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Topics of Stochastic Algebraic Topology

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Abstract

Stochastic algebraic topology studies random or partly known spaces depending on many random parameters. Such spaces typically arise in applications as configuration spaces of large systems. The paper surveys several recent developments of stochastic algebraic topology focusing on three major themes: random 2-dimensional complexes, random Artin and Coxeter groups, and configuration spaces of linkages (known also as polygon spaces) with random length parameters.

Keywords: stochastic topology, random manifold, random complex, random group.

1 Introduction

Problems of mathematical modeling of large systems of various nature (economical, mechanical, ecological, etc) motivate studying random geometric, topological and algebraic objects. For a system of great complexity, it is unrealistic to assume that one may have a precise description of its configuration space which can rather be viewed as a partially known or random space.

Studying random topological and algebraic objects instead of their deterministic analogues has several major advantages. Firstly, random mathematical objects in many cases more adequately reflect reality. Secondly, random objects are often simpler mathematically since exotic and most complicated examples can be ignored

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if they are rare, i.e. appear with very small probability. Finally, random mathematical objects have sometimes a combination of properties which are difficult to arrange in specific examples.

The most developed stochastic–topological object is a random graph. The theory of random graphs, initiated circa 1959 by Erdös and Rényi [15], is nowadays a fast growing branch of applied mathematics, offering a plethora of spectacular results and predictions for various engineering and computer science applications, see [1], [4], [24].

A model for random simplicial complexes of higher dimension was recently introduced and studied by Linial, Meshulam and Wallach, see [27], [28]. The fundamental groups of random 2-complexes are random groups of a fairly general type. Important progress in understanding this model of random groups was made recently by Babson, Hoffman and Kahle [3].

The theory of random groups was initiated by M. Gromov [21], [22]. His model has a density parameter $0 \le d \le 1$, and it is known that for $d < 1/2$ a random group is infinite and hyperbolic, while it is trivial for $d > 1/2$, see [21], page 273. Similar results hold for the random groups produced by the Linial - Meshulam model, see [3]. We refer the reader to [35] and [32] for more details. The article [34] is a recent survey of probabilistic group theory. A different mechanism producing random groups was introduced in [9] and in [6]; it consists in studying right angled Artin and Coxeter groups associated to random graphs.

Configuration spaces of mechanical linkages with random bar lengths were first studied in the papers [16], [20]. These are closed smooth manifolds depending on a large number of independent random parameters. Although the number of homeomorphism types of these manifolds grows extremely fast with the number of bars in the linkage, their topological characteristics can be predicted with high probability when the number of bars tends to infinity.

In this article we survey recent results of stochastic algebraic topology which develop the following three topics: (a) random 2-dimensional complexes; (b) spaces and groups associated to random graphs; (c) configuration spaces of random linkages (random polygon spaces).

2 Random 2-dimensional complexes

2.1 The Linial - Meshulam model

Random two-dimensional complexes of Linial and Meshulam [27] are two-dimensional analogues of Erdös - Rényi random graphs, [15]. The probability space $G(\Delta_n^{(2)}, p)$ of the Linial–Meshulam model is defined as follows. Let Δ_n denote the $(n-1)$-dimensional simplex with vertices $\{1, 2, \dots, n\}$. Then $G(\Delta_n^{(2)}, p)$ denotes the set of all 2-dimensional subcomplexes

$$\Delta_n^{(1)} \subset Y \subset \Delta_n^{(2)},$$

containing the one-dimensional skeleton $\Delta_n^{(1)}$. The probability function $\mathbf{P} : G(\Delta_n^{(2)}, p) \to \mathbf{R}$ is given by the formula

$$\mathbf{P}(Y) = p^{f(Y)}(1-p)^{\binom{n}{3}-f(Y)}, \quad Y \in G(\Delta_n^{(2)}, p),$$

where $f(Y)$ denotes the number of faces in Y. In other words, each of the 2-dimensional simplexes of $\Delta_n^{(2)}$ is included in a random 2-complex Y with probability p, independently of the other 2-simplexes. As in the case of random graphs, $0 < p < 1$ is a probability parameter which may depend on n.

Clearly, the model $G(\Delta_n^{(2)}, p)$ includes all finite 2-dimensional simplicial complexes containing the full 1-skeleton $\Delta_n^{(1)}$; however, the probability of various topological phenomena is dependent on the value of p.

Note that the theory of deterministic 2-complexes itself is a rich and active field of current research with many challenging open questions, see [23].

2.2 Random groups

The fundamental group of a random 2-complex $Y \in G(\Delta_n^{(2)}, p)$ was investigated by Babson, Hoffman, and Kahle [3]. They showed that for [4]

$$p \gg n^{-1/2} \cdot (3 \log n)^{1/2},$$

the group $\pi_1(Y)$ vanishes asymptotically almost surely (a.a.s) [5]. These authors use notions of negative curvature due to Gromov and prove the nontriviality and hyperbolicity of $\pi_1(Y)$ for

$$p \ll n^{-1/2-\epsilon}.$$

The results of [3] carry a strong philosophical message which is very useful for applications: since *"a random group is either trivial or hyperbolic"* it follows that for random groups many delicate algorithmic problems of group theory (such as the word and isomorphism problems) have positive solutions.

2.3 Random complexes in the range $p \ll n^{-1}$

The paper [12] based on preprints [10] and [11], studies random 2-complexes with probability parameter p satisfying $p \ll n^{-1}$. It was shown that in this range the fundamental group of a random 2-complex is free; this statement complements the results of [3]:

Theorem 2.1 [11], [12] *If the probability parameter p satisfies $p \ll n^{-1}$ then a random 2-complex $Y \in G(\Delta_n^{(2)}, p)$ collapses simplicially to a graph, a.a.s. In particular, the fundamental group $\pi_1(Y)$ is free and for any coefficient group G one has $H_2(Y; G) = 0$, a.a.s.*

[4] Recall that the symbol $a_n \ll b_n$ means that $a_n/b_n \to 0$ as $n \to \infty$.

