DL-Lite with Attributes and Sub-Roles (Extended Abstract)

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1 Introduction

The *DL-Lite family* of description logics has recently been proposed and investigated in [5–7] and later extended in [1, 8, 3]. The relevance of the *DL-Lite* family is witnessed by the fact that it forms the basis of OWL 2 QL, one of the three profiles of OWL 2 (http://www.w3.org/TR/owl2-profiles/). According to the official W3C profiles document, the purpose of OWL 2 QL is to be the language of choice for applications that use very large amounts of data.

This paper extends the *DL-Lite* languages of [3] by relaxing the restriction on the interaction between cardinality constraints ($\mathcal{N}$) and role inclusions (or hierarchies, $\mathcal{H}$). We also introduce a new family of languages, $\textit{DL-Lite}_{\alpha}^{\mathcal{HNA}}$, $\alpha \in \{\textit{core}, \textit{krom}, \textit{horn}, \textit{bool}\}$, extending *DL-Lite* with *attributes* ($\mathcal{A}$).

The notion of attributes, borrowed from conceptual modeling formalisms, introduces a distinction between (abstract) objects and application domain values, and consequently, between concepts (sets of objects) and datatypes (sets of values), and between roles (relating objects to objects) and attributes (relating objects to values). The advantage of the presented languages over $\textit{DL-Lite}_{\mathcal{A}}$ [8] is that the range restrictions for attributes can be *local* (and not only *global* as in $\textit{DL-Lite}_{\mathcal{A}}$). Indeed, $\textit{DL-Lite}_{\alpha}^{\mathcal{HNA}}$ has a possibility to express concept inclusion axioms of the form $C \sqsubseteq \forall U.T$, for an attribute U and its datatype T. In this way, we allow re-use of the very same attribute associated to different concepts with different range restrictions. For example, we can say that employees' salary is of type *Integer*, researchers' salary is in the range 35,000–70,000 (enumeration type) and professors' salary in the range 55,000–100,000—while both researchers and professors are employees. Note that local attributes are strictly more expressive than global ones. For example, concept disjointness (or unsatisfiability) can be inferred just from datatype disjointness for the same (existentially qualified) attribute. Since *DL-Lite* languages have been proved useful in capturing conceptual data models [8, 4, 2], the extension with attributes, as presented here, will generalize their use even further.

We aim at establishing computational complexity of knowledge base satisfiability in these new DLs. In particular we prove the following results:

1. We can relax the restrictions presented in [3] limiting the interaction between sub-roles and number restrictions without increasing the complexity of reasoning as far as the problem is limited to TBox satisfiability checking. As

for KB satisfiability, the presence of the ABox should be taken into account if we want to preserve the complexity results.
2. The introduction of *local* range restrictions for attributes is for free for the languages $\textit{DL-Lite}^{\mathcal{NA}}_{bool}$, $\textit{DL-Lite}^{\mathcal{NA}}_{horn}$ and $\textit{DL-Lite}^{\mathcal{NA}}_{core}$.

2 The Description Logic $\textit{DL-Lite}^{\mathcal{HNA}}_{bool}$

The language of $\textit{DL-Lite}^{\mathcal{HNA}}_{bool}$ contains *object names* $a_0, a_1, \ldots$, *value names* $v_1, v_2, \ldots$, *concept names* $A_0, A_1, \ldots$, *role names* $P_0, P_1, \ldots$, *attribute names* $U_0, U_1, \ldots$, and *datatype names* $T_0, T_1, \ldots$. Complex *roles* R and *concepts* C are defined below:

$$\begin{array}{rcl} R & ::= & P_i \mid P_i^-, \\ B & ::= & \top \mid \bot \mid A_i \mid \geq q\, R \mid \geq q\, U_i \\ C & ::= & B \mid \neg C \mid C_1 \sqcap C_2, \end{array}$$

where q is a positive integer. The concepts of the form B are called *basic concepts*.

A $\textit{DL-Lite}^{\mathcal{HNA}}_{bool}$ *TBox*, $\mathcal{T}$, is a finite set of concept, role, attribute and datatype *inclusion axioms* of the form:

$$C_1 \sqsubseteq C_2, \quad C \sqsubseteq \forall U.T, \quad R_1 \sqsubseteq R_2, \quad U \sqsubseteq U', \quad T \sqsubseteq T', \quad T \sqcap T' \sqsubseteq \bot,$$

and an *ABox*, $\mathcal{A}$, is a finite set of assertions of the form:

$$A_k(a_i), \quad \neg A_k(a_i), \quad P_k(a_i, a_j), \quad \neg P_k(a_i, a_j), \quad U_k(a_i, v_j) \quad \text{and} \quad T_k(v_j).$$

We standardly abbreviate $\geq 1\, R$ and $\geq 1\, U$ by $\exists R$ and $\exists U$, respectively. Absence of an attribute (i.e., $C \sqsubseteq \forall U.\bot$) can be expressed as $C \sqcap \exists U \sqsubseteq \bot$.

Together, a TBox $\mathcal{T}$ and an ABox $\mathcal{A}$ constitute the $\textit{DL-Lite}^{\mathcal{HNA}}_{bool}$ *knowledge base* (KB) $\mathcal{K} = (\mathcal{T}, \mathcal{A})$. In the following, we denote by $\textit{role}(\mathcal{K})$ and $\textit{att}(\mathcal{K})$ the sets of role and attribute names occurring in $\mathcal{K}$, respectively; $\textit{role}^{\pm}(\mathcal{K})$ denotes the set $\{P_k, P_k^- \mid P_k \in \textit{role}(\mathcal{K})\}$.

