# Crash Hedging Strategies and Worst–Case Scenario Portfolio Optimization

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**Abstract**

Crash hedging strategies are derived as solutions of non–linear differential equations which itself are consequences of an equilibrium strategy which make the investor indifferent to uncertain (down) jumps. This is done in the situation where the investor has a logarithmic utility and where the market coefficients after a possible crash may change. It is scrutinized when and in which sense the crash hedging strategy is optimal. The situation of an investor with incomplete information is considered as well. Finally, introducing the crash horizon, an implied volatility is derived.

*Keywords:* Optimal portfolios, crash modelling, worst–case scenario, changing market coefficients, implied volatility, crash horizon.

## 1 Introduction

Nowadays it is widely acknowledged that price processes of stocks do have jumps which have to be modelled in some way. In most cases this is done by modelling the price processes of stocks either by some kind of Lévy processes or some processes which are heavy–tailed (see e.g. Aase [1], Merton [11], Eberlein and Keller [4], Embrechts, Klüppelberg and Mikosch [5], or Cont and Tankov [3] just as representatives for various sources). Using another approach, which has been developed in Korn and Wilmott [9], the view will be taken of a semi–specialized stock price process in this paper. More precisely, the distinction will be made between so–called "normal times" where the stock prices are assumed to follow a geometric Brownian motion and "crash times" where the stock price falls suddenly.

This approach puts the emphasis on

- avoiding large losses in *any* possible situation by maximizing the worst–case bound for the utility of terminal wealth.

- the *investment horizon* or the time to maturity, which is very important in crash modelling. However, this variable is neglected in traditional portfolio optimization under the threat of a crash.

- the possible number of crashes within the investment horizon instead of the crash intensity in the traditional crash modelling. Moreover, only a range for the possible crash size is needed and not a specific crash size.

This approach is already looked at in a paper by Korn and Wilmott [9] where the authors determined optimal portfolios under the threat of a crash in the case of logarithmic utility for final wealth. There, the main aim is to show that suitable investment in stocks can still be more profitable than playing safe and investing everything in the riskless bond if a crash of the stock price can occur. The corresponding optimal strategy is found via the solution of a balance problem between obtaining good worst–case bounds in case of a crash on the one hand and also a reasonable performance on the other hand, if no crash occurs at all.

The model has been extended to general utility functions in a recent paper by Korn and Menkens [8]. This paper also introduced the crash hedging strategy and discussed the crash hedging strategy in the case of changing market coefficients after a possible crash and when the investor has a logarithmic utility. However, only the cases are considered where the expected return on the risky asset is larger then the return on the non–risky asset.

Using the approach of Korn and Wilmott [9] the aim of this paper is to generalize the model in various directions and to scrutinize some of its properties. The most important aims are

- determining the crash hedging strategy (see Definition 3.1) for short markets (that is in the case where the expected return on the risky asset is smaller then the return on the non–risky asset).

- calculating the optimal worst–case scenario portfolio strategy which deviates from the crash hedging strategy in two cases substantially.

The paper is organized as follows: Section 2 describes the model only in the case where the investor has a logarithmic utility. The main result is given in section 3 while section 4 gives some examples for this result. Section 5 analyses the situation for an investor with limited information and section 6 introduces the crash horizon and derives an implied volatility. The last section gives a conclusion and an outlook.

# 2 The Set up

As in Korn and Wilmott [9], let us start with the most basic setting and consider a security market consisting of a riskless bond and a single risky security with prices given by

$$dP_{0,0}(t) = P_{0,0}(t)\, r_0\, dt\,, \qquad P_{0,0}(0) = 1\,, \tag{1}$$
$$dP_{0,1}(t) = P_{0,1}(t)\, [\mu_0\, dt + \sigma_0\, dW(t)]\,, \qquad P_{0,1}(0) = p_1\,, \tag{2}$$

with constant market coefficients $\mu_0$, $r_0$ and $\sigma_0 \neq 0$ in "normal times" and where $W$ is a Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$. Assume further that at most one crash can happen within the time horizon $T$. At the "crash time" the stock price suddenly falls. More specific, suppose that the sudden relative fall of the stock price lies in the interval $[k_*, k^*]$, where the constants $0 < k_* < k^* < 1$ ("the lowest and the highest possible crash size, respectively") are given. No probabilistic assumptions are made about the distribution of either the crash time or the crash height. For motivation of this model consider Korn and Wilmott [9].

Assuming that the investor is able to realize that the crash has happened let us model its occurrence via a jump process $N(t)$ which is zero before the jump time and equals one from the jump time onwards. Let us require that $N$ lives also on $(\Omega, \mathcal{F}, P)$. To model the fact that the investor is able to realize that a jump of the stock price has happened it is supposed that the investor's decisions are adapted to the $P$-augmentation $\{\mathcal{F}_t\}$ of the filtration generated by both the Brownian motion $W(t)$ and the jump process $N(t)$.

Let us further suppose that the market conditions change after a possible crash. Let therefore $k$ (with $k \in [k_*, k^*]$) be the arbitrary size of a crash at time $\tau$. The price of the bond and the risky asset is assumed to be

$$dP_{1,0}(t) = P_{1,0}(t)\, r_1\, dt\,, \qquad P_{1,0}(\tau) = P_{0,0}(\tau)\,, \tag{3}$$
$$dP_{1,1}(t) = P_{1,1}(t)\, [\mu_1\, dt + \sigma_1\, dW(t)]\,, \qquad P_{1,1}(\tau) = (1-k)\, P_{0,1}(\tau)\,, \tag{4}$$

with constant market coefficients $r_1$, $\mu_1$ and $\sigma_1 \neq 0$ after a possible crash of size $k$ at time $\tau$.

For simplicity, the initial market will also be called market 0, while the market after a crash will be called market 1.

It is important to keep in mind that the investor does *not* know that a crash will occur, the investor thinks only that it is possible. An investor who knows that a crash will happen within the time horizon $[0, T]$ has additional information and is therefore an insider. The set of possible crash heights of the insider is indeed $K_I := [k_*, k^*]$, while the set of possible crash heights of the investor who thinks that a crash is possible is $K := \{0\} \cup [k_*, k^*]$. In this paper only the portfolio problem of the investor, who thinks a crash is possible, is considered.

**Definition 2.1**

1. *For* $i = 0, 1$, *let* $\boldsymbol{A_i(s,x)}$ *be the* **set of admissible portfolio processes** $\pi(t)$ *corresponding to an initial capital of* $x > 0$ *at time* $s$, *i.e.* $\{\mathcal{F}_t, s \le t \le T\}$*–progressively measurable processes such that*

   *(i) the* **wealth equation** *in market* $i$ *in the usual crash–free setting*

   $$\begin{aligned} dX_i^{\pi,s,x}(t) &= X_i^{\pi,s,x}(t)\left[(r_i + \pi(t)\left[\mu_i - r_i\right])\,dt + \pi(t)\sigma_i\,dW_i(t)\right], &(5)\\ X_i^{\pi,s,x}(s) &= x &(6)\end{aligned}$$

   *has a unique non–negative solution* $X_i^{\pi,s,x}(t)$ *and satisfies*

   $$\int_s^T \left[\pi(t)X_i^{\pi,s,x}(t)\right]^2\,dt < \infty \quad P\text{–}a.s., \tag{7}$$

   *i.e.* $\boldsymbol{X_i^{\pi,s,x}(t)}$ *is the* **wealth process in market** $\boldsymbol{i}$ *in the crash–free world, which uses the portfolio strategy* $\pi$ *and starts at time* $s$ *with initial wealth* $x$.

   *Furthermore,* $X_i^{\pi}(t) := X_i^{\pi,0,x}(t)$ *will be used as an abbreviation.*

   *(ii)* $\pi(t)$ *has left–continuous paths with right limits.*

2. *the corresponding* **wealth process** $\boldsymbol{X^\pi(t)}$ **in the crash model**, *defined as*

   $$X^\pi(t) = \begin{cases} X_0^\pi(t) & \text{for } s \le t < \tau \\ \left[1 - \pi(\tau)k\right] X_1^{\pi,\tau,X_0^\pi(\tau)}(t) & \text{for } t \ge \tau \ge s, \end{cases} \tag{8}$$

   *given the occurrence of a jump of height* $k$ *at time* $\tau$, *is strictly positive. Thereby, it is assumed that the crash time* $\tau$ *is a stopping time, which is supposed to be* $\mathcal{F}_t$*–measurable. The set of admissible portfolio strategies is obviously given by* $A_0(s,x)$ *as long as no crash happens. After a crash at time* $\tau$ *the set is given by* $A_1(\tau,x)$. *Hence,*

   $$A(s,x) := A_0(s,x)\Big|_{[0,\tau]} \cup A_1(\tau,x).$$

3. $A(x)$ *is used as an abbreviation for* $A(0,x)$.