[5] We use the symbol *a.a.s. as an abbreviation for the phrase "asymptotically almost surely".*

Vanishing of the 2-dimensional homology $H_2(Y;G) = 0$ of a random 2-complex Y, in the range $p \ll n^{-1}$, was established by Kozlov [26]. The result of Kozlov was strengthened in a recent preprint of Aronshtam, Linial, Luczak, and Meshulam [2].

Next we remind the concept of *simplicial collapse* which appears in Theorem 2.1. Let Y be a finite 2-dimensional simplicial complex. An edge of Y is called *free* if it is included in exactly one 2-simplex. The *boundary* ∂Y is defined as the union of free edges. We say that a 2-complex Y is *closed* if $\partial Y = \emptyset$.

A 2-simplex of Y is called *free* if at least one of its edges is free. Let $\sigma_1, \dots, \sigma_k$ be all free 2-simplexes in Y, and let $e_1, \dots, e_k$ be free edges with $e_i \subset \sigma_i$. We say that the complex

$$Y' = Y - \cup_{i=1}^{k} \mathrm{int}(\sigma_i) - \cup_{i=1}^{k} \mathrm{int}(e_i)$$

is obtained from Y by collapsing all free 2-simplexes. Clearly $Y' \subset Y$ is a deformation retract. The operation $Y \searrow Y'$ is called a *simplicial collapse*. Note that Y' is not uniquely determined if one of the free simplexes of Y has two free edges; however the pure part of Y' (i.e. the union of 2-simplexes of Y') is uniquely determined.

This process can be iterated $Y' \searrow Y''$, $Y'' \searrow Y'''$, etc. We denote $Y = Y^{(0)}$, $Y' = Y^{(1)}$, $Y'' = Y^{(2)}$ etc. The sequence of subcomplexes $Y^{(0)} \supset Y^{(1)} \supset Y^{(2)} \supset \dots$ is decreasing and there are two possibilities: either (a) for some k, the complex $Y^{(k)}$ is one-dimensional (a graph), or (b) for some k, the complex $Y^{(k)}$ is 2-dimensional and closed, i.e., $\partial Y^{(k)} = \emptyset$.

Definition 2.2 We say that Y is collapsible to a graph in at most k steps if $Y^{(k)}$ is a graph. We say that Y is collapsible to a graph in k steps if $Y^{(k)}$ is a graph and $\dim Y^{(k-1)} = 2$.

A result proven in [11], [12] states:

Theorem 2.3 *If $p > cn^{-1}$ where $c > 3$ is a constant, then a random 2-dimensional complex $Y \in G(\Delta_n^{(2)}, p)$ satisfies $H_2(Y; \mathbf{Z}) \neq 0$, a.a.s.*

Loosely speaking, Theorems 2.1 and 2.3 combined suggest that *a random 2-complex with vanishing 2-dimensional homology is homotopically one-dimensional.*

The main result of [10] (see also [12]) can be stated as follows:

Theorem 2.4 *(a) If for some $k \geq 1$ the probability parameter p satisfies*

$$p \ll n^{-1-\frac{2}{k+1}},$$

then a random 2-complex $Y \in G(\Delta_n^{(2)}, p)$ is collapsible to a graph in at most k steps, a.a.s. (b) If for some $k \geq 1$ the probability parameter p satisfies

$$p \gg n^{-1-\frac{1}{3\cdot 2^{k-1}-1}},$$

then Y is not collapsible to a graph in k or fewer steps, a.a.s.

The method used in [10], [11], [12] is based on studying simplicial embeddings and immersions of polyhedra into random 2-complexes.

2.4 Simplicial embeddings and immersions

The containment problem for subcomplexes of random 2-dimensional complexes is similar to the containment problem for random graphs, see [24], chapter 3. In [11], [12] we also study simplicial immersions, which are more general than simplicial embeddings.

Let S be a 2-dimensional finite simplicial complex. We assume that S is fixed, i.e. independent of n. The set of vertices of S is denoted by $V(S)$.

Definition 2.5 A simplicial embedding $g : S \hookrightarrow Y$, where $Y \in G(\Delta_n^{(2)}, p)$ is a random 2-complex, is defined as an injective map of the set of vertices $V(S)$ of S into the set of vertices $\{1, \dots, n\}$ of Y satisfying the following condition: for any triple of distinct vertices $u_1, u_2, u_3 \in V(S)$ which span a simplex in S, the corresponding points $g(u_1), g(u_2), g(u_3) \in \{1, \dots, n\}$ span a face of Y.

The following definition describes a more general notion.

Definition 2.6 A simplicial immersion $g : S \looparrowright Y$ into a random 2-complex $Y \in G(\Delta_n^{(2)}, p)$ is defined as a map of the set of vertices $V(S)$ of S into the set of vertices $\{1, \dots, n\}$ of Y satisfying the following two conditions:

(a) for any triple of distinct vertices $u_1, u_2, u_3 \in V(S)$ which span a 2-simplex in S, the corresponding points $g(u_1), g(u_2), g(u_3) \in \{1, \dots, n\}$ are pairwise distinct and span a face of Y;

(b) for any pair of distinct 2-simplexes σ and σ' of S, the corresponding 2-simplexes $g(\sigma)$ and $g(\sigma')$ of Y are distinct.

Definition 2.7 For a simplicial 2-complex S let $\mu(S)$ denote

$$\mu(S) = \frac{v}{f} \in \mathbf{Q},$$

where $v = v_S$ and $f = f_S$ are the numbers of vertices and faces in S.

