Auto-Tail Dependence Coefficients for Stationary Solutions of Linear Stochastic Recurrence Equations and for GARCH(1, 1)

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Abstract. We examine the auto-dependence structure of strictly stationary solutions of linear stochastic recurrence equations and of strictly stationary GARCH(1, 1) processes from the point of view of ordinary and generalized tail dependence coefficients. Since such processes can easily be of infinite variance, a substitute for the usual auto-correlation function is needed.

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1. Introduction

In this paper we study serial dependence in time-series from the point of view of ordinary and generalized upper tail dependence coefficients, rather than that of the traditional auto-correlations. More precisely, we will compute such auto-tail dependence coefficients (whose definition we recall in (2.5) and (2.6) below) for strictly stationary solutions of a scalar linear recurrence equation (SRE) and for stationary GARCH(1, 1)-processes. The results of this paper generalize those of [3], in which the author carried out such a study for both stationary and non-stationary ARCH(1)-processes. Here we will focus on the stationary processes, the non-stationary case needing different and slightly more technical proofs.

We briefly recall, from [3], some of the mathematical and statistical arguments for replacing the traditional auto-correlations by alternative auto-dependence measures, when studying a non-linear time-series such as a GARCH.

- For a given stochastic process, auto-covariances and auto-correlations can be ill-defined: as is well known, it is quite easy for stationary GARCH-processes, or for stationary solutions of a SRE, to have infinite variance. An empirically relevant example is that of an IGARCH(1, 1): when modelling a financial return series by a GARCH(1, 1)-process, the hypothesis of in fact having an IGARCH can often not be rejected.

The situation can be worse for auto-correlations of functionals of the process, like the correlations of the squared process. One may want to study these in order to quantify the auto-dependence in the process when, for example, ordinary correlations vanish. A typical example is again that of a GARCH. The auto-correlation function of a squared GARCH requires however the existence of the fourth moment of the stationary process, imposing a yet more stringent condition on the coefficients of the process, a condition which, in empirical applications, will often not be satisfied.

- As an alternative, one may want to use sample auto-correlations; in fact, this is what is mostly done in econometric practice. This presupposes the existence of the limit of such sample auto-correlations as the sample size tends to infinity. It has been shown, however, in a sequence of papers by Davis, Mikosch, Stărică and Basrak, for increasingly general GARCH-processes, that the sample auto-correlations of a stationary GARCH with infinite variance will not tend to a number anymore, but will have as its (weak) limit a random variable whose probability distribution is that of the quotient of two components of some jointly stable random vector. A similar result holds for systems of linear SRE (from which the GARCH-result was in fact deduced).

This does not imply that sample auto-correlations are a priori useless in such cases: if the data set is sufficiently big one could for example study the statistics of an ensemble of auto-correlations computed from different samples drawn from this data set.

- Finally, even when auto-correlations are well-defined, they may not be the most useful dependence measure for the application at hand. In the context of financial risk-management, for example, one of the advantages of the auto-tail dependence coefficients which we will study in this paper is their direct financial significance in terms of the probability of violating a value-at-risk constraint given that such a violation will already have occurred.

We will limit ourselves in this paper to *upper* tail coefficients; the case of lower tail dependence coefficients is completely analogous, and follows easily using simple symmetry arguments. In Section 2 we compute ordinary and generalized auto-upper tail dependence coefficients of the stationary solution of a scalar linear SRE, and in Section 3 those of a strongly stationary GARCH(1, 1). In comparison with ordinary tail dependence coefficients, our results for generalized tail dependence coefficients have both weaker hypotheses and a stronger conclusion, which moreover would be easier to test statistically. As was the case for an ARCH(1) in [3], we expect our results for generalized tail dependence coefficients to generalize to non-stationary GARCH(1, 1)'s and SRE's, though not the ones for ordinary tail dependence coefficient. The vanishing of the lower tail dependence coefficients for non-stationary ARCH(1)'s was an important motivation to introduce the generalized tail dependence coefficients in [3]. Although the generalized auto-tail dependence coefficients may prove to be the more useful ones for practical applications, the question of computing the ordinary auto-tail dependence coefficients is, from a mathematical point of view, a natural one, and merits attention.

Our proofs in the GARCH(1, 1)-case parallel those for scalar SRE. It would have been more satisfactory if we could have derived the auto-tail dependence coefficients of a GARCH(1, 1) directly from those of a related SRE. We only managed to do this in the special case of a symmetrical ARCH(1), as explained at the end of Section 2, where we limited ourselves to ordinary tail dependence coefficients. A similar argument for a GARCH(1, 1), and more generally for a GARCH(p, q), would presumably need a generalization of Theorem 2.1 to systems of linear SRE. Such a generalization does not at this moment seem entirely straightforward.

