, \mu(\tau_k)\}$ if

$$\rho_1 \leq \tau_1 < \tau_2 < \cdots < \tau_k \leq \rho_{nh/2+n} \text{ and } \ell[r(\tau_1) \ldots r(\tau_k)c] = n-k.$$

The stereographic projection of the complex $\mu(AX(c))$ for $W = W(A_3)$ and $W = W(C_3)$ is illustrated in figures 2.3.2 and 2.3.3 respectively. The numbers i in the figure represent the vertex $\mu(\rho_i) = \mu_i$. The fundamental chamber C is the region with vertices $\mu_1, \ldots, \mu_n$.

We briefly describe the classical associahedron which corresponds to the A_n case. We consider a set of $n+3$ elements and the various ways to bracket the product of these elements. For example the 5 different ways to bracket the product $abcd$ are

$$((ab)c)d, (a(bc))d, a((bc)d), (ab)(cd) \text{ and } a(b(cd)).$$

We can construct a graph with complete bracketings as vertices. A cell of dimension k is obtained for each collection of complete bracketings that share a common non-trivial subbracketing of size $n-k$. This graph is the 1-skeleton of the convex polytope which is called the associahedron. It is illustrated in figure 2.3.4 for bracketings of 4 elements.

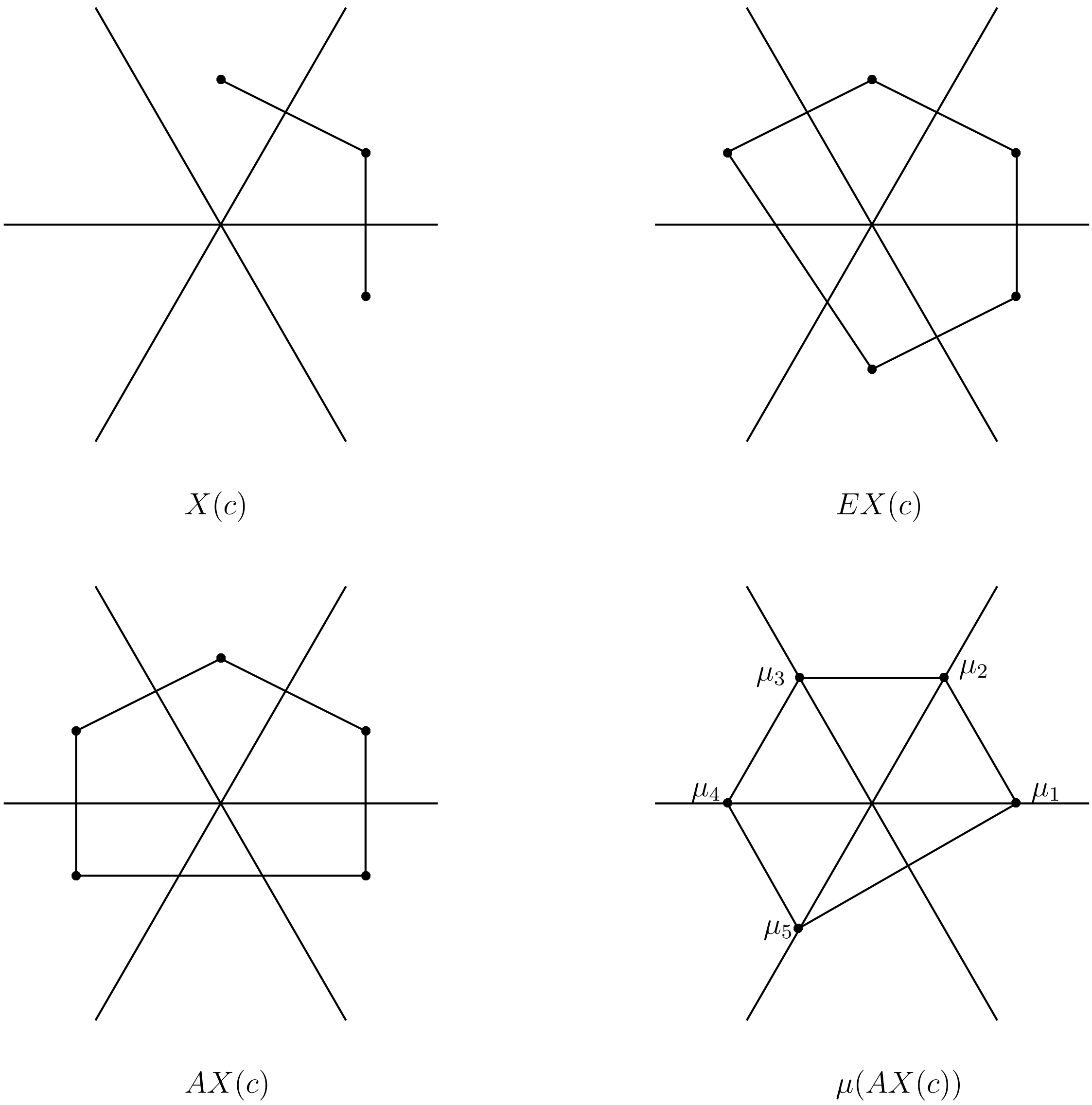


Figure 2.3.1: The complexes $X(c), EX(c), AX(c)$ and $\mu(AX(c))$ for $W(A_2)$

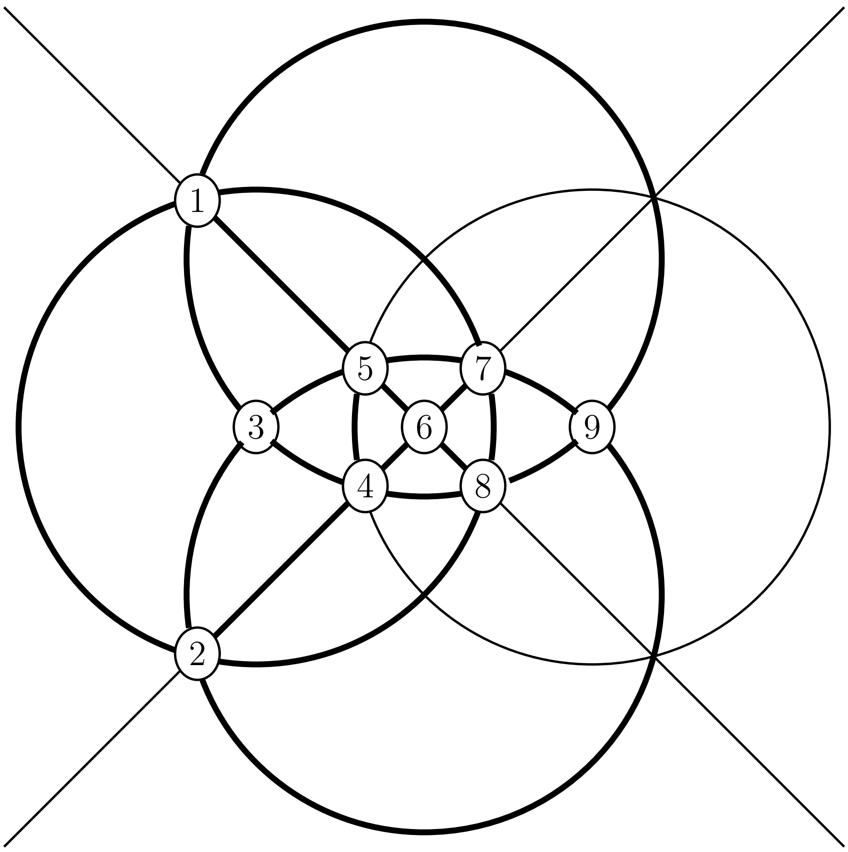


Figure 2.3.2: The complex $\mu(AX(c))$ for $W(A_3)$

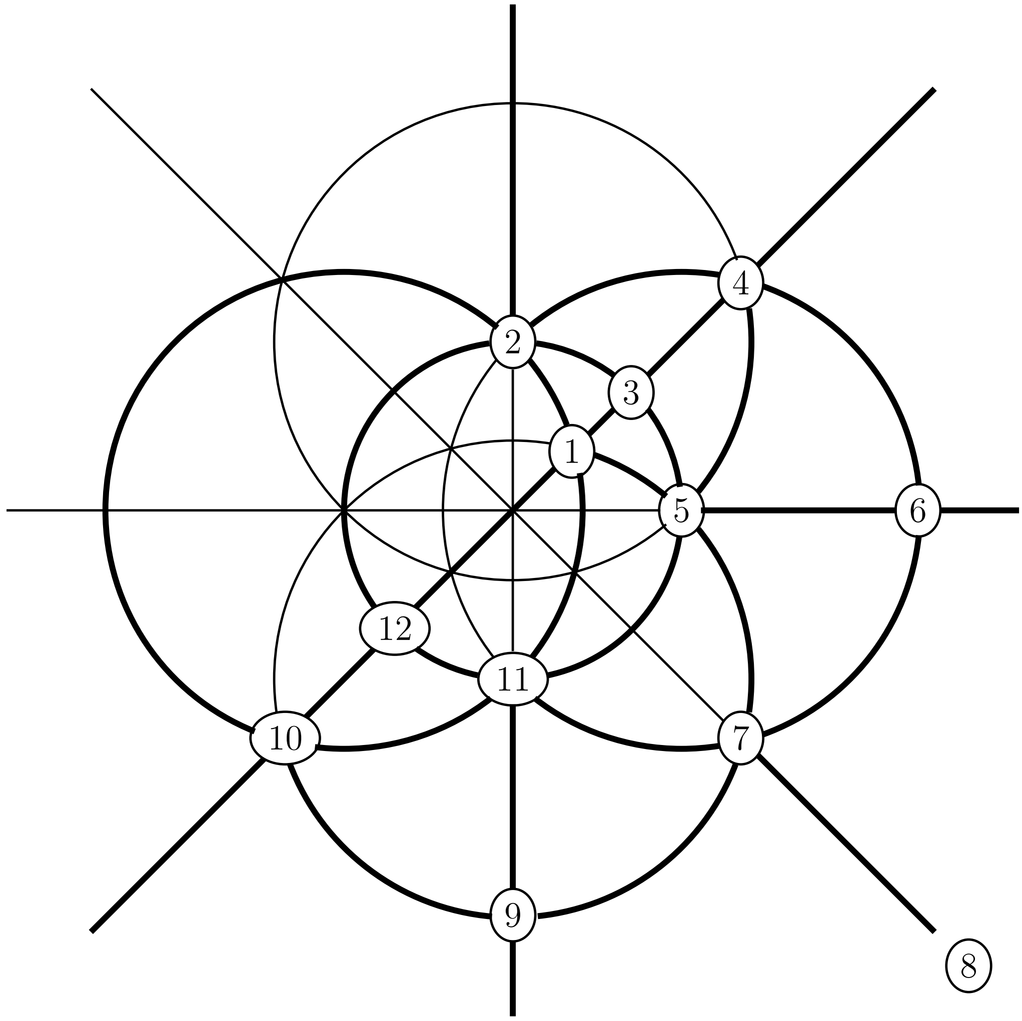


Figure 2.3.3: The complex $\mu(AX(c))$ for $W(C_3)$

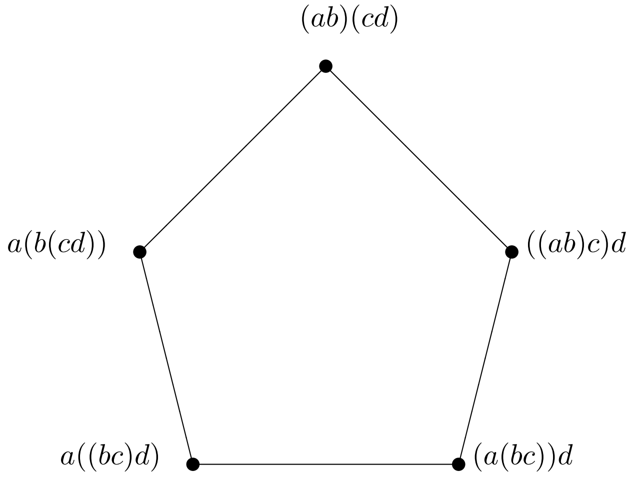


Figure 2.3.4: Graph of bracketings of 4 elements

If we fix a point x in the fundamental chamber C, the W-*permutahedron* is the convex hull of the orbit of x under the action of W. This polytope is the dual of the simplicial W-permutahedron defined in section 2.1.2. In [9] it is shown that the fan determined by the simplicial W-associahedron is a coarsening of the fan determined by the simplicial W-permutahedron.

We can extend the matrix of dot products $\mu_i \cdot \rho_j$ to include $\{\mu_{nh/2+1}, \ldots, \mu_{nh/2+n}\}$. The example below gives details for $W = W(C_4)$.

Example 2.3.6. Let $W = W(C_4), c = [1](23)(12)(34) = [1342]$. Therefore $h = 8$ and $nh/2 = 16$. The simple system is given by

$$\alpha_1 = (1,0,0,0), \alpha_2 = 1/\sqrt{2}(0,-1,1,0), \alpha_3 = 1/\sqrt{2}(-1,1,0,0), \alpha_4 = 1/\sqrt{2}(0,0,-1,1).$$

Set $a = 1/\sqrt{2}$. Then the first 20 roots are given by

$$\rho_1 = (1,0,0,0),\ \ \rho_2 = (0,-a,a,0),\ \ \rho_3 = (a,0,a,0),\ \ \rho_4 = (0,-a,0,a)$$

$$\rho_5 = (0,0,1,0),\ \ \rho_6 = (a,0,0,a),\ \ \rho_7 = (0,0,a,a),\ \ \rho_8 = (a,a,0,0)$$

$$\rho_9 = (0,0,0,1),\ \ \rho_{10} = (0,a,a,0),\ \ \rho_{11} = (0,a,0,a),\ \ \rho_{12} = (-a,0,a,0)$$

$$\rho_{13} = (0, 1, 0, 0), \;\; \rho_{14} = (-a, 0, 0, a), \;\; \rho_{15} = (-a, a, 0, 0), \;\; \rho_{16} = (0, 0, -a, a)$$

$$\rho_{17} = (-1, 0, 0, 0), \;\; \rho_{18} = (0, a, -a, 0), \;\; \rho_{19} = (-a, 0, -a, 0), \;\; \rho_{20} = (0, a, 0, -a).$$

Note that the last 4 roots are the negatives of the first 4 roots. In general the roots $\rho_{nh/2+1}, \ldots, \rho_{nh/2+n}$ are a permutation of the first n roots by [12].