Definition 2.8 Let S be a finite 2-dimensional simplicial complex. Define

$$\tilde{\mu}(S) = \min_{S' \subset S} \mu(S'), \tag{1}$$

where the minimum is formed over all subcomplexes $S' \subset S$ or, equivalently, over all pure subcomplexes $S' \subset S$.

Note that the invariant $\tilde{\mu}$ is monotone decreasing: if S is a subcomplex of T then $\tilde{\mu}(S) \geq \tilde{\mu}(T)$.

Theorem 2.9 [11], [12] *Let S be a finite simplicial complex.*

(A) If $p \ll n^{-\tilde{\mu}(S)}$ then the probability that S admits a simplicial immersion into a random 2-complex $Y \subset G(n, p)$ tends to zero as $n \to \infty$.

(B) If $p \gg n^{-\tilde{\mu}(S)}$ then the probability that S admits a simplicial embedding into a random 2-complex $Y \subset G(n, p)$ tends to one as $n \to \infty$.

The proof uses the first and the second moment methods.

2.5 Balanced and unbalanced triangulations

Definition 2.10 A finite simplicial 2-complex S is called balanced if $\mu(S) = \tilde{\mu}(S)$, i.e. if the quantities defined in Definitions 2.7 and 2.8 coincide. In other words, S is balanced if $\mu(S) \leq \mu(S')$ for any subcomplex $S' \subset S$.

Definition 2.10 is similar to the corresponding notion for random graphs, see [24].

Example 2.11 Let $S = \Sigma_g$ be a triangulated closed orientable surface of genus $g \geq 0$. Then $\chi(S) = 2 - 2g = v - e + f$ where v, e, f denote the numbers of vertices, edges and faces in S correspondingly. Each edge is contained in two faces which gives $3f = 2e$ and therefore

$$\mu(\Sigma_g) = \frac{1}{2} + \frac{2-2g}{f}. \tag{2}$$

Similarly, if $S = N_g$ is a triangulated closed nonorientable surface of genus $g \geq 1$ then $\chi(N_g) = 2 - g$ and

$$\mu(N_g) = \frac{1}{2} + \frac{2-g}{f}. \tag{3}$$

Formulae (2) and (3) give the following:

Corollary 2.12 *The invariant $\mu(\Sigma_g)$ of an orientable triangulated surface Σ_g satisfies:*

(i) $1/2 < \mu(\Sigma_g) \leq 1$ *for* $g = 0$ *(since* $f \geq 4$*);*

(ii) $\mu(\Sigma_g) = 1/2$ *for* $g = 1$ *(the torus);*

(iii) $\mu(\Sigma_g) < 1/2$ *for* $g > 1$*;*

(iv) *If* $f \to \infty$ *(i.e. when the surface is subsequently subdivided) then* $\mu(\Sigma_g) \to 1/2$.

Corollary 2.13 *The invariant $\mu(N_g)$ of a nonorientable triangulated surface N_g satisfies:*

(i) $1/2 < \mu(N_g) \leq 3/5$ *for* $g = 1$ *(since* $f \geq 10$*);*

(ii) $\mu(N_g) = 1/2$ *for* $g = 2$ *(the Klein bottle);*

(iii) $\mu(N_g) < 1/2$ *for* $g > 2$*;*

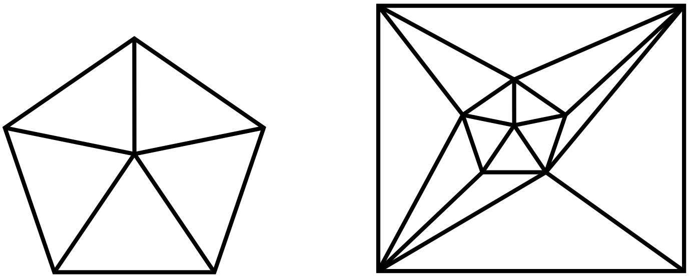
(iv) *If* $f \to \infty$ *(i.e. when the surface is subsequently subdivided) then* $\mu(N_g) \to 1/2$.

Example 2.14 Let S be a triangulated disc. Then $\chi(S) = v - e + f = 1$ and $3f = 2e - e_0$ where e_0 is the number of edges in the boundary ∂S. Substituting $e = (3f + e_0)/2$, one obtains

$$\mu(S) = \frac{1}{2} + \frac{e_0}{2f} + \frac{1}{f}. \tag{4}$$

As a specific example consider the regular n-gon S shown on the figure on the left. Then $v = n + 1$, $f = n$, $e_0 = n$ and

$$\mu(S) = 1 + \frac{1}{n}.$$



In figure 2.14, on the right we have $e_0 = 4$ and the number of faces f equals $f = 2n + 4$. Thus

$$\mu(T) = \frac{1}{2} + \frac{3}{2n+4},$$

converges to $\frac{1}{2}$ as $n \to \infty$.

Corollary 2.15 *For any triangulation S of the disk one has $\mu(S) > 1/2$. There exist triangulations S of D^2 with $\mu(S)$ arbitrarily close to $1/2$.*

Example 2.16 Let S' be such that $\mu(S') < 1$ and suppose that S is obtained from S' by adding a triangle Δ such that $S' \cap \Delta$ is an edge. Then S is not balanced. Indeed, $v_S = v_{S'} + 1$ and $f_S = f_{S'} + 1$ and

$$\mu(S) = \frac{v_{S'} + 1}{f_{S'} + 1} > \frac{v_{S'}}{f_{S'}} = \mu(S').$$

Corollary 2.17 *There exist unbalanced triangulations of the disk.*

Proof. Start with a disk triangulation S' with $\mu(S') < 1$ (for instance, S' can be the square with implanted n-gon, see Example 2.14) and add a triangle $S = S' \cup \Delta$ such that $S' \cap \Delta$ is an edge lying in the boundary $\partial S'$. Then $\mu(S) > \mu(S')$ (see Example 2.16) and S is unbalanced. Clearly, S is homeomorphic to the 2-dimensional disk.□