Semantics. As usual in description logic, an *interpretation*, $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, consists of a nonempty *domain* $\Delta^{\mathcal{I}}$ and an interpretation function $\cdot^{\mathcal{I}}$. The interpretation domain $\Delta^{\mathcal{I}}$ is the union of two non-empty *disjoint* sets: the *domain of objects* $\Delta^{\mathcal{I}}_O$ and the *domain of values* $\Delta^{\mathcal{I}}_V$. We assume that all interpretations agree on the semantics assigned to each datatype T_i, as well as on each constant v_i. In particular, $T_i^{\mathcal{I}} = \textit{val}(T_i) \subseteq \Delta^{\mathcal{I}}_V$ is the set of values of datatype T_i, and each v_i is interpreted as one specific value, denoted $\textit{val}(v_i)$, i.e., $v_i^{\mathcal{I}} = \textit{val}(v_i) \in \textit{val}(T_i)$. Furthermore, $\cdot^{\mathcal{I}}$ assigns to each object name a_i an element $a_i^{\mathcal{I}} \in \Delta^{\mathcal{I}}_O$, to each concept name A_k a subset $A_k^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}_O$ of the domain of objects, to each role name P_k a binary relation $P_k^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}_O \times \Delta^{\mathcal{I}}_O$ over the domain of objects, and to each attribute name U_k a binary relation $U_k^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}_O \times \Delta^{\mathcal{I}}_V$. We adopt here the *unique name assumption* (UNA): $a_i^{\mathcal{I}} \neq a_j^{\mathcal{I}}$, for all $i \neq j$. The role and concept constructs are interpreted in $\mathcal{I}$ in the standard way:

$$
\begin{aligned}
(P_k^-)^{\mathcal{I}} &= \{(y,x) \in \Delta_O^{\mathcal{I}} \times \Delta_O^{\mathcal{I}} \mid (x,y) \in P_k^{\mathcal{I}}\}, && \text{(inverse role)}\\
\top^{\mathcal{I}} &= \Delta_O^{\mathcal{I}}, && \text{(domain of objects)}\\
\bot^{\mathcal{I}} &= \emptyset, && \text{(the empty set)}\\
(\geq q\, R)^{\mathcal{I}} &= \big\{x \in \Delta_O^{\mathcal{I}} \mid \sharp\{y \mid (x,y) \in R^{\mathcal{I}}\} \geq q\big\}, && \text{(at least } q\ R\text{-successors)}\\
(\geq q\, U)^{\mathcal{I}} &= \big\{x \in \Delta_O^{\mathcal{I}} \mid \sharp\{v \mid (x,v) \in U^{\mathcal{I}}\} \geq q\big\}, && \text{(at least } q\ U\text{-attributes)}\\
(\forall U.\, T)^{\mathcal{I}} &= \big\{x \in \Delta_O^{\mathcal{I}} \mid \forall v.\, (x,v) \in U^{\mathcal{I}} \to v \in T^{\mathcal{I}}\big\}, && \text{(attribute value restriction)}\\
(\neg C)^{\mathcal{I}} &= \Delta_O^{\mathcal{I}} \setminus C^{\mathcal{I}}, && \text{(not in } C\text{)}\\
(C_1 \sqcap C_2)^{\mathcal{I}} &= C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}, && \text{(both in } C_1 \text{ and in } C_2\text{)}
\end{aligned}
$$

where $\sharp X$ is the cardinality of X. The *satisfaction relation* $\models$ is also standard:

$$
\begin{array}{rcl@{\qquad}rcl}
\mathcal{I} \models C_1 \sqsubseteq C_2 & \text{iff} & C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}, & \mathcal{I} \models R_1 \sqsubseteq R_2 & \text{iff} & R_1^{\mathcal{I}} \subseteq R_2^{\mathcal{I}},\\
\mathcal{I} \models T_1 \sqsubseteq T_2 & \text{iff} & T_1^{\mathcal{I}} \subseteq T_2^{\mathcal{I}}, & \mathcal{I} \models U_1 \sqsubseteq U_2 & \text{iff} & U_1^{\mathcal{I}} \subseteq U_2^{\mathcal{I}},\\
\mathcal{I} \models T_1 \sqcap T_2 \sqsubseteq \bot & \text{iff} & T_1^{\mathcal{I}} \cap T_2^{\mathcal{I}} = \emptyset, & \mathcal{I} \models P_k(a_i, a_j) & \text{iff} & (a_i^{\mathcal{I}}, a_j^{\mathcal{I}}) \in P_k^{\mathcal{I}},\\
\mathcal{I} \models A_k(a_i) & \text{iff} & a_i^{\mathcal{I}} \in A_k^{\mathcal{I}}, & \mathcal{I} \models \neg P_k(a_i, a_j) & \text{iff} & (a_i^{\mathcal{I}}, a_j^{\mathcal{I}}) \notin P_k^{\mathcal{I}}\\
\mathcal{I} \models \neg A_k(a_i) & \text{iff} & a_i^{\mathcal{I}} \notin A_k^{\mathcal{I}}, & \mathcal{I} \models U_k(a_i, v_j) & \text{iff} & (a_i^{\mathcal{I}}, v_j^{\mathcal{I}}) \in U_k^{\mathcal{I}},\\
 & & & \mathcal{I} \models T_k(v_j) & \text{iff} & v_j^{\mathcal{I}} \in T_k^{\mathcal{I}}.
\end{array}
$$

A KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is said to be *satisfiable* (or *consistent*) if there is an interpretation, $\mathcal{I}$, satisfying all the members of $\mathcal{T}$ and $\mathcal{A}$. In this case we write $\mathcal{I} \models \mathcal{K}$ (as well as $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{A}$) and say that $\mathcal{I}$ is a *model of* $\mathcal{K}$ (of $\mathcal{T}$ and $\mathcal{A}$).

2.1 Fragments of *DL-Lite*$_{bool}^{\mathcal{HNA}}$

We consider various syntactical restrictions on the language of *DL-Lite*$_{bool}^{\mathcal{HNA}}$ along two axes: (i) the Boolean operators ($_{bool}$) on concepts, (ii) the role and attribute inclusions ($\mathcal{H}$). Similarly to classical logic, we adopt the following definitions. A TBox $\mathcal{T}$ will be called a *Krom TBox*[1] if its concept inclusions are restricted to:

$$B_1 \ \sqsubseteq\ B_2, \qquad B_1 \ \sqsubseteq\ \neg B_2 \qquad \text{and} \qquad \neg B_1 \ \sqsubseteq\ B_2, \tag{Krom}$$

(here and below all the B_i and B are basic concepts). $\mathcal{T}$ will be called a *Horn TBox* if its concept inclusions are restricted to:

$$\bigsqcap_k B_k \ \sqsubseteq\ B. \tag{Horn}$$

Finally, we call $\mathcal{T}$ a *core TBox* if its concept inclusions are restricted to:

$$B_1 \ \sqsubseteq\ B_2 \qquad \text{and} \qquad B_1 \sqcap B_2 \ \sqsubseteq\ \bot. \tag{core}$$

[1] The Krom fragment of first-order logic consists of all formulas in prenex normal form whose quantifier-free part is a conjunction of binary clauses.