Therefore the reflections are ordered as follows

$$[1], (23), (1\bar{3}), (24)$$

$$[3], (1\bar{4}), (3\bar{4}), (1\bar{2})$$

$$[4], (2\bar{3}), (2\bar{4}), (13)$$

$$[2], (14), (12), (34)$$

$$-[1], -(23), -(1\bar{3}), -(24)$$

From the roots, we can calculate the μ vectors.

$$\mu_1 = (1, 1, 1, 1), \;\; \mu_2 = (0, 0, 2a, 2a), \;\; \mu_3 = (0, 2a, 2a, 2a), \;\; \mu_4 = (0, 0, 0, 2a)$$

$$\mu_5 = (-1, 1, 1, 1), \;\; \mu_6 = (0, 2a, 0, 2a), \;\; \mu_7 = (-2a, 2a, 0, 2a), \;\; \mu_8 = (0, 2a, 0, 0)$$

$$\mu_9 = (-1, 1, -1, 1), \;\; \mu_{10} = (-2a, 2a, 0, 0), \;\; \mu_{11} = (-2a, 2a, -2a, 0), \;\; \mu_{12} = (-2a, 0, 0, 0)$$

$$\mu_{13} = (-1, 1, -1, -1), \;\; \mu_{14} = (-2a, 0, -2a, 0), \;\; \mu_{15} = (-2a, 0, -2a, -2a), \;\; \mu_{16} = (0, 0, -2a, 0)$$

We arrange the quantities $\rho_i \cdot \mu_j$ into a matrix. This is done for this example in the table below.

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9	ρ_{10}	ρ_{11}	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{15}	ρ_{16}
	$[1]$	(23)	$(1\bar{3})$	(24)	$[3]$	$(1\bar{4})$	$(3\bar{4})$	$(1\bar{2})$	$[4]$	$(2\bar{3})$	$(2\bar{4})$	(13)	$[2]$	(14)	(12)	(34)
μ_1	1	0	$2a$	0	1	$2a$	$2a$	$2a$	1	$2a$	$2a$	0	1	0	0	0
μ_2	0	1	1	1	$2a$	1	2	0	$2a$	1	1	1	0	1	0	0
μ_3	0	0	1	0	$2a$	1	2	1	$2a$	2	2	1	$2a$	1	1	0
μ_4	0	0	0	1	0	1	1	0	$2a$	0	1	0	0	1	0	1
μ_5	-1	0	0	0	1	0	$2a$	0	1	$2a$	$2a$	$2a$	1	$2a$	$2a$	0
μ_6	0	-1	0	0	0	1	1	1	$2a$	1	2	0	$2a$	1	1	1
μ_7	$-2a$	-1	-1	0	0	0	1	0	$2a$	1	2	1	$2a$	2	2	1
μ_8	0	-1	0	-1	0	0	0	1	0	1	1	0	$2a$	0	1	0
μ_9	-1	$-2a$	$-2a$	0	-1	0	0	0	1	0	$2a$	0	1	$2a$	$2a$	$2a$
μ_{10}	$-2a$	-1	-1	-1	0	-1	0	0	0	1	1	1	$2a$	1	2	0
μ_{11}	$-2a$	-2	-2	-1	$-2a$	-1	-1	0	0	0	1	0	$2a$	1	2	1
μ_{12}	$-2a$	0	$-2a$	0	0	-1	0	-1	0	0	0	1	0	1	1	0
μ_{13}	-1	$-2a$	$-2a$	$-2a$	-1	$-2a$	$-2a$	0	-1	0	0	0	1	0	$2a$	0
μ_{14}	$-2a$	-1	-2	0	$-2a$	-1	-1	-1	0	-1	0	0	0	1	1	1
μ_{15}	$-2a$	-1	-2	-1	-1	-2	-2	-1	$-2a$	-1	-1	0	0	0	1	0
μ_{16}	0	-1	-1	0	$-2a$	0	-1	0	0	-1	0	-1	0	0	0	1
$-\mu_1$	-1	0	$-2a$	0	-1	$-2a$	$-2a$	$-2a$	-1	$-2a$	$-2a$	0	-1	0	0	0
$-\mu_2$	0	-1	-1	-1	$-2a$	-1	-2	0	$-2a$	-1	-1	-1	0	-1	0	0
$-\mu_3$	0	0	-1	0	$-2a$	-1	-2	-1	$-2a$	-2	-2	-1	$-2a$	-1	-1	0
$-\mu_4$	0	0	0	-1	0	-1	-1	0	$-2a$	0	-1	0	0	-1	0	-1

2.3.3 Facets of $\mu(AX(c))$

In [1] the first bijection between the elements of NCP_c and facets of $EX(c)$ is constructed. In [9] another bijection is constructed from elements of NCP_c but this time to facets of $\mu(AX(c))$. Before we define a facet, we need to characterise the simple system associated to the length k element $w \in NCP_c$.

Definition 2.3.7. A subset Δ of a root system Φ is a simple system if the elements of Δ form a basis for V and if each $\alpha \in \Phi$ is a linear combination of Δ with coefficients all of the same sign.

Recall that for an element w of NCP_c we define Γ_w to be the set of positive roots whose reflections precede w. We want to find the simple system $\Delta = \{\delta_1, \ldots, \delta_k\}$ associated to Γ_w. By proposition 5.1 of [12], we know that the ordered elements $\delta_1, \ldots, \delta_k$ of the simple system Δ are determined recursively by the fact that δ_k is the last positive root in Γ_w and δ_i is the last positive root in $M(w[r(\delta_k) \ldots r(\delta_{i+1})])$.

Example 2.3.8. As in example 2.3.6, let $W = W(C_4), c = [1342]$. Consider the element $w = (134\bar{2})$. We know that $\ell(w) = 3$. We find that

$$\Gamma_w = \{\rho(1\bar{2}), \rho(2\bar{3}), \rho(2\bar{4}), \rho(13), \rho(14), \rho(34)\}.$$

Therefore $r(\delta_3) = (34)$. To find δ_2, we consider $M(w[r(\delta_3)]) = M((134\bar{2})(34)) = M(13\bar{2})$. We find that

$$\Gamma_{w[r(\delta_3)]} = \{\rho(1\bar{2}), \rho(2\bar{3}), \rho(13)\}.$$

Therefore $r(\delta_2) = (13)$. Finally to find δ_1, consider $M(w[r(\delta_3)r(\delta_2)]) = M((13\bar{2})(13)) = M(1\bar{2})$. It is clear that $r(\delta_1) = (1\bar{2})$.

For $w \in NCP_c$, let $\{\delta_1, \ldots, \delta_k\}$ be the simple system for the element w where $\ell(w) = k$. For $w' = cw^{-1}$, let $\{\theta_1, \ldots, \theta_{n-k}\}$ be the simple system for w'. Then, as in [9], we define the facet

$$F(w) = \{x \in \mathbf{R}^n \mid x \cdot \delta_i \leq 0 \text{ and } x \cdot \theta_j \geq 0\}.$$

In [12], Brady and Watt determine the first facet of $X(w)$ in the lexicographical order for any $w \in NCP_c$ of length k. In [1] there is an analogous description of the lexicographically last facet. We review these here. For each $1 \leq i \leq k$ define the roots

$$\epsilon_i = r(\delta_1)\dots r(\delta_{i-1})\delta_i.$$

We reorder these roots so that $\epsilon_1 < \epsilon_2 < \dots < \epsilon_k$. Then the set $\{\epsilon_1, \epsilon_2, \dots, \epsilon_k\}$ is the vertex set of the first facet of $X(w)$ in the lexicographic order (Proposition 3.6 of [1]).

We can also determine the vertex set of the last facet of $X(w)$ in the lexicographic order. Define the roots η_i for each $1 \leq i \leq k$ as

$$\eta_{n-k+i} = r(\delta_k)\dots r(\delta_{i+1})\delta_i.$$

If we reorder the roots so that $\eta_{n-k+1} < \eta_{n-k+2} < \dots < \eta_n$ then the set $\{\eta_{n-k+1}, \dots, \eta_n\}$ is the vertex set of the last facet of $X(w)$ in the lexicographic order (Proposition 3.14 of [1]).

Recall that we refer to the positive cone on a non-zero vector v in a one-dimensional subspace $X \in L(\mathcal{A})$ as the ray generated by v. We can now determine the rays of $F(w)$ for each $w \in NCP_c$. Let $\{\epsilon_1, \dots, \epsilon_{n-k}\}$ be the vertices of the lexicographically first facet of $X(cw^{-1})$ and let $\{\eta_{n-k+1}, \dots, \eta_n\}$ be the vertices of the lexicographically last facet of $X(w)$. Then by Proposition 5.3 of [9] the rays of $F(w)$ are generated by

$$\{\mu(\epsilon_1), \dots, \mu(\epsilon_{n-k}), \mu[c(\eta_{n-k+1})], \dots, \mu[c(\eta_n)]\}.$$

We will now demonstrate this with some examples.

Example 2.3.9. As in 2.1.3, let $W = W(A_3)$ and $c = (12)(34)(23) = (1243)$. Consider $w = (13)$. We know that $\ell(w) = k = 1$. For $w = (13)$, we find that $\Delta = \{\rho(13)\}$ so therefore $r(\delta_1) = (13)$. The last facet of $X(w)$ has vertex set $\{\eta_n\}$ where $r(\eta_n) = r(\eta_3) = r(\delta_1) = (13)$. We also need to note that $r(c(\eta_1)) = -(12)$.

If $w = (13)$, then $cw^{-1} = (1243)(13) = (243)$ with $\Delta = \{\rho(34), \rho(23)\}$. The vertices of the first facet of $X(cw^{-1})$ are $\{\epsilon_1, \epsilon_2\}$ where $r(\epsilon_1) = (34)$ and $r(\epsilon_2) = (24)$.

Therefore the rays of $F((13))$ are generated by

$$\mu(34), \mu(24), \mu(-(12)).$$

In figure 2.3.5 we have labelled all the facets of $\mu(AX(c))$ for $W = W(A_3)$. If we number the ordered reflections then each facet is labelled by the simple system of the corresponding element. For example 26 corresponds to (243) and 126 corresponds to c.

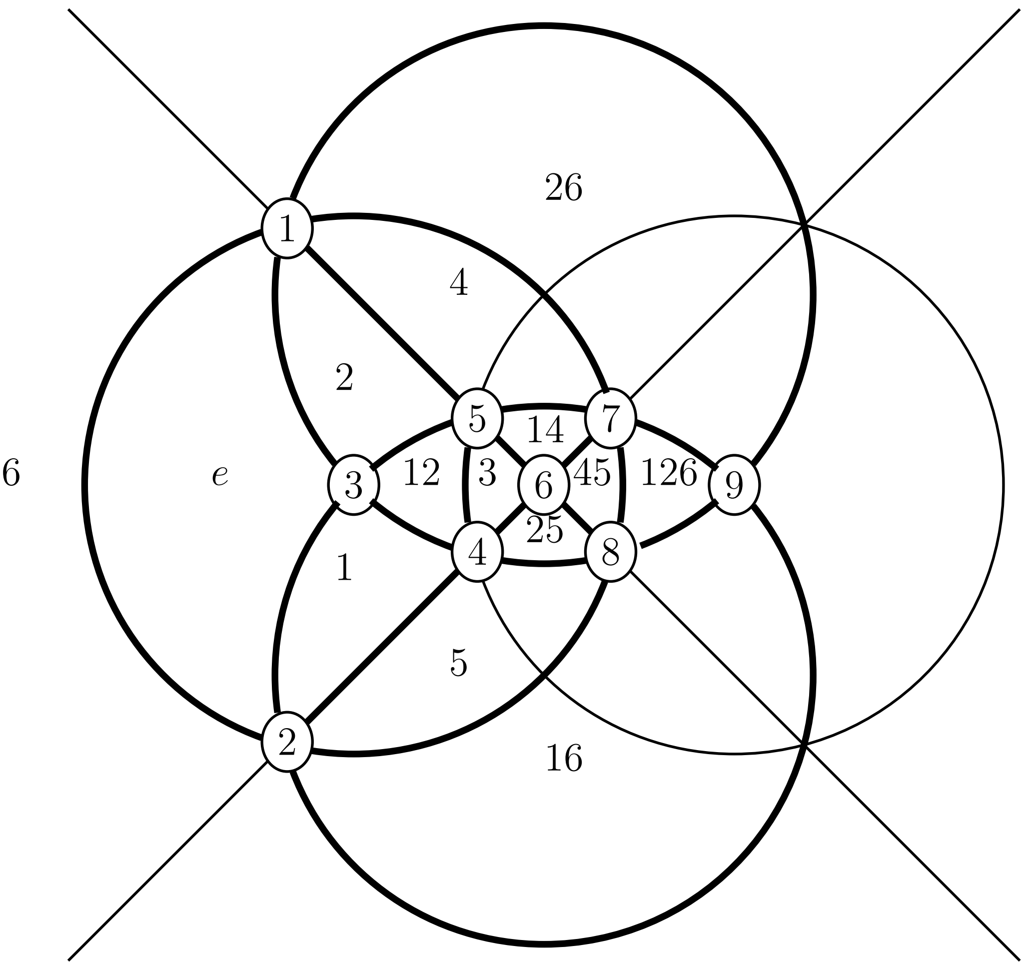


Figure 2.3.5: The complex $\mu(AX(c))$ for $W(A_3)$ with facets labelled

Example 2.3.10. As in 2.3.6, let $W = W(C_4), c = [1342]$ and consider $w = [13]$. We find that $\Gamma_w = \{\rho[1], \rho(1\bar{3}), \rho[3], \rho(13)\}$ so from that we find that $r(\delta_1) = [1]$ and $r(\delta_2) = (13)$. We want the vertex set of the last facet of $X(w)$ so we calculate η_{n-k+i} for $1 \leq i \leq k$. We find that $\eta_{n-k+1} = \eta_3 = r(\delta_2)\delta_1$ so $r(\eta_3) = [3]$ and $\eta_{n-k+2} = \eta_4 = \delta_2$ so $r(\eta_4) = (13)$. Therefore the vertex set is given by $\{\rho[3], \rho(13)\}$.