2. Auto-tail dependence for stationary solutions of stochastic recurrence equations

We limit ourselves to scalar linear stochastic recurrence equations or SRE:

$$X_{n+1} = A_{n+1}X_n + B_{n+1}, \quad n \in \mathbb{Z}, \tag{2.1}$$

with (A_n, B_n) i.i.d. A convenient reference for the theory of such equations, including the systems case, is [1]. It is known that the SRE (2.1) possesses a unique strictly stationary causal solution $(X_n)_{n\in\mathbb{Z}}$ whenever $\mathbb{E}(\log|A_1|) < 0$ and $\mathbb{E}(\max(\log|B_1|,\, 0)) < \infty$. Moreover, by a classical result of Kesten (valid in fact for systems of SRE), this stationary solution has, under certain technical conditions, regularly varying tails. In fact, if $A_1 \geq 0$ a.s., A_1 has a non-lattice distribution and if there exists a positive $\kappa_0 > 0$ such that $\mathbb{E}(A_1^{\kappa_0}) \geq 1$ and $\mathbb{E}\left(A_1^{\kappa_0}\max(\log A_1, 0)\right) < \infty$, then $\lim_{x\to\infty} x^\kappa \overline{F}_{X_n}(x) =: c$ exists, where $\overline{F}_{X_n}(x) = \mathbb{P}(X_n > x)$ and $\kappa > 0$ is the unique positive solution to

$$\mathbb{E}(A_1^\kappa) = 1. \tag{2.2}$$

Stated otherwise,

$$\overline{F}_{X_1}(x) \simeq \frac{c}{x^\kappa}, \quad x \to \infty. \tag{2.3}$$

Goldie [5] has given an alternative proof of Kesten's theorem in the scalar case which provides an explicit formula for the constant c (in terms of the stationary distribution). The precise value of this constant will not be needed in this paper, though, only its existence and the fact that it is non-zero.

For $\alpha \in [0,1]$ let $\overline{q}(\alpha) := \overline{q}_{X_n}(\alpha) := \sup\{y : \overline{F}_{X_n}(y) \geq \alpha\}$ be the α-th upper quantile of $X_n =_d X_1$. It follows from (2.3) that

$$\overline{q}(\alpha) \simeq \left(\frac{c}{\alpha}\right)^{1/\kappa}, \quad \alpha \to 0. \tag{2.4}$$

Recall that the *upper auto-tail dependence coefficient of* X_{n+p} *on* X_n can be defined by

$$\overline{\lambda}_{X_{n+p}|X_n} := \lim_{\alpha\to 0} \mathbb{P}\left(X_{n+p} > \overline{q}(\alpha)|X_n > \overline{q}(\alpha)\right), \tag{2.5}$$

assuming the limit exists. More generally, if $\psi = \psi(\alpha)$ is a function of α defined in some small interval $(0, \delta)$ which satisfies $\lim_{\alpha\to 0} \psi(\alpha) = 0$, we define the *generalized upper auto-tail dependence coefficient* ([3]) by

$$\overline{\lambda}^{\psi}_{X_{n+p}|X_n} := \lim_{\alpha\to 0} \mathbb{P}\left(X_{n+p} > \overline{q}(\psi(\alpha))|X_n > \overline{q}(\alpha)\right). \tag{2.6}$$

We refer to [3] for motivation and further discussion of this concept. As will be clear from our results below, the generalized tail dependence coefficients associated to functions $\psi(\alpha)$ going to 0 at a slower rate than α will often be a stronger indicator of tail dependence than $\overline{\lambda}_{X_{n+p}|X_n}$ itself. We note that (2.5) and (2.6) make sense for any strictly stationary process, and that for a given such process these tail dependence coefficients will only depend on the lag, p, and on ψ. For non-stationary processes the two quantiles in the defining formulas should be those of X_{n+p} and X_n, respectively, and would depend on time $n+p$ and n, respectively. In the case of a non-stationary ARCH(1) with a.s. initial condition $X_0 = x_0 \in \mathbb{R}$, this dependence of quantiles on time is what ultimately causes the ordinary tail dependence coefficients to vanish: cf. [3].

The following two theorems are the main results of this section:

Theorem 2.1. *Let* $(X_n)_n$ *be the unique causal stationary solution of* (2.1)*. Suppose that the conditions of Kesten's theorem are satisfied, and assume that the cumulative probability distribution of* $A_n y + B$ *is continuous for any* $y \in \mathbb{R}$*. Also assume that* A_n *possesses a probability density. Then for* $p \geq 1$*,*

$$\overline{\lambda}_{X_{n+p}|X_n} = \kappa \int_1^\infty \mathbb{P}\left(A_1 A_2 \cdots A_p y > 1\right) \frac{dy}{y^{\kappa+1}}, \tag{2.7}$$

where $\kappa > 0$ *is defined by* (2.2)*.*

Theorem 2.2. *Assuming only that* $F_{A_n y + B_n}$ *is continuous for all* y *(but still assuming that the conditions of Kesten's theorem are met) we have that for all* ψ *satisfying* $\alpha = o(\psi(\alpha))$ *as* $\alpha \to 0$*, the generalized upper tail dependence coefficient* $\overline{\lambda}^{\psi}_{X_{n+p}|X_n} = 1$*, for all* $p \geq 1$ *(and trivially so for* $p = 0$*).*