As $B_1 \sqsubseteq \neg B_2$ is equivalent to $B_1 \sqcap B_2 \sqsubseteq \bot$, core TBoxes can be regarded as sitting in the intersection of Krom and Horn TBoxes. In this paper we study the following logics, for $\alpha \in \{core, krom, horn, bool\}$:

DL-Lite$^{\mathcal{HNA}}_{krom}$*, *DL-Lite$^{\mathcal{HNA}}_{horn}$*, *DL-Lite$^{\mathcal{HNA}}_{core}$ are the fragments of *DL-Lite*$^{\mathcal{HNA}}_{bool}$ with Krom, Horn, and core TBoxes, respectively;

DL-Lite$^{\mathcal{HN}}_{\alpha}$ is the fragment of *DL-Lite*$^{\mathcal{HNA}}_{\alpha}$ without attributes and datatypes;

DL-Lite$^{\mathcal{NA}}_{\alpha}$ is the fragment of *DL-Lite*$^{\mathcal{HNA}}_{\alpha}$ without role and attribute inclusions.

As shown in [3], reasoning in *DL-Lite*$^{\mathcal{HN}}_{\alpha}$ is already rather costly (ExpTime-complete) due to the interaction between role inclusions and number restrictions. However, both of these constructs turn out to be useful for the purposes of conceptual modeling. By limiting their interplay one can get languages with a better computational properties [8, 3]. Before presenting such limitations we need to introduce some notation. For a role R, let $inv(R) = P_k^-$ if $R = P_k$ and $inv(R) = P_k$ if $R = P_k^-$. Given a TBox $\mathcal{T}$ we denote by $\sqsubseteq^*_{\mathcal{T}}$ the reflexive and transitive closure of the relation $\{(R, R'), (inv(R), inv(R')) \mid R \sqsubseteq R' \in \mathcal{T}\}$. We say that $R \equiv^*_{\mathcal{T}} R'$ iff $R \sqsubseteq^*_{\mathcal{T}} R'$ and $R' \sqsubseteq^*_{\mathcal{T}} R$. Say that R' is a *proper sub-role* of R in $\mathcal{T}$ if $R' \sqsubseteq^*_{\mathcal{T}} R$ and $R \not\sqsubseteq^*_{\mathcal{T}} R'$. A proper sub-role R' of R is said to be a *direct sub-role* of R if there is no other proper sub-role R'' of R such that R' is a proper sub-role of R''; the set of direct sub-roles of R is denoted as $dsub_{\mathcal{T}}(R)$.

The language *DL-Lite*$^{(\mathcal{HN})}_{\alpha}$ [3] is the result of imposing the following syntactic restriction on *DL-Lite*$^{\mathcal{HN}}_{\alpha}$ TBoxes $\mathcal{T}$:

(inter) if R has a proper sub-role in $\mathcal{T}$ then $\mathcal{T}$ contains no negative occurrences of number restrictions $\geq q\ R$ or $\geq q\ inv(R)$ with $q \geq 2$

(an occurrence of a concept on the right-hand (left-hand) side of a concept inclusion is called *negative* if it is in the scope of an odd (even) number of negations $\neg$; otherwise it is called *positive*). We will formulate two alternative versions of restriction **(inter)**.

Definition 1. *Given a TBox $\mathcal{T}$ and a role $R \in role^{\pm}(\mathcal{T})$, we define the following parameters:*

$$ubound(R, \mathcal{T}) = \min\big(\{\infty\} \cup \{q - 1 \mid q \geq 2 \text{ and } \geq q\, R \text{ occurs negatively in } \mathcal{T}\}\big),$$
$$lbound(R, \mathcal{T}) = \max\big(\{0\} \cup \{q \mid \geq q\, R \text{ occurs positively in } \mathcal{T}\}\big),$$
$$rank(R, \mathcal{T}) = \max\big(lbound(R, \mathcal{T}), \textstyle\sum_{R' \in dsub_{\mathcal{T}}(R)} rank(R', \mathcal{T})\big),$$
$$rank(R, \mathcal{A}) = \max\big(\{0\} \cup \{n \mid R_i(a, a_i) \in \mathcal{A}, R_i \sqsubseteq^*_{\mathcal{T}} R, \text{ for distinct } a_1, \ldots, a_n\}\big).$$

Consider the languages obtained from *DL-Lite*$^{\mathcal{HN}}_{\alpha}$ by imposing one of the following two restrictions:

(inter1) for every $R \in role^{\pm}(\mathcal{T})$, if R has a proper sub-role in $\mathcal{T}$ then $ubound(R, \mathcal{T}) \geq rank(R, \mathcal{T})$;

(inter2) for every $R \in role^{\pm}(\mathcal{T})$, if R has a proper sub-role in $\mathcal{T}$ then $ubound(R, \mathcal{T}) \geq rank(R, \mathcal{T}) + \max\{1, rank(R, \mathcal{A})\}$.