Now, we want to find the vertex set of the first facet of $X(cw^{-1}) = X([1342][1\bar{3}]) = X(1\bar{4}\bar{2})$. We find that $\Gamma_{cw^{-1}} = \{\rho(24), \rho(1\bar{4}), \rho(1\bar{2})\}$ so $\Delta = \{\rho(24), \rho(1\bar{2})\}$ and $r(\epsilon_1) = (24), r(\epsilon_2) = (1\bar{4})$.

Therefore the rays of $F([13])$ are generated by

$$\{\mu(24), \mu(1\bar{4}), \mu[c[3]], \mu[c(13)]\} = \{\mu(24), \mu(1\bar{4}), \mu[4], \mu(34)\}.$$

Chapter 3

Building the Complex $\mu(AX(c))$

3.1 Simplicial complex $\mu(AX(c))$

Recall that $\mu(AX(c))$ denotes the simplicial complex with vertices $\{\mu_1, \ldots, \mu_{nh/2+n}\}$ and with a simplex on $\{\mu_{i_1}, \ldots, \mu_{i_k}\}$ if and only if

$$1 \leq i_1 < i_2 < \cdots < i_k \leq nh/2 + n \text{ and } l(r(\rho_{i_1}) \ldots r(\rho_{i_k})c) = n - k.$$

In [9], the complex $\mu(AX(c))$ is defined by applying the operator $\mu = 2(I - c)^{-1}$ to $AX(c)$ which, in turn, is isometric to $EX(c)$ which is an extension of $X(c)$. In this section, we will describe an alternative, more direct construction of $\mu(AX(c))$ using intersections of halfspaces. Our construction is analagous to that of the complex $X(c)$ in [12]; however we do not consider the subcomplexes for $w \preceq c$. The properties of the dot products $\mu_i \cdot \rho_j$ will be used in this section. We recall them here for convenience and will refer to them by their number when required.

1. $\mu_i \cdot \rho_i = 1$
2. $\mu_i \cdot \rho_j = -\mu_{j+n} \cdot \rho_i$ for all i and j
3. $\mu_i \cdot \rho_j \geq 0$ for $1 \leq i \leq j \leq nh/2$
4. $\mu_{i+k} \cdot \rho_i = 0$ for $1 \leq k \leq n - 1$ and for all i
5. $\mu_j \cdot \rho_i \leq 0$ for $1 \leq i < j \leq nh/2$

Following [12], we define the positive and negative halfspaces as follows.

Definition 3.1.1. We fix a positive root ρ and define the positive halfspace by

$$\rho^+ = \{x \in \mathbf{R}^n | x \cdot \rho \geq 0\}.$$

The negative halfspace is defined as

$$\rho^- = \{x \in \mathbf{R}^n | x \cdot \rho \leq 0\}.$$

It will be convenient to use the notation $\rho^\perp$ for the hyperplane

$$\rho^\perp = \{x \in \mathbf{R}^n | x \cdot \rho = 0\}.$$

We fix a fundamental chamber C for the W-action where C has inward unit normals $\alpha_1, \ldots, \alpha_n$. Let $r_1, \ldots, r_n$ denote the corresponding reflections. The chamber C has vertices $\mu_1, \ldots, \mu_n$ and is on the positive side of all hyperplanes. It can therefore be expressed as the intersection of the halfspaces

$$\rho_1^+ \cap \rho_2^+ \cap \cdots \cap \rho_{nh/2}^+.$$

Our construction involves beginning with C and building up the complex $\mu(AX(c))$ by adding the vertices $\mu_{n+1}, \ldots, \mu_{nh/2+n}$ in order of ascending indices. We require some definitions and notation.

Definition 3.1.2. For $i \geq n$, we denote by $X(c, \mu_i)$ the set of simplices of $\mu(AX(c))$ whose vertices are in the set $\{\mu_1, \ldots, \mu_i\}$. We will also use X_i as an abbreviation for $\mathrm{cone}|X(c, \mu_i)|$, that is the positive cone on the subcomplex $X(c, \mu_i)$. We note that X_n is the cone on the fundamental chamber C.

Definition 3.1.3. For $i \geq n$, $Y_i = Y(c, \mu_i)$ denotes the positive cone on the set $\{\mu_1, \ldots, \mu_i\}$.

Definition 3.1.4. For $i \geq n$, $Z_i = Z(c, \mu_i) = \rho_{i-n+1}^+ \cap \cdots \cap \rho_{nh/2}^+$.

Note that for $i = nh/2 + n$, Z_i is not on the positive side of any hyperplane. In this case $Z_i = \mathbf{R}^n$.

Our aim is to prove that $X_i = Y_i = Z_i$ for $i \geq n$. We do this using an induction argument. We will first outline the ideas used in the proof. We will see that it suffices to show that $Z_i \subseteq X_i$. We assume that the theorem holds for $i = n + k - 1$ and we will prove that it holds for $i = n + k$. If F is the subcomplex of $\mu(AX(c))$ with $F = X_{n+k-1} \cap \rho_k^{\perp}$ and V is the closure of $|X_{n+k} \setminus X_{n+k-1}|$, then V is the cone on the sphere with base F and apex μ_i. The theorem follows if the closure of $Z_{n+k} \setminus Z_{n+k-1}$ is contained in V.

Example 3.1.5. We will outline the proof of the theorem using an example with $W = W(A_3)$ and $c = (1243)$. Assume that $X_4 = Y_4 = Z_4$. We want to show it holds for $i = 5$. To do this, we let F denote the subcomplex $X_4 \cap \rho_2^{\perp}$ and let V denote the closure of $|X_5 \setminus X_4|$. It is clear in this example that V is a cone with base F and apex μ_5. V is outlined in bold for this example in figure 3.1.1. In this case, V can be expressed as the intersection of halfspaces $\rho_2^- \cap \rho_3^+ \cap \rho_4^+$. Let Z denote the closure of $|Z_5 \setminus Z_4|$ then $Z = \rho_2^- \cap \rho_3^+ \cap \rho_4^+ = V$.

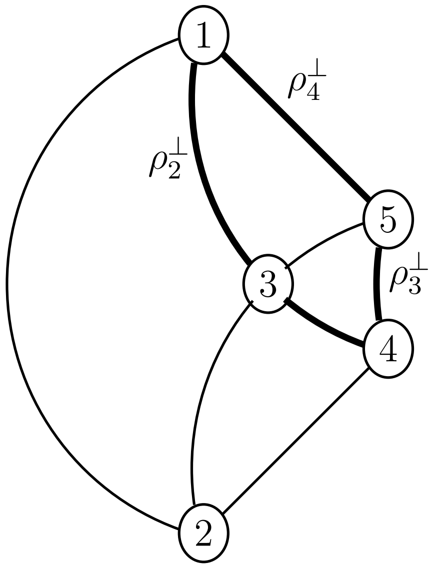


Figure 3.1.1: $X_5 = Y_5 = Z_5$

Theorem 3.1.6. *The equality $X_i = Y_i = Z_i$ holds for $n \leq i \leq nh/2 + n$.*

Proof. $X_i \subseteq Y_i$ by definition. By property 3, we know that $Y_i \subseteq Z_i$. Therefore, it suffices to show that $Z_i \subseteq X_i$. In the case $i = n$, $X_n = \text{cone}|X(c, \mu_n)|$ is the positive cone on the set of simplices of $\mu(AX(c))$ whose vertices precede μ_n and $Z_n = \rho_1^+ \cap \cdots \cap \rho_{nh/2}^+$. In other words, Z_n contains the set of facets on the positive side of all hyperplanes. The only such facet is the fundamental domain. Therefore, $X_n = Y_n = Z_n$.

We assume the theorem holds for $i = n + k - 1$ and show it is true for $i = n + k$. We let $F = X_{n+k-1} \cap \rho_k^\perp$ and V be the closure of $|X_{n+k} \setminus X_{n+k-1}|$. Then V is a cone on the sphere with base F and apex μ_{n+k} since each facet of $X_{n+k} \setminus X_{n+k-1}$ has μ_{n+k} as a vertex.

Let Z denote the closure of $|Z_{n+k} \setminus Z_{n+k-1}| = \rho_k^- \cap \rho_{k+1}^+ \cap \cdots \cap \rho_{nh/2}^+$. We claim that each facet of cone(V) is of the form $\rho_j^\perp \cap V$ for some $j \in \{k, k+1, \ldots, nh/2\}$. If the claim is true the theorem follows because we can express V as the intersection of halfspaces.

$$V = \rho_k^- \cap \rho_{i_1}^+ \cap \ldots \rho_{i_j}^+ \text{ with } \{i_1, \ldots i_j\} \subseteq \{k+1, \ldots, nh/2\}.$$

But we have that

$$\begin{aligned} Z &= \rho_k^- \cap \rho_{k+1}^+ \cap \cdots \cap \rho_{nh/2}^+ \\ &\subseteq \rho_k^- \cap \rho_{i_1}^+ \cap \cdots \cap \rho_{i_j}^+ \\ &= V. \end{aligned}$$

The inclusion above follows from the elementary fact about sets that if $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ then automatically

$$A_1 \cap A_2 \cap \cdots \cap A_n \subseteq A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}.$$

To prove the claim we need to show that each facet of cone(V) apart from F is of the form $\rho_j^\perp \cap V$ for some $j \in \{k+1, \ldots, nh/2\}$. Assume $j < k$ and we will find a contradiction by finding two μ vectors that lie on $\rho_k^\perp$, with the property that they lie on different sides of $\rho_j^\perp$.

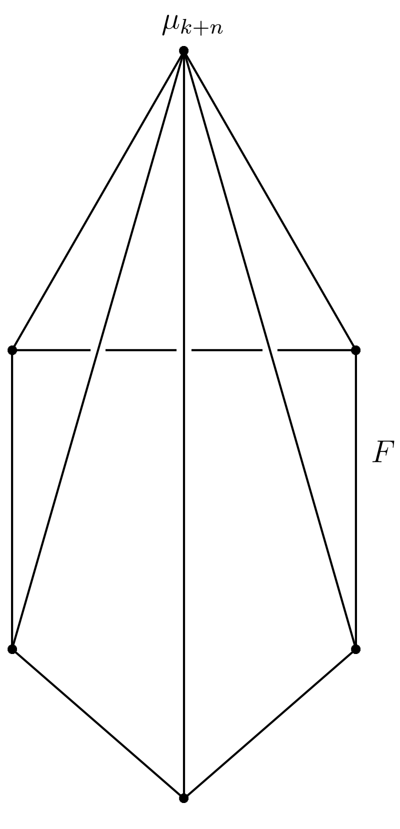


Figure 3.1.2: The cone V

Since V is a cone on μ_{n+k}, each boundary facet of V apart from F must have μ_{n+k} as a vertex. Therefore $\mu_{n+k} \in \rho_j^{\perp}$ and as a result $\mu_{n+k} \perp \rho_j$. Therefore, since $\mu_{n+k} \in F(r(\rho_{n+k})c) = (M(r(\rho_{n+k})c))^{\perp}$,

$$\begin{aligned} r(\rho_j) &\preceq r(\rho_{n+k})c \\ \Rightarrow c &= r(\rho_{n+k})r(\rho_j)\nu \text{ where } \ell(\nu) = n-2 \\ &= r(\rho_j)\nu r(\rho_k) \\ &= r(\rho_j)r(\rho_k)\nu', \end{aligned}$$

for some $\nu' \preceq c$ with $\ell(\nu') = n-2$ since the set of reflections is closed under conjugation. Thus

$$\begin{aligned} r(\rho_k) &\preceq r(\rho_j)c \\ \Rightarrow \rho_k \cdot \mu_j &= 0 \\ \Rightarrow \mu_j &\in \rho_k^{\perp}. \end{aligned}$$

Since $\rho_j \cdot \mu_j = 1$ by property 1, we know that μ_j is on the positive side of $\rho_j^{\perp}$. In other words, it is on the same side as the fundamental domain. Also, since $j < k < n+k$ then μ_j lies on F.