Proof of Theorem 2.1*.* We first prove the theorem for $p = 1$. We have to compute the limit, as $\alpha \to 0$, of

$$\begin{aligned}
& \mathbb{P}\left(X_{n+1} > \overline{q}(\alpha)|X_n > \overline{q}(\alpha)\right) \\
= \ & \frac{1}{\alpha}\mathbb{P}\left(X_{n+1} > \overline{q}(\alpha), X_n > \overline{q}(\alpha)\right) \\
= \ & -\frac{1}{\alpha}\int_{\overline{q}(\alpha)}^\infty \mathbb{P}\left(X_{n+1} > \overline{q}(\alpha)|X_n = x\right)\, d\overline{F}(x) && (2.8) \\
= \ & -\frac{1}{\alpha}\int_{\overline{q}(\alpha)}^\infty \mathbb{P}\left(A_{n+1}x + B_{n+1} > \overline{q}(\alpha)\right)\, d\overline{F}(x), && (2.9)
\end{aligned}$$

where we have written $\overline{F} := \overline{F}_{X_n}$. Note that $\overline{q}(\alpha) \to \infty$ as $\alpha \to 0$; in particular, we can assume without loss of generality that $\overline{q}(\alpha) > 0$. If we now change variables to $x = \overline{q}(\alpha)y$, and let

$$G_\alpha(y) := -\alpha^{-1}\overline{F}(\overline{q}(\alpha)y),$$

Formula (2.9) becomes

$$\int_1^\infty \mathbb{P}\left(A_{n+1}y + \overline{q}(\alpha)^{-1}B_{n+1} > 1\right) dG_\alpha(y). \tag{2.10}$$

Since $\overline{F}$ is right-continuous, $\overline{F}(\overline{q}(\alpha)) \leq \alpha$ (since $\overline{F}_X(x) < \alpha$ for $x > \overline{q}(\alpha)$), and the measures dG_α therefore have total mass bounded by 1. Under the hypotheses of theorem, $\overline{F} = 1 - F$ is continuous, since F, being the stationary distribution, satisfies the integral equation $F(x) = \int \mathbb{P}(Ay + B \leq x)\, dF(y)$, where $(A, B) =_d (A_n, B_n)$, and the probability distribution function of $Ay + B$ is continuous by assumption. It follows that $\overline{F}(\overline{q}(\alpha)) = \alpha$ and, consequently, that dG_α is a probability measure on $[1, \infty)$.

The next two lemmas study the convergence of integrand and measure in (2.10) as $\alpha \to 0$.

Lemma 2.3. *Let*

$$\varphi_\alpha(y) := \mathbb{P}\left(Ay + \overline{q}(\alpha)^{-1}B > 1\right),$$

where $(A, B) =_d (A_n, B_n)$. *Assume* A *has a continuous probability distribution. Then as* $\alpha \to 0$,

$$\varphi_\alpha(y) \to \varphi_0(y) := \mathbb{P}\left(Ay \geq 1\right),$$

uniformly on $y \geq 1$.

Proof. For any fixed $y > 0$, let E_α and E_0 be the events $E_\alpha := E_\alpha(y) := \{(A, B) : Ay + \overline{q}(\alpha)^{-1}B > 1\}$ and $E_0 := E_0(y) := \{A : Ay > 1\}$, respectively. Then $\varphi_\alpha(y) = \mathbb{P}(E_\alpha)$ and $\varphi_0(y) = \mathbb{P}(E_0)$, and

$$|\varphi_\alpha(y) - \varphi_0(y)| \leq \max\left(\mathbb{P}(E_\alpha \setminus E_0), \mathbb{P}(E_0 \setminus E_\alpha)\right) \leq \mathbb{P}(E_\alpha \Delta E_0),$$

where $E_\alpha \Delta E_0 = (E_\alpha \setminus E_0) \cup (E_0 \setminus E_\alpha)$, the symmetric difference. We note that $E_\alpha \Delta E_0$ can be bounded by

$$\begin{aligned} E_\alpha \Delta E_0 &= \left\{1 - \overline{q}(\alpha)^{-1}B < Ay \leq 1\right\} \cup \left\{1 < Ay \leq 1 - \overline{q}(\alpha)^{-1}B\right\} \\ &\subseteq \left\{Ay \in [1 - \overline{q}(\alpha)^{-1}|B|, 1 + \overline{q}(\alpha)^{-1}|B|]\right\} \\ &=: I_\alpha(A, B). \end{aligned}$$

Now let $\varepsilon > 0$ be arbitrary. Since $\lim_{R\to\infty} \mathbb{P}(|B| > R) = 0$, we can find R_ε such that

$$\mathbb{P}\left(Ay \in I_\alpha(A, B), |B| > R_\varepsilon\right) < \varepsilon,$$

uniformly in y. Next, for $y \geq 1$,

$$\begin{aligned} &\mathbb{P}\left(Ay \in I_\alpha(A, B), |B| \leq R_\varepsilon\right) \\ \leq\ &\mathbb{P}\left(A \in [y^{-1}(1 - \overline{q}(\alpha)^{-1}R_\varepsilon), y^{-1}(1 + \overline{q}(\alpha)^{-1}R_\varepsilon)]\right) \\ =\ &F_A\left(y^{-1}(1 + \overline{q}(\alpha)^{-1}R_\varepsilon)\right) - F_A\left(y^{-1}(1 - \overline{q}(\alpha)^{-1}R_\varepsilon)\right), \end{aligned}$$

where $F_A(a) := \mathbb{P}(A \leq a)$ is the cumulative probability distribution function of A. Since $y^{-1} \in [0,1]$ and since F_A is uniformly continuous on compacta, the lemma follows if we use that $\overline{q}(\alpha) \to \infty$. □

The next lemma shows that the family of measures dG_α converges weakly to $\kappa y^{-\kappa-1}dy$, when integrated against continuous functions on $[1,\infty)$ which have an integrable derivative.

