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Graphics calculators and algebra

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Abstract: The personal technology of the graphics calculator is presently the only one likely to be available widely enough to influence curriculum design and implementation on a large scale. The algebra curriculum of the past is overburdened with symbolic manipulation at the expense of understanding for most students. But algebra is much more than just symbolic manipulation. Connections between some aspects of algebra: expressing generality, functions, equations and advanced algebra and some graphics calculator capabilities are briefly described. It is suggested that these kinds of connections need to be taken into account in developing the algebra curriculum as well as in classroom teaching.

The main purpose of this paper is to highlight some of the important connections between algebra in the secondary school and currently available graphics calculators. The focus is on the introductory study of algebra, usually the province of the lower secondary school years in Australia, formally starting in either Year 7 or Year 8 (dependent on the state concerned). In fact, the study of algebra starts much earlier, in the primary school, with a focus on important mathematical ideas associated with patterns and regularities.

The most obvious difference between a graphics calculator and a scientific calculator is the small graphics screen on the former. One of the several uses of the graphics display screen is to draw graphs of functions, so graphics calculator are sometimes called 'graphing' calculators, although this description is too restrictive in outlook. Technology of this kind has been around now since the mid 1980's. Although they are not yet in widespread use in South East Asian countries, graphics calculators are now widely used in parts of Australia, North America and Europe. Two distinctive differences between graphics calculators and other technologies for school mathematics, such as computers, is that the devices were produced mainly for educational use and they are much more portable. Indeed, it has even been suggested that graphics calculators are the first examples of a genuinely personal technology for school mathematics.

Personal technology

As the phrase suggests, 'personal technology' refers to the technology available to individual pupils on a personal and unrestricted basis. In most parts of South East Asia at present, so-called 'personal' computers are not examples of personal technology in schools, despite the use of the term 'personal' to describe them. Their availability in schools is generally at the collective level, such as in a computer laboratory, rather than the individual level. In such situations, their use is controlled by the teacher, rather than by the pupil. They are not permitted for use in examinations, particularly important examinations external to schools. Although some pupils have individual access to a personal computer at home, many others do not. They are still too expensive for any curriculum authority to produce curricula based on the assumption of individual ownership. The situation is similar in Australia, where the mathematics curriculum and common teaching practices of schools have been only slightly affected, if at all, by the increased availability of personal computers in schools.

In contrast, there have been significant changes in school mathematics in a fairly short period of time as a consequence of the availability of personal technology. In some Australian states, students are permitted – in fact, expected – to use a graphics calculator in important external examinations. A consequence of this is that the classroom experience is affected, with students needing to learn how and when and why to use a graphics calculator to help them to think about or to do mathematics.

At present, and for the last twelve years or so, the most mathematically powerful examples of personal technology are graphics calculators. These come in various forms, but at least three are distinguishable when thinking about algebra. The most powerful are those containing Computer Algebra Systems (CAS), such as Texas Instruments' TI-89 and TI-92, Hewlett Packard's HP-48 and Casio's cfx-9970. Given their capabilities, including extensive symbolic manipulation in algebra and calculus, CAS calculators raise a number of significant issues for algebra teaching and learning, some of which are discussed in Kissane, Bradley & Kemp (1997). There is not space in this paper to deal with these issues, although it ought be noted that these sorts of calculators can readily perform all of the symbolic manipulation expected of secondary school mathematics students. This observation alone suggests that such technologies are worth a closer look.

The next set, which I call 'high-end' graphics calculators, are designed to accommodate the needs of students in the later years of schooling and the early undergraduate years. All four manufacturers, Casio, Hewlett Packard, Sharp and Texas Instruments make good examples of these, which are deservedly becoming quite popular in many senior secondary schools in Australia. Algebraically speaking, they are distinguished by having various automated equation solving capabilities and a wider range of function representations (rectangular, parametric, polar and recursive) than the third set of graphics calculators, described below. They are probably the most popular graphics calculators, since they span a wide range of uses over the spectrum of secondary and lower undergraduate education. Students who acquire a graphics calculator of this kind in the secondary school will still find it of use some years later in the early undergraduate years.