The construction of a μ vector on F and on the negative side of $\rho_j^{\perp}$ is more delicate. If we set $\sigma = c[r(\rho_k)]$ then σ is a length $(n-1)$ non-crossing partition. We show that $\mu(\sigma(\rho_j)) \in \rho_k^{\perp}$ as follows

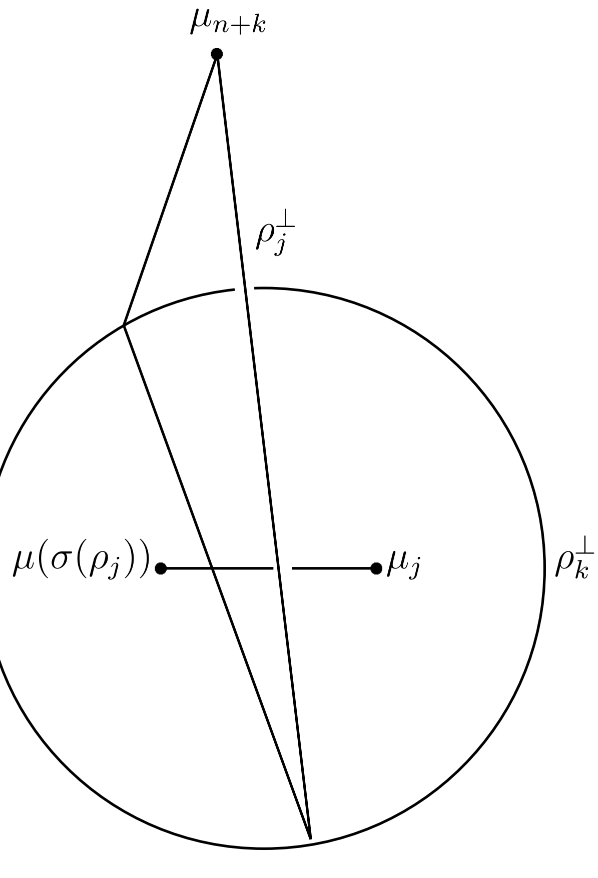


Figure 3.1.3: The facet $\rho_j^\perp$

$$\begin{aligned}
\rho_k \cdot \mu(\sigma(\rho_j)) &= -c[\mu_k] \cdot \sigma(\rho_j) \text{ by property 2} \\
&= -c[\mu_k] \cdot c[r(\rho_k)]\rho_j \\
&= -\mu_k \cdot r(\rho_k)\rho_j \text{ by orthogonality of } c \\
&= -r(\rho_k)\mu_k \cdot \rho_j \\
&= -c[\mu_k] \cdot \rho_j \\
&= -\mu_{n+k} \cdot \rho_j \\
&= 0.
\end{aligned}$$

Also, we can show that $\mu(\sigma(\rho_j))$ is on the negative side of $\rho_j^\perp$, proving that $\rho_j^\perp$ separates $\mu(\sigma(\rho_j))$ and μ_j.

$$\begin{aligned}
\rho_j \cdot \mu(\sigma(\rho_j)) &= -c[\mu_j] \cdot \sigma(\rho_j) \text{ by property 2} \\
&= -c[\mu_j] \cdot c[r(\rho_k)]\rho_j \\
&= -\mu_j \cdot r(\rho_k)\rho_j \text{ by orthogonality of } c \\
&= -r(\rho_k)\mu_j \cdot \rho_j \\
&= -\mu_j \cdot \rho_j \\
&= -1.
\end{aligned}$$

Finally, we need to ensure that $\mu(\sigma(\rho_j)) < \mu_{k+n}$ so that $\mu(\sigma(\rho_i)) \in F$.

$$\begin{aligned}
\mu(\sigma(\rho_j)) < \mu_{k+n} &\Leftrightarrow \sigma(\rho_j) < \rho_{k+n} \\
&\Leftrightarrow [c(r(\rho_k))](\rho_j) < \rho_{k+n} \\
&\Leftrightarrow [r(\rho_k)](\rho_j) < \rho_k.
\end{aligned}$$

Since $\rho_k \cdot \mu_j = 0$, we have that $r(\rho_k) \preceq r(\rho_j)c$. Therefore, $r(\rho_j)r(\rho_k) \preceq c$ where $r(\rho_j)r(\rho_k)$ is a non-crossing partition of length 2 with simple system $\{\rho_j, \rho_k\}$ since $j < k$. Therefore,

$[r(\rho_k)]\rho_j$ is a positive root which precedes ρ_k in the total order by (Proposition in Section 1.4 of [15]). $\square$

We return to example 3.1.5. Recall that V was a cone with base $X_4 \cap \rho_2^{\perp}$ and apex μ_5. For $j < k$ we want to find two μ vectors on either side of $\rho_j^{\perp}$. We can see from figure 3.1.1 that $\mu_5 \in \rho_1^{\perp}$. From the theorem, we know that μ_1 is on the positive side of $\rho_1^{\perp}$. We let $\sigma = c[r(\rho_k)] = (1243)(34) = (124)$. Then by theorem 3.1.6, the vector $\mu(\sigma(\rho_j))$ is on the negative side of $\rho_j^{\perp}$. In this example $\mu((124)[(1,-1,0,0)]) = \mu(0,1,0,-1) = \mu_4$. We can see from the figure that this is in fact the case. This establishes the contradiction since μ_j and $\mu(\sigma(\rho_j))$ are two vertices on F separated by $\rho_j^{\perp}$ so that $V \cap \rho_j^{\perp}$ could not have been a facet of V.

Chapter 4

Existence of an Interval

Recall that we have identified the associahedron with the complex $\mu(AX(c))$ so that associahedron facets are of the form $F(w)$ for $w \in NCP_c$. For a given associahedron facet $F(w)$ we will show that the set of elements $x \in W$, with the property that the permutahedron facet corresponding to x is contained in the associahedron facet $F(w)$, forms an interval in the weak order. We do this by showing that certain sets of reflections associated to w form inversion sets for group elements in W.

4.1 The sets $T_n(w)$, $T_p(w)$, $T_{np}(w)$.

Recall from section 2.3.3 that, for $w \in NCP_c$, we define the facet

$$F(w) = \{\vec{x} \in \mathbf{R}^n \mid \vec{x} \cdot \delta_i \leq 0, \;\; \vec{x} \cdot \theta_j \geq 0\}$$

where $\{\delta_1, \ldots, \delta_k\}$ is the simple system for w and $\{\theta_1, \ldots, \theta_{n-k}\}$ is the simple system for $w' = cw^{-1}$. $F(w)$ is a positive cone on

$$\mu(c[\eta_n]), \ldots, \mu(c[\eta_{n-k+1}]), \mu(\epsilon_{n-k}), \ldots, \mu(\epsilon_1).$$

In [9], there is a geometric characterisation of the regions $F(w)$, which are associahedron facets consisting of those elements whose permutahedron facets lie in $F(w)$ for some $w \in NCP_c$. We can therefore express each associahedron facet $F(w)$ as a union of permutahedron facets. $F(w)$ partitions the set of $nh/2$ hyperplanes into three subsets; those with the facet on the negative side, those with the facet on the positive side and those that split the facet.

Definition 4.1.1. We define

$$T_n(w) = \{t \in T \mid \vec{x} \cdot \rho(t) \leq 0 \text{ for all } \vec{x} \in F(w)\}$$

$$T_p(w) = \{t \in T \mid \vec{x} \cdot \rho(t) \geq 0 \text{ for all } \vec{x} \in F(w)\}$$

$$T_{np}(w) = \{t \in T \mid \text{ there exists } \vec{x} \text{ and } \vec{y} \in F(w) \text{ such that } \vec{x} \cdot \rho_r > 0 \text{ and } \vec{y} \cdot \rho_r < 0\}$$

The set $T_n(w)$ is the set of all reflections whose corresponding hyperplanes have $F(w)$ on their negative side. Similarly, we define $T_{np}(w)$ which is the set of reflections whose hyperplanes split $F(w)$ and $T_p(w)$ which is the set of reflections whose hyperplanes have $F(w)$ on their positive side.

We can determine the elements of $T_n(w)$ from the matrix of dot products $[\rho_i \cdot \mu_j]$ since $r(\rho_i) \in T_n(w)$ if and only if $\rho_i \cdot \mu_j \leq 0$ for each μ_j a corner of $F(w)$. Similarly, it is also possible to determine the elements of $T_{np}(w)$ and $T_p(w)$ from the matrix. The elements of $T_n(w), T_p(w)$ and $T_{np}(w)$ inherit an ordering from the total order on the set of positive roots $\rho_1, \ldots, \rho_{nh/2}$.

Example 4.1.2. Consider $W(A_3), w = (123)$. The element w produces the simple system with $r(\delta_1) = (12)$ and $r(\delta_2) = (23)$. The element $w' = (1243)w^{-1} = (1243)(132) = (34)$ produces the simple system with $r(\theta_1) = (34)$. By definition

$$F(w) = \{\vec{x} \in \mathbf{R}^n \mid \vec{x} \cdot \delta_1 \leq 0,\ \ \vec{x} \cdot \delta_2 \leq 0,\ \ \vec{x} \cdot \theta_1 \geq 0\}$$

To find the corners of the facet $F(w)$, we must find ϵ_1, η_2 and η_3 where $\epsilon_1 = \theta_1$, $\eta_2 = r(\delta_2)\delta_1$ and $\eta_3 = \delta_2$. Therefore $F(w)$ is a positive cone on the vertices $\mu(34), \mu(c[13]), \mu(c[(23)])$. Therefore, $F(w)$ has vertices $\mu(34), \mu(-(12)), \mu(-(14))$. Consider the matrix of dot products for the appropriate μ vectors.

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6
	(12)	(34)	(14)	(24)	(13)	(23)
μ_2	0	1	1	1	0	0
μ_8	-1	0	-1	0	-1	0
μ_9	0	0	-1	-1	-1	-1

It is easy to see from the table that

$$T_n(w) = \{r(\rho_1), r(\rho_5), r(\rho_6)\}, \text{ where } r(\rho_1) = (12), r(\rho_5) = (13), r(\rho_6) = (23),$$

$$T_{np}(w) = \{r(\rho_3), r(\rho_4)\}, \text{ where } r(\rho_3) = (14), r(\rho_4) = (24),$$

$$T_p(w) = \{r(\rho_2)\} \text{ where } r(\rho_2) = (34).$$

Our aim is to prove that $T_n(w)$ is an inversion set for some $x \in W$. We first need to prove a particular property of the positive roots. In the process of doing this, we use the properties of the matrix of dot products $\rho_i \cdot \mu_j$ introduced by Brady and Watt in [12] and which we have reviewed in section 2.3.

It is proved in Corollary 4.7 of [12] that ρ_k does not lie in the positive cone on $\{\rho_{i_1}, \ldots, \rho_{i_m}\}$ for $1 \leq i_1 < i_2 < \cdots < i_m < k \leq nh/2$. This is proved by showing that ρ_k cannot be expressed as a non-negative linear combination of $\{\rho_{i_1}, \ldots, \rho_{i_m}\}$. Analogously, it can be proved that ρ_k does not lie in the positive cone on $\{\rho_{i_1}, \ldots, \rho_{i_m}\}$ for $1 \leq k < i_1 < i_2 < \cdots < i_m \leq nh/2$. Here we prove this result for $m = 2$ in a different way. We prove that if ρ is a positive linear combination of two roots ρ_i and ρ_j with $\rho_i < \rho_j$, then $\rho_i < \rho < \rho_j$ in the total order.

Proposition 4.1.3. *Let ρ_i, ρ_j be positive roots with $\rho_i < \rho_j$. Suppose ρ is a root with $\rho = a\rho_i + b\rho_j$ where $a, b > 0$. Then $\rho_i < \rho < \rho_j$.*

Proof. Consider $\rho \cdot \mu_i$.

$$\begin{aligned}
\rho \cdot \mu_i &= (a\rho_i + b\rho_j) \cdot \mu_i \\
&= a(\rho_i \cdot \mu_i) + b(\rho_j \cdot \mu_i) \\
&= a(1) + b(\rho_j \cdot \mu_i) \text{ by property 1 above.} \\
&> 0 \text{ by property 3.}
\end{aligned}$$

Therefore, $\rho > \rho_i$ by property 5.

Consider $\rho \cdot \mu_{j+n}$.

$$\begin{aligned}
\rho \cdot \mu_{j+n} &= (a\rho_i + b\rho_j) \cdot \mu_{j+n} \\
&= a(\rho_i \cdot \mu_{j+n}) + b(\rho_j \cdot \mu_{j+n}) \\
&= a(\rho_i \cdot \mu_{j+n}) + b(-\rho_j \cdot \mu_j) \text{ by property 2.} \\
&= a(\rho_i \cdot \mu_{j+n}) + b(-1) \text{ by property 1.} \\
&< 0 \text{ by property 5.}
\end{aligned}$$

Therefore, $\rho < \rho_j$ by property 3. □

Now we can prove that $T_n(w)$ is an inversion set. Recall from section 2.1.2 that we can characterise an inversion set as follows. A set $\mathcal{I}$ is an inversion set of a unique element $w \in W$ if $\mathcal{I}$ is closed and its complement in Φ^+ is closed.

Proposition 4.1.4. *$T_n(w)$ is an inversion set for an element $x \in W$.*

Proof. We first prove that the set $T_n(w)$ is closed and we follow this with a proof that $T \setminus T_n(w) = T_{np}(w) \cup T_p(w)$ is closed. Then, by the characterisation of inversion sets in

[17], $T_n(w)$ is an inversion set.

The set $T_n(w)$ is closed: Let $\vec{x}, \vec{y}$ be roots with $r(\vec{x}), r(\vec{y}) \in T_n(w)$. Let $\vec{z}$ be a root of the form $a\vec{x} + b\vec{y}$, with $a, b \in \mathbf{R}_+$. We need to show that $r(\vec{z}) \in T_n(w)$. Let $\vec{u} \in F(w)$. Since $r(\vec{x}), r(\vec{y}) \in T_n(w)$, we know that $\vec{u} \cdot \vec{x} \leq 0$ and $\vec{u} \cdot \vec{y} \leq 0$. We calculate $\vec{u} \cdot \vec{z}$.

$$\begin{aligned} \vec{u} \cdot \vec{z} &= \vec{u} \cdot (a\vec{x} + b\vec{y}) \\ &= a(\vec{u} \cdot \vec{x}) + b(\vec{u} \cdot \vec{y}) \\ &\leq 0 \end{aligned}$$

Therefore, $r(\vec{z}) \in T_n(w)$

The set $T_p(w) \cup T_{np}(w)$ is closed: Since $T \setminus T_n(w) = T_{np}(w) \cup T_p(w)$ elements of $T_p(w) \cup T_{np}(w)$ have the following characterisation.

$$r(\vec{u}) \in T_p(w) \cup T_{np}(w) \Leftrightarrow \text{ there exists a corner } c_u \text{ of } F(w) \text{ with } \vec{u} \cdot c_x > 0$$

Let $\vec{x}, \vec{y}$ be roots where $r(\vec{x}), r(\vec{y}) \in T_p(w) \cup T_{np}(w)$. Let $\vec{z} = a\vec{x} + b\vec{y}$ be a root with $a, b \in \mathbf{R}_+$. We need to show that $r(\vec{z}) \in T_p(w) \cup T_{np}(w)$. We have

$$r(\vec{x}) \in T_p(w) \cup T_{np}(w) \Leftrightarrow \text{ there exists a corner } c_x \text{ of } F(w) \text{ with } \vec{x} \cdot c_x > 0$$

$$r(\vec{y}) \in T_p(w) \cup T_{np}(w) \Leftrightarrow \text{ there exists a corner } c_y \text{ of } F(w) \text{ with } \vec{y} \cdot c_y > 0$$

From proposition 4.1.3, we know that $\vec{x} < \vec{z} < \vec{y}$. Since $\vec{x} \cdot c_x > 0$ and $\vec{x} \cdot \mu_i \leq 0$ for $i > \vec{x}$, we can deduce that $c_x \leq \mu_{\vec{x}} < \mu_{\vec{z}} < \mu_{\vec{y}}$. Property 3 states that $\mu_i \cdot \rho_j \geq 0$ for $1 \leq i \leq j \leq nh/2$. Since $c_x < \mu_{\vec{y}}$, it is clear from this that $c_x \cdot \vec{y} \geq 0$.

$$\begin{aligned}\vec{z} \cdot c_x &= (a\vec{x} + b\vec{y}) \cdot c_x \\ &= a(\vec{x} \cdot c_x) + b(\vec{y} \cdot c_x) \\ &> 0\end{aligned}$$

Since c_x is a corner of $F(w)$, $r(\vec{z}) \in T_p(w) \cup T_{np}(w)$. $\square$

By a similar argument we can prove that $T_p(w)$ is an inversion set. This gives the following.