Theorem 2.18 [11], [12] *Any closed connected triangulated surface S is balanced.*

2.6 Surfaces in random 2-complexes

In the following statement we consider *"small surfaces"*, i.e. triangulated surfaces which do not depend on n. Theorems 2.9, 2.18 and Corollaries 2.12 and 2.13 imply, see [11], [12]:

Corollary 2.19 *One has:*

(i) *If $p \ll n^{-1}$ then a random 2-complex $Y \in G(\Delta_n^{(2)}, p)$ contains*[6] *no small*[7] *closed surfaces, a.a.s.*

(ii) *If $n^{-1} \ll p \ll n^{-3/5}$ then a random 2-complex Y contains small spheres but no small closed surfaces of other topological types, a.a.s.*

[6] In this Corollary the word "contains" means "contains as a simplicial subcomplex."

[7] In this statement one may remove the word "small" as follows from the proof of Theorem 2.1.

(iii) *If $n^{-3/5} \ll p \ll n^{-1/2}$ then a random 2-complex Y contains small spheres and projective planes but no small closed surfaces of higher genera, a.a.s.*

(iv) *If $p \gg n^{-1/2}$ then a random 2-complex Y contains all small spheres, projective planes, tori and Klein bottles, a.a.s.*

(v) *If $p \gg n^{-1/2+\epsilon}$ for some $\epsilon > 0$ then, given a topological type of a closed surface, there exists $f_0 = f_0(\epsilon)$, such that any triangulation of the surface having more than f_0 2-simplexes will be simplicially embeddable into a random 2-complex Y, a.a.s. In particular, if $p \gg n^{-1/2+\epsilon}$, a random 2-complex Y contains small closed orientable and nonorientable surfaces of all possible topological types, a.a.s.*

Corollary 2.20 *For a random 2-complex $Y \in G(\Delta_n^{(2)}, p)$ with $p \gg n^{-1}$ one has*

$$\pi_2(Y) \neq 0, \quad \text{and} \quad H_2(Y; \mathbf{Z}) \neq 0 \tag{5}$$

a.a.s.

3 Aspherical spaces and groups associated to random graphs

Given a finite graph Γ with vertex set V and with the set of edges E, one associates to it a right angled Artin group (also known as a graph group)

$$A_\Gamma = \langle v \in V; vw = wv \quad \text{iff} \quad (v, w) \in E\rangle,$$

see [5], [31]. In the case when Γ is a complete graph A_Γ is a free abelian group of rank $n = |V|$; in the other extreme, when Γ has no edges the group A_Γ is the free group of rank n. In general A_Γ interpolates between the free and free abelian groups.

We are interested in right angled Artin groups associated to *random graphs* Γ. We adopt one of the basic Erdős - Rényi models of random graphs in which each edge of the complete graph on n vertices is included with probability $0 < p < 1$ independently of all other edges. In other words, we consider the probability space $G(\Delta_n^{(1)}, p)$ of all $2^{\binom{n}{2}}$ subgraphs

$$\Delta_n^{(0)} \subset \Gamma \subset \Delta_n^{(1)}$$

of the complete graph on n vertices $\{1, 2, \dots, n\}$ and the probability that a specific graph $\Gamma \in G(\Delta_n^{(1)}, p)$ appears as a result of a random process equals

$$\mathbf{P}(\Gamma) = p^{E_\Gamma}(1-p)^{\binom{n}{2} - E_\Gamma}, \tag{6}$$

where E_Γ denotes the number of edges of Γ, see [24].

The function $\Gamma \mapsto A_\Gamma$ where $\Gamma \in G(\Delta_n^{(1)}, p)$ is a new mechanism producing random groups. In Gromov's model a random group is given by a presentation with randomly generated relations where each of the relations has a fixed length and the total number of relations is also fixed (it depends on the density d). In the case of random right angled Artin groups one generates randomly relations of a special type (namely, the commutation relations). One wants to translate the evolution theory of random graphs to the language of random right angled Artin groups. In [9] we examine statistics of various topological invariants of the group A_Γ associated to a random graph. Each of such invariants is a random function and it is quite natural to ask about its mathematical expectation and distribution function. We assume that $n \to \infty$ and seek results of asymptotic nature.

It is clear that basic properties of Γ are reflected in properties of the group. For example, A_Γ decomposes as a free product if and only if Γ is disconnected. For $\Gamma \in G(\Delta_n^{(1)}, p)$, Erdős and Rényi [15] showed that this holds asymptotically almost surely if $p = \frac{\log n - \omega(n)}{n}$, where $\omega : \mathbb{N} \to \mathbb{R}$ is a function such that $\lim_{n\to\infty} \omega(n) = \infty$.

Likewise, A_Γ decomposes as a direct product if and only if the complementary graph is disconnected, which holds a.a.s. if $1 - p = \frac{\log n - \omega(n)}{n}$.

The Eilenberg-MacLane space K_Γ associated to the right angled Artin group A_Γ of a random graph Γ is a random aspherical space. The spaces K_Γ are closely related to the configuration spaces of braid groups of graphs (see [13]) which indicates the relevance of this study to topological robotics.

3.1 Betti numbers of random graph groups

The cohomology algebra of K_Γ with integral coefficients is the quotient

$$H^*(K_\Gamma; \mathbf{Z}) \simeq E(v_1, \ldots, v_n)/J_\Gamma \tag{7}$$

where $E(v_1, \ldots, v_n)$ is the exterior algebra generated by degree one classes corresponding to the vertices $V = \{v_1, \ldots, v_n\}$ of Γ and the ideal J_Γ is generated by the degree two monomials vw such that the corresponding vertices v, w are not connected by an edge.