With these definitions it is possible to state the worst–case problem. Note that due to the lack of statistical assumptions on the distribution of both the crash height and the crash time, the problem cannot be dealt with by simply maximizing the expected utility of final wealth. However, the crash consequence has to be taken into account in some way. The approach of this paper is to maximize the worst case possible.

**Definition 2.2**

1. *Let the utility function $U$ be given by $U(x) = \ln(x)$. Then the problem to solve*
$$\sup_{\pi(\cdot)\in A(x)} \inf_{\substack{0\le\tau\le T,\\ k\in K}} \mathbb{E}\left[\ln\left(X^{\pi}(T)\right)\right] , \tag{9}$$
*where the final wealth $X^{\pi}(T)$ in the case of a crash of size $k$ at time $\tau$ is given by*
$$X^{\pi}(T) = \left[1-\pi(\tau)k\right] X_1^{\pi,\tau,X_0^{\pi}(\tau)}(T) , \tag{10}$$
*with $X_1^{\pi,\tau,X_0^{\pi}(\tau)}(t)$ as above, is called the* **worst–case scenario portfolio problem***.*

2. *The* **value function** to the above problem *is defined via*
$$\nu_c(t,x) = \sup_{\pi(\cdot)\in A(t,x)} \inf_{\substack{t\le\tau\le T,\\ k\in K}} \mathbb{E}\left[\ln\left(X^{\pi,t,x}(T)\right)\right] . \tag{11}$$

3. *The* **value function** in the crash–free setting **of the market model $\boldsymbol{X_i}$** *will be denoted*
$$\nu_i(t,x) = \sup_{\pi(\cdot)\in A_i(t,x)} \mathbb{E}\left[\ln\left(X_i^{\pi,t,x}(T)\right)\right] .$$

Clearly, the above defined optimization problems are stochastic control problems. A classical approach to solve a stochastic control problem is to derive the corresponding so–called *Hamilton–Jacobi–Bellman equation*, often abbreviated as *HJB–equation*. For an introduction to this method see e.g. Korn [6].

In order to get shorter and more transparent formulae, the following definitions are useful.

**Definition 2.3**
*For $i = 0, 1$ let us name*

1. *the* **optimal portfolio strategy in market $\boldsymbol{i}$***, assuming that no crash will happen, by*
$$\pi_i^* := \frac{\mu_i - r_i}{\sigma_i^2}.$$

2. *Moreover,*
$$\Psi_i := r_i + \frac{1}{2}\left(\frac{\mu_i - r_i}{\sigma_i}\right)^2 = r_i + \frac{\sigma_i^2}{2}\left(\pi_i^*\right)^2$$
*will be called the* **utility growth potential** *or* **earning potential of market $\boldsymbol{i}$***.*

The well–known value function of the crash–free world, given the market coefficients of market $i$, calculates to

$$\begin{aligned} \nu_i(t,x) &= \sup_{\pi(\cdot)\in A_i(t,x)} \mathbb{E}\left[\ln\left(X_i^{\pi,t,x}(T)\right)\right] \\ &= \ln(x) + \left(r_i + \frac{1}{2}\left(\frac{\mu_i - r_i}{\sigma_i^2}\right)^2\right)(T-t) \\ &= \ln(x) + \Psi_i\,(T-t)\,. \end{aligned}$$

In particular, $\nu_1$ is the value–function of the market 1. Hence, $\nu_1$ is the value–function for a crash hedging investor after a crash has happened and no further crash is expected. Moreover, define for an arbitrary admissible portfolio strategy $\pi(t)$

$$\begin{aligned} \nu_\pi(t,x) &:= \mathbb{E}\left[\ln\left(X_0^{\pi,t,x}(T)\right)\right] \\ &= \ln(x) + \mathbb{E}\left[\int_t^T \left[\pi(s)\left(\mu_0 - r_0\right) + r_0 - \frac{1}{2}\pi^2(s)\sigma_0^2\right] ds\right] \\ &= \ln(x) - \frac{\sigma_0^2}{2}\mathbb{E}\left[\int_t^T \left[\left(\pi(s) - \pi_0^*\right)^2 - \frac{2}{\sigma_0^2}\Psi_0\right] ds\right] \\ &= \ln(x) + \mathbb{E}\left[\int_t^T \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\pi(s) - \pi_0^*\right)^2\right] ds\right]. \end{aligned}$$

This is the utility one gets using the portfolio strategy $\pi$ in the initial market. Being in the initial market means that no crash has happened so far. If the portfolio strategy is deterministic, the expectation is redundant.

# 3 The Main Result

In order to get the optimal portfolio strategy for an investor, who wants to maximize her worst–case scenario portfolio problem, it is easier to calculate the portfolio strategy $\hat{\pi}$ first, which makes the investor crash indifferent. Obviously, the investor is indifferent towards a crash, if her maximized expected worst–case final utility before a possible crash is equal to her maximized expected final utility after a crash of the worst possible case. That is, the investor's expected utility is not effected by a crash of the worst possible size. This justifies the following definition, where the convention $\hat{\nu}(t,x) := \nu_{\hat{\pi}}(t,x)$ is used.

**Definition 3.1**

*i) A portfolio strategy $\hat{\pi}$ determined via the equation*

$$\hat{\nu}(t,x) = \left\{ \begin{array}{ll} \nu_1(t, x(1-\hat{\pi}(t)k^*)) & \textit{for } \hat{\pi}(t) \geq 0 \\ \nu_1(t, x(1-\hat{\pi}(t)k_*)) & \textit{for } \hat{\pi}(t) < 0 \end{array} \right\} \qquad \textit{for all } t \in [0,T]$$

*will be called a* **crash hedging strategy**.

*ii) A portfolio strategy $\tilde{\pi}$ is a* **partial crash hedging strategy**, *if there exists an $S \in (0,T)$ such that $\tilde{\pi}$ is a crash hedging strategy on $[0,S]$ and is a solution of the worst–case scenario portfolio problem on $[S,T]$.*

Rewriting the determining equation for the non–negative crash hedging strategy $\hat{\pi}$ gives

$$\begin{aligned} \hat{\nu}(t,x) &= \nu_1(t, x(1-\hat{\pi}(t)k^*)) \\ \Longleftrightarrow \qquad \ln(1-\hat{\pi}(t)k^*) &= \int_t^T \left[ \Psi_0 - \Psi_1 - \frac{\sigma_0^2}{2}(\hat{\pi}(s)-\pi_0^*)^2 \right] ds. \end{aligned} \tag{12}$$

Differentiating with respect to $t$ yields

$$\hat{\pi}'(t) = \left(\hat{\pi}(t) - \frac{1}{k^*}\right) \left[ \frac{\sigma_0^2}{2}(\hat{\pi}(t)-\pi_0^*)^2 + \Psi_1 - \Psi_0 \right].$$

Clearly, $\hat{\pi}(T) = 0$, since the right side of equation (12) is zero for $t = T$ and the left side is only zero for $t = T$, if $\hat{\pi}(T) = 0$. Using $\hat{\pi}(T) = 0$, this gives

$$\hat{\pi}'(T) = -\frac{1}{k^*}(\Psi_1 - r_0) \quad \left\{ \begin{array}{lll} < & 0 & \text{for } \Psi_1 > r_0 \\ = & 0 & \text{for } \Psi_1 = r_0 \\ > & 0 & \text{for } \Psi_1 < r_0 \end{array} \right\}.$$

A close look reveals that $\hat{\pi}'(T) < 0$ implies $\hat{\pi}'(t) \leq 0$ for $t \in [0,T)$. Hence, $\hat{\pi}(t) > 0$ for $t \in [0,T)$.

Moreover, it is straightforward to verify that $\hat{\pi}' \equiv 0$, if $\hat{\pi}'(T) = 0$. Thus, this case yields $\hat{\pi} \equiv 0$. The economic meaning of this being the impossibility to hedge a risky asset if the utility growth potential after a possible crash is only of the size of the initial riskless rate of return.

Finally, the case $\Psi_1 < r_0$ gives

$$\begin{aligned} \nu_\pi(t,x)\Big|_{\pi\equiv 0} &= \ln(x) + r_0(T-t) \\ &> \ln(x) + \Psi_1(T-t) \\ &= \nu_1(t,x) \qquad \text{for } t \in [0,T) \text{ and } x > 0. \end{aligned}$$

Thus, the expected worst–case is given in this situation by an immediate crash, if the portfolio strategy $\pi \equiv 0$ is used. In order to boost the expected worst–case utility, the expected utility after a crash has to be increased. This can only be achieved by going short, i.e. $\pi(t) < 0$ for $t \in [0,T)$. However, if $\hat{\pi}$ is negative, the corresponding differential equation is

$$\hat{\pi}'(t) = \left(\hat{\pi}(t) - \frac{1}{k_*}\right)\left[\frac{\sigma_0^2}{2}\left(\hat{\pi}(t) - \pi_0^*\right)^2 + \Psi_1 - \Psi_0\right],$$

which can be confirmed easily. Note that this differential equation has in $T$ the same behavior as the differential equation for non–negative portfolio strategies. This guarantees that the crash hedging strategy is well–defined. Moreover, $\hat{\pi}'(T) > 0$ implies $\hat{\pi}'(t) \geq 0$ for $t \in [0,T)$. Thus, $\hat{\pi}(t) < 0$ for $t \in [0,T)$.