Proposition 4.1.5. *$T_n(w) \cup T_{np}(w)$ is an inversion set for an element $y \in W$.*

While examining the properties of the sets $T_n(w)$, $T_{np}(w)$ and $T_p(w)$ we proved the following sufficient condition for a root to split $F(w)$ and we record it here.

Proposition 4.1.6. *Suppose ρ_1 precedes ρ_2 in the total order, $r(\rho_1) \in T_p(w), r(\rho_2) \in T_n(w) \cup T_{np}(w)$ and $\rho = a\rho_1 + b\rho_2$ with $a, b > 0$. Then $r(\rho) \in T_{np}(w)$.*

Proof. Since $r(\rho_2) \in T_n(w) \cup T_{np}(w)$, there exists a vertex c_x of $F(w)$ with $\rho_2 \cdot c_x < 0$. Therefore $\mu(\rho_1) < \mu(\rho_2) < c_x$. Since $\rho_1 < \rho_2$, then $\rho_1 \cdot c_x \leq 0$. But $r(\rho_1) \in T_p(w)$ and as a result $\rho_1 \cdot c_x = 0$. We can now calculate $\rho \cdot c_x$.

$$\begin{aligned}\rho \cdot c_x &= (a\rho_1 + b\rho_2) \cdot c_x \\ &= a(0) + b(\rho_2 \cdot c_x) \\ &< 0.\end{aligned}$$

Therefore, $r(\rho) \notin T_p(w)$.

Since $r(\rho_1) \in T_p(w)$, there exists a vertex c_y of $F(w)$ with $\rho_1 \cdot c_y > 0$. Therefore $c_y \leq \mu(\rho_1) < \mu(\rho_2)$. Since $\rho_1 < \rho_2$, then $\rho_2 \cdot c_y \geq 0$.

$$\begin{aligned}\rho \cdot c_y &= (a\rho_1 + b\rho_2) \cdot c_y \\ &= a(\rho_1 \cdot c_y) + b(\rho_2 \cdot c_y) \\ &> 0.\end{aligned}$$

Therefore, $r(\rho) \notin T_n(w)$. Therefore, $r(\rho) \in T_{np}(w)$. □

4.2 Construction of an Interval

We first recall the definition of the weak order from section 2.1.2. For $u, w \in W$ we say that

$$u \leq_R w \Leftrightarrow l(w) = l(u) + l(u^{-1}w)$$

where $l(w)$ denotes the least number k such that w can be written as a word of length k in the alphabet S. In [19], Reading defines a congruence Θ of the weak order and each equivalence class is an interval in the weak order. The Cambrian lattice is defined in [19] as the quotient of the weak order on W modulo Θ. In [21], Reading makes the connection between Cambrian lattices and a set of elements called the Coxeter sortable elements of W and states that the Cambrian lattice is isomorphic to the restriction of the weak order to Coxeter sortable elements.

In [9], there is a geometric characterisation of the congruence classes of Θ. Each congruence class has a region $F(w)$ associated to it for some $w \in NCP_c$ and consists of those elements whose permutahedron facets lie in $F(w)$. The region $F(w)$ is an associahedron facet which can be written as the union of some permutahedra facets. In this section, we want to construct the interval associated to $F(w)$ and calculate its cardinality. We will denote the interval associated to $F(w)$ by I_w and its size by $|I_w|$.

Recall that $T_n(w) = \{t \in T \mid \vec{x} \cdot \rho(t) \leq 0 \text{ for all } \vec{x} \in F(w)\}$. We can determine the elements of $T_n(w)$ from the matrix of dot products since $r(\rho_i) \in T_n(w)$ if and only if $\rho_i \cdot \mu_j \leq 0$ for all μ_j a corner of $F(w)$. We assume the elements of $T_n(w)$ are ordered consistently with the total order on the set of roots $\rho_1, \ldots, \rho_{nh/2}$. We also assume that the elements of $T_p(w) = \{t \in T \mid \vec{x} \cdot \rho(t) \geq 0 \text{ for all } \vec{x} \in F(w)\}$ are ordered.

The facet $F(e)$ is on the positive side of all hyperplanes. If we start at $F(e)$ and cross the walls associated to the reflections in $T_n(w)$ in order, we reach the facet $F(w)$. In particular, we reach the permutahedron facet in $F(w)$ with the minimal inversion set among permutahedron facets in $F(w)$. We will refer to this facet as the minimal facet of $F(w)$. If we start at $F(c)$ (the opposite chamber to C) and cross the reflections in $T_p(w)$ in reverse order, we reach the maximal facet of $F(w)$.

We can therefore associate to w an interval $I_w = [w_1, w_2]$ where w_1 is the minimal facet associated to $F(w)$ and w_2 is the maximal facet. We want to construct this interval and find the size of it in the weak order.

Example 4.2.1. Continuing with the example 2.3.6, let $w = (1\bar{3}\bar{2})$. Therefore $r(\delta_1) = (23)$ and $r(\delta_2) = (1\bar{2})$. We calculate $w' = cw^{-1} = [1342](1\bar{2}\bar{3}) = (24)[3]$

Therefore $r(\theta_1) = (24)$ and $r(\theta_2) = [3]$ and so

$$F(w) = (23)^- \cap (1\bar{2})^- \cap (24)^+ \cap [3]^+$$

Since the elements of $W(C_4)$ are identified with 4×4 signed permutation matrices, we can identify elements with signed permutations. Therefore, for example, the element w which takes the 4-tuple $(1, 2, 3, 4)$ in C to $(2, -4, -1, -3)$ in $w(C)$ can be written $2\bar{4}\bar{1}\bar{3}$.

We find that $T_n(w) = \{[1], (23), (1\bar{3}), (1\bar{2})\}$. Starting at the identity permutation we cross the walls in order:

$$1234 \xrightarrow{[1]} \bar{1}234 \xrightarrow{(23)} \bar{1}324 \xrightarrow{(1\bar{3})} 3\bar{1}24 \xrightarrow{(1\bar{2})} 32\bar{1}4$$

The set $T_p(w) = \{(24), [3], (3\bar{4}), [4], (2\bar{3}), (2\bar{4}), (13), [2], (14), (12), (34)\}$. Starting at the permutation associated to the longest element, we cross the walls in reverse order:

$$\bar{1}\bar{2}\bar{3}\bar{4} \xrightarrow{(34)} \bar{1}\bar{2}\bar{4}\bar{3} \xrightarrow{(12)} \bar{2}\bar{1}\bar{4}\bar{3} \xrightarrow{(14)} \bar{2}\bar{4}\bar{1}\bar{3} \xrightarrow{[2]} 2\bar{4}\bar{1}\bar{3} \xrightarrow{(13)} 2\bar{4}\bar{3}\bar{1} \xrightarrow{(2\bar{4})} \bar{4}2\bar{3}\bar{1}$$

$$\xrightarrow{(2\bar{3})} \bar{4}\bar{3}2\bar{1} \xrightarrow{[4]} 4\bar{3}2\bar{1} \xrightarrow{(3\bar{4})} \bar{3}42\bar{1} \xrightarrow{[3]} 342\bar{1} \xrightarrow{(24)} 324\bar{1}$$

Therefore, $I_w = [32\bar{1}4, 324\bar{1}]$.

Chapter 5

An Algorithm for computing the size of Intervals

Recall from section 4.2 that for each $w \in NCP_c$ there is an interval $I_w = [w_1, w_2]$ associated to $F(w)$, where w_1 is the minimal facet and w_2 is the maximal facet of $F(w)$. In this section, we give an algorithm for recursively calculating the size of the intervals denoted by $|I_w|$. It is possible to order the elements of NCP_c and to reflect each associahedron facet in a certain wall to obtain a union of earlier associahedron facets in the order.

Since reflection is an orthogonal transformation, reflecting an associahedron facet in any one of its walls gives a region with exactly the same number of permutahedron facets. We will see that reflection in the last hyperplane $\delta_k^{\perp}$ gives a region which is also a positive cone on μ vectors and corresponds to a factorisation of c.

5.1 Order on elements of NCP_c

To calculate $|I_w|$, we first put a total order on the elements of NCP_c so that we can calculate the numbers $|I_w|$ in this order. We do this so that when we reflect the facet, we obtain a union of facets whose size is already known. It is proven in [9] that each non-crossing partition gives an associahedron facet with vectors $\mu_{i_1}, \ldots, \mu_{i_n}$ as corners. We have described the construction of the facet in section 2.3.3. The total order on the set of μ vectors induces the order we put on NCP_c. We compare the corners of a facet reverse lexicographically. This means that we say $\mu_{i_1}, \ldots, \mu_{i_n} < \mu_{j_1}, \ldots, \mu_{j_n}$ if

$$\mu_{i_n} < \mu_{j_n}$$

$$\text{or } \mu_{i_n} = \mu_{j_n} \text{ but } \mu_{i_{n-1}} < \mu_{j_{n-1}}$$

$$\vdots$$

$$\text{or } \mu_{i_n} = \mu_{j_n}, \ldots, \mu_{i_2} = \mu_{j_2} \text{ but } \mu_{i_1} < \mu_{j_1}.$$

Example 5.1.1. As in example 2.1.3, let $W = W(A_3)$ and set $c = (1243)$. The table below lists the order of the non-crossing partitions w and the corners of $F(w)$ for each w.

Order	Corners	Element w
1	μ_1, μ_2, μ_3	e
2	μ_2, μ_3, μ_4	(12)
3	μ_1, μ_3, μ_5	(34)
4	μ_3, μ_4, μ_5	(12)(34)
5	μ_4, μ_5, μ_6	(14)
6	μ_1, μ_5, μ_7	(24)
7	μ_5, μ_6, μ_7	(124)
8	μ_2, μ_4, μ_8	(13)
9	μ_4, μ_6, μ_8	(143)
10	μ_6, μ_7, μ_8	(13)(24)
11	μ_1, μ_2, μ_9	(23)
12	μ_1, μ_7, μ_9	(243)
13	μ_2, μ_8, μ_9	(123)
14	μ_7, μ_8, μ_9	(1243)

Once the non-crossing partitions are ordered, we calculate $|I_w|$ in this order. The first facet whose size is calculated is always the facet corresponding to the identity element of W since we know that $|F(e)| = 1$ for all W. $F(e)$ is the facet with vertices $\mu_1, \ldots, \mu_n$. The facet whose size is last calculated is always the facet corresponding to c. $F(c)$ is the antipodal facet to $F(e)$. Its vertices are $\mu_{nh/2}, \ldots, \mu_{nh/2+n}$ and its size is also 1.

5.2 Algorithm

The associahedron facet $F(w)$ can be expressed as an interval I_w in the weak order. By calculating how many permutahedron facets are in the associahedron facet $F(w)$, we are calculating the size of this interval.

Let $w \in NCP_c$ be a length k element with simple system $\{\delta_1, \ldots, \delta_k\}$. We denote by $\{\theta_1, \ldots, \theta_{n-k}\}$ the simple system for the non-crossing partition $w' = cw^{-1}$. Recall from section 2.3.3 that the set $\{\epsilon_1, \ldots, \epsilon_{n-k}\}$ is the vertex set of the first facet of $X(w')$ in lexicographic order and the set $\{\eta_{n-k+1}, \ldots, \eta_n\}$ is the vertex set of the last facet of $X(w)$ in lexicographic order. We will consider the following particular factorisation of c

$$c = r(c[\eta_n]) \ldots r(c[\eta_{n-k+1}])r(\epsilon_{n-k}) \ldots r(\epsilon_1).$$

From [9], we know that since $F(w)$ is a simplicial cone, that is a positive cone on a spherical simplex, all the vertices of $F(w)$ are on $\delta_k^\perp$ except the vertex $\mu(\delta_{n+k})$ which lies on the negative side of $\delta_k^\perp$.

Since δ_k is the last element of the inversion set of w, we reflect $F(w)$ in $\delta_k^\perp$. As a result of the fact that $r(\delta_k)\eta_n = c\eta_n$, we obtain a new factorisation of c when we reflect in $\delta_k^\perp$ where

$$c = r(c\eta_{n-1}) \ldots r(c\eta_{n-k+1})r(\epsilon_{n-k}) \ldots r(\epsilon_1)r(\eta_n).$$

This is the positive cone on the corresponding μ vectors

$$\mu(c\eta_{n-1}), \ldots, \mu(c\eta_{n-k+1}), \mu(\epsilon_{n-k}), \ldots, \mu(\epsilon_1), \mu(\eta_n)$$

If the reflections $r(c\eta_{n-1}), \ldots, r(c\eta_{n-k+1}), r(\epsilon_{n-k}), \ldots, r(\epsilon_1), r(\eta_n)$ are in decreasing order relative to the total induced order on the set of reflections, we know that $r(\delta_k)F(w)$ consists of one associahedron facet by the definition of an associahedron facet from section 2.3.2. Since the elements of NCP_c are ordered, the size of this associahedron facet is already known.