language	**(inter)** [3]	**(inter1)**	**(inter2)**	**non-restrict.**
$DL\text{-}Lite^{\mathcal{HN}}_{core}$	NLogSpace [3]	≥NP [Th.2]	NLogSpace [Th.1]	ExpTime [3]
$DL\text{-}Lite^{\mathcal{HN}}_{horn}$	PTime [3]		PTime [Th.1]	
$DL\text{-}Lite^{\mathcal{HN}}_{krom}$	NLogSpace [3]		NLogSpace [Th.1]	
$DL\text{-}Lite^{\mathcal{HN}}_{bool}$	NP [3]		NP [Th.1]	
$DL\text{-}Lite^{\mathcal{HNA}}_{core}$	NLogSpace [Th.3]	≥NP [Th.2]	NLogSpace [Th.3]	ExpTime
$DL\text{-}Lite^{\mathcal{HNA}}_{horn}$	PTime [Th.3]		PTime [Th.3]	
$DL\text{-}Lite^{\mathcal{HNA}}_{krom}$	NP [Th.4]		NP [Th.4]	
$DL\text{-}Lite^{\mathcal{HNA}}_{bool}$	NP [Th.3]		NP [Th.3]	
$DL\text{-}Lite^{\mathcal{NA}}_{core}$	NA	NA	NA	NLogSpace [Th.3]
$DL\text{-}Lite^{\mathcal{NA}}_{horn}$				PTime [Th.3]
$DL\text{-}Lite^{\mathcal{NA}}_{krom}$				NP [Th. 4]
$DL\text{-}Lite^{\mathcal{NA}}_{bool}$				NP [Th.3]

Table 1: Complexity of *DL-Lite* logics (NA = Non-Applicable).

These new restrictions are in some way weaker than **(inter)** and, for example, allow for the specialization of functional roles: KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with $\mathcal{T} = \{\geq 2\, R \sqsubseteq \bot,\ R_1 \sqsubseteq R_2,\ R_2 \sqsubseteq R\}$, and $\mathcal{A} = \{R(a,b), R_1(a_1,b_1), R_2(a_2,b_2)\}$ does not satisfy **(inter)**, but it satisfies both **(inter1)** and **(inter2)**. Finally, the above restrictions can also be applied to sub-attributes in the languages $DL\text{-}Lite^{\mathcal{HNA}}_{\alpha}$. Table 1 summarizes the obtained complexity results (with numerical parameters q coded in binary).

3 Reasoning in $DL\text{-}Lite^{\mathcal{HN}}_{\alpha}$

In this section, we investigate the complexity of deciding KB satisfiability in languages $DL\text{-}Lite^{\mathcal{HN}}_{\alpha}$ under the restrictions **(inter1)** and **(inter2)**, respectively.

We adapt the proof presented in [3], where a $DL\text{-}Lite^{\mathcal{HN}}_{bool}$ KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is encoded into a sentence $\mathcal{K}^{\ddagger_e}$ in the one-variable first-order logic $\mathcal{QL}^1$. We use a slightly longer but simpler encoding. Every $a_i \in ob(\mathcal{A})$ is associated to the individual constant a_i of $\mathcal{QL}^1$, and every concept name A_i to the unary predicate $A_i(x)$. For each concept $\geq q\, R$ in $\mathcal{K}$ we introduce a fresh unary predicate $E_qR(x)$. For each role name $P_k \in role^{\pm}(\mathcal{K})$, two individual constants dp_k and dp_k^- are introduced, as representatives of the objects in the domain and range of P_k, respectively. The encoding C^* of a concept C is defined inductively:

$$\begin{array}{lll} \bot^* = \bot, & (A_i)^* = A_i(x), & (\geq q\, R)^* = E_qR(x), \\ \top^* = \top, & (\neg C)^* = \neg C^*(x), & (C_1 \sqcap C_2)^* = C_1^*(x) \wedge C_2^*(x). \end{array}$$

The $\mathcal{QL}^1$ sentence encoding the knowledge base $\mathcal{K}$ is defined as follows:

$$\mathcal{K}^{\ddagger_e} \quad = \quad \forall x \big[\mathcal{T}^*(x) \ \wedge\ \mathcal{T}^{\mathcal{R}}(x) \ \wedge \bigwedge_{R \in role^{\pm}(\mathcal{K})} \big(\epsilon_R(x) \ \wedge\ \delta_R(x)\big)\big] \ \wedge\ \mathcal{A}^{\ddagger_e}.$$

Formulas $\mathcal{T}^*(x)$, the $\delta_R(x)$, for $R \in \mathit{role}^{\pm}(\mathcal{K})$, and $T^{\mathcal{R}}(x)$ encode the TBox $\mathcal{T}$:

$$\mathcal{T}^*(x) = \bigwedge_{C_1 \sqsubseteq C_2 \in \mathcal{T}} \big(C_1^*(x) \to C_2^*(x)\big), \qquad \delta_R(x) = \bigwedge_{q,q' \in Q_{\mathcal{T}}^R,\ \ q' > q} \big(E_{q'} R(x) \to E_q R(x)\big),$$

$$\mathcal{T}^{\mathcal{R}}(x) = \bigwedge_{R \sqsubseteq_{\mathcal{T}}^* R'} \ \bigwedge_{q \in Q_{\mathcal{T}}^R} \big(E_q R(x) \to E_q R'(x)\big),$$

where $Q_{\mathcal{T}}^R$ contains 1, all q such that $\geq q\, R$ occurs in $\mathcal{T}$ and all $Q_{\mathcal{T}}^{R'}$, for $R' \sqsubseteq_{\mathcal{T}}^* R$. Sentence $\mathcal{A}^{\ddagger_{\mathbf{e}}}$ encodes the ABox $\mathcal{A}$:

$$\mathcal{A}^{\ddagger_{\mathbf{e}}} = \bigwedge_{A_k(a_i) \in \mathcal{A}} A_k(a_i) \quad \wedge \bigwedge_{\neg A_k(a_i) \in \mathcal{A}} \neg A_k(a_i) \quad \wedge \bigwedge_{\substack{a,a' \in ob(\mathcal{A}) \\ R' \sqsubseteq_{\mathcal{T}}^* R,\ R'(a,a') \in \mathcal{A}}} E_{q_{R,a}^{\mathbf{e}}} R(a) \quad \wedge \bigwedge_{\substack{\neg P_k(a_i,a_j) \in \mathcal{A} \\ R(a_i,a_j) \in \mathcal{A},\ R \sqsubseteq_{\mathcal{T}}^* P_k}} \bot,$$

where $q_{R,a}^{\mathbf{e}}$ is the maximum number in $Q_{\mathcal{T}}^R$ such that there are $q_{R,a}^{\mathbf{e}}$ many distinct a_i with $R_i(a, a_i) \in \mathcal{A}$ and $R_i \sqsubseteq_{\mathcal{T}}^* R$. For each $R \in \mathit{role}^{\pm}(\mathcal{K})$, we also need the following formula expressing the fact that the range of R is not empty whenever its domain is non-empty:

$$\epsilon_R(x) = E_1 R(x) \to \mathit{inv}(E_1 R(dr)),$$

where $\mathit{inv}(E_1 R(dr))$ is $E_1 P_k^-(dp_k^-)$ if $R = P_k$ and $E_1 P_k(dp_k)$ if $R = P_k^-$.