In particular any product $v_{i_1} v_{i_2} \cdots v_{i_r}$ vanishes iff the corresponding vertices $\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ do not form a complete subgraph of Γ. For an integer $r \geq 2$ the r-th Betti number $b_r(A_\Gamma) = b_r(K_\Gamma)$ equals the number of complete subgraphs of size r in Γ. The expectation of the r-th Betti number of the group A_Γ of a random graph Γ, where $r \geq 2$, equals $\mathbf{E}(b_r(A_\Gamma)) = \binom{n}{r} p^{\binom{r}{2}}$. Note that $b_0(A_\Gamma) = 1$ and $b_1(A_\Gamma) = n$ for any graph Γ.

Now we assume that r (the dimension) is fixed and p may depend on n. Asymptotically, the expectation of $b_r(A_\Gamma)$ can be written as

$$\mathbf{E}(b_r(A_\Gamma)) \sim \frac{1}{r!} \left[n p^{\frac{r-1}{2}} \right]^r.$$

The expectation has a positive limit for $n \to \infty$ if and only if

$$n p^{\frac{r-1}{2}} \to c > 0. \tag{8}$$

Under this condition the expectation $\mathbf{E}(b_r(A_\Gamma))$ converges to $\dfrac{c^r}{r!} = \lambda$.

Theorem 3.1 is an interpretation of a theorem of Schürger [33] about complete subgraphs in random graphs.

Theorem 3.1 *Fix an integer $r > 1$ and consider the r-th Betti number of the associated graph group,*

$$b_r : G(\Delta_n^{(1)}, p) \to \mathbf{Z}, \quad b_r(\Gamma) = b_r(A_\Gamma),$$

as a random function of a random graph. If the limit (8) exists and is positive then for any integer $k = 0, 1, \dots,$ the probability $\mathbf{P}(b_r(A_\Gamma) = k)$ converges (as $n \to \infty$) to $e^{-\lambda} \cdot \frac{\lambda^k}{k!}$, where $\lambda = \frac{c^r}{r!}$.

In other words, the limiting distribution is Poisson with mean λ.

3.2 Cohomological dimension of random graph groups

The cohomological dimension of A_Γ equals the size of the maximal clique in Γ. Recall that a clique in a graph is defined as a maximal complete subgraph. The clique number $\mathrm{cl}(\Gamma)$ of a graph Γ is the maximal order of a clique in Γ.

There are many results in the literature about the clique number of random graphs; we may interpret these results as statements about the cohomological dimension of graph groups build from random graphs. Matula [29], [30] discovered that for fixed values of p the distribution of the clique number of a random graph is highly concentrated in the sense that almost all random graphs have about the same clique number. These results were developed further by Bollobás and Erdős; see the monographs of B. Ballobás [4] and of N. Alon and J. Spencer [1].

We restate a result of Matula [30] as a statement about cohomological dimension of random graph groups. Denote

$$z(n, p) = 2\log_q n - 2\log_q \log_q n + 2\log_q(e/2) + 1, \tag{9}$$

where $q = p^{-1}$. We assume that p is independent of n.

Theorem 3.2 *Fix an arbitrary $\epsilon > 0$. Then*

$$\lfloor z(n, p) - \epsilon \rfloor \;\leq\; \mathsf{cd}(A_\Gamma) \;\leq\; \lfloor z(n, p) + \epsilon \rfloor, \tag{10}$$

asymptotically almost surely (a.a.s). In other words, the probability that a graph $\Gamma \in G(\Delta_n^{(1)}, p)$ does not satisfy inequality (10) tends to zero when n tends to infinity.

Here $\lfloor x \rfloor$ denotes the largest integer not exceeding x. We may assume that $\epsilon < 1/2$; then the integers $\lfloor z(n, p) - \epsilon \rfloor$ and $\lfloor z(n, p) + \epsilon \rfloor$ either coincide or differ by 1.

Thus, according to Theorem 3.2, the cohomological dimension $\mathsf{cd}(A_\Gamma)$ for a random graph Γ takes on one of at most two values depending on n and p, with probability approaching 1 as $n \to \infty$.

3.3 Topological complexity of random graph groups

The concept of topological complexity TC(X) was introduced in [16]; it is motivated by the motion planning problem of robotics, see [18]. TC(X) is a homotopy invariant associating an integer to a path-connected topological space X. The paper [9] studies the topological complexity $\mathrm{TC}(\mathrm{K}_\Gamma)$ of the Eilenberg- MacLane space K_Γ associated to a random graph Γ.

Theorem 3.3 [9] *Fix an arbitrary $0 < \epsilon < 1/2$ and assume that the probability parameter $0 < p < 1$ is independent of n. Then for a random graph $\Gamma \in G(\Delta_n^{(1)}, p)$ one has*

$$2 \cdot \lfloor z(n,p) - \epsilon \rfloor + 1 \le \mathrm{TC}(\mathrm{K}_\Gamma) \le 2 \cdot \lfloor \mathrm{z(n,p)} + \epsilon \rfloor + 1, \tag{11}$$

asymptotically almost surely, where $z(n,p)$ is given by (9).

3.4 Automorphism groups of random right-angled Artin groups

Automorphism groups of right-angled Artin groups have been extensively studied in recent years, see, for example, [7,8]. In [6] the authors consider automorphism groups of right-angled Artin groups associated to random graphs.

Theorem 3.4 [6] *Let $\Gamma \in G(\Delta_n^{(1)}, p)$ be a random graph where the probability parameter p is independent on n and satisfies*

$$1 - \frac{1}{\sqrt{2}} < p < 1. \tag{12}$$

Then the right-angled Artin group A_Γ determined by Γ has a finite outer automorphism group $\mathrm{Out}(A_\Gamma)$, a.a.s.

Note that $1 - 1/\sqrt{2} \sim 0.2929$. It would be interesting to know if Theorem 3.4 holds outside the range (12).