If the reflections are not in decreasing order, the sequence must be increasing from $r(\epsilon_1)$ to $r(\eta_n)$. The roots ϵ_1 and η_n are both positive roots. From theorem 5.4 of [12], we know

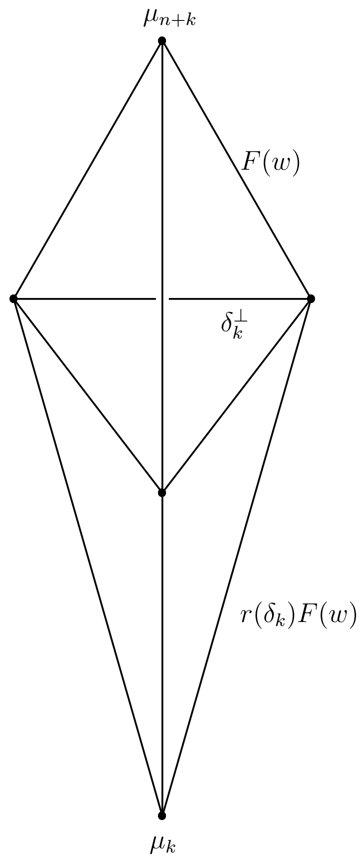


Figure 5.2.1: Reflecting $F(w)$

that they form a simple system for some length 2 element $\sigma \preceq c$. Consider the ordered set $\Gamma_\sigma = \{\tau_1, \dots, \tau_a\}$ of roots that precede σ. Since $r(\epsilon_1)$ and $r(\eta_n)$ form a simple system for σ, we know that $\tau_1 = \epsilon_1$ and $\tau_a = \eta_n$. We can therefore write $r(\delta_k)F(w)$, as a union of $(a-1)$ positive cones. See figure 5.2.4 where $a = 4$.

We denote these cones by $C_1, \dots, C_{a-1}$ where

$$C_1 = \mu(c\eta_{n-1}), \dots, \mu(c\eta_{n-k+1}), \mu(\epsilon_{n-k}), \dots, \mu(\epsilon_2), \mu(\tau_2), \mu(\epsilon_1)$$

$$C_2 = \mu(c\eta_{n-1}), \dots, \mu(c\eta_{n-k+1}), \mu(\epsilon_{n-k}), \dots, \mu(\epsilon_2), \mu(\tau_3), \mu(\tau_2)$$

$$\vdots$$

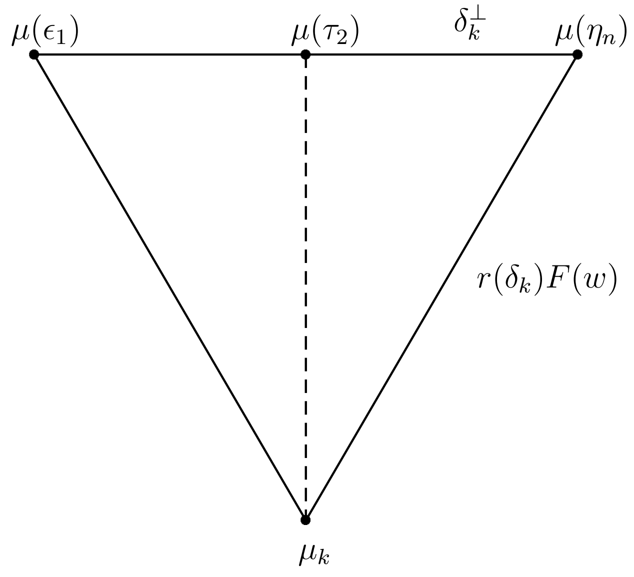


Figure 5.2.2: Increase from $r(\epsilon_1)$ to $r(\eta_n)$ causes a split

$$C_{a-1} = \mu(c\eta_{n-1}), \ldots, \mu(c\eta_{n-k+1}), \mu(\epsilon_{n-k}), \ldots, \mu(\epsilon_2), \mu(\eta_n), \mu(\tau_{a-1}).$$

We examine each of these facets for increasing sequences between $r(\epsilon_2)$ and $r(\tau_i)$ for $i = 2, \ldots, a$. If an increase occurs, we split the positive cone on $r(\epsilon_2)$ and $r(\tau_i)$, for the appropriate i, into a union of positive cones. Assume there is an increase between $r(\epsilon_2)$ and $r(\tau_2)$. We know that they form a simple system for some length 2 element $\sigma' \preceq c$. Consider the ordered set $\Gamma_{\sigma'} = \{\omega_1, \ldots, \omega_b\}$ of roots that precede σ'. Since $r(\epsilon_2)$ and $r(\tau_2)$ form a simple system for σ', we know that $\omega_1 = \epsilon_2$ and $\omega_b = \tau_2$. Therefore the positive cone C_1 on vectors

$$\mu(c[\eta_{n-1}]), \ldots, \mu(c[\eta_{n-k+1}]), \mu(\epsilon_{n-k}), \ldots, \mu(\epsilon_2), \mu(\tau_2), \mu(\epsilon_1)$$

splits further into a union of positive cones $C'_1, \ldots, C'_{b-1}$ where

$$C'_1 = \mu(c[\eta_{n-1}]), \ldots, \mu(c[\eta_{n-k+1}]), \mu(\epsilon_{n-k}), \ldots, \mu(\epsilon_3), \mu(\omega_2), \mu(\epsilon_2), \mu(\epsilon_1),$$

$$C'_2 = \mu(c[\eta_{n-1}]), \ldots, \mu(c[\eta_{n-k+1}]), \mu(\epsilon_{n-k},) \ldots, \mu(\epsilon_3), \mu(\omega_3), \mu(\omega_2), \mu(\epsilon_1)$$

$$\vdots$$

$$C'_{b-1} = \mu(c[\eta_{n-1}]), \ldots, \mu(c[\eta_{n-k+1}]), \mu(\epsilon_{n-k}), \ldots, \mu(\epsilon_3), \mu(\tau_2), \mu(\omega_{b-1}), \mu(\epsilon_1).$$

We examine each of these facets for increasing sequences between $r(\epsilon_3)$ and $r(\omega_i)$ for $i = 2, \ldots, b$ and between $r(\epsilon_1)$ and $r(\omega_j)$ for $j = 2, \ldots, b-1$. If an increase occurs, we split the positive cone and repeat the process where necessary. We continue this until we obtain a union of expressions, all of which are in decreasing order. We therefore have written the facet as a union of associahedron facets.

Since we have calculated the size the facets in order, we have already calculated the size of the intervals in the image of $F(w)$ under δ_k. We can therefore now determine $|I_w|$.

Example 5.2.1. Consider $W = W(A_3)$, $c = (1243)$. Figure 5.2.3 shows the associahedron facets outlined in bold. Consider $F(w)$ with corners μ_2, μ_8 and μ_9. Both μ_2 and μ_8 are on $\rho_6^{\perp}$. When we reflect $F(w)$ in $\rho_6^{\perp}$, we get the facet with corners μ_2, μ_6 and μ_8. This is made up of two associahedron facets. The facet with corners μ_2, μ_4 and μ_8 which is made up of two permutahedron facets and the facet with corners μ_4, μ_6 and μ_8 which consists of one permutahedron facet.

Example 5.2.2. Again, we continue the example 2.3.6 where $W = W(C_4), c = [1342]$. For convenience we recall the order of the reflections.

$$[1], (23), (1\bar{3}), (24)$$

$$[3], (1\bar{4}), (3\bar{4}), (1\bar{2})$$

$$[4], (2\bar{3}), (2\bar{4}), (13)$$

$$[2], (14), (12), (34)$$

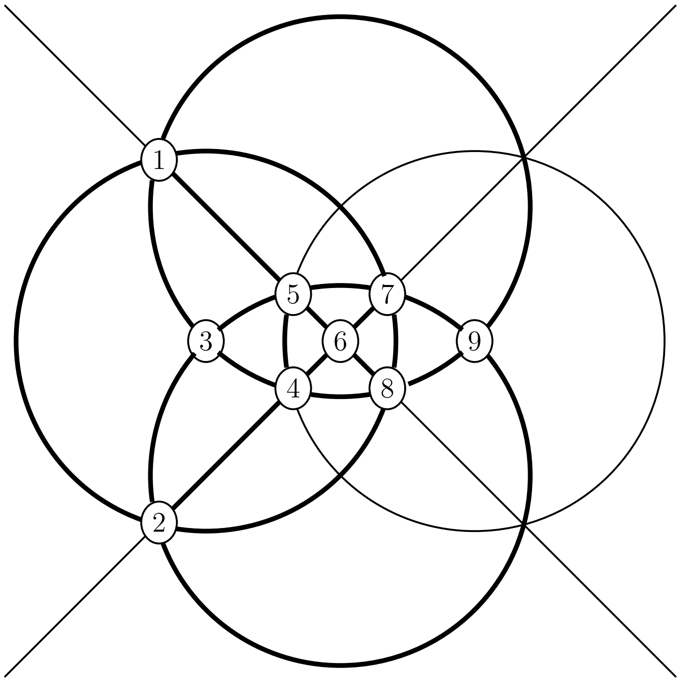


Figure 5.2.3: Example of reflecting an associahedron facet

$$-[1], -(23), -(1\bar{3}), -(24)$$

Let $w = (14)$ then $w' = cw^{-1} = [1342](14) = [12](34)$. We want to determine the vertex set of the first facet of $X(w')$ and the last facet of $X(w)$. Beginning with the first facet of $X(w')$, we find the simple system $\{\theta_1, \theta_2, \theta_3\}$ for w' to be $\{\rho[1], \rho(23), \rho(34)\}$. We want to find $\{r(\epsilon_1), \ldots, r(\epsilon_3)\}$ where $\epsilon_1 = \theta_1$, $\epsilon_2 = r(\theta_1)\theta_2$ and $\epsilon_3 = r(\theta_1)r(\theta_2)\theta_3$. Therefore

$$r(\epsilon_1) = [1], \;\; r(\epsilon_2) = (1\bar{2}), \;\; r(\epsilon_3) = (34).$$

We now find the vertex set of the last facet of $X(w)$. We find that $r(\eta_1) = (14)$. We will require the fact that $r(c[\eta_1]) = -(23)$.

Therefore $F(w)$ is a positive cone on the vertices

$$\mu(-(23)), \mu(34), \mu(1\bar{2}), \mu[1].$$

When we reflect $F(w)$ in $[\rho(14)]^{\perp}$ we get the positive cone on μ vectors

$$\mu(34), \mu(1\bar{2}), \mu[1], \mu(14).$$

This sequence increases between reflections [1] and (14) so the cone splits into a union of positive cones. In order to do this, we must consider the set Γ_σ for the element σ whose simple system is formed by [1] and (14). We find that the ordered set $\Gamma_\sigma = \{\rho[1], \rho(1\bar{4}), \rho[4], \rho(14)\}$. Therefore the cone splits into three positive cones on the following vectors

$$\mu(34), \mu(1\bar{2}), \mu(14), \mu[4] \;\cup\; \mu(34), \mu(1\bar{2}), \mu[4], \mu(1\bar{4}) \;\cup\; \mu(34), \mu(1\bar{2}), \mu(1\bar{4}), \mu[1]$$

We examine these sequences and find an increase in the first one between $(1\bar{2})$ and (14) and in the second one between $(1\bar{2})$ and [4]. The increase in the first sequence occurs between reflections $(1\bar{2})$ and (14). We know that these reflections form a simple system for some element. The reflections that precede that element are $(1\bar{2}), (2\bar{4}), (14)$ in that order. The cone on the corresponding μ vectors can therefore be rewritten as the union of two positive cones. Dealing with the increase in the second sequence, we find the simple system for the element is $\{\rho(1\bar{2}), \rho(14)\}$. Since the elements commute, the cone does not split, but we rewrite its vertices in order. Therefore we get the following union of cones

$$\mu(34), \mu(14), \mu(2\bar{4}), \mu[4] \;\cup\; \mu(34), \mu(2\bar{4}), \mu(1\bar{2}), \mu[4]$$

$$\cup\; \mu(34), \mu[4], \mu(1\bar{2}), \mu(1\bar{4}) \;\cup\; \mu(34), \mu(1\bar{2}), \mu(1\bar{4}), \mu[1]$$

The final increase can be seen in the second facet between $(1\bar{2})$ and [4]. Therefore, $F(w)$ can be written as the following union of positive cones:

$$\mu(34), \mu(14), \mu(2\bar{4}), \mu[4] \;\cup\; \mu(34), \mu(2\bar{4}), \mu[4], \mu(1\bar{2})$$

$$\cup\; \mu(34), \mu[4], \mu(1\bar{2}), \mu(1\bar{4}) \;\cup\; \mu(34), \mu(1\bar{2}), \mu(1\bar{4}), \mu[1]$$

Therefore, $F(w)$ consists of four associahedron facets. This can be seen in figure 5.2.4.

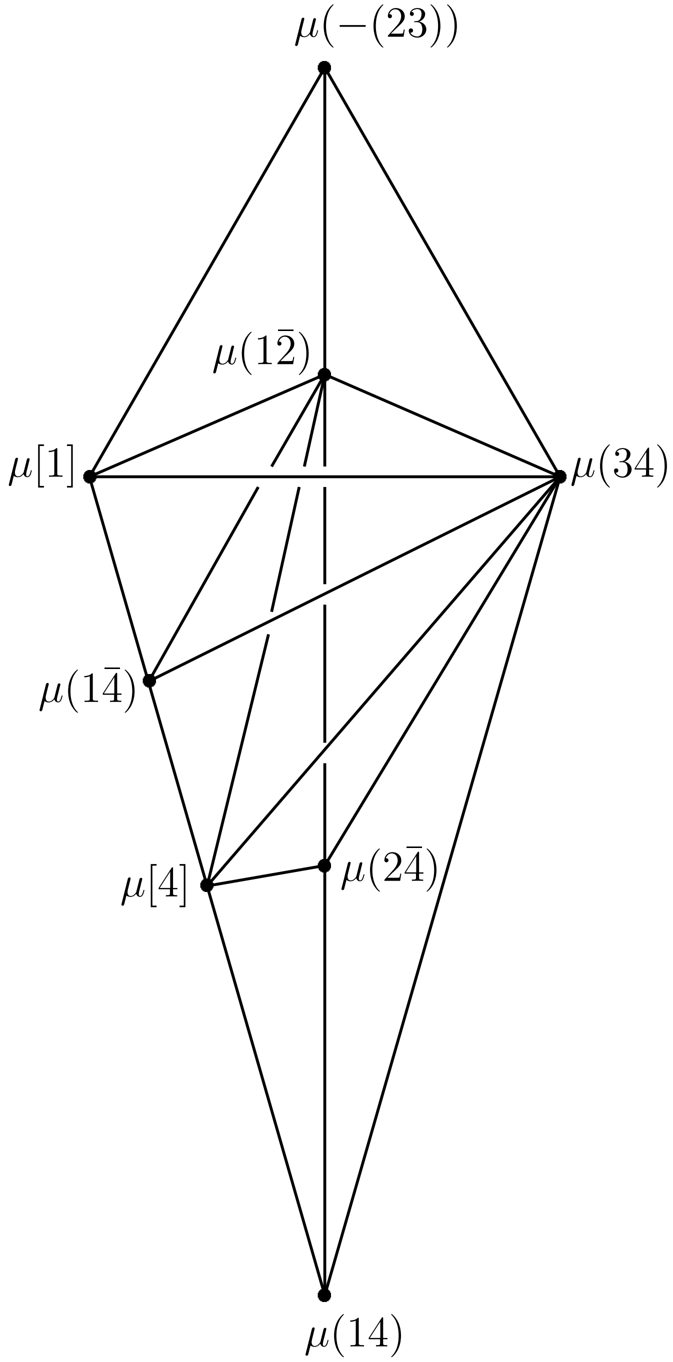


Figure 5.2.4: Associahedron facets for $W = W(A_3)$

Example 5.2.3. Below is the table from example 5.1.1 above with the numbers $|I_w|$ included for each w.