This leads us to the main result of this paper where the first part of the theorem (that is the case $\Psi_1 \geq r_0$) is already presented in Korn and Menkens [8]. For the sake of completeness it is stated here again.

**Theorem 3.2**

1. *If $\Psi_1 \geq r_0$, then there exists a unique crash hedging strategy $\hat{\pi}$, which is given by the solution of the differential equation*

$$\begin{aligned} \hat{\pi}'(t) &= \left(\hat{\pi}(t) - \frac{1}{k^*}\right)\left[\frac{\sigma_0^2}{2}\left(\hat{\pi}(t) - \pi_0^*\right)^2 + \Psi_1 - \Psi_0\right], && (13)\\ \text{and} \quad \hat{\pi}(T) &= 0. && (14)\end{aligned}$$

   *Moreover, this crash hedging strategy is bounded by $0 \leq \hat{\pi} < \frac{1}{k^*}$. Additionally, if $\Psi_1 \leq \Psi_0$ and $\pi_0^* \geq 0$, the crash hedging strategy has another upper bound with $\hat{\pi} < \pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)}$.*

2. *If $\Psi_1 < r_0$, then there exists a unique crash hedging strategy $\hat{\pi}$, which is given by the solution of the differential equation*

$$\begin{aligned} \hat{\pi}'(t) &= \left(\hat{\pi}(t) - \frac{1}{k_*}\right)\left[\frac{\sigma_0^2}{2}\left(\hat{\pi}(t) - \pi_0^*\right)^2 + \Psi_1 - \Psi_0\right], && (15)\\ \text{and} \quad \hat{\pi}(T) &= 0. && (16)\end{aligned}$$

   *Furthermore, this crash hedging strategy is bounded by*

$$\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)} < \hat{\pi}(t) < 0 \qquad \text{for } t \in [0,T).$$

3. *If $\Psi_1 < \Psi_0$ and $\pi_0^* < 0$, there exists a partial crash hedging strategy $\tilde{\pi}$ (which is different from $\hat{\pi}$), if*

$$S := T - \frac{\ln\left(1 - \pi_0^* k_*\right)}{\Psi_0 - \Psi_1} > 0. \qquad (17)$$

*With this, $\tilde{\pi}$ is on $[0, S]$ given by the unique solution of the differential equation*

$$\begin{aligned} \tilde{\pi}'(t) &= \left(\tilde{\pi}(t) - \frac{1}{k_*}\right)\left[\frac{\sigma_0^2}{2}\left(\tilde{\pi}(t) - \pi_0^*\right)^2 + \Psi_1 - \Psi_0\right], & (18) \\ \text{and} \quad \tilde{\pi}(S) &= \pi_0^*. & (19) \end{aligned}$$

*On $[S, T]$ set $\tilde{\pi}(t) := \pi_0^*$. This partial crash hedging strategy is bounded by*

$$\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)} < \tilde{\pi} \leq \pi_0^* < 0.$$

*The optimal portfolio strategy for an investor, who wants to maximize her worst–case scenario portfolio problem, is given by*

$$\bar{\pi}(t) := \min\left\{\hat{\pi}(t), \tilde{\pi}(t), \pi_0^*\right\} \qquad \text{for all } t \in [0, T], \tag{20}$$

*where $\tilde{\pi}(t)$ is only taken into account if it exists. $\bar{\pi}$ will be named the* **optimal crash hedging strategy***.*

**Remark 3.3**

1. It is straightforward to verify that $\hat{\pi}$, $\tilde{\pi}$ and $\bar{\pi}$ are admissible portfolio strategies, since they are bounded as well as continuous.

2. Observe that the optimal crash hedging strategy is independent of the crash time $\tau$.

3. Note that the worst–case utility bound of the crash hedging strategy is given by $\hat{\nu}(t, x)$, or by $\nu_1\left(t, x\left(1 - \hat{\pi}(t)k^*\right)\right)$, which is according to the construction of $\hat{\pi}$ the same. Hence, it is sufficient to show that either $\nu_\pi(t, x) < \hat{\nu}(t, x)$ or $\nu_1\left(t, x\left(1 - \pi(t)k^*\right)\right) < \nu_1\left(t, x\left(1 - \hat{\pi}(t)k^*\right)\right)$ in order to verify that the portfolio strategy $\pi$ has a lower expected worst–case final utility than $\hat{\pi}$.

4. Compare the differential equation (13) with the differential equation Korn and Wilmott [9] got in Corollary 2.2. Rewriting the above differential equation to

$$\hat{\pi}'(t) = \frac{1}{k^*}\left(1 - \hat{\pi}(t)k^*\right)\left[\Psi_0 - \Psi_1 - \frac{\sigma_0^2}{2}\left(\hat{\pi}(t) - \pi_0^*\right)^2\right],$$

   it is easy to see that it is up to the correction term $\Psi_1 - \Psi_0$ the same as the differential equation in Korn and Wilmott [9].

5. Notice that the investor is only indifferent between no crash and a crash of the worst case possible. In general, any other crash of size $k$ with $k_* < k < k^*$ will be favorable for the investor, if the investor uses the crash hedging strategy $\hat{\pi}$.

6. Observe that the investor will $P$–a.s. not go bankrupt, if he uses the portfolio strategy $\hat{\pi}$. Define the pure bond strategy by $\pi_B$ (that is $\pi_B \equiv 0$) then one has

$$\begin{aligned} \nu_{\pi_B}(t,x) < \hat{\nu}(t,x) &= \nu_1\left(t, x\left(1-\hat{\pi}(t)k^*\right)\right) < \nu_1(t,x) \quad \text{if } 0 < \hat{\pi}(t) < \frac{1}{k^*} \\ \hat{\nu}(t,x) &= \nu_1\left(t, x\left(1-\hat{\pi}(t)k_*\right)\right) > \nu_1(t,x) \quad \text{if } \hat{\pi}(t) < 0. \end{aligned}$$

   Since the last case is the classical utility function in the crash–free model where the investor does not go bankrupt, the investor cannot go bankrupt in the other cases as well. Keep in mind that the investor cannot go bankrupt if he pursuit the pure bond strategy.

The following lemmata will prepare the proof of Theorem 3.2 and will reveal some important properties of the crash hedging strategy $\hat{\pi}$.

**Lemma 3.4**
*Any admissible portfolio strategy $\pi$ which satisfies*

$$\mathbb{E}\left[\pi(t)\right] < \hat{\pi}(t) \leq \pi_0^* \qquad \textit{for all } t \in [0,T]$$

*has a lower expected worst–case utility bound than $\hat{\pi}$, the crash hedging strategy.*

**Proof:** Using the Theorem of Fubini and the the fact that

$$\mathbb{E}\left[X^2\right] = \left(\mathbb{E}\left[X\right]\right)^2 + \mathrm{Var}(X) \geq \left(\mathbb{E}\left[X\right]\right)^2$$

for any square integrable random variable $X$, the case of no crash occurring gives

$$\begin{aligned} \nu_\pi(t,x) &= \ln(x) + \mathbb{E}\left[\int_t^T \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\pi(s)-\pi_0^*\right)^2\right] ds\right] \\ &\leq \ln(x) + \int_t^T \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\mathbb{E}\left[\pi(s)\right]-\pi_0^*\right)^2\right] ds \\ &< \ln(x) + \int_t^T \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\hat{\pi}(s)-\pi_0^*\right)^2\right] ds \\ &= \hat{\nu}(t,x), \end{aligned}$$

which shows that $\pi(t)$ has a lower expected worst–case utility bound than $\hat{\pi}$. □

**Lemma 3.5**
*Of all admissible portfolio strategies $\pi$ with*

$$\mathbb{E}\left[\pi(t)\right] < \hat{\pi}(t) \qquad \textit{for all } t \in I := \{t : \hat{\pi}(t) > \pi_0^*\} \tag{21}$$

*$\bar{\pi}$ yields the highest worst–case utility bound (where $\bar{\pi}$ is defined in (20) in Theorem 3.2).*

**Remark 3.6**
Observe that $I$ is a disjoint union of closed intervals, since $\hat{\pi}$ is continuously differentiable. Actually, later on the following can be verified. The case $I \neq \emptyset$ can only happen if $\Psi_1 - \Psi_0 > 0$. However, in this case $\hat{\pi}$ is strictly decreasing, thus $I$ is an interval of the form $I = [0, t_0]$.