Lemma 2.4. *Let $\varphi \in C[0,\infty)$ be differentiable, with derivative in L^1. Then*

$$\int_1^\infty \varphi(y)dG_\alpha(y) \to \kappa \int_1^\infty \varphi(y)\,\frac{dy}{y^{\kappa+1}}.$$

Proof. The hypotheses on φ imply that

$$\int_1^\infty \varphi(y)dG_\alpha(y) = -\varphi(1) - \int_1^\infty \varphi'(y)G_\alpha(y)dy.$$

where we used that $G_\alpha(1) = 1$. Now by (2.4) and (2.3),

$$G_\alpha(y) \simeq -\frac{c}{\alpha(\overline{q}(\alpha))y)^\kappa} = -y^{-\kappa},$$

as $\alpha \to 0$, and therefore

$$\begin{aligned}\int_1^\infty \varphi(y)dG_\alpha(y) &\to -\varphi(1) + \int_1^\infty \varphi'(y)\,y^{-\kappa}\,dy \\ &= \kappa\int_1^\infty \varphi(y)\,y^{-\kappa-1}\,dy,\end{aligned}$$

as was to be shown. □

Proof of Theorem 2.1 *if* $p = 1$*:* using the notation introduced in Lemma 2.3, (2.10) can be written as

$$\int_1^\infty \varphi_\alpha dG_\alpha = \int_1^\infty (\varphi_\alpha - \varphi_0)\,dG_\alpha + \int_1^\infty \varphi_0 dG_\alpha. \tag{2.11}$$

The first term on the right tends to 0, since φ_α converges uniformly to φ_0, by Lemma 2.3, and since dG_α has mass 1, independently of α. Next, Lemma 2.4 implies that the second term tends to

$$\kappa\int_1^\infty \varphi_0(y)\frac{dy}{y^{\kappa+1}} = \kappa\int_1^\infty \mathbb{P}(A_1 y > 1)\frac{dy}{y^{\kappa+1}},$$

where we used that $\varphi_0 = \overline{F}_A(y^{-1})$ is differentiable, with integrable derivative. In fact, $\varphi_0'(y) = y^{-2}f_A(y^{-1})$, where $f_A = -\overline{F}'_A$ is the pdf of A, and

$$\int_1^\infty \varphi_0'(y)dy = \int_0^1 f_A(z)dz \leq 1.$$

Hence (2.11) converges to $\kappa\int_1^\infty \mathbb{P}(A_1 y > 1)\,y^{-\kappa-1}\,dy$, proving Theorem 2.1 when $p = 1$.

The case of arbitrary p follows by observing that the new process $(Y_k)_{k\in\mathbb{Z}}$ defined by $Y_k := X_{kp}$ will be the strictly stationary causal solution of the SRE

$$Y_{k+1} = \widetilde{A}^{(p)}_{k+1} Y_k + \widetilde{B}^{(p)}_{k+1},$$

where

$$\widetilde{A}^{(p)}_{k+1} = A_{(k+1)p} \cdots A_{kp+1} =_d A_p \cdots A_1,$$

and

$$\widetilde{B}^{(p)}_{k+1} = \sum_{j=0}^{p-1} \left(\prod_{\nu=0}^{j-1} A_{(k+1)p-\nu} \right) B_{(k+1)p-j},$$

with the empty product interpreted as the identity. It follows that

$$\overline{\lambda}_{X_p|X_0} = \overline{\lambda}_{Y_1|Y_0} = \kappa \int_1^\infty \mathbb{P}(A_p \cdots A_1 y > 1)\, y^{-\kappa-1} dy.$$

This proves (2.7), since we can without loss of generality take $n = 0$. □

Proof of Theorem 2.2. Starting again with $p = 1$, one now shows by a similar computation to the one which led to (2.10) that

$$\mathbb{P}\left(X_{n+1} > \overline{q}(\psi(\alpha)) | X_n > \overline{q}(\alpha)\right) = \int_1^\infty \widetilde{\varphi}_\alpha(y)\, dG_\alpha(y),$$

where now

$$\widetilde{\varphi}_\alpha(y) := \mathbb{P}\left(Ay + \overline{q}(\alpha)^{-1} B > \overline{q}(\alpha)^{-1} \overline{q}(\psi(\alpha))\right).$$

Since, by (2.4), $\overline{q}(\alpha)^{-1}\overline{q}(\psi(\alpha)) \simeq \left(\psi(\alpha)^{-1}\alpha\right)^{1/\kappa} \to 0$ as $\alpha \to 0$, the arguments of lemma 2.3 now show that the continuity of F_A in 0 implies that $\widetilde{\varphi}_\alpha(y) \to \widetilde{\varphi}_0(y) := \mathbb{P}(Ay \geq 0)$, uniformly for $y \geq 1$. Recall that for Kesten's theorem we need that $A \geq 0$ a.e. It follows that $\widetilde{\varphi}_0(y) = \mathbb{P}(Ay \geq 0) = 1$, for all positive y, and hence

$$\begin{aligned} \mathbb{P}\left(X_{n+1} > \overline{q}(\psi(\alpha)) | X_n > \overline{q}(\alpha)\right) &= \int_1^\infty \widetilde{\varphi}_\alpha dG_\alpha \\ &= \int_1^\infty (\widetilde{\varphi}_\alpha - 1)\, dG_\alpha + \int_1^\infty dG_\alpha \to 1, \end{aligned}$$

as $\alpha \to 0$, as was to be shown. The case of arbitrary positive p follows as before. □