Order	Corners	Element w	Reflected Corners	$\|I_w\|$
1	μ_1, μ_2, μ_3	e		1
2	μ_2, μ_3, μ_4	(12)	μ_1, μ_2, μ_3	1
3	μ_1, μ_3, μ_5	(34)	μ_1, μ_2, μ_3	1
4	μ_3, μ_4, μ_5	(12)(34)	μ_2, μ_3, μ_4	1
5	μ_4, μ_5, μ_6	(14)	μ_3, μ_4, μ_5	1
6	μ_1, μ_5, μ_7	(24)	μ_1, μ_4, μ_5	2
7	μ_5, μ_6, μ_7	(124)	μ_4, μ_5, μ_6	1
8	μ_2, μ_4, μ_8	(13)	μ_2, μ_4, μ_5	2
9	μ_4, μ_6, μ_8	(143)	μ_4, μ_5, μ_6	1
10	μ_6, μ_7, μ_8	(13)(24)	μ_5, μ_6, μ_7	1
11	μ_1, μ_2, μ_9	(23)	μ_1, μ_2, μ_6	5
12	μ_1, μ_7, μ_9	(243)	μ_1, μ_6, μ_7	3
13	μ_2, μ_8, μ_9	(123)	μ_2, μ_6, μ_8	3
14	μ_7, μ_8, μ_9	(1243)	μ_6, μ_7, μ_8	1

Chapter 6

Homology of Non-Crossing Partition Lattices

In this chapter we construct a geometric basis for the homology of the non-crossing partition lattice for any finite, real reflection group W using the geometric model $X(c)$ from [12] which is described in detail in section 2.3. We construct the basis by defining a homotopy equivalence between the proper part of the non-crossing partition lattice and the $(n-2)$-skeleton of $X(c)$.

We relate this to the basis for the homology of the corresponding intersection lattice introduced by Björner and Wachs in [6]. We exhibit an explicit embedding of the homology of the non-crossing partition lattice in the homology of the intersection lattice, using the general construction of a generic affine hyperplane.

6.1 Homotopy Equivalence

The top-dimensional simplices of $X(c)$ are $(n-1)$-dimensional and therefore we consider the barycentric subdivision of the $(n-2)$-skeleton of $X(c)$, denoted by $sd(X^{n-2}(c))$. An $(n-2)$-simplex of $X(c)$ consists of roots $\{\tau_1, \ldots, \tau_{n-1}\}$ with the property that $\ell(r(\tau_1)\ldots r(\tau_{n-1})c) = 1$. Recall that we denote by $P(X^{n-2}(c))$ the poset of simplices of $X^{n-2}(c)$ ordered by inclusion. We begin with the observation that every simplex in $X(c)$ defines a non-crossing partition. Recall from Lemma 4.8 of [12] that if $\{\tau_1, \ldots, \tau_k\}$ is the ordered vertex set of a simplex σ of $X(c)$ then

$$\ell(r(\tau_1)\ldots r(\tau_k)c) = n - k.$$

In particular, $r(\tau_k)\ldots r(\tau_1)$ is a non-crossing partition of length k.

Definition 6.1.1. We define $f : P(X(c)) \to NCP_c$ by

$$f(\sigma) = r(\tau_k)\ldots r(\tau_1)$$

where σ is the simplex of $X(c)$ with ordered vertex set $\{\tau_1, \ldots, \tau_k\}$.

Example 6.1.2. Let $W = W(A_4)$ and $c = (12)(34)(23)(45) = (12453)$. The total order on the set of roots induces the following order on reflections

$$(12), (34), (14), (35), (24), (15), (25), (13), (45), (23).$$

Let $\sigma = \{(12), (15), (13)\} \in P(X(c))$. Then $f(\sigma) = (13)(15)(12) = (1253)$.

Lemma 6.1.3. *The map f is a poset map.*

Proof. Let $\sigma = \{\tau_1, \dots, \tau_k\} \in P(X(c))$ and let $\theta \preceq \sigma$. Therefore, $\theta = \{\tau_{i_1}, \dots, \tau_{i_l}\}$ for some $1 \leq i_1 < \cdots < i_l \leq k$. Note that since the set of reflections is closed under conjugation we have $r(\rho)r(\tau) = r(\tau)r(\rho')$, where $\rho' = r(\tau)[\rho]$ for any roots ρ and τ. We can use this equality to conjugate the reflections in $f(\theta)$ to the beginning of the expression for $f(\sigma)$. Therefore $f(\theta) = r(\tau_{i_l}) \dots r(\tau_{i_1}) \preceq r(\tau_k) \dots r(\tau_1) = f(\sigma)$. □

Example 6.1.4. Continuing from example 6.1.2 above, let $W = W(A_4)$, $c = (12)(34)(23)(45) = (12453)$ and $\sigma = \{(12), (15), (13)\}$. If $\theta = \{(12), (13)\}$ then it satisfies $\theta \preceq \sigma$. We find that $f(\theta) = (13)(12) = (123)$.

$$\begin{aligned} f(\sigma) &= (1253) \\ &= (13)(15)(12) \\ &= (13)(12)(25) \end{aligned}$$

Therefore, $f(\theta) \preceq f(\sigma)$.

By definition of f, $f^{-1}(c)$ is the set of maximal elements in $P(X(c))$ and $f^{-1}(e)$ is empty. We therefore can consider the induced map,

$$\hat{f} : \hat{P}(X(c)) \to \overline{NCP_c}$$

where $\hat{P}(X(c))$ is the poset obtained from $P(X(c))$ by removing the maximal elements. Note that $\hat{P}(X(c))$ is the poset of simplices of the $(n-2)$-skeleton of $X(c)$.

Theorem 6.1.5. *The map $\hat{f}$ is a homotopy equivalence.*

Proof. Since f is a poset map by lemma 6.1.3, $\hat{f} : \hat{P}(X(c)) \to \overline{NCP_c}$ is a poset map. We intend to apply Quillen's Fibre Lemma [18] to this map $\hat{f}$. Following notation from [18], we define the subposet $\hat{f}_{\preceq w}$ of $\hat{P}(X(c))$ for $w \in \overline{NCP_c}$ by

$$\hat{f}_{\preceq w} = \{\sigma \in \hat{P}(X(c)) : \hat{f}(\sigma) \preceq w\}.$$

We claim that $\hat{f}_{\preceq w} = P(X(w))$. Assuming the claim, the theorem follows from Proposition 1.6 of [18] if $|P(X(w))|$ is contractible. It is shown in Corollary 7.7 of [12] that $X(w)$ is contractible for all $w \in NCP_c$. Since $X(w)$ and $sd(X(w))$ are homeomorphic (by [18] for example) and $|P(X(w))| = sd(X(w))$, it follows that $|P(X(w))|$ is contractible.

To prove the claim we first show that $\hat{f}_{\preceq w} \subseteq P(X(w))$. If $\sigma \in \hat{f}_{\preceq w}$, then $e \prec \hat{f}(\sigma) \preceq w \prec c$ by definition of $\hat{f}_{\preceq w}$. By applying lemma 6.1.3 to the reflections corresponding to vertices of σ, it follows that $\sigma \in P(X(w))$. To show that $P(X(w)) \subseteq \hat{f}_{\preceq w}$, let $\sigma \in P(X(w))$. If σ has ordered vertex set $\{\tau_1, \ldots, \tau_k\}$, then $r(\tau_i) \preceq w$ for each i by definition of $X(w)$. Then $\hat{f}(\sigma) = r(\tau_k) \ldots r(\tau_1) \preceq c$. From equation 3.4 of [12], we know that since $\hat{f}(\sigma) \preceq c$, $w \preceq c$ and each $r(\tau_i) \preceq w$ then $\hat{f}(\sigma) = r(\tau_k) \ldots r(\tau_1) \preceq w$. Therefore, $\sigma \in \hat{f}_{\preceq w}$. □

Example 6.1.6. Let $W = W(A_3), c = (12)(34)(24) = (1234)$. Then, the figure 6.1.1 illustrates that $|P(X^{n-2}(w))|$ for $w = (123)$ is homotopy equivalent to the subposet of $|(\overline{NCP_c})|$ that consists of the elements in W that precede w.

Corollary 6.1.7. $|\overline{NCP_c}|$ *has the homotopy type of a wedge of spheres, one for each facet of* $X(c)$.

Proof. The map $\hat{f}$ induces a homotopy equivalence $|\hat{f}| : |\hat{P}(X(c))| \to |\overline{NCP_c}|$. The simplicial complex $X(c)$ is a spherical complex that is convex and hence contractible (Corollary

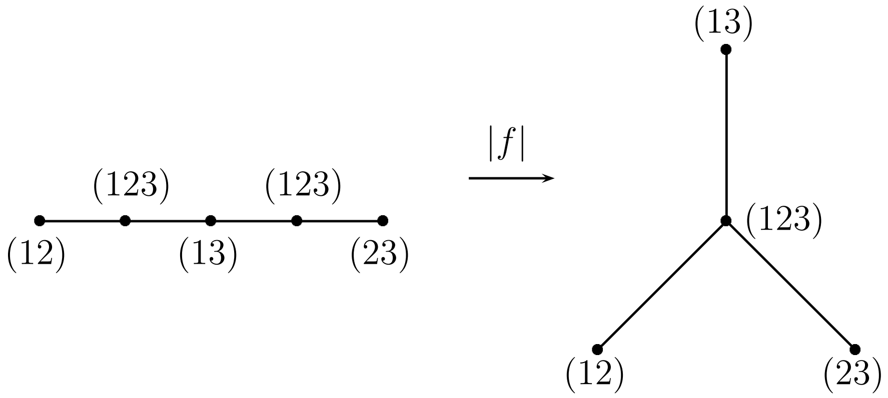


Figure 6.1.1: Homotopy Equivalence

7.7 of [12]). Let Y denote the subspace of $X(c)$ obtained by removing a point from the interior of each facet. Then $|\hat{P}(X(c))|$ is a deformation retract of Y and therefore has the homotopy type of a wedge of $(n-2)$ spheres. The number of such spheres is equal to the number of facets of $X(c)$. $\square$

Note 6.1.8. This is a more direct proof of the result in corollary 4.4 of [2] where it is proved that for a crystallographic root system, the Möbius number of NCP_c is equal to $(-1)^n$ times the number of maximal simplices of $|NCP_c|$, which can also be viewed as positive clusters corresponding to the root system.

Note 6.1.9. After this work was completed, we became aware of [3] which proves a more general version of theorem 6.1.5.

6.2 Homology Embedding

We now briefly review the results in [6] where geometric bases for the homology of intersection lattices are constructed. Let $\mathcal{A}$ be a central and essential hyperplane arrangement in $\mathbf{R}^n$. We refer to the connected components of $\mathbf{R}^n \setminus \mathcal{A}$ as regions. Recall that the intersection lattice $L_{\mathcal{A}}$ of $\mathcal{A}$ denotes the set of intersections of subfamilies of $\mathcal{A}$, partially ordered by reverse inclusion.

Homology generators are found by using a non-zero vector $\mathbf{v}$ such that the hyperplane $H_{\mathbf{v}}$, which is through $\mathbf{v}$ and normal to $\mathbf{v}$, is generic. This means that $\dim(H_{\mathbf{v}} \cap X) = \dim(X) - 1$ for all $X \in L_{\mathcal{A}}$. In Theorem 4.2 of [6], it is proven that the collection of cycles g_R corresponding to regions R such that $R \cap H$ is nonempty and bounded, form a basis of $\tilde{H}_{d-2}(\bar{L}_{\mathcal{A}})$ where H is an affine hyperplane, generic with respect to $\mathcal{A}$. Lemma 4.3 of [6] states that for each region R, the affine slice $R \cap H_{\mathbf{v}}$ is nonempty and bounded if and only if $\mathbf{v} \cdot \mathbf{x} > 0$ for all $\mathbf{x} \in R$. At this point, we refer the reader to figure 6.2.1 which illustrates this basis for $W = W(C_3)$. The figure shows the stereographic projection of the open hemisphere satisfying $\mathbf{v} \cdot \mathbf{x} > 0$ and is combinatorially equivalent to the projection onto $H_{\mathbf{v}}$. The actual projection onto $H_{\mathbf{v}}$ is shown in figure 6.2.2. Each region in the figure which is non-empty and bounded contributes a generator to the basis for the homology of the intersection lattice.

The fact that the hyperplane $H_{\mathbf{v}}$ is generic is equivalent to the fact that $0 \notin H_{\mathbf{v}}$ and $H \cap X \neq \emptyset$ for all 1-dimensional subspaces $X \in L_{\mathcal{A}}$ (Section 4 of [6]). It is therefore sufficient to check that $H_{\mathbf{v}}$ is generic with respect to the set of rays. In section 6.2.1, we describe for any W, the general construction of a vector $\mathbf{v}$ with $H_{\mathbf{v}}$ generic. In section 6.2.2, we use the construction of $\mathbf{v}$ to explicitly embed the homology of the non-crossing partition lattice in the homology of the intersection lattice.