3.5 Hyperbolicity of random Coxeter groups

Given a graph Γ with vertex set V and with the set of edges E one associates to it a right angled Coxeter group

$$W_\Gamma = \langle v \in V; v^2 = 1, \, vw = wv \quad \text{iff} \quad (v,w) \in E \rangle.$$

Assuming that $\Gamma \in G(\Delta_n^{(1)}, p)$ is a random graph of Erdős and Rényi [15], the function $\Gamma \mapsto W_\Gamma$ gives another mechanism producing random groups.

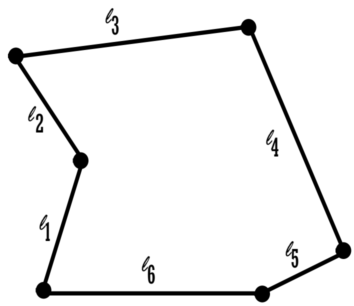
Theorem 3.5 [6] *The right-angled Coxeter group W_Γ corresponding to a random Erdős - Rényi graph $\Gamma \in G(n,p)$ is hyperbolic a.a.s. if either $(1-p)n^2 \to 0$ or $pn \to 0$. If however $pn \to \infty$ and $(1-p)n^2 \to \infty$ then the random right-angled Coxeter group W_Γ is not hyperbolic, a.a.s.*

In [6] a more general theorem of this kind is proven.

We observe that for random Coxeter groups the hyperbolicity is quite a rare phenomenon unlike for random group models of Gromov and Linial-Meshulam.

4 Statistical theory of linkages and polygon spaces

Interesting examples of random topological spaces are provided by configuration spaces of mechanical linkages and polygon spaces. Recall that a *linkage* is a simple mechanism consisting of n bars in $\mathbf{R}^3$ having fixed lengths $l_1, \dots, l_n$ which are cyclically connected by revolving joints forming a closed polygonal chain. Angles between bars of the linkage may vary, the only condition is that the links do not become disconnected from each other. We consider a pair of configurations of the



linkage as being identical if one can be obtained from the other by a rigid motion of the space $\mathbf{R}^3$. The configuration space of the linkage

$$N_\ell = \{(u_1, \dots, u_n) \in S^2 \times \cdots \times S^2; \sum_{i=1}^{n} l_i u_i = 0 \in \mathbf{R}^3\} / \mathrm{SO}(3),$$

parameterizes all possible configurations. Here $\ell = (l_1, l_2, \dots, l_n) \in \mathbf{R}^n_+$ is the n-tuple of the bar lengths, called the *length vector*. Spaces N_ℓ are also known as *polygon spaces* as they parameterize shapes of all n-gons in $\mathbf{R}^3$ having sides of length $l_1, \dots, l_n$.

The spaces N_ℓ appear in molecular biology where they represent shapes of long molecules. Clearly, information about topological properties of these spaces may lead to interesting new effects in molecular and chemical design. Statistical shape theory is another area where the spaces N_ℓ play an interesting role: they describe the space of shapes having certain geometric properties with respect to the central point. Having in mind these applications it is quite natural to assume that the number of links n is large, $n \to \infty$, and that the numbers $l_i > 0$ are not entirely known or are known with some random error.

Let us now recall some basic facts concerning the topology of N_ℓ and its dependence on the length vector $\ell \in \mathbf{R}^3_+$. For a *generic* ℓ the space N_ℓ is a closed smooth manifold of dimension $2(n-3)$. If ℓ is not generic then N_ℓ is a compact manifold with finitely many singularities. Clearly, $N_\ell = N_{t\ell}$ for any $t > 0$. Hence we may consider ℓ as lying in the quotient space A^{n-1} of $\mathbf{R}^n_+$ modulo the action of $\mathbf{R}_+$ by scalar multiplication. A^{n-1} can be identified with the interior of the standard simplex, i.e. the set given by the inequalities $l_1 > 0, \dots, l_n > 0$ and $\sum l_i = 1$.

We know that N_ℓ is diffeomorphic to $N_{\ell'}$ if ℓ' is obtained from ℓ by permuting coordinates. Let Σ_n denote the permutation group of n symbols. Clearly Σ_n acts on

$\mathbf{R}^n$ and on A^{n-1} by permuting coordinates and the manifold N_ℓ depends only on the Σ_n-orbit of the vector ℓ. To explain further the character of the dependence of N_ℓ on ℓ we need to recall the concept of a *chamber* (see [19]). A chamber is a connected component of the complement $\mathbf{R}^n_+ - \bigcup_J H_J$, where for any subset $J \subset \{1, \dots, n\}$ we denote by $H_J \subset \mathbf{R}^n$ the hyperplane $\sum_{i \in J} l_i = \sum_{i \notin J} l_i$. *Generic* length vectors are defined as those lying in chambers, not on hyperplanes H_J.

It was proven in [19] that there is a one-to-one correspondence between the orbits of chambers and diffeomorphism type of manifolds [8] N_ℓ.

The following picture summarizes our description of *the field of topological spaces* $\ell \mapsto N_\ell$ viewed as a single object. The open simplex A^{n-1} is divided into a huge number of tiny chambers, each representing a diffeomorphism type of manifolds N_ℓ. The symmetric group Σ_n acts on the simplex A^{n-1} mapping chambers to chambers and, for $n \neq 4$, two manifolds N_ℓ and $N_{\ell'}$ are diffeomorphic if and only if the vectors ℓ, ℓ' lie in chambers belonging to the same Σ_n-orbit.