Lemma 1. *A DL-Lite$_{bool}^{\mathcal{HN}}$ knowledge base under restriction* **(inter2)** *is satisfiable iff the $\mathcal{QL}^1$-sentence $\mathcal{K}^{\ddagger_{\mathbf{e}}}$ is satisfiable.*

Proof. (Sketch) The only challenging direction is ($\Leftarrow$). To prove it, we adapt the proofs of Theorem 5.2 and Lemma 5.14 in [3]. The idea of the proof is to construct a *DL-Lite$_{bool}^{\mathcal{HN}}$* interpretation $\mathcal{I}$, from $\mathfrak{M}$, the minimal Herbrand model of $\mathcal{K}^{\ddagger_{\mathbf{e}}}$. We denote the interpretations of unary predicates P and constants a of $\mathcal{QL}^1$ in $\mathfrak{M}$ by $P^{\mathfrak{M}}$ and $a^{\mathfrak{M}}$, respectively. Let $D = ob(\mathcal{A}) \cup \{dp_k, dp_k^- \mid P_k \in \mathit{role}(\mathcal{K})\}$ be the domain of $\mathfrak{M}$. Then $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is defined inductively: $\Delta^{\mathcal{I}} = \bigcup_{m=0}^{\infty} W_m$, such that W_0 is the set $D_0 = ob(\mathcal{A})$, and for every $a_i \in ob(\mathcal{A})$, $a_i^{\mathcal{I}} = a_i^{\mathfrak{M}}$. Each set W_{m+1}, $m \geq 0$, is constructed by adding to W_m fresh *copies* of certain elements from $D \setminus ob(\mathcal{A})$. The extensions $A_k^{\mathcal{I}}$ of concept names A_k are defined by taking

$$A_k^{\mathcal{I}} = \{w \in \Delta^{\mathcal{I}} \mid \mathfrak{M} \models A_k^*[cp(w)]\}, \tag{1}$$

where $cp(w)$ is the element $d \in D$ of which w is a copy.

The interpretation for each role P_k, is defined inductively as $P_k^{\mathcal{I}} = \bigcup_{m=0}^{\infty} P_k^m$, where $P_k^m \subseteq W_m \times W_m$, along with the construction of $\Delta^{\mathcal{I}}$. The initial interpretation for each role name P_k is defined as follows:

$$P_k^0 = \{(a_i^{\mathfrak{M}}, a_j^{\mathfrak{M}}) \in W_0 \times W_0 \mid R(a_i, a_j) \in \mathcal{A} \text{ and } R \sqsubseteq_{\mathcal{T}}^* P_k\}. \tag{2}$$

For every $R \in \mathit{role}^{\pm}(\mathcal{K})$, the *required R-rank* $r(R, d)$ of $d \in D$ is defined by taking $r(R, d) = \max\big(\{0\} \cup \{q \in Q_{\mathcal{T}}^R \mid \mathfrak{M} \models E_q R[d]\}\big)$. The *actual R-rank* $r_m(R, w)$ of a point $w \in \Delta^{\mathcal{I}}$ at step m is

$$r_m(R, w) = \begin{cases} \sharp\{w' \in W_{m+1} \mid (w, w') \in P_k^{m+1}\}, & \text{if } R = P_k, \\ \sharp\{w' \in W_{m+1} \mid (w', w) \in P_k^{m+1}\}, & \text{if } R = P_k^-. \end{cases}$$

Assume that W_m and P_k^m, $m \geq 0$, have been already defined. Let $W_{m+1} = \emptyset$ and $P_k^{m+1} = \emptyset$, for each role name P_k. If we had $r_m(R, w) = r(R, cp(w))$, for each role R and $w \in W_m$, then the interpretation we need would be constructed. However, the actual rank of some points could still be smaller than the required rank. We cure these defects by adding R-successors for them. Note that the 'curing' process for a given w and R, not only increases the actual R-rank of w, but also all its R'-ranks, for all $R \sqsubseteq^*_{\mathcal{T}} R'$. At this point we adapt the construction in [3] to obtain the interpretation $\mathcal{I}$ we are intending. For each $P_k \in \mathit{role}(\mathcal{K})$, we consider two sets of defects in P_k^m: $\Lambda_k^m = \{w \in W_m \setminus W_{m-1} \mid r_m(P_k, w) < r(P_k, cp(w))\}$ and $\Lambda_k^{m-} = \{w \in W_m \setminus W_{m-1} \mid r_m(P_k^-, w) < r(P_k^-, cp(w))\}$.

In each equivalence class $[R] = \{S \mid S \equiv^*_{\mathcal{T}} R\}$ we select a single role, a *representative*. Let $G = (\mathit{Rep}^*_{\mathcal{T}}, E)$ be a directed graph such that $\mathit{Rep}^*_{\mathcal{T}}$ is the set of representatives and $(R, R') \in E$ iff R is a proper sub-role of R'. Clearly, G is a directed acyclic graph and so, by a topological sort, one can assign to each representative a unique number smaller than the number of all its descendants in G. We use the ascending total order induced on G when choosing an element P_k in $\mathit{Rep}^*_{\mathcal{T}}$, and extend in that way W_m and P_k^m to W_{m+1} and P_k^{m+1}, respectively.