**Proof:** For any admissible portfolio strategy $\pi$ with

$$\mathbb{E}[\pi(t)] < 2\pi_0^* - \hat{\pi}(t) \qquad \text{for all } t \in I$$

Lemma 3.4 applies basically analog. Hence, let us restrict to portfolio strategies $\pi$, which satisfy

$$2\pi_0^* - \hat{\pi}(t) < \mathbb{E}[\pi(t)] < \hat{\pi}(t) \qquad \text{for some } t \in I.$$

Without loss of generality, let us assume that

$$2\pi_0^* - \hat{\pi}(t) < \mathbb{E}[\pi(t)] < \hat{\pi}(t) \qquad \text{for all } t \in I.$$

For simplicity, let us suppose that $I$ is of the form $I = [t_0, t_1]$. Choosing $t \in I$, the above inequality implies

$$\begin{aligned}
&\ln(x) + \mathbb{E}\left[\int_t^{t_1} \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\pi(s) - \pi_0^*\right)^2\right] ds\right] \\
&\quad = \ln(x) + \int_t^{t_1} \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\mathbb{E}[\pi(s)] - \pi_0^*\right)^2 - \frac{\sigma_0^2}{2}\mathrm{Var}(\pi(s))\right] ds \\
&\quad > \ln(x) + \int_t^{t_1} \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\hat{\pi}(s) - \pi_0^*\right)^2 - \frac{\sigma_0^2}{2}\mathrm{Var}(\pi(s))\right] ds. \qquad (22)
\end{aligned}$$

The last inequality shows that any portfolio strategy $\pi(t)$ satisfying (21) and with a variance small enough has a higher expected final utility than the crash hedging strategy $\hat{\pi}(t)$, if no crash occurs and if $t \in I$. It is straightforward to verify that of these strategies only $\pi_0^*$ maximizes the expected final utility in the interval $I$, if no crash happens.

Lemma 3.4 justifies that – without loss of generality – it is possible to set $\pi(t) = \hat{\pi}(t)$ for $t \notin I$. Using this together with the above inequality (22) gives

$$\nu_\pi(t, x) > \hat{\nu}(t, x) - \frac{\sigma_0^2}{2}\int_t^{t_1} \mathrm{Var}(\pi(s))\, ds,$$

which shows that any portfolio strategy $\pi$ satisfying (21) and with a variance small enough has a higher expected final utility than the crash hedging strategy

$\hat{\pi}$, if no crash occurs. It is straightforward to verify that of these strategies only $\bar{\pi}$ maximizes the expected final utility, if no crash happens. In order to show that this maximizes the expected worst–case utility bound, one has to consider the following two cases.

i) $\pi_0^* \geq 0$:
It suffices to verify that for all $t \in I$

$$\begin{aligned} \nu_{\bar{\pi}}(t,x) &\leq \nu_1(t, x(1-\pi_0^* k^*)) \\ \iff \quad \int_t^T \left[\Psi_0 - \Psi_1 - \frac{\sigma^2}{2}\left(\bar{\pi}(s) - \pi_0^*\right)^2\right] ds &\leq \ln\left(1 - \pi_0^* k^*\right). \end{aligned}$$

Since $\hat{\pi}$ is decreasing and ending at $\hat{\pi}(T) = 0$, there exists an $S \in [0, T)$ such that $\bar{\pi}(s) = \hat{\pi}(s)$ for $s \in [S, T]$. The important case which has to be considered is $S \in (t, T)$. Hence, the above reduces to

$$\begin{aligned} \left(\Psi_0 - \Psi_1\right)(S - t) + \underbrace{\int_S^T \left[\Psi_0 - \Psi_1 - \frac{\sigma^2}{2}\left(\hat{\pi}(s) - \pi_0^*\right)^2\right] ds}_{=\ln\left(1-\pi_0^* k^*\right)} &\leq \ln\left(1 - \pi_0^* k^*\right) \\ \iff \quad \Psi_1 &\geq \Psi_0. \end{aligned}$$

This last inequality can be verified easily, since it is only possible that $\hat{\pi} \geq \pi_0^*$, if $\Psi_1 \geq \Psi_0$.

ii) $\pi_0^* < 0$:
If $\Psi_1 \geq \Psi_0$ than it is straightforward to verify that

$$\nu_\pi(t,x) \leq \nu_1(t, x(1 - \pi(t) k_*))$$

for any admissible portfolio strategy $\pi$ with $\pi \leq 0$. Thus, let us consider the case $\Psi_1 < \Psi_0$. It remains to confirm that

$$\begin{aligned} \nu_{\bar{\pi}}(t,x) &\leq \nu_1(t, x(1-\bar{\pi}(t) k_*)) \\ \iff \quad \int_t^T \left[\Psi_0 - \Psi_1 - \frac{\sigma^2}{2}\left(\bar{\pi}(s) - \pi_0^*\right)^2\right] ds &\leq \ln\left(1 - \bar{\pi}(t) k_*\right). \end{aligned}$$

Assume that $\bar{\pi} \equiv \pi_0^*$. This leads to

$$\begin{aligned} \left(\Psi_0 - \Psi_1\right)(T - t) &\leq \ln\left(1 - \bar{\pi}(t) k_*\right) \\ \iff \quad t &\geq T - \frac{\ln\left(1 - \bar{\pi}(t) k_*\right)}{\Psi_0 - \Psi_1}. \end{aligned}$$

The right side of the last inequality has been defined in (17) to be $S$. This inequality shows that $\pi_0^*$ is an optimal worst–case portfolio strategy on $[\max(S,0),T]$. Obviously, at time $S$ equality holds. Suppose that $S>0$, then

$$\nu_{\pi_0^*}(t,x) > \nu_1(t,x(1-\pi_0^* k_*)) \qquad \text{for } t\in[0,S).$$

In this situation the worst case is given by an immediate crash. However, it is clearly possible to improve this worst–case utility by reducing $\pi_0^*$. Equality is by construction reached for $\tilde{\pi}$. Since $\nu_\pi$ is strictly increasing for $\pi<\pi_0^*$ and $\nu_1(t,x(1-\pi k_*))$ is strictly decreasing for $\pi$, it is straightforward to verify that $\tilde{\pi}$ is optimal in this situation.

This concludes the assertion. □

**Lemma 3.7**
*Any admissible portfolio strategy $\pi$ which satisfies*

$$\mathbb{E}[\pi(t)] > \bar{\pi}(t) \qquad \textit{for some } t\in[0,T] \tag{23}$$

*has a lower worst–case utility bound than the optimal crash hedging strategy $\bar{\pi}$.*

**Proof:** First, let us suppose that $\mathbb{E}[\pi(0)]\le\bar{\pi}(0)$ as well as $\mathbb{E}[\pi(T)]\le\bar{\pi}(T)$. Hence, there exists $t^\star\in[0,T)$ and $\varepsilon>0$ such that

$$\mathbb{E}[\pi(t)]\le\bar{\pi}(t) \quad \text{for } t\in[0,t^\star] \qquad \text{and} \qquad \mathbb{E}[\pi(t)]>\bar{\pi}(t) \quad \text{for } t\in(t^\star,\varepsilon].$$

Such a construction is always possible due to assumption (1ii) in Definition 2.1. Without loss of generality, let us suppose that $\pi(t)=\bar{\pi}(t)$ for $t\in[0,t^\star]$.

It suffices to show that

$$\nu_1(t,x(1-\pi(t)k^*)) < \nu_1(t,x(1-\bar{\pi}(t)k^*)) \qquad \text{for } t\in(t^\star,\varepsilon].$$

However, this can be seen straightforward

$$\begin{aligned}
\nu_1(t,x(1-\pi(t)k^*)) &= \ln(x)+\mathbb{E}[\ln(1-\pi(t)k^*)]+\Psi_1(T-t)\\
&\le \ln(x)+\ln(1-\mathbb{E}[\pi(t)]k^*)+\Psi_1(T-t)\\
&< \ln(x)+\ln(1-\bar{\pi}(t)k^*)+\Psi_1(T-t)\\
&= \nu_1(t,x(1-\bar{\pi}(t)k^*)).
\end{aligned} \tag{24}$$

Hence, an immediate crash of the worst possible size at time $t$ gives a lower final expected utility for the portfolio strategy $\pi$ than for the crash hedging strategy $\hat{\pi}$.

It is easy to verify that inequality (24) holds for $t=0$ or $t=T$ as well, if $\mathbb{E}[\pi(0)]>\bar{\pi}(0)$ or $\mathbb{E}[\pi(T)]>\bar{\pi}(T)$, respectively.

Since $\nu_1(t, x(1 - \bar{\pi}(t)k^*))$ is already the worst–case utility bound for $\bar{\pi}$, this proves the assertion. □

**Proof of Theorem 3.2:** The differential equations have been derived above. Furthermore, it is straightforward to verify that any crash hedging strategy has to satisfy $\hat{\pi}(T) = 0$.

$$\begin{aligned} & \hat{\nu}(T, x) &=& \nu_1(T, x(1 - \hat{\pi}(T)k^*)) \\ \Longleftrightarrow \quad & \ln(x) &=& \ln(x) + \ln(1 - \hat{\pi}(T)k^*) \,. \end{aligned}$$

Since $k^* \neq 0$, this is only possible for $\hat{\pi}(T) = 0$.