We end this section by showing how the auto-tail dependence coefficients of a stationary ARCH(1) found in [3] can be re-derived from Theorem 2.1, if we furthermore assume that the ARCH(1) is symmetric. Recall that an ARCH(1)-process is defined by the non-linear stochastic recursion

$$X_{n+1} = \sqrt{\omega + aX_n^2}\, \epsilon_{n+1},$$

with $(\epsilon_n)_n$ i.i.d. and $a, \omega \geq 0$. We will assume that ϵ_n has a symmetric probability density. If $(X_n)_n$ is an ARCH(1), then $(X_n^2)_n$ will solve the linear SRE $X_{n+1}^2 = A_{n+1} X_n^2 + B_{n+1}$, with $A_{n+1} = a\epsilon_{n+1}^2$ and $B_{n+1} = \omega\epsilon_{n+1}^2$. Stationarity and regular tail-variation for $(X_n)_n$ then follow easily from an application of Kesten's theorem to $(X_n)_n$, cf. [4].

Let $(X_n)_n$ be a strictly stationary causal ARCH(1), with $X_n =_d X$. If $x > 0$, then it follows from the symmetry of ϵ_n that $\overline{F}_X(x) = \frac{1}{2}\overline{F}_{|X|}(x)$. Hence $\overline{q}_X(\alpha) = \overline{q}_{|X|}(2\alpha)$ for α's less than 1/2. We also note that $\overline{q}_{|X|}(\alpha) = \sqrt{\overline{q}_{X^2}(\alpha)}$. Using that $|X_{n+p}|$ does not depend on the sign of X_n, we then find that

$$\begin{aligned}
& \mathbb{P}\left(X_{n+p} > \overline{q}_X(\alpha) | X_n > \overline{q}_X(\alpha)\right) \\
= \; & \frac{1}{2}\mathbb{P}\left(|X_{n+p}| > \overline{q}_X(\alpha) \,|\, X_n > \overline{q}_X(\alpha)\right) \\
= \; & \frac{1}{2}\mathbb{P}\left(|X_{n+p}| > \overline{q}_{|X|}(2\alpha) \,|\, |X_n| > \overline{q}_X(\alpha), X_n > 0\right) \\
= \; & \frac{1}{2}\mathbb{P}\left(|X_{n+p}| > \overline{q}_{|X|}(2\alpha) \,|\, |X_n| > \overline{q}_X(\alpha)\right) \\
= \; & \frac{1}{2}\mathbb{P}\left(|X_{n+p}| > \overline{q}_{|X|}(2\alpha) \,|\, |X_n| > \overline{q}_{|X|}(2\alpha)\right) \\
= \; & \frac{1}{2}\mathbb{P}\left(X_{n+p}^2 > \overline{q}_{X^2}(2\alpha) \,|\, X_n^2 > \overline{q}_{X^2}(\alpha)\right).
\end{aligned}$$

Taking the limit of $\alpha \to 0$, and using Theorem 2.1, we see that if we let κ_{X^2} denote the tail-index of X_n^2, then

$$\begin{aligned}
\overline{\lambda}_{X_{n+p}|X_n} &= \frac{1}{2}\,\kappa_{X^2} \int_1^\infty \mathbb{P}\left(a^p \prod_{j=1}^{p} \epsilon_{n+j}^2\, y > 1\right) \frac{dy}{y^{\kappa_{X^2}+1}} \\
&= \frac{1}{2} \cdot 2\kappa_{X^2} \int_1^\infty \mathbb{P}\left(a^{p/2} \prod_{j=1}^{p} |\epsilon_{n+j}|\, z > 1\right) \frac{dz}{z^{2\kappa_{X^2}+1}} \\
&= \kappa_X \int_1^\infty \mathbb{P}\left(a^{p/2}\epsilon_{n+p} \prod_{j=1}^{p-1} |\epsilon_{n+j}| z > 1\right) \frac{dz}{z^{\kappa_X+1}}
\end{aligned}$$

where $\kappa_X = 2\kappa_{X^2}$ is the tail-index of X_n. Letting f_ϵ denote the pdf of ϵ_n, this integral can be evaluated as

$$\kappa_X \int_1^\infty \int_{\mathbb{R}^{p-1}} \overline{F}_\epsilon\left(a^{-p/2}\left(\prod_{j=1}^{p-1} |z_j|^{-1}\right) z^{-1}\right) \prod_{j=1}^{p-1} f_\epsilon(z_j) \prod_{j=1}^{p-1} dz_j\, \frac{dz}{z^{\kappa_X+1}}, \quad (2.12)$$

which is the analogue for upper tails of theorem 2 of [3]. Apart from the symmetry assumption, this establishes $\overline{\lambda}_{X_{n+p}|X_n}$ under slightly weaker hypotheses than those in [3], since we needed neither continuity nor boundedness of f_ϵ, only existence.