(Λ_k^m) Let $w \in \Lambda_k^m, q = r(P_k, cp(w)) - r_m(P_k, w)$, $d = cp(w)$. There is $q' \geq q > 0$ with $\mathfrak{M} \models E_{q'} P_k[d]$. Then, $\mathfrak{M} \models E_1 P_k[d]$ and $\mathfrak{M} \models E_1 P_k^-[dp_k^-]$. In this case we take q *fresh* copies $w_1', \dots, w_q'$ of dp_k^-, add them to W_{m+1} and for each $1 \leq i \leq q$, set $cp(w_i') = dp_k^-$, add the pairs (w, w_i') to each P_j^{m+1} with $P_k \sqsubseteq^*_{\mathcal{T}} P_j$ and the pairs (w_i', w) to each P_j^{m+1} with $P_k^- \sqsubseteq^*_{\mathcal{T}} P_j$ (note that by adding pairs to P_j^{m+1} we change its the actual rank);

(Λ_k^{m-}) This rule is the mirror image of (Λ_k^m): P_k and dp_k^- are replaced with P_k^- and dp_k, respectively.

We need to show that, for all $w \in \Delta^{\mathcal{I}}$ and all $\geq q\, R$ in $\mathcal{T}$,

($\mathbf{a}_1$) if $\geq q\, R$ occurs positively in $\mathcal{T}$ then $\mathfrak{M} \models E_q R[cp(w)]$ implies $w \in (\geq q\, R)^{\mathcal{I}}$;
($\mathbf{a}_2$) if $\geq q\, R$ occurs negatively in $\mathcal{T}$ then $w \in (\geq q\, R)^{\mathcal{I}}$ implies $\mathfrak{M} \models E_q R[cp(w)]$.

Consider first $w \in W_0$. It should be clear that actual R-rank of w

$$r_0(R, w) \quad \leq \quad \mathit{rank}(R, \mathcal{A}) + \textstyle\sum_{R' \in \mathit{dsub}_{\mathcal{T}}(R)} \mathit{rank}(R', \mathcal{T})$$

and so, by **(inter2)**, the total number of R-successors before we cure the defects does not exceed $\mathit{ubound}(R, \mathcal{T})$. If $\mathit{ubound}(R, \mathcal{T}) = \infty$ then there are no negative occurrences of $\geq q\, R$ with $q \geq 2$ and, although may have $r_m(R, w) \geq r(R, cp(w))$ after curing the defects of R, both **($\mathbf{a}_1$)** and **($\mathbf{a}_2$)** hold. Otherwise, we have $\mathit{ubound}(R, \mathcal{T}) + 1 \in Q^R_{\mathcal{T}}$ and so, by **(inter2)**, $\max Q^R_{\mathcal{T}} > \mathit{rank}(R, \mathcal{T}) + \mathit{rank}(R, \mathcal{A})$, whence $r_0(R, w) < \max Q^R_{\mathcal{T}}$. So, as $r(R, cp(w)) \leq \mathit{lbound}(R, \mathcal{T})$ and $\mathit{lbound}(R, \mathcal{T}) < \mathit{ubound}(R, \mathcal{T}) < \max Q^R_{\mathcal{T}}$, after curing the defect, we will have $r_m(R, w) = r(R, cp(w))$, for all $m > 0$, and both **($\mathbf{a}_1$)** and **($\mathbf{a}_2$)** hold. The case with $w \in W_{m_0} \setminus W_{m_0 - 1}$, for $m_0 > 0$ is similar, only now

$$r_{m_0}(R, w) \quad \leq \quad 1 + \textstyle\sum_{R' \in \mathit{dsub}_{\mathcal{T}}(R)} \mathit{rank}(R', \mathcal{T}).$$

Finally, we show that $\mathcal{I} \models \varphi$ for each $\varphi \in \mathcal{K}$. For $\varphi = A_k(a_i)$, $\varphi = \neg A_k(a_i)$ the claim is by the definition of $A_k^{\mathcal{I}}$. For $\varphi = \neg P_k(a_i, a_j)$, we have $(a_i, a_j) \in P_k^{\mathcal{I}}$ iff $(a_i, a_j) \in P_k^0$ iff $R(a_i, a_j) \in \mathcal{A}$ and $R \sqsubseteq_{\mathcal{T}}^* P_k$. By induction on the structure of concepts and (a_1) and (a_2), one can show that $\mathcal{I} \models C_1 \sqsubseteq C_2$ whenever $\mathfrak{M} \models \forall x(C_1^*(x) \to C_2^*(x))$, for each $\varphi = C_1 \sqsubseteq C_2$. Finally, $\mathcal{I} \models \varphi$ holds by definition in case $\varphi = R_1 \sqsubseteq R_2 \in \mathcal{T}$.

Theorem 1. *Under restriction* **(inter2)***, checking KB satisfiability is* NP-*complete in DL-Lite*$_{bool}^{\mathcal{HN}}$, PTIME-*complete in DL-Lite*$_{horn}^{\mathcal{HN}}$ *and* NLOGSPACE-*complete in both DL-Lite*$_{krom}^{\mathcal{HN}}$ *and DL-Lite*$_{core}^{\mathcal{HN}}$.

We now consider the case where the restriction **(inter1)** is imposed on the interaction between sub-roles and number restrictions. In presence of an ABox, **(inter2)** restricts the number of R-successors in the ABox, which appears to be a strong constraint on the instances of the ABox. On the other hand, the less restrictive condition **(inter1)**, which does not impose any bound on R-successors in the ABox, does not come for free, as shown by the following theorem:

Theorem 2. *Under restriction* **(inter1)***, checking KB satisfiability is* NP-*hard even in DL-Lite*$_{core}^{\mathcal{HN}}$.

Proof. We show that graph 3-colorability can be reduced to KB satisfiability. Let $G = (V, E)$ be a graph with vertices V and edges E and $\{r, g, b\}$ be three colors. Consider the following KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with role names v_i and w and object names o, r, g, b and the x_i, for each vertex $v_i \in V$:

$$\begin{aligned}\mathcal{T} =& \{\geq (|V| + 4)\, w \sqsubseteq \bot\} \cup \{v_i \sqsubseteq w,\ B_1 \sqsubseteq \exists v_i,\ B_2 \sqcap \exists v_i^- \sqsubseteq \bot \mid v_i \in V\} \cup \\ & \{\exists v_i^- \sqcap \exists v_j^- \sqsubseteq \bot \mid (v_i, v_j) \in E\}, \\ \mathcal{A} =& \{B_1(o), w(o, r), w(o, g), w(o, b)\} \cup \{w(o, x_i), B_2(x_i) \mid v_i \in V\}.\end{aligned}$$

It can be shown that $\mathcal{K}$ is satisfiable iff G is 3-colorable.