Let us prove that $\hat{\pi}$ is bounded. The upper bound $\frac{1}{k^*}$ of $\hat{\pi}$ can be verified in equation (12) due to the fact that this equation has a pole for $\hat{\pi}(t) = \frac{1}{k^*}$ for an arbitrary $t \in [0, T]$. The economic meaning of this being that the investor would be bankrupt in case of $\hat{\pi}(t) \geq \frac{1}{k^*}$ and a crash of size $k^*$ at time $t$. The bound $\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)}$ is due to the fact that it is a zero of the right side of (13) or (15), if $\Psi_1 \leq \Psi_0$. This bound will never be reached by the continuous crash hedging strategy, since this would imply that the continuous crash hedging strategy would eventually become constant, which can not be the case. The other bounds have been shown above.

In order to prove the existence and uniqueness of a solution for the differential equation let us denote

$$F(t, y) := \left(y - \frac{1}{k^*}\right)\left[\frac{\sigma_0^2}{2}(y - \pi_0^*)^2 + \Psi_1 - \Psi_0\right].$$

Clearly, $F$ is continuously partial differentiable and therefore especially locally Lipschitz–continuous with respect to $y$. This gives then already the uniqueness of the solution for the differential equation (13) with terminal value (14).

Moreover, since $F$ is continuous and locally Lipschitz–continuous on $[0, T] \times \mathbb{R}$ the theorem of Picard–Lindelöf gives the existence of a solution for the differential equation (13) with terminal value (14) on a suitable neighbourhood of any arbitrary point $y \in \mathbb{R}$. This gives the existence of a solution for the differential equation on any compact set $[a, b] \subset \mathbb{R}$ with $a < b$. Choosing $a$ and $b$ sufficiently large, this proves the existence of a solution for the differential equation (13), since $\hat{\pi}$ is bounded.

Replacing $k^*$ by $k_*$ shows that the same statement holds for the differential equation (15) with terminal value (16).

Applying Lemma 3.4, Lemma 3.5 and Lemma 3.7 gives the optimality of $\bar{\pi}$ in both cases. All in all, this concludes the Theorem. □

# 4 Examples and Further Remarks

In order to compare the results in this paper with the results of Korn and Wilmott [9] let us name the optimal portfolio strategy of the market $i$ given that the market conditions do not change after a crash $\hat{\phi}_i$. This is the situation of Korn and Wilmott [9].

The cases

1. $\boldsymbol{\Psi_1 = \Psi_0}$ ***and*** $\boldsymbol{\pi_0^* \geq 0}$
2. $\boldsymbol{\Psi_1 > \Psi_0}$ ***and*** $\boldsymbol{\pi_0^* \geq 0}$
3. $\boldsymbol{r_0 \leq \Psi_1 \leq \Psi_0}$ ***and*** $\boldsymbol{\pi_0^* \geq 0}$

will not be scrutinized in the following since they have already been discussed in Korn and Menkens [8]. Thus, only the following remaining cases will be considered.

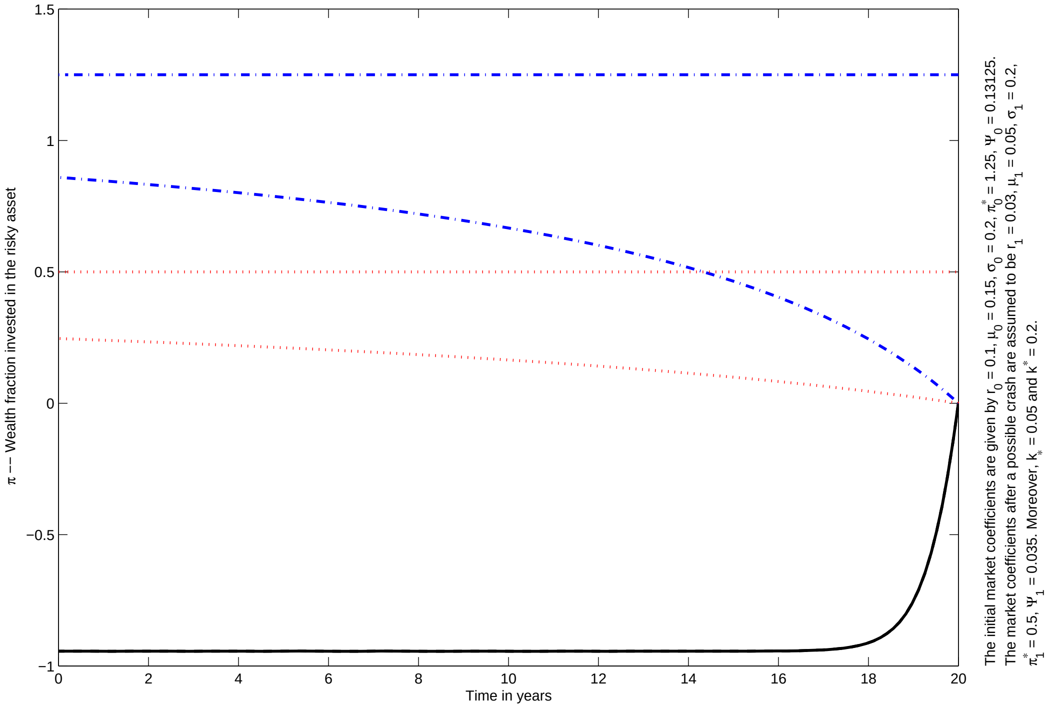
4. $\boldsymbol{\Psi_1 < r_0}$ ***and*** $\boldsymbol{\pi_0^* \geq 0}$
   The crash hedging strategy which is also an optimal crash hedging strategy is negative. This means that the investor goes short of the risky asset in this situation (see Figure 1). This is due to the fact that the earning potential after a crash $\Psi_1$ is even less as the risk free interest rate today $r_0$. This implies that the investor can increase his expected worst–case utility by going short and thus takes substantial losses into account as long as no crash happens. However, if a crash happens the investor is able to transfer some of his utility to the market 1 by using a short strategy which give him substantial gains in utility.

   It is amazing that it is optimal for the investor to go short even if the probability of a crash is by no means sure.

5. $\boldsymbol{\Psi_1 > \Psi_0}$ ***and*** $\boldsymbol{\pi_0^* < 0}$
   The crash hedging strategy in this case is positive and greater or equal than $\hat{\phi}_0$ (see Figure 2). This is even though $\hat{\phi}_1$ is only greater or equal than either the crash hedging strategy or $\hat{\phi}_0$ close to the terminal investment period. The optimal crash hedging strategy is given by $\pi_0^*$.

6. $\boldsymbol{r_0 \leq \Psi_1 \leq \Psi_0}$ ***and*** $\boldsymbol{\pi_0^* < 0}$
   In this situation the crash hedging strategy is positive (see Figure 3). Clearly, this is not optimal since it is like throwing away money (or utility for that reason). Hence, it is optimal for the investor to take the portfolio strategy $\pi_0^*$ at the end of her investment period and favoring a crash. However, if the investment period is so large that $S$ defined in (17) is positive, the partial crash hedging strategy $\tilde{\pi}$, given by the solution of (18) and (19), is an optimal crash hedging strategy and makes the investor crash indifferent on $[0, S]$. However, on $(S, T]$ a crash is favorable for the investor.