The third set might be described as 'low-end' graphics calculators. They appear to have been produced mainly with middle school pupils in mind and are manufactured by Casio, Sharp and Texas Instruments. Importantly for the notion of personal technology, they are very much less expensive than typical high-end calculators, usually somewhere between half and a third of the price. Indeed, the price of this sort of technology is now comparable with that of scientific calculators of the kind that have been routinely purchased by many, if not most, Australian secondary school students for almost two decades. While still not cheap, low-end graphics calculators are comparable in price with other adolescent purchases in affluent countries, such as a pair of shoes or two or three modern CD's, and thus are already affordable to the great majority of Australian families. Many schools have found that the most affordable way of introducing graphics calculators into their curriculum is through the purchase of a class set, so that each student (or pair of students) in a class has regular access to a calculator. In fact, the majority of Australian students do not require a much more powerful calculator than a low-end graphics calculator to meet their mathematical needs.

There is an urgent need to reconsider the secondary school algebra curriculum in the light of what technology is potentially available, either through ownership or long-term personal loan, to every single pupil. For at least the next few years, it seems likely that only graphics calculators will fit this description. Curriculum development in schools and school systems throughout South East Asia ought to be informed about the relationships between this kind of technology and the mathematics curriculum.

Algebra

Most secondary school students have encountered some algebra in school. Evidence from many sources, over many years, from the anecdotal to the more carefully researched, suggests that the meeting has often been characterised by limited success and even dread (on the part of pupils and teachers alike). The algebra offered by schools, until very recently, appears to most students to have been preoccupied with routines for symbolic manipulation, of dubious utility and devoid of much meaning beyond the confines of the mathematics classroom. These routines have included the 'collection of like terms', 'expanding', 'simplifying', factorising expressions and solving (a remarkably small repertoire of) equations. Even in 1999, I have no doubt that many students interpret algebra in such a procedural way. Although we have managed to produce a small subset of pupils with technical competence at such manipulations, very few of these have gained much insight into what algebra is (and isn't), what it is for or why it is important. For most students, much of the time, algebra mainly comprises a collection of symbolic manipulation procedures, rather than also including a richly intertwined collection of concepts and strategies. The noted mathematics educator, Robert Davis, was less than complimentary about such an emphasis:

> At one extreme, we have the most familiar type of course, where the student is asked to master rituals for manipulating symbols written on paper. The topics in such a course have names like “removing parentheses,” “changing signs,” “collecting like terms,” “simplifying,” and so on. It should be immediately clear that a course of this type, focussing mainly on meaningless notation, would be entirely inappropriate for elementary school children; many of us would argue that this type of course, although exceedingly common, is in fact inappropriate for all students. (1989, p 268)

A recent attempt to try to inject meaning into the algebra curriculum and to focus more carefully on the important ideas of algebra was provided by Lowe et al (1993-4). In part, this work was informed by the seminal work in both *A National Statement on Mathematics for Australian Schools* and the *National Mathematics Profile*, which identified three broad dimensions of algebra and indicated how these might develop over the early years of the algebra curriculum. The three 'substrands' of the 'Algebra' strand were labelled 'Expressing generality', 'Functions' and 'Equations'. Later refinements of these documents built upon the same structure, identifying for example that the study of functions involves both relationships and graphs, and that inequalities and equations ought to be considered together.

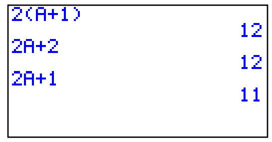
Connections

The essential connection between personal technologies such as graphics calculators and the algebra curriculum is that the technology provides fresh opportunities for pupils to learn about algebra. The key to these is the capacity of the calculator to enable *exploration* of key concepts – related to the metaphor of the calculator as a laboratory (Kissane, 1995). Space precludes an exhaustive listing of the kinds of explorations made possible, many of which are contained in publications such as Kissane (1997). However, a few of the connections are described briefly below.