The main idea (first proposed in [20]) is to use methods of probability theory and statistics in dealing with the variety of diffeomorphism types of configuration spaces N_ℓ for n large. We may view the length vector $\ell \in A^{n-1}$ as a random variable whose statistical behavior is characterized by a probability measure ν_n. Topological invariants of N_ℓ become random functions; for example one may consider the *average* or *expected* Betti numbers [9]

$$\mathbf{E}(b_{2p}(N_\ell)) = \int_{A^{n-1}} b_{2p}(N_\ell) d\nu_n \tag{13}$$

where the integration is understood with respect to ℓ. One of the main results (see below) states that for p fixed and n large this average $2p$-dimensional Betti number can be calculated explicitly up to an exponentially small error. More precisely, we prove that

$$\mathbf{E}(b_{2p}(N_\ell)) = \int_{A^{n-1}} b_{2p}(N_\ell) d\nu_n \sim \sum_{i=0}^{p} \binom{n-1}{i};$$

see Theorem 4.2 below for a precise statement. It might appear surprising that the asymptotic value of the average Betti number $b_{2p}(N_\ell)$ does not depend of the sequence of probability measures ν_n which are allowed to vary in an ample class of *admissible* probability measures described below.

One may also study the average Betti numbers $b_p(M_\ell)$ of configuration spaces of *planar* polygon spaces. Recall that the space of planar n-gons with a fixed length vector is defined as

$$M_\ell = \{(u_1, \dots, u_n) \in S^1 \times \cdots \times S^1; \sum_{i=1}^{n} l_i u_i = 0 \in \mathbf{R}^2\} / \mathrm{SO}(2).$$

Paper [20] calculated the asymptotic values of the average Betti numbers

$$\mathbf{E}(b_p(M_\ell)) = \int_{A^{n-1}} b_p(M_\ell) d\nu_n \tag{14}$$

8 Results of this kind are known for the polygon spaces $M_\ell, \bar{M}_\ell$ as well.

9 It is known that all odd-dimensional Betti numbers of N_ℓ vanish.

for two different sequences of probability measures ν_n on $\mathbf{R}^n_+$. It was discovered in [20] that for large n the answers for these two distinct measures were equal. In [17] *a universality phenomenon* was established for a large class of admissible probability measures.

4.1 Admissible sequences of probability measures

For a vector $\ell = (l_1, \ldots, l_n)$ we denote by $|\ell| = \max\{|l_1|, \ldots, |l_n|\}$ the maximum of the absolute values of coordinates. The symbol A^{n-1} denotes the open unit simplex, i.e. the set of all vectors $\ell = (l_1, l_2, \ldots, l_n) \in \mathbf{R}^n$ such that $l_i > 0$ and $l_1 + \cdots + l_n = 1$. Let μ_n denote the Lebesgue measure on A^{n-1} normalized so that $\mu_n(A^{n-1}) = 1$. For an integer $p \geq 1$ we denote by

$$\Lambda_p = \{\ell \in A^{n-1}; |\ell| \geq (2p)^{-1}\}. \tag{15}$$

Clearly, $\Lambda_p \subset \Lambda_q$ for $p \leq q$ and $\Lambda_p = A^{n-1}$ for $2p \geq n$.

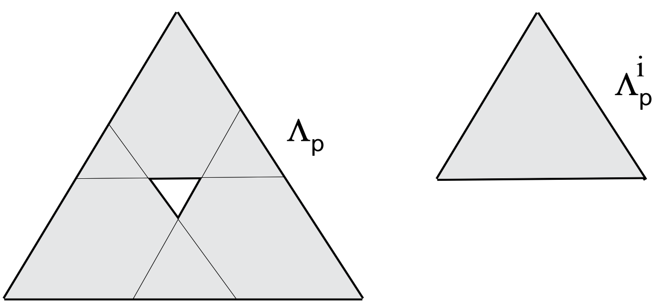
It will be helpful to think of p being fixed and of n being large, say, tending to ∞. The set Λ_p is shown on the picture on the left. It is the union of n domains Λ_p^i defined as

$$\Lambda_p^i = \{\ell = (l_1, \ldots, l_n) \in A^{n-1}; l_i \geq (2p)^{-1}\},$$

where $i = 1, \ldots, n$. Each Λ_p^i is homothetic to A^{n-1} with factor $(1 - \frac{1}{2p})$ and hence $\mu_n(\Lambda_p^i) = (1 - \frac{1}{2p})^{n-1}$. It follows that

$$\mu_n(\Lambda_p) \leq n \cdot (1 - \frac{1}{2p})^{n-1}. \tag{16}$$

We see that the normalized Lebesgue measure of Λ_p is exponentially small for large n.



Definition 4.1 Consider a sequence of probability measures ν_n on A^{n-1} where $n = 1, 2, \ldots$. It is called *admissible* if

$$\nu_n = f_n \cdot \mu_n$$

where $f_n : A^{n-1} \to \mathbf{R}$ is a sequence of Lebesgue measurable functions satisfying:

(i) $f_n \geq 0$,

(ii) $\int_{A^{n-1}} f_n d\mu_n = 1$,

(iii) for any $p \geq 1$ there exist constants $K > 0$ and $0 < b < 2$ such that

$$f_n(\ell) \leq K \cdot b^n \tag{17}$$

for any n and any $\ell \in \Lambda_p \subset A^{n-1}$.

Examples. It is obvious that the sequence $\nu_n = \mu_n$ is admissible.

As another example of an admissible sequence of measures can be constructed as follows. Consider the unit cube $\Box^n \subset \mathbf{R}^n_+$ given by the inequalities $0 \leq l_i \leq 1$ for $i = 1, \ldots, n$. Let χ_n be the probability measure on $\mathbf{R}^n_+$ supported on $\Box^n \subset \mathbf{R}^n_+$ such that the restriction $\chi_n|_{\Box^n}$ is the Lebesgue measure, $\chi_n(\Box^n) = 1$. Consider the sequence of induced measures

$$\nu_n = q_*(\chi_n) \tag{18}$$

on simplices A^{n-1} where

$$q : \mathbf{R}^n_+ \to A^{n-1}, \quad q(\ell) = \frac{1}{l_1 + \cdots + l_n} \cdot \ell \tag{19}$$

is the normalization map The measures ν_n have a very clear geometric meaning: it is the probability distribution in the case when the bar lengths l_i are independent and are uniformly distributed in the unit interval $[0, 1]$, see [20].