4 Reasoning with Attributes

In this section we study the effect of extending *DL-Lite* with attributes. In particular, we show that for the Bool, Horn and core cases the addition of attributes does not change the complexity of KB satisfiability.

Theorem 3. *KB satisfiability is* NP-*complete in DL-Lite*$_{bool}^{\mathcal{NA}}$, PTIME-*complete in DL-Lite*$_{horn}^{\mathcal{NA}}$ *and* NLOGSPACE-*complete in DL-Lite*$_{core}^{\mathcal{NA}}$.

Under restriction **(inter2)***, checking KB satisfiability is* NP-*complete in DL-Lite*$_{bool}^{\mathcal{HNA}}$, PTIME-*complete in DL-Lite*$_{horn}^{\mathcal{HNA}}$ *and* NLOGSPACE-*complete in DL-Lite*$_{core}^{\mathcal{HNA}}$.

Proof. (Sketch) We encode a *DL-Lite*$_{\alpha}^{\mathcal{HNA}}$ KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ in a $\mathcal{QL}^1$ sentence $\mathcal{K}^{\ddagger_{\mathbf{a}}}$ in a way similar to the translation used in Lemma 1. Denote by $val(\mathcal{A})$ the set of all value names that occur in $\mathcal{A}$. Similarly to roles, we define the sets $Q_{\mathcal{T}}^U$

of natural numbers for all occurrences of $\geq q\, U$ (including sub-attributes). We need a unary predicate $E_q U(x)$, for each attribute name U and $q \in Q^U_{\mathcal{T}}$, denoting the set of objects with at least q values of attribute U. We also need, for each attribute name U and each datatype T, a unary predicates $UT(x)$, denoting all objects that may have attribute U values only of datatype T. Following this intuition, we extend $\cdot^*$ by the following two statements:

$$(\geq q\, U)^* = E_q U(x) \qquad \text{and} \qquad (\forall U.T)^* = UT(x).$$

The $\mathcal{QL}^1$ sentence encoding the KB $\mathcal{K}$ is defined as follows:

$$\mathcal{K}^{\ddagger_{\mathbf{a}}} \;=\; \mathcal{K}^{\ddagger_{\mathbf{e}}} \;\wedge\; \forall x \big[\mathcal{T}^{\mathcal{U}}(x) \wedge \bigwedge_{U \in att(\mathcal{K})} \big(\delta_U(x) \wedge \alpha^1_U(x) \wedge \alpha^2_U(x)\big)\big] \;\wedge\; \mathcal{A}^{\ddagger_{\mathbf{a}}} \;\wedge\; \mathcal{A}^{\ddagger^2_{\mathbf{a}}},$$

where $\mathcal{K}^{\ddagger_{\mathbf{e}}}$ is as before, $\mathcal{T}^{\mathcal{U}}(x)$, $\delta_U(x)$ and $\mathcal{A}^{\ddagger_{\mathbf{a}}}$ are similar to $\mathcal{T}^{\mathcal{R}}(x)$, $\delta_R(x)$ and $\mathcal{A}^{\ddagger_{\mathbf{e}}}$, but rephrased for attributes and their inclusions. The new types of ABox assertions require the following formula:

$$\mathcal{A}^{\ddagger^2_{\mathbf{a}}} = \bigwedge_{U_k(a_i,v_j) \in \mathcal{A}} \; \bigwedge_{\text{datatype } T} \big(UT(a_i) \to Tv_j\big) \quad \wedge \quad \bigwedge_{T(v_j) \in \mathcal{A}} Tv_j,$$

where Tv_j is a propositional variable for each datatype T and each $v_j \in val(\mathcal{A})$. The two additional formulas, $\alpha^1_U(x)$ and $\alpha^2_U(x)$, capturing datatype inclusions and disjointness constraints are:

$$\begin{aligned}
\alpha^1_U(x) \quad &= \bigwedge_{T \sqsubseteq T' \in \mathcal{T}} \big(UT(x) \to UT'(x)\big),\\
\alpha^2_U(x) \quad &= \bigwedge_{T \sqcap T' \sqsubseteq \bot \in \mathcal{T}} \big[\big(UT(x) \wedge UT'(x) \wedge E_1 U(x) \to \bot\big) \wedge \bigwedge_{v \in val(\mathcal{A})} \big(Tv \wedge T'v \to \bot\big)\big].
\end{aligned}$$

We would like to note here that the formula $\alpha^2_U(x)$ for disjoint datatypes demonstrates a subtle interaction between attribute range constraints $\forall U.T$ and minimal cardinality constraints $\exists U$.

We show that $\mathcal{K}$ is satisfiable iff the $\mathcal{QL}^1$-sentence $\mathcal{K}^{\ddagger_{\mathbf{a}}}$ is satisfiable. For ($\Leftarrow$), let $\mathfrak{M} \models \mathcal{K}^{\ddagger_{\mathbf{a}}}$. We construct a model $\mathcal{I} = (\Delta^{\mathcal{I}}_O \cup \Delta^{\mathcal{I}}_V, \cdot^{\mathcal{I}})$ of $\mathcal{K}$ similarly to the way we proved Lemma 1 but this time datatypes will have to be taken into account: let $\Delta^{\mathcal{I}}_O$ be defined inductively as before and $\Delta^{\mathcal{I}}_O = val(\mathcal{A}) \cup V$. The set V will be constructed starting from $val(\mathcal{A})$ in order to 'cure' the attribute successors as follows. For each datatype T and each attribute U, let

$$T^0 = \{v \in val(\mathcal{A}) \mid \mathfrak{M} \models Tv\} \quad \text{and} \quad U^0 = \{(a,v) \mid U(a,v) \in \mathcal{A}\}.$$