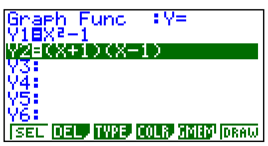
Expressing generality

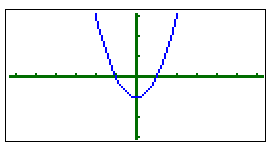
Graphics calculators use the standard conventions of algebraic representation; for example, $2AB^2$

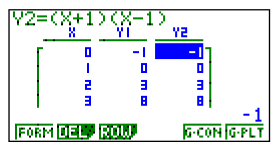
calculator with alphabetic memories storing numbers seems likely to help students come to terms with the notion of variables as place holders. In the same way, $2(A + 1)$ and $2A + 2$ will give the same numerical value on a calculator, regardless of the value of A, while $2(A + 1)$ and $2A + 1$ will (usually) give different values.Some of these characteristics are shown in the screens below, in which A has been given the value 5 and B the value 7. (These and other screens in this paper were produced on a Casio cfx-9850G calculator, a good example of a 'high-end' graphics calculator.)



Equivalence transformations such as factorising, collecting like terms and expanding can be represented either numerically and graphically to enhance meaning. For example, graphs of $y = x^2 - 1$ and of $y = (x + 1)(x - 1)$ are identical, as are their associated numerical tables of values. The critical concept of algebraic identity is representable both numerically and graphically; in contrast, student work with identities has often been restricted to the symbolic in the past. The following screens show some of these kinds of connections.



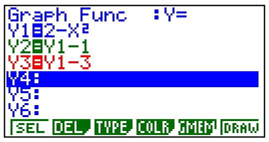


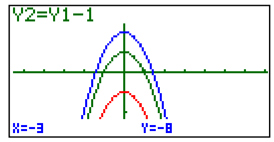


Functions

Aspects of the study of functions are also positively affected by the capabilities of graphics calculators to represent relationships symbolically, numerically and graphically – the so-called 'rule of three'. Explorations of functions represented graphically or numerically (in tables) are easily undertaken by pupils on graphics calculators with minimal prior experience. While movements among representations were available before personal technologies like this were invented, they were frequently hindered by the time and error-prone complexity of producing them (by hand graphing or numerical substitution, or both). Thus, students can now readily produce the family of graphs shown below and focus on how and why the graphs differ.

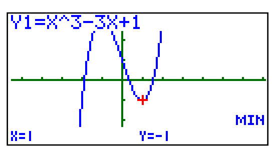
As well as movement among representations, personal technology permits pupils to readily explore families of functions and thus the crucial notion of transformation. The screens below show an example of a graphics calculator used as a function transformer (Kissane, 1997), to permit students to see the effects of transforming, in this case by the subtraction of a constant.





A graphics calculator allows the algebra curriculum to focus on classes of relationships of obvious importance that were traditionally neglected until later algebra study (such as exponential functions). Although exponential functions are very important because of their usefulness to model growth situations, they are usually not dealt with in introductory algebra courses, because students find it difficult to deal with them. However, using a graphics calculator, exponential functions are no more difficult to graph or to tabulate than are quadratic functions.

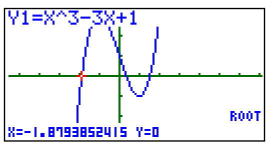
Automatic graphical exploration capabilities of calculators allow students to deal numerically with questions which were previously not accessible until the calculus had been studied. For example, the screens below show how a calculator can locate a relative minimum point of a function graphically.