3. Tail dependence coefficients for GARCH(1,1)

Let $(X_n, \sigma_n)_{n\in\mathbb{Z}}$ be a strictly stationary GARCH(1, 1), that is, $(X_n)_n$ is the strictly stationary causal solution of the following system of non-linear stochastic recursion equations:

$$\begin{cases} X_{n+1} = \sigma_{n+1}\epsilon_{n+1} \\ \sigma_{n+1}^2 = \omega + aX_n^2 + b\sigma_n^2, \end{cases} \tag{3.1}$$

where $(\epsilon_n)_n$ is i.i.d. and where $\omega, a, b > 0$. The system (3.1) can be linearized into a 2×2-system of linear SRE for (X_n^2, σ_n^2) (with in fact a deterministic equation for σ_n^2). We can also find a linear SRE for the coefficients σ_n^2 by themselves: if we substitute $X_n = \sigma_n\epsilon_n$ into the second equation of (3.1), then

$$\sigma_{n+1}^2 = A_n\sigma_n^2 + B_n, \tag{3.2}$$

with $A_n := a\epsilon_n^2 + b$ and $B_n := \omega$; cf. [7]. By Kesten's theorem, and under appropriate conditions on ϵ_n^2, the stationary solution σ_n^2 will have a regularly varying tail of index κ_{σ^2}, where $\kappa = \kappa_{\sigma^2} > 0$ is the unique positive solution of $\mathbb{E}\left((a\epsilon^2 + b)^\kappa\right) = 1$. It follows that

$$\overline{F}_\sigma(s) := \mathbb{P}(\sigma_n > s) \simeq c\, s^{-2\kappa_{\sigma^2}}, \quad s \to \infty,$$

for some positive constant c. The tail-behavior of $X_n = \sigma_n\epsilon_n$ can be found from Breiman's lemma (cf. [2, 7]) which states that if Y and Z are independent non-negative random variables with Y regularly varying of index α and Z satisfying $\mathbb{E}(Z^{\alpha+\varepsilon}) < \infty$ for some $\varepsilon > 0$, then $\mathbb{P}(YZ > x) \simeq \mathbb{E}(Z^\alpha) \cdot \mathbb{P}(Y > x)$, for $x \to \infty$. Applying this with $Y = \sigma_n$ and $Z = \max(\epsilon_n, 0)$, we conclude that X_n has a regularly varying upper tail of index $\kappa_X =: 2\kappa_{\sigma^2}$, for if $x > 0$, then

$$\begin{aligned} \mathbb{P}(X_n > x) &= \mathbb{P}(\sigma_n \max(\epsilon_n, 0) > x) \\ &\simeq c\,\mathbb{E}\left(\max(\epsilon_n, 0)^{\kappa_X}\right) \cdot x^{-\kappa_X}, \quad x \to \infty. \end{aligned} \tag{3.3}$$

(There is of course a similar result for the lower tails.) It follows that the upper quantiles of X_n behave asymptotically as

$$\overline{q}_X(\alpha) := \overline{q}_{X_n}(\alpha) \simeq \left(c\,\mathbb{E}(\max(\epsilon, 0)^{\kappa_X})\right)^{1/\kappa_X} \alpha^{-1/\kappa_X}, \ \alpha \to 0. \tag{3.4}$$

To compute the upper tail dependence coefficients of our stationary GARCH(1, 1), we start again with the case of lag $p = 1$. We assume that the ϵ_n have a pdf, f_ϵ. If $\overline{\lambda}_{X_{n+1}|X_n}(\alpha) := \mathbb{P}\left(X_{n+1} > \overline{q}_X(\alpha) | X_n > \overline{q}_X(\alpha)\right)$ we find, by conditioning on the pair of independent random variables (σ_n, ϵ_n), which is independent of ϵ_{n+1} also, that

$$\begin{aligned} &\overline{\lambda}_{X_{n+1}|X_n}(\alpha) = \alpha^{-1}\,\mathbb{P}\left(\sqrt{\omega + (a\epsilon_n^2 + b)\sigma_n^2}\,\epsilon_{n+1} > \overline{q}_X(\alpha),\ \sigma_n\epsilon_n > \overline{q}_X(\alpha)\right) \\ = &\ -\alpha^{-1}\int\int_{\{sz > \overline{q}_X(\alpha), s>0\}} \mathbb{P}\left(\sqrt{\omega + (az^2 + b)s^2}\epsilon_{n+1} > \overline{q}_X(\alpha)\right) d\overline{F}_\sigma(s)\, f_\epsilon(z)dz \\ = &\ \int_0^\infty \int_{s=z^{-1}}^\infty \mathbb{P}\left(\sqrt{\overline{q}_X(\alpha)^{-2}\omega + (az^2 + b)s^2}\,\epsilon_{n+1} > 1\right) dG_\alpha(s)\, f_\epsilon(z)dz, \end{aligned}$$

where F_σ is the stationary distribution of σ_n, and where $G_\alpha(s) := -\alpha^{-1}\overline{F}_\sigma(\overline{q}_X(\alpha)s)$, as before. If we let

$$\varphi_\alpha(s, z) := \mathbb{P}\left(\sqrt{\overline{q}_X(\alpha)^{-2}\omega + (az^2 + b)s^2}\,\epsilon_{n+1} > 1\right),$$

then we find using Lemma 2.3 with $A = (az^2 + b)\epsilon_{n+1}^2$ and $B := \omega\epsilon_{n+1}^2$ that for any fixed $z > 0$,

$$\varphi_\alpha(s, z) \to \varphi_0(s, z) := \mathbb{P}\left(\sqrt{(az^2 + b)s^2}\,\epsilon_{n+1} > 1\right), \quad \alpha \to 0,$$