For every attribute $U \in att(\mathcal{K})$, we can define the required U-rank $r(U,d)$ of $d \in D$ and the actual U-rank $r_0(R,w)$ of $w \in \Delta^{\mathcal{I}}_O$ as before, treating U as a role name (the only difference is that there will be only one step, and so, the actual rank is needed only for step 0). We can also consider the equivalence relation induced by the sub-attribute relation in $\mathcal{T}$, then we can choose representatives and a linear order on them respecting the sub-attribute relation of $\mathcal{T}$. We can start from the smaller attributes and 'cure' their defects. Let $w \in \Delta_O^{\mathcal{I}}$ and $q = r(U, cp(w)) - r_0(U, w) > 0$. Take q fresh elements $v_1, \ldots, v_q$, add those fresh values to V, add pairs $(w, v_1), \ldots, (w, v_q)$ to U^0 and add $v_1, \ldots, v_q$ to T^0 for each datatype T with $\mathfrak{M} \models UT[cp(w)]$. Let $U^{\mathcal{I}}$ and $T^{\mathcal{I}}$ be the resulting relations. Now, it can be shown that if $\mathfrak{M} \models \mathcal{K}^{\ddagger\mathbf{a}}$ then $\mathcal{I} \models \varphi$ for every $\varphi \in \mathcal{K}$. We only note here that fresh values v_j cannot be added to two disjoint datatypes T and T' because of formula $\alpha_U^2(x)$.

Now, given a KB with a Bool or Horn TBox, $\mathcal{K}^{\ddagger\mathbf{a}}$ is a universal one-variable formula or a universal one-variable Horn formula, respectively, which immediately gives the NP and PTime upper complexity bounds for the Bool and Horn fragments. The NLogSpace upper bound for KBs with core TBoxes is not so straightforward because $\alpha_U^2(x)$ is not a binary clause. In this case we note that $\mathcal{K}^{\ddagger\mathbf{a}}$ is still a universal one-variable Horn formula and therefore, $\mathcal{K}^{\ddagger\mathbf{a}}$ is satisfiable iff it is true in the 'minimal' model. The minimal model can be constructed in the bottom-to-top fashion by using only positive clauses of $\mathcal{K}^{\ddagger\mathbf{a}}$ (i.e., clauses of the form $\forall x\,(B_1(x) \land \cdots \land B_k(x) \to H(x))$) and then checking whether the negative clauses of $\mathcal{K}^{\ddagger\mathbf{a}}$ (i.e., clauses of he form $\forall x\,(B_1(x) \land \cdots \land B_k(x) \to \bot)$) hold in the constructed model. By inspection of the structure of $\mathcal{K}^{\ddagger\mathbf{a}}$, one can see that all its positive clauses are in fact binary, and therefore, whether an atom is true in its minimal model or not can be checked in NLogSpace.

It is of interest to note that the complexity of KB satisfiability increases in the case of Krom TBoxes:

Theorem 4. *KB satisfiability is* NP*-complete in DL-Lite*$_{krom}^{\mathcal{NA}}$*, and so, in DL-Lite*$_{krom}^{\mathcal{HNA}}$ *even under* **(inter)** *and* **(inter2)**.

Proof. (Sketch) The proof exploits the ternary disjointness formula $\alpha_U^2(x)$ in $\mathcal{K}^{\ddagger\mathbf{a}}$. In fact, if $T \sqcap T' \sqsubseteq \bot \in \mathcal{T}$ then the following concept inclusion, although not in the syntax of *DL-Lite*$_{krom}^{\mathcal{NA}}$, is a logical consequence of $\mathcal{T}$ (cf. $\alpha_U^2(x)$):

$$\forall U.T \sqcap \forall U.T' \sqcap \exists U \sqsubseteq \bot.$$

Using such ternary intersections one can encode 3SAT. Let $\varphi = \bigwedge_{i=1}^m C_i$ be a 3CNF, where the C_i are ternary clauses over variables $p_1, \ldots, p_n$. Now, suppose $p_{j_i^1} \lor \neg p_{j_i^2} \lor p_{j_i^3}$ is the ith clause of φ. It is equivalent to $\neg p_{j_i^1} \land p_{j_i^2} \land \neg p_{j_i^3} \to \bot$ and so, can be encoded as follows:

$$T_i^1 \sqcap T_i^2 \sqsubseteq \bot, \qquad \neg A_{j_i^1} \sqsubseteq \forall U_i.T_i^1, \qquad A_{j_i^2} \sqsubseteq \forall U_i.T_i^2, \qquad \neg A_{j_i^3} \sqsubseteq \exists U_i,$$

where the $A_1, \ldots, A_n$ are concept names for the variables $p_1, \ldots, p_n$, and U_i is an attribute and T_i^1 and T_i^2 are datatypes for the ith clause (note that Krom concept inclusions of the form $\neg B \sqsubseteq B'$ are required, which is not allowed in the core TBoxes). Let $\mathcal{T}$ consist of all such inclusions for clauses in φ. It can be seen that φ is satisfiable iff $\mathcal{T}$ is satisfiable.

5 Conclusions

We studied two different extensions of the *DL-Lite* logics. First, *local attributes* allow to use the same attribute associated to different concepts with different datatype range restrictions. We showed that the extension with attributes is harmless with the only notable exception of the Krom fragment, where the complexity rises from NLogSpace to NP.

Second, we consider weak syntactic restrictions on interaction between cardinality constraints and role inclusions and study their impact on the complexity of satisfiability. For example, under **(inter)** [3], roles with sub-roles cannot have maximum cardinality constraints. We present two alternative restrictions, which coincide without ABoxes, and show that the complexity of TBox satisfiability under them coincides with the complexity of TBox satisfiability without role inclusions. However, if we want to preserve complexity of KB reasoning, condition **(inter2)** imposes a bound on the number R-successors in the ABox. Indeed, under the weaker condition **(inter1)** complexity of KB satisfiability rises to at least NP (even for the core fragment).

As a future work, we intend to fill the gaps in Table 1 and, in particular, to see whether the NP-hardness results have a matching upper bound. We are also working on query answering in the languages with attributes.

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