4.2 Expectations of Betti Numbers

The next Theorem describes the asymptotic values (as $n \to \infty$) of the expectations of the Betti numbers of polygon spaces in $\mathbf{R}^3$:

Theorem 4.2 [17] *Fix an admissible sequence of probability measures ν_n and an integer $p \geq 0$, and consider the $2p$-dimensional Betti number (13) of polygon spaces N_ℓ in $\mathbf{R}^3$ as a random variable on A^{n-1}, for large $n \to \infty$. Then there exist constants $C > 0$ and $0 < a < 1$ (depending on the sequence of measures ν_n and on the number p but independent of n) such that the average Betti numbers (13) satisfy*

$$\left| \int_{A^{n-1}} b_{2p}(N_\ell) d\nu_n - \sum_{k=0}^{p} \binom{n-1}{k} \right| < C \cdot a^n \tag{20}$$

for all n.

The following theorem describes the asymptotic values of the expectations of Betti numbers in the case of the planar polygon spaces M_ℓ.

Theorem 4.3 [17] *Fix an admissible sequence of probability measures ν_n and an integer $p \geq 0$, and consider the average p-dimensional Betti number (14) of planar polygon spaces for large $n \to \infty$. Then there exist constants $C > 0$ and $0 < a < 1$ (depending on the sequence of measures ν_n and on the number p but independent of n) such that*

$$\left| \int_{A^{n-1}} b_p(M_\ell) d\nu_n - \binom{n-1}{p} \right| < C \cdot a^n \tag{21}$$

for all n.

4.3 Expectations of middle-dimensional and total Betti numbers

C. Dombry and C. Mazza in their paper [14] studied the average Betti numbers of unbounded dimension, the average total Betti numbers as well as the average Poincaré polynomials of the polygon spaces M_ℓ and N_ℓ.

The main assumption concerning the probability measure μ_n on the simplex A^{n-1} made in the work [14] is the following. Let μ be a diffuse probability measure on $\mathbf{R}_+ = (0, \infty)$ such that

$$\int e^{\eta x} d\mu(x) < \infty \quad \text{for some} \quad \eta > 0.$$

Recall that a measure is said to be diffuse if it satisfies $\mu(\{x\}) = 0$ for all $x \in \mathbf{R}_+$. The product measure $\mu^{\otimes n}$ on $\mathbf{R}_+^n$ describes the situation when each of the components l_i of the length vector is distributed according to the measure μ, independently of all other components. Then μ_n is the push-forward measure

$$\mu_n = q_*(\mu^{\otimes n})$$

where $q : \mathbf{R}_+^n \to A^{n-1}$ is the central projection (19).

Theorem 4.4 [14] *Let $(p_n)_{n\geq 3}$ be a sequence of integers satisfying $0 \leq p_n \leq n-3$.*

(1) If $\limsup n^{-1}p_n < 1/2$, then[10]

$$\int_{A^{n-1}} b_{p_n}(N_\ell) d\mu_n \sim \sum_{k=0}^{p_n} \binom{n-1}{k} \tag{22}$$

as $n \to \infty$.

(2) If $\liminf n^{-1}p_n > 1/2$, then

$$\int_{A^{n-1}} b_{p_n}(N_\ell) d\mu_n \sim \sum_{k=0}^{n-p_n-3} \binom{n-1}{k} \tag{23}$$

as $n \to \infty$.

(3) If $\lim n^{-1/2}(p_n - n/2) = \alpha$, then

$$\int_{A^{n-1}} b_{p_n}(M_\ell) d\mu_n \sim C(\alpha)2^{n-1} \tag{24}$$

as $n \to \infty$, with

$$C(\alpha) = \int_{2|\alpha|}^{\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \mathbf{P}(|Z| < \frac{um}{\sigma}) \mathrm{d}u,$$

where $m = \mathbf{E}(\ell)$, $\sigma^2 = \mathrm{Var}(\ell)$, and Z is standard normal.

Finally we mention results concerning the average total Betti numbers of polygon spaces. We denote them by

$$B(M_\ell) = \sum_{p=0}^{n-3} b_p(M_\ell), \quad B(N_\ell) = \sum_{p=0}^{n-3} b_{2p}(N_\ell),$$

[10] In this theorem and in (25) the symbol $a_n \sim b_n$ means $a_n/b_n \to 1$ as $n \to \infty$.

correspondingly. Dombry and Mazza [14] proved that the average total Betti numbers satisfy the following asymptotic formulae

$$\begin{aligned} &\int_{A^{n-1}} B(M_\ell)d\mu_n \sim 2^{n-1}, \\ &\int_{A^{n-1}} B(N_\ell)d\mu_n \sim n2^{n-2}. \end{aligned} \tag{25}$$

We know that for the equilateral planar linkage one has $B(M_{(1,1,\dots,1)}) \sim 2^{n-1}$. Thus the first asymptotic formula in (25) states that in the planar case the average total Betti number is asymptotically achieved on the total Betti number of the equilateral linkage. However the total Betti number for equilateral linkage in the space $\mathbf{R}^3$ for $n = 2r+1$ odd[11] equals

$$B(N_{(1,1,\dots,1)}) = \sum_{k=0}^{r-1}(r-k)\binom{2r}{k} \sim \sqrt{\frac{n}{2\pi}} \cdot 2^{n-2}.$$

This deviates from the second asymptotic formula in (25), i.e. the average total Betti number for spatial linkages is less than the total Betti number of the equilateral linkage.

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