uniformly on $\{s \geq z^{-1}\}$. Note that in the present situation, dG_α is a positive measure on $(0, \infty)$, not just on $[1, \infty)$. However, the total mass of G_α on each fixed interval $[z^{-1}, \infty)$ with $z > 0$ still stays uniformly bounded as $\alpha \to 0$, since $-G_\alpha(z^{-1}) = \alpha^{-1}\overline{F}(\overline{q}(\alpha)z^{-1})) \leq Cz^\kappa$, by (3.3) and (3.4). Hence

$$\int_{z^{-1}}^\infty (\varphi_\alpha(s, z) - \varphi_0(s, z))\, dG_\alpha(s) \to 0,$$

as $\alpha \to 0$, for all $z > 0$. We next observe that

$$\left|\int_{z^{-1}}^\infty (\varphi_\alpha(s, z) - \varphi_0(s, z)) dG_\alpha(s)\right| \leq 2|G_\alpha(z^{-1})| \leq Cz^{\kappa_X}.$$

Since $\int_0^\infty z^{\kappa_X} f_\epsilon(z) dz < \infty$, dominated convergence implies that

$$\int_0^\infty \int_{z^{-1}}^\infty (\varphi_\alpha(s, z) - \varphi_0(s, z))\; f_\epsilon(z)\, dG_\alpha(s) dz \to 0,$$

first as an iterated integral and then, by Fubini, as a double integral. Next, arguing as in the proof of Lemma 2.4, we find that as $\alpha > 0$,

$$\int_a^\infty \varphi(s)\, dG_\alpha(s) \;\to\; \mathbb{E}\left(\max(\epsilon, 0)^{\kappa_X}\right)^{-1}\; \kappa_X \int_a^\infty \varphi(s)\, s^{-\kappa_X - 1} ds,$$

for any $a > 0$ and any continuous function φ on $[a, \infty)$ having an integrable derivative. In particular, for any fixed $z > 0$,

$$\int_{z^{-1}}^\infty \varphi_0(s, z)\, dG_\alpha(s) \to \kappa_X \left(\max(\epsilon, 0)^{\kappa_X}\right)^{-1} \int_{z^{-1}}^\infty \varphi_0(s, z)\, s^{-\kappa_X - 1} ds,$$

since $\varphi_0(s, z) = \overline{F}_\varepsilon\left((az^2 + b)^{-1/2} s^{-1/2}\right)$ is differentiable in s, with integrable derivative on $[z^{-1}, \infty)$. It then follows easily, by dominated convergence again, that

$$\lim_{\alpha \to 0} \overline{\lambda}_{X_{n+1}|X_n}(\alpha) = \kappa_X\, \mathbb{E}\left(\max(\epsilon, 0)^{\kappa_X}\right)^{-1} \cdot \int\int_{sz > 1, s > 0} \mathbb{P}\left(s\sqrt{az^2 + b}\,\epsilon_{n+1} > 1\right)\; f_\epsilon(z)\, s^{-\kappa_X - 1}\, ds\, dz. \tag{3.5}$$

We briefly check the convergence of (3.5): the integral is equal to

$$\frac{\kappa_X}{\mathbb{E}\left(\max(\epsilon, 0)^{\kappa_X}\right)} \int_0^\infty \left(\int_{z^{-1}}^\infty \mathbb{P}(s\sqrt{az^2 + b}\epsilon_{n+1} > 1)\, s^{-\kappa_X - 1} ds\right) f_\epsilon(z)\, dz,$$

and since probabilities are bounded by 1, the inner integral is bounded by $\kappa_X^{-1} z^\kappa$, and the whole expression by

$$\frac{1}{\mathbb{E}\left(\max(\epsilon, 0)\right)^\kappa} \int_0^\infty z^\kappa f_\epsilon(z)\, dz = 1,$$

as of course it should, since (3.5) represents a limit of probabilities.

By Fubini, we can also express $\overline{\lambda}_{X_{n+1}|X_n}$ by the alternative formula

$$\overline{\lambda}_{X_{n+1}|X_n} = \frac{\kappa_X}{\mathbb{E}\left(\max(\epsilon,0)^{\kappa_X}\right)} \int_0^\infty \mathbb{P}\left(\epsilon_{n+1}\sqrt{a\epsilon_n^2+b} > s^{-1}, \epsilon_n > s^{-1}\right) \frac{ds}{s^{\kappa_X+1}}.$$

In this form the formula easily generalizes to arbitrary lags, and we can state the following theorem:

Theorem 3.1. *Let (X_n, σ_n) be a strictly stationary GARCH(1, 1) such that the SRE for σ_n^2 satisfies the conditions for Kesten's theorem, with tail-index κ_{σ^2}. Suppose that the GARCH's innovations ϵ_n possess a probability density, and let $\kappa_X := 2\kappa_{\sigma^2}$. Then*

$$\overline{\lambda}_{X_{n+p}|X_n} = \frac{\kappa_X}{\mathbb{E}\left(\max(\epsilon,0)^{\kappa_X}\right)} \int_0^\infty \mathbb{P}\left(\epsilon_{n+p}\prod_{j=0}^{p-1}(a\epsilon_{n+j}^2+b)^{1/2} > s^{-1},\, \epsilon_n > s^{-1}\right) \frac{ds}{s^{\kappa_X+1}}.$$