6.2.1 Construction of a generic vector for general finite reflection groups

Let $\{\tau_1, \ldots, \tau_n\}$ be an arbitrary set of linearly independant roots. Since the number of roots is finite and rays occur at the intersection of hyperplanes, it follows that the number of unit rays is finite. Hence, the set $\{\mathbf{r} \cdot \rho \mid \mathbf{r} \text{ a unit ray}, \rho \text{ a root}\}$ is finite and

$$\lambda = \min\{|\mathbf{r} \cdot \rho| : \mathbf{r} \text{ a unit ray}, \rho \text{ a root and } \mathbf{r} \cdot \rho \neq 0\}$$

is a well defined, positive, real number. It will be convenient to use the auxiliary quantity $a = 1 + 1/\lambda$.

Proposition 6.2.1. *Let $\mathbf{v} = \tau_1 + a\tau_2 + a^2\tau_3 + \cdots + a^{n-1}\tau_n$ and $\mathbf{r}$ be a unit length ray.*

Then $|\mathbf{r}\cdot\mathbf{v}| \geq \lambda$. *In particular,* $H_{\mathbf{v}}$ *is generic.*

Proof. Let $\mathbf{r}$ denote a unit length ray. Since $\{\tau_1, \dots, \tau_n\}$ is a linearly independant set, $\mathbf{r}\cdot\tau_k \neq 0$ for some τ_k. Let k be the index with $1 \leq k \leq n$ satisfying

$$\mathbf{r}\cdot\tau_k \neq 0, \text{ and } \mathbf{r}\cdot\tau_{k+1} = 0, \dots, \mathbf{r}\cdot\tau_n = 0.$$

By replacing $\mathbf{r}$ by $-\mathbf{r}$ if necessary, we can assume that $\mathbf{r}\cdot\tau_k > 0$ and hence $\mathbf{r}\cdot\tau_k \geq \lambda$ by the definition of λ. We now compute $\mathbf{r}\cdot\mathbf{v}$.

$$\begin{aligned}
\mathbf{r}\cdot\mathbf{v} &= \mathbf{r}\cdot(\tau_1 + a\tau_2 + a^2\tau_3 + \cdots + a^{n-1}\tau_n) \\
&= \mathbf{r}\cdot\tau_1 + a(\mathbf{r}\cdot\tau_2) + a^2(\mathbf{r}\cdot\tau_3) + \cdots + a^{n-1}(\mathbf{r}\cdot\tau_n) \\
&= \mathbf{r}\cdot\tau_1 + a(\mathbf{r}\cdot\tau_2) + a^2(\mathbf{r}\cdot\tau_3) + \cdots + a^{k-1}(\mathbf{r}\cdot\tau_k) + 0 \\
&\geq -1 + a(-1) + a^2(-1) + \cdots + a^{k-2}(-1) + a^{k-1}(\lambda) \\
&= -1(1 + a + a^2 + \cdots + a^{k-2}) + a^{k-1}(\lambda) \\
&= \lambda.
\end{aligned}$$

The last equality follows from the formula for the sum of a geometric series and the fact that $\lambda = 1/(a-1)$. □

6.2.2 Specialising the generic hyperplane

In order to relate the homology basis for non-crossing partition lattices to the homology basis for the corresponding intersection lattice, we apply the operator $\mu = 2(I-c)^{-1}$ from [12] to $X(c)$ to obtain the complex which we call $\mu(X(c))$ and which is the positive part of the complex $\mu(AX(c))$ studied in [9] and reviewed in section 2.3. Recall that the complex $\mu(X(c))$ has vertices $\mu(\rho_1), \dots, \mu(\rho_{nh/2})$ and a simplex on $\mu(\rho_{i_1}), \dots, \mu(\rho_{i_k})$ if

$$\rho_1 \leq \rho_{i_1} < \cdots < \rho_{i_k} \leq \rho_{nh/2} \text{ and } \ell(r(\rho_{i_1})\dots r(\rho_{i_k})c) = n-k.$$

The walls of the facets of $\mu(AX(c))$ are hyperplanes. Since regions considered in [6] are bounded by reflection hyperplanes, this provides the connection between the two and explains why we use $\mu(X(c))$ instead of $X(c)$ for comparing the two homology bases.

We now apply Proposition 6.2.1 to the case where $\tau_1, \ldots, \tau_n$ are the last n positive roots. Thus we set $\tau_i = \rho_{nh/2-n+i}$. Since $\{\tau_1, \ldots, \tau_n\}$ is a set of consecutive roots and $r(\tau_n) \ldots r(\tau_1) = c$, the set $\{\tau_1, \ldots, \tau_n\}$ is linearly independent by note 3.1 of [12].

Proposition 6.2.2. *For $\tau_i = \rho_{nh/2-n+i}$ and*

$$\mathbf{v} = \tau_1 + a\tau_2 + a^2\tau_3 + \cdots + a^{n-1}\tau_n,$$

$\mu(\rho_i) \cdot \mathbf{v} > 0$ *for all* $1 \leq i \leq nh/2$.

Proof. Recall from proposition 4.6 of [12] and section 2.3 that the following properties hold.

$$\mu(\rho_i) \cdot \rho_j \geq 0 \text{ for } 1 \leq i \leq j \leq nh/2.$$

$$\mu(\rho_{i+t}) \cdot \rho_i = 0 \text{ for } 1 \leq t \leq n-1 \text{ and for all } i.$$

Since $\tau_1, \ldots, \tau_n$ are the last n positive roots, it follows that $\mu(\rho_i) \cdot \tau_j \geq 0$. Furthermore for each ρ_i, there is at least one τ_j with $\mu(\rho_i) \cdot \tau_j > 0$ by linear independence of $\{\tau_1, \ldots, \tau_n\}$. Since all the coefficients of $\mathbf{v}$ are strictly positive, $\mu(\rho_i) \cdot \mathbf{v} > 0$. □

Proposition 6.2.3. *The projection of $\mu(X(c))$ onto the affine hyperplane $H_{\mathbf{v}}$ where $\mathbf{v}$ is as in proposition 6.2.2 induces an embedding of the homology of the non-crossing partition lattice into the homology of the corresponding intersection lattice.*

Proof. Recall from section 6.1 that homology generators for the non-crossing partition lattice are identified with the boundaries of facets of $X(c)$ and hence with boundaries of facets of $\mu(X(c))$. On the other hand, we can use the generic vector $\mathbf{v}$ to identify homology generators of the intersection lattice with cycles g_R corresponding to regions R such that $R \cap H$ is nonempty and bounded. From [9], the boundary of each facet of $\mu(X(c))$ is a union of pieces of reflection hyperplanes. It follows that vertices $\mu(\rho_i)$ for $1 \leq i \leq nh/2$ are rays and each facet of $\mu(X(c))$ projects to a union of affine slices of the form $R \cap H$. Furthermore, the projection of distinct $\mu(X(c))$ facets have disjoint interiors.

We denote the projection map by $p : \mu(X(c)) \to H$ and by p_* the induced map from the homology of the non-crossing partition lattice to the homology of the intersection lattice. Then p_* takes the homology generator g'_F corresponding to a facet F of $\mu(X(c))$ to the sum of the intersection lattice homology generators g_R corresponding to the affine slices $R \cap H$ contained in $p(F)$. That is $p_*(g'_F) = \Sigma b_R g_R$ where $b_R = 1$ if $R \cap H$ is contained in $p(F)$ and 0 otherwise.

To establish injectivity of p_*, we observe that $p_*(\Sigma a_F g'_F) = \Sigma c_R g_R$ where $c_R = 0$ if R is not contained in $p(\mu(X(c)))$ and $c_R = a_F$ if F is the unique facet satisfying $R \subseteq p(F)$. Thus $\Sigma a_F g'_F$ is an element of $\mathrm{Ker}(p_*)$ if and only if $a_F = 0$ for all F. $\square$

Example 6.2.4. For $W = W(C_3)$ and for appropriate choices of fundamental domain and simple system, the relevant regions are shown in figure 6.2.1 where i represents $\mu(\rho_i)$.

The basis for homology of the intersection lattice is formed by cycles corresponding to regions in figure 6.2.1 which are non-empty and bounded. For this example, there are 15 such regions.

Homology generators for the non-crossing partition lattice are identified with the boundaries of facets of $\mu(X(c))$, of which there are 10 in this example. These facets are outlined in bold. Note that the facet with corners $\mu(\rho_2), \mu(\rho_4), \mu(\rho_8)$ is a union of two facets of the

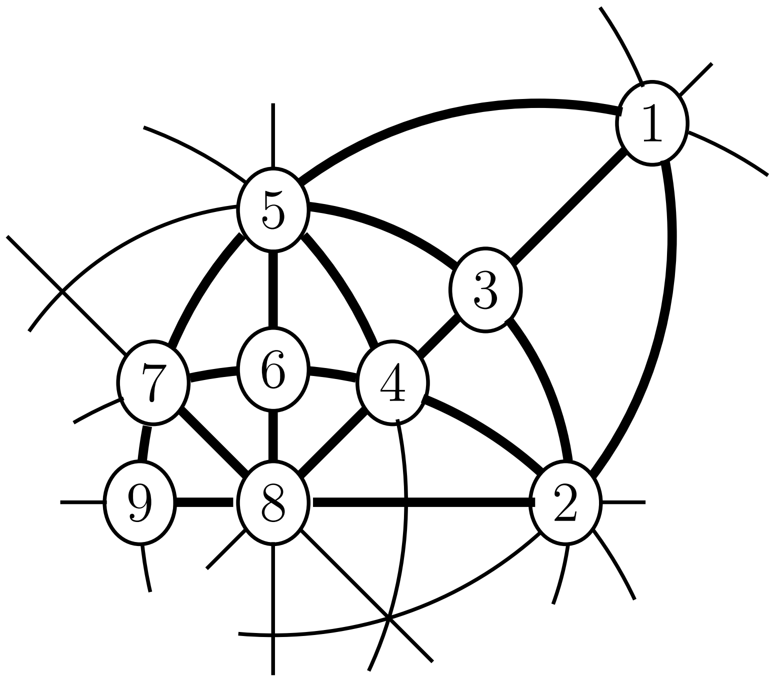


Figure 6.2.1: Homology

Coxeter complex and therefore the embedding maps the homology element associated to this facet to the sum of the two corresponding generators in the homology of the intersection lattice.

The corresponding affine projection of $\mu(X(c))$ is shown in bold in figure 6.2.2.

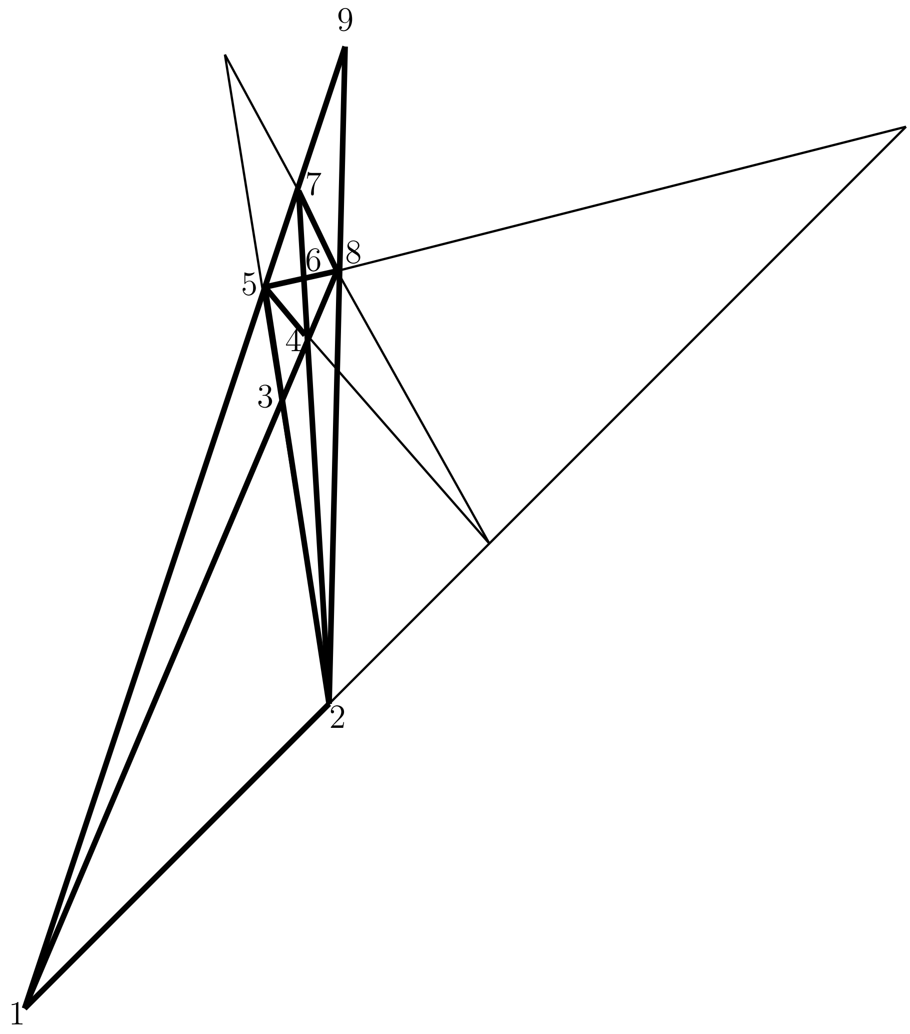


Figure 6.2.2: Homology

Chapter 7

Conclusion

We conclude the thesis by considering some future directions of our results. Firstly, we would like to further investigate the algorithm described in chapter 5. Specifically, there is the question of which intervals arise in this manner. Since the computation of interval sizes in finite reflection groups is difficult in general, this seems worthwhile exploring.

We would also like to consider the link between the minimal elements of our interval and Coxeter sortable elements and show that they are equivalent. We expect that the minimal elements can be written as products of the elements in the inversion set in increasing order. Reading's characterisation is in terms of reduced decompositions in terms of simple reflections and the connection between the two is unclear.

Finally, figure 6.2.2 shows a particular non-crossing partition lattice in bold inside a geometric lattice. We would like to characterise which sublattices of geometric lattices arise as non-crossing partition lattices.

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