Figure 1: Example $\Psi_1 < r_0$ and $\pi_0^* \geq 0$



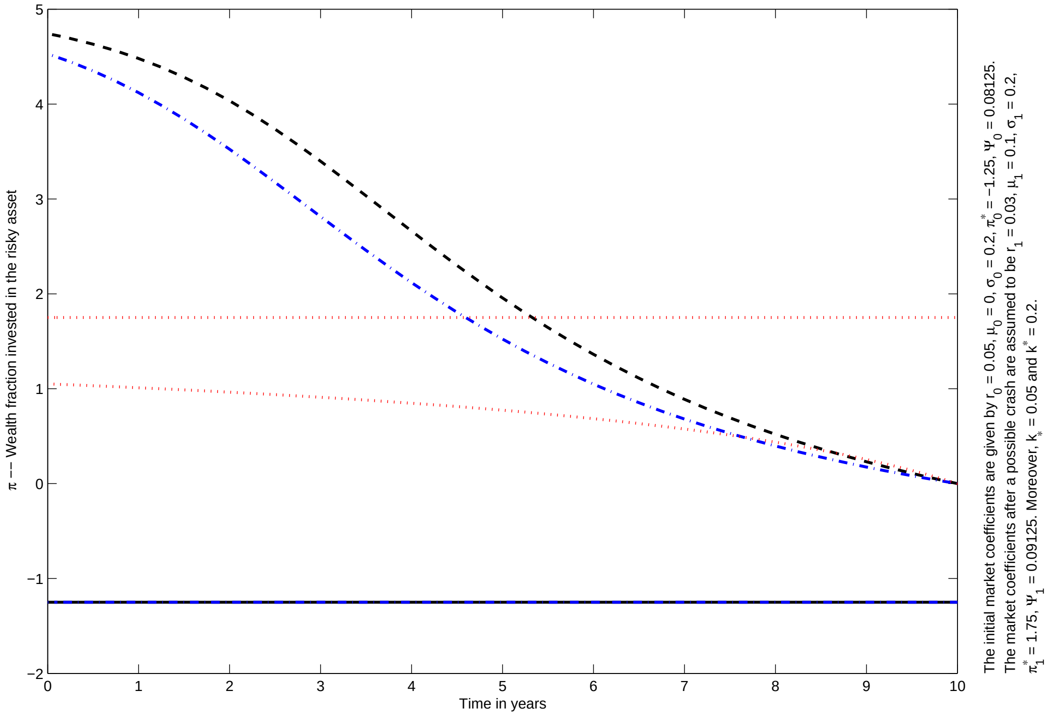

This graphic shows $\hat{\pi} = \bar{\pi}$ (solid line), $\hat{\phi}_0$ (dash–dotted line), $\hat{\phi}_1$ (dotted line), $\pi_0^*$ (constant dash–dotted line), and $\pi_1^*$ (constant dotted line).

7. $\boldsymbol{\Psi_1 < r_0}$ ***and*** $\boldsymbol{\pi_0^* < 0}$

The crash hedging strategy is negative (see Figure 4 and Figure 5). However, it is still not the optimal worst–case portfolio strategy. As in the last case the optimal worst–case portfolio strategy is given by $\tilde{\pi}$. For $S > 0$ the optimal crash hedging strategy is given by $\pi_0^*$ on $[S, T]$ and by $\tilde{\pi}$ on $[0, S]$. Again, as in case 6, the investor is crash indifferent on $[0, S]$ and favors a crash on $(S, T]$ if she uses the optimal crash hedging strategy.

This case (as well as case 6) shows the *Bellman principle* or *optimality principle* nicely. The Bellman principle (or optimality principle) asserts that a section of an optimal trajectory is also an optimal trajectory (see Bellman [2], compare also with Korn [6]). Without knowing the Bellman principle, one might — *wrongly* — guess that min $\{\hat{\pi}, \pi_0^*\}$ is the optimal crash hedging strategy. Since $\hat{\pi}$ is not optimal on $[S, T]$, it can neither be optimal on $[0, S]$, which is due to the Bellman principle. Therefore, applying the Bellman principle leads to the solution $\tilde{\pi}$.

Figure 2: Example $\Psi_1 > \Psi_0$ and $\pi_0^* < 0$




This graphic shows $\hat{\pi}$ (dashed line), $\bar{\pi} = \pi_0^*$ (solid line), $\hat{\phi}_0$ (dash–dotted line), $\hat{\phi}_1$ (dotted line), and $\pi_1^*$ (constant dotted line).

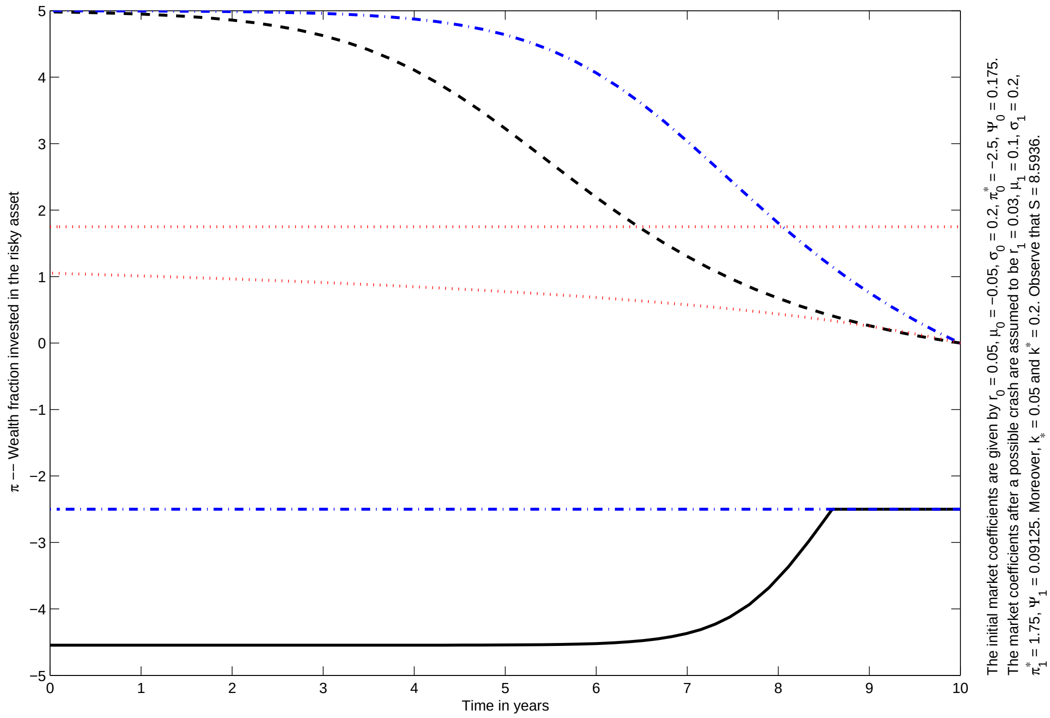
# 5 The Investor with Blurred Information

Assume now that the investor knows the market coefficients of the initial market 0 (e.g. by observation or by estimation). However, the investor does not know the market coefficients after a possible crash. Instead he does only know possible ranges for the market coefficients after a possible crash. More specific, let us suppose that the investor thinks that the market coefficients of market 1, that is $r_1$, $\mu_1$, and $\sigma_1$ will be within the range of $[r_{1*}, r_1^*]$, $[\mu_{1*}, \mu_1^*]$, and $[\sigma_{1*}, \sigma_1^*]$ with $\sigma_{1*} > 0$, respectively. An investor with such information will be called **an investor with blurred information** and his crash hedging strategy will be named $\hat{\pi}_{bi}$. Hence, the **worst–case scenario portfolio problem of the investor with blurred information** is

$$\inf_{\substack{r_1 \in \left[r_{1*}, r_1^*\right], \\ \mu_1 \in \left[\mu_{1*}, \mu_1^*\right], \sigma_1 \in \left[\sigma_{1*}, \sigma_1^*\right]}} \sup_{\pi(\cdot) \in A(x)} \inf_{\substack{0 \le \tau \le T, \\ k \in K}} \mathbb{E}\left[\ln\left(X^{\pi}(T)\right)\right] . \tag{25}$$

Observe that this situation is most relevant in practise since in general the investor has only a notion of the whereabouts of the market coefficients after a possible

Figure 3: Example $r_0 \leq \Psi_1 \leq \Psi_0$ and $\pi_0^* < 0$




This graphic shows $\hat{\pi}$ (dashed line), $\bar{\pi} = \tilde{\pi}$ (solid line), $\hat{\phi}_0$ (dash–dotted line), $\hat{\phi}_1$ (dotted line), $\pi_0^*$ (constant dash–dotted line), and $\pi_1^*$ (constant dotted line).

crash.

**Proposition 5.1**

1. *If* $\Psi_{1_{\min}} \geq r_0$*, then there exists a unique crash hedging strategy* $\hat{\pi}_{bi}$*, which is given by the solution of the differential equation*

$$\hat{\pi}'_{bi}(t) = \left(\hat{\pi}_{bi}(t) - \frac{1}{k^*}\right)\left[\frac{\sigma_0^2}{2}\left(\hat{\pi}_{bi}(t) - \pi_0^*\right)^2 + \Psi_{1_{\min}} - \Psi_0\right], \quad (26)$$