Proof. It is probably easiest to first observe that $X_{n+p} = \sigma_{n+p}\epsilon_{n+p}$, where σ_{n+p}^2 is related to σ_n by an linear SRE of the form

$$\sigma_{n+p}^2 = A_{n+p;n}\sigma_n^2 + B_{n+p;n},$$

(found by iterating (3.2)), and then repeat our computation of $\overline{\lambda}_{X_{n+1}|X_n}$ above. Note that

$$A_{n+p;n} = \prod_{j=0}^{p-1} A_{n+j} = \prod_{j=0}^{p-1}(a\epsilon_{n+j}^2+b);$$

we will not need the explicit expression for $B_{n+p;n}$. Arguing as before we then find that

$$\begin{aligned}\overline{\lambda}_{X_{n+p}|X_n} \quad = \quad & \frac{\kappa_X}{\mathbb{E}\left(\max(\epsilon,0)^{\kappa_X}\right)} \int_0^\infty \int_{s^{-1}}^\infty \\ & \mathbb{P}\left(s\,\epsilon_{n+p}\prod_{j=1}^{p-1}(a\epsilon_{n+j}^2+b)^{1/2}(az_n^2+b)^{1/2} > 1\right) f_{\epsilon_n}(z_n)\,dz_n\,\frac{ds}{s^{\kappa_X+1}},\end{aligned}$$

which is Theorem 3.1. □

Remark 3.2. *We can write $\overline{\lambda}_{X_{n+p}|X_n}$ more explicitly as*

$$\begin{aligned}\frac{\kappa_X}{\mathbb{E}\left(\max(\epsilon,0)^{\kappa_X}\right)} \int_{s>0}\int_{z_n>s^{-1}}\int_{\mathbb{R}^{p-1}} \quad & \overline{F}_\epsilon\left(s^{-1}\prod_{j=0}^{p-1}(az_{n+j}^2+b)^{-1/2}\right) \\ & \prod_{j=0}^{p-1} f_\epsilon(z_{n+j}) \prod_{j=0}^{p-1} dz_{n+j}\,\frac{ds}{s^{\kappa_X+1}}.\end{aligned}$$

We verify that for $b = 0$, this formula reduces to formula (2.12) *for an ARCH(1). In fact, setting $b = 0$, and changing variables to $y = z_n s$, the integral becomes*

$$\frac{\kappa_X}{\mathbb{E}\left(\max(\epsilon,0)^{\kappa_X}\right)} \int_0^\infty \int_1^\infty \int_{\mathbb{R}^{p-1}} \overline{F}_\epsilon \left(a^{-p/2} \left(\prod_{j=1}^{p-1} |z_{n+j}|^{-1} \right) y^{-1} \right) f_\epsilon(s^{-1}y) \prod_{j=1}^{p-1} f_\epsilon(z_{n+j}) \prod_{j=1}^{p-1} dz_{n+j}\, dy \, \frac{ds}{s^{\kappa_X+2}},$$

and we can carry out the s-integration:

$$\begin{aligned} \int_0^\infty f_\epsilon(s^{-1}y) \frac{ds}{s^{\kappa_X+2}} &= y^{-\kappa_X-1} \int_0^\infty f_\epsilon(w) w^{\kappa_X} \, dw \\ &= \mathbb{E}\left(\max(\epsilon,0)^{\kappa_X}\right) y^{-\kappa_X-1}. \end{aligned}$$

We therefore find that $\overline{\lambda}_{X_{n+p}|X_n}$ equals

$$\kappa_X \int_1^\infty \int_{\mathbb{R}^{p-1}} \overline{F}_\epsilon \left(a^{-p/2} \left(\prod_{j=1}^{p-1} |z_{n+j}|^{-1} \right) y^{-1} \right) \prod_{j=1}^{p-1} f_\epsilon(z_{n+j}) \prod_{j=1}^{p-1} dz_{n+j} \frac{dy}{y^{\kappa_X+1}},$$

which is (2.12).

We finally note the following analogue of Theorem 2.2 for stationary GARCH(1,1) processes which generalizes [3, Theorem 5] for ARCH(1)'s:

Theorem 3.3. *For a strictly stationary GARCH(1,1) as in Theorem* 3.1 *but now with ϵ_n only required to have a continuous cumulative probability distribution, we have that $\overline{\lambda}^{\psi}_{X_{n+p}|X_n} = \mathbb{P}(\epsilon > 0)$, for all ψ such that $\alpha = o(\psi(\alpha))$ for $\alpha \to 0$ and all $p \geq 1$.*

The proof is similar to the proof of Theorem 3.1, starting off from

$$\begin{aligned} &\mathbb{P}\left(X_{n+1} > \overline{q}_X(\psi(\alpha)) \,|\, X_n > \overline{q}_X(\alpha)\right) \\ = &\int_0^\infty \int_{z^{-1}}^\infty \mathbb{P}\left(\sqrt{\overline{q}_X(\alpha)^{-2}\omega + (az^2+b)s^2}\, \epsilon_{n+1} > \frac{\overline{q}_X(\psi(\alpha))}{\overline{q}_X(\alpha)} \right) dG_\alpha(s)\, f_\epsilon(z) dz, \end{aligned}$$

and using that the probability in the integrand of this integral now tends to $\mathbb{P}(\epsilon_{n+1} > 0)$. Since

$$\int_{z^{-1}}^\infty dG_\alpha(s) \to \left(\max(\epsilon,0)^{\kappa_X}\right)^{-1} z^\kappa,$$

the theorem follows. Details are left to the reader.

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