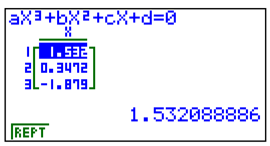


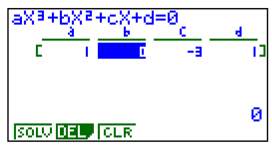
With a graphics calculator available, students might be expected to learn to *use* graphs of functions rather than merely to draw them; we might even expect that students will decide for themselves when and why a graph would be appropriate, rather than relying on teachers and textbooks to tell them. The difference is of considerable practical importance to the algebra curriculum.

Equations and inequalities

Elementary equations and inequalities can be explored profitably by pupils making use of calculator capabilities for graphing and numerical tabulation. Unlike the conventional equation solving algorithms using symbolic manipulation, such explorations are not restricted to the linear and the quadratic. Armed with graphics calculators, students might be expected to explore in new ways relationships between functions, equations and graphs and to develop a repertoire of ways of dealing with equations and inequalities, rather than the 'one best way' characteristic of the past. (See Kissane (1995) for an extended example of this.) It would be reasonable to expect that pupils will be able to solve a particular equation in several ways, and will develop the acumen to choose the most appropriate method for a particular circumstance. To give an elementary example, the screens below show two ways in which (approximate) solutions of the cubic equation $x^3 - 3x = 1$ can be obtained on a Casio cfx-9850G.



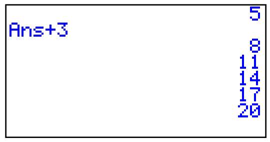




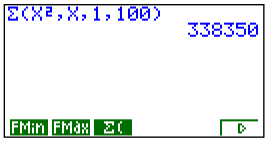
It is clear that the significance of factorising quadratic expressions and of the quadratic formula is altered by the availability of technology of these kinds.

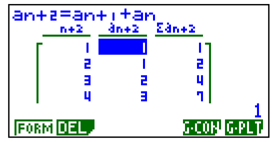
Advanced algebra

Some aspects of the study of elementary algebra have traditionally not appeared until relatively late in students' education. However, access to technology may make them accessible much earlier. An example concerns dealing with sequences and series. Even the least sophisticated graphics calculators allow for recursively defined sequences to be generated easily. The calculator screen below shows successive terms of an arithmetic progression with common difference 3. Each press of the "Execute" key has the effect of generating the next term. This process makes it relatively easy for students to find directly a certain term or to find which term has a certain value.

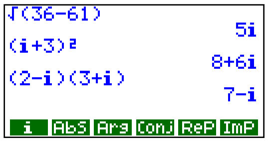


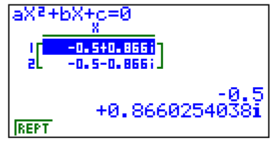
More sophisticated calculators such as the Casio cfx-9850G allow for series to be numerically evaluated and even for Fibonacci series to be studied, as shown in the next pair of screens.





To give another example, complex numbers are dealt with routinely by the Casio cfx-9850G, as shown below.





Conclusion

Connections of the kinds illustrated above between elementary algebra and graphics calculators deserve attention in secondary school curriculum development, even in countries in which adequate access to technology is not yet available. Although computers are much more powerful forms of technology, graphics calculators offer powerful new ways of dealing with the problems traditionally addressed by secondary school algebra. It seems important to re-evaluate the significance of different aspects of school algebra, particularly the focus on symbolic manipulation. It might be argued, for example, that we concentrate more attention than in the past on *expressing* relationships algebraically rather than on manipulating the expressions themselves; similarly, we may focus more attention on *formulating* equations and *interpreting* solutions, rather than on the algebraic manipulations required to solve equations. The development of the graphics calculator demands that we take a fresh look at the algebra curriculum, how it is taught and how it is learned, under an assumption of continuing and self-directed personal access to technology. Both Lowe et al (1993-4) and Kissane (1997) offer many specific examples of this kind of thinking.

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