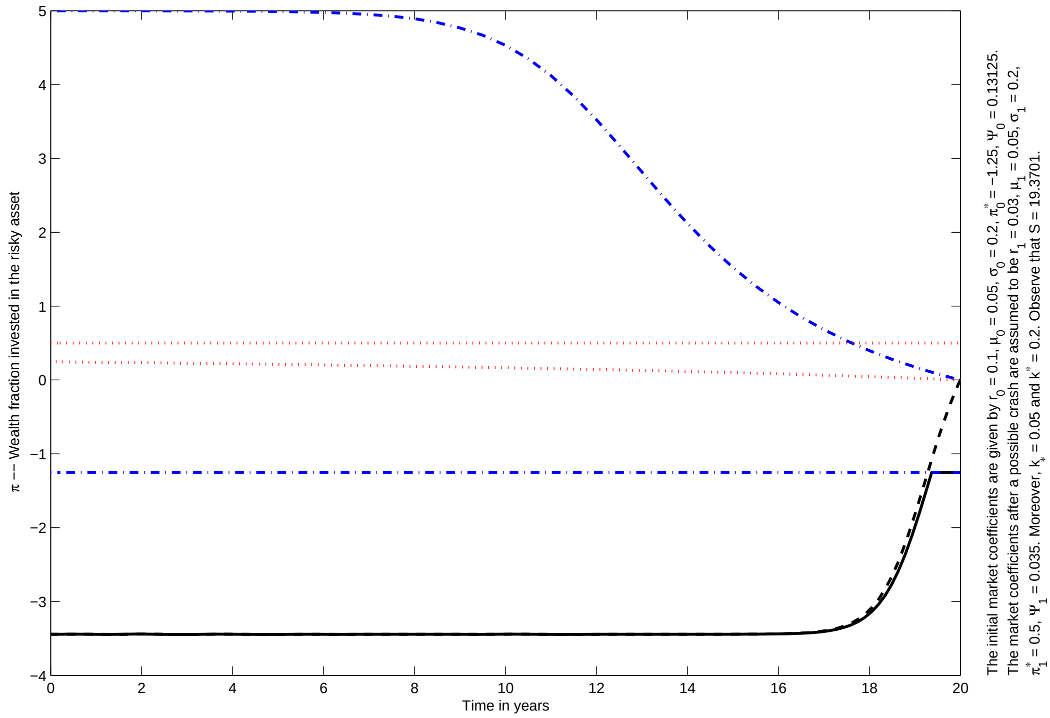
$$\text{and} \quad \hat{\pi}_{bi}(T) = 0. \quad (27)$$

*Moreover, this crash hedging strategy is bounded by* $0 \leq \hat{\pi}_{bi} \leq \frac{1}{k^*}$*, if* $\Psi_{1_{\min}} > \Psi_0$*. In the case of* $\Psi_{1_{\min}} \leq \Psi_0$*, the crash hedging strategy is additionally bounded by*

$$0 \leq \hat{\pi}_{bi} \leq \pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_{1_{\min}}\right)}.$$

2. *If* $\Psi_{1_{\min}} < r_0$*, then there exists a unique crash hedging strategy* $\hat{\pi}_{bi}$*, which is*

Figure 4: Example $\Psi_1 < r_0$ and $\pi_0^* < 0$, the long term behaviour




This graphic shows $\hat{\pi}$ (dashed line), $\bar{\pi} = \tilde{\pi}$ (solid line), $\hat{\phi}_0$ (dash–dotted line), $\hat{\phi}_1$ (dotted line), $\pi_0^*$ (constant dash–dotted line), and $\pi_1^*$ (constant dotted line).

*given by the solution of the differential equation*

$$\hat{\pi}_{bi}'(t) = \left(\hat{\pi}_{bi}(t) - \frac{1}{k_*}\right)\left[\frac{\sigma_0^2}{2}\left(\hat{\pi}_{bi}(t) - \pi_0^*\right)^2 + \Psi_{1_{\min}} - \Psi_0\right], \quad (28)$$

$$\text{and} \quad \hat{\pi}_{bi}(T) = 0. \quad (29)$$

*Furthermore, this crash hedging strategy is bounded by*

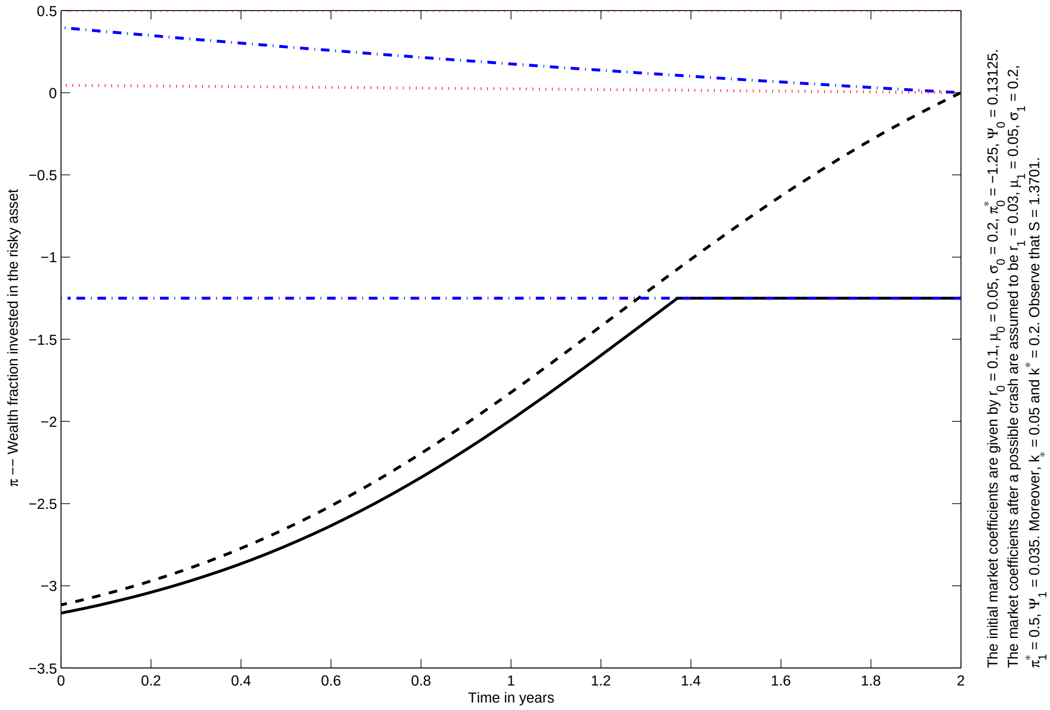
$$\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_{1_{\min}}\right)} \leq \hat{\pi}_{bi}(t) < 0 \qquad \text{for } t \in [0, T).$$

3. *If $\Psi_{1_{\min}} < \Psi_0$ and $\pi_0^* < 0$, there exists a partial crash hedging strategy $\tilde{\pi}_{bi}$ (which is different from $\hat{\pi}_{bi}$), if*

$$S := T - \frac{\ln\left(1 - \pi_0^* k_*\right)}{\Psi_0 - \Psi_{1_{\min}}} > 0. \quad (30)$$

*With this, $\tilde{\pi}_{bi}$ is on $[0, S]$ given by the unique solution of the differential*

Figure 5: Example $\Psi_1 < r_0$ and $\pi_0^* < 0$




This graphic shows $\hat{\pi}$ (dashed line), $\bar{\pi} = \tilde{\pi}$ (solid line), $\hat{\phi}_0$ (dash–dotted line), $\hat{\phi}_1$ (dotted line), $\pi_0^*$ (constant dash–dotted line), and $\pi_1^*$ (constant dotted line).

*equation*

$$\begin{aligned} \tilde{\pi}'_{bi}(t) &= \left(\tilde{\pi}_{bi}(t) - \frac{1}{k_*}\right)\left[\frac{\sigma_0^2}{2}\left(\tilde{\pi}_{bi}(t) - \pi_0^*\right)^2 + \Psi_{1_{\min}} - \Psi_0\right], \quad (31)\\ \textit{and} \quad \tilde{\pi}_{bi}(S) &= \pi_0^*. \quad (32) \end{aligned}$$

*On $[S,T]$ set $\tilde{\pi}_{bi}(t) := \pi_0^*$. This partial crash hedging strategy is bounded by*

$$\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_{1_{\min}}\right)} \leq \tilde{\pi}_{bi} \leq \pi_0^* < 0.$$

*The optimal portfolio strategy for an investor, who wants to maximize his worst–case scenario portfolio problem, is given by*

$$\bar{\pi}_{bi}(t) := \min\left\{\hat{\pi}_{bi}(t), \tilde{\pi}_{bi}(t), \pi_0^*\right\}, \quad (33)$$

*where $\tilde{\pi}_{bi}(t)$ is only taken into account if it exists. $\bar{\pi}_{bi}$ will be denoted the* **optimal crash hedging strategy** *of the investor with blurred information.*

**Proof:** Considering the market 1, which eventually reigns after a crash, the worst case for an investor is that $\Psi_1$ – the utility growth potential – will be minimal. Defining

$$\Psi_{1_{\min}} := \min \left\{ \Psi_1 \mid r_1 \in \left[r_{1*}, r_1^*\right], \mu_1 \in \left[\mu_{1*}, \mu_1^*\right], \sigma_1 \in \left[\sigma_{1*}, \sigma_1^*\right] \right\},$$

the proof follows now as the proof of Theorem 3.2. □

A special case is the **clueless** investor. The clueless investor has only a notion of what the interest rate might at least be. However, the clueless investor has no idea about neither the expected rate of return nor the volatility. Hence, in this situation the range of the market coefficients are $r_1 \in [r_{1*}, \infty)$, $\mu_1 \in \mathbb{R}$, and $\sigma_1 \in (0, \infty)$. It is straightforward to verify that the minimal utility growth potential in market 1 is given by $\Psi_{1_{\min}} = r_{1*}$. Thus the crash hedging strategy of the clueless investor which will be named $\hat{\pi}_{cl}$ calculates as in Proposition 5.1, but with $\Psi_{1_{\min}} = r_{1*}$.

**Remark 5.2**
Note that $r_{1*}$ can be either positive or negative. However, the cases $r_{1*} = 0$ and $r_{1*} = r_0$ are probably the most important ones.

1. Given that the clueless investor assumes that $r_{1*} = r_0$, which implies that $\hat{\pi}_{cl} \equiv 0$, this theory can explain why most people are not investing into the stock market. *No other portfolio theory can explain this fact.*

   However, if the clueless investor supposes that $r_{1*} = 0 < r_0$, which implies that $\hat{\pi}_{cl} < 0$, the clueless investor should go short in the stock market. This cannot be observed in practise.

2. Also if the case $\hat{\pi}_{cl} > \pi_0^*$ is for $\pi_0^* > 0$ theoretical possible, it is practically irrelevant. Moreover, in general, it is valid that $\hat{\pi}_{cl} \ll \pi_0^*$.

# 6 Crash Horizon and Implied Volatility

So far we have assumed that the time horizon $T$ is the investment horizon. The assumption was then, that within this investment horizon, there exists the possibility of a crash.

Let us suppose now that the investment horizon is $T$ and the time horizon for a possible crash is $S_c$ with $S_c < T$. This means that the investor expects to see a crash in the time interval $[0, S_c]$. Thus, $S_c$ will be called the possible ***crash horizon***. The smaller $S_c$ is, the more imminent a crash is considered possible from the point of view of the investor.

Observe that the crash hedging strategy changes over time $t$, since the investment horizon changes over time. In other words, the crash hedging strategy

$\hat{\pi}(t)$ belongs to the investment horizon $T - t$. Thus, if the investment horizon is $T + t$ for some $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ with $t_0 \in (0, T)$ and $\varepsilon > 0$ sufficiently small, the investment horizon would stay at $T$ for that period. Correspondingly, the crash hedging strategy would be constantly staying at $\hat{\pi}(t_0)$ as long as the time is in the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$.

This observation justifies the following approach for the crash horizon. As long as the crash horizon $S_c$ does not change and is smaller than the investment horizon $T$, the investor keeps a constant portfolio. More precisely, the investor keeps $\hat{\pi}(0; S_c)$ with a hypothetical investment horizon $S_c$.

$$\pi_c^*(S_c) := \hat{\pi}(0; S_c) := \hat{\pi}(0), \tag{34}$$

where $\hat{\pi}(0)$ is the initial investment strategy of an investor who has the investment horizon $S_c$. Using that the optimal portfolio process in the crash free model is given by $\pi_0^*$, equation (34) can be used to calculate an implied volatility.

$$\hat{\sigma}_c^2(S_c) := \frac{\mu_0 - r_0}{\hat{\pi}(0; S_c)}. \tag{35}$$

This is the implied variance in the crash model given a specified utility function and a crash horizon $S_c$. Subtracting the variance of the risky asset, one gets the implied variance (read risk) of a crash.

$$\sigma_c^2(S_c) := \left|\hat{\sigma}_c^2(S_c) - \sigma_0^2\right|.$$

Using the fact that

$$\hat{\pi}(0; S_c) = \hat{\pi}(T - S_c; T) \qquad \text{for arbitrary } T \geq S_c,$$

one can rewrite equation (35) as follows

$$\hat{\sigma}_c^2(S_c) := \frac{\mu_0 - r_0}{\hat{\pi}(T - S_c; T)} \qquad \text{for } T \geq S_c.$$

Therefore, it is possible to differentiate $\hat{\sigma}_c^2$ with respect to $S_c$

$$\frac{d\hat{\sigma}_c^2(S_c)}{dS_c} = \frac{\mu_0 - r_0}{\hat{\pi}^2(T - S_c; T)} \hat{\pi}'(T - S_c; T).$$

If $\pi'$ is continuous from the right, one can take the limit $T \downarrow S_c$, yielding

$$\frac{d\hat{\sigma}_c^2(S_c)}{dS_c} = \frac{\mu_0 - r_0}{\hat{\pi}^2(0; S_c)} \hat{\pi}'(0; S_c),$$

thus showing the differentiability of the implied variance $\hat{\sigma}_c^2(S_c)$.

Using the differential equation (13), the derivative calculates to

$$\begin{aligned}\frac{d\hat{\sigma}_c^2(S_c)}{dS_c} &= \frac{\mu_0 - r_0}{\hat{\pi}^2(0;S_c)}\hat{\pi}'(0;S_c)\\ &= \frac{\mu_0 - r_0}{\hat{\pi}^2(0;S_c)}\left(\hat{\pi}(0;S_c) - \frac{1}{k^*}\right)\left[\frac{\sigma_0^2}{2}\left(\hat{\pi}(0;S_c) - \pi_0^*\right)^2 + \Psi_1 - \Psi_0\right].\end{aligned}$$

The derivative of $\hat{\sigma}_c^2$ is decreasing for $\pi_0^* > 0$ and $\Psi_1 > r_0$, which is in accordance with intuition. The implied variance gets lower as the crash horizon gets farther away. The weaker threat of a possible crash is favorable for the investor. The same is true for $\pi_0^* < 0$ and $\Psi_1 < r_0$. Here, the threat of a crash comes not from the crash itself (since $\pi_0^* < 0$ it is acually favorable for the investor), but from the very bad earning potential after a crash $\Psi_1 < r_0$.

However, the derivative of $\hat{\sigma}_c^2$ is increasing for $\pi_0^* < 0$ and $\Psi_1 > r_0$. The implied variance gets greater as the crash horizon gets farther away. In this situation the investor would favor a crash happening. This is due to the fact that $\pi_0^* < 0$. The same is true for $\pi_0^* > 0$ and $\Psi_1 < r_0$. However, there is no explanation for this behavior.

Notice that it is also possible to define an implied volatility via the optimal crash hedging strategy $\bar{\pi}$ instead of the crash hedging strategy $\hat{\pi}$. Denoting this implied variance by $\bar{\sigma}_c^2$, it measures only the one–sided risk of a possible crash. $\bar{\sigma}_c^2$ does not measure both sides of a possible crash as $\hat{\sigma}_c^2$ does. Using $\bar{\sigma}_c^2$ the investor seems to be indifferent in certain situations to a change in the crash horizon which is clearly not the case.

This variance depends on the investors risk behavior as well as on the crash horizon, which is also fixed by the investor. Moreover, the implied variance depends on the market coefficient after a possible crash. Thus, this is only the individually perceived risk of a crash.

However, this model gives an explanation for the observed change in the volatility over time, which is not possible within the Black–Scholes model.

Furthermore, this calculated implied volatility can be used for option pricing. Another application might be the possibility to calculate the intrinsic crash horizon of the market.

For another approach to value options in a jump diffusion model, see Merton [12]. There, however, one need the knowledge of the distribution of the jumps, while in the worst–case scenario model one need to know only the crash horizon. However, both approaches make use of the utility function of the investor.

# 7 Conclusion and Outlook

Considering changing market conditions leads to interesting phenomena (see Section 4). Just to mention one phenomenon, in the case of an initial short market (namely $\pi_0^* < 0$), it can happen that the optimal portfolio strategy is completely different from the corresponding crash hedging strategy. Scrutinizing these phenomena analytically, the definition of the *utility growth potential* (see Definition 2.3) is very useful (see Theorem 3.2).

It has already been pointed out that the investment horizon is very important in crash modelling. This paper does so by defining the *crash hedging strategy* to be the portfolio strategy which balances out – at *any* time where the investor is invested – the expected utility of the final wealth of the investor between the case of no crash occurring and the one of a crash occurring.

Next to the optimal portfolio under the threat of a crash, developed in Korn and Wilmott [9], this is the only approach known to the author which takes the investment horizon into account. By doing so, this model does not need the *worst–case concept* developed by Korn and Wilmott [9]. However, examining the optimality of the crash hedging strategy, it shows that the crash hedging strategy is — at most — optimal in the sense of the worst–case concept, thereby revealing its close relationship with the worst–case scenario portfolio model.

The investment strategies which can be observed in practice (e.g. the investment schemes of German pension funds) are taking their investment horizon into account. These strategies are similar to the portfolio strategies developed in this paper as well as in Korn and Wilmott [9] and Korn and Menkens [8]. Korn [7] applies the worst–case scenario concept to the optimization problem of insurers.

Another interesting approach has been developed in Lui et al. [10] where they consider price processes for the stock with stochastic volatility and where both the price process as well as the stochastic volatility process have jumps. They conclude that within this setting it is not optimal for the investor to take leveraged or short positions. However, they still derive an optimal portfolio strategy which is independent from the investment horizon. Therefore, it might be interesting to try to merge this approach with the approach used in this paper.

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